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Chemical potential standard for atomic Bose–Einstein condensates

Sigmund Kohler¹ and Fernando Sols²

¹ Institut für Physik, Universität Augsburg, Universitätsstraße 1, D-86135 Augsburg, Germany

² Departamento de Física Teórica de la Materia Condensada and Instituto ‘Nicolás Cabrera’, Universidad Autónoma de Madrid, E-28049 Madrid, Spain
E-mail: sigmund.kohler@physik.uni-augsburg.de and fernando.sols@uam.es

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Abstract. When subject to an external time-periodic perturbation of frequency f , a Josephson-coupled two-state Bose–Einstein condensate may respond with a constant chemical potential difference $\Delta\mu = khf$, where h is Planck’s constant and k is an integer. We propose an experimental procedure to produce ac-driven atomic Josephson devices that can be used to define a standard of chemical potential. We investigate how to circumvent some of the specific problems derived from the present lack of advanced atom circuit technology. We include the effect of dissipation due to quasiparticles, which is essential to help the system relax towards the exact Shapiro resonance, and set limits to the range of values which the various physical quantities must have in order to favour the achievement of a stable and accurate chemical potential difference between the macroscopic condensates.

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1. Introduction

The realization of Bose–Einstein condensation in dilute atomic gases [1]–[3] has opened up the possibility of investigating the macroscopic behaviour of coherent quantum matter in a new class of physical system different from superconductors and helium liquids. Together with vortex formation, the Josephson effect between weakly coupled Bose condensates is probably the most characteristic signature of superfluidity. Initially predicted [4] and observed [5, 6] in the context of superconductivity, the Josephson effect has also been observed in superfluid ^3He [7, 8], a fact that underlies the fundamental nature of quantum behaviour at the macroscopic scale. Atomic Bose–Einstein condensates (BECs) have in common with superconductors and superfluids that their most remarkable properties are derived from the existence of a macroscopic wavefunction. Because of the profound analogies between these gauge-symmetry broken systems, the possibility of observing the Josephson effect in trapped atomic gases was recognized early [9]–[12]. Although some preliminary evidence for the existence of the Josephson effect in atomic gases already existed [13, 14], it is fair to say that, compared with its superconducting and superfluid counterparts, the exploration of the physics of weakly linked atomic condensates is still in its infancy. Given the potential richness and the convenient tunability of BEC set-ups, it is clear that decided progress must be made to investigate this whole new class of problems. At present however, an important limitation is that the technology of atom circuits (also called ‘atomtronics’) is still only moderately developed, although a bright future can already be foreseen [15, 16]. In this sense, proposals that do not rely on the feasibility of controlled atom transport are particularly welcome.

One of the most remarkable properties of superconducting devices is the existence of Shapiro resonances. When two weakly coupled superconductors are subject to a voltage difference that is the sum of a dc component V and a periodic signal $v \sin(2\pi ft)$, a continuous range of nonzero dc currents are possible if

$$V = \frac{h}{2e}kf, \quad (1)$$

where $2e$ is the Cooper pair charge, h is Planck’s constant and k is an integer [6, 17]. A more common set-up is one in which, due to the existence of a large impedance in series, an external current source is applied. If the external current consists of a dc contribution and a weak ac perturbation of frequency f , the Josephson link displays a dc V – I characteristic with voltage plateaus at values satisfying the resonance condition (1). The height of the Shapiro steps linking the voltage plateaus provides a method to measure the constant of nature $2e/h$ with such precision and universality [18, 19] that, since 1972, the opposite view has been adopted whereby $2e/h$ is assumed to be known and equation (1) is used to define a standard unit of voltage [17, 20, 21].

In this paper we address the question of whether it is possible to prepare a BEC Josephson junction (BJJ) that satisfies the resonance condition (1) in a stable form. Another way of phrasing the problem is whether macroscopic self-trapping of a large population imbalance [22, 23] may be stabilized against dissipation by an oscillating perturbation. A condensate in a time-dependent trap [24], and particularly a double condensate [25], has been shown to provide a convenient test ground for quantum chaos. Here we face a different regime which predominantly involves regular motion with dissipation. The challenge is to identify the conditions under which dissipation can be exploited to let the system evolve towards one of the stable resonance islands.

At present we have a limited understanding of the dissipation mechanisms operating in a BJJ, which render its macroscopic phase-number dynamics non-conservative. We know that

contributions may arise at least from incoherent exchange of normal atoms [11], creation of quasiparticles by the fluctuating condensate [23] or spontaneous atom losses [26, 27]. As long as the net atom loss stays relatively small, its effect on the phase dynamics is similar in many respects to that of incoherent particle exchange, both yielding ohmic dissipation under a wide range of circumstances.

We wish to stress that the physics discussed in this paper applies both to double-well condensates, displaying the external Josephson effect, and to optically coupled two-component atomic condensates, which exhibit the internal Josephson effect [12]. A central concept in the forthcoming discussion is the application of a time-dependent external potential. This may be achieved by the application of suitably designed time-dependent magnetic fields or dipole forces. Another essential ingredient, not considered in [22], is that the Josephson current can counteract the dissipative current, thereby permitting the existence of stationary Shapiro resonances. Moreover, by providing an effective friction to the phase dynamics, the dissipative current ensures that the system gets arbitrarily close to the exact resonance at sufficiently long times.

We want to remark that no fundamental reason prevents the possibility of exploring this novel approach to Shapiro resonances in the context of superconductors. There, capacitive couplings might be used to control or measure voltage differences.

Section 2 is devoted to a quantitative formulation of the problem. We know that each phase-locked solution generates an attractive basin in phase space. We raise the question of how dissipation can be exploited to let the system land successfully on that region, never to escape from it except for quantum tunnelling decay. The solution is offered in section 3, where we investigate the dissipative dynamics in phase space. Later, in section 4, we provide an estimate of the probability that a system prepared with a large chemical potential difference evolves in such a way that, in the process of decaying, it is trapped in a classically stable well. The stochasticity comes from the intrinsically random nature of the choice of phase made by the double-condensate system shortly after the connection [23]. Section 5 is devoted to the effect of quantum fluctuations, which may cause macroscopic tunnelling decay. The combined analysis of sections 2–5 yields a set of constraints that must be fulfilled to achieve an optimal realization of the predicted Shapiro resonances. A concluding discussion is provided in section 6.

2. Formulation of the problem

We consider N Bose-condensed atoms in a double-well trapping potential $V(x, \xi)$ that depends on a harmonically time-dependent control parameter $\xi(t) = \xi_0 \cos(\Omega t)$. In a situation sufficiently close to equilibrium, a split condensate is formed which can be described within a two-mode approximation. Thus, we use for the field operator the ansatz $\psi(x) = \varphi_A(x)a + \varphi_B(x)b$, where the orbital functions $\varphi_{A,B}$ are normalized to unity. This yields the two-mode Hamiltonian [10, 12]

$$H = -\frac{\hbar\omega_R}{2}(a^\dagger b + b^\dagger a) + E_A(N_A, \xi) + E_B(N_B, \xi). \quad (2)$$

Here, ω_R denotes the effective Rabi frequency of the two-mode problem, which is proportional to the Josephson coupling energy [12]. It is important to note that, in the case of optically coupled two-component BECs (internal Josephson effect), ω_R does not generally coincide with the Rabi frequency Ω_R governing atomic transitions. Rather, one has $\omega_R = \Omega_R s_{AB}$, where $s_{AB} \equiv \int dx \varphi_A^* \varphi_B$ [28]. It is only when the trapping configuration and the interactions are such

that $\varphi_A(x) = \varphi_B(x)$, that the two frequencies become identical. The Gross–Pitaevskii energy of fragment i :

$$E_i(N_i, \xi) = N_i \int dx \varphi_i^*(x) \left(-\frac{\hbar^2}{2m} \Delta + V(x, \xi) + \frac{gN_i}{2} |\varphi_i(x)|^2 \right) \varphi_i(x), \quad (3)$$

$i = A, B$, has inherited a time dependence from the control parameter $\xi(t)$.

For an analysis of the system in its classical limit, we replace in the Heisenberg equations of motion the operators a, b by $\sqrt{N_A} e^{-i\phi_A}, \sqrt{N_B} e^{-i\phi_B}$ and expand the Gross–Pitaevskii energies for small ξ and small number imbalance compared with the equilibrium values N_A^0 and N_B^0 (usually taken to be $N/2$). For the variables $\phi \equiv (\phi_A - \phi_B)$ and $z \equiv (N_A - N_B)/N \equiv 2n/N$, with time measured in units of $1/\omega_R$, this yields

$$\dot{\phi} = \frac{z}{\sqrt{1-z^2}} \cos \phi + \Lambda z + \varepsilon \cos(\Omega t) = \Delta\mu/\hbar, \quad (4)$$

$$\dot{z} = -\sqrt{1-z^2} \sin \phi - \gamma \dot{\phi}, \quad (5)$$

with the scaled driving amplitude

$$\varepsilon = \frac{\xi_0}{\hbar\omega_R} \left[\frac{\partial^2 E_A(N_A, \xi)}{\partial \xi \partial N_A} - \frac{\partial^2 E_B(N_B, \xi)}{\partial \xi \partial N_B} \right]_{N_A=N_B=N/2, \xi=0} \quad (6)$$

and the effective interaction constant

$$\Lambda = \frac{N}{2\hbar\omega_R} \left[\frac{\partial^2 E_A(N_A, 0)}{\partial N_A^2} + \frac{\partial^2 E_B(N_B, 0)}{\partial N_B^2} \right]_{N_A=N_B=N/2}. \quad (7)$$

Equations (4) and (5) can also be obtained from the classical non-rigid pendulum Hamiltonian [10, 12] (hereafter energies are expressed in units of $\hbar\omega_R$):

$$H(z, \phi, \lambda) = -\sqrt{1-z^2} \cos \phi + \frac{1}{2} \Lambda z^2 + \varepsilon z \cos(\Omega t), \quad (8)$$

where (z, ϕ) are canonically conjugate coordinates. For a simplification, we have assumed a symmetric situation with $E_A(N/2, \xi) = E_B(N/2, -\xi)$ —the generalization is, of course, straightforward and results in an additional phase drift. The last term in the equation of motion (5) has been introduced phenomenologically to describe a dissipative current, i.e. an incoherent exchange of atoms. For high temperatures, $k_B T \gg \Delta\mu$, this current is ohmic, i.e. proportional to the chemical potential difference, $\dot{n} = -G \Delta\mu$ [11]. Following the reasoning by Josephson [4], the chemical potential difference is given by the time derivative of the relative phase, $\dot{\phi} = \Delta\mu/\hbar$, and thus we obtain the dissipative term in equation (5) with³ $\gamma = 2\hbar G/N$. We will also analyse the more general case in which \dot{n} may not be a linear function of $\dot{\phi}$, a situation which is likely to appear at low temperatures.

In order to put the central discussion into the proper context, it is convenient to review the dynamics of the relative phase after two independently prepared condensates are connected [23]. Before the connection, the phase is completely undefined, which means that the Fock state of fixed particle number is in a coherent superposition of different phase states. Upon connection, various mechanisms involving quasiparticle dynamics intervene to destroy the coherence between the different phase states. As a result, the reduced density matrix of the macroscopic phase-number

³ At first sight there seems to be an ambiguity as to whether dissipation should come as $-\gamma\dot{\phi}$ or rather $-\gamma'z$. That the former is the correct approach can also be inferred from a careful study of quantum dissipation models [29]. In the present context, however, the choice is without practical consequences.

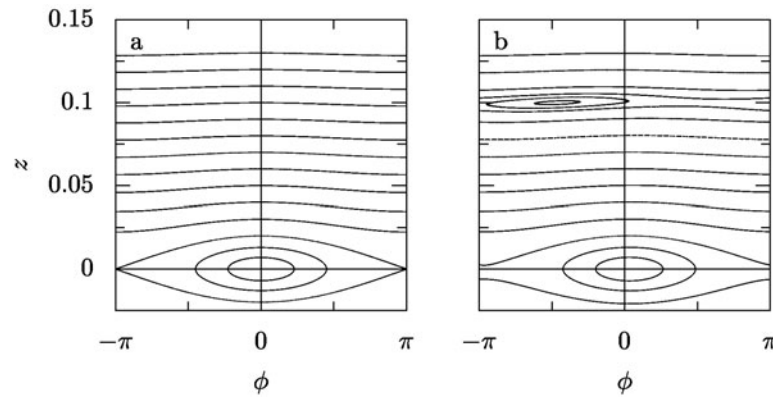


Figure 1. Phase space portrait of the undriven ($\varepsilon = 0$ (a)) and the driven ($\varepsilon = 100$ (b)) undamped momentum-shortened pendulum Hamiltonian (8) at stroboscopic times $t \bmod 2\pi k/\Omega = \pi/2$. The interaction is $\Lambda = 10^4$ and the driving frequency $\Omega = 1000$.

system becomes quasi-diagonal in the phase representation, which is to say that the phase is effectively measured among a menu of values uniformly distributed between 0 and 2π . After this quick definition of the phase, a phase-number Gaussian wavepacket forms that evolves semiclassically in the parabolic tight-binding lattice formed by the different number eigenstates. Due to the interaction term, the system experiences a tilted lattice and, thus, undergoes Bloch oscillations, which may be viewed as the ac Josephson response to an approximately constant $\Delta\mu$ created by the number imbalance with the possible concurrence of other factors. Its trajectory follows one of the running solutions in the phase space diagram shown in figure 1(a). Thus, the system oscillates around a nonzero number average \bar{z} displaying the so-called macroscopic quantum self-trapping (MQST) [22]. However, these oscillations are not stable, since the fluctuating condensate atom number in each well couples to the many-quasiparticle field and experiences dissipation. Thus the energy stored in the macroscopic degree of freedom decreases and \bar{z} decays slowly towards its equilibrium value [23].

In the presence of an ac driving, the situation may change qualitatively, since classically stable resonance islands form in phase space for values of z, ϕ such that

$$\Lambda \bar{z} = k\Omega, \quad d\bar{z}/dt = 0, \quad (9)$$

where k is an integer (see figure 1(b)). Here, the bars indicate the time average over fast Bloch or external driving oscillations, of frequencies $k\Omega$ and Ω , respectively, and the time derivative is meant over a longer timescale. In general, the system displays running solutions weakly oscillating around an average value \bar{z} which slowly decays because of dissipation. Such a decaying trajectory may or may not be trapped in one of the attractive basins around the Shapiro resonances.

For a provisional answer to the central question formulated above, we integrate numerically the equations of motion (4) and (5) in the Josephson regime ($\Lambda \gg 1$; we take $\Lambda = 10^4$), where the BJJ displays, in the absence of dissipation, stable running MQST solutions. When starting with an initial imbalance $z(0)$ that lies above a resonance island, we find two qualitatively different types of solutions: the first type (the lower curve in figure 2) is what one would typically expect, namely that, owing to the ohmic current, the imbalance decays until it leaves the MQST

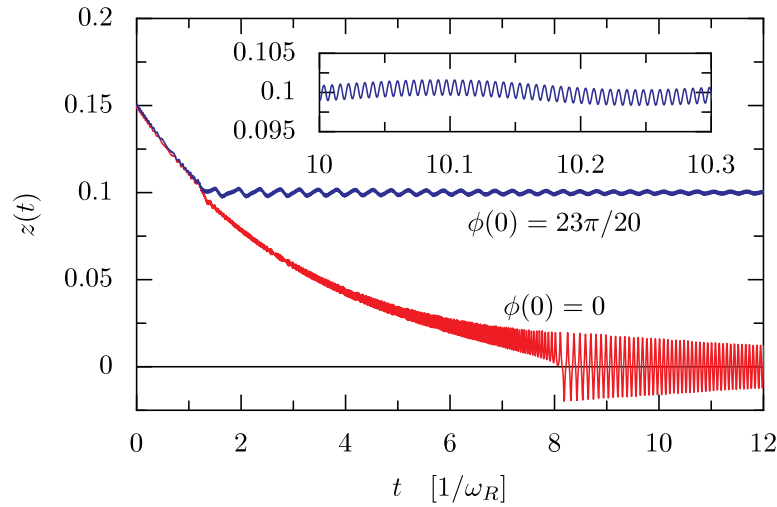


Figure 2. Population imbalance $z(t)$ for interaction $\Lambda = 10^4$, damping $\gamma = 3 \times 10^{-5}$, driving amplitude $\varepsilon = 100$ and frequency $\Omega = 1000$ for two different initial values of the phase. The saturation time-averaged value $\bar{z} = 0.1$ of the upper solution corresponds to the chemical potential difference $\hbar\Omega$ ($k = 1$). The inset is a blow-up that resolves the fast underlying Bloch (or MQST) oscillations.

regime and ultimately exhibits damped plasma oscillations. Except for the small kink displayed when crossing the resonance zone, the dynamics is qualitatively the same as in the undriven case. However, for the same parameters but different initial phase, there exist also solutions with an intriguing behaviour (see the upper curve in figure 2): there, the system gets trapped in the resonance island and the transient decay comes to a standstill instead of continuing towards the true equilibrium solution ($z = 0$). At long times, the number imbalance z oscillates with amplitude ε around a non-zero value $z_1 = \Omega/\Lambda$. We will show that then the average chemical potential difference is precisely a multiple of the driving frequency. More specifically, we find (in scaled units)

$$\Delta\mu = k\Omega + \varepsilon \cos(\Omega t). \quad (10)$$

We wish to remark that the BEC set-up considered here differs substantially from its superconducting analogues in that there the stabilization of a voltage difference at the Shapiro resonance is achieved with the concurrence of an externally imposed current, i.e. in an experiment a current source is needed. Here by contrast, the stabilization of a chemical potential difference cannot rely on such external current sources which for BECs are not (yet) available. As a drawback, the present type of ‘voltage’ standard can be obtained only at the price that a single run of the experiment does not warrant a stable solution. In the following, we provide a detailed analysis of this type of Shapiro resonance, estimating their parameter dependence and, eventually, the probability that they can be spontaneously realized.

3. Realization of phase-locked solutions

Our central interest lies in the existence and the stability of so-called phase-locked solutions, i.e. solutions $\phi(t)$ that have only small fluctuations around a long-time average $\bar{\phi} \approx k\Omega t$. Then,

since the main chemical potential difference between the two condensate fragments comes from the interaction energy, the average number imbalance \bar{z} settles down at a value $z_k \equiv k\Omega/\Lambda$. This motivates the ansatz

$$\phi = k\Omega t + \frac{\varepsilon}{\Omega} \sin(\Omega t) + \delta\phi, \quad (11)$$

$$z = \frac{k\Omega}{\Lambda} + \alpha \cos(k\Omega t) + \delta z, \quad (12)$$

where α denotes the yet unknown amplitude of the fast residual Bloch oscillations that can be appreciated in the inset of figure 2. We would like to choose α such that the resulting $\delta\phi$ and δz display only slow, decaying oscillations around a stationary solution $\delta\phi = \text{constant}$, $\delta z = 0$ to which they tend at long times. In other words, α should be such that, in the absence of δz and with $\delta\phi$ replaced by a constant, equations (11) and (12) correctly capture the asymptotic system dynamics.

The equations of motion for $\delta\phi$ and δz are obtained by inserting (11) and (12) into the original equations (4) and (5). The resulting system of equations involves rapidly oscillating coefficients of period $2\pi/\Omega$. It can be shown that, if $\alpha = 1/k\Omega$ and $\Omega \gg 1$, then $\delta\phi$ and δz vary more slowly than those coefficients (a result which we have confirmed by numerical studies), so that the different timescales can be separated consistently and all time-dependent coefficients can be replaced by their *time averages*. On the other hand, we are interested in the case of two condensate fragments which are comparable in size ($|z| \ll 1$). Thus we neglect within this analytic discussion the momentum shortening and replace the square root in the Hamiltonian (8) by unity [12]. Finally we obtain the equations of motion

$$\frac{d}{dt}\delta\phi = \Lambda \delta z, \quad (13)$$

$$\frac{d}{dt}\delta z = -J_k(\varepsilon/\Omega) \sin(\delta\phi) - \gamma k\Omega - \gamma \Lambda \delta z, \quad (14)$$

which describe a dissipative particle in the static tilted washboard potential sketched in figure 3. They can be obtained from the Hamiltonian

$$H(\delta\phi, \delta z) = \frac{1}{2}\Lambda \delta z^2 - J_k(\varepsilon/\Omega) \cos(\delta\phi) + \gamma k\Omega \delta\phi \quad (15)$$

together with the dissipative force $F_{\text{diss}} = -\gamma \Lambda \delta z$. The new canonical coordinates δz and $\delta\phi$ represent momentum and position, respectively, and J_k denotes the k th Bessel function of the first kind. We emphasize two differences between these equations and the undriven rigid pendulum Hamiltonian: first, there is a tilt $\gamma k\Omega$ which originates from a constant dissipative current caused by a finite number imbalance. Despite its physical origin, it appears formally as a conservative force. Second, the Josephson coupling energy for the phase shift $\delta\phi$ is *renormalized* by a factor $J_k(\varepsilon/\Omega)$ which, for $\varepsilon \ll \Omega$, renders it much smaller than the original Josephson energy. Correspondingly, the plasma frequency is renormalized by a factor⁴ $\sqrt{J_k(\varepsilon/\Omega)}$. Once $\delta\phi$ is trapped within one specific well, it exhibits damped plasma oscillations with the renormalized frequency until it ultimately comes to rest. In this stationary solution one has $\delta\dot{\phi} = \delta\dot{z} = 0$, which is possible thanks to the cancellation of the first two terms in the rhs of equation (14). Therefore, one finds from equation (11) and the Josephson relation $\dot{\phi} = \Delta\mu/\hbar$ the chemical potential difference (10).

⁴ If we had relaxed the assumption $|z| \ll 1$, then $J_k(\varepsilon/\Omega)$ would have appeared multiplied by a factor $\sqrt{1 - \bar{z}^2}$. However, one must remember that, if $|z|$ were to become comparable to unity, then the two-mode approximation might have to be revised.

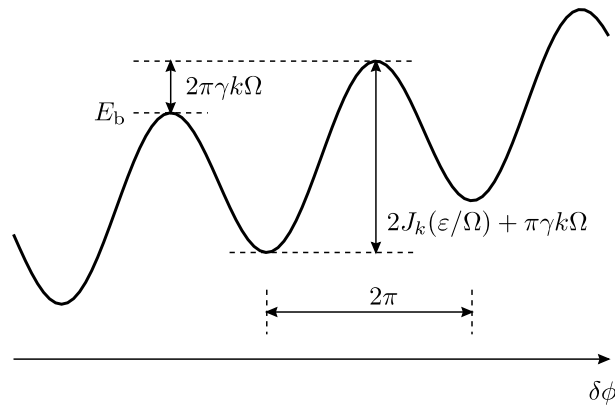


Figure 3. Sketch of the effective washboard potential (15) as a function of the phase difference $\delta\phi$ for $\gamma k\Omega < J_k(\epsilon/\Omega)$.

Having mapped the long-time dynamics of the originally time-dependent situation to an equivalent static problem, we are now able to derive the conditions under which phase-locked solutions exist. This results in the restrictions of the parameters summarized in table 1. Obviously, the effective washboard potential in the Hamiltonian (15) possesses minima only for $\gamma k\Omega < J_k(\epsilon/\Omega)$ which amounts for small driving amplitudes $\epsilon \ll \Omega$ to $\gamma k\Omega \ll (\epsilon/2\Omega)^k$. This is most easily satisfied for $k = 1$, where the condition for the existence of wells is $\epsilon > 2\gamma\Omega^2$. Since, on the other hand, the driving should be weak, $\epsilon \ll \Omega$, both conditions on the driving amplitude ϵ can be fulfilled simultaneously only for sufficiently small dissipation, $\gamma \ll 1/\Omega$.

Also the interaction strength Λ obeys restrictions which are obtained in the following way: physically, the idea behind our scheme is to counterbalance a dissipative current with the help of ac driving, thereby stabilizing a MQST solution against friction. The condition for operating clearly in a regime where MQST dominates is $2/\sqrt{\Lambda} \ll \bar{z}$ [22, 23]. On the other hand, the two condensate fragments should not differ too much in size, thus, $\bar{z} \ll 1$. Since we aim at stabilizing the imbalance $\bar{z} = \Omega/\Lambda$, fulfilling both conditions requires a sufficiently large interaction strength, typically $\Lambda \gtrsim 10^3$.

A further physical reason for operating the Bose–Josephson junction in the (interaction-dominated) Josephson limit stems from an important property of phase-locked solutions, namely that generally the centre of an attractive basin is not truly at rest but follows the trajectory of a particle in the absence of driving and dissipation [30]. In the Josephson regime, such a behaviour corresponds to the MQST solutions exhibiting Bloch oscillations with amplitude $\alpha = 1/\Lambda\bar{z}$ around an average value \bar{z} , as can be appreciated in equation (12). At resonance, their relative amplitude becomes $\alpha/\bar{z} = 1/\Lambda\bar{z}^2 = \Lambda/\Omega^2$. Ideally, for the realization of stable phase-locked solutions, this amplitude should be small, as is the case in the Josephson regime ($\Lambda \gg 1$). On the contrary, in the (non-interacting) Rabi regime ($\Lambda < 1$) [12], the undamped and undriven pendulum exhibits large number (Rabi) oscillations around $z = 0$, i.e. there is no MQST. This results in large oscillations of any attractive basin which render BJJs in the Rabi regime useless for the present purposes.

We conclude this section with a comparison between Shapiro resonances and another well-known, apparently similar dynamical phenomenon. It has been shown [31] that for a high-frequency driving with $\Omega \gg \Delta\mu/\hbar$, the Josephson energy acquires a factor $J_0(\epsilon/\Omega)$ and that, consequently, the coherent exchange of atoms is brought to a standstill if ϵ/Ω is chosen as

Table 1. Conditions for the existence of phase-locked solutions which are suitable for a chemical potential standard with $\Delta\mu = \hbar\Omega$ (i.e. $\bar{z} = \Omega/\Lambda$ and $k = 1$).

Restriction	Physical significance
$\varepsilon \ll \Omega$	Chemical potential difference much larger than its modulation
$\varepsilon > 2\gamma\Omega^2$	Existence of stable wells in the washboard potential
$\Lambda \ll \Omega^2$	Operation within MQST regime, Bloch oscillations small
$\Lambda \gg \Omega$	Number imbalance small

a zero of the Bessel function J_0 . Such an effect amounts to an interacting version of the so-called coherent destruction of tunnelling (CDT) predicted for single particles in bistable potentials [32]. We emphasize that the present phenomenon is different in two respects: first, CDT takes place at zeros of Bessel functions and, thus, requires large driving amplitudes, while here, the driving amplitude is much lower. Second, CDT fades out under the influence of dissipation, while in our case, moderate dissipation is essential for the convergence towards a phase-locked solution.

4. Locking probability

After investigating which is the parameter regime that permits stable phase-locked solutions, we turn to a remaining intriguing question: what is the probability that such a solution is hit by starting with a random initial phase? To give a crude estimate for the answer, we employ again the analogy of the dissipative motion in a tilted washboard potential.

Let us assume that we start out of resonance with an imbalance $z_1 < z < z_2$, with $z_k = k\Omega/\Lambda$, that corresponds to $\delta z > 0$, i.e. to a particle moving uphill in the tilted washboard of figure 3. Having only finite energy, the particle will bounce in one specific well. At the entry point, the energy will be in the range $(E_b, E_b + 2\pi\gamma k\Omega)$, where E_b is the potential energy at the top of the lower barrier. While moving within the well during one cycle, the particle dissipates the energy

$$E_{\text{diss}} \approx - \int F_{\text{diss}} d\phi \approx 2\pi\gamma\Lambda\sqrt{2\varepsilon/\Lambda\Omega}. \quad (16)$$

Here, we have estimated the maximum value of the dissipative force $F_{\text{diss}} = -\gamma\Lambda\delta z$ (cf equation (14)) from the maximum kinetic energy $\frac{1}{2}\Lambda\delta z^2 \approx 2J_k(\varepsilon/\Omega) \approx \varepsilon/\Omega$ for $k = 1$ and $\gamma \ll \varepsilon/\Omega^2$. Assuming for the particle's phase space trajectory the shape of an ellipse yields the rhs of equation (16). If the initial energy minus the dissipated energy lies below the barrier, the particle will end up at rest in the well. By assuming that the random initial phases translate into equally distributed initial energies, we find that this happens with probability

$$w = \frac{E_{\text{diss}}}{2\pi\gamma\Omega} \approx \sqrt{\frac{2\varepsilon\Lambda}{\Omega^3}}, \quad (17)$$

if $E_{\text{diss}} < 2\pi\gamma\Omega$, and $w = 1$ otherwise. Ideally, one would like to have $w = 1$ to ensure phase locking in all runs. However, it is easy to show from the constraints in table 1 that this would require $\Omega \ll \varepsilon$ which, for reasons already indicated, is not of physical interest. Thus one must

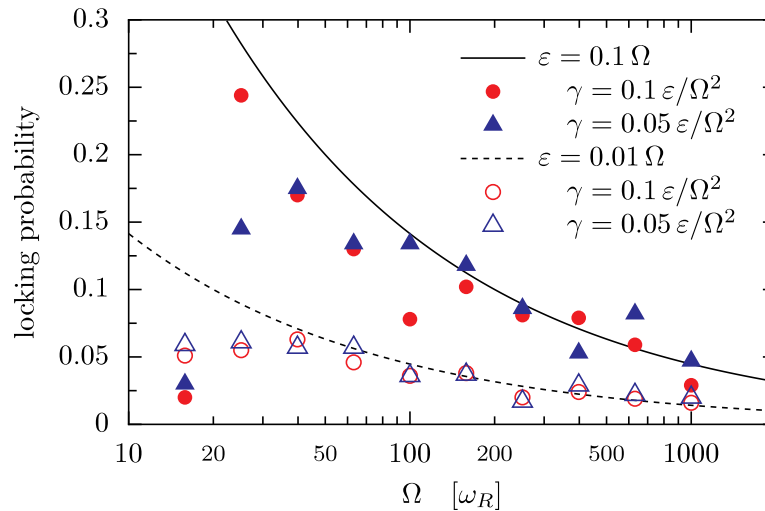


Figure 4. Estimate (17) for the locking probability compared to the fraction of phase-locked solutions (symbols) obtained from numerical integrations of equations (4) and (5) with random initial phase. The driving amplitude is $\varepsilon = 0.1\Omega$ (full curve, filled symbols) and $\varepsilon = 0.01\Omega$ (broken curve, open symbols). The interaction is $\Lambda = \Omega/z_1$, where $z_1 = 0.1$, and the dissipation $\gamma = 0.1\varepsilon/\Omega^2$ (circles) and $\gamma = 0.05\varepsilon/\Omega^2$ (triangles). Each data point is obtained from 1000 simulation runs.

content oneself with identifying a range of parameters that make the probability of landing in a resonance non-negligible.

One might conceive situations where the control of the relative particle number before the connection (as well as of the interaction parameters) were so good that the system could be forced to be within the attractive basin from the start. While designing $z(0)$ to be very close to the resonance value $\bar{z} = \Omega/\Lambda$ would surely increase the probabilities of relaxing towards the Shapiro resonance, we wish to emphasize here that success can never be fully guaranteed because of the *intrinsically random nature* of the initial phase $\phi(0)$, as can be clearly inferred from inspection of figure 1(b).

To confirm the analytical estimate (17) for the locking probability, we have integrated numerically the equations of motion (4) and (5) starting with a number imbalance $z(0) = 1.5\Omega/\Lambda$ and a random initial phase $\phi(0)$. The parameters have always been chosen such that $\varepsilon \ll \Omega \ll \Lambda$ and $\gamma \ll 1/\Omega$ according to the requirements derived in the last section. Figure 4 compares the analytical estimate (17) to the fraction of numerical runs that converge to a phase-locked solution. The numerical and analytical results agree well in the regime $\Omega \gtrsim 30$. For $\Omega = 100$, the locking probability typically assumes values of the order of 10%. We attribute the worse agreement between theory and simulation for $\Omega \lesssim 30$ to the fact that, in such a range, Ω becomes so small that the requirement $\Lambda \ll \Omega^2$ (see table 1) can no longer be satisfied if, as is the case in figure 4, Λ is constrained to be Ω/z_1 with $z_1 = 0.1$. An extreme case of disagreement is found for the lowest frequency considered when $\varepsilon = 0.1\Omega$.

The ohmic dissipation which we have assumed in our previous analysis is only justified for temperatures well above the chemical potential difference [11]. For a typical condensate, however, the chemical potential is of the same order as the temperature and thus the dissipative

current may not follow a linear law. Therefore, as a last item within the classical analysis, we consider the more general dissipative current $\dot{z}|_{\text{diss}} = -g(z)$, which might depend on various parameters like, for example, the interaction strength and the temperature. Using the ansatz (12), we linearize $g(z) \approx -g(z_k) - g'(z_k) \delta z$ and repeat the analysis from above. We find for $k = 1$ again the locking probability (17) but with an additional factor $z_1 g'(z_1)/g(z_1) = z_1 [\ln g(z_1)]'$, $z_1 = \Omega/\Lambda$. On the other hand, the slope of the washboard, i.e. the average dissipative current, acquires a factor $g(z_1)$. For very low temperatures $k_B T \ll \Delta\mu$, for instance, the quasiparticle-excitation decay mechanism discussed in [23] yields $g(z) \propto z^2$, and therefore the locking probability is enhanced by a factor of 2, but is still of the same order.

5. Quantum fluctuations

The classical treatment of the Bose–Josephson junction given above, has its limitations due to quantum fluctuations of the number n and the phase ϕ . They come from the commutation relation $[n, e^{i\phi}] = e^{i\phi}$ which amounts for small phase uncertainty to

$$[z, \phi] = -2i/N. \quad (18)$$

This leads basically to two constraints for the washboard potential: first, the potential wells of a washboard potential are quantum mechanically metastable since a particle will tunnel out at a rate [33]

$$\kappa = \omega_0 e^{-2\pi E_0/\hbar\omega_0}. \quad (19)$$

To justify the classical treatment from above, $1/\kappa$ must be larger than all the other timescales of the problem. A second, related point is that the phase-space region corresponding to the metastable well must support sufficiently many quantum states to treat both the number and the phase as continuous classical variables. For a well of depth E_0 with curvature ω_0^2 , the number of quantum states can be estimated as $m = E_0/\hbar\omega_0$. From both arguments, we conclude that the ratio of the potential depth and the energy quantum of a small oscillation at the bottom of the well determines whether we operate in the classical limit $E_0 \gg \hbar\omega_0$.

To determine the connection between E_0 , $\hbar\omega_0$ and the non-standard commutation relation (18) and our scaled parameters for $k = 1$, we have to accomplish the replacements

$$E_0 \rightarrow \frac{\varepsilon}{\Omega}, \quad \hbar \rightarrow \frac{2}{N}, \quad \omega_0^2 \rightarrow \frac{\varepsilon\Lambda}{2\Omega}. \quad (20)$$

This results in

$$\frac{E_0}{\hbar\omega_0} = N \sqrt{\frac{\varepsilon}{2\Omega\Lambda}}, \quad (21)$$

which is essentially the ratio between the Josephson coupling energy and the renormalized plasma frequency in the metastable well. For typical parameters used above ($\varepsilon = 0.01\Omega$, $\Lambda = 10\Omega$, $\Omega \gtrsim 100$), a condensate consisting of $N \gtrsim 10^5$ atoms supports $m \approx 100$ states and yields an escape rate that is practically zero. Therefore, we do not expect any relevant quantum correction.

6. Conclusions

We have investigated a realistic set-up that may provide the basis for a standard of chemical potential difference between weakly connected atomic BECs. Due to the still rudimentary development of atom circuit technology, we have focused on schemes that do not require coupling to an external circuit, thus staying away from straightforward analogues of well-tested superconducting devices. In particular, we have investigated the possibility of connecting two separate condensates in the presence of ac driving in such a way that, as the system relaxes towards equilibrium, it has an appreciable probability of being trapped in a Shapiro resonance for which $\Delta\mu$ is exactly an integer multiple of hf , with f the driving frequency. If the resolution of the imaging process is good enough, a frequency $\Omega = 2\pi f$ can be found satisfying simultaneously

$$\Lambda \Delta z < \Omega \ll \Lambda. \quad (22)$$

While the second inequality ensures a small number imbalance (see table 1), the first one expresses the experimental ability to resolve two different Shapiro plateaus, $\Delta z < z_{k+1} - z_k = \Omega/\Lambda$. Therefore, under any reasonable visibility conditions ($\Delta z \ll 1$), there exists a range of frequencies for which it is possible to identify the integer ratio $\Delta\mu/hf$, i.e. for which we can know the precise Shapiro step where the double condensate has become locked. In particular, we can tell whether the phase-locking has failed and the experimental run needs to be repeated. Therefore, we see that the main requirement of a precision measurement is satisfied, namely that, with the help of an accurately controlled frequency, a *poor* measurement of a physical quantity (the atom number imbalance) enables a *fine* measurement of another physical quantity (the chemical potential difference).

Instead of being a hindrance, dissipation provides here a crucial help, since it allows the system to spontaneously relax towards a stable resonance where the chemical potential difference is guaranteed to have a precise value. We have identified a number of parameter constraints that must be satisfied in order to have an optimum control of the Shapiro resonance. These include the need to have a weak driving signal, the operation within the collective Josephson regime, the need to keep the average number imbalance small, the requirement of even smaller Bloch oscillations, the necessity to optimize the probability of landing in the desired Shapiro step out of a random initial phase and the robustness against escape by quantum tunnelling from the metastable state.

Fortunately, this set of constraints does still allow for a window of realistic parameters within which accurate chemical potential differences could be realized. Let us, for instance, consider $N = 10^6$ ^{23}Na atoms in a split trap with frequency $\omega_{\text{ho}} = 2\pi \times 100$ Hz. The bulk properties of the condensate are readily estimated within the Thomas–Fermi approximation to yield the chemical potential $\mu \approx 70\hbar\omega_{\text{ho}}$ and the mean field energy $E_A(N) \approx \hbar N^{7/5} \times 0.01 \text{ s}^{-1}$. Estimating the Josephson coupling energy, respectively the Rabi frequency, is more tedious and relies sensitively on the assumptions made for the shape and size of the barrier. We assume here a Gaussian shape of width $6 \mu\text{m}$ and height of $1.05 \mu\text{m}$ such that, for a driving amplitude $\varepsilon = 0.01\Omega$, the top of the barrier lies always above the instantaneous chemical potential of both wells. Along the lines of [11, 12], we obtain the effective action $S_0 = 1.7\hbar$ and, thus, the Rabi frequency $\omega_R \approx 2\pi \times 0.05$ Hz. This finally yields the effective interaction $\Lambda \approx 10^4$ that we have assumed above. The (bare) Josephson plasma frequency is approximately $\omega_{\text{JP}} = 2\pi \times 5$ Hz; for the resonant solution in figure 2, phase locking sets in after 5 s.

One might wonder what may be the use of a ‘voltage’ standard in the absence of an atom circuit. First of all, no fundamental reason prevents the use of the present scheme in future atom

circuits. Then, this concept will have to be tested against ideas more directly borrowed from the technology of superconductors. Second, already within the currently limited possibilities of condensate transport, the realization of a Shapiro resonance may provide a convenient playground for the investigation of novel BEC scenarios. For instance, it seems possible to extend the experimental work of [14] to include an external ac driving that locks a pair of wells into resonance. The static behaviour of that pair of condensates should contrast with the dynamic behaviour of atoms in neighbouring wells. That would provide a most direct test of the Josephson effect in atomic condensates. Third, a fine control of the chemical potential may open up the possibility of detailed checks of our current understanding of the novel many-body problem posed by trapped quantum gases. Quite likely, the effect of interactions will be easily isolated from that of gravity or other well-controlled external fields. Finally, there is the hope that the present work may provide the basis for more efficient or practical concepts that will make the emergent ‘atomtronics’ a precise technology.

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