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ORIGINAL ARTICLE

Controllability of periodic linear systems, the Poincaré sphere, and quasi-affine systems

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Abstract

For periodic linear control systems with bounded control range, an autonomized system is introduced by adding the phase to the state of the system. Here, a unique control set (i.e., a maximal set of approximate controllability) with nonvoid interior exists. It is determined by the spectral subspaces of the homogeneous part which is a periodic linear differential equation. Using the Poincaré sphere, one obtains a compactification of the state space allowing us to describe the behavior "near infinity" of the original control system. Furthermore, an application to quasi-affine systems yields a unique control set with nonvoid interior.

Keywords Periodic linear control system · Quasi-affine control system · Control set · Floquet theory

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1 Introduction

We study controllability properties for periodic linear control systems and give an application to quasi-affine control systems. Periodic linear control systems have the form

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad u(t) \in U, \tag{1}$$

where $A \in L^\infty(\mathbb{R}, \mathbb{R}^{d\times d})$ and $B \in L^\infty(\mathbb{R}, \mathbb{R}^{d\times m})$ are T-periodic for some $T > 0$. We suppose that the controls $u = (u_1, \ldots, u_m)$ have values in a bounded convex neighborhood U of the origin in $\mathbb{R}^m$. The set of admissible controls is

$$\mathcal{U} = \{u \in L^\infty(\mathbb{R}, \mathbb{R}^m) \,|u(t) \in U \text{ for almost all } t\}.$$

We denote the solutions (in the Carathéodory sense) of (1) with initial condition $x(t_0) = x_0$ by $\varphi(t; t_0, x_0, u),\ t \in \mathbb{R}$. The homogeneous part of (1) is the (uncontrolled) homogeneous periodic differential equation

$$\dot{x}(t) = A(t)x(t). \tag{2}$$

Nonautonomous control systems can be autonomized by including time in the state of the system. This is useful, if recurrence properties can be exploited; cf. Johnson and Nerurkar [16]. In the T-periodic case, it suffices to add the phases $\tau \in [0, T)$ to the states in $\mathbb{R}^d$ (cf. Gayer [15] for general periodic nonlinear systems) and we follow this approach. We give a spectral characterization of the reachable sets generalizing Sontag [24, Corollary 3.6.7] for autonomous linear control systems; cf. also [24, p. 139] for some historical remarks. The proof also uses arguments from Colonius, Cossich, and Santana [9, Theorem 15] for autonomous discrete-time systems. This yields a characterization of the unique control set (i.e., a maximal set of approximate controllability) with nonvoid interior. The Poincaré sphere from the global theory of nonlinear differential equations (introduced by Poincaré [23] for polynomial differential equations) provides a compactification of the state space; cf. the monograph Perko [22, Section 3.10] and, e.g., Valls [26] for a recent contribution. This leads us to a description of the behavior "near infinity" of the original control system. Since the induced system on the Poincaré sphere is obtained by projection of a homogeneous system it suffices to consider its restriction to the upper hemisphere. Alternatively one might consider the induced system on projective space. In [11], we have used the latter approach for autonomous affine control systems. We remark that Da Silva [12] has generalized [24, Corollary 3.6.7] in another direction, for linear control systems on solvable Lie groups. General background on control of periodic linear systems is contained in Bittanti and Colaneri [2].

The present paper may also be considered as a contribution to a Floquet theory of periodic control systems. They involve two T-periodic matrix functions $A(\cdot)$ and $B(\cdot)$ and a periodic coordinate change can transform only one of them to a constant matrix, hence periodic linear systems cannot be conjugated to autonomous linear systems. But the formulation of Floquet theory in the framework of linear skew product flows can be generalized (cf., e.g., Colonius and Kliemann [8, Chapter 7], and Kloeden and

Rasmussen [19] for the general theory of skew product flows). The spectral subspaces (the stable, center, and unstable subspaces) of (2) depending on the phase $\tau \in [0, T)$ characterize controllability properties.

In the last part of this paper, we introduce quasi-affine control systems which have the form

$$\dot{x}(t) = A(v(t))x(t) + B(v(t))u(t), \tag{3}$$

with $A(v) := A_0 + \sum_{i=1}^{p} v_i A_i$ for $v \in V \subset \mathbb{R}^p$, where $A_0, A_1, \ldots, A_p \in \mathbb{R}^{d\times d}$, and $B : V \to \mathbb{R}^{d\times m}$ is continuous. The controls (u, v) have values in a compact convex neighborhood $U \times V \subset \mathbb{R}^m \times \mathbb{R}^p$ of $(0, 0)$, and the set of admissible controls is

$$\mathcal{U} \times \mathcal{V} = \{(u, v) \in L^\infty(\mathbb{R}, \mathbb{R}^m) \times L^\infty(\mathbb{R}, \mathbb{R}^p) \,| u(t) \in U \text{ and } v(t) \in V \text{ for a.a. } t\}.$$

Quasi-affine systems look similar to linear control systems, but the coefficient matrices in front of x and u may depend on the additional controls v. If a periodic $v \in \mathcal{V}$ is fixed, one obtains a periodic linear control system with controls u. We use this relation to prove results for control sets of quasi-affine systems. A special case is affine control systems with separated additive and multiplicative control terms,

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{p} v_i(t) A_i x(t) + Bu(t). \tag{4}$$

Controllability properties of affine systems are a classical topic in control theory. We only refer to the monographs Mohler [21], Elliott [14], and Jurdjevic [17]. Our recent paper [11] proves results on control sets of general affine systems; cf. also [10] for control sets about equilibria.

The contents of this paper are the following. After preliminaries in Sect. 2 on T-periodic linear control systems, Sect. 3 introduces the autonomized control system with state space $\mathbb{S}^1 \times \mathbb{R}^d$, where the unit circle $\mathbb{S}^1$ is parametrized by $\tau \in [0, T)$. In Sect. 4, Theorem 11 characterizes the reachable and controllable subsets using the spectral subbundles of the periodic differential equation (2). Theorem 13 shows that a unique control set $D^a \subset \mathbb{S}^1 \times \mathbb{R}^d$ with nonvoid interior exists. It is unbounded if the center subbundle is nontrivial. Section 5 projects the control system to the open upper hemisphere $\mathbb{S}^{d,+}$ of the Poincaré sphere. Together with the equator $\mathbb{S}^{d,0}$, this constitutes a compactification where the behavior "near infinity" is mapped onto the behavior near the equator. The control set D^a on $\mathbb{S}^1 \times \mathbb{R}^d$ projects onto the control set D_P^a on $\mathbb{S}^1 \times \mathbb{S}^{d,+}$ and the intersection of $\overline{\operatorname{int} D_P^a}$ with $\mathbb{S}^1 \times \mathbb{S}^{d,0}$ is determined by the image of the center subbundle of (2). These results are also new for autonomous linear control systems. Section 6 presents some low-dimensional examples, and, finally, Sect. 7 introduces quasi-affine systems. Theorem 17 characterizes their unique control set with nonvoid interior using the control sets of the periodic linear control systems for fixed periodic $v \in \mathcal{V}$.

Notation: For a matrix $A \in \mathbb{R}^{d\times d}$, the set of eigenvalues is denoted by $\operatorname{spec}(A)$ and for $\mu \in \operatorname{spec}(A)$ the generalized eigenspace is $\ker(\mu I - A)^k$, where k is the dimension of the largest Jordan block for μ. By $GE(A, \mu)$, we denote the real generalized eigenspace of μ, which is the real part of the generalized eigenspace. The interior of a set M in a metric space is $\mathrm{int}M$ and $\mathbb{N} = \{0, 1, 2, \ldots\}$.

2 Preliminaries

In this section, we introduce some notation and discuss consequences of the T-periodicity property. In particular, we recall a result on controllability for periodic systems without control restrictions.

The principal fundamental solution $X(t, s) \in \mathbb{R}^{d\times d}$, $s, t \in \mathbb{R}$, of the homogeneous equation (2) is the matrix solution of

$$\frac{d}{\mathrm{d}t} X(t, s) = A(t)X(t, s) \text{ with } X(s, s) = I_d.$$

Here, $X(t, r)X(r, s) = X(t, s)$, $t, r, s \in \mathbb{R}$, and by T-periodicity $X(t+kT, s+kT) = X(t, s)$ for all $k \in \mathbb{Z}$. The variation-of-parameters formula for the solutions of (1) yields

$$\varphi(t; t_0, x, u) = X(t, t_0)x + \int_{t_0}^{t} X(t, s)B(s)u(s)\mathrm{d}s. \tag{5}$$

Denote for $x \in \mathbb{R}^d$ the reachable set for $t \geq t_0$ and the controllable set for $t \leq t_0$ of (1) by

$$\begin{aligned} \mathbf{R}_t(t_0, x) &= \{\varphi(t; t_0, x, u) \,|u \in \mathcal{U}\}, \\ \mathbf{C}_t(t_0, x) &= \{y \in \mathbb{R}^d \,|\exists u \in \mathcal{U} : \varphi(t_0; t, y, u) = x\}, \end{aligned} \tag{6}$$

resp., and let the reachable set and the controllable set be

$$\mathbf{R}(t_0, x) := \bigcup\nolimits_{t\geq t_0} \mathbf{R}_t(t_0, x) \text{ and } \mathbf{C}(t_0, x) := \bigcup\nolimits_{t\leq t_0} \mathbf{C}_t(t_0, x).$$

Lemma 1 *Let $x \in \mathbb{R}^d$, $t \geq t_0$, and $k \in \mathbb{N}$.*
(i) The reachable sets are convex and satisfy $\mathbf{R}_t(t_0, x) = \mathbf{R}_{t+kT}(t_0 + kT, x)$.
(ii) The reachable sets $\mathbf{R}_{kT+t_0}(t_0, 0)$ *are increasing with* $k \in \mathbb{N}$.

Proof Convexity of $\mathbf{R}_t(t_0, x)$ holds since the control range U is convex. The equality in (i) follows from

$$\begin{aligned} \varphi(t + kT; t_0 + kT, x, u) &= X(t + kT, t_0 + kT)x + \int_{t_0+kT}^{t+kT} X(t + kT, s)B(s)u(s)\mathrm{d}s \\ &= X(t, t_0)x + \int_{t_0}^{t} X(t + kT, s + kT)B(s + kT)u(s + kT)\mathrm{d}s \\ &= X(t, t_0)x + \int_{t_0}^{t} X(t, s)B(s)u(s + kT)\mathrm{d}s \\ &= \varphi(t; t_0, x, u(\cdot + kT)), \end{aligned}$$

where $u(\cdot + kT)(s) := u(s + kT)$, $s \in \mathbb{R}$.

For assertion (ii), let $k \geq \ell$ and consider $x \in \mathbf{R}_{\ell T+t_0}(t_0, 0)$ with

$$x = \varphi(\ell T + t_0; t_0, 0, u) = \int_{t_0}^{\ell T+t_0} X(\ell T + t_0, s)B(s)u(s)\mathrm{d}s.$$

Define

$$v(t) = \begin{cases} 0 & \text{for } t \in [t_0, (k-\ell)T + t_0) \\ u(t - (k-\ell)T) & \text{for } t \in [(k-\ell)T + t_0, kT + t_0] \end{cases}.$$

Then, one obtains

$$\begin{aligned} &\varphi(kT + t_0; t_0, 0, v) = \int_{t_0}^{kT+t_0} X(kT + t_0, s)B(s)v(s)\mathrm{d}s \\ &= \int_{t_0}^{(k-\ell)T+t_0} X(kT + t_0, s)B(s)v(s)\mathrm{d}s + \int_{(k-\ell)T+t_0}^{kT+t_0} X(kT + t_0, s)B(s)v(s)\mathrm{d}s \\ &= 0 + \int_{t_0}^{\ell T+t_0} X(kT + t_0, s + (k-\ell)T)B(s + (k-\ell)T)v(s + (k-\ell)T)\mathrm{d}s \\ &= \int_{t_0}^{\ell T+t_0} X(\ell T + t_0, s)B(s)u(s)\mathrm{d}s = x, \end{aligned}$$

hence $\mathbf{R}_{\ell T+t_0}(t_0, 0) \subset \mathbf{R}_{kT+t_0}(t_0, 0)$. □

In order to clarify the relationship between the reachable and the controllable sets of the considered nonautonomous control systems, it is convenient to introduce the following time-reversed systems (cf. Sontag [24, Definition 2.6.7 and Lemma 2.6.8]). The reversal of (1) at $\mu \in \mathbb{R}$ is

$$\dot{y}(t) = -A(\mu - t)y(t) - B(\mu - t)u(t), \quad u \in \mathcal{U}, \tag{7}$$

with trajectories denoted by $\varphi_\mu^-(t; t_0, x_0, u),\ t \in \mathbb{R}$.

Lemma 2 *For $t_1 < t_0$, the controllable set $\mathbf{C}_{t_1}(t_0, x)$ of (1) coincides with the reachable set $\mathbf{R}_{t_0}^-(t_1, x)$ of the time-reversed system (7) at $\mu = t_0 + t_1$.*

Proof Let $y \in \mathbf{C}_{t_1}(t_0, x)$ with $\varphi(t_0; t_1, y, u) = x$. Define the control $u^-(t) := u(t_0 + t_1 - t)$, $t \in \mathbb{R}$. The function $y(t) := \varphi(t_0 + t_1 - t; t_1, y, u)$, $t \in [t_1, t_0]$, satisfies the differential equation

$$\dot{y}(t) = -A(t_0 + t_1 - t)y(t) - B(t_0 + t_1 - t)u(t_0 + t_1 - t),$$

and $y(t_0) = y$ and $y(t_1) = x$. Thus, $y(t) = \varphi_{t_0+t_1}^-(t; t_1, x, u^-)$, $t \in [t_1, t_0]$, and the assertion follows. □

Since $\mathbf{C}_{-kT+t_0}(t_0, x) = \mathbf{R}^-_{t_0}(-kT + t_0, x) = \mathbf{R}^-_{kT+t_0}(t_0, x)$ Lemma 1 implies that also the controllable sets are convex and for $k \geq \ell$ in $\mathbb{N}$ the inclusion $\mathbf{C}_{-\ell T+t_0}(t_0, 0) \subset \mathbf{C}_{-kT+t_0}(t_0, 0)$ holds.

Proposition 3 *For $\tau \in [0, T]$ consider $x \in \mathbf{R}_{kT+\tau}(\tau, 0)$ and $y \in \mathbf{R}_{\ell T+\tau}(\tau, 0)$ where $k, \ell \in \mathbb{N}$. Then, it follows that*

$$x + X(kT + \tau, \tau)y = x + X(T + \tau, \tau)^k y \in \mathbf{R}_{(k+\ell)T+\tau}(\tau, 0). \tag{8}$$

Proof There are $u, v \in \mathcal{U}$ with

$$\begin{aligned} x &= \varphi(kT + \tau; \tau, 0, u) = \int_\tau^{kT+\tau} X(kT + \tau, s)B(s)u(s)\mathrm{d}s, \\ y &= \varphi(\ell T + \tau; \tau, 0, v) = \int_\tau^{\ell T+\tau} X(\ell T + \tau, s)B(s)v(s)\mathrm{d}s. \end{aligned}$$

Define

$$w(t) = \begin{cases} v(t) & \text{for } t \in [\tau, \ell T + \tau] \\ u(t - \ell T) & \text{for } t \in (\ell T + \tau, (k + \ell)T + \tau] \end{cases}.$$

Then, one computes

$$\begin{aligned} \varphi((k + \ell)T + \tau; \tau, 0, w) &= \int_\tau^{(k+\ell)T+\tau} X((k + \ell)T + \tau, s)B(s)w(s)\mathrm{d}s \\ &= \int_\tau^{\ell T+\tau} X((k + \ell)T + \tau, s)B(s)v(s)\mathrm{d}s \\ &\quad + \int_{\ell T+\tau}^{(k+\ell)T+\tau} X((k + \ell)T + \tau, s)B(s)u(s - \ell T)\mathrm{d}s \\ &= \int_\tau^{\ell T+\tau} X((k + \ell)T + \tau, \ell T + \tau)X(\ell T + \tau, s)B(s)v(s)\mathrm{d}s \\ &\quad + \int_\tau^{kT+\tau} X((k + \ell)T + \tau, \ell T + s)B(\ell T + s)u(s)\mathrm{d}s \\ &= X(kT + \tau, \tau) \int_\tau^{\ell T+\tau} X(\ell T + \tau, s)B(s)v(s)\mathrm{d}s \\ &+ \int_\tau^{kT+\tau} X(kT + \tau, s)B(s)u(s)\mathrm{d}s \\ &= X(kT + \tau, \tau)y + x. \end{aligned}$$

Thus, (8) holds. □

Controllability criteria for periodic linear systems without control constraints are well known. The following theorem is due to Brunovsky [5], slightly reformulated.

Theorem 4 *For the periodic linear system in (1) without control restrictions, the following properties are equivalent:*

(i) For any two points $x_1, x_2 \in \mathbb{R}^d$ and any $t_0 \in \mathbb{R}$, there are $t_1 > t_0$ and $u \in L^\infty([t_0, t_1], \mathbb{R}^m)$ such that $\varphi(t_1; t_0, x_1, u) = x_2$.

(ii) For any two points $x_1, x_2 \in \mathbb{R}^d$ and $\tau \in [0, T]$, there is $u \in L^\infty([\tau, dT + \tau], \mathbb{R}^m)$ such that $\varphi(dT + \tau; \tau, x_1, u) = x_2$.

(iii) The rows of the matrix function $X(t, 0)^{-1}B(t)$, $t \in [0, dT]$, are linearly independent.

If any of the equivalent conditions above is satisfied, the system in (1) without control restrictions is called controllable.

Proof Brunovsky [5, Proposition 3] shows that conditions (i) and (iii) are equivalent. By T-periodicity, condition (ii) implies (i). Conversely, cf. the proof of [5, Proposition 3], condition (i) implies (ii) for $\tau = 0$. If (i) holds for the system with matrix functions $A(t)$ and $B(t)$, $t \in \mathbb{R}$, it also holds for the system with $A(\tau + t)$ and $B(\tau + t)$, $t \in \mathbb{R}$, with $\tau \in [0, T]$. Hence, condition (ii) follows for all $\tau \in [0, T]$. □

Remark 1 Condition (iii) above generalizes the Kalman criterion for controllability of autonomous systems. It is equivalent to assignability of the spectrum by T-periodic state feedbacks, [5, Theorem on p. 302]. As shown in Bittanti, Guarbadassi, Mafezzoni, and Silverman [3] and Bittanti, Colaneri, and Guarbadassi [4], a criterion generalizing the Hautus–Popov spectral characterization for controllability is only equivalent to null-controllability.

Theorem 4 implies the following first result on controllability properties of the system with control restrictions.

Proposition 5 *Assume that the periodic linear system in (1) without control restrictions is controllable, and let $\tau \in [0, T]$. Then for system (1) with controls $u \in \mathcal{U}$, the reachable set $\mathbf{R}_{dT+\tau}(\tau, 0)$ and the controllable set $\mathbf{C}_{-dT+\tau}(\tau, 0)$ of (1) are convex and contain an ε-ball $\mathbf{B}(0; \varepsilon)$ with $\varepsilon > 0$ around $0 \in \mathbb{R}^d$. The sets*

$$\mathbf{R}_{\mathbb{N}T+\tau}(\tau, 0) := \bigcup_{k\in\mathbb{N}} \mathbf{R}_{kT+\tau}(\tau, 0) \text{ and } \mathbf{C}_{-\mathbb{N}T+\tau}(\tau, 0) := \bigcup_{k\in\mathbb{N}} \mathbf{C}_{-kT+\tau}(\tau, 0),\ \tau \in [0, T],$$

are convex and open. Furthermore also the sets

$$\mathbf{R}_{\mathbb{N}T+\tau}(0, 0),\ \mathrm{int}\mathbf{R}_{\mathbb{N}T+\tau}(0, 0),\ \mathbf{C}_{-\mathbb{N}T+\tau}(0, 0),\ \text{and } \mathrm{int}\mathbf{C}_{-\mathbb{N}T+\tau}(0, 0)$$

are convex, and

$$\mathbf{R}_{\mathbb{N}T+\tau}(\tau, 0) \subset \mathrm{int}\mathbf{R}_{\mathbb{N}T+\tau}(0, 0) \text{ and } \mathbf{C}_{-\mathbb{N}T+\tau}(\tau, 0) \subset \mathrm{int}\mathbf{C}_{-\mathbb{N}T+\tau}(0, 0).$$

Proof Convexity of $\mathbf{R}_{\mathbb{N}T+\tau}(\tau, 0)$ follows from Lemma 1. Fix a basis $y_1, \ldots, y_d$ of $\mathbb{R}^d$. By Theorem 4 for every $\tau \in [0, T]$, there are $u_i^\tau \in L^\infty([0, (d+1)T], \mathbb{R}^m)$ with

$$y_i = \varphi(dT + \tau; \tau, 0, u_i^\tau) = \int_\tau^{dT+\tau} X(dT + \tau, s)B(s)u_i^\tau(s)\mathrm{d}s \text{ for } i = 1, \ldots, d,$$

and $\varphi(dT + \tau; \tau, 0, u)$ depends continuously on $(\tau, u) \in [0, T] \times L^\infty([0, (d+1)T], \mathbb{R}^m)$. Let $\varepsilon_0 > 0$ be small enough such that $z_1, \ldots, z_d$ form a basis of $\mathbb{R}^d$ for any $z_i \in \mathbf{B}(y_i; \varepsilon_0)$. By continuity, there is for every $\tau_0 \in [0, T]$ a $\delta_0 > 0$ such that $\varphi(dT + \tau; \tau, 0, u_i^{\tau_0}) \in \mathbf{B}(y_i; \varepsilon_0)$ for $|\tau - \tau_0| < \delta_0$. Now compactness of $[0, T]$ shows that there are finitely many $\tau_j \in [0, T]$ such that $\varphi(dT + \tau; \tau, 0, u_i^{\tau_j}), i = 1, \ldots, d$, form a basis of $\mathbb{R}^d$. By linearity, there is $\alpha > 0$ such that also $\varphi(dT + \tau; \tau, 0, \alpha u_i^{\tau_j})$ form a basis of $\mathbb{R}^d$ and $\alpha u_i^{\tau_j} \in \mathcal{U}$ for all i, j. This shows that there is a ball $\mathbf{B}(0; \varepsilon)$ contained in $\mathbf{R}_{\mathrm{d}T+\tau}(\tau, 0) \subset \mathbf{R}_{\mathbb{N}T+\tau}(\tau, 0)$ for all $\tau \in [0, T]$.

Let $x = \varphi(kT + \tau; \tau, 0, u) \in \mathbf{R}_{\mathbb{N}T+\tau}(\tau, 0)$ for some $k \in \mathbb{N}$ and $u \in \mathcal{U}$. The set $X(kT + \tau, \tau)\mathbf{B}(0; \varepsilon)$ is open and Proposition 3 implies that for each $y \in \mathbf{B}(0; \varepsilon)$

$$x + X(kT + \tau, \tau)y \in \mathbf{R}_{(k+d)T+\tau}(\tau, 0) \subset \mathbf{R}_{\mathbb{N}T+\tau}(\tau, 0).$$

This shows that $\mathbf{R}_{\mathbb{N}T+\tau}(\tau, 0)$ is open. The control

$$v(t) := \begin{cases} 0 & \text{for } t \in [0, \tau) \\ u(t) & \text{for } t \in [\tau, kT + \tau] \end{cases}$$

yields $x = \varphi(kT + \tau; \tau, 0, u) = \varphi(kT + \tau; 0, 0, v) \in \mathbf{R}_{\mathbb{N}T+\tau}(0, 0)$, hence the inclusion $\mathbf{R}_{\mathbb{N}T+\tau}(\tau, 0) \subset \mathrm{int}\mathbf{R}_{\mathbb{N}T+\tau}(0, 0)$ holds. For convexity of $\mathbf{R}_{\mathbb{N}T+\tau}(0, 0)$ let for $i = 1, 2$

$$y_i = \varphi(\tau; 0, x_i, u_i) \in \mathbf{R}_{\mathbb{N}T+\tau}(0, 0) \text{ with } x_i \in \mathbf{R}_{\mathbb{N}T}(0, 0), u_i \in \mathcal{U}.$$

Then, linearity implies for $\alpha \in [0, 1]$ that

$$\begin{aligned} \alpha y_1 + (1-\alpha)y_2 &= \alpha\varphi(\tau; 0, x_1, u_1) + (1-\alpha)\varphi(\tau; 0, x_2, u_2) \\ &= \varphi(\tau; 0, \alpha x_1 + (1-\alpha)x_2, \alpha u_1 + (1-\alpha)u_2) \in \mathbf{R}_{\mathbb{N}T+\tau}(0, 0). \end{aligned}$$

Since $\mathbf{R}_{\mathbb{N}T+\tau}(0, 0)$ is convex also $\mathrm{int}\mathbf{R}_{\mathbb{N}T+\tau}(0, 0)$ is convex; cf. Dunford and Schwartz [13, Theorem V.2.1].

The assertions for the controllable sets follow by time reversal from Lemma 2 and Lemma 1(i). □

3 The autonomized system

First some results from Floquet theory are recalled (cf. Chicone [6], Teschl [25], and Colonius and Kliemann [8, Section 7.2]). Then, we introduce the autonomized system.

Consider the unit circle $\mathbb{S}^1$ parametrized by $\tau \in [0, T)$ and define the shift

$$\theta : \mathbb{R} \times \mathbb{S}^1 \to \mathbb{S}^1,\ \theta(t; \tau) = t + \tau \mod T \text{ for } t \in \mathbb{R}, \tau \in \mathbb{S}^1.$$

Here, $\tau + t \mod T$ denotes the unique element $\tau + t - kT \in [0, T)$ for some $k \in \mathbb{Z}$. Let $\psi(t; \tau_0, x_0)$ be the solution of (2) with initial condition $x(\tau_0) = x_0$ and define

$$\Psi = (\theta, \psi) : \mathbb{R} \times \mathbb{S}^1 \times \mathbb{R}^d \to \mathbb{S}^1 \times \mathbb{R}^d,\ \Psi(t; \tau_0, x_0) = (\theta(t; \tau_0), \psi(t; \tau_0, x_0)). \quad (9)$$

Then, Ψ is a continuous dynamical system. The first component does not depend on the second component and $\Psi(t; \tau_0, x_0)$ is linear in the argument x_0, hence Ψ is a linear skew product flow; cf. Kloeden and Rasmussen [19, Section 2.2].

The Floquet multipliers of equation (2) are the eigenvalues μ of

$$X(T + \tau, \tau) = X(T + \tau, T)X(T, 0)X(0, \tau) = X(\tau, 0)X(T, 0)X(\tau, 0)^{-1},\ \tau \in [0, T).$$

The Floquet exponents are $\lambda_j := \frac{1}{T} \log |\mu|$ (the Floquet exponents as defined here are the real parts of the Floquet exponents defined in [6] and [25]). Note that $\lambda_j < 0$ if and only if $|\mu| < 1$. The following result is [8, Theorem 7.2.9].

Theorem 6 *Let $\Psi = (\theta, \psi) : \mathbb{R} \times \mathbb{S}^1 \times \mathbb{R}^d \longrightarrow \mathbb{S}^1 \times \mathbb{R}^d$ be the linear skew product flow associated with the T-periodic linear differential equation (2). For each $\tau \in \mathbb{S}^1$, there exists a decomposition*

$$\mathbb{R}^d = L(\lambda_1, \tau) \oplus \cdots \oplus L(\lambda_\ell, \tau)$$

into linear subspaces $L(\lambda_j, \tau)$, called the Floquet (or Lyapunov) spaces, with the following properties:

(i) The Floquet spaces have dimension $d_j := \dim L(\lambda_j, \tau)$ independent of $\tau \in \mathbb{S}^1$.

(ii) They are invariant under multiplication by the principal fundamental matrix in the following sense:

$$X(t + \tau, \tau)L(\lambda_j, \tau) = L(\lambda_j, \theta(t; \tau)) \text{ for all } t \in \mathbb{R} \text{ and } \tau \in \mathbb{S}^1.$$

(iii) For every $\tau \in \mathbb{S}^1$, the Floquet (or Lyapunov) exponents satisfy

$$\lambda(x, \tau) := \lim_{t \to \pm\infty} \frac{1}{t} \log \|\psi(t; \tau, x)\| = \lambda_j \text{ if and only if } 0 \neq x \in L(\lambda_j, \tau).$$

The Floquet space $L(\lambda_j, \tau)$ is the direct sum of the real generalized eigenspaces for all Floquet multipliers μ with $\frac{1}{T} \log |\mu| = \lambda_j$,

$$L(\lambda_j, \tau) = \bigoplus_\mu GE(X(T + \tau, \tau), \mu).$$

Define for $\tau \in [0, T)$ the stable, the center, and the unstable subspaces, resp., by

$$E_\tau^- = \bigoplus_{\lambda_j < 0} L(\lambda_j, \tau),\ E_\tau^0 := L(0, \tau),\ \text{and } E_\tau^+ := \bigoplus_{\lambda_j > 0} L(\lambda_j, \tau).$$

Then, $E_\tau^\pm = X(\tau, 0)E_0^\pm$ and $E_\tau^0 = X(\tau, 0)E_0^0$, and $\mathbb{S}^1 \times \mathbb{R}^d$ splits into the Whitney sum $\mathcal{E}^- \oplus \mathcal{E}^0 \oplus \mathcal{E}^+$ of the stable, the center, and the unstable subbundles

$$\mathcal{E}^\pm = \left\{(\tau, x) \in \mathbb{S}^1 \times \mathbb{R}^d \,\middle|\, x \in E_\tau^\pm\right\}, \ \mathcal{E}^0 = \left\{(\tau, x) \in \mathbb{S}^1 \times \mathbb{R}^d \,\middle|\, x \in E_\tau^0\right\}, \quad (10)$$

resp. We also introduce analogous subbundles for the center-stable subspaces and the center-unstable subspaces given by

$$E_\tau^{-,0} := \bigoplus\nolimits_{\lambda_j \le 0} L(\lambda_j, \tau) \text{ and } E_\tau^{+,0} = \bigoplus\nolimits_{\lambda_j \ge 0} L(\lambda_j, \tau), \ \tau \in [0, T), \text{ resp.}$$

Similarly as for periodic differential equations, it is convenient for linear periodic control systems of the form (1) to extend the state space by adding the phase $\tau \in [0, T)$ to the state in order to get an autonomous system. We obtain the following autonomized control system on $\mathbb{S}^1 \times \mathbb{R}^d$,

$$\dot{\tau}(t) = 1 \bmod T, \quad \dot{x}(t) = A(\tau(t))x(t) + B(\tau(t))u(t), \qquad u \in \mathcal{U}, \qquad (11)$$

with solutions

$$\varphi^a(t; (\tau_0, x_0), u) = (\tau_0 + t \bmod T, \varphi(\tau_0 + t; \tau_0, x_0, u)), \ t \in \mathbb{R}.$$

Observe that (11) is not a linear control system.

Remark 2 If the matrix functions $A(\cdot)$ and $B(\cdot)$ are merely measurable, the general existence theory of ordinary differential equations does not apply to equation (11). Nevertheless, the solutions are well defined.

Denote the reachable and controllable sets for $t \ge 0$ of (11) by

$$\begin{aligned}
&\mathbf{R}_t^a(\tau, x) = \left\{\varphi^a(t; (\tau, x), u) \,|u \in \mathcal{U}\right\}, \\
&\mathbf{C}_t^a(\tau, x) = \{(\sigma, y) \in \mathbb{S}^1 \times \mathbb{R}^d \,\big|\exists u \in \mathcal{U} : \varphi^a(t; (\sigma, y), u) = (\tau, x)\}, \\
&\mathbf{R}^a(\tau, x) = \bigcup\nolimits_{t\ge 0} \mathbf{R}_t^a(\tau, x) \text{ and } \mathbf{C}^a(\tau, x) = \bigcup\nolimits_{t\ge 0} \mathbf{C}_t^a(\tau, x),
\end{aligned}$$

resp. The time-reversed autonomous system is

$$\dot{\tau}(t) = -1 \bmod T, \quad \dot{y}(t) = -A(\tau(t))y(t) - B(\tau(t))u(t), \qquad u \in \mathcal{U}. \qquad (12)$$

The reachable sets $\mathbf{R}_t^{a,-}(\tau, x)$ of the time-reversed autonomized system (12) coincide with the controllable sets $\mathbf{C}_t^a(\tau, x)$ of system (11). Note the following relation to the reachable and controllable sets defined in (6) for the periodic system (1).

Lemma 7 *For $(\tau, x) \in \mathbb{S}^1 \times \mathbb{R}^d$ and $t \ge 0$ the following assertions hold:*

$$\begin{aligned}
&\mathbf{R}_t^a(\tau, x) = \{(\tau + t \bmod T, y) \,|y \in \mathbf{R}_{\tau+t}(\tau, x)\}, \\
&\mathbf{C}_t^a(\tau, x) = \{(\sigma, y) \in \mathbb{S}^1 \times \mathbb{R}^d \,|\sigma + t = \tau \bmod T \textit{ and } y \in \mathbf{C}_\sigma(\sigma + t, x)\},
\end{aligned}$$

$$\mathbf{R}^a(0,0) = \{(\tau,x) \in \mathbb{S}^1 \times \mathbb{R}^d \,|x \in \mathbf{R}_{\mathbb{N}\mathbb{T}+\tau}(0,0)\},$$
$$\mathbf{C}^a(0,0) = \{(\tau,x) \in \mathbb{S}^1 \times \mathbb{R}^d \,|x \in \mathbf{C}_\tau(kT,0), k \geq 1\}.$$

Proof By definition one has $(\sigma, y) = \varphi^a(t;(\tau,x),u) \in \mathbf{R}^a_t(\tau,x)$ if and only if $\sigma = \tau + t \bmod T$ and $\varphi(\tau+t;\tau,x,u) = y$. This shows that $y \in \mathbf{R}_{\tau+t}(\tau,x)$ and the first assertion follows. By definition $(\sigma, y) \in \mathbf{C}^a_t(\tau,x)$ means that $\sigma + t = \tau \bmod T$ and $\varphi(\sigma+t;\sigma,y,u) = x$ for some $u \in \mathcal{U}$, hence $y \in \mathbf{C}_\sigma(\sigma+t,x)$. Furthermore, one finds

$$\begin{aligned}
\mathbf{R}^a(0,0) &= \bigcup\nolimits_{t\geq 0}\{(t \bmod T, x)\,|x \in \mathbf{R}_t(0,0)\} \\
&= \{(\tau,x) \in \mathbb{S}^1 \times \mathbb{R}^d \,|x \in \mathbf{R}_{\mathbb{N}T+\tau}(0,0)\}, \\
\mathbf{C}^a(0,0) &= \bigcup\nolimits_{t\geq 0}\{(\tau,x) \in \mathbb{S}^1 \times \mathbb{R}^d \,|\tau + t = 0 \bmod T \text{ and } x \in \mathbf{C}_\tau(\tau+t,0)\}.
\end{aligned}$$

□

In particular, Lemma 7 shows that $\mathbf{R}^a(0,0) = \mathbb{S}^1 \times X$ for a subset $X \subset \mathbb{R}^d$ if and only if $X = \mathbf{R}_{\mathbb{N}\mathbb{T}+\tau}(0,0)$ for all $\tau \in \mathbb{S}^1$. The next lemma provides additional information about the reachable and controllable sets for $x = 0$ of the autonomized system.

Lemma 8 *If the periodic linear system in* (1) *without control restrictions is controllable, then*

$$\mathbb{S}^1 \times \{0\} \subset \operatorname{int}\mathbf{R}^a(\tau,0) \cap \operatorname{int}\mathbf{C}^a(\tau,0),\ \ \tau \in [0,T).$$

Proof For $\tau \in \mathbb{S}^1$, Lemma 7 implies

$$\begin{aligned}
\mathbf{R}^a(\tau,0) &= \bigcup_{t\geq 0}\mathbf{R}^a_t(\tau,0) = \bigcup_{t\geq 0}\{(\tau+t \bmod T, x)\,|x \in \mathbf{R}_{\tau+t}(\tau,0)\} \\
&= \{(\sigma,x) \in \mathbb{S}^1 \times \mathbb{R}^d \,|\sigma \in [\tau,T), x \in \mathbf{R}_{\mathbb{N}T+\sigma}(\tau,0)\} \\
&\quad \cup \{(\sigma,x) \in \mathbb{S}^1 \times \mathbb{R}^d \,|\sigma \in [0,\tau), x \in \mathbf{R}_{kT+\sigma}(\tau,0), k \geq 1\}.
\end{aligned}$$

By Proposition 5, the set $\mathbf{R}_{\mathrm{d}T+\tau}(\tau,0)$ contains an ε-ball $\mathbf{B}(0;\varepsilon)$ around 0 for some $\varepsilon > 0$. For $y \in \mathbf{B}(0;\varepsilon)$, there is some $u \in \mathcal{U}$ with $y = \varphi(dT+\tau;\tau,0,u)$. Define for $\sigma \in [0,T]$

$$v(t) := \begin{cases} u(t) & \text{for } t \in [\tau, dT+\tau) \\ 0 & \text{for } t \in [dT+\tau,(d+1)T+\sigma] \end{cases}.$$

With the invertible matrices $Y(\sigma) := X((d+1)T+\sigma, dT+\tau)$, it follows that

$$\varphi((d+1)T+\sigma;\tau,0,v) = X((d+1)T+\sigma, dT+\tau)y = Y(\sigma)y,$$

showing that $Y(\sigma)\mathbf{B}(0;\varepsilon) \subset \mathbf{R}_{(d+1)T+\sigma}(\tau,0)$. The matrices $Y(\sigma)$ and hence also their singular values $0 < \delta_1(\sigma) \leq \cdots \leq \delta_d(\sigma)$ depend continuously on $\sigma \in [0,T]$

(cf. Sontag [24, Corollary A.4.4]). In particular, the minimal singular values $\delta_1(\sigma)$ are bounded away from 0, since $[0, T]$ is compact. Now recall the geometric interpretation of the singular values of a matrix A (cf., e.g., Arnold [1, p. 118]): δ_i is the length of the ith principal axis of the ellipsoid $A(\mathbb{S}^{d-1})$ obtained as the image of the unit sphere $\mathbb{S}^{d-1}$ under the linear mapping A. It follows that there is a ball $\mathbf{B}(0;\varepsilon_0)$ with $\varepsilon_0 > 0$ contained in every set $Y(\sigma)\mathbf{B}(0;\varepsilon)$, $\sigma \in [0, T]$. This proves that $\mathbf{B}(0;\varepsilon_0)$ is contained in every set $\bigcup_{k\geq 1} \mathbf{R}_{kT+\sigma}(\tau, 0)$ and it follows that $\mathbb{S}^1 \times \{0\} \subset \mathrm{int}\mathbf{R}^a(\tau, 0)$.

The assertion for $\mathbf{C}^a(\tau, 0)$ follows by time reversal. □

4 Spectral characterization of reachable and controllable sets

In this section, we characterize the reachable and the controllable sets of the autonomized system (11) by the spectral bundles of the homogeneous part (2) introduced in Theorem 6.

We start with the following technical lemma.

Lemma 9 *Let $\delta > 0$ and $\mu \in \mathbb{C}$ with $|\mu| \geq 1$. Then, there are $n_k \to \infty$ and $a_{n_k} \in \mathbb{C}$ with $\left|a_{n_k}\right| < \delta$ such that $\mu^{n_k} a_{n_k} \in \mathbb{R}$ and $\left|\mu^{n_k} a_{n_k}\right| \geq \frac{\delta}{2}$ for all k.*

Proof With $\mu^n = x_n + \imath y_n$ and $a = \alpha + \imath\beta$, we have

$$\mu^n a = (x_n + \imath y_n)(\alpha + \imath\beta) = x_n\alpha - y_n\beta + \imath(x_n\beta + y_n\alpha).$$

If $x_n = 0$ choose $\alpha_n := 0$, $\beta_n := \frac{\delta}{2}$ to obtain $\mu_n a_n = -y_n\beta_n \in \mathbb{R}$ and

$$\left|\mu^n a_n\right| = |\mu|^n |a_n| \geq |a_n| = \frac{\delta}{2}.$$

If $x_n \neq 0$ the product $\mu^n a$ is real if and only if $\beta = -\alpha\frac{y_n}{x_n}$. According to Colonius, Cossich, and Santana [9, Lemma 13], there are $n_k \to \infty$ such that $\left|\frac{\mathrm{Im}(\mu^{n_k})}{\mathrm{Re}(\mu^{n_k})}\right| \to 0$ and hence, with $\alpha_{n_k} := \frac{\delta}{2}$, $\beta_{n_k} := -\alpha_{n_k}\frac{y_{n_k}}{x_{n_k}}$, and k large enough,

$$\left|\beta_{n_k}\right| = \alpha_{n_k}\left|\frac{y_{n_k}}{x_{n_k}}\right| = \frac{\delta}{2}\left|\frac{\mathrm{Im}(\mu^{n_k})}{\mathrm{Re}(\mu^{n_k})}\right| < \frac{\delta}{2}.$$

It follows for $a_{n_k} := \alpha_{n_k} + \imath\beta_{n_k}$ that

$$\left|a_{n_k}\right|^2 = \alpha_{n_k}^2 + \beta_{n_k}^2 < \frac{1}{4}\delta^2 + \frac{1}{4}\delta^2, \text{ and hence } \left|a_{n_k}\right| < \delta.$$

This choice of a_{n_k} guarantees $\mu^{n_k} a_{n_k} \in \mathbb{R}$ and using $|\mu| \geq 1$

$$\left|\mu^{n_k} a_{n_k}\right| = |\mu|^{n_k}\left|a_{n_k}\right| \geq \left|a_{n_k}\right| \geq \left|\alpha_{n_k}\right| = \frac{\delta}{2}.$$

□

The next lemma relates the reachable sets and the center-unstable subspaces of the homogeneous part.

Lemma 10 *Assume that the periodic linear system in* (1) *without control restrictions is controllable. Then for every* $\tau \in [0, T)$*, the center-unstable subspace* $E_\tau^{+,0}$ *of the homogeneous part* (2) *and the reachable sets of system* (1) *with controls* $u \in \mathcal{U}$ *satisfy*

$$E_\tau^{+,0} \subset \mathbf{R}_{\mathbb{N}T+\tau}(\tau, 0) \subset \mathrm{int}\mathbf{R}_{\mathbb{N}T+\tau}(0, 0).$$

Proof The second inclusion follows from Proposition 5. It remains to prove the first inclusion. Since by Proposition 5 $\mathbf{R}_{\mathbb{N}T+\tau}(\tau, 0)$ is convex, it suffices to prove that the real generalized eigenspaces for the eigenvalues (the Floquet multipliers) with absolute value greater than or equal to 1 are contained in $\mathbf{R}_{\mathbb{N}T+\tau}(\tau, 0)$. For each eigenvalue μ of $X(T+\tau, \tau)$ and $q \in \mathbb{N}$, let $J_q(\mu) := \ker(\mu I - X(T+\tau, \tau)^q)$ and denote the set of real parts by

$$J_q^{\mathbb{R}}(\mu) := \mathrm{Re}(J_q(\mu)) = \{\mathrm{Re}\, v \,\big|\, v \in J_q(\mu)\}.$$

Note that $J_q^{\mathbb{R}}(\mu) \subset J_{q+1}^{\mathbb{R}}(\mu)$. Since $\mathbb{C}^d$ splits into the direct sum of the generalized eigenspaces $\bigcup_{q\in\{0,1,\ldots,d\}} \ker(\mu I - X(T+\tau, \tau)^q)$ and $X(T+\tau, \tau)$ is real it follows that $\mathbb{R}^d$ splits into the direct sum of the subspaces

$$\bigcup_{q\in\{0,1,\ldots,d\}} J_q^{\mathbb{R}}(\mu) \text{ for } \mu \in \mathrm{spec}(X(T+\tau, \tau)).$$

Fix an eigenvalue $\mu = x + \imath y$ of $X(T+\tau, \tau)$ with $|\mu| \geq 1$. It suffices to show that $J_q^{\mathbb{R}}(\mu) \subset \mathbf{R}_{\mathbb{N}T+\tau}(\tau, 0)$ for all q.

We prove the statement by induction on q, the case $q = 0$ being trivial. So assume that $J_{q-1}^{\mathbb{R}}(\mu) \subset \mathbf{R}_{\mathbb{N}T+\tau}(\tau, 0)$ and take any $w = w_1 + \imath w_2 \in J_q(\mu)$. We must show that $w_1 \in \mathbf{R}_{\mathbb{N}T+\tau}(\tau, 0)$. Note that $w_1, w_2 \in J_q^{\mathbb{R}}(\mu)$ (cf. Sontag [24, p. 119]).

For $a \in \mathbb{C}$ and $n \geq 1$, one computes

$$\begin{aligned} X(T+\tau, \tau)^n aw &= (X(T+\tau, \tau) - \mu I + \mu I)^n aw \\ &= \sum_{j=0}^{n} \binom{n}{j} (X(T+\tau, \tau) - \mu I)^{n-j} \mu^j aw = \mu^n aw + z(n), \end{aligned} \tag{13}$$

where $z(n) := \sum_{j=0}^{n-1} \binom{n}{j} (X(T+\tau, \tau) - \mu I)^{n-j} \mu^j aw$. Since $aw \in J_q(\mu)$ it follows that $(X(T+\tau, \tau) - \mu I)^i aw \in J_{q-1}(\mu)$ for all $i \geq 1$, hence $z(n) \in J_{q-1}(\mu), n \geq 1$. Equality (13) implies

$$\mu^n aw = X(T+\tau, \tau)^n aw - z(n). \tag{14}$$

One finds with $a = \alpha + \imath\beta$

$$\mathrm{Re}(aw) = \mathrm{Re}((\alpha + \imath\beta)(w_1 + \imath w_2)) = \alpha w_1 - \beta w_2,$$

hence

$$\|\mathrm{Re}(aw)\| \leq 2\max(|\alpha|, |\beta|)\max(\|w_1\|, \|w_2\|) \leq 2\,|a|\max(\|w_1\|, \|w_2\|).$$

According to Proposition 5, one has $0 \in \mathrm{int}\mathbf{R}_{\mathrm{d}T+\tau}(\tau, 0)$. Thus, there is $\delta > 0$ such that $\mathrm{Re}(aw) \in \mathbf{R}_{\mathrm{d}T+\tau}(\tau, 0)$ for all $a \in \mathbb{C}$ with $|a| < \delta$.

By Lemma 9, there are a sequence $(n_k)_{k\in\mathbb{N}}$ with $n_k \to \infty$ and $a_{n_k} \in \mathbb{C}$ with $\left|a_{n_k}\right| < \delta$ such that $\mu^{n_k} a_{n_k} \in \mathbb{R}$ and $\left|\mu^{n_k} a_{n_k}\right| \geq \frac{\delta}{2}$. Then, it follows that $\mathrm{Re}(a_{n_k} w) \in \mathbf{R}_{\mathrm{d}T+\tau}(\tau, 0)$ for all k.

Now choose $\ell \in \mathbb{N}$ with $\ell \geq 2/\delta$. Taking real parts in (14) and choosing $a = a_{n_k}$, one obtains

$$\mu^{n_k} a_{n_k} w_1 = X(T+\tau, \tau)^{n_k}\,\mathrm{Re}(a_{n_k} w) - \mathrm{Re}\, z(n_k), \tag{15}$$

where $\mathrm{Re}\, z(n_k) \in J^{\mathbb{R}}_{q-1}(\mu)$ and $\mathrm{Re}(a_{n_k} w) \in \mathbf{R}_{\mathrm{d}T+\tau}(\tau, 0)$. For $k = 1$ the variation-of-parameters formula (5) with $u = 0$ implies

$$X(T+\tau, \tau)^{n_1}\,\mathrm{Re}(a_{n_1} w) = X(n_1 T + \tau, \tau)\,\mathrm{Re}(a_{n_1} w) \in \mathbf{R}_{(n_1+d)T+\tau}(\tau, 0).$$

We may assume that $n_2 \geq n_1 + d$ and obtain

$$\begin{aligned} &X(T+\tau, \tau)^{n_1}\,\mathrm{Re}(a_{n_1} w) + X(T+\tau, \tau)^{n_2}\,\mathrm{Re}(a_{n_2} w) \\ &\quad = X(T+\tau, \tau)^{n_1}\left[\mathrm{Re}(a_{n_1} w) + X(T+\tau, \tau)^{d} X(T+\tau, \tau)^{n_2-n_1-d}\,\mathrm{Re}(a_{n_2} w)\right]. \end{aligned}$$

With $x = \mathrm{Re}(a_{n_1} w) \in \mathbf{R}_{\mathrm{d}T+\tau}(\tau, 0)$ and

$$y = X(T+\tau, \tau)^{n_2-n_1-d}\,\mathrm{Re}(a_{n_2} w) \in \mathbf{R}_{(n_2-n_1)T+\tau}(\tau, 0),$$

Proposition 3 yields

$$x + X(T+\tau, \tau)^{d} y \in \mathbf{R}_{(d+n_2-n_1)T+\tau}(\tau, 0).$$

Hence, using again formula (5) with $u = 0$ and Lemma 1(ii),

$$\begin{aligned} &X(T+\tau, \tau)^{n_1}\,\mathrm{Re}(a_{n_1} w) + X(T+\tau, \tau)^{n_2}\,\mathrm{Re}(a_{n_2} w) \\ &\quad \in X(T+\tau, \tau)^{n_1}\mathbf{R}_{(d+n_2-n_1)T+\tau}(\tau, 0) \subset \mathbf{R}_{(d+n_2)T+\tau}(\tau, 0). \end{aligned}$$

In the next step, we obtain for $n_3 \geq n_2 + d$

$$\begin{aligned} &X(T+\tau, \tau)^{n_1}\,\mathrm{Re}(a_{n_1} w) + X(T+\tau, \tau)^{n_2}\,\mathrm{Re}(a_{n_2} w) + X(T+\tau, \tau)^{n_3}\,\mathrm{Re}(a_{n_3} w) \\ &\quad \subset \mathbf{R}_{(2d+n_3)T+\tau}(\tau, 0). \end{aligned}$$

Proceeding in this way, we arrive at

$$\sum_{k=1}^{\ell} X(T+\tau, \tau)^{n_k}\,\mathrm{Re}(a_{n_k} w) \in \mathbf{R}_{((\ell-1)d+n_\ell)T+\tau}(\tau, 0).$$

Summing (15) from $k = 1$ to ℓ this yields

$$\sum_{k=1}^{\ell} \mu^{n_k} a_{n_k} w_1 = \sum_{k=1}^{\ell} \left[X(T+\tau,\tau)^{n_k} \operatorname{Re}(a_{n_k} w) - \operatorname{Re} z(n_k) \right]$$
$$\in \mathbf{R}_{((\ell-1)d+n_\ell)T+\tau}(\tau,0) + J_{q-1}^{\mathbb{R}}(\mu) \subset \mathbf{R}_{\mathbb{N}T+\tau}(\tau,0) + J_{q-1}^{\mathbb{R}}(\mu).$$

By the induction hypothesis, the linear subspace $J_{q-1}^{\mathbb{R}}(\mu)$ is contained in the convex set $\mathbf{R}_{\mathbb{N}T+\tau}(\tau,0)$, which is open by Proposition 5. This implies (cf. Sontag [24, Lemma 3.6.4]) that $\mathbf{R}_{\mathbb{N}T+\tau}(\tau,0) + J_{q-1}^{\mathbb{R}}(\mu) = \mathbf{R}_{\mathbb{N}T+\tau}(\tau,0)$. If $\mu^{n_k} a_{n_k} > 0$ for all $k \in \{1,\ldots,\ell\}$, then the real number $\rho := \sum_{k=1}^{\ell} \mu^{n_k} a_{n_k} > \ell \cdot \delta/2 \geq 1$. For the k with $\mu^{n_k} a_{n_k} < 0$, replace a_{n_k} by $-a_{n_k}$ to get the same conclusion. It follows that w_1 is a convex combination of the points 0 and ρw_1 in the convex set $\mathbf{R}_{\mathbb{N}T+\tau}(\tau,0)$:

$$w_1 = \left(1-\rho^{-1}\right)\cdot 0 + \rho^{-1}\cdot \rho w_1.$$

It follows that $w_1 \in \mathbf{R}_{\mathbb{N}T+\tau}(\tau,0)$ completing the induction step. We have shown that $E_\tau^{+,0} \subset \mathbf{R}_{\mathbb{N}T+\tau}(\tau,0)$ proving the lemma. □

The following result characterizes the reachable and controllable sets of the autonomized system (11) by spectral properties of the homogeneous part (2). Recall that we denote the spectral subbundles of the T-periodic linear differential equation (2) by $\mathcal{E}^-, \mathcal{E}^+, \mathcal{E}^{+,0}$, and $\mathcal{E}^{-,0}$. For subsets $K_\tau \subset \mathbb{R}^d$ and matrices $Y(\tau)$, $\tau \in \mathbb{S}^1 = [0,T)$, we use the following notation:

$$\mathcal{K} := \bigcup_{\tau\in[0,T)} \left\{(\tau,x)\in \mathbb{S}^1\times\mathbb{R}^d \,|x\in K_\tau\right\},\; Y\mathcal{K} := \bigcup_{\tau\in[0,T)} \{(\tau,x)\,|x\in Y(\tau)K_\tau\}.$$

Theorem 11 *Suppose that the periodic system in* (1) *with unconstrained controls is controllable and consider the autonomized system* (11) *with controls* $u \in \mathcal{U}$.

(i) Then, the reachable set $\mathbf{R}^a(0,0) \subset \mathbb{S}^1 \times \mathbb{R}^d$ *satisfies* $\mathbb{S}^1 \times \{0\} \subset \operatorname{int}\mathbf{R}^a(0,0)$ *and*

$$X(dT+\cdot,\cdot)\mathcal{K}^- \oplus \mathcal{E}^{+,0} \subset \operatorname{int}\mathbf{R}^a(0,0) \subset \mathcal{K}^- \oplus \mathcal{E}^{+,0},$$

with uniformly bounded convex sets $K_\tau^- := \operatorname{int}\mathbf{R}_{\mathbb{N}T+\tau}(0,0) \cap E_\tau^-$, $\tau \in \mathbb{S}^1$.

(ii) The controllable set $\mathbf{C}^a(0,0)$ *satisfies* $\mathbb{S}^1 \times \{0\} \subset \operatorname{int}\mathbf{C}^a(0,0)$ *and*

$$\mathcal{E}^{-,0} \oplus X(-dT+\cdot,\cdot)\mathcal{K}^+ \subset \operatorname{int}\mathbf{C}^a(0,0) \subset \mathcal{E}^{-,0}\oplus\mathcal{K}^+$$

with uniformly bounded convex sets $K_\tau^+ := \operatorname{int}\mathbf{C}_{-\mathbb{N}T+\tau}(0,0) \cap E_\tau^+$, $\tau \in \mathbb{S}^1$.

Proof (i) Lemma 7 and Lemma 8 imply that $\mathbb{S}^1 \times \{0\} \subset \operatorname{int}\mathbf{R}^a(0,0)$ and

$$\mathbf{R}^a(0,0) = \bigcup\nolimits_{\tau\in[0,T)} \left\{(\tau,x)\in\mathbb{S}^1\times\mathbb{R}^d \,|x\in \mathbf{R}_{\mathbb{N}T+\tau}(0,0)\right\}. \tag{16}$$

We claim that

$$\operatorname{int}\mathbf{R}_{\mathbb{N}T+\tau}(0,0) = K_\tau^- + E_\tau^{+,0} \text{ for every } \tau \in [0,T). \tag{17}$$

Recall that by Proposition 5 the set $\operatorname{int}\mathbf{R}_{\mathbb{N}T+\tau}(0,0)$ is convex. Lemma 10 shows that $E_\tau^{+,0} \subset \mathbf{R}_{\mathbb{N}T+\tau}(\tau,0) \subset \operatorname{int}\mathbf{R}_{\mathbb{N}T+\tau}(0,0)$. By Sontag [24, Lemma 3.6.4], it follows that

$$K_\tau^- + E_\tau^{+,0} \subset \operatorname{int}\mathbf{R}_{\mathbb{N}T+\tau}(0,0) + E_\tau^{+,0} = \operatorname{int}\mathbf{R}_{\mathbb{N}T+\tau}(0,0).$$

For the converse inclusion, write $x \in \operatorname{int}\mathbf{R}_{\mathbb{N}T+\tau}(0,0)$ as $x = y \oplus z$ with $y \in E_\tau^-$ and $z \in E_\tau^{+,0}$. Again by [24, Lemma 3.6.4] it follows that

$$y = x - z \in \operatorname{int}\mathbf{R}_{\mathbb{N}T+\tau}(0,0) + E_\tau^{+,0} = \operatorname{int}\mathbf{R}_{\mathbb{N}T+\tau}(0,0),$$

which proves that $y \in K_\tau^-$ and therefore $x \in K_\tau^- + E_\tau^{+,0}$. This proves (17).

By (16), it follows that $\operatorname{int}\mathbf{R}^a(0,0) \subset \mathcal{K}^- \oplus \mathcal{E}^{+,0}$ proving the second inclusion in (i).

For the first inclusion in (i) consider $x = \varphi(kT+\tau;0,0,u) \in \mathbf{R}_{\mathbb{N}T+\tau}(0,0)$ and recall that by Proposition 5 there is a ball $\mathbf{B}(0;\varepsilon) \subset \mathbf{R}_{\mathrm{d}T+\tau}(\tau,0)$ for all $\tau \in \mathbb{S}^1$. For $y = \varphi(dT+\tau;\tau,0,v) \in \mathbf{B}(0;\varepsilon)$ define

$$w(t) = \begin{cases} u(t) & \text{for } t \in [0,kT+\tau) \\ v(t-kT) & \text{for } t \in [kT+\tau,(d+k)T+\tau] \end{cases}.$$

This implies

$$\begin{aligned}
&\varphi((k+d)T+\tau;0,0,w) = \varphi((k+d)T+\tau;kT+\tau,x,w)\\
&\quad = X((k+d)T+\tau,kT+\tau)x + \int_{kT+\tau}^{(k+d)T+\tau} X((k+d)T+\tau,s)B(s)w(s)\mathrm{d}s\\
&\quad = X(dT+\tau,\tau)x + \int_\tau^{dT+\tau} X(dT+\tau,s)B(s)v(s)ds = X(\mathrm{d}T+\tau,\tau)x + y,
\end{aligned}$$

showing that

$$X(dT+\tau,\tau)x + \mathbf{B}(0;\varepsilon) \subset \mathbf{R}_{\mathbb{N}T+\tau}(0,0),\ \tau \in [0,T).$$

It follows that

$$\left\{(\tau,x) \in \mathbb{S}^1 \times \mathbb{R}^d \,|x \in X(dT+\tau,\tau)\mathbf{R}_{\mathbb{N}T+\tau}(0,0)\right\} \subset \operatorname{int}\mathbf{R}^a(0,0),\ \tau \in \mathbb{S}^1.$$

Since $X(dT+\tau,\tau)E_\tau^{+,0} = E_\tau^{+,0}$ equality (17) implies

$$X(dT+\tau,\tau)\operatorname{int}\mathbf{R}_{\mathbb{N}T+\tau}(0,0) = X(dT+\tau,\tau)(K_\tau^- + E_\tau^{+,0}) = X(dT+\tau,\tau)K_\tau^- + E_\tau^{+,0}$$

and $X(dT+\tau,\tau)K_\tau^- \subset E_\tau^-$. This shows the first inclusion in assertion (i),

$$X(dT+\tau,\tau)\mathcal{K}^- \oplus \mathcal{E}^{+,0} = \bigcup_{\tau\in[0,T)} \left\{(\tau,x)\in\mathbb{S}^1\times\mathbb{R}^d \,\middle|\, x\in X(dT+\tau,\tau)K_\tau^- + E_\tau^{+,0}\right\}$$
$$\subset \mathrm{int}\mathbf{R}^a(0,0).$$

In order to prove that K_τ^- is bounded, let $x=\varphi(kT+\tau;0,0,u)\in\mathbf{R}_{\mathbb{N}T+\tau}(0,0)\cap E_\tau^-$. Then using linearity

$$\begin{aligned} x &= \varphi(kT+\tau;0,0,u) = \varphi(kT+\tau;\tau,\varphi(\tau,0,0,u),u) \\ &= X(kT+\tau,\tau)\varphi(\tau;0,0,u) + \varphi(kT+\tau;\tau,0,u). \end{aligned} \tag{18}$$

Define for $\tau\in\mathbb{S}^1$ a bounded linear map by

$$\mathcal{B}_\tau : L^\infty([0,T],\mathbb{R}^m)\to\mathbb{R}^d,\ \mathcal{B}_\tau(u') = \int_0^T X(T+\tau,\tau+s)B(\tau+s)u'(s)\mathrm{d}s.$$

Using the variation-of-parameters formula (5) and periodicity, one computes

$$\begin{aligned} \varphi(kT+\tau;\tau,0,u) &= \int_\tau^{kT+\tau} X(kT+\tau,s)B(s)u(s)\mathrm{d}s \\ &= \sum_{j=0}^{k-1}\int_{jT+\tau}^{(j+1)T+\tau} X(kT+\tau,s)B(s)u(s)\mathrm{d}s \\ &= \sum_{j=0}^{k-1} X(T+\tau,\tau)^{k-j}\int_0^T X((j+1)T+\tau,jT+\tau+s) \\ &\quad B(jT+\tau+s)u(jT+\tau+s)\mathrm{d}s \\ &= \sum_{j=0}^{k-1} X(T+\tau,\tau)^{k-j}\int_0^T X(T+\tau,\tau+s)B(\tau+s)u(jT+\tau+s)\mathrm{d}s \\ &= \sum_{j=0}^{k-1} X(T+\tau,\tau)^{k-j}\mathcal{B}_\tau(u(jT+\tau+\cdot)). \end{aligned}$$

Consider the projection $\pi_\tau : \mathbb{R}^d = E_\tau^-\oplus E_\tau^{+,0}\to E_\tau^-$ along $E_\tau^{+,0}$. By Theorem 6(ii), the subspaces E_τ^- and $E_\tau^{+,0}$ are $X(T+\tau,\tau)$-invariant, hence π_τ commutes with $X(T+\tau,\tau)$. With (18) this yields

$$\begin{aligned} x &= \pi_\tau x = \pi_\tau X(kT+\tau,\tau)\varphi(\tau;0,0,u) + \pi_\tau\varphi(kT+\tau;\tau,0,u) \\ &= X(T+\tau,\tau)^k\pi_\tau\varphi(\tau;0,0,u) + \sum_{j=0}^{k-1} X(T+\tau,\tau)^{k-j}\pi_\tau\mathcal{B}_\tau(u(jT+\tau+\cdot)). \end{aligned}$$

Since $X(T+\tau,\tau)|_{E_\tau^-}$ is a linear contraction there exist constants $a \in (0,1)$ and $c \geq 1$ such that $\|X(T+\tau,\tau)^n x'\| \leq ca^n \|x'\|$ for all $n \in \mathbb{N}$ and $x' \in E_\tau^-$. Due to the compactness of $\mathbb{S}^1$ and the continuity of the solutions, these constants may be chosen independently of $\tau \in \mathbb{S}^1$. It follows that

$$\left\| X(T+\tau,\tau)^k \pi_\tau \varphi(\tau;0,0,u) \right\| \leq ca^k \left\| \pi_\tau \varphi(\tau;0,0,u) \right\|$$

and

$$\| \sum_{j=0}^{k-1} X(T+\tau,\tau)^{k-j} \pi_\tau \mathcal{B}_\tau(u(jT+\tau+\cdot)) \| \leq \sum_{j=0}^{k-1} ca^{k-j} \left\| \pi_\tau \mathcal{B}_\tau(u(jT+\tau+\cdot)) \right\|.$$

Since U is compact, there is $M > 0$ such that $\left\| \pi_\tau \varphi(\tau,0,0,u') \right\|, \|\pi_\tau \mathcal{B}_\tau(u')\| \leq M$ for all $\tau \in \mathbb{S}^1$ and $u' \in \mathcal{U}$. Thus K_τ^- is bounded by

$$\|x\| = \|\varphi(kT+\tau;0,0,u)\| \leq ca^k M + cM \sum_{j=0}^{k-1} a^{k-j} \leq \frac{2cM}{1-a}.$$

Assertion (ii) follows by considering the time-reversed systems. □

Next we define subsets of complete approximate controllability.

Definition 1 A nonvoid set $D^a \subset \mathbb{S}^1 \times \mathbb{R}^d$ is a control set of the autonomized system (11) on $\mathbb{S}^1 \times \mathbb{R}^d$ if it has the following properties: (i) for all $(\tau,x) \in D^a$ there is a control $u \in \mathcal{U}$ such that $\varphi^a(t,(\tau,x),u) \in D^a$ for all $t \geq 0$, (ii) for all $(\tau,x) \in D^a$ one has $D^a \subset \overline{\mathbf{R}^a(\tau,x)}$, and (iii) D^a is maximal with these properties, that is, if $D' \supset D^a$ satisfies conditions (i) and (ii), then $D' = D^a$.

The following lemma shows that there is a control set around (0, 0).

Lemma 12 *Suppose that the periodic system in (1) with unconstrained controls is controllable. Then, $D^a := \overline{\mathbf{R}^a(0,0)} \cap \mathbf{C}^a(0,0)$ is a control set and $\mathbb{S}^1 \times \{0\} \subset \mathrm{int} D^a$.*

Proof Theorem 11 shows that $\mathbb{S}^1 \times \{0\} \subset \mathrm{int}\mathbf{R}^a(0,0) \cap \mathbf{C}^a(0,0)$. Consider $(\tau,x),(\sigma,y) \in \overline{\mathbf{R}^a(0,0)} \cap \mathbf{C}^a(0,0)$ and let $\varepsilon > 0$. Then, there are $t_1, t_2 \geq 0$ and $u_1, u_2 \in \mathcal{U}$ with

$$\varphi^a(t_1;(\sigma,y),u_1) = (0,0) \text{ and } d(\varphi^a(t_2;(0,0),u_2),(\tau,x)) < \varepsilon.$$

Define a control v by

$$v(t) := \begin{cases} u_1(t) & \text{for } t \in [0,t_1] \\ u_2(t-t_1) & \text{for } t \in (t_1, t_1+t_2] \end{cases}.$$

Then, it follows that $d(\varphi^a(t_1+t_2,(\sigma,y),v),(\tau,x)) < \varepsilon$. This shows that

$$\overline{\mathbf{R}^a(0,0)} \cap \mathbf{C}^a(0,0) \subset \overline{\mathbf{R}^a(\tau,x)} \text{ for all } (\tau,x) \in \overline{\mathbf{R}^a(0,0)} \cap \mathbf{C}^a(0,0). \tag{19}$$

Define D^a as the union of all sets D' with $D' \subset \overline{\mathbf{R}^a(\tau, x)}$ for all $(\tau, x) \in D'$ and $\overline{\mathbf{R}^a(0,0)} \cap \mathbf{C}^a(0,0) \subset D'$. Then, any $(\tau, x) \in D^a$ is in some set D' and $(0,0) \in \overline{\mathbf{R}^a(0,0)} \cap \mathrm{int}\mathbf{C}^a(0,0)$ implies that there are $t > 0$ and $u \in \mathcal{U}$ with $\varphi^a(t; (\tau, x), u) \in \mathbf{C}^a(0,0)$ and $(\tau, x) \in \mathbf{C}^a(0,0)$ follows. This shows that $D^a \subset \mathbf{C}^a(0,0)$. Since $D^a \subset \overline{\mathbf{R}^a(0,0)}$ this also implies $D^a \subset \overline{\mathbf{R}^a(\tau, x)}$ proving $D^a = \overline{\mathbf{R}^a(0,0)} \cap \mathbf{C}^a(0,0)$. It also follows that D^a is a maximal set with $D^a \subset \overline{\mathbf{R}^a(\tau, x)}$ and $\mathrm{int}D^a \neq \varnothing$. Hence, Kawan [18, Proposition 1.20] yields that D^a is a control set. □

The following theorem proves the existence of a unique control set with nonvoid interior of the autonomized system. Recall that the center subbundle $\mathcal{E}^0$ of the periodic linear differential equation (2) is nontrivial if and only if 0 is a Floquet exponent. This holds if and only if there is a Floquet multiplier of modulus 1, i.e., $\mathrm{spec}(X(T,0)) \cap \mathbb{S}^1 \neq \varnothing$.

Theorem 13 *Suppose that the periodic system in (1) with unconstrained controls is controllable. Then, there exists a unique control set D^a with nonvoid interior of the autonomized system (11) with controls $u \in \mathcal{U}$.*

It is given by $D^a = \overline{\mathbf{R}^a(0,0)} \cap \mathbf{C}^a(0,0)$ and satisfies $\mathbb{S}^1 \times \{0\} \subset \mathrm{int}D^a$ and, with $\mathcal{K}^- \subset \mathcal{E}^-$ and $\mathcal{K}^+ \subset \mathcal{E}^+$ defined in Theorem 11 and $Y(\tau) := X(dT + \tau, \tau)$, $\tau \in \mathbb{S}^1$,

$$Y(\cdot)\mathcal{K}^- \oplus \mathcal{E}^0 \oplus Y(\cdot)^{-1}\mathcal{K}^+ \subset \mathrm{int}D^a \subset D^a \subset Y(\cdot)^{-1}\overline{\mathcal{K}^-} \oplus \mathcal{E}^0 \oplus Y(\cdot)\mathcal{K}^+. \quad (20)$$

In particular, $\mathrm{int}D^a$ *is unbounded if and only if the center subbundle $\mathcal{E}^0$ is nontrivial.*

Proof The inclusion $\mathbb{S}^1 \times \{0\} \subset \mathrm{int}D^a$ and $D^a = \overline{\mathbf{R}^a(0,0)} \cap \mathbf{C}^a(0,0)$ follow by Lemma 12. Furthermore, the inclusions (20) imply the last assertion since $X(dT + \tau, \tau)$, $\tau \in \mathbb{S}^1$, as well as $\mathcal{K}^-$ and $\mathcal{K}^+$ are bounded. Theorem 11 implies

$$X(dT + \cdot, \cdot)\mathcal{K}^- \oplus \mathcal{E}^+ \oplus \mathcal{E}^0 \subset \mathrm{int}\mathbf{R}^a(0,0),$$
$$\mathcal{E}^- \oplus \mathcal{E}^0 \oplus X(-dT + \cdot, \cdot)\mathcal{K}^+ \subset \mathrm{int}\mathbf{C}^a(0,0).$$

Since $\mathrm{int}D^a \subset \mathrm{int}\mathbf{R}^a(0,0) \cap \mathrm{int}\mathbf{C}^a(0,0)$ and $X(-dT + \tau, \tau) = X(\tau, -dT + \tau)^{-1} = X(dT + \tau, \tau)^{-1}$ for $\tau \in \mathbb{S}^1$ the first inclusion in (20) follows. In order to prove the third inclusion, let $(\tau, y) \in \mathbf{R}^a(0,0)$ be given by

$$(\tau, y) = \varphi^a(t; (0,0), u) = (t \bmod T, \varphi(t; 0, 0, u) = (\tau, \varphi(\ell T + \tau; 0, u))$$

with $y = \varphi(\ell T + \tau; 0, u) \in \mathbf{R}_{\mathbb{N}T+\tau}(0,0)$. By Proposition 5, there is a ball $\mathbf{B}(0; \varepsilon) \subset \mathbf{R}_{\mathrm{d}T+\tau}(\tau, 0)$. Hence, Proposition 3 implies

$$\mathbf{B}(0; \varepsilon) + X(dT + \tau, \tau)y \subset \mathbf{R}_{(d+\ell)T+\tau}(\tau, 0) \subset \mathbf{R}_{\mathbb{N}T+\tau}(\tau, 0).$$

Since $\varepsilon > 0$ is independent of $\tau \in \mathbb{S}^1$ it follows that $X(\mathrm{d}T + \cdot, \cdot)\mathbf{R}^a(0,0) \subset \mathrm{int}\mathbf{R}^a(0,0)$, and hence

$$X(dT + \cdot, \cdot)\overline{\mathbf{R}^a(0,0)} \subset \overline{\mathrm{int}\mathbf{R}^a(0,0)}.$$

Analogously it follows that

$$X(-dT+\cdot,\cdot)\mathbf{C}^a(0,0)\subset \mathrm{int}\mathbf{C}^a(0,0).$$

By Theorem 11 we obtain for $x\in D^a=\overline{\mathbf{R}^a(0,0)}\cap \mathbf{C}^a(0,0)$,

$$X(dT+\cdot,\cdot)x\in \overline{\mathrm{int}\mathbf{R}^a(0,0)}\subset \overline{\mathcal{K}^-\oplus\mathcal{E}^{+,0}}=\overline{\mathcal{K}^-}\oplus\mathcal{E}^0\oplus\mathcal{E}^+,$$
$$X(dT+\cdot,\cdot)^{-1}x=X(-dT+\cdot,\cdot)x\in \mathrm{int}\mathbf{C}^a(0,0)\subset \mathcal{E}^-\oplus\mathcal{E}^0\oplus\mathcal{K}^+.$$

This implies

$$x\in X(dT+\cdot,\cdot)^{-1}\left(\overline{\mathcal{K}^-}\oplus\mathcal{E}^0\oplus\mathcal{E}^+\right)\cap X(dT+\cdot,\cdot)\left(\mathcal{E}^-\oplus\mathcal{E}^0\oplus\mathcal{K}^+\right).$$

By Theorem 6, the subbundles $\mathcal{E}^0$ and $\mathcal{E}^\pm$ are invariant under $X(dT+\cdot,\cdot)$ and hence

$$D^a\subset X(dT+\cdot,\cdot)^{-1}\overline{\mathcal{K}^-}\oplus\mathcal{E}^0\oplus X(dT+\cdot,\cdot)\mathcal{K}^+=Y(\cdot)^{-1}\overline{\mathcal{K}^-}\oplus\mathcal{E}^0\oplus Y(\cdot)\mathcal{K}^+.$$

This proves the third inclusion in (20).

It remains to show uniqueness. Let $E\subset\mathbb{S}^1\times\mathbb{R}^d$ be an arbitrary control set with nonvoid interior. Then for every $\alpha\in(0,1]$, the set $E_\alpha:=\{(\tau,\alpha x)\,|(\tau,x)\in E\}$ satisfies

$$\mathrm{int}E_\alpha=\{(\tau,\alpha x)\,|(\tau,x)\in\mathrm{int}E\}. \tag{21}$$

The solutions $\varphi^a(t;(\tau_0,x_0),u)$ of the autonomized control system (11) are linear in (x_0,u), hence it follows for all $\alpha\in[0,1]$ and $t>0$, $(\tau,x_0)\in\mathbb{S}^1\times\mathbb{R}^d$, $u\in\mathcal{U}$ that

$$\varphi^a(t;(\tau,\alpha x_0),\alpha u)=\alpha\varphi^a(t;(\tau,\alpha x_0),\alpha u).$$

Here, $\alpha u\in\mathcal{U}$ since by assumption the control range U is convex and $0\in U$. It follows that the set E_α satisfies conditions (i) and (ii) for control sets in Definition 1 since they hold for E. Thus, E_α is contained in some control set D_α and $\mathrm{int}E_\alpha\subset\mathrm{int}D_\alpha$.

It is clear that for α near 1 the control sets D_α and E coincide since their intersection is nonvoid. Now choose any $(\tau,x)\in\mathrm{int}E$ and suppose, by way of contradiction, that

$$\alpha_0:=\inf\{\alpha\in(0,1]\,|\forall\beta\in(\alpha,1]:(\tau,\beta x)\in\mathrm{int}E\}>0.$$

Then, $(\tau,\alpha_0x)\in\partial E$ and $(\tau,\alpha_0x)\in\mathrm{int}E_{\alpha_0}\subset\mathrm{int}D_{\alpha_0}$ by (21). Therefore, $E\cap\mathrm{int}D_{\alpha_0}\neq\varnothing$, and it follows that $E=D_{\alpha_0}$ and $(\tau,\alpha_0x)\in\mathrm{int}D_{\alpha_0}=\mathrm{int}E$. This is a contradiction and so $\alpha_0=0$. Choosing $\alpha>0$ small enough such that $(\tau,\alpha x)\in D^a$ we obtain $(\tau,\alpha x)\in E\cap D^a$ and it follows that $E=D^a$. □

Remark 3 A control system is called locally accessible, if the reachable and controllable sets up to time $t>0$ have nonvoid interior for every $t>0$. If this holds for the autonomized system (11), then Colonius and Kliemann [7, Lemma 3.2.13(i)] implies that $\overline{D^a}=\overline{\mathrm{int}D^a}$. Even if the system in (1) without control restrictions is controllable, the autonomized system (11) need not satisfy $\mathrm{int}\mathbf{R}^a_t(\tau,x)\neq\varnothing$ for small $t>0$, hence, in general, it is not locally accessible. The example in Bittanti, Guarbadassi, Mafezzoni, and Silverman [3, p. 38] is a counterexample.

Remark 4 Gayer [15, Theorem 3] relates the control sets of autonomized (general nonlinear) control systems to control sets of discrete-time systems depending on $\tau \in \mathbb{S}^1$ defined by Poincaré maps. For system (11), these systems are defined by

$$\Phi_\tau^u : \mathbb{R}^d \to \mathbb{R}^d,\ \Phi_\tau^u(\cdot) = \varphi^a(T, (\tau, \cdot), u) = \varphi(T + \tau; \tau, \cdot, u) \quad u \in \mathcal{U}.$$

5 The Poincaré sphere

This section describes the global controllability behavior of periodic linear control systems of the form (1) with homogeneous part (2) by projection to the Poincaré sphere. This allows us to determine the behavior "near infinity" by the induced system near the equator.

The system on the Poincaré sphere is obtained by attaching the state space $\mathbb{R}^d$ to the north pole $(0, 1) \in \mathbb{R}^d \times \mathbb{R}$ of the unit sphere $\mathbb{S}^d$ in $\mathbb{R}^{d+1}$ and then taking the stereographic projection to $\mathbb{S}^d$. More formally, the extended system with scalar part $\dot{z} = 0$ is defined as

$$\begin{pmatrix} \dot{x}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{pmatrix} A(t) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ z(t) \end{pmatrix} + \sum_{i=1}^m u_i(t) \begin{pmatrix} 0 & b_i(t) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ z(t) \end{pmatrix}, \tag{22}$$

where $b_i(t)$ denote the columns of $B(t)$. For $z \equiv 1$, we get a copy of the original system (1). Abbreviate

$$\hat{A}(t) = \begin{pmatrix} A(t) & 0 \\ 0 & 0 \end{pmatrix},\ \hat{B}_i(t) = \begin{pmatrix} 0 & b_i(t) \\ 0 & 0 \end{pmatrix},\ \sum_{i=1}^m u_i(t)\hat{B}_i(t) = \begin{pmatrix} 0 & B(t)u(t) \\ 0 & 0 \end{pmatrix}.$$

The projection of the homogeneous control system (22) on $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ has the form (omitting the argument t)

$$\begin{aligned} \begin{pmatrix} \dot{s} \\ \dot{s}_{d+1} \end{pmatrix} &= [\hat{A} - (s^\top, s_{d+1})\hat{A}(s^\top, s_{d+1})^\top \cdot I_{d+1}](s^\top, s_{d+1})^\top \\ &\quad + \sum_{i=1}^m u_i[\hat{B}_i - (s^\top, s_{d+1})\hat{B}_i(s^\top, s_{d+1})^\top \cdot I_{d+1}](s^\top, s_{d+1})^\top. \end{aligned}$$

This is obtained by subtracting the radial components of the linear vector fields $\hat{A}(t)$ and $\hat{B}_i(t)$.

We compute

$$\begin{pmatrix} \dot{s} \\ \dot{s}_{d+1} \end{pmatrix} = \left[\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} - (s^\top, s_{d+1}) \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} s \\ s_{d+1} \end{pmatrix} \cdot I_{d+1} \right] \begin{pmatrix} s \\ s_{d+1} \end{pmatrix}$$

$$
\begin{aligned}
&+\sum_{i=1}^{m} u_i \left[\begin{pmatrix} 0 & b_i \\ 0 & 0 \end{pmatrix} - (s^\top, s_{d+1}) \begin{pmatrix} 0 & b_i \\ 0 & 0 \end{pmatrix} \begin{pmatrix} s \\ s_{d+1} \end{pmatrix} \cdot I_{d+1}\right] \begin{pmatrix} s \\ s_{d+1} \end{pmatrix} \\
&= \begin{pmatrix} A - s^\top As \cdot I_d & 0 \\ 0 & -s^\top As \end{pmatrix} \begin{pmatrix} s \\ s_{d+1} \end{pmatrix} \\
&+\sum_{i=1}^{m} u_i \begin{pmatrix} -s^\top b_i s_{d+1} \cdot I_d & b_i \\ 0 & -s^\top b_i s_{d+1} \end{pmatrix} \begin{pmatrix} s \\ s_{d+1} \end{pmatrix} \\
&= \begin{pmatrix} \left[A - s^\top As \cdot I_d\right] s \\ -s^\top As\, s_{d+1} \end{pmatrix} + \sum_{i=1}^{m} u_i \begin{pmatrix} -s^\top b_i s_{d+1} s + b_i s_{d+1} \\ -s^\top b_i s_{d+1}^2 \end{pmatrix}. \qquad (23)
\end{aligned}
$$

This is the system equation for the induced control system on the Poincaré sphere. By adding the phase $\tau \in \mathbb{S}^1$, this induces an autonomous control system on $\mathbb{S}^1 \times \mathbb{S}^d$.

Remark 5 The homogeneous control system (23) also induces a control system on projective space $\mathbb{P}^d$ and a corresponding autonomized control system on $\mathbb{S}^1 \times \mathbb{P}^d$. Parallel to the following developments on the unit sphere $\mathbb{S}^d$, one may also work with $\mathbb{P}^d$. Here, we prefer to work on the sphere since this allows us to write down everything explicitly.

On the "equator" of the sphere $\mathbb{S}^d$ given by

$$
\mathbb{S}^{d,0} := \{s = (s_1, \dots, s_d, s_{d+1}) \in \mathbb{S}^d \,|s_{d+1} = 0\},
$$

the first d components of (23) reduce to the (uncontrolled) differential equation

$$
\dot{s}(t) = (A(t) - s(t)^\top A(t)s(t) \cdot I_d)s(t). \qquad (24)
$$

The flow of this differential equation leaves $\mathbb{S}^{d-1} \subset \mathbb{R}^d$ invariant. This coincides with the periodic differential equation obtained by projecting the homogeneous part (2) to $\mathbb{S}^{d-1}$. Furthermore, the equator is invariant, hence also the upper hemisphere $\mathbb{S}^{d,+} := \{s = (s_1, \dots, s_d, s_{d+1}) \in \mathbb{S}^d \,|s_{d+1} > 0\}$ is invariant. When the phases $\tau \in [0, T)$ are added to the states, the periodic differential equations (2) and (24) induce autonomous differential equations on $\mathbb{S}^1 \times \mathbb{R}^d$ and $\mathbb{S}^1 \times \mathbb{S}^{d-1}$, resp.

A conjugacy of (autonomous) control systems

$$
\begin{aligned}
\dot{x}(t) &= f(x(t), u(t)) \text{ on } M \text{ with } u(t) \in U, \\
\dot{y}(t) &= g(y(t), u(t)) \text{ on } N \text{ with } u(t) \in U,
\end{aligned}
$$

on manifolds M and N can be defined as a map $h : M \to N$ which together with its inverse h^{-1} is C^∞ such that the trajectories $\varphi(t; x_0, u), t \in \mathbb{R}$, on M and $\psi(t; y_0, u), t \in \mathbb{R}$, on N with initial conditions $\varphi(0; x_0, u) = x_0$ and $\psi(0; y_0, u) = y_0$ (assumed to exist) satisfy

$$
h(\varphi(t; x_0, u)) = \psi(t; h(x_0), u) \text{ for all } t \in \mathbb{R} \text{ and } x \in M, u \in \mathcal{U}.
$$

Analogously, one can define conjugacies of differential equations. It is clear that reachable sets, controllable sets, and control sets are preserved under conjugacies.

In the following, we slightly abuse notation by identifying vectors and their transposes when it is clear from the context what is meant.

Proposition 14 *(i) The map*

$$e_P : \mathbb{S}^1 \times \mathbb{R}^d \to \mathbb{S}^1 \times \mathbb{S}^{d,+},\ (\tau, x) \mapsto \left(\tau, \frac{(x,1)}{\|(x,1)\|}\right) = \left(\tau, \frac{(x,1)}{\sqrt{1+\|x\|^2}}\right)$$

is a conjugacy of the autonomized control system (11) *on* $\mathbb{S}^1 \times \mathbb{R}^d$ *and the restriction to* $\mathbb{S}^1 \times \mathbb{S}^{d,+}$ *of the autonomized system induced by* (23).

(ii) The map $e_{\mathbb{S}} : \mathbb{S}^1 \times \mathbb{S}^{d-1} \to \mathbb{S}^1 \times \mathbb{S}^{d,0},\ (\tau, s) \mapsto (\tau, s, 0) \in \mathbb{S}^1 \times \mathbb{S}^{d,0}$ *is a conjugacy of the autonomized differential equation induced by* (24) *on* $\mathbb{S}^1 \times \mathbb{S}^{d-1}$ *and the restriction to* $\mathbb{S}^1 \times \mathbb{S}^{d,0}$ *of the autonomized control system corresponding to* (23).

Proof (i) The map e_P is C^∞ (even analytic) with C^∞ inverse given by

$$(e_P)^{-1}(\tau, s_1, \ldots, s_d, s_{d+1}) = \left(\tau, \frac{s_1}{s_{d+1}}, \ldots, \frac{s_d}{s_{d+1}}\right)$$
$$\text{for } (\tau, s_1, \ldots, s_d, s_{d+1}) \in \mathbb{S}^1 \times \mathbb{S}^{d,+}.$$

In fact, one verifies

$$e_P((e_P)^{-1}(\tau, s)) = \left(\tau, \frac{\left(\frac{s_1}{s_{d+1}}, \ldots, \frac{s_d}{s_{d+1}}, 1\right)}{\sqrt{1 + \frac{s_1^2}{s_{d+1}^2} + \cdots + \frac{s_d^2}{s_{d+1}^2}}}\right) = (\tau, s).$$

The conjugacy property follows from

$$(\tau(t), s(t)) = \left(\tau(t), \frac{(x(t), 1)}{1 + \|(x(t), 1)\|}\right) = e_P(\tau(t), x(t)),\ t \in \mathbb{R}.$$

(ii) This trivially holds since the solutions on $\mathbb{S}^{d,0}$ are obtained by adding the last component 0 to the solutions on $\mathbb{S}^{d-1}$. □

Since the image of the map e_P is contained in the (open) upper hemisphere $\mathbb{S}^{d,+}$, a converging sequence of points $e_P(\tau_k, x_k)$ in the image converges to an element of $\mathbb{S}^1 \times \mathbb{S}^{d,0}$ if and only if $\|x_k\| \to \infty$. Hence, Proposition 14 shows that the behavior near the equator reflects the behavior near infinity.

Next we discuss the projection of the reachable and controllable sets to the Poincaré sphere. Note that under the map $x \mapsto \frac{(x,1)}{\|(x,1)\|}$ the origin $x = 0 \in \mathbb{R}^d$ is mapped to the north pole $(0, 1) \in \mathbb{S}^{d,+} \subset \mathbb{R}^d \times \mathbb{R}$. The following theorem shows that for the autonomized system the closure of the reachable set from the north pole intersects the equator in the image of the center-unstable subbundle, and the closure of the controllable set to the north pole intersects the equator in the image of the center-stable subbundle. Furthermore, the closure of the unique control set with nonvoid interior on $\mathbb{S}^1 \times \mathbb{S}^{d,+}$ intersects the equator in the image of the center subbundle.

Theorem 15 *Suppose that the periodic system in (1) with unconstrained controls is controllable.*

(i) Then, the projections to $\mathbb{S}^1 \times \mathbb{S}^{d,+}$ of the reachable and controllable sets, resp., of the autonomized system (11) satisfy

$$\overline{e_P(\mathrm{int}\mathbf{R}^a(0,0))} \cap \left(\mathbb{S}^1 \times \mathbb{S}^{d,0}\right) = \overline{e_P(\mathcal{E}^{+,0})} \cap \left(\mathbb{S}^1 \times \mathbb{S}^{d,0}\right),$$
$$\overline{e_P(\mathrm{int}\mathbf{C}^a(0,0))} \cap \left(\mathbb{S}^1 \times \mathbb{S}^{d,0}\right) = \overline{e_P(\mathcal{E}^{-,0})} \cap \left(\mathbb{S}^1 \times \mathbb{S}^{d,0}\right).$$

(ii) The induced system on $\mathbb{S}^1 \times \mathbb{S}^{d,+}$ has a unique control set with nonvoid interior given by $D_P^a = e_P(D^a)$ satisfying

$$\overline{\mathrm{int} D_P^a} \cap (\mathbb{S}^1 \times \mathbb{S}^{d,0}) = \overline{e_P\left(\mathcal{E}^0\right)} \cap \left(\mathbb{S}^1 \times \mathbb{S}^{d,0}\right).$$

In particular, $\mathrm{int} D^a$ *is bounded if and only if* $\overline{\mathrm{int} D_P^a} \subset \mathbb{S}^1 \times \mathbb{S}^{d,+}$.

Proof (i) The conjugacy property from Proposition 14(i) shows that $e_P(\mathbf{R}^a(0,0))$ and $e_P(\mathbf{C}^a(0,0))$ are the reachable and controllable set, resp., of $(0,(0,1)) = e_P(0,0)$ in $\mathbb{S}^1 \times \mathbb{S}^{d,+}$, where $(0,1)$ is the north pole, and $D_P^a := e_P(D^a)$ is the unique control set with nonvoid interior satisfying $\mathrm{int}(e_P(D^a)) = e_P(\mathrm{int} D^a)$. The inclusion $e_P(\mathcal{E}^{+,0}) \subset \mathrm{int}\mathbf{R}^a(0,0)$ follows from Theorem 11 implying the inclusion "$\supset$". For the converse, let $(\tau, z, 0) \in \overline{e_P(\mathrm{int}\mathbf{R}^a(0,0))} \cap \left(\mathbb{S}^1 \times \mathbb{S}^{d,0}\right)$. By Theorem 11, there are $b_k \in K_{\tau_k}^-$ and $x_k \in E_{\tau_k}^{+,0}$ with $e_P(\tau_k, b_k + x_k) \to (\tau, z, 0)$ and $\|x_k\| \to \infty$. This implies $\|b_k + x_k\| \to \infty$ and

$$e_P(\tau_k, b_k + x_k) = \left(\tau_k, \frac{b_k}{\|(b_k + x_k, 1)\|} + \frac{x_k}{\|(b_k + x_k, 1)\|}, \frac{1}{\|(b_k + x_k, 1)\|}\right).$$

Using that the b_k remain bounded, one finds

$$\frac{b_k}{\|(b_k + x_k, 1)\|} \to 0, \ \frac{1}{\|(b_k + x_k, 1)\|} \to 0, \text{ and } \frac{x_k}{\|(b_k + x_k, 1)\|} - \frac{x_k}{\|(x_k, 1)\|} \to 0.$$

Then, it follows that

$$e_P(\tau_k, x_k) = \left(\tau_k, \frac{(x_k, 1)}{\|(x_k, 1)\|}\right) \to (\tau, z, 0).$$

This shows that $(\tau, z, 0) \in \overline{e_P(\mathcal{E}^{+,0})} \cap \left(\mathbb{S}^1 \times \mathbb{S}^{d,0}\right)$. The assertions for the controllable set follow similarly.

(ii) By Theorem 13, it follows that the unbounded part of $\mathrm{int} D^a$ is $\mathcal{E}^0$. ◻

Remark 6 Theorem 15(i) shows that for the autonomized system on the Poincaré sphere bundle $\mathbb{S}^1 \times \mathbb{S}^d$ the closure of the reachable set from the north pole $e_P(0, 0) = (0, (0, 1)) \in \mathbb{S}^1 \times \mathbb{S}^d$ intersects the "equator" $\mathbb{S}^1 \times \mathbb{S}^{d,0}$ in the image under e_P of the center-stable subbundle $\mathcal{E}^{+,0}$. A closer look at the dynamics on the equator reveals a finer picture: Consider the Floquet bundles $\{(\tau, x) \in \mathbb{S}^1 \times \mathbb{R}^d \,|x \in L(\lambda_j, \tau)\}$. The projected flow on the projective bundle $\mathbb{S}^1 \times \mathbb{P}^{d-1}$ goes from the projected Floquet bundle for λ_j to the projected Floquet bundles with $\lambda_i > \lambda_j$. This can be made precise by some notions from topological dynamics: the projected Floquet bundles form the finest Morse decomposition, in particular, they coincide with the chain recurrent components (cf. Colonius and Kliemann [8, Section 7.2 and Theorem 8.3.3]). For the relation to the flow on $\mathbb{S}^1 \times \mathbb{S}^{d-1}$, one can prove that for every chain recurrent component on $\mathbb{S}^1 \times \mathbb{P}^{d-1}$ there are at most two chain recurrent components on $\mathbb{S}^1 \times \mathbb{S}^{d-1}$ projecting to it. By Proposition 14 (ii), this also describes the flow on the "equator" $\mathbb{S}^1 \times \mathbb{S}^{d,0}$. Examples 2 and 3 illustrate some of these claims.

6 Examples

First we note the following consequence of Theorem 13. In the scalar case with $d = 1$, one obtains from the inclusions in (20) that one of the following cases holds: The set $\mathrm{int} D^a$ is contained either in $\mathcal{K}^-$ or in $\mathcal{K}^+$ (if the Floquet exponent is negative or positive, resp.) or $D^a = \mathcal{E}^0 = \mathbb{S}^1 \times \mathbb{R}$ (if 0 is the Floquet exponent). In the first two cases D^a is bounded, and in the third case, it is unbounded.

Example 1 Consider the periodic scalar example

$$\dot{x}(t) = a(t)x(t) + u(t), \quad u(t) \in U = [-1, 1], \tag{25}$$

with $a(t) := -1$ for $t \in [0, 1]$ and $a(t) := -2$ for $t \in (1, 2]$ extended to a 2-periodic function on $\mathbb{R}$. Note that for $t \geq s$, $x_0 \in \mathbb{R}$, and $u \in \mathcal{U}$ the solution is

$$\varphi(t; s, x_0, u) = X(t, s)x_0 + \int_s^t X(t, \sigma)u(\sigma)\mathrm{d}\sigma \text{ with } X(t, s) = e^{\int_s^t a(\sigma)\mathrm{d}\sigma} > 0.$$

The system with unconstrained controls is controllable, and the stable subspace is $E_\tau^- = \mathbb{R}$ for all $\tau \in [0, 2]$. Lemma 12 implies that $\mathbb{S}^1 \times \{0\} \subset \mathrm{int}\overline{\mathbf{C}^a(0, 0)}$ for the autonomized system, and taking $u \equiv 0$ one sees that $\mathbb{S}^1 \times \mathbb{R} = \overline{\mathbf{C}^a(0, 0)}$, hence $\mathbf{C}^a(0, 0) = \mathbb{S}^1 \times \mathbb{R}$. By Theorem 13, there is a unique control set D^a with nonvoid interior and $\mathbb{S}^1 \times \{0\} \subset \mathrm{int} D^a$. This yields $D^a = \overline{\mathbf{R}^a(0, 0)}$. Recall from Lemma 7 that

$$\mathbf{R}^a(0, 0) = \{(\tau, x) \in \mathbb{S}^1 \times \mathbb{R}^d \,|x \in \mathbf{R}_{2\mathbb{N}+\tau}(0, 0)\}. \tag{26}$$

The solutions satisfy, for $t \geq s \geq 0$ and $u \in \mathcal{U}$,

$$\varphi(t; s, x_0, -1) \leq \varphi(t; s, x_0, u) = -\varphi(t; s, -x_0, -u) \leq \varphi(t; s, x_0, 1).$$

This implies that $\varphi(t; 0, 0, u) \le \varphi(t; 0, 0, 1)$ for all $t \ge 0$ and $u \in \mathcal{U}$. Since $U = -U$ the equation above with $x_0 = 0$ implies that the reachable sets $\mathbf{R}_t(0, 0)$ are symmetric around 0. Together with (26), this shows that for the computation of D^a it suffices to determine $\overline{\mathbf{R}^a(0, 0)} \cap [0, \infty)$. By Proposition 5 $\mathbf{R}_{2\mathbb{N}+\tau}(0, 0)$ is convex. Using $u \equiv 0$ and $u \equiv 1$, one finds that $\mathbf{R}_{2k+\tau}(0, 0) \cap [0, \infty) = [0, \varphi(2k + \tau; 0, 0, 1)]$ for all $k \in \mathbb{N}$ and $\tau \in \mathbb{S}^1 = [0, 2)$.

Claim: For fixed $k \in \mathbb{N}$, the reachable sets $\mathbf{R}_{2k+\tau}(0, 0)$ are increasing with $\tau \in [0, 1]$ and decreasing with $\tau \in [1, 2)$. For fixed $\tau \in [0, 2)$, they are increasing with $k \in \mathbb{N}$ and they are given by

$$\mathbf{R}_{2\mathbb{N}+\tau}(0, 0) = \bigcup_{k\in\mathbb{N}} \mathbf{R}_{2k+\tau}(0, 0) = \left(-\frac{r(\tau)}{1 - e^{-3}}, \frac{r(\tau)}{1 - e^{-3}}\right), \tag{27}$$

where

$$r(\tau) := \begin{cases} \frac{1}{2}e^{-2-\tau} - e^{-3} - \frac{1}{2}e^{-\tau} + 1 & \text{for } \tau \in [0, 1] \\ \frac{1}{2}e^{2-2\tau}\left(1 - e^{-1} - e^{2\tau-5}\right) + \frac{1}{2} & \text{for } \tau \in [1, 2) \end{cases}.$$

Since in the proof of this claim we always take control $u \equiv 1$, we suppress this argument in φ. Let $x_0 \in \mathbb{R}$ and compute for $\tau \in [0, 1]$ using 2-periodicity

$$\begin{aligned} \varphi(2 + \tau; \tau, x_0) &= \varphi(2 + \tau; 2, \varphi(2; \tau, x_0)) = \varphi(\tau; 0, \varphi(2; \tau, x_0)) \\ &= \varphi(\tau; 0, \varphi(2; 1, \varphi(1; \tau, x_0))) = e^{-3}x_0 + r(\tau). \end{aligned} \tag{28}$$

For $\tau \in [1, 2]$ compute

$$\varphi(\tau; 0, 0) = \varphi(\tau; 1, \varphi(1; 0, 0)) = \frac{1}{2}e^{-2\tau+2} - e^{-2\tau+1} + \frac{1}{2},$$
$$\varphi(2 + \tau; \tau, x_0) = \varphi(2 + \tau; 3, \varphi(3; \tau, x_0)) = \varphi(\tau; 1, \varphi(3; \tau, x_0)) = e^{-3}x_0 + r(\tau).$$

Repeated use of these formulas, periodicity, and induction show for $k \in \mathbb{N}$ and $\tau \in [0, 2]$

$$\begin{aligned} \varphi(2(k + 1) + \tau; 0, 0) &= \varphi(2 + \tau; \tau, \varphi(2k + \tau; 0, 0)) = e^{-3}\varphi(2k + \tau; 0, 0) + r(\tau) \\ &= e^{-3(k+1)}\varphi(\tau; 0, 0) + \sum\nolimits_{j=0}^{k} e^{-3j} r(\tau). \end{aligned} \tag{29}$$

Equation (29) implies $\lim_{k\to\infty} \varphi(2k + \tau; 0, 0) = \frac{r(\tau)}{1-e^{-3}}$ proving (27).

The sets $\mathbf{R}_{2k+\tau}(0, 0)$ are increasing with k since

$$\begin{aligned} &\varphi(2(k + 1) + \tau; 0, 0) - \varphi(2k + \tau; 0, 0) = (e^{-3(k+1)} - e^{-3k})\varphi(\tau; 0, 0) + e^{-3k}r(\tau) \\ &= \begin{cases} e^{-3k-3-\tau}\left(\frac{1}{2}e^3 + \frac{1}{2}e - 1\right) > 0 & \text{for } \tau \in [0, 1] \\ e^{-2\tau+1}\left[e^{-3}\left(\frac{1}{2}e - 1\right) + \frac{1}{2}\right] > 0 & \text{for } \tau \in [1, 2] \end{cases}. \end{aligned}$$

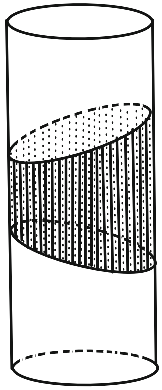

Fig. 1 Control set D^a in Example 1

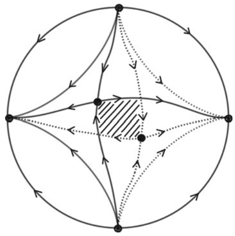

Fig. 2 Control set $e_p(D^a)$ and phase portraits for $u = 1, u = -1$ in Example 2

The sets $\mathbf{R}_{2\mathbb{N}+\tau}(0, 0)$ are increasing with $\tau \in [0, 1]$ since for $0 \leq \sigma \leq \tau \leq 1$

$$\varphi(2k + \tau; 0, 0) = \varphi(\tau; 0, \varphi(2k; 0, 0)) = e^{-\tau}(\varphi(2k; 0, 0) - 1) + 1 \leq \varphi(2k + \sigma; 0, 0).$$

Here, we use that $e^{-\tau} \leq e^{-\sigma}$ and that for $x \geq 1$ one has $a(t)x + u \leq 0$ for all $u \in U$ implying $\varphi(2k; 0, 0) - 1 \leq 0$.

The sets $\mathbf{R}_{2\mathbb{N}+\tau}(0, 0)$ are decreasing with $\tau \in [1, 2]$ since for $1 \leq \sigma \leq \tau \leq 2$

$$\begin{aligned}\varphi(2k + \tau; 0, 0) = \varphi(\tau; 1, \varphi(2k + 1; 0, 0)) &= e^{2-2\tau}\varphi(2k + 1; 0, 0) + \int_1^\tau e^{-2(\tau-s)}\mathrm{d}s\\ &= e^{2-2\tau}(\varphi(2k + 1; 0, 0) - \frac{1}{2}) + \frac{1}{2} \leq \varphi(2k + \sigma; 0, 0).\end{aligned}$$

Here, we use $e^{2-2\tau} \leq e^{2-2\sigma}$ and $\varphi(2k + 1; 0, 0) \geq \varphi(1; 0, 0) = 1 - e^{-1} > \frac{1}{2}$.

Figure 1 presents a sketch of the control set D^a in $\mathbb{S}^1 \times \mathbb{R}$.

The following two examples are autonomous two-dimensional linear control systems. Hence, it is not necessary to autonomize the system and the results on the control sets in $\mathbb{R}^2$ follow from Sontag [24, Corollary 3.6.7]. These examples serve as illustrations for the projection to the Poincaré sphere.

Example 2 Consider

$$\dot{x}(t) = x(t) + u(t), \quad \dot{y}(t) = -y(t) + u(t), \tag{30}$$

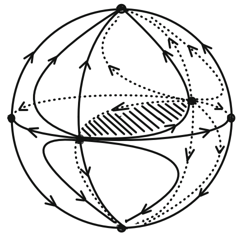

Fig. 3 Control set $e_p(D^a)$ and phase portraits for $u = 1, u = -1$ in Example 3

with $u(t) \in U = [-1, 1]$. Here, the origin is a saddle for the uncontrolled system. For the control system induced on the Poincaré sphere $\mathbb{S}^2$, a computation based on (23) yields

$$\begin{aligned}\dot{s}_1 &= \left[1 - s_1^2 + s_2^2 - u(s_1s_3 + s_2s_3)\right] s_1 + us_3 \\ \dot{s}_2 &= \left[-1 - s_1^2 + s_2^2 - u(s_1s_3 + s_2s_3))\right] s_2 + us_3 \\ \dot{s}_3 &= \left[-s_1^2 + s_2^2 - u(s_1s_3 + s_2s_3)\right] s_3. \end{aligned} \tag{31}$$

For $u = 0$ the north pole $(0, 0, 1)$ is the only equilibrium, and for $u \neq 0$ the equilibria move away from the north pole. Theorem 13 implies that there is a unique control set $D^a \subset \mathbb{S}^1 \times \mathbb{R}^2$ with nonvoid interior and that it is bounded. By Theorem 15(ii), $D_P^a = e_P(D^a)$ is the unique control set with nonvoid interior on the upper hemisphere $\mathbb{S}^{2,+}$. On the equator one has $s_3 = 0$ and the equation reduces to

$$\dot{s}_1 = 2s_2^2 s_1, \quad \dot{s}_2 = -2s_1^2 s_2.$$

This coincides with the projection of the homogeneous part of the original equation in $\mathbb{R}^2$ onto the unit circle $\mathbb{S}^1$. The equilibria are $(\pm 1, 0, 0)$ and $(0, \pm 1, 0)$. Linearization on the equator $\mathbb{S}^{2,0}$ yields in $e^1 = (1, 0, 0)$ and $e^2 = (0, 1, 0)$

$$\dot{x}_2 = -2x_2 \text{ and } \dot{x}_1 = 2x_1, \text{ resp.}$$

If we linearize on the sphere $\mathbb{S}^2$ we have to linearize (31) in e^1 and e^2 with respect to the second and third arguments only. We obtain $\begin{pmatrix} \mp 2 & u \\ 0 & -1 \end{pmatrix}$ with eigenvalues ∓ 2 and ∓ 1 with eigenvectors given by $(x, 0)^\top$ and $(\pm u \cdot x, x)^\top$, $x \neq 0$, resp.

The orthogonal projection of the system on the upper hemisphere $\mathbb{S}^{2,+}$ to the unit disk yields the global phase portrait with control set $e_P(D^a)$ sketched in Fig. 2. Observe that near the equator s_3 is close to 0, hence the control vector field in (31) goes to 0 for $s_3 \to 0$.

Remark 7 Perko [22] considers the differential equation (30) with $u = 0$. In this case the formulas derived above coincide with his results. The global phase portrait in Fig. 2 is similar to [22, Figure 5 on p. 275] with the additional feature that around the north pole of $\mathbb{S}^2$ the image of the control set occurs. Perko [22], as well as Lefschetz [20, pp. 202], actually, does these computations for differential forms, i.e., in the cotangent bundle of the sphere.

The following example is a slight modification of Example 2. It illustrates Remark 6 since the flow on the intersection of $\overline{e_P(\mathbf{R}(0))}$ with the equator is nontrivial.

Example 3 Consider the autonomous system given by

$$\dot{x}(t) = x(t) + u(t), \quad \dot{y}(t) = 2y(t) + u(t),$$

with $u(t) \in U = [-1, 1]$. Note that for constant u the equilibrium given by $(-u, -u/2)$ is an unstable knot. Since the eigenvalues 1 and 2 are positive, the control set D^a with nonvoid interior is bounded. The reachable set from the origin coincides with the unstable subspace and satisfies $\mathbf{R}(0) = E^+ = \mathbb{R}^2$. For the projection to the Poincaré sphere one obtains

$$\overline{e_P(\mathbf{R}(0))} \cap \mathbb{S}^{2,0} = \overline{e_P(E^+)} \cap \mathbb{S}^{2,0} = \mathbb{S}^{2,0}.$$

On the other hand, Proposition 14(ii) shows that the flow on the equator $\mathbb{S}^{2,0}$ is determined by the flow on the unit circle $\mathbb{S}^1$ induced by the homogeneous part (with $u \equiv 0$). The Floquet subspaces $L(1) = \mathbb{R} \times \{0\}$ and $L(2) = \{0\} \times \mathbb{R}$ are given by the eigenspaces and intersect $\mathbb{S}^1$ in the equilibria $(\pm 1, 0)$ and $(0, \pm 1)$, resp. All other points $s_0 \in \mathbb{S}^1$ satisfy $\lim_{t \to -\infty} s(t, s_0) = (\pm 1, 0)$ and $\lim_{t \to \infty} s(t, s_0) = (0, \pm 1)$. The orthogonal projection of the system on the upper hemisphere $\mathbb{S}^{2,+}$ to the unit disk yields the global phase portrait with control set $e_P(D^a)$ sketched in Fig. 3.

7 Controllability properties of quasi-affine systems

In this section, we apply the results above to the study of controllability properties for quasi-affine control systems of the form (3). Explicitly, system (3) may be written as

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{p} v_i(t) A_i x(t) + B(v(t))u(t), \quad (u, v) \in \mathcal{U} \times \mathcal{V}. \tag{32}$$

We denote the solutions of (32) with initial condition $x(0) = x_0 \in \mathbb{R}^d$ by $\psi(t; x_0, u, v),\ t \in \mathbb{R}$. The homogeneous part of (32) is the bilinear control system

$$\dot{x}(t) = A(v(t))x(t), \quad v \in \mathcal{V}, \tag{33}$$

and we denote the solutions of (33) with $x(0) = x_0$ by $\psi_{\text{hom}}(t; x_0, v),\ t \in \mathbb{R}$. Control systems (32) and (33) come with associated flows given by

$$\begin{aligned} &\Psi : \mathbb{R} \times \mathcal{U} \times \mathcal{V} \times \mathbb{R}^d \to \mathcal{U} \times \mathcal{V} \times \mathbb{R}^d, \text{ with} \\ &\quad \Psi(t; u, v, x) := (u(t + \cdot), v(t + \cdot), \psi(t; x, u, v)) \text{ and} \end{aligned}$$

$$\Psi_{\text{hom}} : \mathbb{R} \times \mathcal{V} \times \mathbb{R}^d \to \mathcal{V} \times \mathbb{R}^d, \text{ with}$$
$$\Psi_{\text{hom}}(t; v, x) := (v(t+\cdot), \psi_{\text{hom}}(t; x, v)),$$

resp. Here, $u(t+\cdot)(s) := u(t+s)$ and $v(t+\cdot)(s) := v(t+s)$, $s \in \mathbb{R}$, are the right shifts and $\mathcal{U} \subset L^\infty(\mathbb{R}, \mathbb{R}^m)$ and $\mathcal{V} \subset L^\infty(\mathbb{R}, \mathbb{R}^p)$ are endowed with a metric for the weak* topology. Then, $\mathcal{U}$ and $\mathcal{V}$ are compact and the shifts are chain transitive; cf. Colonius and Kliemann [7, Chapter 4] or Kawan [18, Section 1.4]. The flow Ψ_{hom} is a continuous linear skew product flow on the vector bundle $\mathcal{V} \times \mathbb{R}^d$ since (33) is control-affine. On the other hand, the affine flow Ψ on the vector bundle $(\mathcal{U} \times \mathcal{V}) \times \mathbb{R}^d$ is not continuous, in general, even if we suppose that $B(v) := B_0 + \sum_{i=1}^p v_i B_i$ with $B_0, B_1, \ldots, B_p \in \mathbb{R}^{d\times m}$. In fact, if products $v_i u_j$ occur on the right hand side of (32), the system is not control-affine, and hence continuity does not hold.

For any periodic $v \in \mathcal{V}$, one obtains a periodic linear control system

$$\dot{x}(t) = A(v(t))x(t) + B(v(t))u(t), \quad u \in \mathcal{U}. \tag{34}$$

Fix a T_v-periodic control $v \in \mathcal{V}$ and parametrize the unit circle $\mathbb{S}^1$ by $\tau \in [0, T_v)$. A corresponding augmented autonomous control system on $\mathbb{S}^1 \times \mathbb{R}^d$ is defined by

$$\psi_v^a(t; (\tau_0, x_0), u) = (t + \tau_0 \bmod T_v, \psi(t; x_0, u, v(\tau_0 + \cdot))), \quad u \in \mathcal{U}. \tag{35}$$

The reachable set of $(\tau_0, x_0) \in \mathbb{S}^1 \times \mathbb{R}^d$ is

$$\mathbf{R}_v^a(\tau_0, x_0) := \{\psi_v^a(t; (\tau_0, x_0), u) \,|t \ge 0 \text{ and } u \in \mathcal{U}\}.$$

Analogously, the controllable sets $\mathbf{C}_v^a(\tau_0, x_0)$ are defined. If the system in (34) without control restriction is controllable, Theorem 13 shows that one finds a unique control set $D_v^a = \overline{\mathbf{R}_v^a(0,0)} \cap \mathbf{C}_v^a(0,0)$ with nonvoid interior of the autonomized system (35) and $\mathbb{S}^1 \times \{0\} \subset \operatorname{int} D_v^a$.

Let $\pi_2 : \mathbb{S}^1 \times \mathbb{R}^d \to \mathbb{R}^d$, $\pi_2(\tau, x) = x$ for $(\tau, x) \in \mathbb{S}^1 \times \mathbb{R}^d$. The following theorem establishes the existence of a control set for quasi-affine systems, defined analogously as in Definition 1, containing all $\pi_2(D_v^a)$ for the control sets D_v^a for periodic $v \in \mathcal{V}$.

Theorem 16 *Suppose that the following assumptions hold:*

(i) for every periodic $v \in \mathcal{V}$ the periodic linear system in (34) with unconstrained controls $u \in L^\infty(\mathbb{R}, \mathbb{R}^m)$ is controllable;

(ii) the quasi-affine system (32) is locally accessible, i.e., $\mathbf{R}_{\le S}(x)$ and $\mathbf{C}_{\le S}(x)$ have nonvoid interiors for all $S > 0$ and all $x \in \mathbb{R}^d$.

Then, the quasi-affine system (32) has a control set D with nonvoid interior such that for all periodic $v \in \mathcal{V}$ the control sets D_v^a of the autonomized periodic linear control system (35) satisfy $\pi_2(D_v^a) \subset D$.

Proof Fix a T_v-periodic control $v \in \mathcal{V}$. The set $\pi_2(D_v^a)$ is a neighborhood of $0 \in \mathbb{R}^d$ and for all $x, y \in \pi_2(D_v^a)$ there are $\tau_x, \tau_y \in \mathbb{S}^1$ with $(\tau_x, x), (\tau_y, y) \in D_v^a$ and $(\tau_y, y) \in \overline{\mathbf{R}_v(\tau_x, x)}$. This means that there are $t_n \ge 0$ and $u_n \in \mathcal{U}$ with

$$\psi_v^a(t_n; (\tau_x, x), u_n) = (t_n + \tau_x \bmod T_v, \psi(t_n; x, u_n, v(\tau_x + \cdot))) \to (\tau_y, y) \text{ for } n \to \infty.$$

In particular, this shows that $y \in \overline{\mathbf{R}(x)}$, where $\mathbf{R}(x)$ is the reachable set from x of the quasi-affine system (32) given by

$$\mathbf{R}_{\leq S}(x) := \{\psi(t; x, u', v') \,|\, t \in [0, S],\ (u', v') \in \mathcal{U} \times \mathcal{V}\} \text{ and } \mathbf{R}(x) := \bigcup\nolimits_{S>0} \mathbf{R}_{\leq S}(x).$$

Define D as the union of all sets D' satisfying $D' \subset \overline{\mathbf{R}(x)}$ for all $x' \in D'$ and containing $\pi_2(D_v^a)$. We claim that $D \subset \overline{\mathbf{R}(x)}$ for all $x \in D$. For the proof of the claim, let $x, y \in D$. Then, there are sets D' and D'' with $\pi_2(D_v^a) \subset D' \cap D''$ and $x \in D'$, $y \in D''$. We know that $0 \in \mathrm{int}\pi_2(D_v^a)$. By local accessibility of the quasi-affine system there is $S > 0$ with $\varnothing \neq \mathrm{int}\mathbf{C}_{\leq S}(0) \subset \pi_2(D_v^a) \subset D'$. Then, the inclusion $D' \subset \overline{\mathbf{R}(x)}$ implies $0 \in \mathbf{R}(x)$. Since $0, y \in D''$ the claim follows from $y \in \overline{\mathbf{R}(0)} \subset \overline{\mathbf{R}(x)}$. Thus D is a maximal set with the property that for all $x \in D$ one has $D \subset \overline{\mathbf{R}(x)}$. Since $\mathrm{int}D \neq \varnothing$ Kawan [18, Proposition 1.20] implies that D is a control set.

For every periodic control $v \in \mathcal{V}$, the projected set $\pi_2(D_v^a)$ contains $0 \in \mathbb{R}^d$. Thus, the maximality property of control sets implies that the control set D is independent of v and hence contains $\pi_2(D_v^a)$ for every periodic $v \in \mathcal{V}$. □

Next we show that under some additional assumptions the control set D of the quasi-affine system coincides (up to closure) with the union of the projected control sets D_v^a. Thus, the control set D can be obtained by fixing periodic controls v and determining the control sets D_v of the corresponding autonomized systems (11).

Theorem 17 *Suppose that the following assumptions hold:*

(i) for every periodic $v \in \mathcal{V}$ the periodic linear system in (34) *with unconstrained controls $u \in L^\infty(\mathbb{R}, \mathbb{R}^m)$ is controllable;*

(ii) the quasi-affine system (32) *is locally accessible;*

(iii) for all periodic $v \in \mathcal{V}$ all Floquet exponents of the periodic homogeneous part (33) *of* (32) *are different from* 0*;*

Then, the quasi-affine system (32) *has a unique control set D with nonvoid interior, and it satisfies*

$$\overline{D} = \overline{\bigcup\nolimits_{v \in \mathcal{V}\ periodic} \pi_2(\mathrm{int}D_v^a)}.$$

Proof The control set D from Theorem 16 contains all $\pi_2(\mathrm{int}D_v^a)$. Let E be an arbitrary control set with nonvoid interior of (32). By local accessibility, Colonius and Kliemann [7, Lemma 3.2.13(i)] shows that $\overline{E} = \overline{\mathrm{int}E}$. Hence, it suffices to prove that $\mathrm{int}E \subset \overline{\bigcup_{v\in\mathcal{V}\text{ periodic}} \pi_2(\mathrm{int}D_v^a)}$, which also implies $E = D$. Fix a point $x_0 \in \mathrm{int}E$. Denote by $\mathrm{int}_\infty(\mathcal{U})$ the interior of $\mathcal{U}$ with respect to the L^∞-norm, and note that $\mathrm{int}_\infty(\mathcal{U})$ is dense in $\mathcal{U}$ in this norm.

Claim. For every $\varepsilon > 0$, there are $y \in \mathbb{R}^d$, $T > 0$, and $(u, v) \in \mathrm{int}_\infty(\mathcal{U}) \times \mathcal{V}$ with $\|y - x_0\| < \varepsilon$ and $\psi(dT; y, u, v) = y$.

For the proof of the claim note first that by local accessibility [7, Lemma 3.2.13(iii)] implies $\mathrm{int}E \subset \mathbf{R}(x_0)$ and hence there are $u^0 \in \mathcal{U}$, $v \in \mathcal{V}$, and $T > 0$ with $x_0 = \varphi(T; x_0, u^0, v)$. We may suppose that u^0 and v are T-periodic functions. Thus, $T = T_v$

for the T-periodic control $v \in \mathcal{V}$ and we obtain $x_0 = \varphi(\mathrm{d}T_v; x_0, u^0, v)$. Since $\mathrm{int}_\infty(\mathcal{U})$ is dense in $\mathcal{U}$ one finds for all $\varepsilon > 0$ a $\mathrm{d}T_v$-periodic control u with $\|u - u^0\|_{L^\infty} < \varepsilon$. By the hyperbolicity assumption (ii), Colonius, Santana, Setti [11, Proposition 2.9(i)] implies that the $\mathrm{d}T_v$-periodic inhomogeneous differential equation (34) has a unique d_v-periodic solution with initial value y at time 0. With the principal fundamental solution denoted by $X_v(t, s)$, it is given by

$$y = [I_d - X_v(\mathrm{d}T_v, 0)]^{-1} \int_0^{\mathrm{d}T_v} X_v(\mathrm{d}T_v, s) B(v(s)) u(s) \mathrm{d}s.$$

By [11, Proposition 2.9(iv)], the initial values y of these periodic solutions converge to x_0 for u converging to u^0 in $L^\infty([0, \mathrm{d}T_v], \mathbb{R}^m)$. This proves the **claim**.

It remains to prove that x_0 is in $\overline{\pi_2(D_v^a)}$. This follows if we can show that $y \in \pi_2(D_v^a)$ since y is arbitrarily close to x_0. Assumption (i) and Theorem 4(ii) imply that for all points $z \in \mathbb{R}^d$ there is $u' \in L^\infty([0, \mathrm{d}T_v], \mathbb{R}^m)$ such that

$$z = \psi(\mathrm{d}T_v; 0, u', v) = \int_0^{\mathrm{d}T_v} X_v(\mathrm{d}T_v, s) B(v(s)) u'(s) \mathrm{d}s.$$

Since $u \in \mathrm{int}_\infty(\mathcal{U})$ it follows that for all z in a neighborhood $N_1(y)$ of y there is $u' \in \mathcal{U}$ with

$$z - X_v(\mathrm{d}T_v, 0)y = \int_0^{\mathrm{d}T_v} X_v(\mathrm{d}T_v, s) B(v(s)) u'(s) \mathrm{d}s, \text{ hence } z = \psi(\mathrm{d}T_v; y, u', v).$$

With $\mathrm{d}T_v = 0 \bmod T_v$, this means by Lemma 7 that the points $(0, z) \in \mathbb{S}^1 \times N_1(y)$ are contained in the reachable set

$$\mathbf{R}^a_{v,\mathrm{d}T_v}(0, y) = \{(0, y') \,|\, y' \in \mathbf{R}_{v,\mathrm{d}T_v}(0, y)\}$$

of the autonomized system (35) for the T_v-periodic v. Applying the same arguments to the time-reversed system, one finds that all $(0, z) \in \mathbb{S}^1 \times \mathbb{R}^d$ with z in a neighborhood $N_2(y)$ of y are in the controllable set $\mathbf{C}^a_{v,\mathrm{d}T_v}(0, y)$. Every point $(0, z)$ with $z \in N_1(y) \cap N_2(y)$ can be steered to $(0, y)$ and then to any other point in this intersection. This implies that $(0, y)$ is in the interior of a control set of the autonomized system (35). The only control set with nonvoid interior of this system is D_v^a, hence it follows that $(0, y) \in \mathrm{int} D_v^a$ and $y \in \pi_2(\mathrm{int} D_v^a)$ and concludes the proof. □

Similarly as the periodic linear system (4) also the quasi-affine system (32) can be projected to the upper hemisphere $\mathbb{S}^{d,+}$ of the Poincaré sphere by a conjugacy $e_P^0 : \mathbb{R}^d \to \mathbb{S}^{d,+}$. We obtain the following corollary.

Corollary 18 *Under the assumptions of Theorem 17, the control set D of the quasi-affine system (32) projects to the unique control set $e_P^0(D)$ with nonvoid interior for the system induced on $\mathbb{S}^{d,+}$.*

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