Berichte aus der Mathematik

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Cohomology and Connections on Fiber Bundles

D 17 (Diss. TH Darmstadt)

Verlag Shaker
Aachen 1995

Die Deutsche Bibliothek - CIP-Einheitsaufnahme

Gross, Christian:
Cohomology and Connections on Fiber Bundles / Christian Gross.
- Als Ms. gedr. -
Aachen : Shaker, 1995
(Berichte aus der Mathematik)
Zugl.: Darmstadt, Techn. Hochsch., Diss., 1995
ISBN 3-8265-1005-4

ISBN 3-8265-1005-4
ISSN 0945-0882

Shaker Verlag GmbH, Hubertusstr. 40, 52064 Aachen
Telefon: 0241 / 406351 - Telefax: 0241 / 406354

Preface

This dissertation is inspired both from mathematics and physics. It deals with one of the links of both sciences: the notion of a fiber bundle. All field theories in theoretical physics are based on fiber bundles. E. g., electromagnetism can be modelled by a principal $\mathbf{S}^1$-bundle and a MAXWELL connection on it. This bundle also allows for the description of magnetic monopoles, which have gained greater significance nowadays in grand unification theories. For another example, take the skyrmion bundle in theoretical nuclear physics. As a generalization of the ungauged SKYRME model, the skyrmion bundle is associated with the monopole bundle and treats interactions between mesons, baryons and electromagnetic fields. In both cases the structure group $\mathbf{S}^1$ of the bundle is abelian. Yet in YANG-MILLS theories also fiber bundles with non-abelian structure groups such as SU_n are considered. This is the setting for the dissertation in hand. It generalizes the results on $\mathbf{S}^1$-bundles to fiber bundles with non-abelian structure groups and combines the cohomology of a bundle with connections given on it.

There are many parallels between the definition of a fiber bundle and that of a manifold. Manifolds are generalizations of the Euclidean spaces. E. g., the n-sphere $\mathbf{S}^n$, the prototype for a manifold, locally looks like (an open subset of) $\mathbf{R}^n$, but globally has a nontrivial structure. Analogously for fiber bundles: these are generalizations of direct products of manifolds. *Locally* a bundle B looks like the direct product $U_\alpha \times F$, where the U_α are subsets of the base manifold M covering $M = \bigcup_{\alpha\in A} U_\alpha$, and F denotes the fiber. *Globally* a bundle will be more complicated, only the trivial bundle also is a global direct product $M \times F$.

Thus in contrast to $M \times F$, where two projections $\mathrm{pr}_M: M \times F \to M$ and $\mathrm{pr}_F: M \times F \to F$ are given, we have only one global projection $\pi: B \to M$ from a bundle onto its base space, whereas projections onto the fiber are merely defined locally: $\pi_\alpha: \pi^{-1}(U_\alpha) \to F$. For every bundle we have a bundle atlas — cf. again the analogy to manifolds — that consists of charts $\psi_\alpha: \pi^{-1}(U_\alpha) \to U_\alpha \times F$, i. e., diffeomorphisms with $\mathrm{pr}_{U_\alpha} \circ \psi_\alpha = \pi|_{U_\alpha}$ and $\mathrm{pr}_F \circ \psi_\alpha = \pi_\alpha$.

The global structure of a fiber bundle can be determined if one knows how to change from one bundle chart to another. For every point x in an overlap region $U_{\alpha\beta} := U_\alpha \cap U_\beta$, this change of the bundle chart defines a diffeomorphism of the fiber $g_{\alpha\beta}(x) := \psi_\alpha|_{\pi^{-1}(x)} \circ \left(\psi_\beta|_{\pi^{-1}(x)}\right)^{-1}: F \to F$. At this point LIE theory is involved. Bundles are equipped with a structure group G, i. e., a LIE group with a left action $L: G \times F \to F$. Since L is required to be effective, we may think of G as of a subgroup of the group of all diffeomorphisms of F. With his identification all transition functions are supposed to be differentiable maps $g_{\alpha\beta}: U_{\alpha\beta} \to G$. The structure group G and the maps $g_{\alpha\beta}$ indeed determine the structure of the bundle, e. g., if G consists of just one element then the bundle is necessarily trivial.

For a principal fiber bundle, $G = F$ and the left action is simply multiplication λ with elements from the left. Thus given any fiber bundle one can construct the so-called associated principal bundle by replacing F by G and L by λ. Vice versa, given any principal bundle P and a left action $L: G \times F \to F$, one can construct an associated fiber bundle with fiber F.

The concepts of the structure group and the associated bundles become more apparent for the prototypes of nontrivial bundles, the MOEBIUS band and the KLEIN bottle. Physically, the MOEBIUS band consists of a strip of paper whose ends are glued together after a 180° flip of one end. Thus its base is a 1-sphere $\mathbf{S}^1$ and its fiber is an interval. Everything stays the same for the KLEIN bottle, only that its fiber is not an interval but another $\mathbf{S}^1$. Thus the KLEIN bottle is a cylinder whose ends are glued together after a 180° flip of one end. (That this construction is impossible in threedimensional space and thus the KLEIN bottle cannot be embedded into $\mathbf{R}^3$ as a manifold, shall not bother us here.) Both examples are associated: their structure group is isomorphic to $\mathbf{Z}_2$ and consists of the identity transformation of the fiber and the 180° flip. Thus the fibers of their associated principal bundle P consist of just two elements and P is a two-fold cover of its base space $\mathbf{S}^1$.

In order to illustrate the notion of the so-called DE RHAM cohomology of a manifold, we have to introduce vector fields and differential forms. A vector field $\mathcal{X}$ associates with every point x on a manifold M an element $\mathcal{X}_x$ in the tangent space $T_x(M)$ of the manifold in the point $x \in M$. The set of vector fields will be denoted by $\mathcal{D}^1(M)$. A p-form is an alternating p-linear map $\phi_p: \mathcal{D}^1(M) \times \cdots \times \mathcal{D}^1(M) \to C^\infty(M)$. The set of p-forms on M will be denoted by $\mathcal{A}_p(M)$ and the set of all forms on M by $\mathcal{A}(M) := \bigoplus_{p=0}^\infty \mathcal{A}_p(M)$. Besides these formal definitions it is quite instructive to think of forms as of integrands of integrals over submanifolds of M: p-forms are integrands of integrals over p-dimensional submanifolds. E. g., for a n-dimensional oriented manifold we have its volume form $dV \in \mathcal{A}_n(M)$ with $\mathrm{Vol}(M) = \int_M dV$.

From vector analysis the notions of the gradient, the rotation and the divergence of a vector field may be familiar, as well as the theorems of GAUSS, STOKES, etc., connected with these operations. Using forms we can present all these theorems in a very compact way. We have an operator $d: \mathcal{A}(M) \to \mathcal{A}(M), \mathcal{A}_{p-1}(M) \to \mathcal{A}_p(M)$, the so-called exterior derivative of forms, and if ∂M denotes the $(n-1)$-dimensional boundary of a n-dimensional manifold M and ω is a $(n-1)$-form on M, then

$$\int_{\partial M} \omega = \int_M d\omega.$$

Just as $\partial(\partial M) = \emptyset$, i.e., the boundary of a boundary of a manifold is empty, d is a differential operator, i. e., $d^2 := d \circ d = 0$. The forms in the kernel of d are called *closed* forms, and the forms in the image of d are called *exact* forms. $d^2 = 0$ yields that all exact forms are closed and that the vector space of the closed modulo the exact p-forms is well-defined. This quotient space is called the p-th DE RHAM cohomology group $H^p(M)$ and the (total) cohomology of M means the direct sum $H^*(M) := \bigoplus_{p=0}^\infty H^p(M)$.

Although the DE RHAM cohomology is defined by differential geometric means, it is a ***topological*** invariant: whenever two manifolds are homeomorphic, their cohomology is necessarily isomorphic. Thus the cohomology can be used to distinguish manifolds and it is quite important to know a manifolds cohomology. Especially for bundles, the question arises whether the cohomology of a bundle can be computed from $H^*(M)$ and $H^*(F)$.

Every differentiable map $f: M \to N$ between two manifolds M and N canonically induces a homomorphism on the forms in the opposite direction, the so-called pullback $f^*: \mathcal{A}(N) \to \mathcal{A}(M)$. Thus using the pullback π^* we may lift every form ω on the base space onto the bundle. We may think of $\pi^*\omega$ as of being invariant along the fibers. π also induces a homomorphism in cohomology $[\pi^*]: H^*(M) \to H^*(B)$. For a direct product $M \times F$ this also works for $H^*(F)$ and leads to the KÜNNETH formula

$$H^*(M \times F) \cong H^*(M) \otimes H^*(F).$$

For a nontrivial bundle the situation becomes much more complicated and leads to the theory of spectral sequences. Spectral sequences compute $H^*(B)$ from $H^*(M)$ and $H^*(F)$. They also answer the question which closed forms on the fiber can be extended to closed forms on the bundle. We call these forms 0-transgressive.

This exactly is a problem that occurs quite often in theoretical physics if one tries to "gauge" a theory that is defined for a manifold F. One constructs a fiber bundle with gauge (resp., structure) group G, fiber F and (mostly) space-time as the base manifold. For computations it is then necessary to "generalize" the given closed differential forms $\phi \in \mathcal{A}(F)$ to the bundle case: one needs a closed form $\psi \in \mathcal{A}(B)$ such that ψ reproduces ϕ when restricted to the fibers: $\psi|_{\pi^{-1}(x)} = \phi$ for all $x \in M$.

As mentioned, spectral sequences tell us for which forms ϕ such a ψ exists. If this is the case, they also provide a formula for such a ψ. Nevertheless this formula involves a partition of unity subordinate to the given cover $\{U_\alpha\}_{\alpha\in A}$ of M. For any such partition the formula gives a different form ψ within the generated cohomology class. (Note that, a priori, ψ is not unique but defined only up to an exact form on B, whose restriction to the fibers is zero.)

From the physicists point of view, this situation is quite unsatisfactory since a partition of unity does not bear any physical meaning and there is no reason why one partition — and the corresponding form ψ — should be better than another. In fact one would like to obtain a representative ψ for the generated cohomology class that can be associated with the physics in question, that is the gauge potentials and the gauge fields of the field theory.

This takes us to the notion of connections on fiber bundles. Again we start with the case of a direct product $M \times F$. Here for every tangent space, a horizontal direction (tangential to M) and a vertical direction (tangential to F) are given naturally and we thus have canonical horizontal and vertical projections of vector fields: $\mathcal{D}^1(M\times F) = h\mathcal{D}^1(M\times F) \oplus v\mathcal{D}^1(M\times F)$. For a fiber bundle, only the vertical direction tangential to the fiber is given naturally. Every local bundle chart defines another horizontal direction. The definition of a global horizontal complement to the vertical space thus requires an additional structure, and this is exactly what a connection Γ is: it defines global horizontal and vertical projections of vector fields such that $\mathcal{D}^1(B) = h\mathcal{D}^1(B) \oplus v\mathcal{D}^1(B)$. On a principal bundle, such a connection is closely related to the gauge potentials and the gauge fields (cf. below). Once such a connection is defined on a principal bundle, it also defines connections on all associated fiber bundles.

In addition, a connection defines lifts of vector fields on the base onto horizontal fields on the bundle and projections of forms on the bundle. These lifts and projections now can be used for the desired extensions of forms to the fiber. In fact, for every differential form $\phi \in \mathcal{A}(F)$ that is invariant under the given left action L, there exists exactly one vertical form on the bundle, say $\phi v \in \mathcal{A}(B)$, such that $\phi v|_{\pi^{-1}(x)} = \phi$. From the physicists point of view, this seems to be a satisfactory generalization, but unfortunately we are not done with that, since the following diagram does *not* commute:

$$\begin{array}{ccc} \phi & \longmapsto & \phi v \\ \downarrow & & \downarrow \\ d\phi & \longmapsto & (d\phi)v \neq d(\phi v). \end{array}$$

Thus although we start with a closed form ϕ, the generated vertical form ϕv needs not be closed. In general, we are not able to find a vertical representative for this cohomology class generated by a 0-transgressive form, but we need to admit horizontal terms. Thus the question will be whether we can find such a representative where these horizontal terms are "naturally" given by the connection Γ, in fact, by the gauge fields. In that case, we call the resulting form adapted to Γ. Those forms are candidates for the desired generalizations of closed forms in field theories.

So much for a general introduction into the main topics of this dissertation. We proceed as follows:

In Chapter 1 we introduce tensor fields and differential forms on manifolds. To this purpose we first present some elementary results on modules and algebras, and on their homomorphisms and derivations. Then the wedge product of forms and their exterior differentiation d are defined. In the second section we extend these operations to vector valued forms. We introduce pullbacks and push-outs and examine the interior product of forms with respect to a vector field and the LIE differentiation of tensor fields.

The third section is devoted to the "bullet operator" of forms, $\chi \bullet \phi$, a generalization of the wedge product. We discuss elementary properties of this new operator such as associativity and its behavior under pullbacks and push-outs. The examination of expressions $\chi \bullet (\phi + \psi)$ and $d(\chi \bullet \phi)$ will then lead us to what we call "triangle operators." In the next section we discuss differential forms on LIE groups, introduce invariant vector fields and forms and derive the MAURER-CARTAN identities.

Finally we examine LIE group actions S on manifolds in Section 1.5. We generalize the notion of invariant forms and define equivariant forms and the vector fields that are induced by elements of the LIE algebra. With the aid of these induced vector fields, the expressions $S^{\mathrm{i}}_{\bullet}\phi$, $S^{\vee}_{\bullet}\phi$ and $\phi \circledS \theta$ for differential forms ϕ and θ are introduced. We prove some quite voluminous formulae on their exterior derivative, in order to prepare several theorems in the following chapter.

Chapter 2 treats fiber bundles and connections on them. In the first section we give the basic definitions for principal and associated bundles, list several examples, discuss sections of bundles and cite the main theorems on the triviality of bundles from literature. Next we introduce connections on principal bundles in the second section and examine the connection 1-form ω^Γ and its exterior covariant derivative, the curvature 2-form Ω^Γ. These take us to the structure equations and BIANCHI'S identities. We also introduce pseudotensorial and tensorial forms as equivariant, resp., horizontal equivariant forms and compute their exterior covariant derivative. Section 2.2 closes with the examination of the gauge potentials A^α and the gauge fields F^α for a cover $\mathfrak{U} = \{U_\alpha\}_{\alpha\in A}$ of the base manifold. The forms A^α and F^α are pullbacks of ω^Γ, resp., Ω^Γ under local sections. We derive the equations of motion for these forms.

The third section then defines connections on associated bundles. To this purpose, we use the lifts of vector fields in order to gain global expressions for the projections of fields and forms. These then enable us to determine which forms on the fiber can be naturally extended to the bundle. In fact, these will be invariant forms and "bullet combinations" of equivariant forms with pseudotensorial forms on the principal bundle. In addition, we introduce the covariant derivative of sections in a vector bundle.

For the sake of completeness we then digress to linear connections of a manifold in Section 2.4. Treating tensor fields as sections in the tensor algebra bundle of the manifold, we obtain the covariant derivative of tensor fields. We define the torsion and the curvature field and prove BIANCHI'S identities and the structure equations for linear connections. In particular, we discuss the most important example of a linear connection, the LEVI-CIVITA connection on pseudo-Riemannian manifolds.

Since very often bundles are defined merely by a bundle atlas and transition functions for the change of bundle charts, there is a need for the local evaluation of connections. This is done in Section 2.5: we prove several formulae for the behavior of fields and forms under a change of bundle charts and for their local projections. In combination with our results in Section 1.5, these formulae then enable us to compute the exterior derivative of the extended forms from Section 2.3. Finally we specialize to bundles with abelian structure groups and — even more specially — with one-dimensional abelian structure groups. The results give new insights into the treatment of the skyrmion bundle.

In Chapter 3 we introduce differential complexes and their cohomologies, as well as spectral sequences to compute the latter. Especially, we develop spectral sequences of fiber bundles and combine their cohomology with connections. As always, we start with the very definitions of complexes, subcomplexes, double complexes and augmented complexes in Section 3.1. We also illustrate the significance of homotopy operators which provide sufficient conditions for two cohomologies to be isomorphic. In Section 3.2 we then give a survey over the DE RHAM cohomology $H^\bullet(M)$. In particular, we compute $H^\bullet(\mathbb{R}^n)$ and $H^\bullet(\mathrm{S}^n)$. Moreover, we specialize to the subcomplexes of invariant, resp., equivariant forms and their cohomologies $H^\bullet_{\mathrm{inv}}(M)$, resp.,

$H^{\bullet}_{\text{equiv}}(M)$, and derive first results on the DE RHAM cohomology of a LIE group G.

The third section is devoted to the LIE algebra cohomology $H^{\bullet}(\mathfrak{g})$. It will prove a great help in computing $H^{\bullet}(G)$, indeed, $H^{\bullet}(\mathfrak{g}) \cong H^{\bullet}(G)$ for compact connected LIE groups G with LIE algebra $\mathfrak{g}$. We also cite the definition of primitive elements in the exterior algebra of the dual $\mathfrak{g}^{\bullet}$ for reductive LIE algebras, that enable us to compute $H^{\bullet}(G)$ for the classical LIE groups.

In the next section we examine the ČECH-DE RHAM complex $C(\mathfrak{U}, \mathcal{A})$ for a cover $\mathfrak{U}$ of a manifold M. The generalized MAYER-VIETORIS principle proves that $H^{\bullet}(M) \cong H^{\bullet}_{D}(C(\mathfrak{U}, \mathcal{A}))$. Then we introduce spectral sequences in the following section to compute the cohomology of a double complex like $C(\mathfrak{U}, \mathcal{A})$. We also give the notion of transgressive and 0-transgressive forms and show that the latter are exactly those closed forms on the fiber that define a cohomology class in $H^{\bullet}(B)$.

In Section 3.6 we then combine the cohomology of a fiber bundle with a given connection Γ. To this purpose we introduce Γ-adapted and G-transgressive forms and examine whether a cohomology class can be represented by a form that is adapted to Γ. We prove that every G-transgressive form is 0-transgressive and that the generated cohomology class can be represented by a form adapted to Γ. Moreover, this holds for any bundle that comes along with the given left action of the structure group G on the fiber F. As a corollary for semisimple LIE groups G, we prove that every closed invariant n-form on the fiber is G-transgressive for $n \leq 2$. This yields that for any bundle $B(M, F, G)$ the cohomology groups $H^n(B)$ contain subgroups isomorphic to $H^n_{\text{inv}}(F)$. Finally we apply our results to the skyrmion bundle and to the non-abelian YANG-MILLS theories.

This dissertation continues the research presented in our theses for a mathematics and a physics degree. The former [1] dealt with the mathematical treatment of electromagnetism via differential forms. We also examined the principal $\mathbb{S}^1$-bundle and the MAXWELL connection on it that allow for the description of magnetic monopoles. In our thesis for a physics degree [2] we presented the skyrmion bundle and computed its homotopy and cohomology groups.

Acknowledgements

Such a dissertation never is the merit of a single person but requires assistance from many sides. Thus I would like to thank everybody who aided or encouraged me during the last years. First of all I am indebted to Prof. Dr. K. H. Hofmann for his care and support and his assistance with several essays I wrote in that time. For the same reasons I would like to thank Prof. Dr. F. Beck and Prof. Dr. K. Keimel for his encouragement and his readiness to refer this dissertation.

Sincere thanks to Prof. Dr. Karl-Hermann Neeb for numerous discussions and helpful ideas and special thanks to Torsten-Karl Strempel and Werner Plank for friendship, board and lodging.

Finally I am indebted to my family for everything and last but not least to the "Studienstiftung des deutschen Volkes" for the financial support during my college time.

Contents

Chapter 1

Foundations of Fields and Forms

1.1 Tensor Algebra and GRASSMANN Algebra

There are several ways of introducing vector fields, tensor fields and differential forms on a finite dimensional manifold M. E. g., one can define them as sections in (tangent, cotangent, etc.) bundles over M. Instead, according to HELGASON, we introduce them as derivations (cf. [3, p. 8]):

Definition 1.1 *Let* $\mathbb{A}$ *and* $\mathbb{B}$ *be algebras over a field* $\mathbb{K}$. *We call a map* $D: \mathbb{A} \to \mathbb{B}$ *a* derivation of $\mathbb{A}$ into $\mathbb{B}$ along an algebra homomorphism $h: \mathbb{A} \to \mathbb{B}$ *if*

$$(\forall x, y \in \mathbb{K},\ f, g \in \mathbb{A})\ \ D(xf+yg) = xD(f)+yD(g),\ D(fg) = (Df)h(g)+h(f)(Dg).$$

A map $D: \mathbb{A} \to \mathbb{A}$ *is called a* derivation of $\mathbb{A}$ *if it is a derivation along the identity morphism* $\mathrm{id}_{\mathbb{A}}: \mathbb{A} \to \mathbb{A}$. *We denote by* $\mathrm{der}_h(\mathbb{A}, \mathbb{B})$, *resp.,* $\mathrm{der}\,\mathbb{A}$ *the set of all derivations of* $\mathbb{A}$ *into* $\mathbb{B}$ *along* h, *resp., derivations of* $\mathbb{A}$. *If* $Z(\mathbb{A})$ *denotes the center of* $\mathbb{A}$, *then* $\mathrm{der}\,\mathbb{A}$ *is an* $Z(\mathbb{A})$*-module, where for all* $f \in Z(\mathbb{A})$, $g \in \mathbb{A}$ *and* $D, D' \in \mathrm{der}\,\mathbb{A}$

$$(fD)g := f(Dg), \qquad\qquad (D + D')g := Dg + D'g.$$

Moreover, $\mathrm{der}\,\mathbb{A}$ *is a* LIE *algebra with commutator* $[D, D'] := D \circ D' - D' \circ D \in \mathrm{der}\,\mathbb{A}$.

Analogously, for graded $\mathbb{A} = \bigoplus_{r=0}^{\infty} \mathbb{A}_r$, *(where* $fg \in \mathbb{A}_{r+s}$ *if* $f \in \mathbb{A}_r$ *and* $g \in \mathbb{A}_s$,*) a linear mapping* $S: \mathbb{A} \to \mathbb{A}$ *is called a* skew-derivation *of* $\mathbb{A}$ *if for all* $f \in \mathbb{A}_r$, $g \in \mathbb{A}$

$$S(fg) = (Sf)g + (-1)^r f(Sg).$$

A (skew-)derivation S *of* $\mathbb{A} = \bigoplus_{r=0}^{\infty} \mathbb{A}_r$ *is* of degree $k \in \mathbb{Z}$, *if* $S: \mathbb{A}_r \to \mathbb{A}_{r+k}$ *for all* r.

For all $f, g \in Z(\mathbb{A})$ and $D, D' \in \mathrm{der}\,\mathbb{A}$ we have $Dg, D'f \in Z(\mathbb{A})$ and

$$[fD, gD'] = fg[D, D'] + f(Dg)D' - g(D'f)D. \tag{1}$$

Lemma 1.2 *Let* $\mathbb{A}$ *be a graded algebra,* D, D' *derivations of degree* k, *resp.,* k' *and* S, S' *skew-derivations of degree* k, *resp.,* k'.

1. $[D, D']$ *is a derivation of degree* $k + k'$.

2. *$[D, S']$ is a skew-derivation of degree $k + k'$, if k is even.*

3. *$[S, S']$ is a derivation of degree $k + k'$, if k and k' are even.*

4. *$S \circ S' + S' \circ S$ is a derivation of degree $k + k'$, if k and k' are odd.*

Definition 1.3 *For any real C^∞-manifold M, $C^\infty(M)$ means the algebra of all differentiable maps from M to $\mathbb{R}$ (equipped with pointwise addition and multiplication).*

Let $\mathcal{D}^1(M) := \operatorname{der} C^\infty(M)$, its elements ($\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}, \ldots$) are called(contravariant) vector fields *on M. For every $x \in M$, $\mathcal{X} \in \mathcal{D}^1(M)$ defines an element $\mathcal{X}_x \in T_x(M)$ of the tangent space of M at x by $\mathcal{X}_x(f) := \mathcal{X}(f)(x) \in \mathbb{R}$ for all $f \in C^\infty(M)$.*

$\mathcal{D}_1(M)$ denotes the dual of $\mathcal{D}^1(M)$, it is the $C^\infty(M)$-module of covariant vector fields *on M. By* tensor fields of type (r, s) *$(r, s \geq 0)$ we mean the elements of $\mathcal{D}^r_s(M)$, which denotes the $C^\infty(M)$-module of all $C^\infty(M)$-multilinear mappings of $\prod_{i=1}^r \mathcal{D}_1(M) \times \prod_{j=1}^s \mathcal{D}^1(M)$ to $C^\infty(M)$ (where we put $\mathcal{D}^0_0(M) := C^\infty(M)$). Define $\mathcal{D}^r(M) := \mathcal{D}^r_0(M)$, $\mathcal{D}_s(M) := \mathcal{D}^0_s(M)$, $\mathcal{D}(M) := \bigoplus_{r,s=0}^\infty \mathcal{D}^r_s(M)$, $\mathcal{D}^*(M) := \bigoplus_{r=0}^\infty \mathcal{D}^r(M)$ and $\mathcal{D}_*(M) := \bigoplus_{s=0}^\infty \mathcal{D}_s(M)$.*

The last definitions are legitimate because of the following lemma (cf. [3, p. 12]):

Lemma 1.4 *$\mathcal{D}^1(M)$ and $\mathcal{D}_1(M)$, resp., $\mathcal{D}^r_s(M)$ and $\mathcal{D}^s_r(M)$ are dual to each other.*

$\mathcal{D}(M)$ can be given a tensor product structure: let $a \in \mathcal{D}^p_q(M)$, $b \in \mathcal{D}^r_s(M)$, $\mathcal{X}^i, \mathcal{Y}^i \in \mathcal{D}^1(M)$ and $\mathcal{X}_j, \mathcal{Y}_j \in \mathcal{D}_1(M)$. Then $a \otimes b \in \mathcal{D}^{p+r}_{q+s}(M)$ is defined by

$$\begin{aligned}(a \otimes b)(\mathcal{X}_1, \ldots, \mathcal{X}_p, \mathcal{Y}_1, \ldots, \mathcal{Y}_r; \mathcal{X}^1, \ldots, \mathcal{X}^q, \mathcal{Y}^1, \ldots, \mathcal{Y}^s) :=& \\ = \; a(\mathcal{X}_1, \ldots, \mathcal{X}_p; \mathcal{X}^1, \ldots, \mathcal{X}^q)\, b(\mathcal{Y}_1 \ldots, \mathcal{Y}_r; \mathcal{Y}^1, \ldots, \mathcal{Y}^s).& \qquad (2)\end{aligned}$$

This turns $\mathcal{D}(M)$ into an associative algebra over the ring $C^\infty(M)$, the so-called *mixed tensor algebra over M*, with subalgebras $\mathcal{D}^*(M)$ and $\mathcal{D}_*(M)$. Lemma 1.4 also yields that $\mathcal{D}^r_s(M)$ and $\operatorname{Hom}(D^s(M), D^r(M))$ are isomorphic for all $r, s \in \mathbb{N}_0$.

Definition 1.5 *For any $p \in \mathbb{N}$, $\mathcal{A}_p(M) \subseteq \mathcal{D}_p(M)$ denotes the submodule of all alternating $C^\infty(M)$-p-linear maps from $\prod_{i=1}^p \mathcal{D}^1(M)$ to $C^\infty(M)$ (i. e. of all alternating $C^\infty(M)$-linear maps from $\mathcal{D}^p(M)$ to $C^\infty(M)$), its elements are called* p-forms *on M. $\mathcal{A}_0(M) := C^\infty(M)$ and $\mathcal{A}(M) := \bigoplus_{p=0}^\infty \mathcal{A}_p(M)$, we call its elements* (exterior) differential forms *on M.*

For any vector spaces V, W let $\operatorname{Alt}_p(V, W)$ denote the vector space of alternating p-linear maps from V^p to W and $\operatorname{Alt}(V, W) := \bigoplus_{p=0}^\infty \operatorname{Alt}_p(V, W)$, where $\operatorname{Alt}_0(V, W) := W$. Then $\omega \in \mathcal{A}_p(M)$ defines an element $\omega_x \in \operatorname{Alt}_p(T_x(M), \mathbb{R})$ for all $x \in M$ and for $\mathcal{X}^i \in \mathcal{D}^1(M)$ we have $\omega(\mathcal{X}^1, \ldots, \mathcal{X}^p)(x) = \omega_x(\mathcal{X}^1_x, \ldots, \mathcal{X}^p_x)$.

For differential forms we have two important mappings: the wedge product and the exterior differentiation. We will introduce the former in a more general setting: Let $\mathbb{A}$ be an associative, commutative algebra over $\mathbb{Q}$ and E an $\mathbb{A}$-module. For any permutation $\rho \in S_p$, $p \in \mathbb{N}$, with $(-1)^\rho = \operatorname{sgn}(\rho)$ we have an obvious representation on $\bigotimes^p E$, resp., $\prod^p E$: for $e_i \in E$ define

$$\bar{\rho}\colon \bigotimes^p E \to \bigotimes^p E\colon \quad \bar{\rho}(e_1 \otimes \cdots \otimes e_p) := e_{\rho^{-1}(1)} \otimes \cdots \otimes e_{\rho^{-1}(p)}.$$

For $p \in \mathbb{N}_0$, consider the linear transformation $\mathfrak{A}_p\colon \bigotimes^p E \to \bigotimes^p E$ given by

$$\mathfrak{A}_0 := \mathrm{id}_{\mathbf{A}}, \qquad \mathfrak{A}_p := \frac{1}{p!} \sum_{\rho \in S_p} (-1)^\rho \bar{\rho}, \quad p \geq 1, \tag{3}$$

(here we need $\mathbb{Q}$ in the domain of scalars, the rest holds for any commutative ring) and extend $\mathfrak{A}_p$ naturally to an $\mathbf{A}$-linear map $\mathfrak{A}^{(E)}\colon \mathcal{T}(E) \to \mathcal{T}(E)$ (where $\mathcal{T}(E) := \bigoplus_{p=0}^\infty \bigotimes_{\mathbf{A}}^p E$ denotes the tensor algebra of an $\mathbf{A}$-module E). Then $\mathfrak{A}^2 = \mathfrak{A}$ and $\mathfrak{A}_p \circ \bar{\rho} = \bar{\rho} \circ \mathfrak{A}_p = (-1)^\rho \mathfrak{A}_p$ for any $\rho \in S_p$. We put $\bigwedge^p(E) := \mathfrak{A}_p(\bigotimes^p E)$, $\bigwedge(E) := \mathfrak{A}(\mathcal{T}(E))$, thus $\mathfrak{A}$ is a projection of $\mathcal{T}(E)$ onto $\bigwedge(E)$, called *alternation*. If N denotes the kernel of $\mathfrak{A}$ then $\mathfrak{A}(a) + N = a + N$ for all $a \in \mathcal{T}(E)$ and although $\mathfrak{A}$ is not an algebra endomorphism, N is a (two-sided) ideal in $\mathcal{T}(E)$ generated by $\{e \otimes e | e \in E\}$. This yields

$$\mathfrak{A}(a \otimes b) = \mathfrak{A}(\mathfrak{A}(a) \otimes \mathfrak{A}(b)) \qquad \text{for all} \quad a, b \in \mathcal{T}(E) \tag{4}$$

($\neq \mathfrak{A}(a) \otimes \mathfrak{A}(b)$ in general), so the algebra $\mathcal{T}(E)/N$ is defined. (If $\mathbf{A}$ is just a commutative ring, $\bigwedge(E) := \mathcal{T}(E)/N$ by definition.)

Definition 1.6 *For any $a, b \in \bigwedge(E)$ the* wedge *or* exterior product *is defined by*

$$a \wedge b := \mathfrak{A}(a \otimes b).$$

Thus $\wedge$ makes the following diagram commutative, where $\otimes_N$ denotes the multiplication on the quotient ring:

$$\begin{array}{ccc}
\mathcal{T}(E)/N \times \mathcal{T}(E)/N & \xrightarrow{\ \otimes_N\ } & \mathcal{T}(E)/N \\
\Big\downarrow \mathfrak{A} \times \mathfrak{A} & & \Big\downarrow \mathfrak{A} \\
\bigwedge(E) \times \bigwedge(E) & \xrightarrow{\ \wedge\ } & \bigwedge(E)
\end{array}$$

This turns $\bigwedge(E) \cong \mathcal{T}(E)/N$ into an associative algebra over $\mathbf{A}$: the *exterior algebra* or GRASSMANN *algebra* of E. If E is generated by n elements then $\bigwedge^p(E) = \{0\}$ for $p > n$, cf. BOURBAKI, [4, III, p. 80].

Note 1.7 Everything works well not only on $\mathcal{T}(E)$, but also on its completion $\overline{\mathcal{T}}(E) := \prod_{p=0}^\infty \bigotimes^p E$, where we get the associative algebra $\overline{\bigwedge}(E) \cong \overline{\mathcal{T}}(E)/\overline{N}$.

Definition 1.8 *For any associative, commutative $\mathbb{Q}$-algebra $\mathbf{A}$ and any $\mathbf{A}$-module E we define the symmetrization $\mathfrak{S}$ analogously to $\mathfrak{A}$ by dropping $(-1)^\rho$ in (3). The* symmetric algebra $\mathrm{S}(E) = \bigoplus_{p=0}^\infty \mathrm{S}^p(E)$, *which is defined by* $\mathrm{S}^p(E) ;= \mathfrak{S}_p(\bigotimes^p E)$, $\mathrm{S}(E) := \mathfrak{S}(\mathcal{T}(E))$, *is a commutative algebra with* $\mathrm{S}(E) \cong \mathcal{T}(E)/\ker \mathfrak{S}$, *since the* *two-sided ideal* $\ker \mathfrak{S} = \mathcal{T}(E)' \leq \mathcal{T}(E)$ *is generated by* $\{e \otimes f - f \otimes e | e, f \in E\}$. *If* $\mathbb{A}$ *is just a commutative ring, one defines* $\mathrm{S}(E) := \mathcal{T}(E)/\mathcal{T}(E)'$. *The* symmetric product *will be denoted by*

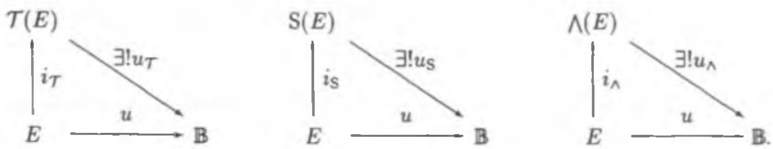
$$a \vee b := \mathfrak{S}(a \otimes b) \qquad \textit{for all} \quad a, b \in \mathrm{S}(E).$$

Analogously to Definition 1.5, for any vector spaces V, W, $\mathrm{Sym}_p(V, W)$ *denotes the vector space of symmetric p-linear maps from* V^p *to* W *and* $\mathrm{Sym}(V, W) := \bigoplus_{p=0}^{\infty} \mathrm{Sym}_p(V, W)$, *where* $\mathrm{Sym}_0(V, W) := W$.

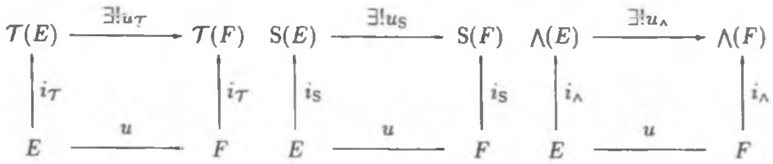
For convenience we define Sym^ς, S^ς *and* $\mathfrak{S}^\varsigma$ *for* $\varsigma = \pm 1$ *by* $\mathrm{Sym}^+ := \mathrm{Sym}$, $\mathrm{Sym}^- := \mathrm{Alt}$, $\mathrm{S}^+ := \mathrm{S}$, $\mathrm{S}^- := \bigwedge$ *and* $\mathfrak{S}^+ := \mathfrak{S}$, $\mathfrak{S}^- := \mathfrak{A}$.

We collect some elementary results on tensor products and homomorphisms from [4, II and III]. Let $E^* = \mathrm{Hom}(E, \mathbb{A})$ denote the dual of the $\mathbb{A}$-module E.

Lemma 1.9 (Universal properties of $\mathcal{T}(E)$, $\mathrm{S}(E)$ and $\bigwedge(E)$) *Let* $\mathbb{A}$ *be a commutative ring,* $\mathbb{B}$ *an* $\mathbb{A}$*-algebra,* E *an* $\mathbb{A}$*-module and* $u: E \to \mathbb{B}$ *any* $\mathbb{A}$*-module homomorphism. Denote the natural injections of* E *into* $A := \mathcal{T}(E)$, $\mathrm{S}(E)$, *resp.,* $\bigwedge(E)$ *by* $i_A: E \to A$. *For* $A = \mathrm{S}(E)$ *suppose* $u(e) \cdot u(f) = u(f) \cdot u(e)$, *and for* $A = \bigwedge(E)$ *suppose* $u(e) \cdot u(e) = 0$ *for all* $e, f \in E$. *Then* u *extends to a unique* $\mathbb{A}$*-algebra homomorphism* $u_A: A \to \mathbb{B}$ *such that* $u = u_A \circ i_A$, *i. e., the following diagrams commute:*



If F is a second $\mathbb{A}$-module and $u: E \to F$ is an $\mathbb{A}$-module homomorphism, we obtain unique homomorphisms $u_\mathcal{T}: \mathcal{T}(E) \to \mathcal{T}(F)$, $u_\mathrm{S}: \mathrm{S}(E) \to \mathrm{S}(F)$, resp., $u_\wedge: \bigwedge(E) \to \bigwedge(F)$ of graded algebras such that the following diagrams commute:



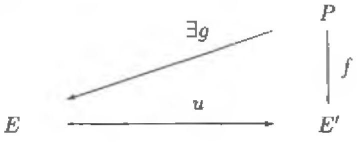
Proposition 1.10 *Let* $\mathbb{A}$ *be a commutative ring and* E *an* $\mathbb{A}$*-module.*

1. *Any* $\mathbb{A}$*-module homomorphism* $u: E \to \bigotimes^p E$ *extends to a unique derivation* $D_u: \mathcal{T}(E) \to \mathcal{T}(E)$ *of degree* $p - 1$.

2. *Any* $\mathbb{A}$*-module homomorphism* $u: E \to \mathrm{S}^p E$ *extends to a unique derivation* $D_u: \mathrm{S}(E) \to \mathrm{S}(E)$ *of degree* $p - 1$.

3. *Any* A*-module homomorphism* $u: E \to \bigwedge^p E$ *extends to a unique derivation, resp., skew-derivation* $D_u: \bigwedge(E) \to \bigwedge(E)$ *of degree* $p-1$ *if* p *is odd, resp., even.*

Definition 1.11 *An* A*-module* P *is called projective, if for any surjective* A*-module homomorphism* $u: E \to E'$ *and any homomorphism* $f: P \to E'$*, there exists a homomorphism* $g: P \to E$ *with* $f = u \circ g$*.*



P is projective iff P is a direct summand of a free module $F = P \oplus \tilde{F}$.

Lemma 1.12 *If* E *is a projective module, then* $\mathcal{T}(E)$, $\mathrm{S}(E)$ *and* $\bigwedge(E)$ *are projective, too. If* E *and* F *are finitely generated projective modules, then* $\mathrm{Hom}(E, F)$ *is finitely generated projective, too, thus* E^* *is finitely generated projective if* E *is so.*

Lemma 1.13 *Let* A *be a commutative ring and* E_i,F_i,G *be* A*-modules.*

1. *We have canonical* A*-module isomorphisms*

$$\begin{aligned} \mathrm{Hom}(E \otimes F, G) &\cong \mathrm{Hom}(E, \mathrm{Hom}(F, G)) \cong \mathrm{Hom}(F, \mathrm{Hom}(E, G)), \\ (E \otimes F)^* &\cong \mathrm{Hom}(E, F^*) \cong \mathrm{Hom}(F, E^*) \qquad (G = \mathrm{A}). \end{aligned}$$

2. *We have a canonical* A*-module morphism*

$$\mathrm{Hom}(E_1, F_1) \otimes \mathrm{Hom}(E_2, F_2) \to \mathrm{Hom}(E_1 \otimes E_2, F_1 \otimes F_2),$$

which is bijective if any of the pairs (E_1, E_2), (E_1, F_1) *or* (E_2, F_2) *consists of finitely generated projective* A*-modules.*

3. *The canonical* A*-module morphism* $\nu: \mathrm{Hom}(E, G) \otimes F \to \mathrm{Hom}(E, G \otimes F)$ *with* $\nu(\gamma \otimes f) := (e \mapsto \gamma(e) \otimes f)$ *is injective if* F *is projective, it is bijective if* E *or* F *is finitely generated projective.*

4. *The canonical* A*-module morphism* $\theta: E^* \otimes F \to \mathrm{Hom}(E, F)$ *with* $\theta(e^* \otimes f) := (e \mapsto e^*(e) \otimes f)$ *is injective if* F *is projective, it is bijective if* E *or* F *is finitely generated projective.*

5. *The canonical evaluation morphism* $\jmath_E: E \to E^{**}$ *is injective if* E *is projective, it is bijective if* E *is finitely generated projective.*

6. *The canonical* A*-module morphism* $\theta' := \theta \circ (\jmath_E \otimes \mathrm{id}_F): E \otimes F \to \mathrm{Hom}(E^*, F)$ *is injective if* E *and* F *are projective, it is bijective if* E *is finitely generated projective.*

7. *The canonical* $\mathbb{A}$*-module morphism* $\mu\colon E^* \otimes F^* \to (E \otimes F)^*$ *with* $\mu(e^* \otimes f^*) := (e \otimes f \mapsto e^*(e) f^*(f))$ *is bijective if* E *or* F *is finitely generated projective.*

For any $\mathbb{A}$-module F, $\mathrm{Hom}(\cdot, F)$ is a contravariant functor in the category of $\mathbb{A}$-modules ($\mathcal{T}$ is a covariant one) and thus defines an alternation and a symmetrization $\mathrm{Hom}(\mathfrak{S}^\varsigma, F) = \cdot \circ \mathfrak{S}^\varsigma$ on $\mathrm{Hom}(\mathcal{T}(E), F)$. So e. g.,

$$\begin{array}{ccccccc} \bigotimes^p E & \xrightarrow{\mathfrak{S}^\varsigma_p} & \bigotimes^p E & & \mathrm{Hom}(\bigotimes^p E, F) & \xleftarrow{\cdot \circ \mathfrak{S}^\varsigma_p} & \mathrm{Hom}(\bigotimes^p E, F) \\ \big\downarrow \tilde\rho & & \big\downarrow \tilde\rho & \xrightarrow{\mathrm{Hom}(\cdot, F)} & \big\uparrow \cdot \circ \tilde\rho & & \big\uparrow \cdot \circ \tilde\rho \\ \bigotimes^p E & \xrightarrow{\mathfrak{S}^\varsigma_p} & \bigotimes^p E & & \mathrm{Hom}(\bigotimes^p E, F) & \xleftarrow{\cdot \circ \mathfrak{S}^\varsigma_p} & \mathrm{Hom}(\bigotimes^p E, F) \end{array}$$

are commutative diagrams for any $\rho \in S_p$. We obtain (cf. [4, pp. 70, 80]):

$$\mathrm{Hom}(\mathfrak{S}^\varsigma_p, F)(\mathrm{Hom}(\otimes^p E, F)) \cong \mathrm{Hom}((\mathrm{S}^\varsigma)^p(E), F).$$

In the category of $\mathbb{R}$-vector spaces we thus have alternations and symmetrizations on $\mathrm{Hom}(\mathcal{T}(V), W)$ with $\mathrm{Sym}^\varsigma_p(\mathrm{Hom}(\bigotimes^p V, W)) = \mathrm{Sym}^\varsigma_p(V, W) \cong \mathrm{Hom}((\mathrm{S}^\varsigma)^p(V), W)$ for all $p \in \mathbb{N}_0$ and vector spaces V, W.

For $F = \mathbb{A}$ we have a canonical homomorphism $J\colon \mathcal{T}(E^*) \to \mathcal{T}(E)^*$. Analogously to (2), $J_p\colon \bigotimes^p E^* \to (\bigotimes^p E)^*$ is naturally given by

$$(e_1^* \otimes \cdots \otimes e_p^*)(e_1 \otimes \cdots \otimes e_p) := e_1^*(e_1) \cdots e_p^*(e_p)$$

and obeys $\mathrm{Hom}((\mathfrak{S}^\varsigma)^E_p, \mathbb{A}) \circ J_p = J_p \circ (\mathfrak{S}^\varsigma)^{E^*}_p$. By Lemma 1.13, J_p is an isomorphism if E is finitely generated projective. This is the case, if we deal with finite dimensional vector spaces or, by the following theorem, with vector fields on finite dimensional manifolds. Then both alternations, resp., symmetrizations coincide: $\mathrm{Hom}((\mathfrak{S}^\varsigma)^E, \mathbb{A}) = (\mathfrak{S}^\varsigma)^{E^*}$ on $\mathcal{T}(E^*)$ and $\mathrm{Hom}((\mathfrak{S}^\varsigma)^E, \mathbb{A})(\mathcal{T}(E^*)) = \mathrm{S}^\varsigma(E^*)$.

Lemma 1.14 *Let* $N \in \mathbb{N}$ *and suppose that for all* $i \in I$, E_i *are finitely generated projective* $\mathbb{A}_i$*-modules such that* $\tilde E_i$ *exist with* $E_i \oplus \tilde E_i \cong \mathbb{A}^N$. *Define* $\mathbb{A} := \prod_{i\in I} \mathbb{A}_i$, $E := \prod_{i \in I} E_i$ *and* $\tilde E := \prod_{i\in I} \tilde E_i$ *with componentwise multiplication. Then* E *is a finitely generated projective* $\mathbb{A}$*-module with* $E \oplus \tilde E \cong \mathbb{A}^N$.

Proof. $\phi\colon \prod_{i\in I} E_i \oplus \prod_{i\in I} \tilde E_i \to \prod_{i\in I}(E_i \oplus \tilde E_i)$, $[(e_i)_{i\in I}, (\tilde e_i)_{i\in I}] \mapsto ([e_i, \tilde e_i])_{i\in I}$ is an isomorphism of $\mathbb{A}$-modules. □

Theorem 1.15 (Swan's theorem) *For every* n*-dimensional paracompact manifold* M, $\mathcal{D}^1(M)$ *is a finitely generated projective* $C^\infty(M)$*-module. As a consequence*

$$\mathcal{D}^\bullet(M) = \mathcal{T}(\mathcal{D}^1(M)), \quad \mathcal{D}_\bullet(M) = \mathcal{T}(\mathcal{D}_1(M)), \quad \mathcal{D}(M) = \mathcal{T}(\mathcal{D}^1(M)) \otimes \mathcal{T}(\mathcal{D}_1(M)).$$

Proof. For connected M_0, see GREUB, HALPERIN, VANSTONE, [5, I p. 107]; tracing their proof shows that one can always choose $N = n(n+1)$ vector fields generating $\mathcal{D}^1(M_0)$. On an arbitrary paracompact manifold $M = \dot{\bigcup}_{i\in I} M_i$ this holds for any component M_i, such that we may find $C^\infty(M_i)$-modules $\tilde{\mathcal{D}}(M_i)$ with $\mathcal{D}^1(M_i) \oplus \tilde{\mathcal{D}}(M_i) = C^\infty(M_i)^N$. Since $C^\infty(M) = \prod_{i\in I} C^\infty(M_i)$ and $\mathcal{D}^1(M) = \prod_{i\in I} \mathcal{D}^1(M_i)$, the statement follows from Lemma 1.14. □

Note 1.16 Some remarks on topological properties of manifolds: by definition every finite dimensional manifold M is locally compact and locally arcwise connected. The latter ensures that the connected and the arcwise connected components are identical, thus M is connected iff it is arcwise connected.

For connected M, it is equivalent to say that M satisfies the second axiom of countability (i. e. has a countable basis), that a Riemannian metric on M exists, that M is metrizable or that M is paracompact, cf. KOBAYASHI, NOMIZU, [6, p. 271]. This yields equivalence also for manifolds with countably many components.

So in the general case, the second axiom of countability implies the other three properties. These are equivalent for finite dimensional manifolds: every metrizable topological space is paracompact, every paracompact manifold admits a Riemannian metric using the partition of unity subordinate to the atlas of M, and the Riemannian metric in turn guaranties a metric d_i on each component $M_i \subseteq M$. Combined with the discrete metric between the components we obtain a metric on M: by the axiom of choice, we may pick $c_i \in M_i$ for all $i \in I$ and define for $x_i \in M_i$, $y_j \in M_j$:

$$d(x_i, y_j) := \begin{cases} d_i(x_i, y_i), & \text{if } i = j, \\ d_i(x_i, c_i) + d_j(y_j, c_j) + 1, & \text{if } i \neq j. \end{cases}$$

For fields we denote $A_p := \mathfrak{A}_p^{\mathcal{D}_1(M)} = \mathrm{Hom}(\mathfrak{A}_p^{\mathcal{D}^1(M)}, C^\infty(M))$. Then

$$\mathcal{A}_p(M) = A_p(\mathcal{D}_p(M)) = \bigwedge\nolimits^p \mathcal{A}_1(M) \qquad \text{for all} \quad p \geq 1, \tag{5}$$

thus every p-form on a paracompact manifold can be represented as a sum of wedge products of 1-forms. $\mathcal{A}(M) = A(\mathcal{D}_*(M)) = \bigwedge(\mathcal{D}_1(M))$ is the GRASSMANN *algebra of the manifold M.*

For all $f, g \in C^\infty(M)$, $\alpha_r \in \mathcal{A}_r(M)$, $\beta_s \in \mathcal{A}_s(M)$ and $\mathcal{X}^i \in \mathcal{D}^1(M)$ we have

$$f \wedge \alpha_r = \alpha_r \wedge f = f \cdot \alpha_r, \qquad f \wedge g = f \cdot g \qquad \text{and} \tag{6}$$

$$\alpha_r \wedge \beta_s(\mathcal{X}^1, \ldots, \mathcal{X}^{r+s}) = \frac{1}{(r+s)!} \sum_{\rho \in S_{r+s}} (-1)^\rho \left(\alpha_r(\mathcal{X}^{\rho(1)}, \ldots, \mathcal{X}^{\rho(r)})\right) \cdot \cdot\left(\beta_s(\mathcal{X}^{\rho(r+1)}, \ldots, \mathcal{X}^{\rho(r+s)})\right). \tag{7}$$

Analogous formulae hold for $\alpha^1_{r_1} \wedge \cdots \wedge \alpha^k_{r_k}, k \in \mathbb{N}$ with $\frac{1}{r!}$ and $\rho \in S_r$, where $r := \sum_{i=1}^k r_i$; using the cycle $\tau = (123\cdots(r+s))^s \in S_{r+s}$ with $(-1)^\tau = (-1)^{rs}$ one proves (for any $\mathbb{A}$-module E)

$$\alpha_r \wedge \beta_s = (-1)^{rs} \beta_s \wedge \alpha_r. \tag{8}$$

Obviously, (7) also holds for the exterior product of alternating maps $\alpha_r \in \mathrm{Alt}_r(V, \mathbb{R})$ and $\beta_s \in \mathrm{Alt}_s(V, \mathbb{R})$, where V is a finite dimensional vector space.

Definition 1.17 *Let $p \in \mathbb{N}_0$, $\omega \in \mathcal{A}_p(M)$, $\mathcal{X}^i \in \mathcal{D}^1(M)$. Then on $\mathcal{A}_p(M)$ the* exterior differentiation *$d: \mathcal{A}(M) \to \mathcal{A}(M)$ is defined by ($\widehat{\ }$ denotes omission)*

$$\begin{aligned}(p+1)d\omega(\mathcal{X}^1,\dots,\mathcal{X}^{p+1}) &= \sum_{i=1}^{p+1}(-1)^{i+1}\mathcal{X}^i(\omega(\mathcal{X}^1,\dots,\widehat{\mathcal{X}^i},\dots,\mathcal{X}^{p+1})) \\ &+\sum_{i=1}^{p}\sum_{j=i+1}^{p+1}(-1)^{i+j}\omega([\mathcal{X}^i,\mathcal{X}^j],\mathcal{X}^1,\dots,\widehat{\mathcal{X}^i},\dots,\widehat{\mathcal{X}^j},\dots,\mathcal{X}^{p+1}). \qquad (9)\end{aligned}$$

By the following proposition d is a differential operator, cf. [3, p. 20]:

Proposition 1.18 *d is a $\mathbb{R}$-linear mapping with the following properties:*

1. $d(\mathcal{A}_p(M)) \subseteq \mathcal{A}_{p+1}(M)$ *for all* $p \in \mathbb{N}_0$,
2. $f \in \mathcal{A}_0(M) \implies df(\mathcal{X}) = \mathcal{X}(f)$ *for all* $\mathcal{X} \in \mathcal{D}^1(M)$,
3. $d^2 := d \circ d = 0$,
4. $d(\alpha_p \wedge \omega) = d\alpha_p \wedge \omega + (-1)^p \alpha_p \wedge d\omega$, *if* $\alpha_p \in \mathcal{A}_p(M)$, $\omega \in \mathcal{A}(M)$.

These properties define d uniquely.

According to Definition 1.1, d thus is a skew-derivation of $\mathcal{A}(M)$ of degree 1.

Before we concentrate on the GRASSMANN algebra and introduce vector-valued forms, we close this section with a remark on derivations of the *mixed* tensor algebra.

Definition 1.19 *For a finitely generated projective* $\mathbf{A}$*-module E the* mixed tensor algebra *$\mathcal{T}^\bullet_\bullet(E)$ is defined by*

$$\mathcal{T}^\bullet_\bullet(E) = \bigoplus_{r,s=0}^{\infty} \mathcal{T}^r_s(E) \quad \text{and} \quad \mathcal{T}^r_s(E) = \underbrace{E\otimes\cdots\otimes E}_{r}\otimes\underbrace{E^*\otimes\cdots\otimes E^*}_{s}.$$

(We have proved that $\mathcal{T}^r_s(E) \cong [\mathcal{T}^s_r(E)]^*$.) For all $k \le r, l \le s \in \mathbb{N}$, $\mathbf{A}$-module homomorphisms $C^k_l: \mathcal{T}^r_s(E) \to \mathcal{T}^{r-1}_{s-1}(E)$ are uniquely defined by the requirement that for all $e^i \in E$, $e^*_j \in E^*$

$$C^k_l(e^1\otimes\cdots\otimes e^r\otimes e^*_1\otimes\cdots\otimes e^*_s) := e^*_l(e^k)\cdot e^1\otimes\cdots\widehat{e^k}\cdots\otimes e^r\otimes e^*_1\otimes\cdots\widehat{e^*_l}\cdots\otimes e^*_s.$$

C^k_l is called the contraction of the k-th contravariant and the l-th covariant index. $\underline{\mathrm{der}\,\mathcal{T}^\bullet_\bullet(E)}$ *denotes the* LIE *subalgebra of all derivations of the mixed tensor algebra that preserve type and commute with all contractions, i. e. for all $D \in \underline{\mathrm{der}\,\mathcal{T}^\bullet_\bullet(E)}$:*

$$\begin{aligned} D(K\otimes K') &= (DK)\otimes K' + K\otimes(DK') \quad \text{for all } K, K' \in \mathcal{T}^\bullet_\bullet(E), \qquad (10)\\ D(\mathcal{T}^r_s(E)) &\subseteq \mathcal{T}^r_s(E) \quad \text{and} \quad D\circ C = C\circ D \quad \text{for all contractions } C. \qquad (11)\end{aligned}$$

$\underline{\mathrm{der}\,\mathcal{T}^\bullet_\bullet(E)}_0$ *denotes the* LIE *subalgebra of all $D \in \underline{\mathrm{der}\,\mathcal{T}^\bullet_\bullet(E)}$ with $D|_{\mathbf{A}} = 0$.*

We then have the following proposition (cf. [6, p. 25]):

Proposition 1.20 *For any finitely generated projective* $\mathbb{A}$*-module* E *the restriction map* $|_E: \underline{\operatorname{der} \mathcal{T}^*_*(E)_0} \to \operatorname{End}(E)$ *is an isomorphism of* LIE *algebras.*

Proof: (cf. [6, p. 25]). Obviously $|_E$ is a homomorphism of LIE algebras. Let $D \in \underline{\operatorname{der} \mathcal{T}^*_*(E)_0}$ and $B := D|_E: E \to E$. Then for all $e \in E$ and $e^* \in E^*$,

$$0 = D(e^*(e)) = C(D(e \otimes e^*)) = C[(De \otimes e^*) + (e \otimes De^*)] = e^*(Be) + (De^*)(e),$$

and thus $D|_{E^*} = -B^*$, where B^* is the transpose of B. Since $\mathcal{T}^*_*(E)$ is generated by $\mathbb{A}$, E and E^*, D is determined by its restriction to $\mathbb{A}$, E and E^* and thus $|_E$ is injective. Conversely, given any $B \in \operatorname{End}(E)$, we define $D|_{\mathbb{A}} = 0$, $D|_E = B$ and $D|_{E^*} = -B^*$ and extend D to a derivation of $\mathcal{T}^*_*(E)$ by (10). The existence of D is then a consequence of the universal factorization property of the tensor product. □

Corollary 1.21 *The restriction map* $|_V: \underline{\operatorname{der} \mathcal{T}^*_*(V)} \to \operatorname{End}(V)$ *is an isomorphism for any finite dimensional vector space* V.

Proof. (10) yields $D1 = 0$, thus $D|_{\mathbb{K}} = D|_{\mathcal{T}^0_0(V)} = 0$. □

1.2 Vector Valued Differential Forms

For any real vector space V the algebraic tensor products

$$C^\infty(M) \otimes V, \quad \mathcal{D}_p(M) \otimes V, \quad \mathcal{D}_*(M) \otimes V, \quad \mathcal{A}_p(M) \otimes V \quad \text{and} \quad \mathcal{A}(M) \otimes V$$

are $C^\infty(M)$-modules (trivial in the second factor). Let

$$C^\infty(M,V), \quad \mathcal{D}_p(M,V), \quad \mathcal{D}_*(M,V), \quad \mathcal{A}_p(M,V) \quad \text{and} \quad \mathcal{A}(M,V)$$

denote the $C^\infty(M)$-modules of all weakly differentiable maps from M to V and of the corresponding V-valued covariant fields and forms on M: $C^\infty(M,V)$ contains all maps $f: M \to V$ with $\omega \circ f \in C^\infty(M)$ for every linear functional $\omega: V \to \mathbb{R}$, $\mathcal{A}(M,V)$ contains all alternating $C^\infty(M)$-linear maps $\alpha: \mathcal{D}^*(M) \to C^\infty(M,V)$, etc. The canonical embedding $\imath: C^\infty(M) \otimes V \to C^\infty(M,V)$, defined by $[\imath(f \otimes v)](x) := f(x)v \in V$ for all $f \in C^\infty(M)$, $x \in M$ and $v \in V$, is injective and induces canonical embeddings of $\mathcal{D}_*(M) \otimes V$ into $\mathcal{D}_*(M,V)$, resp., of $\mathcal{A}(M) \otimes V$ into $\mathcal{A}(M,V)$.

$A := \operatorname{Hom}(\mathfrak{A}^{\mathcal{D}^1(M)}, C^\infty(M,V))$ defines the alternation $A: \mathcal{D}_*(M,V) \to \mathcal{A}(M,V)$. If $V \cong \mathbb{R}^n$ with its natural differential structure then $C^\infty(M,V)$, resp., $\mathcal{A}(M,V)$ exactly contain the differentiable maps from M to V, resp., differential forms on M with values in V and the embeddings are bijective. This enables us to identify $\mathcal{A}(M) \otimes V$ with $\mathcal{A}(M,V)$, etc. We also identify $\mathcal{A}(M,\mathbb{R})$ and $\mathcal{A}(M)$, etc. For infinite dimensional V the tensor products represent only the submodules of those maps f, resp., forms α, where $f(M)$, resp., $\alpha(\mathcal{D}^*(M))$ spans only a finite subspace in V. Omitting $\imath$ we write:

Definition 1.22 *For $\mathcal{X}, \mathcal{X}^i \in \mathcal{D}^1(M)$, $f \in C^\infty(M)$, $\omega \in \mathcal{A}_p(M)$, $x \in M$ and $v \in V$ define*

$$\begin{aligned} \mathcal{X}(f \otimes v) &:= \mathcal{X} f \otimes v, & (f \otimes v)(x) &:= f(x) v \in V, \\ d(\omega \otimes v) &:= d\omega \otimes v, & (\omega \otimes v)(\mathcal{X}^1, \ldots, \mathcal{X}^p) &:= \omega(\mathcal{X}^1, \ldots, \mathcal{X}^p) \otimes v, \\ (\omega \otimes v)_x(\mathcal{X}^1_x, \ldots, \mathcal{X}^p_x) &:= (\omega \otimes v)(\mathcal{X}^1, \ldots, \mathcal{X}^p)(x) &= \omega_x(\mathcal{X}^1_x, \ldots, \mathcal{X}^p_x) \otimes v \in V. \end{aligned}$$

For bilinear $\phi: V \times W \to Z$ define $\wedge_\phi: (\mathcal{A}(M) \otimes V) \times (\mathcal{A}(M) \otimes W) \to (\mathcal{A}(M) \otimes Z)$ by

$$(\alpha \otimes v) \wedge_\phi (\beta \otimes w) := (\alpha \wedge \beta) \otimes \phi(v, w) \qquad \text{for all} \quad \alpha, \beta \in \mathcal{A}(M), \quad v \in V, w \in W.$$

For a bilinear mapping $\phi: V \times V \to V$, we will use $\wedge_V$ rather than $\wedge_\phi$ and for a Lie *algebra $\mathfrak{g}$ the notation $\wedge_\mathfrak{g}$ will imply $\phi(X, Z) := [X, Y]$.*

$\wedge_V$ turns $\mathcal{A}(M) \otimes V$ into a (non-associative) algebra. We immediately get:

Lemma 1.23 *Let $\alpha_r \in \mathcal{A}_r(M) \otimes V$, $\beta_s \in \mathcal{A}_s(M) \otimes V$, $\phi: V \times V \to V$ bilinear.*

1. *If ϕ is associative then $\wedge_V$ is so, too.*
2. *If ϕ is commutative then* $\quad \alpha_r \wedge_V \beta_s = (-1)^{rs} \beta_s \wedge_V \alpha_r$.
3. *If ϕ is anticommutative then $\alpha_r \wedge_V \beta_s = (-1)^{rs+1} \beta_s \wedge_V \alpha_r$.*

If $\mathbf{A}$ is an algebra and $.: \mathbf{A} \times V \to V$ is a (left) representation on a vector space V with respect to the multiplication ϕ in $\mathbf{A}$ (i. e., $\phi(a,b).v = a.(b.v)$ for all $a, b \in \mathbf{A}$, $v \in V$), then for all $\alpha, \beta, \gamma \in \mathcal{A}(M)$

$$[(\alpha \otimes a) \wedge_\mathbf{A} (\beta \otimes b)] \wedge_. (\gamma \otimes v) = (\alpha \otimes a) \wedge_. [(\beta \otimes b)] \wedge_. (\gamma \otimes v)] = (\alpha \wedge \beta \wedge \gamma) \otimes [(\phi(a,b).v].$$

We will use $\wedge_.$ for the wedge product of $\mathrm{gl}(\mathbb{R}^n)$-valued and $\mathbb{R}^n$-valued forms.

As a special case of $\wedge_\phi$ for $\mathbf{A} = \mathbb{R}$, $\mathcal{A}(M, V)$ also is an $\mathcal{A}(M)$-bimodule; denote this module multiplication with $\wedge$, too. Using the $C^\infty(M)$-module structure of $C^\infty(M, V)$, we get formulae like (6), (7) and (8) for $\alpha_r \in \mathcal{A}_r(M, V)$, $\beta_s \in \mathcal{A}_s(M)$ and vice versa, etc. For $\mathcal{A}(M) \otimes V$ we have

$$(\alpha \otimes v) \wedge \beta = (\alpha \wedge \beta) \otimes v = \alpha \wedge (\beta \otimes v) \qquad \text{for all} \quad \alpha, \beta \in \mathcal{A}(M),\ v \in V. \tag{12}$$

Theorem 1.24 *On a finite dimensional paracompact manifold M we have the following $C^\infty(M)$-module isomorphisms for any vector space V:*

$$\begin{aligned} \mathcal{D}_*(M, V) &\cong \mathcal{D}_*(M) \otimes C^\infty(M, V) &\cong \mathcal{T}(D_1(M)) \otimes C^\infty(M, V), \\ \mathcal{A}(M, V) &\cong \mathcal{A}(M) \otimes C^\infty(M, V) &\cong \bigwedge(D_1(M)) \otimes C^\infty(M, V). \end{aligned}$$

If $\mathbf{A}$ is an associative algebra with unit $\mathbb{1}$ we get $\mathcal{A}(M) \otimes \mathbf{A} = \bigwedge_\mathbf{A}(\mathcal{A}_1(M) \otimes \mathbf{A})$.

Proof. By SWAN'S Theorem 1.15 there exists a $C^\infty(M)$-module $\tilde{\mathcal{D}}(M)$ such that $\mathcal{D}^1(M) \oplus \tilde{\mathcal{D}}(M) = C^\infty(M)^N$. Let $i: \mathcal{D}^1(M) \to C^\infty(M)^N$ and $\rho: C^\infty(M)^N \to \mathcal{D}^1(M)$ be the module homomorphisms with $\rho \circ i = \mathrm{id}_{\mathcal{D}^1(M)}$ and fix a basis $\{E^j\}_{j=1,\dots,N}$ for $C^\infty(M)^N$ with dual basis $\{e_j\}_{j=1,\dots,N}$. For $j \leq N$ let $\omega_j := e_j \circ i \in \mathcal{D}_1(M)$, define $\tilde{a} \in \mathrm{Hom}_{C^\infty(M)}(\otimes^p[C^\infty(M)^N], C^\infty(M,V))$ for any $a \in \mathcal{D}_p(M,V)$ by $\tilde{a} := a \circ \otimes^p \rho$ and let $a^{j_1\cdots j_p} := \tilde{a}(E^{j_1} \otimes \cdots \otimes E^{j_p}) \in C^\infty(M,V)$ for all $j_i \leq N$. Then $\tilde{a} = \sum_{j_1\cdots j_p=1}^N e_{j_1} \otimes \cdots \otimes e_{j_p} \otimes a^{j_1\cdots j_p}$ whence $a = \tilde{a} \circ \otimes^p i = \sum_{j_1\cdots j_p=1}^N \omega_{j_1} \otimes \cdots \otimes \omega_{j_p} \otimes a^{j_1\cdots j_p} \in \otimes^p \mathcal{D}_1(M) \otimes C^\infty(M,V)$. The reverse direction is trivial, so $\mathcal{D}_*(M,V) \cong \mathcal{D}_*(M) \otimes C^\infty(M,V)$. From this the statement for $\mathcal{A}(M,V)$ follows immediately; the last is a consequence of $a = \mathbb{1}^{(1)} \otimes \cdots \otimes \mathbb{1}^{(s)} \otimes a$ for any $a \in \mathrm{A}$ and any $s \in \mathbb{N}$. □

Lemma 1.25 *Proposition 1.18 holds for $\mathcal{A}(M) \otimes V$ as well, not only for $\wedge$ in the sense of* (12) *but also for $\wedge_\phi$ and $\wedge_V$: whenever $\wedge_V$ is defined, d is a skew-derivation of degree 1 of $\mathcal{A}(M) \otimes V$.*

Definition 1.26 (Pullbacks and push-outs) *If $f: M \to N$ is differentiable, we denote the differential of f at $x \in M$ by df_x. We have $[df_x(\mathcal{X}_x)]g = \mathcal{X}_x(g \circ f)$ for all $\mathcal{X}_x \in T_x(M)$, $g \in C^\infty(N)$.*

For $\alpha \in \mathcal{D}_r(N,V), r \in \mathbb{N}$ and $X_i \in T_x(M)$, the pullback *$f^*\alpha \in \mathcal{A}_r(M,V)$ is defined by $(f^*\alpha)_x(X_1,\dots,X_r) = \alpha_{f(x)}(df_x(X_1),\dots,df_x(X_r))$. For $\alpha \in C^\infty(N,V)$ we have $f^*\alpha := \alpha \circ f$, linear extension defines the pullback on $\mathcal{D}_*(N,V)$. Obviously $f^*(\mathcal{A}(N,V)) \subseteq \mathcal{A}(M,V)$ and — if we insert $\mathcal{D}_*(M) \otimes V$ into $\mathcal{D}_*(M,V)$ — $f^*(\mathcal{D}_*(N) \otimes V) \subseteq \mathcal{D}_*(M) \otimes V$ and $f^*(\mathcal{A}(N) \otimes V) \subseteq \mathcal{A}(M) \otimes V$.*

If f is a diffeomorphism then for $\mathcal{X} \in \mathcal{D}^1(M)$ the push-out *$f_*\mathcal{X} \in \mathcal{D}^1(N)$ is defined by $(f_*\mathcal{X})_{f(x)} = df_x(\mathcal{X}_x)$ for all $x \in M$.*

Analogously, every linear map $F: V \to W$ defines a pullback $F^* = \mathrm{Hom}(\mathcal{T}(F), Z)$: $\mathrm{Hom}(\mathcal{T}(W), Z) \to \mathrm{Hom}(\mathcal{T}(V), Z)$: for $K \in \mathrm{Hom}(\otimes^p W, Z)$, $p \in \mathbb{N}$ and $X_i \in V$ we have $F^*K(X_1,\dots,X_p) := K(F(X_1),\dots,F(X_p))$, so $F^*(\mathrm{Alt}(W,Z)) \subseteq \mathrm{Alt}(V,Z)$. $F_\circ = \mathrm{Hom}(\mathcal{T}(Z), F)$: $\mathrm{Hom}(\mathcal{T}(Z), V) \to \mathrm{Hom}(\mathcal{T}(Z), W)$ is defined by $F_\circ K = F \circ K$, so $F_\circ(\mathrm{Alt}(Z,V)) \subseteq \mathrm{Alt}(Z,W)$.

Finally F defines the push-out $F_*: \mathcal{D}_*(M,V) \to \mathcal{D}_*(M,W)$ by $F_*\omega = F \circ \omega$. Again $F_*(\mathcal{A}(M,V)) \subseteq \mathcal{A}(M,W)$ and $F_*(\mathcal{D}_*(M) \otimes V) \subseteq \mathcal{D}_*(M) \otimes W$, where we have $F_*(\alpha \otimes v) = \alpha \otimes F(v)$ for all $\alpha \in \mathcal{D}_*(M)$, $v \in V$.

Note 1.27 There seems to be an ambiguity in the definition of df_x for $x \in M$ and $f \in C^\infty(M) \otimes V$: df_x can be interpreted as differential $df_x: T_x(M) \to T_{f(x)}(V)$ and as value of the 1-form $df \in \mathcal{A}_1(M) \otimes V$ in $x \in M$ in the sense of the Definitions 1.5 and 1.22. But if we naturally identify the tangent spaces of V with V: $T_v(V) = V$ for all $v \in V$, the ambiguity vanishes, since we have for the differential

$$df_x(\mathcal{X}_x) = \mathcal{X}_x(f) = \mathcal{X}(f)(x) = df(X)(x) \in V \quad \text{for all} \quad \mathcal{X} \in \mathcal{D}^1(M).$$

Pullbacks and push-outs obey $(f \circ g)_* = f_* \circ g_*$, $(f \circ g)^* = g^* \circ f^*$, which one may prove using the chain rule $d(f \circ g)_x = df_{g(x)} \circ dg_x$. We have, cf. [3, p. 24]:

Lemma 1.28 *If $f: M \to N$ is a diffeomorphism then $f_*: \mathcal{D}^1(M) \to \mathcal{D}^1(N)$ is an isomorphism of* LIE *algebras, so*

$$f_*[\mathcal{X},\mathcal{Y}] = [f_*\mathcal{X}, f_*\mathcal{Y}] \qquad \text{for all} \quad \mathcal{X},\mathcal{Y} \in \mathcal{D}^1(M).$$

Lemma 1.29 *If $f: M \to N$ is differentiable, $F: V \to W$ and $G: X \to Y$ linear, $\alpha, \beta \in \mathcal{A}(N) \otimes V$, $\gamma \in \mathcal{A}(N) \otimes W$, $\omega \in \mathcal{A}(N)$ and $K \in \mathrm{Hom}(\mathcal{T}(W), X)$ then*

1. *f^* and F_* commute: $f^*(F_*\alpha) = F_*(f^*\alpha)$, analogously $F^*(G_\circ K) = G_\circ(F^*K)$;*
2. *f^* and F_* commute with d: $\quad d(f^*\alpha) = f^*(d\alpha), \quad d(F_*\alpha) = F_*(d\alpha)$;*
3. *$f^*(\omega \wedge \alpha) = (f^*\omega) \wedge (f^*\alpha), \quad F_*(\omega \wedge \alpha) = \omega \wedge (F_*\alpha)$;*
4. *$f^*(\alpha \wedge_\phi \gamma) = (f^*\alpha) \wedge_\phi (f^*\gamma)$, for any bilinear $\phi: V \times W \to Z$;*
5. *$f^*(\alpha \wedge_V \beta) = (f^*\alpha) \wedge_V (f^*\beta)$, i. e., f^* is an algebra homomorphism;*
6. *$F_*(\alpha \wedge_V \beta) = (F_*\alpha) \wedge_W (F_*\beta)$, if in addition $F \circ \phi_V = \phi_W \circ (F \times F)$, thus F_* is an algebra homomorphism, if F is one.*

Lemma 1.30 *For every differentiable map $m: P_1 \times P_2 \to N$ the mappings $m_p := m(p,\cdot): P_2 \to N$ and $m^q := m(\cdot, q): P_1 \to N$ are differentiable for all $p \in P_1$, $q \in P_2$. Identifying $T_{(p,q)}(P_1 \times P_2)$ and $T_p(P_1) \oplus T_q(P_2)$, we have*

$$dm_{(p,q)}(X,Y) = (dm_p)_q(Y) + (dm^q)_p(X) \quad \text{for all } X \in T_p(P_1),\, Y \in T_q(P_2). \tag{13}$$

For differentiable $f: M \to P_1$, $g: M \to P_2$ and $h = m \circ (f,g): M \to N$ this yields

$$(h^*\omega)_x = [f^*(m^{g(x)})^*\omega]_x + [g^*(m_{f(x)})^*\omega]_x \quad \text{for all } x \in M,\, \omega \in \mathcal{A}_1(N,V). \tag{14}$$

Analogously to Definition 1.19 we define contractions of tensor fields:

Definition 1.31 *For all $k \leq r, l \leq s \in \mathbb{N}$, $C^\infty(M)$-linear maps $C^k_l: \mathcal{D}^r_s(M) \to \mathcal{D}^{r-1}_{s-1}(M)$ are uniquely defined by the requirement that for all $\mathcal{X}^i \in \mathcal{D}^1(M)$, $\mathcal{Y}_j \in \mathcal{D}_1(M)$*

$$C^k_l(\mathcal{X}^1 \otimes \cdots \otimes \mathcal{X}^r \otimes \mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_s) := \mathcal{Y}_l(\mathcal{X}^k) \cdot \mathcal{X}^1 \otimes \cdots \widehat{\mathcal{X}^k} \cdots \otimes \mathcal{X}^r \otimes \mathcal{Y}_1 \otimes \cdots \widehat{\mathcal{Y}_l} \cdots \otimes \mathcal{Y}_s.$$

C^k_l is called the contraction of the k-th contravariant and the l-th covariant index.

Definition 1.32 *For each $\mathcal{X} \in \mathcal{D}^1(M)$ the* interior product with respect to $\mathcal{X}$, *$\imath_\mathcal{X}: \mathcal{D}_\bullet(M,V) \to \mathcal{D}_\bullet(M,V)$, $\mathcal{D}_p(M,V) \to \mathcal{D}_{p-1}(M,V)$, is defined in the following way: for each $\omega_p \in \mathcal{D}_p(M,V)$ and $\mathcal{Y}^i \in \mathcal{D}^1(M)$,*

$$(\imath_\mathcal{X}\omega_p)(\mathcal{Y}^1, \ldots, \mathcal{Y}^{p-1}) := p\, \omega_p(\mathcal{X}, \mathcal{Y}^1, \ldots, \mathcal{Y}^{p-1}).$$

Thus $\imath_\mathcal{X} f = 0$ for all $f \in C^\infty(M,V)$ and we may write $\imath_\mathcal{X}\omega_p = p\, C^1_1(\mathcal{X} \otimes \omega_p)$.

Obviously $\imath_\mathcal{X}(\mathcal{D}_p(M) \otimes V) \subseteq \mathcal{D}_{p-1}(M) \otimes V$, $\imath_\mathcal{X}(\mathcal{A}_p(M,V)) \subseteq \mathcal{A}_{p-1}(M,V)$ and $\imath_\mathcal{X}(\mathcal{A}_p(M) \otimes V) \subseteq \mathcal{A}_{p-1}(M) \otimes V$ and one easily proves:

Lemma 1.33 *For all $\mathcal{X}, \mathcal{Y} \in \mathcal{D}^1(M)$ and $f \in C^\infty(M)$, the interior product satisfies:*

1. *$\imath_\mathcal{X}$ is a skew-derivation of degree -1 of $\mathcal{A}(M)$ (and of $\mathcal{A}(M) \otimes V$, whenever $\wedge_V$ is defined): it even is $C^\infty(M)$-linear and obeys*
$$\imath_\mathcal{X}(\alpha_p \wedge \omega) = \imath_\mathcal{X}\alpha_p \wedge \omega + (-1)^p \alpha_p \wedge \imath_\mathcal{X}\omega, \qquad \text{if} \quad \alpha_p \in \mathcal{A}_p(M),\ \omega \in \mathcal{A}(M);$$
2. $\imath_{\mathcal{X}+\mathcal{Y}} = \imath_\mathcal{X} + \imath_\mathcal{Y}, \qquad \imath_{f\mathcal{X}} = f \cdot \imath_\mathcal{X};$
3. *$\imath_\mathcal{X} \circ \imath_\mathcal{Y} = -\imath_\mathcal{Y} \circ \imath_\mathcal{X}$, thus $\imath_\mathcal{X}$ is a differential operator on $\mathcal{A}(M)$, resp., $\mathcal{A}(M) \otimes V$: $(\imath_\mathcal{X})^2 = \imath_\mathcal{X} \circ \imath_\mathcal{X} = 0$.*

Definition 1.34 *A* one-parameter group of (differentiable) transformations *on a manifold M is a mapping $\varphi: \mathbb{R} \times M \to M$ with $\varphi(t,x) = \varphi_t(x)$, where $\varphi_t: M \to M$ is a diffeomorphism for all $t \in \mathbb{R}$ and satisfies $\varphi_{t+s} = \varphi_t \circ \varphi_s$ for all $s, t \in \mathbb{R}$.*

A local *one-parameter group of* local *transformations is defined in the same way, except that $\varphi_t(x)$ is defined only for t in a neighborhood of 0 and x in an open set $U \in M$.*

For such one-parameter groups one proves [6, pp. 12 – 16]:

Proposition 1.35 *Every one-parameter group of transformations φ on M induces a vector field $\mathcal{X} \in \mathcal{D}^1(M)$ by:*
$$\mathcal{X}_x(f) := \frac{d}{dt} f(\varphi(t,x))|_{t=0} \qquad \text{for all} \quad f \in C^\infty(M),\ x \in M.$$
For all $x \in M$ the orbit *$\varphi^x: \mathbb{R} \to M$ is then an* integral curve *of $\mathcal{X}$, i. e., $\mathcal{X}_{\varphi(t,x)}$ is tangential to φ^x for all $t \in \mathbb{R}$. We have $\mathcal{X}_{\varphi_s(x)} = d\varphi_s(\mathcal{X}_x)$ for all $s \in \mathbb{R}$, $x \in M$ and*
$$[\mathcal{X}, \mathcal{Y}]_x = \lim_{t \to 0} \frac{1}{t}\{\mathcal{Y}_x - ((\varphi_t)_* \mathcal{Y})_x\} = \lim_{t \to 0} \frac{1}{t}\{((\varphi_{-t})_* \mathcal{Y})_x - \mathcal{Y}_x\} \quad \text{for all} \quad \mathcal{Y} \in \mathcal{D}^1(M).$$
Analogous statements hold for local one-parameter groups of local transformations with induced vector field $\mathcal{X} \in \mathcal{D}^1(U)$.

Proposition 1.36 *For every $\mathcal{X} \in \mathcal{D}^1(M)$ and every $x \in M$ there exists a neighborhood U of x, $\epsilon > 0$ and a local one-parameter group of local transformations $\varphi:]-\epsilon, \epsilon[\times U \to M$ which induces $\mathcal{X}$.*

$\mathcal{X} \in \mathcal{D}^1(M)$ is called *complete* if there exists a *global* one-parameter group of transformations that induces $\mathcal{X}$. On a *compact* manifold every vector field is complete.

Let $\mathrm{Diff}(M)$ denote the group of diffeomorphisms of the manifold M. For any $f \in \mathrm{Diff}(M)$ and $x \in M$, $df_x: T_x(M) \to T_{f(x)}(M)$ is a linear isomorphism and induces an isomorphism of the tensor algebras $\tilde{f}_x: \mathcal{T}(T_x(M)) \to \mathcal{T}(T_{f(x)}(M))$. We thus get an algebra automorphism $\tilde{f}: \mathcal{D}(M) \to \mathcal{D}(M)$ defined by
$$(\tilde{f}K)_x := \tilde{f}_{f^{-1}(x)}(K_{f^{-1}(x)}) \qquad \text{for all} \quad K \in \mathcal{D}(M),\ x \in M. \tag{15}$$
$\tilde{f}$ preserves type and commutes with contractions (cf. [6, p. 28]). For $\omega \in \mathcal{D}_*(M)$ we have $\tilde{f}\omega = (f^{-1})^*\omega$, so $\tilde{f}(\mathcal{A}(M)) = \mathcal{A}(M)$.

Definition 1.37 *Let φ be the (local) one-parameter group of transformations generated by a vector field $\mathcal{X} \in \mathcal{D}^1(M)$ according to Proposition 1.36. Then the* LIE differentiation $L_{\mathcal{X}}: \mathcal{D}(M) \to \mathcal{D}(M)$ with respect to $\mathcal{X}$ *is defined by*

$$(L_{\mathcal{X}}K)_x := \lim_{t\to 0}\frac{1}{t}\{K_x - (\widetilde{\varphi}_t K)_x\} = -\frac{d}{dt}(\widetilde{\varphi}_t K)_x|_{t=0} \qquad \text{for all} \quad K \in \mathcal{D}(M),\ x \in M.$$

$L_{\mathcal{X}}K$ *is called the* LIE derivative of the tensor field K with respect to $\mathcal{X}$. *Defining* $\frac{d}{dt}(\widetilde{\varphi}_t K) \in \mathcal{D}(M)$ *pointwise for all* $x \in M$*, we thus have* $L_{\mathcal{X}}K = -\frac{d}{dt}(\widetilde{\varphi}_t K)|_{t=0}$.

$$L_{\mathcal{X}}(\omega \otimes v) := L_{\mathcal{X}}\omega \otimes v = [\frac{d}{dt}(\varphi_t^*\omega)|_{t=0}] \otimes v \qquad \text{for all} \quad \omega \in \mathcal{D}_*(M),\ v \in V \quad (16)$$

defines the LIE *differentiation on* $\mathcal{D}_*(M) \otimes V$.

The following propositions hold [6, pp. 29 – 35]:

Proposition 1.38 *For all* $\mathcal{X}, \mathcal{Y} \in \mathcal{D}^1(M)$*, the* LIE *differentiation satisfies:*

1. $L_{\mathcal{X}}$ *is a derivation of* $\mathcal{D}(M)$*, thus it is linear and obeys*
$$L_{\mathcal{X}}(K \otimes K') = (L_{\mathcal{X}}K) \otimes K' + K \otimes (L_{\mathcal{X}}K') \quad \text{for all } K, K' \in \mathcal{D}(M);$$
2. $L_{\mathcal{X}}$ *preserves type:* $L_{\mathcal{X}}(\mathcal{D}^r_s(M)) \subseteq \mathcal{D}^r_s(M)$ *and commutes with contractions;*
3. $[L_{\mathcal{X}}, L_{\mathcal{Y}}] = L_{[\mathcal{X},\mathcal{Y}]}$ *and* $L_{\lambda\mathcal{X}+\mu\mathcal{Y}} = \lambda L_{\mathcal{X}} + \mu L_{\mathcal{Y}}$ *for all* $\lambda, \mu \in \mathbb{R}$*, which means that* $\{L_{\mathcal{X}} | \mathcal{X} \in \mathcal{D}^1(M)\}$ *is a* LIE *subalgebra of* $\operatorname{der}\mathcal{D}(M)$;
4. $L_{\mathcal{X}}f = \mathcal{X}f$ *for all* $f \in C^\infty(M)$, $\qquad L_{\mathcal{X}}\mathcal{Y} = [\mathcal{X}, \mathcal{Y}]$;
5. *if* $\omega \in \mathcal{D}_n(M) \otimes V$, $K \in \mathcal{D}^1_n(M) \cong \operatorname{Hom}(\mathcal{D}^n(M), \mathcal{D}^1(M))$ *and* $\mathcal{Y}^j \in \mathcal{D}^1(M)$,
$$\begin{aligned}(L_{\mathcal{X}}\omega)(\mathcal{Y}^1,\dots,\mathcal{Y}^n) &= \mathcal{X}(\omega(\mathcal{Y}^1,\dots,\mathcal{Y}^n)) - \sum_{i=1}^n \omega(\mathcal{Y}^1,\dots,[\mathcal{X},\mathcal{Y}^i],\dots,\mathcal{Y}^n), \quad (17)\\ (L_{\mathcal{X}}K)(\mathcal{Y}^1,\dots,\mathcal{Y}^n) &= [\mathcal{X}, K(\mathcal{Y}^1,\dots,\mathcal{Y}^n)] - \sum_{i=1}^n K(\mathcal{Y}^1,\dots,[\mathcal{X},\mathcal{Y}^i],\dots,\mathcal{Y}^n); \quad (18)\end{aligned}$$
6. $[L_{\mathcal{X}}, \imath_{\mathcal{Y}}] = \imath_{[\mathcal{X},\mathcal{Y}]}$ *on* $\mathcal{D}_*(M) \otimes V$*, thus* $L_{\mathcal{X}}$ *commutes with* $\imath_{\mathcal{X}}$.

$L_{\mathcal{X}}(\mathcal{A}(M) \otimes V) \subseteq \mathcal{A}(M) \otimes V$, since $L_{\mathcal{X}}$ commutes with alternations, cf. (17). For $\omega \in \mathcal{A}(M) \otimes V$ we deduce from (17) and (1) that for all $f \in C^\infty(M)$:

$$L_{f\mathcal{X}}\omega = f \cdot L_{\mathcal{X}}\omega + df \wedge \imath_{\mathcal{X}}\omega.$$

Moreover, since $\widetilde{\varphi}_t\omega = (\varphi_{-t})^*\omega$, d and $L_{\mathcal{X}}$ commute and we have (cf. Lemma 1.2)

Proposition 1.39 *For every* $\mathcal{X} \in \mathcal{D}^1(M)$, $L_{\mathcal{X}}$ *is a derivation of degree 0 of* $\mathcal{A}(M)$ *(and of* $\mathcal{A}(M) \otimes V$*, whenever* $\wedge_V$ *is defined), which commutes with* d *and* $\imath_{\mathcal{X}}$. *Conversely, every derivation of degree 0 of* $\mathcal{A}(M)$ *commuting with* d *is equal to* $L_{\mathcal{X}}$ *for some* $\mathcal{X} \in \mathcal{D}^1(M)$.

Finally, the homotopy identity $L_{\mathcal{X}} = d \circ \imath_{\mathcal{X}} + \imath_{\mathcal{X}} \circ d$ *holds on* $\mathcal{A}(M) \otimes V$.

Using $L_{\mathcal{X}} = d \circ \imath_{\mathcal{X}} + \imath_{\mathcal{X}} \circ d$ one easily proves:

Lemma 1.40 *Let $f\colon M \to N$ be a diffeomorphism. Then on $\mathcal{A}(M) \otimes V$ we have*

$$\imath_{\mathcal{X}} \circ f^* = f^* \circ \imath_{f_*\mathcal{X}}, \qquad d \circ f^* = f^* \circ d, \qquad L_{\mathcal{X}} \circ f^* = L_{f_*\mathcal{X}} \qquad \textit{for all} \quad \mathcal{X} \in \mathcal{D}^1(M).$$

Recall that every tensor field $S \in \mathcal{D}^1_1(M)$ can be viewed as a linear endomorphism of $\mathcal{D}^1(M)$. Analogous to Proposition 1.20, S uniquely defines a derivation S' of $\mathcal{D}(M)$ with the following properties:

1. $S' \in \underline{\operatorname{der}\mathcal{D}(M)}$, i. e., S' preserves type and commutes with contractions;
2. $S'(f) = 0$ for all $f \in C^\infty(M)$, thus $S' \in \underline{\operatorname{der}\mathcal{D}(M)}_0$;
3. $[S'(\mathcal{X})](\omega) = S(\mathcal{X}, \omega)$ for all $\mathcal{X} \in \mathcal{D}^1(M)$, $\omega \in \mathcal{D}_1(M)$;
4. $\{S' | S \in \mathcal{D}^1_1(M)\}$ is an ideal in $\underline{\operatorname{der}\mathcal{D}(M)}$.

Proposition 1.41 *Every derivation $D \in \underline{\operatorname{der}\mathcal{D}(M)}$ can be decomposed uniquely as:*

$$D = L_{\mathcal{X}} + S',$$

where $\mathcal{X} \in \mathcal{D}^1(M)$ and $S \in \mathcal{D}^1_1(M)$. Two derivations $D_1, D_2 \in \underline{\operatorname{der}\mathcal{D}(M)}$ coincide iff they coincide on $C^\infty(M)$ and $\mathcal{D}^1(M)$.

1.3 Bullets and Triangles

Definition 1.42 *For any $\chi^s_r \in \mathcal{A}_r(M, \operatorname{Hom}(\bigotimes^s W, Z))$, where $s \in \mathbb{N}$, $r \in \mathbb{N}_0$, and $F_j \in \operatorname{Hom}(\bigotimes^q V, W)$, $j = 1, \ldots, s$, we define $\chi^{F_1,\ldots,F_s}_r \in \mathcal{A}_r(M, \operatorname{Hom}(\bigotimes^{sq} V, Z))$ by*

$$\chi^{F_1,\ldots,F_s}_r = [(F_1 \otimes \cdots \otimes F_s)^*]_* \chi^s_r$$

Thus if $\chi^s_r \in \mathcal{A}_r(M) \otimes \operatorname{Hom}(\bigotimes^s W, Z)$ then $\chi^{F_1,\ldots,F_s}_r \in \mathcal{A}_r(M) \otimes \operatorname{Hom}(\bigotimes^{sq} V, Z)$.

Since $(F_1 \otimes \cdots \otimes F_s) \in \operatorname{Hom}(\bigotimes^{sq} V, \bigotimes^q W)$, $\chi^{F_1,\ldots,F_s}_r(\mathcal{X}^1, \ldots, \mathcal{X}^r)$ is well-defined according to Definition 1.26. It is multilinear in F_j: for all $\lambda, \mu \in \mathbb{K}$ and all $j \leq s$

$$\chi_r^{F_1,\ldots,\lambda F_j + \mu F'_j,\ldots,F_s} = \lambda \chi_r^{F_1,\ldots,F_j,\ldots,F_s} + \mu \chi_r^{F_1,\ldots,F'_j,\ldots,F_s}. \tag{19}$$

Definition 1.43 *For $\chi^s_r \in \mathcal{A}_r(M, \operatorname{Hom}(\bigotimes^s W, Z))$ and $\phi^q_p \in \mathcal{A}_p(M) \otimes \operatorname{Hom}(\bigotimes^q V, W)$, $p, q, r, s{-}1 \in \mathbb{N}_0$, let $d^{sq}_{r+sp} \in \mathcal{D}_{r+sp}(M, \operatorname{Hom}(\bigotimes^{sq} V, Z))$ with $d^{sq}_{r+sp}(\mathcal{X}^1, \ldots, \mathcal{X}^{r+sp})(x) :=$*

$$[\chi_x(\mathcal{X}^1_x, \ldots, \mathcal{X}^r_x)] \circ [\phi_x(\mathcal{X}^{r+1}_x, \ldots, \mathcal{X}^{r+p}_x) \otimes \cdots \otimes \phi_x(\mathcal{X}^{r+(s-1)p+1}_x, \ldots, \mathcal{X}^{r+sp}_x)]$$

for all $x \in M$ and define $\chi^s_r \bullet \phi^q_p := A_{r+sp}(d_{r+sp}) \in \mathcal{A}_{r+sp}(M, \operatorname{Hom}(\bigotimes^{sq} V, Z))$. $\chi^0_r \bullet \phi^q_p := \chi^0_r$ and linear extension defines $\chi \bullet \phi^q_p \in \mathcal{A}(M, \operatorname{Hom}(\mathcal{T}(V), Z))$ for all $\chi \in \mathcal{A}(M, \operatorname{Hom}(\mathcal{T}(W), Z))$.

Roughly speaking, the bullet operator means the following: for any $x \in M$ and $\mathcal{X}^i \in \mathcal{D}^1(M)$, $\chi_x(\mathcal{X}_x^1, \ldots, \mathcal{X}_x^r)$ defines an element in $\mathrm{Hom}(\otimes^s W, Z)$. Instead of using s vectors in W as input for this map, we may also use s maps in $\mathrm{Hom}(\otimes^q V, W)$ as input to obtain an element in $\mathrm{Hom}(\otimes^{sq} V, Z)$. But again for any $x \in M$ and $\mathcal{Y}^i \in \mathcal{D}^1(M)$, $\phi_x(\mathcal{Y}_x^1, \ldots, \mathcal{Y}_x^p)$ defines such a map in $\mathrm{Hom}(\otimes^q V, W)$. Altogether the combination of χ and s factors ϕ defines an element $d_{r+sp}^{sq} \in \mathcal{D}_{r+sp}(M, \mathrm{Hom}(\otimes^{sq} V, Z))$. Using the alternation A_{r+sp}, we finally obtain a form in $\mathcal{A}_{r+sp}(M, \mathrm{Hom}(\otimes^{sq} V, Z))$.

Lemma 1.44 *For $p, q, r, s-1 \in \mathbb{N}_0$ and $\phi_p^q = \sum_{i=1}^{m} \phi^i \otimes F_i \in \mathcal{A}_p(M) \otimes \mathrm{Hom}(\otimes^q V, W)$,*

$$\chi_r^s \bullet \phi_p^q = \sum_{i_1, \ldots, i_s = 1}^{m} \chi_r^{F_{i_1}, \ldots, F_{i_s}} \wedge \phi^{i_1} \wedge \cdots \wedge \phi^{i_s}.$$

Thus if $\chi_r^s \in \mathcal{A}_r(M) \otimes \mathrm{Hom}(\otimes^s W, Z)$ then also $\chi_r^s \bullet \phi_p^q \in \mathcal{A}_{r+sp}(M) \otimes \mathrm{Hom}(\otimes^{sq} V, Z)$.

Lemma 1.45 *For $p, q, r, s-1 \in \mathbb{N}_0$, p odd, and $\phi_p^q = \sum_{i=1}^m \phi^i \otimes F_i$, we have*

$$\begin{aligned} \chi_r^s \bullet \phi_p^q &= \sum_{1 \le i_1 < \cdots < i_s \le m} \sum_{\rho \in S_s} \chi_r^{F_{i_{\rho(1)}}, \ldots, F_{i_{\rho(s)}}} \wedge \phi^{i_{\rho(1)}} \wedge \cdots \wedge \phi^{i_{\rho(s)}} \\ &= \sum_{1 \le i_1 < \cdots < i_s \le m} \Big(\sum_{\rho \in S_s} (-1)^\rho \, \chi_r^{F_{i_{\rho(1)}}, \ldots, F_{i_{\rho(s)}}} \Big) \wedge \phi^{i_1} \wedge \cdots \wedge \phi^{i_s}. \end{aligned} \tag{20}$$

Thus $\chi_r^s \bullet \phi_p^q = 0$ if $s > m$; if V and W are finite dimensional and $s > \dim W (\dim V)^q$, then $\chi_r^s \bullet \phi_p^q = 0$ for all $\phi_p^q \in \mathcal{A}_p(M) \otimes \mathrm{Hom}(\otimes^q V, W)$.

Proof. $\phi^i \wedge \phi^i = 0$, because p odd, and $\dim \mathrm{Hom}(\otimes^q V, W) = \dim W (\dim V)^q$. □

Recall Sym^ς from Definition 1.8. If $\chi \in \mathcal{A}(M, \mathrm{Sym}^\varsigma(W, Z))$ (e. g., if $\chi = \chi_r^s$ with $s = 0, 1$), it is quite natural to ask for a resulting form $\chi \bullet \phi_p^q \in \mathcal{A}(M, \mathrm{Sym}^\varsigma(V, Z))$. We can achieve this by $(\mathrm{Sym}^\varsigma)_*(\chi \bullet \phi_p^q)$ according to Definition 1.26. Define

$$\ell := \varsigma^{q+1}(-1)^p = \pm 1, \tag{21}$$

then the following lemma holds:

Lemma 1.46 *For $p, q, r, s-1 \in \mathbb{N}_0$, $\phi_p^q = \sum_{i=1}^m \phi^i \otimes F_i \in \mathcal{A}_p(M) \otimes \mathrm{Hom}(\otimes^q V, W)$ and $\chi_r^s \in \mathcal{A}_r(M, \mathrm{Sym}_s^\varsigma(W, Z))$, we have*

$$\begin{aligned} (\mathrm{Sym}_{sq}^\varsigma)_*(\chi_r^s \bullet \phi_p^q) &= \sum_{i_1, \ldots, i_s = 1}^{m} (\mathrm{Sym}_{sq}^\varsigma)_*(\chi_r^{F_{i_1}, \ldots, F_{i_s}}) \wedge \phi^{i_1} \wedge \cdots \wedge \phi^{i_s}, \\ \text{if } (-1)^p = \varsigma^{q+1} = -1: \quad &= s! \sum_{1 \le i_1 < \cdots < i_s \le m} (\mathrm{Sym}_{sq}^\varsigma)_*(\chi_r^{F_{i_1}, \ldots, F_{i_s}}) \wedge \phi^{i_1} \wedge \cdots \wedge \phi^{i_s}, \\ \text{if } s > 1 \text{ and } \ell = -1: \quad &= 0. \end{aligned}$$

Proof. The first equation is trivial from Lemmas 1.44 and 1.29.3. Now for $s > 1$, $\phi^{i_1} \wedge \cdots \wedge \phi^{i_j} \wedge \cdots \wedge \phi^{i_k} \wedge \cdots \wedge \phi^{i_s} = (-1)^p \phi^{i_1} \wedge \cdots \wedge \phi^{i_k} \wedge \cdots \wedge \phi^{i_j} \wedge \cdots \wedge \phi^{i_s}$ and

$$(\mathrm{Sym}^{\varsigma}_{sq})_*(\chi_r^{F_{i_1},\dots,F_{i_j},\dots,F_{i_k},\dots,F_{i_s}}) = \varsigma^{q+1}(\mathrm{Sym}^{\varsigma}_{sq})_*(\chi_r^{F_{i_1},\dots,F_{i_k},\dots,F_{i_j},\dots,F_{i_s}}) \qquad (22)$$

yield $$(\mathrm{Sym}^{\varsigma}_{sq})_*(\chi_r^{F_{i_1},\dots,F_{i_j},\dots,F_{i_k},\dots,F_{i_s}}) \wedge \phi^{i_1} \wedge \cdots \wedge \phi^{i_j} \wedge \cdots \wedge \phi^{i_k} \wedge \cdots \wedge \phi^{i_s} =$$

$$= \ell \ (\mathrm{Sym}^{\varsigma}_{sq})_*(\chi_r^{F_{i_1},\dots,F_{i_k},\dots,F_{i_j},\dots,F_{i_s}}) \wedge \phi^{i_1} \wedge \cdots \wedge \phi^{i_k} \wedge \cdots \wedge \phi^{i_j} \wedge \cdots \wedge \phi^{i_s}. \qquad (23)$$

Thus evaluating $\sum_{\rho \in S_s}$ in (20) proves the rest. □

Let us derive some properties of the bullet operator. First we will look for associativity and its behavior under pullbacks and push-outs.

Proposition 1.47 *Let $\kappa^u_t \in \mathcal{A}_t(M, \mathrm{Hom}(\bigotimes^u X, Y))$, $\chi^s_r \in \mathcal{A}_r(M) \otimes \mathrm{Hom}(\bigotimes^s W, X)$ and $\phi^q_p \in \mathcal{A}_p(M) \otimes \mathrm{Hom}(\bigotimes^q V, W)$ for $p, q, r, s, t, u \in \mathbb{N}_0$. Then*

$$\kappa^u_t \bullet (\chi^s_r \bullet \phi^q_p) = (-1)^{prs\frac{u(u-1)}{2}} \ (\kappa^u_t \bullet \chi^s_r) \bullet \phi^q_p \ \in \mathcal{A}_{t+ur+usp}(M, \mathrm{Hom}(\textstyle\bigotimes^{usq} V, Y)). \quad (24)$$

Proof. Let $\chi^s_r = \sum_{j=1}^n \chi^j \otimes G_j$ and $\phi^q_p = \sum_{i=1}^m \phi^i \otimes F_i$. By Lemma 1.44 we find

$$\begin{aligned} \kappa^u_t \bullet (\chi^s_r \bullet \phi^q_p) &= \sum_{j_1,\dots,j_u=1}^{n} \ \sum_{i_{11},\dots,i_{su}=1}^{m} \kappa_t^{G_{j_1}\circ(F_{i_{11}}\otimes\cdots\otimes F_{i_{s1}}),\dots,G_{j_u}\circ(F_{i_{1u}}\otimes\cdots\otimes F_{i_{su}})} \wedge \\ &\quad \wedge \chi^{j_1} \wedge \phi^{i_{11}} \wedge \cdots \wedge \phi^{i_{s1}} \wedge \cdots \wedge \chi^{j_u} \wedge \phi^{i_{1u}} \wedge \cdots \wedge \phi^{i_{su}}, \end{aligned}$$

while $$\begin{aligned} (\kappa^u_t \bullet \chi^s_r) \bullet \phi^q_p &= \sum_{j_1,\dots,j_u=1}^{n} \ \sum_{i_{11},\dots,i_{su}=1}^{m} (\kappa_t^{G_{j_1},\dots,G_{j_u}})_t^{F_{i_{11}},\dots,F_{i_{s1}},\dots,F_{i_{1u}},\dots,F_{i_{su}}} \wedge \\ &\quad \wedge \chi^{j_1} \wedge \cdots \wedge \chi^{j_u} \wedge \phi^{i_{11}} \wedge \cdots \wedge \phi^{i_{s1}} \wedge \cdots \wedge \phi^{i_{1u}} \wedge \cdots \wedge \phi^{i_{su}}. \end{aligned}$$

Now $$\begin{aligned} &\chi^{j_1} \wedge \phi^{i_{11}} \wedge \cdots \wedge \phi^{i_{s1}} \wedge \cdots \wedge \chi^{j_u} \wedge \phi^{i_{1u}} \wedge \cdots \wedge \phi^{i_{su}} = \\ &= (-1)^{prs(1+2+\cdots+(u-1))} \chi^{j_1} \wedge \cdots \wedge \chi^{j_u} \wedge \phi^{i_{11}} \wedge \cdots \wedge \phi^{i_{s1}} \wedge \cdots \wedge \phi^{i_{1u}} \wedge \cdots \wedge \phi^{i_{su}} \\ &= (-1)^{prs\frac{u(u-1)}{2}} \chi^{j_1} \wedge \cdots \wedge \chi^{j_u} \wedge \phi^{i_{11}} \wedge \cdots \wedge \phi^{i_{s1}} \wedge \cdots \wedge \phi^{i_{1u}} \wedge \cdots \wedge \phi^{i_{su}}. \end{aligned}$$

On the other hand $(F_{i_{11}} \otimes \cdots \otimes F_{i_{s1}} \otimes \cdots \otimes F_{i_{1u}} \otimes \cdots \otimes F_{i_{su}})^* \circ (G_{j_1} \otimes \cdots \otimes G_{j_u})^* = [G_{j_1} \circ (F_{i_{11}} \otimes \cdots \otimes F_{i_{s1}}), \dots, G_{j_u} \circ (F_{i_{1u}} \otimes \cdots \otimes F_{i_{su}})]^*$, so both κ-expressions are identical for each set of indices. □

Corollary 1.48 *If $\kappa \in \mathcal{A}(M, \mathrm{Alt}(X, Y))$, then for p, q, r or s even, esp. $q = 0$:*

$$\kappa \bullet (\chi^s_r \bullet \phi^q_p) = (\kappa \bullet \chi^s_r) \bullet \phi^q_p. \qquad (25)$$

Proof. Whenever for a κ^u_t in (24) $prs\frac{u(u-1)}{2}$ is odd, $r + sp$ and sq are even and $u > 1$, thus the left side of (24) vanishes by Lemma 1.46. □

Lemma 1.49 *Let M, N be C^∞-manifolds and V, W, Y, Z vector spaces.*
1. If $f\colon M \to N$ is differentiable and $\chi \in \mathcal{A}(N, \mathrm{Hom}(\mathcal{T}(W), Z))$ then

$$(\forall\, \phi^q_p \in \mathcal{A}_p(N) \otimes \mathrm{Hom}(\textstyle\bigotimes^q V, W)) \ f^*(\chi \bullet \phi^q_p) = (f^*\chi) \bullet (f^*\phi^q_p) \in \mathcal{A}(M, \mathrm{Hom}(\mathcal{T}(V), Z));$$

2. If $A: W \to Y$ is linear and $\chi \in \mathcal{A}(M, \mathrm{Hom}(\mathcal{T}(Y), Z))$ then

$$(\forall \phi_p^q \in \mathcal{A}_p(M) \otimes \mathrm{Hom}(\otimes^q V, W))\ \chi \bullet [(A_\circ)_* \phi_p^q] = [(A^*)_* \chi] \bullet \phi_p^q \in \mathcal{A}(M, \mathrm{Hom}(\mathcal{T}(V), Z)),$$

$$(\forall\, \theta_p \in \mathcal{A}_p(M) \otimes W) \quad \chi \bullet (A_* \theta_p) = [(A^*)_* \chi] \bullet \theta_p \in \mathcal{A}(M, Z);$$

3. If $B: Y \to Z$ linear and $\chi \in \mathcal{A}(M, \mathrm{Hom}(\mathcal{T}(W), Y))$ then

$$(\forall \phi_p^q \in \mathcal{A}_p(M) \otimes \mathrm{Hom}(\otimes^q V, W))\ (B_\circ)_* (\chi \bullet \phi_p^q) = [(B_\circ)_* \chi] \bullet \phi_p^q \in \mathcal{A}(M, \mathrm{Hom}(\mathcal{T}(V), Z)),$$

$$(\forall\, \theta_p \in \mathcal{A}_p(M) \otimes W) \quad B_* (\chi \bullet \theta_p) = [(B_\circ)_* \chi] \bullet \theta_p \in \mathcal{A}(M, Z).$$

Analogous results hold for (anti)symmetrized forms in $\mathcal{A}(M, \mathrm{Sym}^{\epsilon}(W, Z))$, etc. If in 1. we have $\chi \in \mathcal{A}(N) \otimes \mathrm{Hom}(\mathcal{T}(W), Z)$, the result will be in $\mathcal{A}(M) \otimes \mathrm{Hom}(\mathcal{T}(V), Z)$, etc.

Proof. *1.* follows from Lemmas 1.29 and 1.44; *2.* and *3.* are easily proved directly or by Proposition 1.47: let $a_0^1 := 1 \otimes A \in \mathcal{A}_0(M) \otimes \mathrm{Alt}_1(W, Y)$, then $[(A_\circ)_* \phi_p^q] = a_0^1 \bullet \phi_p^q$ and $[(A^*)_* \chi] = \chi \bullet a_0^1$; analogously with $b_0^1 := 1 \otimes B \in \mathcal{A}_0(M) \otimes \mathrm{Alt}_1(Y, Z)$, $[(B_\circ)_* \chi] = b_0^1 \bullet \chi$, which is well-defined in this special case. □

Obviously $\chi \bullet \phi_p^q$ is $\mathcal{A}(M)$-linear only in χ. If $\chi \in \mathcal{A}(M, \mathrm{Hom}(\otimes^s W, Z))$, then

$$\chi \bullet (f\, \phi_p^q) = f^s\, (\chi \bullet \phi_p^q) \qquad \text{for all} \quad f \in C^\infty(M). \tag{26}$$

We would like to give an expression for $\chi \bullet (\phi_p^q + \psi_p^q)$. First we observe that every $\chi_r^s \in \mathcal{A}_r(M, \mathrm{Sym}_s^{\epsilon}(W, Z))$ naturally defines

$$\chi_r^{s';s''} \in \mathcal{A}_r(M, \mathrm{Sym}_{s'}^{\epsilon}(W, \mathrm{Sym}_{s''}^{\epsilon}(W, Z))) \quad \text{for all} \ s', s'' \in \mathbb{N}_0,\ s' + s'' = s. \tag{27}$$

For any such combination of s' and s'', $\chi_r^s \bullet (\phi_p^q + \psi_p^q)$ will contain terms, where s' factors of ϕ_p^q and s'' terms of ψ_p^q serve as input for χ_r^s. In order to cover this situation, we need the following two definitions.

Definition 1.50 *For any $\chi_r^{s';s''} \in \mathcal{A}_r(M, \mathrm{Hom}(\otimes^{s'} W', \mathrm{Hom}(\otimes^{s''} W'', Z)))$, where $s', s'' \in \mathbb{N}$, $r \in \mathbb{N}_0$, and $G_i \in \mathrm{Hom}(\otimes^q V, W')$, $i = 1, \ldots, s'$, $H_j \in \mathrm{Hom}(\otimes^q V, W'')$, $j = 1, \ldots, s''$, we define:*

$$\chi_r^{G_1,\ldots,G_{s'};s''} := [(G_1 \otimes \cdots \otimes G_{s'})^*]_* \chi_r^{s';s''} \in \mathcal{A}_r(M, \mathrm{Hom}(\otimes^{s'q} V, \mathrm{Hom}(\otimes^{s''} W'', Z)))$$
$$\chi_r^{s';H_1,\ldots,H_{s''}} := [((H_1 \otimes \cdots \otimes H_{s''})^*)_\circ]_* \chi_r^{s';s''} \in \mathcal{A}_r(M, \mathrm{Hom}(\otimes^{s'} W', \mathrm{Hom}(\otimes^{s''q} V, Z)))$$

If $\chi_r^{s';s''} \in \mathcal{A}_r(M) \otimes \mathrm{Hom}(\otimes^{s'} W', \mathrm{Hom}(\otimes^{s''} W'', Z))$ then $\chi_r^{G_1,\ldots,G_{s'};s''} \in \mathcal{A}_r(M) \otimes \mathrm{Hom}(\otimes^{s'q} V, \mathrm{Hom}(\otimes^{s''} W'', Z)$, $\chi_r^{s';H_1,\ldots,H_{s''}} \in \mathcal{A}_r(M) \otimes \mathrm{Hom}(\otimes^{s'} W', \mathrm{Hom}(\otimes^{s''q} V, Z))$.

Definition 1.51 *For every $\chi_r^{s';s''} \in \mathcal{A}_r(M, \mathrm{Hom}(\otimes^{s'} W', \mathrm{Hom}(\otimes^{s''} W'', Z)))$ and $\phi_p^q \in \mathcal{A}_p(M) \otimes \mathrm{Hom}(\otimes^q V, W')$, where $p, q, r, s', s'' \in \mathbb{N}_0$, let $Z' := \mathrm{Hom}(\otimes^{s''} W'', Z)$ and $\tilde{\chi}_r^{s'} := \chi_r^{s';s''} \in \mathcal{A}_r(M, \mathrm{Hom}(\otimes^{s'} W', Z'))$, and define*

$$\chi_r^{s';s''} \blacktriangleleft \phi_p^q := \tilde{\chi}_r^{s'} \bullet \phi_p^q \in \mathcal{A}_{r+s'p}(M, \mathrm{Hom}(\otimes^{s'q} V, \mathrm{Hom}(\otimes^{s''} W'', Z))).$$

Be $\psi_p^q \in \mathcal{A}_p(M) \otimes \mathrm{Hom}(\otimes^q V, W'')$ and $\jmath: \otimes^{s'} W' \to [\mathrm{Hom}(\otimes^{s'} W', Z) \to Z]$ the evaluation morphism. Define $\chi_r^{s';s''} \blacktriangleright \psi_p^q \in \mathcal{A}_{r+s''p}(M, \mathrm{Hom}(\otimes^{s'} W', \mathrm{Hom}(\otimes^{s''q} V, Z)))$ by

$$\jmath(w^1 \otimes \cdots \otimes w^{s'})_* (\chi_r^{s';s''} \blacktriangleright \psi_p^q) := [\jmath(w^1 \otimes \cdots \otimes w^{s'})_* \chi_r^{s';s''}] \bullet \psi_p^q \quad \text{for all} \ w^i \in W'.$$

Thus for $\chi \in \mathcal{A}_r(M, \mathrm{Hom}(\bigotimes^s W, Z))$, the direction of the triangle operators indicates whether the second form is used as input for the first s' or the last s'' factors in $\chi_x(\mathcal{X}_x^1, \ldots, \mathcal{X}_x^r) \in \mathrm{Hom}(\bigotimes^s W, Z)$.

Lemma 1.52 *Using the notation of the previous definitions, we have*

$$\begin{aligned} \chi_r^{s';s''} \blacktriangleleft \phi_p^q &= \sum_{j_1,\ldots,j_{s'}=1}^{m} \chi_r^{G_{j_1},\ldots,G_{j_{s'}};s''} \wedge \phi^{j_1} \wedge \cdots \wedge \phi^{j_{s'}} \quad \text{if } \phi_p^q = \sum_{j=1}^{m} \phi^j \otimes G_j, \\ \chi_r^{s';s''} \blacktriangleright \psi_p^q &= \sum_{k_1,\ldots,k_{s''}=1}^{m} \chi_r^{s';H_{k_1},\ldots,H_{k_{s''}}} \wedge \psi^{k_1} \wedge \cdots \wedge \psi^{k_{s''}} \quad \text{if } \psi_p^q = \sum_{k=1}^{m} \psi^k \otimes H_k. \end{aligned}$$

$\chi_r^{s';s''} \blacktriangleleft \phi_p^q \in \mathcal{A}_{r+s'p}(M) \otimes \mathrm{Hom}(\bigotimes^{s'q} V, \mathrm{Hom}(\bigotimes^{s''} W'', Z))$ *and* $\chi_r^{s';s''} \blacktriangleright \psi_p^q \in \mathcal{A}_{r+s''p}(M) \otimes \mathrm{Hom}(\bigotimes^{s'} W', \mathrm{Hom}(\bigotimes^{s''q} V, Z))$ *if* $\chi_r^{s';s''} \in \mathcal{A}_r(M) \otimes \mathrm{Hom}(\bigotimes^{s'} W', \mathrm{Hom}(\bigotimes^{s''} W'', Z))$.

Lemma 1.53 *Let* $\chi_r^{s';s''}$, $\phi_{p'}^q$ *and* $\psi_{p''}^q$ *be defined as before. Then*

$$(\chi_r^{s';s''} \blacktriangleleft \phi_{p'}^q) \blacktriangleright \psi_{p''}^q = (-1)^{p'p''s's''} (\chi_r^{s';s''} \blacktriangleright \psi_{p''}^q) \blacktriangleleft \phi_{p'}^q \in \mathcal{A}_{r+s'p'+s''p''}(M, \mathrm{Hom}(\textstyle\bigotimes^{sq} V, Z))$$

For $\chi_r^s \in \mathcal{A}_r(M, \mathrm{Sym}_s^\varsigma(W, Z))$ *with* $\chi_r^{s';s''}$ *from* (27),

$$(\mathrm{Sym}_{sq}^\varsigma)_*[(\chi_r^{s';s''} \blacktriangleleft \phi_{p'}^q) \blacktriangleright \psi_{p''}^q] = \varsigma^{(q+1)s's''} (\mathrm{Sym}_{sq}^\varsigma)_*[(\chi_r^{s'';s'} \blacktriangleright \phi_{p'}^q) \blacktriangleleft \psi_{p''}^q].$$

Proof. With the previous notation, the first two terms are both equal to

$$\begin{gathered} \textstyle\sum_{i_1,\ldots,i_{s'+s''}=1}^{m} \chi_r^{G_{i_1},\ldots,G_{i_{s'}};H_{i_{s'+1}},\ldots,H_{i_{s'+s''}}} \wedge \phi^{i_1} \wedge \cdots \wedge \phi^{i_{s'}} \wedge \psi^{i_{s'+1}} \wedge \cdots \wedge \psi^{i_{s'+s''}}; \\ (\mathrm{Sym}_{sq}^\varsigma)_*(\chi_r^{G_{i_1},\ldots,G_{i_{s'}},H_{i_{s'+1}},\ldots,H_{i_s}}) = \varsigma^{(q+1)s's''} (\mathrm{Sym}_{sq}^\varsigma)_*(\chi_r^{H_{i_{s'+1}},\ldots,H_{i_s},G_{i_1},\ldots,G_{i_{s'}}}) \end{gathered}$$

from (22) proves the second equation. □

Proposition 1.54 *For* $p, q, r, s \in \mathbb{N}_0$, *let* $\phi_p^q, \psi_p^q \in \mathcal{A}_p(M) \otimes \mathrm{Hom}(\bigotimes^q V, W)$ *and* $\chi_r^s \in \mathcal{A}_r(M, \mathrm{Sym}_s^\varsigma(W, Z))$. *Define* ℓ *as in* (21). *Then* $(\mathrm{Sym}_{sq}^\varsigma)_*[\chi_r^s \bullet (\phi_p^q + \psi_p^q)] =$

$$\begin{aligned} &= \sum_{k=0}^{s} \binom{s}{k}_\ell (\mathrm{Sym}_{sq}^\varsigma)_*[(\chi_r^{k;s-k} \blacktriangleleft \phi_p^q) \blacktriangleright \psi_p^q] = \sum_{k=0}^{s} (-1)^{pk(s-1)} \binom{s}{k}_\ell (\mathrm{Sym}_{sq}^\varsigma)_*[(\chi_r^{k;s-k} \blacktriangleright \psi_p^q) \blacktriangleleft \phi_p^q] \\ &= \sum_{k=0}^{s} \binom{s}{k}_\ell (\mathrm{Sym}_{sq}^\varsigma)_*[(\chi_r^{k;s-k} \blacktriangleleft \psi_p^q) \blacktriangleright \phi_p^q] = \sum_{k=0}^{s} (-1)^{pk(s-1)} \binom{s}{k}_\ell (\mathrm{Sym}_{sq}^\varsigma)_*[(\chi_r^{k;s-k} \blacktriangleright \phi_p^q) \blacktriangleleft \psi_p^q]. \end{aligned}$$

$\binom{s}{k}_\ell = \binom{s}{s-k}_\ell$: *whenever* $(\mathrm{Sym}_{sq}^\varsigma)_*[\chi_r^s \bullet (\phi_p^q + \psi_p^q)]$ *is nonzero according to Lemma 1.46,*
$\binom{s}{k}_\ell = \binom{s}{k}_+ := \binom{s}{k}$, *while* $\binom{s}{k}_- := \begin{cases} 0, & \text{if } s \text{ even and } k \text{ odd}, \\ \binom{[s/2]}{[k/2]}, & \text{else (for } r \in \mathbb{R},\ [r] := \max_{z \in \mathbb{Z}}\{z \le r\}). \end{cases}$

Proof. The equations are trivial for $s = 0$ and $s = 1$, so assume $s > 1$. Let $\phi_p^q = \sum_{i=1}^m \phi^i \otimes F_i$ and $\psi_p^q = \sum_{i=1}^m \psi^i \otimes F_i$. Then with $\tilde{\chi}_r^{i_1,\dots,i_s} := (\mathrm{Sym}^\varsigma_{sq})_*(\chi_r^{F_{i_1},\dots,F_{i_s}})$,

$$(\mathrm{Sym}^\varsigma_{sq})_*(\chi_r^s \bullet (\phi_p^q + \psi_p^q)) = \sum_{i_1,\dots,i_s=1}^m \tilde{\chi}_r^{i_1,\dots,i_s} \wedge (\phi^{i_1} + \psi^{i_1}) \wedge \dots \wedge (\phi^{i_s} + \psi^{i_s}), \text{ and}$$

$$(\mathrm{Sym}^\varsigma_{sq})_*[(\chi_r^{k;s-k} \blacktriangleleft \phi_p^q) \blacktriangleright \psi_p^q] = \sum_{i_1,\dots,i_s=1}^m \tilde{\chi}_r^{i_1,\dots,i_s} \wedge \phi^{i_1} \wedge \dots \wedge \phi^{i_k} \wedge \psi^{i_{k+1}} \wedge \dots \wedge \psi^{i_s}.$$

We proceed by induction on s. Thus $(\mathrm{Sym}^\varsigma_{sq})_*(\chi_r^s \bullet (\phi_p^q + \psi_p^q)) =$

$$\begin{aligned}
&= \sum_{i_s=1}^m (\mathrm{Sym}^\varsigma_{sq})_*(\mathrm{Sym}^\varsigma_{(s-1)q})_*(\chi_r^{s-1;F_{i_s}} \bullet (\phi_p^q + \psi_p^q)) \wedge (\phi^{i_s} + \psi^{i_s}) \\
&= \sum_{k=0}^{s-1} \binom{s-1}{k}_\ell \sum_{i_1,\dots,i_s=1}^m \tilde{\chi}_r^{i_1,\dots,i_s} \wedge \phi^{i_1} \wedge \dots \wedge \phi^{i_k} \wedge \psi^{i_{k+1}} \wedge \dots \wedge \psi^{i_{s-1}} \wedge (\phi^{i_s} + \psi^{i_s}) \\
&= \sum_{k=0}^{s} \left[\binom{s-1}{k}_\ell + \ell^{s-k}\binom{s-1}{k-1}_\ell\right] \sum_{i_1,\dots,i_s=1}^m \tilde{\chi}_r^{i_1,\dots,i_s} \wedge \phi^{i_1} \wedge \dots \wedge \phi^{i_k} \wedge \psi^{i_{k+1}} \wedge \dots \wedge \psi^{i_s},
\end{aligned}$$

where we have used (23). Recursion $\binom{s}{k}_\ell = \binom{s-1}{k}_\ell + \ell^{s-k}\binom{s-1}{k-1}_\ell$ proves the formulae for $\binom{s}{k}_\ell$. Lemma 1.53 and interchanging ϕ_p^q and ψ_p^q finally yield the rest. □

We will also need a formula for the exterior derivative of $\chi_r^s \bullet \phi_p^q$:

Proposition 1.55 *Let* $\phi_p^q \in \mathcal{A}_p(M) \otimes \mathrm{Hom}(\bigotimes^q V, W)$ *and* $\chi_r^s \in \mathcal{A}_r(M) \otimes \mathrm{Sym}^\varsigma_s(W, Z)$ *for* $p, q, r, s \in \mathbb{N}_0$. *Define* $\binom{s}{1}_\ell$ *as in* Proposition 1.54. *Then* $d[(\mathrm{Sym}^\varsigma_{sq})_*(\chi_r^s \bullet \phi_p^q)] =$

$$\begin{aligned}
&= (\mathrm{Sym}^\varsigma_{sq})_*[(d\chi)_{r+1}^s \bullet \phi_p^q] + (-1)^r \binom{s}{1}_\ell (\mathrm{Sym}^\varsigma_{sq})_*[(\chi_r^{1;s-1} \blacktriangleleft (d\phi)_{p+1}^q) \blacktriangleright \phi_p^q] \\
&= (\mathrm{Sym}^\varsigma_{sq})_*[(d\chi)_{r+1}^s \bullet \phi_p^q] + (-1)^{r+p(s-1)} \binom{s}{1}_\ell (\mathrm{Sym}^\varsigma_{sq})_*[(\chi_r^{s-1;1} \blacktriangleleft \phi_p^q) \blacktriangleright (d\phi)_{p+1}^q].
\end{aligned}$$

Proof. With the notation of the previous proof, Lemmas 1.46, 1.25 and 1.29 yield

$$\begin{aligned}
d[(\mathrm{Sym}^\varsigma_{sq})_*(\chi_r^s \bullet \phi_p^q)] &= \sum_{i_1,\dots,i_s=1}^m d\tilde{\chi}_r^{i_1,\dots,i_s} \wedge \phi^{i_1} \wedge \dots \wedge \phi^{i_s} + \\
&+ \sum_{j=1}^s \sum_{i_1,\dots,i_s=1}^m (-1)^{r+p(j-1)} \tilde{\chi}_r^{i_1,\dots,i_s} \wedge \phi^{i_1} \wedge \dots \wedge \phi^{i_{j-1}} \wedge d\phi^{i_j} \wedge \phi^{i_{j+1}} \wedge \dots \wedge \phi^{i_s} = \\
&= (\mathrm{Sym}^\varsigma_{sq})_*[(d\chi)_{r+1}^s \bullet \phi_p^q] + (-1)^r \sum_{j=1}^s \sum_{i_1,\dots,i_s=1}^m \ell^{j-1}\, \tilde{\chi}_r^{i_1,\dots,i_s} \wedge d\phi^{i_1} \wedge \phi^{i_2} \wedge \dots \wedge \phi^{i_s} \\
&= (\mathrm{Sym}^\varsigma_{sq})_*[(d\chi)_{r+1}^s \bullet \phi_p^q] + (-1)^r \binom{s}{1}_\ell \sum_{i_1,\dots,i_s=1}^m \tilde{\chi}_r^{i_1,\dots,i_s} \wedge d\phi^{i_1} \wedge \phi^{i_2} \wedge \dots \wedge \phi^{i_s},
\end{aligned}$$

where we used (21) and (22) in the second step. Lemma 1.53 proves the rest. □

Proposition 1.55 also holds for $\imath_X$ instead of d, and for L_X, if one drops $(-1)^r$. Tracing the previous proof we get for the general case:

Corollary 1.56 *If* $\phi_p^q \in \mathcal{A}_p(M) \otimes \mathrm{Hom}(\otimes^q V, W)$ *and* $\chi_r^s \in \mathcal{A}_r(M) \otimes \mathrm{Hom}(\otimes^s W, Z)$,

$$d(\chi_r^s \bullet \phi_p^q) = (d\chi)_{r+1}^s \bullet \phi_p^q + (-1)^r \sum_{j=0}^{s-1} (-1)^{jp} [(\chi_r^{j;s-j} \blacktriangleleft \phi_p^q)^{1;s-j-1} \blacktriangleleft (d\phi)_{p+1}^q] \blacktriangleright \phi_p^q. \quad (28)$$

1.4 Differential Forms on Lie Groups

We only consider finite dimensional Lie groups G and their Lie algebras $\mathfrak{g} = \mathbf{L}(G)$.

Definition 1.57 *For any* Lie *group* G *with multiplication* $\mu: G \times G \to G$, $\mu(g,h) = gh$, *inversion* $\eta: G \to G$, $\eta(g) = g^{-1}$, *neutral element* e *and* $\mathfrak{g} = \mathbf{L}(G) = T_e(G)$ *let*

1. $\lambda_g: G \to G$, $\lambda_g(h) = \mu_g(h) = gh$ *be the* left multiplication *with* $g \in G$,
2. $\rho_g: G \to G$, $\rho_g(h) = \mu^g(h) = hg$ *be the* right multiplication *with* $g \in G$,
3. $I_g: G \to G$, $I_g(h) = ghg^{-1}$ *be the* conjugation *with* $g \in G$,
4. $\mathrm{Ad}: G \to \mathrm{Aut}(\mathfrak{g}), \mathrm{Ad}(g) = (dI_g)_e: \mathfrak{g} \to \mathfrak{g}, \mathrm{Ad}(g)X = d\lambda_g d\rho_{g^{-1}}(X)$ *and*
5. $\mathrm{ad} = d(\mathrm{Ad})_e: \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}), \mathrm{ad}(X)(Y) = [X,Y]$ *be the* adjoint actions.

Definition 1.58 *For any* Lie *group* G *let* $\mathcal{D}_S^1(G) \subseteq \mathcal{D}^1(G)$ *for* $S = L, R$ *defined by*

$$\begin{aligned} \mathcal{D}_L^1(G) &:= \{\mathcal{X} \in \mathcal{D}^1(G) | (\forall g \in G)\ (\lambda_g)_*\mathcal{X} = \mathcal{X}\}, \\ \mathcal{D}_R^1(G) &:= \{\mathcal{X} \in \mathcal{D}^1(G) | (\forall g \in G)\ (\rho_g)_*\mathcal{X} = \mathcal{X}\} \end{aligned}$$

denote the Lie *subalgebras of* left, *resp.*, right invariant vector fields *on* G.

Lemma 1.59 *For* $X \in \mathfrak{g}$ *define* $\mathcal{L}_X \in \mathcal{D}_L^1(G)$ *and* $\mathcal{R}_X \in \mathcal{D}_R^1(G)$ *by*

$$(\mathcal{L}_X)_g := d\lambda_g(X), \quad \textit{resp.,} \quad (\mathcal{R}_X)_g = d\rho_g(X) \quad \textit{for all} \quad g \in G.$$

Then $\mathcal{L}: \mathfrak{g} \to \mathcal{D}_L^1(G)$ *and* $-\mathcal{R}: \mathfrak{g} \to \mathcal{D}_R^1(G)$ *are* Lie *algebra isomorphisms with* $\eta_*\mathcal{L}_X = -\mathcal{R}_X$ *and*

$$\begin{aligned} (\forall X, Y \in \mathfrak{g}) \qquad & [\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}, \qquad [\mathcal{R}_X, \mathcal{R}_Y] = \mathcal{R}_{[Y,X]} = -\mathcal{R}_{[X,Y]}, \\ (\forall g \in G,\ \forall X \in \mathfrak{g}) \qquad & (\rho_{g^{-1}})_*\mathcal{L}_X = \mathcal{L}_{\mathrm{Ad}(g)X}, \qquad (\lambda_g)_*\mathcal{R}_X = \mathcal{R}_{\mathrm{Ad}(g)X}, \\ \mathcal{D}_L^1(G) \cap \mathcal{D}_R^1(G) \ &= \ \mathcal{L}(\{X \in \mathfrak{g} | \mathrm{Ad}(G)X = X\}) \ = \ \mathcal{L}(\mathbf{L}(Z(G))), \end{aligned}$$

where $Z(G)$ *denotes the center of* G.

Proof. $\eta \circ \lambda_g = \rho_{g^{-1}} \circ \eta$, thus $(\eta_*\mathcal{L}_X)_{g^{-1}} = d\rho_{g^{-1}} d\eta_e(X) = d\rho_{g^{-1}}(-X) = -(\mathcal{R}_X)_{g^{-1}}$. $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$ is just the definition of the commutator in $\mathfrak{g} = \mathbf{L}(G)$. By Lemma 1.28 $[\mathcal{R}_X, \mathcal{R}_Y] = [-\eta_*\mathcal{L}_X, -\eta_*\mathcal{L}_Y] = \eta_*[\mathcal{L}_X, \mathcal{L}_Y] = \eta_*\mathcal{L}_{[X,Y]} = -\mathcal{R}_{[X,Y]}$. As λ_g and ρ_h commute, $(\rho_{g^{-1}})_*\mathcal{X} \in \mathcal{D}_L^1(G)$ for all $\mathcal{X} \in \mathcal{D}_L^1(G)$, $g \in G$ and $(\lambda_g)_*\mathcal{Y} \in \mathcal{D}_R^1(G)$ for all $\mathcal{Y} \in \mathcal{D}_R^1(G)$. Now $\mathcal{L}^{-1}((\rho_{g^{-1}})_*\mathcal{L}_X) = (\rho_{g^{-1}})_*\mathcal{L}_X(e) = d\rho_{g^{-1}}(g) d\lambda_g(e)X = \mathrm{Ad}(g)X$ and $\mathcal{R}^{-1}((\lambda_g)_*\mathcal{R}_X) = (\lambda_g)_*\mathcal{R}_X(e) = \mathrm{Ad}(g)X$. From this the last claim follows immediately because $\mathrm{Ad}(g)X = X \iff I_G(\exp(tX) = \exp(tX)$ for all $t \in \mathbb{R} \iff X \in \mathbf{L}(Z(G))$. □

Definition 1.60 *For any* LIE *group* G *let* $\mathcal{A}^S(G,V)\subseteq\mathcal{A}(G,V)$ *for* $S=L,R$ *with*

$$\begin{aligned}\mathcal{A}^L(G,V) &:= \{\omega\in\mathcal{A}(G,V)|(\forall g\in G)\ \lambda_g^*\omega=\omega\},\\ \mathcal{A}^R(G,V) &:= \{\omega\in\mathcal{A}(G,V)|(\forall g\in G)\ \rho_g^*\omega=\omega\} \quad \text{and}\\ \mathcal{A}^I(G,V) &:= \mathcal{A}^L(G,V)\cap\mathcal{A}^R(G,V)\end{aligned}$$

denote the vector spaces of left, right, *resp.*, bi-invariant V-valued differential forms *on* G. *The submodules* $\mathcal{A}^S(G)\otimes V$ *and* $\mathcal{A}^I(G)\otimes V$ *are defined analogously.*

Note 1.61 By Lemma 1.29 we have $d(\mathcal{A}_p^S(G)\otimes V)\subseteq\mathcal{A}_{p+1}^S(G)\otimes V$ and if $\wedge_V$ is given, $\mathcal{A}^S(G)\otimes V$ are subalgebras of $\mathcal{A}(G)\otimes V$. If G is compact with normalized HAAR measure μ and $\omega\in\mathcal{A}(G,V)$ we have projections $\omega\mapsto\omega^S\in\mathcal{A}^S(G,V)$ (note $(\omega^S)^S=\omega^S$):

$$\omega^L := \int_G \lambda_g^*\omega\, d\mu(g), \qquad \omega^R := \int_G \rho_g^*\omega\, d\mu(g). \tag{29}$$

For $X_1,\dots,X_p\in T_g(G)$ we have

$$\begin{aligned}(\forall\omega\in\mathcal{A}_p^L(G,V)) \quad \omega_g(X_1,\dots,X_p) &= \omega_e((d\lambda_{g^{-1}})_gX_1,\dots,(d\lambda_{g^{-1}})_gX_p),\\ \text{resp.,}\quad (\forall\omega\in\mathcal{A}_p^R(G,V)) \quad \omega_g(X_1,\dots,X_p) &= \omega_e((d\rho_{g^{-1}})_gX_1,\dots,(d\rho_{g^{-1}})_gX_p).\end{aligned}$$

Thus ω_e determines $\omega\in\mathcal{A}^S(G,V)$ completely, which yields the following lemma:

Lemma 1.62 ψ^S: $\mathrm{Alt}(\mathfrak{g},V)\to\mathcal{A}^S(G,V)$, $\mathrm{Alt}_p(\mathfrak{g},V)\to\mathcal{A}_p^S(G,V)$: $\omega_e\mapsto\omega$ *are isomorphisms of vector spaces and* $\mathcal{A}^S(G,V)=\mathcal{A}^S(G)\otimes V$, $\mathcal{A}_p^S(G,V)=\mathcal{A}_p^S(G)\otimes V$. *Also* $\mathcal{A}^I(G,V)=\mathcal{A}^I(G)\otimes V$. *For* $V=\mathbb{R}$, *the functions* ψ^S *are isomorphisms of graded algebras.*

Corollary 1.63 *If* $g,h\in G$ *and* $K\in\mathrm{Alt}_p(\mathfrak{g},V)$ *we have* $\eta^*\psi^L(K)=(-1)^p\psi^R(K)$,

$$\rho_g^*\psi^L(K)=\psi^L(\mathrm{Ad}(g^{-1})^*K) \qquad \text{and} \qquad \lambda_g^*\psi^R(K)=\psi^R(\mathrm{Ad}(g)^*K);$$

especially for $K:=\mathrm{Ad}(h)\in\mathrm{Aut}(\mathfrak{g})$ *this means* $\eta^*\psi^L(\mathrm{Ad}(h))=-\psi^R(\mathrm{Ad}(h))$,

$$\rho_g^*\psi^L(\mathrm{Ad}(h))=\psi^L(\mathrm{Ad}(hg^{-1})) \qquad \text{and} \qquad \lambda_g^*\psi^R(\mathrm{Ad}(h))=\psi^R(\mathrm{Ad}(hg)).$$

Corollary 1.64 *For every bi-invariant* $\omega\in\mathcal{A}^I(G,V)$, *we have* $d\omega=0$.

$$\psi^I\colon \mathrm{Alt}(\mathfrak{g},V)_{\mathrm{inv}} := \{K\in\mathrm{Alt}(\mathfrak{g},V)|(\forall g\in G)\ \mathrm{Ad}(g^{-1})^*K=K\}\to\mathcal{A}^I(G,V)$$

is an isomorphism of vector spaces, resp., of graded algebras if $V=\mathbb{R}$.

Proof. Corollary 1.63 yields $\eta^*\omega=(-1)^p\omega$ for $\omega\in\mathcal{A}_p^I(G,V)$. Since d commutes with pullbacks, $d\omega\in\mathcal{A}_{p+1}^I(G,V)$ and $d\omega=(-1)^pd\eta^*\omega=(-1)^p\eta^*d\omega=-d\omega$. □

Definition 1.65 $\Theta_g^L(\mathcal{X}_g):=d\lambda_{g^{-1}}(\mathcal{X}_g)\in\mathfrak{g}$ *for all* $g\in G$ *defines the unique* left canonical 1-form $\Theta^L=\psi^L(\mathrm{id}_{\mathfrak{g}})\in\mathcal{A}_1^L(G,\mathfrak{g})$ *with* $\Theta_e^L=\mathrm{id}_{\mathfrak{g}}$. *In analogy* $\Theta^R=\psi^R(\mathrm{id}_{\mathfrak{g}})\in\mathcal{A}_1^R(G,\mathfrak{g})$ *is defined by* $\Theta_g^R(\mathcal{X}_g):=d\rho_{g^{-1}}(\mathcal{X}_g)$. *Obviously* $\Theta_g^R=\mathrm{Ad}(g)\circ\Theta_g^L$ *for all* $g\in G$ *and thus* $\Theta^R=\mathrm{Ad}\bullet\Theta^L$.

Corollary 1.63 yields that $\eta^*\Theta^L = -\Theta^R$, $\rho_g^*\Theta^L = \psi^L(\mathrm{Ad}(g^{-1}))$, $\lambda_g^*\Theta^R = \psi^R(\mathrm{Ad}(g))$ for all $g \in G$. From these we can recover the relations in Corollary 1.63 using Lemma 1.49 and (26) because

$$\psi^S(K) = (1 \otimes K) \bullet \Theta^S \in \mathcal{A}^S(G,V) \qquad \text{for all} \quad K \in \mathrm{Alt}(\mathfrak{g}, V). \tag{30}$$

Note that for any linear $\Lambda: V \to W$, Lemma 1.49.3 combined with (30) yields

$$\Lambda_*\psi^S(K) = \Lambda_*[(1 \otimes K) \bullet \Theta^S] = [1 \otimes (\Lambda_\circ K)] \bullet \Theta^S = \psi^S(\Lambda_\circ K) \in \mathcal{A}^S(G, W). \tag{31}$$

Lemma 1.66 *If $h: G \to H$ is a* LIE *group homomorphism, then*

$$h^*\Theta_H^S = dh_e \Theta_G^S, \qquad h^*(\psi_H^S(K)) = \psi_G^S(dh_e^* K) \quad \textit{for all} \quad K \in \mathrm{Alt}(\mathfrak{h}, V).$$

Proof. Because $\lambda_{h(g)} \circ h = h \circ \lambda_g$ for all $g \in G$,

$$(h^*\Theta_H^L)_g = (d\lambda_{h(g)^{-1}})_{h(g)} dh_g = dh_e (d\lambda_{g^{-1}})_g = dh_e \circ (\Theta_G^L)_g.$$

$\rho_{h(g)} \circ h = h \circ \rho_g$ proves the result for $S = R$, the rest follows from Lemma 1.49. □

Definition 1.67 *For any differentiable map $f: M \to G$ we call $f^*\Theta^L \in \mathcal{A}_1(M, \mathfrak{g})$ the* left *and $f^*\Theta^R \in \mathcal{A}_1(M, \mathfrak{g})$ the* right differential *of f.*

So $f^*\Theta^R = (\mathrm{Ad} \circ f) \bullet f^*\Theta^L$, $f^*\Theta^L = (\mathrm{Ad} \circ \eta \circ f) \bullet f^*\Theta^R$. Lemma 1.66 yields:

Corollary 1.68 *If $h: G \to H$ is a* LIE *group homomorphism, then for every differentiable $f: M \to G$ and every $K \in \mathrm{Alt}(\mathfrak{h}, V)$ we have*

$$(h \circ f)^*\Theta_H^S = dh_e \circ f^*\Theta_G^S, \qquad (h \circ f)^*\psi_H^S(K) = f^*\Theta_G^S(dh_e^* K) \in \mathcal{A}(M) \otimes V.$$

Definition 1.69 *For any $f, g: M \to G$ we define $f \cdot g, f^{-1}: M \to G$ "pointwise": $f \cdot g := \mu \circ (f, g)$, $f^{-1} := \eta \circ f$.*

Theorem 1.70 *For all differentiable $f, g: M \to G$ and all $h \in G$ we have*

$$\begin{aligned}
(f \cdot g)^*\psi^L(\mathrm{Ad}(h)) &= (\mathrm{Ad} \circ I_h \circ g^{-1}) \bullet f^*\psi^L(\mathrm{Ad}(h)) + g^*\psi^L(\mathrm{Ad}(h)), \\
(f \cdot g)^*\psi^R(\mathrm{Ad}(h)) &= f^*\psi^R(\mathrm{Ad}(h)) + (\mathrm{Ad} \circ I_h \circ f) \bullet g^*\psi^R(\mathrm{Ad}(h)), \\
(f^{-1})^*\psi^L(\mathrm{Ad}(h)) &= -(\mathrm{Ad} \circ I_h \circ f) \bullet f^*\psi^L(\mathrm{Ad}(h)) \;=\; -f^*\psi^R(\mathrm{Ad}(h)), \\
(f^{-1})^*\psi^R(\mathrm{Ad}(h)) &= -(\mathrm{Ad} \circ I_h \circ f^{-1}) \bullet f^*\psi^R(\mathrm{Ad}(h)) \;=\; -f^*\psi^L(\mathrm{Ad}(h)); \\
(f \cdot g)^*\Theta^L &= (\mathrm{Ad} \circ g^{-1}) \bullet f^*\Theta^L + g^*\Theta^L, \\
(f \cdot g)^*\Theta^R &= f^*\Theta^R + (\mathrm{Ad} \circ f) \bullet g^*\Theta^R, \\
(f^{-1})^*\Theta^L &= -(\mathrm{Ad} \circ f) \bullet f^*\Theta^L \;=\; -f^*\Theta^R, \\
(f^{-1})^*\Theta^R &= -(\mathrm{Ad} \circ f^{-1}) \bullet f^*\Theta^R \;=\; -f^*\Theta^L.
\end{aligned}$$

Proof. To prove Theorem 1.70 directly, observe that (13) for $m = \mu$ yields the generalized product rule $d(f \cdot g)_x = (d\rho_{g(x)})_{f(x)} df_x + (d\lambda_{f(x)})_{g(x)} dg_x$ for all $x \in M$. On the other hand, Theorem 1.70 immediately follows from Corollary 1.105, which we will state below. □

Corollary 1.71 *For all differentiable $f, g: M \to G$ and all $K \in \mathrm{Alt}_p(\mathfrak{g}, V)$ we have*

$$\begin{aligned}(f\cdot g)^*\psi^L(K) &= (1\otimes K)\bullet[(\mathrm{Ad}\circ g^{-1})\bullet f^*\Theta^L + g^*\Theta^L],\\ (f\cdot g)^*\psi^R(K) &= (1\otimes K)\bullet[f^*\Theta^R + (\mathrm{Ad}\circ f)\bullet g^*\Theta^R],\\ (f^{-1})^*\psi^L(K) &= (-1)^p f^*\psi^R(K).\end{aligned}$$

Proof. The first two equations follow from the results for Θ^S in connection with Lemma 1.49 and (30); Corollary 1.63 and $f^{-1} = \eta\circ f$ yield the third one. □

Corollary 1.72 *Let $c: M \to \{c\} \subseteq G$ be constant then for all $f: M \to G$ and $K \in \mathrm{Alt}(\mathfrak{g}, V)$:*

$$\begin{aligned}(c\cdot f)^*\Theta^L &= (\lambda_c\circ f)^*\Theta^L = f^*\Theta^L, & (f\cdot c)^*\Theta^L &= f^*\rho_c^*\Theta^L = f^*\psi^L(\mathrm{Ad}(c^{-1})),\\ (f\cdot c)^*\Theta^R &= (\rho_c\circ f)^*\Theta^R = f^*\Theta^R, & (c\cdot f)^*\Theta^R &= f^*\lambda_c^*\Theta^R = f^*\psi^R(\mathrm{Ad}(c));\\ (c\cdot f)^*\psi^L(K) &= f^*\psi^L(K), & (f\cdot c)^*\psi^L(K) &= f^*\psi^L(\mathrm{Ad}(c^{-1})^*K),\\ (f\cdot c)^*\psi^R(K) &= f^*\psi^R(K), & (c\cdot f)^*\Theta^R &= f^*\psi^R(\mathrm{Ad}(c)^*K).\end{aligned}$$

Corollary 1.73 *For all differentiable $f, g: M \to G$ and $K \in \mathrm{Alt}(\mathfrak{g}, V)$ we have:*

$$\begin{aligned}(\forall K)\ f^*\psi^L(K) = g^*\psi^L(K) &\iff f^*\Theta^L = g^*\Theta^L \iff f\cdot g^{-1} \text{ locally constant},\\ (\forall K)\ f^*\psi^R(K) = g^*\psi^R(K) &\iff f^*\Theta^R = g^*\Theta^R \iff f^{-1}\cdot g \text{ locally constant}.\end{aligned}$$

Proof. Again we only show the former equivalences proving $A \Longrightarrow B \Longrightarrow C \Longrightarrow A$. Firstly, $A \Longrightarrow B$ is trivial. For $B \Longrightarrow C$, Theorem 1.70 yields

$$(f\cdot g^{-1})^*\Theta^L = (\mathrm{Ad}\circ g)\circ f^*\Theta_L + (g^{-1})^*\Theta^L = (\mathrm{Ad}\circ g)\circ(f^*\Theta^L - g^*\Theta^L) = 0,$$

if $f^*\Theta^L = g^*\Theta^L$. Thus $d(f\cdot g^{-1}) = 0$, so $f\cdot g^{-1}$ is locally constant. Finally, if $h := f\cdot g^{-1}$ is locally constant, i. e. $dh = 0$, then $h^*\Theta^L = 0$ and $f^*\psi^L(K) = (h\cdot g)^*\psi^L(K) = g^*\psi^L(K)$ by Theorem 1.70, which proves $C \Longrightarrow A$. □

Lemma 1.74 *For any differentiable $f: M \to G$ and $\mathfrak{g}$-valued forms $\omega, \phi \in \mathcal{A}(M, \mathfrak{g})$,*

$$\begin{aligned}(\mathrm{Ad}\circ f)\bullet(\omega\wedge_{\mathfrak{g}}\phi) &= [(\mathrm{Ad}\circ f)\bullet\omega]\wedge_{\mathfrak{g}}[(\mathrm{Ad}\circ f)\bullet\phi],\\ d[(\mathrm{Ad}\circ f)\bullet\phi) &= (\mathrm{Ad}\circ f)\bullet(f^*\Theta^L\wedge_{\mathfrak{g}}\phi + d\phi).\end{aligned}$$

Proof. This is a corollary to Lemma 1.96 below. □

Suppose $\Phi: G \to \mathbb{A}$ is a homomorphism of a group G into the multiplicative semigroup of an algebra $\mathbb{A}$. Then $\Phi(G) \subset \mathbb{A}$ is a group and we thus can define $\Phi^{-1}: G \to \Phi(G)$ analogously to Definition 1.69 by $\Phi^{-1} = \Phi\circ\eta_G = \eta_{\Phi(G)}\circ\Phi$.

Lemma 1.75 *Let G be a* LIE *group, $\mathbb{A}$ an algebra and $\Phi: G \to \mathbb{A}$ a C^∞-homomorphism into the multiplicative semigroup of $\mathbb{A}$. Then $S^\Phi := (d\Phi_e)_*\Theta^S \in \mathcal{A}^S(G, \mathbb{A})$ with $S_e^\Phi = d\Phi_e$ for $S = L, R$, and $L^\Phi = \Phi^{-1}\cdot d\Phi$, $R^\Phi = d\Phi\cdot\Phi^{-1}$.*

Proof. By Lemma 1.29 $\lambda_g^* L^\Phi = (d\Phi_e)_*(\lambda_g^*\Theta^L) = (d\Phi_e)_*\Theta^L = L^\Phi$ and $\rho_g^* R^\Phi = R^\Phi$ for all $g \in G$. For the last assertions confer Note 1.27 and observe $L_g^\Phi = d\Phi_e \circ d\lambda_{g^{-1}}^G = d\lambda_{\Phi(g)^{-1}}^{\Phi(G)} \circ d\Phi_g = \Phi(g)^{-1} \cdot d\Phi_g$ and $R_g^\Phi = d\rho_{\Phi(g)^{-1}}^{\Phi(G)} \circ d\Phi_g = d\Phi_g \cdot \Phi(g)^{-1}$ because Φ is a homomorphism and multiplication in $\mathbb{A}$ is linear and thus may be identified with its differential. □

Lemma 1.75 applies to representations $\Phi: G \to \mathrm{Gl}(\mathbb{C}^n)$ (cf. Definition 1.85). If we take $\mathbb{A} = \mathrm{End}(\mathbb{C}^n)$ then $L^\Phi = \Phi^*\Theta^L_{\mathrm{Gl}(\mathbb{C}^n)}$ is the left and $R^\Phi = \Phi^*\Theta^R_{\mathrm{Gl}(\mathbb{C}^n)}$ is the right differential of Φ by Lemma 1.66. If $G < \mathrm{Gl}(\mathbb{C}^n)$ and Φ is the embedding, we shall write $L = U^{-1}dU$, $R = (dU)U^{-1}$ and $\wedge: (\mathcal{A}(M) \otimes \mathbb{A}) \times (\mathcal{A}(M) \otimes \mathbb{A}) \to (\mathcal{A}(M) \otimes \mathbb{A})$ for the wedge product induced by multiplication $\cdot$ in $\mathrm{End}(\mathbb{C}^n)$. In that case we have $T_U(G) = U \cdot \mathfrak{g} = \mathfrak{g} \cdot U$ for all $U \in G$. So for $\mathcal{X} \in \mathcal{D}^1(G)$ we have $\mathcal{X}_U = UX$ with a suitable $X \in \mathfrak{g}$ and $L_U(\mathcal{X}_U) = U^{-1}\mathcal{X}_U = X$, resp., $R_U(\mathcal{X}_U) = \mathcal{X}_U U^{-1} = \mathrm{Ad}(U)X \in \mathfrak{g}$. Physicists call L and R *left*, resp., *right invariant currents*.

Every $Q \in \mathrm{End}(\mathbb{C}^n)$ defines a linear form Tr_Q on $\mathrm{End}(\mathbb{C}^n)$ by $\mathrm{Tr}_Q(U) := \mathrm{Tr}(QU)$. For $\alpha \in \mathcal{A}(M, \mathrm{End}(\mathbb{C}^n))$, let $\alpha^k := \underbrace{\alpha \wedge \cdots \wedge \alpha}_{k}$. Then $S^k \in \mathcal{A}_k^S(G, \mathrm{End}(\mathbb{C}^n))$,

$$\lambda_k^Q := (\mathrm{Tr}_Q)_* L^k = \mathrm{Tr}(QL^k) \in \mathcal{A}_k^L(G,\mathbb{C}), \quad \rho_k^Q := (\mathrm{Tr}_Q)_* R^k = \mathrm{Tr}(QR^k) \in \mathcal{A}_k^R(G,\mathbb{C})$$

with $d\lambda_k^Q = \mathrm{Tr}(Q\, dL^k)$, $d\rho_k^Q = \mathrm{Tr}(Q\, dR^k)$ by Lemma 1.29. For $Q = \mathbb{1}$ we have the bi-invariant

$$\omega_k := \lambda_k^{\mathbb{1}} = \rho_k^{\mathbb{1}} = \mathrm{Tr}(L^k) = \mathrm{Tr}(R^k) \in \mathcal{A}_k^I(G,\mathbb{C}). \tag{32}$$

Lemma 1.76 *If $k, l \in \mathbb{N}$, $\alpha \in \mathcal{A}_{2k-1}(M, \mathrm{End}(\mathbb{C}^n))$, then $\mathrm{Tr}(\alpha^{2l}) = 0$, especially*

$$\mathrm{Tr}[(L^\Phi)^{2l}] = \mathrm{Tr}(R^\Phi)^{2l}] = 0 \in \mathcal{A}_{2l}(G,\mathbb{C})$$

for any representation $\Phi: G \to \mathrm{Gl}(\mathbb{C}^n)$, and thus $\omega_{2l} = 0$.

Proof. Let $\tau = (12\ldots 2l(2k-1))^{2k-1} \in S^{2l(2k-1)}$ with $(-1)^\tau = -1$. Then $\mathrm{Tr}(\alpha^{2l}) = -\mathrm{Tr}(\alpha^{2l}) \circ \tilde{\tau}$, being a form, but $\mathrm{Tr}(\alpha^{2l}) = \mathrm{Tr}(\alpha^{2l}) \circ \tilde{\tau}$ by symmetry of the trace. □

Let $\mathcal{U}(\mathfrak{g}) = \mathcal{T}(\mathfrak{g})/J_{\mathrm{M}}$ the universal enveloping algebra of $\mathfrak{g}$, where J_{M} is the ideal generated by (HILGERT, NEEB [7, p. 167])

$$\mathrm{M} = \{a \in \mathcal{T}(\mathfrak{g}) | a = X \otimes Y - Y \otimes X - [X,Y];\ X,Y \in \mathfrak{g}\}.$$

For $\mathfrak{g} \neq 0$, $\mathcal{U}(\mathfrak{g})$ has an infinite base, $\dim \mathfrak{g} \leq \aleph_0$ yields $\dim \mathcal{U}(\mathfrak{g}) = \aleph_0$ (POINCARÉ-BIRKHOFF-WITT theorem in [7, p. 170]). If $\sigma: \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ means the canonical embedding, using the associative bilinear mapping given on $\mathcal{U} := \mathcal{U}(\mathfrak{g})$, we define $(\sigma_*\alpha)^k := \sigma_*\alpha \wedge_{\mathcal{U}} \cdots \wedge_{\mathcal{U}} \sigma_*\alpha \in \mathcal{A}(M) \otimes \mathcal{U}(\mathfrak{g})$ for all $\alpha \in \mathcal{A}(M, \mathfrak{g})$. This yields $(S^\Phi)^k = (d\Phi)'_*(\sigma_*\Theta^S)^k$ by Lemma 1.29, where $d\Phi': \mathcal{U}(\mathfrak{g}) \to \mathrm{End}(\mathbb{C}^n)$ is the unique algebra homomorphism with $d\Phi' \circ \sigma = d\Phi$ and $d\Phi'(1) = \mathbb{1}$ that exists by the universal property of $\mathcal{U}(\mathfrak{g})$ (cf. [7, p. 167]):

Lemma 1.77 *For an associative algebra $\mathbb{A}$ with unit $\mathbb{1}$ let $\mathbb{A}_{\mathrm{Lie}}$ denote the* LIE *algebra one obtains from $\mathbb{A}$ defining $[A, B] := A \cdot B - B \cdot A$ for all $A, B \in \mathbb{A}$. Then for any* LIE *algebra homomorphism $\pi: \mathfrak{g} \to \mathbb{A}_{\mathrm{Lie}}$, there exists one unique homomorphism of associative algebras $\pi': \mathcal{U}(\mathfrak{g}) \to \mathbb{A}$ with $\pi' \circ \sigma = \pi$ and $\pi'(1) = \mathbb{1}$.*

Thus for a representation $\pi\colon \mathfrak{g} \to \mathfrak{gl}(\mathbb{C}^n) = \mathrm{End}(\mathbb{C}^n)$, the universal property of $\mathcal{U}(\mathfrak{g})$ yields a unique algebra morphism $\pi'\colon \mathcal{U}(\mathfrak{g}) \to \mathrm{End}(\mathbb{C}^n)$ "extending π." Then there is a unique morphism $\pi_* = \mathrm{id} \otimes \pi'\colon \mathcal{A}(M) \otimes \mathcal{U}(\mathfrak{g}) \to \mathcal{A}(M) \otimes \mathrm{End}(\mathbb{C}^n)$ of associative $\mathcal{A}(M)$-algebras (with the algebra multiplications $\wedge_{\mathcal{U}}$, resp., $\wedge = \wedge_{\mathrm{End}(\mathbb{C}^n)}$).

In order to achieve such an associative wedge product for $\mathfrak{g}$-valued forms, one needs to embed $\mathfrak{g}$ into $\mathcal{U}(\mathfrak{g})$ since $\wedge_{\mathfrak{g}}$ is not associative. Moreover, the following lemma yields that $(\alpha_p \wedge_{\mathfrak{g}} \alpha_p) \wedge_{\mathfrak{g}} \alpha_p$ and $\alpha_p \wedge_{\mathfrak{g}} (\alpha_p \wedge_{\mathfrak{g}} \alpha_p)$ are zero for all $\alpha_p \in \mathcal{A}_p(M, \mathfrak{g})$.

Lemma 1.78 *Let* $\alpha_r \in \mathcal{A}_r(M,\mathfrak{g}), \beta_s \in \mathcal{A}_s(M,\mathfrak{g}), \gamma_t \in \mathcal{A}_t(M,\mathfrak{g})$ *then*

$$(\alpha_r \wedge_{\mathfrak{g}} \beta_s) \wedge_{\mathfrak{g}} \gamma_t + (-1)^{r(s+t)}(\beta_s \wedge_{\mathfrak{g}} \gamma_t) \wedge_{\mathfrak{g}} \alpha_r + (-1)^{t(r+s)}(\gamma_t \wedge_{\mathfrak{g}} \alpha_r) \wedge_{\mathfrak{g}} \beta_s = 0,$$
$$\textit{resp.,}\quad (\alpha_r \wedge_{\mathfrak{g}} \beta_s) \wedge_{\mathfrak{g}} \gamma_t - \alpha_r \wedge_{\mathfrak{g}} (\beta_s \wedge_{\mathfrak{g}} \gamma_t) = (-1)^{ts}(\alpha_r \wedge_{\mathfrak{g}} \gamma_t) \wedge_{\mathfrak{g}} \beta_s.$$

Proof: straightforward by Lemma 1.23 and JACOBI identity, cf. [1, p. 43]. □

We already know that $\omega \in \mathcal{A}^S(G,V)$ yields $d\omega \in \mathcal{A}^S(G,V)$ since d commutes with pullbacks. One quickly verifies using (17) that if $\mathcal{S}_X = \mathcal{L}_X$, resp., $\mathcal{S}_X = \mathcal{R}_X$ for $X \in \mathfrak{g}$ denotes a left, resp., right invariant vector field, then also $\imath_{\mathcal{S}_X}\omega$ and $L_{\mathcal{S}_X}\omega \in \mathcal{A}^S(G,V)$. Thus Lemma 1.62 yields that $\imath_{\mathcal{S}_X}$, $L_{\mathcal{S}_X}$ and d induce operators $\imath^S_X$, L^S_X and d^S on $\mathrm{Alt}(\mathfrak{g},V)$, such that the following diagram

$$\begin{array}{ccccccc}
\mathrm{Alt}(\mathfrak{g},V) & \xrightarrow{\imath^S_X} & \mathrm{Alt}(\mathfrak{g},V) & \xrightarrow{L^S_X} & \mathrm{Alt}(\mathfrak{g},V) & \xrightarrow{d^S} & \mathrm{Alt}(\mathfrak{g},V) \\
\downarrow{\scriptstyle \psi^S} & & \downarrow{\scriptstyle \psi^S} & & \downarrow{\scriptstyle \psi^S} & & \downarrow{\scriptstyle \psi^S} \\
\mathcal{A}^S(G,V) & \xrightarrow{\imath_{\mathcal{S}_X}} & \mathcal{A}^S(G,V) & \xrightarrow{L_{\mathcal{S}_X}} & \mathcal{A}^S(G,V) & \xrightarrow{d} & \mathcal{A}^S(G,V)
\end{array}$$

commutes. We write $\mathrm{sgn}(S) = \begin{cases} -1 & S = L \\ +1 & S = R \end{cases}$ and obtain:

Proposition 1.79 *For* $X, X_i \in \mathfrak{g}$, $p \in \mathbb{N}_0$ *and* $K \in \mathrm{Alt}_p(\mathfrak{g},V)$, *we have*

$$(\imath^S_X K)(X_1,\ldots,X_{p-1}) = p\,K(X,X_1,\ldots,X_{p-1}), \tag{33}$$

$$(L^S_X K)(X_1,\ldots,X_p) = \mathrm{sgn}(S)\sum_{i=1}^{p} K(X_1,\ldots,[X,X_i],\ldots,X_p), \tag{34}$$

$$(d^S K)(X_1,\ldots,X_{p+1}) = \frac{-\,\mathrm{sgn}(S)}{p+1}\sum_{i=1}^{p}\sum_{j=i+1}^{p+1}(-1)^{i+j}K([X_i,X_j],X_1,\ldots,\widehat{X_i},\ldots,\widehat{X_j},\ldots,X_{p+1}). \tag{35}$$

The following identities hold for any $X, Y \in \mathfrak{g}$:

$$d^S \circ d^S = \imath^S_X \circ \imath^S_X = 0,\quad L^S_X = d^S \circ \imath^S_X + \imath^S_X \circ d^S,\quad [L^S_X, \imath^S_Y] = \imath^S_{[X,Y]},\quad [L^S_X, L^S_Y] = L^S_{[X,Y]}.$$

Proof. (33) is obvious. Using left, resp., right invariance we only need to prove $\psi^S \circ L^S_X = L_{\mathcal{S}_X} \circ \psi^S$ and $\psi^S \circ d^S = d \circ \psi^S$ at e. Now for any $K \in \mathrm{Alt}_p(\mathfrak{g},V)$, $\psi^S(K)_e(X_1,\ldots,X_p) = [\psi^S(K)(\mathcal{S}_{X_1},\ldots,\mathcal{S}_{X_p})](e)$ and these maps are constant on G, whence $\mathcal{X}(\psi^S(K)(\mathcal{S}_{X_1},\ldots,\mathcal{S}_{X_p})) = 0$ for all $\mathcal{X} \in \mathcal{D}^1(G)$ follows. So the corresponding terms in (17) and (9) vanish and we obtain (34) and (35) from $[\mathcal{S}_X, \mathcal{S}_Y] = -\,\mathrm{sgn}(S)\mathcal{S}_{[X,Y]}$. The rest is immediate by the properties of d, $\imath_{\mathcal{S}_X}$ and $L_{\mathcal{S}_X}$. □

Definition 1.80

$$\begin{aligned} \mathrm{Alt}(\mathfrak{g},V)_{\mathfrak{g}-\mathrm{inv}} &:= \{K \in \mathrm{Alt}(\mathfrak{g},V) | (\forall X \in \mathfrak{g})\ L^L_X K = L^R_X K = 0\}, \\ \mathcal{A}^S(G,V)_{\mathfrak{g}-\mathrm{inv}} &:= \{\omega \in \mathcal{A}^S(\mathfrak{g},V) | (\forall X \in \mathfrak{g})\ L_{S_X}\omega = 0\}. \end{aligned}$$

We call the elements of $\mathcal{A}^S(G,V)_{\mathfrak{g}-\mathrm{inv}}$ $\mathfrak{g}$-invariant forms on G.

By definition of L^S_X, the restricted map $\psi^S\colon \mathrm{Alt}(\mathfrak{g},V)_{\mathfrak{g}-\mathrm{inv}} \to \mathcal{A}^S(G,V)_{\mathfrak{g}-\mathrm{inv}}$ is an isomorphism of vector spaces. Observe that for any $K \in \mathrm{Alt}(\mathfrak{g},V)$ and $X_i \in \mathfrak{g}$,

$$(d^S K)(X_1,\dots,X_{p+1}) = \frac{-1}{2(p+1)}\sum_{i=1}^{p+1}(-1)^i (L^S_{X_i}K)(X_1,\dots,\widehat{X_i},\dots,X_{p+1}). \quad (36)$$

So $d^S(\mathrm{Alt}(\mathfrak{g},V)_{\mathfrak{g}-\mathrm{inv}}) = 0$. We obtain the following generalization of Corollary 1.64:

Proposition 1.81 *For every $\mathfrak{g}$-invariant $\omega \in \mathcal{A}^S(G,V)$, $S = L,R$, we have $d\omega = 0$. Every bi-invariant form is $\mathfrak{g}$-invariant: $\mathcal{A}^S(G,V)_{\mathrm{inv}} \leq \mathcal{A}^S(G,V)_{\mathfrak{g}-\mathrm{inv}}$ and analogously $\mathrm{Alt}(\mathfrak{g},V)_{\mathrm{inv}} \leq \mathrm{Alt}(\mathfrak{g},V)_{\mathfrak{g}-\mathrm{inv}}$. If G is connected, all these vector spaces are isomorphic.*

Proof. The first statement has just been proved. The others are corollaries to Proposition 1.93 and Lemma 1.98 below. □

If K obeys $K([X,Y]) = [K(X),K(Y)]$ for all $X,Y \in \mathfrak{g}$, Proposition 1.79 yields

Corollary 1.82 *If G is a* LIE *group and $K \in \mathrm{End}(\mathfrak{g})$ then*

$$\begin{aligned} d(\psi^S(K)) &= \mathrm{sgn}(S)\frac{1}{2}\psi^S(K) \wedge_{\mathfrak{g}} \psi^S(K) \in \mathcal{A}^S_2(G,\mathfrak{g}), \\ d(\sigma_*\psi^S(K)) &= \mathrm{sgn}(S)\sigma_*\psi^S(K) \wedge_{\mathcal{U}} \sigma_*\psi^S(K) \in \mathcal{A}^S_2(G,\mathcal{U}(\mathfrak{g})). \end{aligned}$$

Proof. Again, we only need to prove the identities at e. Now $K \in \mathrm{End}(\mathfrak{g})$ yields $2(d\psi^S(K))_e(X_1,X_2) = \mathrm{sgn}(S)\psi^S(K)([S_{X_1},S_{X_2}])(e) = \mathrm{sgn}(S)K([X_1,X_2]) = \mathrm{sgn}(S)[K(X_1),K(X_2)] = \mathrm{sgn}(S)(\psi^S(K) \wedge_{\mathfrak{g}} \psi^S(K))_e(X_1,X_2)$. The second equation follows from $\sigma_*(\frac{1}{2}\alpha \wedge_{\mathfrak{g}} \alpha) = \sigma_*\alpha \wedge_{\mathcal{U}} \sigma_*\alpha \in \mathcal{A}_2(M) \otimes \mathcal{U}(\mathfrak{g})$ for all $\alpha \in \mathcal{A}_1(M,\mathfrak{g})$. □

Taking $K = \mathrm{id}_{\mathfrak{g}}$ we obtain

Corollary 1.83 (Maurer-Cartan identities) *If G is a* LIE *group, then*

$$\begin{aligned} d\Theta^S &= \mathrm{sgn}(S)\frac{1}{2}\Theta^S \wedge_{\mathfrak{g}} \Theta^S \in \mathcal{A}^S_2(G,\mathfrak{g}), \\ d(\sigma_*\Theta^S) &= \mathrm{sgn}(S)\sigma_*\Theta^S \wedge_{\mathcal{U}} \sigma_*\Theta^S \in \mathcal{A}^S_2(G,\mathcal{U}(\mathfrak{g})). \end{aligned}$$

Lemma 1.29 yields $dL = -L \wedge L$, $dR = +R \wedge R$, and by Lemma 1.25 we get:

Corollary 1.84 *For $l \in \mathbb{N}_0$ and $G < \mathrm{Gl}(\mathbb{C}^n)$, the left and the right differential obey the following rules:*

$$\begin{aligned} dL^{2l+1} &= -L^{2l+2}, & dR^{2l+1} &= R^{2l+2}, & dL^{2l+2} &= dR^{2l+2} = 0, \\ d(UL^{2l}) &= UL^{2l+1}, & d(L^{2l}U^{-1}) &= -L^{2l+1}U^{-1}, & d(UL^{2l+1}) &= d(L^{2l+1}U^{-1}) = 0. \end{aligned}$$

Thus $d\omega_{2l+1} = 0,\ d\lambda^Q_{2l+1} = -\lambda^Q_{2l+2},\ d\rho^Q_{2l+1} = \rho^Q_{2l+2},\ d\lambda^Q_{2l+2} = \rho^Q_{2l+2} = 0.$ (37)

Analogous relations hold for the left and right differentials of any $f\colon M \to G$.

1.5 LIE Transformation Groups

Recall the notation of S^p and S_g from Lemma 1.30 and $\tilde{f}$ from (15).

Definition 1.85 *By a* left, *resp.,* right LIE transformation group *of a manifold P we mean a* LIE *group G acting on P from the left, resp., right, and this action $S: G \times P \to P$ (where $S := L$, resp., $S := R$) is differentiable. Then by Lemma 1.30 all S_g and S^p are differentiable for all $g \in G$, $p \in P$. The function S will also be called a* LIE group action *on P. If P is a vector space and the action is linear, we speak of a* representation *of G. The* trivial *action means the natural projection $\mathrm{pr}_P: G \times V \to V$.*

An action is called effective *if $S_g = \mathrm{id}_P$ only for $g = e$. In that case G may be thought of as a subgroup of* Diff(P). *An action is called* free *if (in addition) $S_g(p) = p$ only for $g = e$ for all $p \in P$, i. e., if for the pre-images $\bigcup_{p \in P}(S^p)^{-1}(p) = \{e\}$ holds. Finally, G acts* transitively *if for all $p_1, p_2 \in P$ there exists $g \in G$ with $S_g(p_1) = p_2$. In that case, P is called a* homogeneous manifold *of G.*

A tensor field $K \in \mathcal{D}(P)$ is called invariant (under S), *if $(\widetilde{S_g})_* K = K$ for all $g \in G$. A vector field $\mathcal{X} \in \mathcal{D}^1(P)$, resp., a differential form $\omega \in \mathcal{A}(P, V)$, is invariant if $(S_g)_*\mathcal{X} = \mathcal{X}$, resp., $S_g^*\omega = \omega$ for all $g \in G$. Denote the sets of these by $\mathcal{D}(P)_{\mathrm{inv}}$, $\mathcal{D}^1(P)_{\mathrm{inv}}$, resp., $\mathcal{A}(P, V)_{\mathrm{inv}}$.*

*For any subgroup $H < G$, we define $\mathcal{D}(P)_{H-\mathrm{inv}}$, resp., $\mathcal{A}(P, V)_{H-\mathrm{inv}}$ to be the sets of those tensor fields, resp., forms that are invariant under the restriction of S onto $H \times P$. Especially we will use this notation for G_1-*invariant forms, *where G_1 is the connected component of the neutral element in G.*

Via λ and ρ every LIE group acts freely and transitively on itself, and Ad is a representation of G on $\mathfrak{g}$ (resp., the underlying vector space) from the left.

For any action of a group G on a manifold P and all $g \in G$, $p \in P$, we have

$$L^p \circ \rho_g = L^{L(g,p)}, \qquad \text{resp.,} \qquad R^p \circ \lambda_g = R^{R(g,p)}; \tag{38}$$

$$L^p \circ \lambda_g = L_g \circ L^p, \qquad \text{resp.,} \qquad R^p \circ \rho_g = R_g \circ R^p. \tag{39}$$

For any vector field $\mathcal{X} \in \mathcal{D}^1(P)$, Lemma 1.40 yields that on $\mathcal{A}(P) \otimes V$:

$$\imath_{\mathcal{X}} \circ S_g^* = S_g^* \circ \imath_{(S_g)_*\mathcal{X}}, \qquad d \circ S_g^* = S_g^* \circ d, \qquad L_{\mathcal{X}} \circ S_g^* = L_{(S_g)_*\mathcal{X}}.$$

Thus $\mathcal{A}(P)_{\mathrm{inv}}$ and $\mathcal{A}(P)_{\mathrm{inv}} \otimes V$ are subalgebras of $\mathcal{A}(P)$, resp., $\mathcal{A}(P) \otimes V$ (whenever $\wedge_V$ is defined), with $d(\mathcal{A}(P)_{\mathrm{inv}}) \subseteq \mathcal{A}(P)_{\mathrm{inv}}$. Analogous statements hold for $\mathcal{A}(P)_{H-\mathrm{inv}}$ and $\mathcal{A}(P)_{H-\mathrm{inv}} \otimes V$, which are modules of $\mathcal{A}(P)_{\mathrm{inv}}$. Obviously $\mathcal{A}(P)_{\mathrm{inv}} \subseteq \mathcal{A}(P)_{H-\mathrm{inv}}$ and $\mathcal{A}(P, V)_{\mathrm{inv}} \subseteq \mathcal{A}(P, V)_{H-\mathrm{inv}}$ for any subgroup $H < G$.

Lemma 1.86 *Any* LIE *group action S defines a* LIE *group action $S \circ \eta$ on the opposite side by $(S \circ \eta)_g := S_{g^{-1}}$ for all $g \in G$.*

Note 1.87 *If S and S' are two commuting* LIE *group actions of G on P, i. e., if $S'_g(S_h(p)) = S_h(S'_g(p))$ for all $g, h \in G$ and $p \in P$, and if $S_g(p_0) = S'_g(p_0)$ for a $p_0 \in P$ and all $g \in G$, then at least on the orbit $S_G(p_0) = S'_G(p_0)$, the two actions S and S' act from opposite sides.*

Proof. E. g., if $S = L$, then $S'_{gh}L_l(p_0) = L_l S'_{gh}(p_0) = L_l L_g L_h(p_0) = L_l L_g S'_h(p_0) = L_l S'_h L_g(p_0) = L_l S'_h S'_g(p_0) = S'_h S'_g L_l(p_0)$, so S' acts from the right. □

Lemma 1.88 *If $S: G \times P \to P$ is a* LIE *group action then $S_*: G \times \mathcal{D}^1(P) \to \mathcal{D}^1(P)$, $S^* \circ \eta: G \times \mathcal{A}(P,V) \to \mathcal{A}(P,V)$, $S': G \times \mathcal{A}(P, \mathrm{Hom}(\mathcal{T}(\mathfrak{g}), V)) \to \mathcal{A}(P, \mathrm{Hom}(\mathcal{T}(\mathfrak{g}), V))$ and $S'': G \times \mathcal{A}(P, \mathfrak{g}) \to \mathcal{A}(P, \mathfrak{g})$ defined by*

$$\begin{aligned} (S_*)_g(\mathcal{X}) &:= (S_g)_*\mathcal{X} && \textit{for all} \quad \mathcal{X} \in \mathcal{D}^1(P), \\ (S^* \circ \eta)_g(\omega) &:= (S_{g^{-1}})^*\omega && \textit{for all} \quad \omega \in \mathcal{A}(P,V), \\ S'_g(\chi) &:= (S_{g^{-1}})^*(\mathrm{Ad}(g^{\mathrm{sgn}(S)})^*)_*\chi && \textit{for all} \quad \chi \in \mathcal{A}(P, \mathrm{Hom}(\mathcal{T}(\mathfrak{g}), V)) \quad \textit{and} \\ S''_g(\varphi) &:= (S_{g^{-1}})^* \mathrm{Ad}(g^{-\mathrm{sgn}(S)})_*\varphi && \textit{for all} \quad \varphi \in \mathcal{A}(P, \mathfrak{g}), \end{aligned}$$

are all representations of G on the same side.

Thus push-outs preserve the side while pullbacks change them.

Definition 1.89 *Let S, S' be two actions of G on spaces X, resp., X' on the same side. A mapping $f: X \to X'$ is called* G-equivariant, *if*

$$\begin{array}{ccc} G \times X & \xrightarrow{\quad S \quad} & X \\ \Big\downarrow \mathrm{id} \times f & & \Big\downarrow f \\ G \times X' & \xrightarrow{\quad S' \quad} & X' \end{array}$$

commutes, i. e., if $S'(g, f(x)) = f(S(g,x))$ for all $x \in X$ and $g \in G$.

If S is a LIE *group action on a manifold P, then — referring to the right action* Ad^* *on* $\mathrm{Hom}(\mathcal{T}(\mathfrak{g}), V)$ *— we call the differential form $\chi \in \mathcal{A}(P, \mathrm{Hom}(\mathcal{T}(\mathfrak{g}), V))$ G-equivariant, if χ is invariant under S'. Analogously, $\varphi \in \mathcal{A}(P, \mathfrak{g})$ will be called G-equivariant if φ is invariant under S''. We denote the sets of these invariant forms by $\mathcal{A}(P, \mathrm{Hom}(\mathcal{T}(\mathfrak{g}), V))_{\mathrm{equiv}}$, resp., $\mathcal{A}(P, \mathfrak{g})_{\mathrm{equiv}}$. They are modules over $\mathcal{A}(P)_{\mathrm{inv}}$.*

Thus $\omega \in \mathcal{A}(P,V)$ (with $V \neq \mathfrak{g}$) is G-equivariant iff it is invariant under S. Definition 1.89 should be compared to Note 1.87: R and Ad^* are actions on the same sides, while R^* and $(\mathrm{Ad}^*)_*$ are commuting representations on $\mathcal{A}(P, \mathrm{Hom}(\mathcal{T}(\mathfrak{g}), V))$ on opposite sides. Analogous statements hold for L and $(\mathrm{Ad} \circ \eta)^*$, resp., L^* and $((\mathrm{Ad} \circ \eta)^*)_*$.

Lemma 1.90 *Let $S: G \times P \to P$ be a* LIE *group action and $L': G \to \mathrm{Gl}(W)$ be a left representation. If $\varphi_r \in \mathcal{A}_r(P, W)$ and $\chi \in \mathcal{A}(P, \mathrm{Hom}(\mathcal{T}(W), V))$ are equivariant in the sense that $S_g^*\varphi_r = L'(g^{-\mathrm{sgn}(S)})_*\varphi_r$ and $S_g^*\chi = (L'(g^{\mathrm{sgn}(S)})^*)_*\chi$ for all $g \in G$, then $\chi \bullet \varphi_r$ is invariant. E. g., if $\chi \in \mathcal{A}(P, \mathrm{Hom}(\mathcal{T}(\mathfrak{g}), V))_{\mathrm{equiv}}$ and $\varphi_r \in \mathcal{A}_r(P, \mathfrak{g})_{\mathrm{equiv}}$ then $\chi \bullet \varphi_r$ is invariant.*

Proof. $S_g^*(\chi \bullet \varphi_r) = (S_g^*\chi) \bullet (S_g^*\varphi_r) = \chi \bullet [(L'(g^{\operatorname{sgn}(S)})^*)_* S_g^*\varphi_r] = \chi \bullet \varphi_r$, where the first equality follows from Lemma 1.49.1 and the second from 1.49.2 □

The forms φ_r and χ are also called *pseudotensorial forms* of type (L', W), resp., of type $((L' \circ \eta)^*, \operatorname{Hom}(\mathcal{T}(W), V))$, cf. Definition 2.46.

If G is compact with HAAR measure μ we have a projection onto G-equivariant forms defined in the following way (cf. Note 1.61):

$$\chi_{\text{equiv}} := \int_G (\operatorname{Ad}(g^{-\operatorname{sgn}(S)})^*)_* S_g^* \chi \, d\mu(g) \quad \text{for all } \chi \in \mathcal{A}(P, \operatorname{Hom}(\mathcal{T}(\mathfrak{g}), V)), \tag{40}$$

$$\varphi_{\text{equiv}} := \int_G \operatorname{Ad}(g^{\operatorname{sgn}(S)})_* S_g^* \varphi \, d\mu(g) \qquad \text{for all } \varphi \in \mathcal{A}(P, \mathfrak{g}). \tag{41}$$

If G is a LIE transformation group of P then every $X \in \mathfrak{g}$ induces a one-parameter group φ of transformations on P by $\varphi(t,p) := S(e^{tX}, p)$. Thus Propositions 1.35, 1.38.4, 1.38.6 and 1.39 yield, analogously to Lemma 1.59:

Lemma 1.91 *Let G be a LIE transformation group of P with LIE group action $S = L$, resp., $S = R$. Every $X \in \mathfrak{g}$ induces $\mathcal{S}_X \in \mathcal{D}^1(P)$ by $(\mathcal{S}_X)_p := (dS^p)_e(X)$, so*

$$(\mathcal{S}_X)_p(f) = (dS^p)_e(X)(f) = \frac{d}{dt} f(S_{e^{tX}}(p))|_{t=0} \quad \text{for all } f \in C^\infty(P),\ p \in P,$$

$$[\mathcal{S}_X, \mathcal{Y}]_p = \lim_{t\to 0} \frac{1}{t}\{\mathcal{Y}_p - ((S_{e^{tX}})_*\mathcal{Y})_p\} = \lim_{t\to 0} \frac{1}{t}\{((S_{e^{-tX}})_*\mathcal{Y})_p - \mathcal{Y}_p\} \text{ for all } \mathcal{Y} \in \mathcal{D}^1(P).$$

$\mathcal{R}\colon \mathfrak{g} \to \mathcal{D}^1(P)$ and $-\mathcal{L}\colon \mathfrak{g} \to \mathcal{D}^1(P)$ are LIE algebra homomorphisms and

$$[\mathcal{R}_X, \mathcal{R}_Y] = \mathcal{R}_{[X,Y]}, \quad [\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[Y,X]} = -\mathcal{L}_{[X,Y]} \quad \text{for all } X, Y \in \mathfrak{g},$$

$$(R_{g^{-1}})_* \mathcal{R}_X = \mathcal{R}_{\operatorname{Ad}(g)X}, \quad (L_g)_* \mathcal{L}_X = \mathcal{L}_{\operatorname{Ad}(g)X} \quad \text{for all } g \in G,\ X \in \mathfrak{g}.$$

For the interior product and the LIE differentiation, we get for all $X, Y \in \mathfrak{g}$:

$$[L_{\mathcal{S}_X}, L_{\mathcal{S}_Y}] = \operatorname{sgn}(S) L_{\mathcal{S}_{[X,Y]}}, \quad [L_{\mathcal{S}_X}, d] = 0,$$

$$[L_{\mathcal{S}_X}, \imath_{\mathcal{S}_Y}] = \operatorname{sgn}(S) \imath_{\mathcal{S}_{[Y,X]}}, \quad L_{\mathcal{S}_X} = \imath_{\mathcal{S}_X} \circ d + d \circ \imath_{\mathcal{S}_X}.$$

Definition 1.92 *We call a tensor field K, resp., a differential form ω* $\mathfrak{g}$-invariant *if $L_{\mathcal{S}_X} K = 0$, resp., $L_{\mathcal{S}_X}\omega = 0$ for all $X \in \mathfrak{g}$. Analogously, ω will be called* horizontal *if $\imath_{\mathcal{S}_X}\omega = 0$ for all $X \in \mathfrak{g}$. Denote their sets by $\mathcal{D}(P)_{\mathfrak{g}-\text{inv}}$, $\mathcal{A}(P)_{\mathfrak{g}-\text{inv}}$, resp., $\mathcal{A}(P)h$ and let $\mathcal{A}(P)h_{\mathfrak{g}-\text{inv}} := \mathcal{A}(P)_{\mathfrak{g}-\text{inv}} \cap \mathcal{A}(P)h$.*

The notion of "horizontal" forms will become apparent in Section 2.2.

Proposition 1.93 *$\mathcal{A}(P)_{\mathfrak{g}-\text{inv}}$, $\mathcal{A}(P)h$ and $\mathcal{A}(P)h_{\mathfrak{g}-\text{inv}}$ are graded subalgebras of $\mathcal{A}(P)$ with $d(\mathcal{A}(P)_{\mathfrak{g}-\text{inv}}) \subseteq \mathcal{A}(P)_{\mathfrak{g}-\text{inv}}$ and $d(\mathcal{A}(P)h_{\mathfrak{g}-\text{inv}}) \subseteq \mathcal{A}(P)h_{\mathfrak{g}-\text{inv}}$. Analogous statements hold for $\mathcal{A}(P)_{\mathfrak{g}-\text{inv}} \otimes V$ and $\wedge_V$, etc.*

$\mathcal{A}(P)_{\text{inv}} \otimes V \subseteq \mathcal{A}(P)_{\mathfrak{g}-\text{inv}} \otimes V = \mathcal{A}(P)_{G_1-\text{inv}} \otimes V$ for every vector space V. If G is connected then $\mathcal{A}(P)_{\text{inv}} \otimes V = \mathcal{A}(P)_{\mathfrak{g}-\text{inv}} \otimes V$.

Proof: use Lemma 1.91 and the fact that $\imath_X$ and L_X are (skew-)derivations of $\mathcal{A}(P)$. The last statements follow from $G_1 = \langle \exp \mathfrak{g} \rangle$, cf. [5, II p. 126]. □

Lemma 1.94 *$\mathcal{S}\colon \mathfrak{g} \to \mathcal{D}^1(P)$ induces a G-equivariant $C^\infty(P)$-module homomorphism $\mathcal{S}'\colon C^\infty(P,\mathfrak{g}) \to C^\infty(P)\mathcal{S}(\mathfrak{g}) \subseteq \mathcal{D}^1(P)$ (with respect to S'' and S_*). If G acts effectively on P then $\mathcal{S}$ is injective. If G acts freely on P then even $(dS^p)_e$ is injective for all $p \in P$, thus $X \neq 0$ yields $(\mathcal{S}_X)_p \neq 0$ for all $p \in P$; for every basis $\{E_i\}_{1,\ldots,\dim\mathfrak{g}}$ for $\mathfrak{g}$, $\{\mathcal{S}_{E_i}\}_{1,\ldots,\dim\mathfrak{g}}$ is then a basis for the free $C^\infty(P)$-module $C^\infty(P)\mathcal{S}(\mathfrak{g})$ and the induced $\mathcal{S}'$ is an isomorphism of free $C^\infty(P)$-modules.*

Proof. Assume that G acts effectively. Let $X \in \mathfrak{g}$ and suppose $(\mathcal{S}_X)_p(f) = 0$ for all $f \in C^\infty(P)$ and all $p \in P$. For $p = S(e^{sX}, p')$ this yields $\frac{d}{dt} f(S_{e^{(t+s)X}}(p'))|_{t=0} = \frac{d}{dt} f(S_{e^{tX}}(p'))|_{t=s} = 0$ for all $f \in C^\infty(P)$, $p' \in P$ and $s \in \mathbb{R}$. Thus $S(e^{tX}, p') = p'$ for all $p' \in P$ and $t \in \mathbb{R}$, and thus $X = 0$ since S is effective. Analogously for a free action, one proves injectivity of $(dS^p)_e$ for all $p \in P$ using $(\mathcal{S}_X)_{S(e^{sX},p)} = dS_{e^{sX}}(\mathcal{S}_X)_p$ from Proposition 1.35. But then all $\mathcal{S}_{E_i}$ are independent over $C^\infty(P)$, since they are independent for all $p \in P$. □

Observe that, with respect to Lemma 1.59, we have changed the notation for $\mathcal{L}$ and $\mathcal{R}$, because in general we do not get invariant vector fields on P. Note that on G itself, where $R^g = \lambda_g$, the results for $\mathcal{R}$ in Lemma 1.91 (resp., for $\mathcal{L}$ in Lemma 1.59) yield for all $X, Y \in \mathfrak{g}$:

$$[X,Y] = [\mathcal{R}_X, \mathcal{R}_Y]_e = \lim_{t\to 0} \frac{1}{t}\{Y - ((\rho_{e^{tX}})_* \mathcal{R}_Y)_e\} = \lim_{t\to 0} \frac{1}{t}\{Y - \mathrm{Ad}(e^{-tX})Y\}. \quad (42)$$

Note 1.95 Just as $\mathrm{Ad}\colon G \to \mathrm{Gl}(\mathfrak{g})$ induces the representation $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{gl}(\mathfrak{g})$ in (42), every representation $L'\colon G \to \mathrm{Gl}(V)$ of a LIE group G induces a representation $l' = dL'_e\colon \mathfrak{g} \to \mathrm{gl}(V)$ such that $L' \circ \exp X = e^{l'X}$ for all $X \in \mathfrak{g}$. Thus any (left) linear action L induces a bilinear mapping $l\colon \mathfrak{g} \times V \to V$ with $l_{[X,Y]} = [l_X, l_Y]$ and we obtain l by

$$l(X,v) = (dL^v)_e(X) = \lim_{t\to 0} \frac{1}{t}\{L(e^{tX}, v) - v\} = \lim_{t\to 0} \frac{1}{t}\{v - L(e^{-tX}, v)\} \quad (43)$$

for all $X \in \mathfrak{g}$, $v \in V$. Analogous statements hold for right representations R: the left action $R \circ \eta$ induces a LIE algebra homomorphism $-r'$: $r_{[X,Y]} = [r_Y, r_X]$, where

$$r(X,v) = (d(R \circ \eta)^v)_e(-X) = \lim_{t\to 0} \frac{1}{t}\{R(e^{tX}, v) - v\} = \lim_{t\to 0} \frac{1}{t}\{v - R(e^{-tX}, v)\}. \quad (44)$$

From this point of view, $\mathcal{R}$ and $-\mathcal{L}\colon \mathfrak{g} \to \mathcal{D}^1(P) = \mathrm{der}\, C^\infty(P)$ are the (infinite dimensional) representations induced by the LIE group representations $(R^*)'$ and $(L^* \circ \eta)'\colon G \to \mathrm{Aut}(C^\infty(P))$. Note that $\exp \circ \mathrm{Ad}(g) = I_g \circ \exp$ yields for $s = l$, resp., $s = r$, and all $g \in G, X \in \mathfrak{g}, v \in V$ the following relations:

$$s(X, S(g,v)) = S(g, s(\mathrm{Ad}(g^{\mathrm{sgn}(S)})X, v)), \quad S(g, s(X,v)) = s(\mathrm{Ad}(g^{-\mathrm{sgn}(S)})X, S(g,v)).$$

Identifying L and L', resp., R and R', we thus get the following lemma:

Lemma 1.96 *Let $S\colon G \to \mathrm{Gl}(V)$ be a representation and $s\colon \mathfrak{g} \times V \to V$ be the induced bilinear map according to* Note 1.95. *Then for any differentiable $f\colon M \to G$ and forms $\omega \in \mathcal{A}(M,\mathfrak{g})$ and $\phi \in \mathcal{A}(M) \otimes V$,*

$$(S \circ f) \bullet (\omega \wedge_s \phi) = [(\mathrm{Ad} \circ f^{-\mathrm{sgn}(S)}) \bullet \omega] \wedge_s [(S \circ f) \bullet \phi], \quad (45)$$

$$d[(S \circ f) \bullet \phi] = (S \circ f) \bullet (f^*\Theta^S \wedge_s \phi + d\phi). \quad (46)$$

Proof. Only (46) still needs to be proved. For $S = L$, observe that for all $g \in G$, $L \circ \lambda_g = \lambda'_{L(g)} \circ L$ with $\lambda'_{L(g)}: \mathrm{Gl}(V) \to \mathrm{Gl}(V): A \mapsto L(g) \circ A$. For $\mathcal{X} \in \mathcal{D}^1(M)$ and $x \in M$, this yields $[d(L \circ f)]\mathcal{X}(x) = dL_{f(x)} \circ d\lambda_{f(x)}(f^*\Theta^L)_x \mathcal{X}_x = \lambda'_{L(f(x))} \circ l' \circ (f^*\Theta^L)_x \mathcal{X}_x$ with l' from Note 1.95, and thus $[d(L \circ f)] \bullet \phi = (L \circ f) \bullet (f^*\Theta^L \wedge_l \phi)$. Analogous arguments hold for $S = R$. □

Generally, for a left representiation $L: G \to \mathrm{Gl}(V)$ and induced representation $l: \mathfrak{g} \to \mathrm{gl}(V)$ we compute analogously to Proposition 1.93, cf. [5, II p. 128]:

Proposition 1.97 *If a differential form $\chi \in \mathcal{A}(P) \otimes V$ is G-equivariant in the sense that $S_g^*\chi = L(g^{-\mathrm{sgn}(S)})_*\chi$ for all $g \in G$, then*

$$L_{S_X}\chi = -\,\mathrm{sgn}(S)l(X)_*\chi \qquad \textit{for all} \quad X \in \mathfrak{g},$$

i. e., χ is $\mathfrak{g}$-equivariant. *Forms are $\mathfrak{g}$-equivariant iff they are G_1-equivariant, thus if G is connected, $\mathfrak{g}$-equivariance is equivalent to G-equivariance.*

We will denote the vector space of $\mathfrak{g}$-equivariant forms by $\mathcal{A}_{\mathfrak{g}-\mathrm{equiv}}(P) \otimes V$. It is a $\mathcal{A}_{\mathfrak{g}-\mathrm{inv}}(P)$-module.

Recall $L^S: \mathfrak{g} \times \mathrm{Alt}(\mathfrak{g}, V) \to \mathrm{Alt}(\mathfrak{g}, V)$ from Proposition 1.79. We will use L^S also for the corresponding map $L^S: \mathfrak{g} \times \mathrm{Hom}(\mathcal{T}(\mathfrak{g}), V) \to \mathrm{Hom}(\mathcal{T}(\mathfrak{g}), V)$, that is defined in total analogy to (34).

Lemma 1.98 *According to* Note 1.95, *L^L, resp., L^R are the bilinear mappings induced by* $(\mathrm{Ad} \circ \eta)^*$, *resp.,* $(\mathrm{Ad})^*: G \times \mathrm{Hom}(\mathcal{T}(\mathfrak{g}), V) \to \mathrm{Hom}(\mathcal{T}(\mathfrak{g}), V)$.

Proof. For $X, E_i \in \mathfrak{g}$ and $K \in \mathrm{Alt}_s(\mathfrak{g}, V)$, $[\frac{d}{dt}\mathrm{Ad}(e^{\mathrm{sgn}(S)tX})^* K|_{t=0}](E_1 \otimes \cdots \otimes E_s) =$

$$\begin{aligned}
&= \frac{d}{dt}[K(\mathrm{Ad}(e^{\mathrm{sgn}(S)tX})E_1 \otimes \cdots \otimes \mathrm{Ad}(e^{\mathrm{sgn}(S)tX})E_s)]|_{t=0} \\
&= \mathrm{sgn}(S)\sum_{j=1}^{s} K(E_1 \otimes \cdots \otimes E_{j-1} \otimes \{\frac{d}{dt}[\mathrm{Ad}(e^{tX})E_j]|_{t=0}\} \otimes E_{j+1} \otimes \cdots \otimes E_s) \\
&= \mathrm{sgn}(S)\sum_{j=1}^{s} K(E_1 \otimes \cdots \otimes E_{j-1} \otimes [X, E_j] \otimes E_{j+1} \otimes \cdots \otimes E_s)
\end{aligned}$$

$= (L_X^S K)(E_1 \otimes \cdots \otimes E_s)$ by (42) and (34). □

Let us now return to the induced, complete vector fields $\mathcal{S}_X$. (16) and (17) yield:

Proposition 1.99 *Let $\omega \in \mathcal{A}_n(P) \otimes V$, $X \in \mathfrak{g}$ and $\mathcal{P}^i \in \mathcal{D}^1(P)$. Then*

$$\mathcal{S}_X(\omega(\mathcal{P}^1, \ldots, \mathcal{P}^n)) = [\frac{d}{dt}((S_{e^{tX}})^*\omega)|_{t=0}](\mathcal{P}^1, \ldots, \mathcal{P}^n) + \sum_{i=1}^{n} \omega(\mathcal{P}^1, \ldots, [\mathcal{S}_X, \mathcal{P}^i], \ldots, \mathcal{P}^n).$$

Corollary 1.100 *If $\omega \in \mathcal{A}_n(P)_{\mathfrak{g}-\mathrm{inv}} \otimes V$, then for all $X \in \mathfrak{g}$*

$$\mathcal{S}_X(\omega(\mathcal{P}^1, \ldots, \mathcal{P}^n)) = \sum_{i=1}^{n} \omega(\mathcal{P}^1, \ldots, [\mathcal{S}_X, \mathcal{P}^i], \ldots, \mathcal{P}^n).$$

Lemma 1.98 proves the following corollary to Propositions 1.97 and 1.99:

Corollary 1.101 *Let $\chi_n^s \in \mathcal{A}_n(P) \otimes \mathrm{Hom}(\bigotimes^s \mathfrak{g}, V)$ be $\mathfrak{g}$-equivariant. Then for all $p \in P$, $\mathcal{P}^i \in \mathcal{D}^1(P)$ and $X, E_i \in \mathfrak{g}$:*

$$
\begin{aligned}
&(L_{\mathcal{S}_X}\chi_n^s)(\mathcal{P}^1, \ldots, \mathcal{P}^n)(p)(E_1 \otimes \cdots \otimes E_s) = \\
&= \left\{\mathcal{S}_X(\chi_n^s(\mathcal{P}^1, \ldots, \mathcal{P}^n)) - \sum_{i=1}^{n} \chi_n^s(\mathcal{P}^1, \ldots, [\mathcal{S}_X, \mathcal{P}^i], \ldots, \mathcal{P}^n)\right\}(p)(E_1 \otimes \cdots \otimes E_s) \\
&= \operatorname{sgn}(S) \sum_{j=1}^{s} \chi_n^s(\mathcal{P}^1, \ldots, \mathcal{P}^n)(p)(E_1 \otimes \cdots \otimes E_{j-1} \otimes [X, E_j] \otimes E_{j+1} \otimes \cdots \otimes E_s).
\end{aligned}
$$

Definition 1.102 *Let S be a* LIE *group action of G on P and $\omega_n \in \mathcal{A}_n(P, V)$. We define $S_\bullet^i\omega_n \in \mathcal{A}_{n-i}(P, \mathrm{Alt}_i(\mathfrak{g}, V))$, $i \le n$, for all $\mathcal{P}^j \in \mathcal{D}^1(P)$, $E_k \in \mathfrak{g}$ and $p \in P$ by*

$$[(S_\bullet^i\omega_n)(\mathcal{P}^1, \ldots, \mathcal{P}^{n-i})(p)](E_1, \ldots, E_i) := \tfrac{n!}{(n-i)!}\omega_n(\mathcal{S}^1, \ldots, \mathcal{S}^i, \mathcal{P}^1, \ldots, \mathcal{P}^{n-i})(p) \in V,$$

where $\mathcal{S}^i := \mathcal{S}_{E_i}$. Thus $S_\bullet^i\omega_n \in \mathcal{A}_{n-i}(P) \otimes \mathrm{Alt}_i(\mathfrak{g}, V)$ if $\omega_n \in \mathcal{A}_n(P) \otimes V$. For $i > n$ we put $S_\bullet^i\omega_n = 0$.

$S_\bullet^i\omega_n$ is well-defined: since $\mathfrak{g}$ is finite dimensional, $\mathrm{Alt}_i(\mathfrak{g}, V)^* \cong \bigwedge^i \mathfrak{g} \otimes V^*$ by Lemma 1.13, so every $\varphi \in \mathrm{Alt}_i(\mathfrak{g}, V)^*$ may be written as $\sum_{k=0}^{N}(E_1^k \wedge \cdots \wedge E_i^k) \otimes v_k^*$ with $v_k^* \in V^*$ and $N = \binom{\dim \mathfrak{g}}{i}$. But $[((E_1 \wedge \cdots \wedge E_i) \otimes v)^* \circ (S_\bullet^i\omega_n)(\mathcal{P}^1, \ldots, \mathcal{P}^{n-i})](p) =$

$$= v^*[(S_\bullet^i\omega_n)(\mathcal{P}^1, \ldots, \mathcal{P}^{n-i})(p)](E_1, \ldots, E_i) = \tfrac{n!}{(n-i)!}[v^* \circ \omega_n(\mathcal{S}^1, \ldots, \mathcal{S}^i, \mathcal{P}^1, \ldots, \mathcal{P}^{n-i})](p),$$

so $\omega_n \in \mathcal{A}_n(P, V)$ yields $\varphi \circ (S_\bullet^i\omega_n)(\mathcal{P}^1, \ldots, \mathcal{P}^{n-i}) \in C^\infty(M)$ for all $\varphi \in \mathrm{Alt}_i(\mathfrak{g}, V)^*$. If $\{E_k\}$ is a base for $\mathfrak{g}$, we obtain for $\omega \in \mathcal{A}_n(P)$ and $v \in V$:

$$[S_\bullet^i(\omega \otimes v)](\mathcal{P}^1, \ldots, \mathcal{P}^{n-i}) = \tfrac{n!}{(n-i)!}\sum_{k_1 < \cdots < k_i} \omega(\mathcal{S}^{k_1}, \ldots, \mathcal{S}^{k_i}, \mathcal{P}^1, \ldots, \mathcal{P}^{n-i}) \otimes [(E_{k_1} \wedge \cdots \wedge E_{k_i}) \mapsto v].$$

Lemma 1.103 *For all $i \le n$, $S_\bullet^i\colon \mathcal{A}_n(P, V) \to \mathcal{A}_{n-i}(P, \mathrm{Alt}_i(\mathfrak{g}, V))$ is $C^\infty(P)$-linear. For $\omega_n \in \mathcal{A}_n(P, V)$, $\chi_n^s \in \mathcal{A}_n(P, \mathrm{Alt}_s(\mathfrak{g}, V))$ and $i + j \le n$, we have*

$$
\begin{aligned}
S_\bullet^0\omega_n &= \omega_n, \qquad (S_\bullet^n\omega_n)(p) = n!\,[(S^p)^*\omega_n]_e \quad \textit{for all} \quad p \in P, && (47)\\
S_\bullet^i(\Lambda_*\omega_n) &= (\Lambda_\circ)_*(S_\bullet^i\omega_n) \qquad \textit{for all} \quad \Lambda \in \mathrm{Hom}(V, W), && (48)\\
S_g^*(S_\bullet^i\omega_n) &= (\mathrm{Ad}(g^{\operatorname{sgn}(S)})^*)_*[S_\bullet^i(S_g^*\omega_n)], \qquad \textit{thus} && (49)\\
S_g^*(S_\bullet^i\chi_n^s) &= (\mathrm{Ad}(g^{\operatorname{sgn}(S)})^*)_*(S_\bullet^i\chi_n^s), \quad \textit{if} \quad S_g^*\chi_n^s = (\mathrm{Ad}(g^{\operatorname{sgn}(S)})^*)_*\chi_n^s. && (50)
\end{aligned}
$$

Let $f^{i,j}\colon \mathrm{Alt}_{i+j}(\mathfrak{g}, V) \hookrightarrow \mathrm{Alt}_i(\mathfrak{g}, \mathrm{Alt}_j(\mathfrak{g}, V))$ denote the injection defined by

$$[f^{i,j}(a)](E_1, \ldots, E_i)(F_1, \ldots, F_j) := a(E_1, \ldots, E_i, F_1, \ldots, F_j) \quad \textit{for all } a \in \mathrm{Alt}_{i+j}(\mathfrak{g}, V)$$

(cf. the canonical isomorphism from Lemma 1.13.1). Then

$$f_*^{i,j}(S_\bullet^{i+j}\omega_n) = (-1)^{ij} S_\bullet^i(S_\bullet^j\omega_n). \tag{51}$$

(50) yields that $S^i_\bullet\omega_n$ is G-equivariant if ω_n is invariant under S, thus restriction of S to $G_1 \times P$ proves that $S^i_\bullet\omega_n$ is $\mathfrak{g}$-equivariant if ω_n is $\mathfrak{g}$-invariant.

Theorem 1.104 *Let S be a* LIE *group action of G on P, S' a representation of G on V on the same side and $\omega \in \mathcal{A}_n(P,V)$ with $S_h^*\omega = (S'_h)_*\omega$ for all $h \in G$. If $g\colon M \to G$ and $f\colon M \to P$ are differentiable, then $[S \circ (g,f)]^*\omega_n =$*

$$\sum_{i=0}^{n} \frac{(-1)^{i(n-i)}}{i!} S'_g \bullet [f^*(S^i_\bullet\omega_n) \bullet g^*\Theta_G^S] = \sum_{i=0}^{n} \frac{(-1)^{i(n-i)}}{i!} [S'_g \bullet f^*(S^i_\bullet\omega_n)] \bullet g^*\Theta_G^S.$$

Proof. Let $\mathcal{X}^i \in \mathcal{D}^1(M)$ and $x \in M$. Then by (13) $[S\circ(g,f)]^*\omega_n(\mathcal{X}^1,\ldots,\mathcal{X}^n)(x) =$

$$\begin{aligned}
&= \omega_{S(g(x),f(x))}(\ldots,[(dS_{g(x)})_{f(x)}df_x + (dS^{f(x)})_{g(x)}dg_x]\mathcal{X}^i_x,\ldots)\\
&= (S^*_{g(x)}\omega)_{f(x)}(\ldots,df_x\mathcal{X}^i_x + d(S_{g^{-1}(x)} \circ S^{f(x)})_{g(x)}dg_x\mathcal{X}^i_x,\ldots)\\
&= S'_{g(x)} \circ [\omega_{f(x)}(\ldots,df_x\mathcal{X}^i_x + (dS^{f(x)})_e(g^*\Theta_G^S)_x\mathcal{X}^i_x,\ldots)]\\
&= S'_{g(x)} \circ \Big[\sum_{i=0}^{n}\binom{n}{i}\sum_{\rho\in S_n}\tfrac{(-1)^\rho}{n!}\,\omega_{f(x)}((dS^{f(x)})_e(g^*\Theta_G^S)_x\mathcal{X}_x^{\rho(1)},\ldots,df_x\mathcal{X}_x^{\rho(i+1)},\ldots,df_x\mathcal{X}_x^{\rho(n)})\Big]\\
&= \sum_{i=0}^{n}\tfrac{1}{i!}S'_{g(x)} \circ \Big\{\sum_{\rho\in S_n}\tfrac{(-1)^\rho}{n!}(S^i_\bullet\omega)_{f(x)}(df_x\mathcal{X}_x^{\rho(i+1)},\ldots,df_x\mathcal{X}_x^{\rho(n)})[(g^*\Theta_G^S)_x\mathcal{X}_x^{\rho(1)},\ldots]\Big\}\\
&= \sum_{i=0}^{n}\tfrac{(-1)^{i(n-i)}}{i!}S'_{g(x)} \circ \{[f^*(S^i_\bullet\omega) \bullet g^*\Theta_G^S](\mathcal{X}^1,\ldots,\mathcal{X}^n)(x)\}\\
&= \Big\{\sum_{i=0}^{n}\tfrac{(-1)^{i(n-i)}}{i!}S'_g \bullet [f^*(S^i_\bullet\omega) \bullet g^*\Theta_G^S]\Big\}(\mathcal{X}^1,\ldots,\mathcal{X}^n)(x).
\end{aligned}$$

In the third step we used (39) and Definition 1.65, then antisymmetry of ω, linearity of S' and finally Definition 1.43. Now the other equality follows from (25). □

Suppose that under the previous conditions, $(S^p)^*\omega$ is independent of $p \in P$. Then by (38), $(S^p)^*\omega \in \mathcal{A}_n(G,V)$ is invariant: $(L^p)^*\omega = \psi^R(K)$, resp., $(R^p)^*\omega = \psi^L(K)$ for a $K \in \mathrm{Alt}_n(\mathfrak{g},V)$. Moreover, (39) in combination with (31) yields $(S'_g)_\circ K = \mathrm{Ad}(g^{-\operatorname{sgn} S})^*K$, so for the $i = n$ term in Theorem 1.104 we get from (47) and Lemma 1.49 ($-S := R$ for $S = L$, and vice versa):

$$\begin{aligned}
S'_g \bullet [f^*(S^n_\bullet\omega_n) \bullet g^*\Theta_G^S] &= [n! \otimes ((S'_g)_\circ K)] \bullet g^*\Theta_G^S = [(n! \otimes (\mathrm{Ad}\circ g^{-\operatorname{sgn} S})^*K] \bullet g^*\Theta_G^S\\
&= n!\, g^*\psi^{-S}(K).
\end{aligned}$$

The $i = 0$ term reads $S'_g \bullet f^*\omega$, so for $\omega \in \mathcal{A}_1(G,\mathfrak{g})$ we obtain

Corollary 1.105 *Let $L, R\colon G \times P \to P$ be a left, resp., right action of G on P and $f\colon M \to P$ and $g\colon M \to G$ be differentiable; $K \in \mathrm{Alt}_1(\mathfrak{g},\mathfrak{g})$ be invertible and $\omega \in \mathcal{A}_1(P,\mathfrak{g})$. Then $K(\mathrm{Ad}\circ g)K^{-1} \in \mathcal{A}_0(M,\mathrm{Alt}_1(\mathfrak{g},\mathfrak{g}))$ and we have*

1. *If $(L^p)^*\omega = \psi^R(K)$ and $L_c^*\omega = K\,\mathrm{Ad}(c)K^{-1} \circ \omega$ for all $p \in P$, $c \in G$, then*

$$[L \circ (g,f)]^*\omega = K(\mathrm{Ad}\circ g)K^{-1} \bullet f^*\omega + g^*\psi^R(K).$$

2. If $(R^p)^*\omega = \psi^L(K)$ *and* $R_c^*\omega = K\,\mathrm{Ad}(c^{-1})K^{-1}\circ\omega$ *for all* $p \in P$, $c \in G$, *then*

$$[R\circ(g,f)]^*\omega = K(\mathrm{Ad}\circ\eta\circ g)K^{-1}\bullet f^*\omega + g^*\psi^L(K).$$

Proof. If $K \in \mathrm{Alt}_1(\mathfrak{g},\mathfrak{g})$ is invertible, $S'_g = K\circ\mathrm{Ad}(g^{-\mathrm{sgn}(S)})\circ K^{-1}$. □

Corollary 1.105 gives a proof for Theorem 1.70 above: put $P = G$ and $K = \mathrm{Ad}(h)$, resp., $K = \mathrm{id}_{\mathfrak{g}}$ and observe that $(f\cdot f^{-1})^* = e^* = 0$, where $e\colon M \to \{e\} \subseteq G$ is the constant map onto the neutral element.

Recall $(S^i_\bullet\omega_n)^{E_1,\dots,E_i}_{n-i} \in \mathcal{A}_{n-i}(P,V)$ for $E_k \in \mathfrak{g}$ from Definition 1.42. We have

$$(S^i_\bullet\omega_n)^{E_1,\dots,E_i}_{n-i} = (\imath_{S^i}\circ\cdots\circ\imath_{S^1})\omega_n. \tag{52}$$

Lemma 1.106 *Let S be a* LIE *group action of G on P. For all $\omega_n \in \mathcal{A}_n(P)\otimes V$, $i \le n+1$ and $E_k \in \mathfrak{g}$ we have* $\{S^i_\bullet(d\omega_n) - (-1)^i d(S^i_\bullet\omega_n)\}^{E_1,\dots,E_i}_{n+1-i} =$

$$= -\sum_{j=1}^{i}(-1)^j\Big\{[S^{i-1}_\bullet(L_{S^j}\omega_n)]^{E_1,\dots,\widehat{E_j},\dots,E_i}_{n+1-i} + \mathrm{sgn}(S)\sum_{k=j+1}^{i}(S^{i-1}_\bullet\omega_n)^{E_1,\dots,\widehat{E_j},\dots,[E_j,E_k],\dots,E_i}_{n+1-i}\Big\}$$

$$= -\sum_{j=1}^{i}(-1)^j\Big\{[L_{S^j}(S^{i-1}_\bullet\omega_n)]^{E_1,\dots,\widehat{E_j},\dots,E_i}_{n+1-i} - \mathrm{sgn}(S)\sum_{k=j+1}^{i}(S^{i-1}_\bullet\omega_n)^{E_1,\dots,\widehat{E_j},\dots,[E_j,E_k],\dots,E_i}_{n+1-i}\Big\}.$$

Proof. Since $d\circ\imath_{\mathcal{X}}+\imath_{\mathcal{X}}\circ d = L_{\mathcal{X}}$ and $[L_{\mathcal{X}},\imath_{\mathcal{Y}}] = \imath_{[\mathcal{X},\mathcal{Y}]}$ for all $\mathcal{X},\mathcal{Y} \in \mathcal{D}^1(P)$, we get by induction: $\{S^i_\bullet(d\omega_n)-(-1)^i d(S^i_\bullet\omega_n)\}^{E_1,\dots,E_i}_{n+1-i} = -\sum_{j=1}^{i}(-1)^j(\imath_{S^i}\circ\cdots\circ L_{S^j}\circ\cdots\circ\imath_{S^1})\omega_n$

$$= -\sum_{j=1}^{i}(-1)^j(\cdots\circ\widehat{\imath_{S^j}}\circ\cdots)(L_{S^j}\omega_n) - \sum_{j=1}^{i}(-1)^j\sum_{k=1}^{j-1}(\imath_{S^i}\circ\cdots\circ\widehat{\imath_{S^j}}\circ\cdots\circ\imath_{[S^j,S^k]}\circ\cdots\circ\imath_{S^1})\omega_n.$$

Interchanging j and k in the last sum and $[S^j,S^k] = \mathrm{sgn}(S)\mathcal{S}_{[E_j,E_k]}$ from Lemma 1.91 yield the first equation. The second is proved analogously. □

If $\chi^s_n \in \mathcal{A}_n(P)\otimes\mathrm{Hom}(\bigotimes^s\mathfrak{g},V)$ is $\mathfrak{g}$-equivariant, Corollary 1.101 yields

$$[S^{i-1}_\bullet(L_{S^j}\chi^s_n)]^{E_1,\dots,\widehat{E_j},\dots,E_{i+s}}_{n+1-i} = \mathrm{sgn}(S)\sum_{k=1}^{s}(S^{i-1}_\bullet\chi^s_n)^{E_1,\dots,\widehat{E_j},\dots,[E_j,E_{i+k}],\dots,E_{i+s}},$$

(we again identify $\mathrm{Hom}(\bigotimes^{i+s}\mathfrak{g},V)$ and $\mathrm{Hom}(\bigotimes^i\mathfrak{g},\mathrm{Hom}(\bigotimes^s\mathfrak{g},V))$). Thus we have:

Corollary 1.107 *For all $\mathfrak{g}$-equivariant $\chi^s_n \in \mathcal{A}_n(P)\otimes\mathrm{Hom}(\bigotimes^s\mathfrak{g},V)$ and $i \le n+1$,*

$$\begin{aligned}\{[S^i_\bullet(d\chi^s_n)] - (-1)^i d(S^i_\bullet\chi^s_n)\}^{E_1,\dots,E_{i+s}}_{n+1-i} &= \\ = -\mathrm{sgn}(S)\sum_{j=1}^{i}\sum_{k=j+1}^{i+s}(-1)^j(S^{i-1}_\bullet\chi^s_n)^{E_1,\dots,\widehat{E_j},\dots,[E_j,E_k],\dots,E_{i+s}}_{n+1-i}&.\end{aligned}$$

Thus for $\mathfrak{g}$-invariant ω_n, $d\omega_n = 0$ yields $d(S_\bullet\omega_n) = 0$, too.

Analogously one proves:

Lemma 1.108 *If $\omega_n \in \mathcal{A}_n(P) \otimes V$ and $i \leq n$, then for all $\mathcal{X} \in \mathcal{D}^1(P)$ and $E_k \in \mathfrak{g}$*

$$\begin{aligned}
[S_\bullet^i(\imath_{\mathcal{X}}\omega_n)]_{n-1-i}^{E_1,\ldots,E_i} &= (-1)^i [\imath_{\mathcal{X}}(S_\bullet^i\omega_n)]_{n-1-i}^{E_1,\ldots,E_i}, \\
[S_\bullet^i(L_{\mathcal{X}}\omega_n) - L_{\mathcal{X}}(S_\bullet^i\omega_n)]_{n-i}^{E_1,\ldots,E_i} &= \sum_{j=1}^{i}(-1)^j [S_\bullet^{i-1}(\imath_{[\mathcal{X},S_j]}\omega_n)]_{n-i}^{E_1,\ldots,\widehat{E_j},\ldots,E_i}.
\end{aligned}$$

If $\mathcal{X} = \mathcal{S}_X$ with $X \in \mathfrak{g}$, we get $[S_\bullet^i(\imath_{\mathcal{S}_X}\omega_n)]_{n-1-i}^{E_1,\ldots,E_i} = (S_\bullet^{i+1}\omega_n)_{n-1-i}^{X,E_1,\ldots,E_i}$,

$$[S_\bullet^i(L_{\mathcal{S}_X}\omega_n) - L_{\mathcal{S}_X}(S_\bullet^i\omega_n)]_{n-i}^{E_1,\ldots,E_i} = -\operatorname{sgn}(S)\sum_{j=1}^{i}(S_\bullet^i\omega_n)_{n-i}^{E_1,\ldots,[X,E_j],\ldots,E_i}.$$

Lemma 1.109 *Let $\chi_n^1 \in \mathcal{A}_n(P, \mathrm{Hom}(\mathfrak{g}, V))$ and $\{E_k\}_{k=1,\ldots,\dim\mathfrak{g}}$ be a basis for $\mathfrak{g}$. Then for $\theta_q = \sum_{k=1}^{\dim\mathfrak{g}} \theta_q^k \otimes E_k \in \mathcal{A}_q(P,\mathfrak{g})$ and $\phi_p = \sum_{l=1}^{\dim\mathfrak{g}} \phi_p^l \otimes E_l \in \mathcal{A}_p(P,\mathfrak{g})$,*

$$\chi_n^1 \bullet (\theta_q \wedge_{\mathfrak{g}} \phi_p) = \sum_{j=1}^{\dim\mathfrak{g}} \chi_n^{E_j} \wedge (\theta_q \wedge_{\mathfrak{g}} \phi_p)^j = \sum_{k,l=1}^{\dim\mathfrak{g}} \chi_n^{[E_k,E_l]} \wedge \theta_q^k \wedge \phi_p^l. \tag{53}$$

Proof. Be $[E_k, E_l] = \sum_{j=1}^{\dim\mathfrak{g}} c_{kl}^j E_j$ with structure constants c_{kl}^j. Then by Definition 1.22, $\theta_q \wedge_{\mathfrak{g}} \phi_p = \sum_{k,l=1}^{\dim\mathfrak{g}} \theta_q^k \wedge \phi_p^l \otimes [E_k, E_l] = \sum_{j,k,l=1}^{\dim\mathfrak{g}} c_{kl}^j \theta_q^k \wedge \phi_p^l \otimes E_j$, thus

$$\chi_n^1 \bullet (\theta_q \wedge_{\mathfrak{g}} \phi_p) = \sum_{j=1}^{\dim\mathfrak{g}} \chi_n^{E_j} \wedge (\theta_q \wedge_{\mathfrak{g}} \phi_p)^j = \sum_{j,k,l=1}^{\dim\mathfrak{g}} c_{kl}^j \chi_n^{E_j} \wedge \theta_q^k \wedge \phi_p^l = \sum_{k,l=1}^{\dim\mathfrak{g}} \chi_n^{\sum_{j=1}^{\dim\mathfrak{g}} c_{kl}^j E_j} \wedge \theta_q^k \wedge \phi_p^l,$$

where we used Lemma 1.44 and (19). □

Proposition 1.110 *Let S be a* Lie *group action of G on P, $\theta_q \in \mathcal{A}_q(P,\mathfrak{g})$, $\phi_p \in \mathcal{A}_p(P,\mathfrak{g})$ and $\chi_n^s \in \mathcal{A}_n(P) \otimes \mathrm{Hom}(\bigotimes^s \mathfrak{g}, V)$ $\mathfrak{g}$-equivariant. Then for all $i \leq n+1$ with $\ell = (-1)^{q-1}$*

$$\begin{aligned}
&\Big\{\Big[d(S_\bullet^i\chi_n^s) - (-1)^i S_\bullet^i(d\chi_n^s)\Big]_{n+1}^{i;s} \blacktriangleleft \theta_q\Big\}^s \bullet \phi_p = \\
&= \operatorname{sgn}(S)\Big\{ -\tbinom{i}{2}_\ell \Big\{\Big[(S_\bullet^{i-1}\chi_n^s)_{n+1-i}^{i-2;s+1} \blacktriangleleft \theta_q\Big]^{1;s} \blacktriangleleft (\theta_q \wedge_{\mathfrak{g}} \theta_q)\Big\}^s \bullet \phi_p + \\
&\quad + \tbinom{i}{1}_\ell \sum_{k=1}^{s}(-1)^{qp(k-1)}\Big\{\Big[[(S_\bullet^{i-1}\chi_n^s)_{n+1-i}^{i-1;s} \blacktriangleleft \theta_q]^{k-1;s-k+1} \blacktriangleleft \phi_p\Big]^{1;s-k} \blacktriangleleft (\theta_q \wedge_{\mathfrak{g}} \phi_p)\Big\}^{s-k} \bullet \phi_p\Big\}.
\end{aligned}$$

Proof. With the notation of the previous lemma, we evaluate the left side using Lemma 1.44. Then by Corollary 1.107,

$$\begin{aligned}
&\sum_{l_1,\ldots,l_{i+s}}^{\dim\mathfrak{g}} \{d(S_\bullet^i\chi_n^s) - (-1)^i S_\bullet^i(d\chi_n^s)\}_{n+1-i}^{E_{l_1},\ldots,E_{l_{i+s}}} \wedge \cdots \wedge \theta_q^{l_i} \wedge \phi_p^{l_{i+1}} \wedge \cdots \wedge \phi_p^{l_{i+s}} = \\
&= \operatorname{sgn}(S)\sum_{j=1}^{i}\sum_{k=j+1}^{i+s}(-1)^{i+j}\sum_{l_1,\ldots,l_{i+s}}^{\dim\mathfrak{g}}(S_\bullet^{i-1}\chi_n^s)_{n+1-i}^{E_{l_1},\ldots,\widehat{E_{l_j}},\ldots,[E_{l_j},E_{l_k}],\ldots,E_{l_{i+s}}} \wedge \cdots \wedge \theta_q^{l_i} \wedge \phi_p^{l_{i+1}} \wedge \cdots
\end{aligned}$$

$$
\begin{aligned}
= -\operatorname{sgn}(S)\sum_{j=1}^{i}\sum_{k=j+1}^{i}\ell^{k-j+1}\sum_{l_1,\dots,l_{i+s}}^{\dim\mathfrak{g}}&(S_\bullet^{i-1}\chi_n^s)_{n+1-i}^{E_{l_1},\dots,\widehat{E_{l_j}},\dots,\widehat{E_{l_k}},\dots,E_{l_i},[E_{l_j},E_{l_k}];\dots,E_{l_{i+s}}}\wedge\\
&\wedge\theta_q^{l_1}\wedge\cdots\widehat{\theta_q^{l_j}}\cdots\widehat{\theta_q^{l_k}}\cdots\wedge\theta_q^{l_i}\wedge\theta_q^{l_j}\wedge\theta_q^{l_k}\wedge\phi_p^{l_{i+1}}\wedge\cdots\wedge\phi_p^{l_{i+s}}\\
+\operatorname{sgn}(S)\sum_{j=1}^{i}\ell^{i-j}\sum_{k=1}^{s}(-1)^{qp(k-1)}\sum_{l_1,\dots,l_{i+s}}^{\dim\mathfrak{g}}&(S_\bullet^{i-1}\chi_n^s)_{n+1-i}^{E_{l_1},\dots,\widehat{E_{l_j}},\dots,E_{l_i};\dots,[E_{l_j},E_{l_{i+k}}],\dots,E_{l_{i+s}}}\wedge\\
&\wedge\theta_q^{l_1}\wedge\cdots\widehat{\theta_q^{l_j}}\cdots\wedge\theta_q^{l_i}\wedge\phi_p^{l_{i+1}}\wedge\cdots\wedge\theta_q^{l_j}\wedge\phi_p^{l_{i+k}}\wedge\cdots\wedge\phi_p^{l_{i+s}}\\
= -\operatorname{sgn}(S)\sum_{j=1}^{i}\sum_{k=j+1}^{i}\ell^{k-j+1}\sum_{l_1,\dots,l_{i+s-1}}^{\dim\mathfrak{g}}&(S_\bullet^{i-1}\chi_n^s)_{n+1-i}^{E_{l_1},\dots,E_{l_{i-1}};E_{l_i},\dots,E_{l_{i+s}}}\wedge\\
&\wedge\theta_q^{l_1}\wedge\cdots\wedge\theta_q^{l_{i-2}}\wedge(\theta_q\wedge_{\mathfrak{g}}\theta_q)^{l_{i-1}}\wedge\phi_p^{l_i}\wedge\cdots\wedge\phi_p^{l_{i+s-1}}\\
+\operatorname{sgn}(S)\sum_{j=1}^{i}\ell^{i-j}\sum_{k=1}^{s}(-1)^{qp(k-1)}\sum_{l_1,\dots,l_{i+s-1}}^{\dim\mathfrak{g}}&(S_\bullet^{i-1}\chi_n^s)_{n+1-i}^{\bar{E}_{l_1},\dots,\bar{E}_{l_{i-1}};E_{l_i},\dots,E_{l_{i+s}}}\wedge\\
&\wedge\theta_q^{l_1}\wedge\cdots\wedge\theta_q^{l_{i-1}}\wedge\phi_p^{l_i}\wedge\cdots\wedge(\theta_q\wedge_{\mathfrak{g}}\phi_p)^{l_{i+k-1}}\wedge\cdots\wedge\phi_p^{l_{i+s-1}},
\end{aligned}
$$

by (53). Since $\sum_{j=1}^{i}\sum_{k=j+1}^{i}\ell^{k-j+1}=\binom{i}{2}_\ell$ and $\sum_{j=1}^{i}\ell^{i-j}=\binom{i}{1}_\ell$, all follows from Lemma 1.52. □

Corollary 1.111 *Suppose $\theta\in\mathcal{A}_1(P,\mathfrak{g})$ and $\chi_n^s\in\mathcal{A}_n(P)\otimes\operatorname{Sym}_s^\varsigma(\mathfrak{g},V)$ in* Proposition 1.110, *then with $\ell=\varsigma(-1)^p$ for all $i\leq n+1$*

$$
\begin{aligned}
\Big\{\big[d(S_\bullet^i\chi_n^s)\big]^{i;s}\blacktriangleleft\theta\Big\}^s\bullet\phi_p-(-1)^i\Big\{\big[S_\bullet^i(d\chi_n^s)\big]^{i;s}\blacktriangleleft\theta\Big\}^s\bullet\phi_p &=\\
= \ -\operatorname{sgn}(S)\ \binom{i}{2}_\ell\Big\{\big[(S_\bullet^{i-1}\chi_n^s)_{n+1-i}^{i-2;s+1}\blacktriangleleft\theta\big]^{1;s}\blacktriangleleft(\theta\wedge_{\mathfrak{g}}\theta)\Big\}^s\bullet\phi_p&\\
+\operatorname{sgn}(S)\ i\ \binom{s}{1}_\ell\Big\{\big[(S_\bullet^{i-1}\chi_n^s)^{i-1;s}\blacktriangleleft\theta\big]^{1;s-1}\blacktriangleleft(\theta\wedge_{\mathfrak{g}}\phi_p)\Big\}^{s-1}\bullet\phi_p.&
\end{aligned}
$$

Proof: immediately from Lemma 1.53 and $\sum_{k=1}^{s}\ell^{k-1}=\binom{s}{1}_\ell$. □

Definition 1.112 *Let S be a* LIE *group action of G on P. Then for $\omega_n\in\mathcal{A}_n(P,V)$ and $\theta\in\mathcal{A}_1(P,\mathfrak{g})$ we define*

$$\omega_n\circledS\theta:=\sum_{i=0}^{n}\frac{(-1)^{i(n-i)}}{i!}(S_\bullet^i\omega_n)\bullet\theta\in\mathcal{A}_n(P,V).$$

Analogously, for $f\colon M\to P$ and $\theta\in\mathcal{A}_1(M,\mathfrak{g})$, resp., linear $\Lambda\colon V\to W$ we write

$$
\begin{aligned}
(f^*\omega_n)\circledS\theta &:= \sum_{i=0}^{n}\frac{(-1)^{i(n-i)}}{i!}f^*(S_\bullet^i\omega_n)\bullet\theta\in\mathcal{A}_n(M,V), \quad \textit{resp.,}\\
(\Lambda_*\omega_n)\circledS\theta &:= \sum_{i=0}^{n}\frac{(-1)^{i(n-i)}}{i!}\Lambda_*[(S_\bullet^i\omega_n)\bullet\theta]\in\mathcal{A}_n(P,W), \quad \textit{etc.}
\end{aligned}
$$

Thus the result from Theorem 1.104 may be written as

$$[S\circ(g,f)]^*\omega_n \ = \ (S'_g\bullet f^*\omega_n)\circledS g^*\Theta_G^S. \tag{54}$$

Theorem 1.113 *Let S be a* LIE *group action of G on P, $\theta \in \mathcal{A}_1(P,\mathfrak{g})$, $\phi_p \in \mathcal{A}_p(P,\mathfrak{g})$ and $\chi_n^s \in \mathcal{A}_n(P) \otimes \mathrm{Sym}_s^\varsigma(\mathfrak{g},V)$ $\mathfrak{g}$-equivariant. If $\ell := \varsigma(-1)^p$, then*

$$\begin{aligned} d[(\chi_n^s \circledS \theta) \bullet \phi_p] - [(d\chi_n^s) \circledS \theta] \bullet \phi_p &= \\ = \ &\Big\{\Big[(S_\bullet \chi_n^s) \circledS \theta\Big]^{1;s} \blacktriangleleft (d\theta - \mathrm{sgn}(S)\, \tfrac{1}{2}\, \theta \wedge_{\mathfrak{g}} \theta)\Big\}^s \bullet \phi_p \\ &+(-1)^n \tbinom{s}{1}_\ell \Big[(\chi_n^s \circledS \theta)^{1;s-1} \blacktriangleleft (d\phi_p - \mathrm{sgn}(S)\, \theta \wedge_{\mathfrak{g}} \phi_p)\Big]^{s-1} \bullet \phi_p . \end{aligned}$$

Proof. By linearity of d and $\bullet$ in its left argument we obtain for the left side

$$\begin{aligned} &\sum_{i=0}^{n} \tfrac{(-1)^{in-i}}{i!}\, d\{[(S_\bullet^i \chi_n^s)^{i;s} \blacktriangleleft \theta] \bullet \phi_p\} - \sum_{i=0}^{n+1} \tfrac{(-1)^{in}}{i!} \{[S_\bullet^i (d\chi_n^s)]^{i;s} \blacktriangleleft \theta\} \bullet \phi_p = \\ &= \sum_{i=0}^{n} \tfrac{(-1)^{in-i}}{i!} [d(S_\bullet^i \chi_n^s)^{i;s} \blacktriangleleft \theta] \bullet \phi_p + \tbinom{s}{1}_\ell \sum_{i=0}^{n} \tfrac{(-1)^{in-n-i}}{i!} \{[(S_\bullet^i \chi_n^s)^{i;s} \blacktriangleleft \theta]^{1;s-1} \blacktriangleleft d\phi_p\} \bullet \phi_p \\ &\quad - \sum_{i=1}^{n} \tfrac{(-1)^{in-n-i}}{(i-1)!} \{[(S_\bullet^i \chi_n^s)^{i-1;s+1} \blacktriangleleft \theta]^{1;s} \blacktriangleleft d\theta\} \bullet \phi_p - \sum_{i=0}^{n+1} \tfrac{(-1)^{in}}{i!} \{[S_\bullet^i (d\chi_n^s)]^{i;s} \blacktriangleleft \theta\} \bullet \phi_p \end{aligned}$$

by Proposition 1.55. With Corollary 1.111 we get

$$\begin{aligned} &\sum_{i=0}^{n} \tfrac{(-1)^{in-i}}{i!} [d(S_\bullet^i \chi_n^s)^{i;s} \blacktriangleleft \theta] \bullet \phi_p - \sum_{i=0}^{n+1} \tfrac{(-1)^{in}}{i!} \{[S_\bullet^i (d\chi_n^s)]^{i;s} \blacktriangleleft \theta\} \bullet \phi_p = \\ &= \ - \sum_{i=2}^{n+1} \tfrac{(-1)^{in-i}}{(i-2)!} \{[(S_\bullet^{i-1} \chi_n^s)^{i-2;s+1} \blacktriangleleft \theta]^{1;s} \blacktriangleleft (\mathrm{sgn}(S)\, \tfrac{1}{2}\, \theta \wedge_{\mathfrak{g}} \theta)\} \bullet \phi_p \\ &\quad + \tbinom{s}{1}_\ell \sum_{i=1}^{n+1} \tfrac{(-1)^{in-i}}{(i-1)!} \{[(S_\bullet^{i-1} \chi_n^s)^{i-1;s} \blacktriangleleft \theta]^{1;s-1} \blacktriangleleft (\mathrm{sgn}(S)\, \theta \wedge_{\mathfrak{g}} \phi_p)\} \bullet \phi_p . \end{aligned}$$

Finally we put all together and use $S_\bullet^{i+1} \chi_n^s = (-1)^i S_\bullet^i (S_\bullet \chi_n^s)$ from (51). □

For $\chi_n^s \in \mathcal{A}_n(P) \otimes \mathrm{Hom}(\bigotimes^s \mathfrak{g}, V)$, the last term in Theorem 1.113 reads

$$\sum_{k=1}^{s} (-1)^{n+p(k-1)} \Big\{\Big[[(\chi_n^s \circledS \theta)^{k-1;s-k+1} \blacktriangleleft \phi_p]^{1;s-k} \blacktriangleleft (d\phi_p - \mathrm{sgn}(S)\, \theta \wedge_{\mathfrak{g}} \phi_p)\Big]^{s-k} \bullet \phi_p\Big\}$$

as a consequence of Proposition 1.110, cf. (28). In any case we get the following

Corollary 1.114 *If S is a* LIE *group action of G on P, $\chi_n^s \in \mathcal{A}_n(P) \otimes \mathrm{Hom}(\bigotimes^s \mathfrak{g}, V)$ $\mathfrak{g}$-equivariant, and $\theta \in \mathcal{A}_1(P,\mathfrak{g})$, $\phi_p \in \mathcal{A}_p(P,\mathfrak{g})$ with $d\phi_p = \mathrm{sgn}(S)\, \theta \wedge_{\mathfrak{g}} \phi_p$, then*

$$d[(\chi_n^s \circledS \theta) \bullet \phi_p] = [(d\chi_n^s) \circledS \theta] \bullet \phi_p + \Big\{\Big[(S_\bullet \chi_n^s) \circledS \theta\Big]^{1;s} \blacktriangleleft (d\theta - \mathrm{sgn}(S) \tfrac{1}{2} \theta \wedge_{\mathfrak{g}} \theta)\Big\}^s \bullet \phi_p .$$

Now suppose, θ is a pullback of an invariant 1-form on G. Then Corollary 1.82, resp., the MAURER-CARTAN identities 1.83 give

Corollary 1.115 *Let S be a* LIE *group action of G on P, $f\colon P \to G$ differentiable, $K \in \mathrm{End}(\mathfrak{g})$ and $\chi_n^s \in \mathcal{A}_n(P) \otimes \mathrm{Hom}(\bigotimes^s \mathfrak{g}, V)$ $\mathfrak{g}$-equivariant.*

1. *If* $\chi_n^s \in \mathcal{A}_n(P) \otimes \mathrm{Sym}_s^c(\mathfrak{g}, V)$ *and* $\phi_p \in \mathcal{A}_p(P, \mathfrak{g})$, *then*

$$d[(\chi_n^s \circledcirc f^*\psi^S(K)) \bullet \phi_p] = [(d\chi_n^s) \circledcirc f^*\psi^S(K)] \bullet \phi_p$$
$$+(-1)^n \binom{s}{1}_\ell \left[(\chi_n^s \circledcirc f^*\psi^S(K))^{1;s-1} \blacktriangleleft (d\phi_p - \mathrm{sgn}(S)\, f^*\psi^S(K) \wedge_{\mathfrak{g}} \phi_p)\right]^{s-1} \bullet \phi_p,$$
$$d[(\chi_n^s \circledcirc f^*\Theta^S) \bullet \phi_p] = [(d\chi_n^s) \circledcirc f^*\Theta^S] \bullet \phi_p$$
$$+(-1)^n \binom{s}{1}_\ell \left[(\chi_n^s \circledcirc f^*\Theta^S)^{1;s-1} \blacktriangleleft (d\phi_p - \mathrm{sgn}(S) f^*\Theta^S \wedge_{\mathfrak{g}} \phi_p)\right]^{s-1} \bullet \phi_p.$$

2. *For* $\phi_p \in \mathcal{A}_p(P, \mathfrak{g})$ *with* $d\phi_p = \mathrm{sgn}(S) f^*\psi^S(K) \wedge_{\mathfrak{g}} \phi_p$, *e. g. for* $\phi_2 = d(f^*\psi^S(K))$,

$$d[(\chi_n^s \circledcirc f^*\psi^S(K)) \bullet \phi_p] = [(d\chi_n^s) \circledcirc f^*\psi^S(K)] \bullet \phi_p,$$
$$d[(\chi_n^s \circledcirc f^*\Theta^S) \bullet \phi_p] = [(d\chi_n^s) \circledcirc f^*\Theta^S] \bullet \phi_p.$$

Finally, in the case $s = 0$, Theorem 1.113 yields

Corollary 1.116 *If* S *is a* LIE *group action of* G *on* P *and* $\omega_n \in \mathcal{A}_n(P) \otimes V$ *is* $\mathfrak{g}$*-invariant, then for all* $\theta \in \mathcal{A}_1(P, \mathfrak{g})$

$$d(\omega_n \circledcirc \theta) = (d\omega_n) \circledcirc \theta + [(S_\bullet \omega_n) \circledcirc \theta]^1 \blacktriangleleft (d\theta - \tfrac{1}{2}\mathrm{sgn}(S)\,\theta \wedge_{\mathfrak{g}} \theta).$$

For any $f\colon P \to G$, $K \in \mathrm{End}(\mathfrak{g})$, *especially* $K = \mathrm{id}_{\mathfrak{g}}$, *we thus obtain*

$$d(\omega_n \circledcirc f^*\psi^S(K)) = (d\omega_n) \circledcirc f^*\psi^S(K), \qquad d(\omega_n \circledcirc f^*\Theta^S) = (d\omega_n) \circledcirc f^*\Theta^S.$$

In the next chapter we will be interested especially in the case where $\phi_2 = d\theta - \frac{1}{2}\mathrm{sgn}(S)\,\theta \wedge_{\mathfrak{g}} \theta$. Using Lemma 1.23.3 and $\theta \wedge_{\mathfrak{g}} (\theta \wedge_{\mathfrak{g}} \theta) = 0$ from Lemma 1.78 one easily checks that this yields $d\phi_2 = \mathrm{sgn}(S)\,\theta \wedge_{\mathfrak{g}} \phi_2$. Thus Corollary 1.114 reads

$$d[(\chi_n^s \circledcirc \theta) \bullet \phi_2] = [(d\chi_n^s) \circledcirc \theta] \bullet \phi_2 + \left[(S_\bullet \chi_n^s) \circledcirc \theta\right] \bullet \phi_2.$$

Now $S_\bullet \chi_n^s \in \mathcal{A}_{n-1}(P, \mathrm{Hom}(\mathfrak{g}, \mathrm{Hom}(\bigotimes^s \mathfrak{g}, V))) \cong \mathcal{A}_{n-1}(P, \mathrm{Hom}(\bigotimes^{s+1} \mathfrak{g}, V))$. Since ϕ_2 has even degree, only the symmetric part of $\mathrm{Hom}(\bigotimes^{s+1}\mathfrak{g}, V)$ counts (e.g., confer Lemma 1.44). So $\left[(S_\bullet \chi_n^s) \circledcirc \theta\right] \bullet \phi_2 = \mathrm{Sym}_*\left[(S_\bullet \chi_n^s) \circledcirc \theta\right] \bullet \phi_2 = \left[\mathrm{Sym}_*(S_\bullet \chi_n^s) \circledcirc \theta\right] \bullet \phi_2$, because $\circledcirc$ only acts on $\mathcal{A}(P)$ and commutes with any operation on $\mathrm{Hom}(\bigotimes^{s+1}\mathfrak{g}, V)$. This leads to the following definition:

Definition 1.117 *For* $\chi_n^s \in \mathcal{A}_n(P, \mathrm{Hom}(\bigotimes^s \mathfrak{g}, V))$ *and any* LIE *group action* S *of* G *on* P, *we define*

$$S_\bullet^\vee \chi_n^s := \mathrm{Sym}_*(S_\bullet \chi_n^s) \in \mathcal{A}_{n-1}(P, \mathrm{Sym}_{s+1}(\mathfrak{g}, V)).$$

Corollary 1.118 *If* S *is a* LIE *group action of* G *on* P, $\chi_n^s \in \mathcal{A}_n(P) \otimes \mathrm{Hom}(\bigotimes^s \mathfrak{g}, V)$ $\mathfrak{g}$*-equivariant,* $\theta \in \mathcal{A}_1(P, \mathfrak{g})$ *and* $\phi_2 = d\theta - \frac{1}{2}\mathrm{sgn}(S)\,\theta \wedge_{\mathfrak{g}} \theta \in \mathcal{A}_2(P, \mathfrak{g})$, *then*

$$d[(\chi_n^s \circledcirc \theta) \bullet \phi_2] = [(d\chi_n^s) \circledcirc \theta] \bullet \phi_2 + \left[(S_\bullet^\vee \chi_n^s) \circledcirc \theta\right] \bullet \phi_2.$$

Extend the symmetric product $\vee$ in $\mathrm{Sym}(\mathfrak{g}, \mathbb{R}) \cong S(\mathfrak{g}^*)$ to $\mathrm{Sym}(\mathfrak{g}, V)$, whenever a bilinear map $\phi\colon V \times V \to V$ is given. Equip $\mathcal{A}(P) \otimes \mathrm{Sym}(\mathfrak{g}, V)$ with the gradation induced by $\mathcal{A}(P)$, then we obtain from Lemma 1.33.1 and (52):

Lemma 1.119 $S_\bullet^\vee$ *is a skew-derivation of degree* -1 *of* $\mathcal{A}(P)_{\mathrm{equiv}} \otimes \mathrm{Sym}(\mathfrak{g}, V)$ *and* $\mathcal{A}(P) \otimes \mathrm{Sym}(\mathfrak{g}, V)$. *E. g. for all* $\alpha_n \in \mathcal{A}_n(P) \otimes \mathrm{Sym}(\mathfrak{g}, V)$ *and* $\omega \in \mathcal{A}(P) \otimes \mathrm{Sym}(\mathfrak{g}, V)$,

$$S_\bullet^\vee(\alpha_n \wedge_\vee \omega) = (S_\bullet^\vee \alpha_n) \wedge_\vee \omega + (-1)^n \alpha_n \wedge_\vee (S_\bullet^\vee \omega).$$

Chapter 2

Principles of Bundles and Principal Bundles

Fiber bundles are generalizations of the direct product of two given topological spaces. Their concept is crucial for a lot of applications in mathematics and physics, reaching from differential geometry, topological algebra and LIE groups to gauge theories in theoretical physics. As already mentioned in the preface, the definition of a bundle is analogous to the one of a manifold: we have a bundle atlas consisting of charts which enable us to describe the bundle locally as a direct product of the base space and the fiber, while the global structure of the bundle may be more complicated. In contrast to a global direct product, only one global projection exists: the one onto the base, whereas projections onto the fiber typically only exist locally.

2.1 Basic Definitions

For our purposes, we only consider bundles that consist of C^∞-manifolds. The following definition is due to STEENROD (cf. [8, p. 7]) and POOR (cf. [9, p. 1]):

Definition 2.1 *A* (fiber) bundle $B(M, F, G)$ *consists of*

1. *a* C^∞*-manifold* B *called the* bundle (manifold),
2. *a* C^∞*-manifold* M *called the* base (manifold),
3. *a* C^∞*-manifold* F *called the* (standard) fiber,
4. *a left* LIE *group action* $L: G \times F \to F$*: if* L *is effective,* G *is called the* (structure) group of the bundle,
5. *a* C^∞-projection $\pi: B \to M$ *of the bundle onto the base,*
6. *a* bundle atlas $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in A}$ *with* bundle charts (U_α, ψ_α)*, where* $\mathfrak{U} = \{U_\alpha\}_{\alpha\in A}$ *is an open cover of* M *and* $\psi_\alpha: \pi^{-1}(U_\alpha) \to U_\alpha \times F: b \mapsto (\pi(b), \pi_\alpha(b))$ *are* local trivializations*(i. e. diffeomorphisms) with* local projections $\pi_\alpha: \pi^{-1}(U_\alpha) \to F$ *onto the fiber (we write* $U_{\alpha_1\cdots\alpha_n} := U_{\alpha_1} \cap \cdots \cap U_{\alpha_n}$ *for all* $\alpha_i \in A$*),*

7. *a family* $\{g_{\beta\alpha}: U_{\alpha\beta} \to G \,|\, \alpha, \beta \in A, U_{\alpha\beta} \neq \emptyset\}$ *of differentiable* transition functions $g_{\beta\alpha}$, *such that* $L_{g_{\beta\alpha}(x)} = T_{\beta\alpha}|_{\{x\}\times F}$ *holds for all* $x \in U_{\alpha\beta} \neq \emptyset$, *where* $T_{\beta\alpha} = (\psi_\beta|_{\pi^{-1}(U_{\alpha\beta})}) \circ (\psi_\alpha|_{\pi^{-1}(U_{\alpha\beta})})^{-1}: U_{\alpha\beta} \times F \to U_{\alpha\beta} \times F$.

Two bundles B and B' will be identified, if they have the same bundle manifold, base, fiber, group and projection and their bundle atlases are compatible to each other in the sense that for all $x \in U_\alpha \cap U'_\beta$

$$\overline{g_{\beta\alpha}}(x) := (\psi'_\beta|_{\pi^{-1}(\{x\})}) \circ (\psi_\alpha|_{\pi^{-1}(\{x\})})^{-1}$$

coincides with the operation of an element of G and the map $\overline{g_{\beta\alpha}}: U_\alpha \cap U'_\beta \to G$ so obtained is C^∞. Briefly, we identify two bundles if the union of the two bundle atlases is again a bundle atlas. Thus we may regard a fiber bundle to be equipped with a "maximal" bundle atlas and assume that all U_α in $\mathfrak{U}$ are Euclidean neighborhoods in M. In view of this maximal atlas, the original bundle atlas is sometimes called a *pre-atlas*, but we will not make this distinction.

Definition 2.2 *Two bundles B and B' having the same base, fiber and group are said to be* equivalent *if there exists a fiber preserving diffeomorphism $B \to B'$ inducing the identity on M.*

Note 2.3 Even in the general case when B, M, F are just topological spaces, many topological properties of M and F carry over to B: If M and F are HAUSDORFF then B is HAUSDORFF, the same holds for (local) compactness, (local) connectedness, arcwise connectedness and the axioms of countability (first axiom: every point has a countable basis for its neighborhoods, second axiom: a countable basis for the topology exists), cf. [8, p. 13]. We also deduce that B is a manifold if M and F are manifolds and that B is paracompact if M and F are paracompact (cf. Note 1.16).

Recall Definition 1.69. By construction, the transition functions $g_{\alpha\beta}$ obey

$$g_{\alpha\beta}|_{U_{\alpha\beta\gamma}} \cdot g_{\beta\gamma}|_{U_{\alpha\beta\gamma}} = g_{\alpha\gamma}|_{U_{\alpha\beta\gamma}} \qquad \text{for all} \quad \alpha, \beta, \gamma \in A, \quad \text{where} \quad U_{\alpha\beta\gamma} \neq \emptyset \qquad (55)$$

From (55) we easily deduce $g_{\alpha\alpha} = e$ and $g_{\alpha\beta} = (g_{\beta\alpha})^{-1}$ for all $\alpha, \beta \in A$.

Note 2.4 The transition functions $g_{\alpha\beta}$ are crucial for the global structure of the bundle. If M with an open cover $\mathfrak{U} = \{U_\alpha\}_{\alpha\in A}$ and the fiber F are given then the $g_{\alpha\beta}$ define the whole bundle up to equivalences (cf. [8, p. 14]):

Theorem 2.5 (Existence theorem) *If $L: G \times F \to F$ is a left* LIE *group action, $\mathfrak{U} = \{U_\alpha\}_{\alpha\in A}$ is an open cover of a manifold M and $\{g_{\alpha\beta}\}_{\alpha,\beta\in A}$ is a family of C^∞-maps $g_{\alpha\beta}: U_{\alpha\beta} \to G$ such that* (55) *holds, then there exists a bundle $B(M, F, G)$ with these transition functions $g_{\alpha\beta}$. Any two such bundles are equivalent.*

Well-known examples for bundles are the MOEBIUS band and the KLEIN bottle. The *tangent bundle* $T(M)$ of a manifold M consists of all tangent vectors at all points in M, where M is the n-dimensional base manifold, $\mathbb{R}^n$ is the fiber and $\mathrm{Gl}(\mathbb{R}^n)$ is the group of the bundle. Analogously we have the *cotangent bundle* $T^*(M)$, that consists of all cotangent vectors at all points in M. $T(M)$ and its dual $T^*(M)$ are *vector bundles*: the fiber is a (finite dimensional) vector space and the group action is linear. Given two vector bundles $E_i(M, V_i, \mathrm{Gl}(V_i))$, $i = 1, 2$, one can define the *tensor product bundle* $(E_1 \otimes E_2)(M, V_1 \otimes V_2, \mathrm{Gl}(V_1 \otimes V_2))$ and the *homomorphism bundle* $\mathrm{Hom}(E_1, E_2)(M, \mathrm{Hom}(V_1, V_2), \mathrm{Gl}(\mathrm{Hom}(V_1, V_2))) \cong E_1^* \otimes E_2$ in a natural way. $(E_1 \oplus E_2)(M, V_1 \oplus V_2, \mathrm{Gl}(V_1 \oplus V_2))$ is called the WHITNEY *sum* of E_1 and E_2.

One also has *algebra bundles*, where the fiber is an algebra and G consists of algebra isomorphisms. Examples are the *tensor algebra bundle* $\bigotimes E(M, \bigotimes V, G)$ and the *exterior algebra bundle* $\bigwedge E(M, \bigwedge V, G)$, cf. [9, pp. 23 – 24].

Definition 2.6 (Cross-)sections *are* C^∞*-maps* $\sigma: M \to B: x \mapsto \sigma(x) \in \pi^{-1}(\{x\})$. *Thus* $\pi \circ \sigma = \mathrm{id}_M$. *Their set is denoted by* ΓB.

Normally only local sections exist: we have $\sigma_{\alpha,y}: U_\alpha \to \pi^{-1}(U_\alpha): x \mapsto \psi_\alpha^{-1}(x, y)$ fixing $y \in F$, but for vector bundles global sections always exist, e. g. the "zero section." The sections of $T(M)$, resp., $T^*(M)$ are exactly the vector fields, resp., the 1-forms on M. That $\mathcal{D}^1(M)$ and $\mathcal{D}_1(M)$ are $C^\infty(M)$-modules also follows from:

Lemma 2.7 *If* E *is a vector bundle over* M *then* ΓE *is a* $C^\infty(M)$*-module. For* $f \in C^\infty(M)$, $\sigma \in \Gamma E$ *and* $x \in U_\alpha$ *we have* $\psi_\alpha[(f\sigma)(x)] = (x, f(x)\pi_\alpha(\sigma(x))$.

Definition 2.8 *The* trivial bundle *or* product bundle $M \times F$ *is the direct product of the two manifolds with natural projection* $\mathrm{pr}_M: M \times F \to M$, $\mathfrak{U} = \{M\}$ *and trivial group* $G = \{\mathrm{id}_F\}$.

For any finite dimensional V, the $C^\infty(M)$-module $\mathcal{A}(M, V) \cong \mathcal{A}(M) \otimes V$ contains the sections of $\bigwedge T^*(M) \otimes (M \times V) \cong \mathrm{Hom}(\bigwedge T(M), M \times V)$.

Whenever the group of the bundle consists of the identity alone, then the bundle is equivalent to a trivial bundle (cf. [8, p. 16]). We will also say that a bundle with group G is equivalent to the trivial bundle, if it is equivalent to a bundle with this group G such that for this bundle all $g_{\alpha\beta} = e$.

Analogously, if $H < G$, we say that the group G of a bundle B can be *reduced to* H, if B is equivalent to a bundle, where all $g_{\alpha\beta}$ take their values in H.

Let $B_1(M_1, F_1, G)$ be a fiber bundle and F_2 be a submanifold of F_1. Suppose G may be reduced to a subgroup H where F_2 is invariant under H: there exists $A' \subseteq A$ such that $\mathfrak{U}' = \{U_\alpha\}_{\alpha \in A'}$ covers M_1 and for all α, β in A' with $U_{\alpha\beta} \neq \emptyset$, $g_{\alpha\beta}$ maps into H. Then $\psi_\alpha^{-1}(\{x\} \times F_2) = \psi_\beta^{-1}(\{x\} \times F_2)$ for all $\alpha, \beta \in A'$ and $x \in U_{\alpha\beta}$. Let $B_2 \subseteq B_1$ denote the union of all subspaces $\psi_\alpha^{-1}(\{x\} \times F_2)$ for all $\alpha \in A'$ and $x \in M_2$, where M_2 is a submanifold of M_1. Then B_2 is a submanifold of B_1 and the functions $\psi_\alpha^{-1}|_{(U_\alpha \cap M_2) \times F_2}$ determine a bundle structure for B_2 with fiber F_2, base M_2 and restricted bundle charts. If H does not act effectively on F_2, the group of B_2 is a factor group H' of G (cf. [8, p. 24] versus [9, p. 6]).

Definition 2.9 $B_2(M_2, F_2, H')$ *is called a* subbundle *of* $B_1(M_1, F_1, G)$.

Some examples: For every submanifold M' of M, $[\pi^{-1}(M')](M', F, G)$ is a subbundle of $B(M, F, G)$; it is equivalent to the trivial bundle if $M' = U \in \mathfrak{U}$.

Consider the projection $\pi: B \to M$ and recall Definition 1.26: since π is not a diffeomorphism, a map $\pi_*: \mathcal{D}^1(B) \to \mathcal{D}^1(M)$ is not defined, yet π induces a mapping of the tangent bundles $d\pi: T(B) \to T(M)$. We may reduce the group of $T(B)$ to those tangent space isomorphisms induced by (fiber preserving) bundle diffeomorphisms. Then

$$V(B) := (d\pi)^{-1}(0) = \bigcup_{b\in B} \ker d\pi_b =: \bigcup_{b\in B} V_b(B) \subsetneq T(B)$$

defines a subbundle of $T(B)$ consisting of all vectors tangent to the fiber. This subbundle is called the *vertical bundle* $V(B)$, its sections are named *vertical vector fields*, their set is denoted by $v\mathcal{D}^1(B)$. This, in turn, defines a subbundle $V(B)^\perp$ of $T^*(B)$ consisting of all covectors cotangent to the base. Finally this defines $\bigwedge V(B)^\perp \otimes (M \times V)$ as a subbundle of $\bigwedge T^*(B) \otimes (M \times V)$ for any finite dimensional vector space V. Its sections are called *horizontal V-valued forms* on B, and their set is denoted by $\mathcal{A}(B, V)h$.

Definition 2.10 *By a* principal bundle $P(M, G)$ *we mean a bundle, where $G = F$ acting on itself by left multiplication. In addition we have a free fiber preserving right* LIE *group action $R: G \times P \to P$ defined by*

$$R(g,p) := \psi_\alpha^{-1}(\pi(p), \pi_\alpha(p) \cdot g) \qquad \text{for all} \quad p \in P,\, g \in G, \quad \text{where} \quad \pi(p) \in U_\alpha,$$

which is independent of the choice of α since left and right multiplication commute.

Given any bundle $B(M, F, G)$ one can construct the *associated principal bundle P* by taking $M = \bigcup_{\alpha\in A} U_\alpha$, the structure group G and the maps $g_{\alpha\beta}$ but choosing G as fiber (cf. Note 2.4). E. g., the principal bundle associated with the tangent bundle $T(M)(M, \mathbf{R}^n, \mathrm{Gl}(\mathbf{R}^n))$ is the so-called *frame bundle* $L(M)$ with $\mathrm{Gl}(\mathbf{R}^n)$ as fiber. Its sections differentiably associate with any point $x \in M$ a basis for the tangent space $T_x(M)$.

As another example for principal bundles, take $G = \mathbf{S}^1 \cong \mathbb{R}/\mathbb{Z}$ and choose $M = \mathbf{S}^2$ with cover $\mathfrak{U} = \{U_+, U_-\}$, where U_+ and U_- cover the northern, resp., southern hemisphere and the intersection U_{+-} is a ring $\mathbf{S}^1\times]-\epsilon,\epsilon[$. We define $g_{-+}, g_{+-}: U_{+-} \to G$ by $g_{-+} = -g_{+-} := m \cdot \mathrm{pr}_{\mathbf{S}^1}$ with $m \in \mathbf{Z}$. Together with $g_{++} = g_{--} = 0$, these functions $g_{\alpha\beta}$ obey (55) and thus define a unique (up to equivalences) principal bundle $P_m(\mathbf{S}^2, \mathbf{S}^1)$. One can show ([1], [2]) that the bundles P_m and P_{-m} are isomorphic (via reflecting $\mathbf{S}^2$ at its equator) and that

$$P_0 \cong \mathbf{S}^2 \times \mathbf{S}^1, \qquad P_1 \cong \mathbf{S}^3, \qquad P_m \cong \mathbf{S}^3/\mathbf{Z}_{|m|} \quad \text{for} \quad m > 1, \tag{56}$$

with the finite subgroups $\mathbf{Z}_{|m|}$ of $\mathbf{S}^1$, which itself is a closed subgroup of $\mathbf{S}^3 \cong \mathrm{SU}_2$. The quotient map $\pi: \mathbf{S}^3 \to \mathbf{S}^2 = \mathbf{S}^3/\mathbf{S}^1$ is known as the HOPF *fibering* of the $\mathbf{S}^3$.

Definition 2.11 *Two bundles having the same base and group are said to be* associated bundles *if their associated principal bundles are equivalent.*

Lemma 2.12 *Any bundle and its associated principal bundle are associated. Equivalent bundles are associated. In addition, if two associated bundles have the same fiber and the same action of the group on the fiber, then they are equivalent. Being associated is an equivalence relation on the set of all fiber bundles.*

Proof: [8, p. 43] □

Instead of defining general bundles first and then specifying to principal bundles, we can also start with the latter. [6, p. 50] gives the following equivalent definition:

Definition 2.13 *A* principal bundle $P(M,G)$ *consists of*

1. *a* C^∞*-manifold* P *called the* bundle (manifold),
2. *a* C^∞*-manifold* M *called the* base (manifold),
3. *a* LIE *transformation group* G *called the* group of the bundle, *acting on* P *from the right, such that this action* $R: G \times P \to P$ *is free,* M *is the quotient manifold* P/G *and the canonical projection* $\pi: P \to M$ *is* C^∞,
4. *a* bundle atlas $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in A}$ *with* bundle charts (U_α, ψ_α), *where* $\mathfrak{U} = \{U_\alpha\}_{\alpha\in A}$ *is an open cover of* M *and* $\psi_\alpha: \pi^{-1}(U_\alpha) \to U_\alpha \times G: p \mapsto (\pi(p), \pi_\alpha(p))$ *are* local trivializations *(i. e. diffeomorphisms) with* local projections $\pi_\alpha: \pi^{-1}(U_\alpha) \to G$ *onto the group satisfying* $\pi_\alpha(R(g,p)) = \pi_\alpha(p)\cdot g$ *for all* $p \in P$, $g \in G$.

For all $x \in U_{\alpha\beta} \neq \emptyset$ and $p \in \pi^{-1}(\{x\})$, we have $g_{\beta\alpha}(x) = \pi_\beta(p)\cdot\pi_\alpha(p)^{-1}$, since $\pi_\beta(R(g,p))\cdot\pi_\alpha(R(g,p))^{-1} = \pi_\beta(p)\cdot\pi_\alpha(p)^{-1}$ for all $g \in G$.

Lemma 2.14 *If* $U_{\alpha\beta} \neq \emptyset$ *and* $\sigma_{\alpha,g}$, $\sigma_{\beta,h}$ *denote local sections on* U_α, *resp.,* U_β *then*

$$\begin{aligned} \sigma_{\alpha,g}|_{U_{\alpha\beta}} &= R\circ(h^{-1}g_{\beta\alpha}g, \sigma_{\beta,h}|_{U_{\alpha\beta}}) \qquad \text{for all} \quad g,h \in G, \\ \text{thus} \qquad \sigma_{\alpha,e}|_{U_{\alpha\beta}} &= R\circ(g_{\beta\alpha}, \sigma_{\beta,e}|_{U_{\alpha\beta}}). \end{aligned}$$

Proof. For $x \in U_{\alpha\beta}$ and $g,h \in G$, $\sigma_{\alpha,g}(x) = \psi_\alpha^{-1}(x,g) = \psi_\beta^{-1}\psi_\beta\psi_\alpha^{-1}(x,g) = \psi_\beta^{-1}(x, g_{\beta\alpha}(x)g) = \psi_\beta^{-1}(x, hh^{-1}g_{\beta\alpha}(x)g) = R(h^{-1}g_{\beta\alpha}(x)g, \sigma_{\beta,h}(x))$. □

Definition 2.15 *The* trivial principal bundle $M\times G$ *is the product manifold* $M\times G$ *with projection* pr_M, $\mathfrak{U} = \{M\}$ *and* $R_g(x,h) := (x, hg)$ *for all* $x \in M$ *and* $g,h \in G$.

Proposition 2.16 *Let* G *be a* LIE *group and* H *a closed subgroup of* G. *Then* H *acts on* G *on the right by multiplication and* $G(G/H, H)$ *is a principal bundle.*

Proof: [6, p. 55] □

The following definition of associated bundles is also due to [6, p. 54]:

Definition 2.17 *Let $P(M,G)$ be a principal bundle and $L: G \times F \to F$ be a left* LIE *group action of G on a manifold F. We define a free right* LIE *group action $\tilde{R}$ of G on the product manifold $P \times F$ as follows:*

$$\tilde{R}_g(p,f) := (R_g(p), L_{g^{-1}}(f)) \qquad \text{for all} \quad p \in P,\ f \in F,\ g \in G.$$

The quotient space $P \times_G F$ by this action $\tilde{R}$ can be endowed with a differentiable structure such that $\hat{\pi}: P \times_G F \to M$, which is induced by $\pi \circ \mathrm{pr}_P: P \times F \to M$, becomes C^∞. We call $P \times_G F$ the fiber bundle with fiber F associated with *P. If (U_α, ψ_α) is a bundle chart for P and $p \in \pi^{-1}(U_\alpha)$, then $(U_\alpha, \widehat{\psi_\alpha})$, where $\widehat{\psi_\alpha}((p,f)G) := (\pi(p), L(\pi_\alpha(p), f))$ for all $f \in F$, is a bundle chart for the associated bundle $P \times_G F$.*

Both definitions of associated bundles are equivalent: for any bundle $B(M,F,G)$ with associated principal bundle $P(M,G)$, we have $B(M,F,G) \cong P(M,G) \times_G F$. We will denote the canonical projection by $\tilde{\pi}: P \times F \to B$. Definition 2.17 yields:

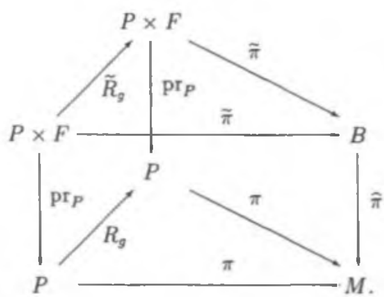
$$\hat{\pi} \circ \tilde{\pi} = \pi \circ \mathrm{pr}_P, \qquad \hat{\pi}_\alpha \circ \tilde{\pi} = L \circ (\pi_\alpha \circ \mathrm{pr}_P, \mathrm{pr}_F). \tag{57}$$

Lemma 2.18 *If $B = P \times_G F$ is associated with the principal bundle $P(M,G)$ then $(P \times F)(B,G)$ is a principal bundle over B with right action $\tilde{R}$, cover $\hat{\pi}^{-1}\mathfrak{U}$ of B and local trivializations*

$$\widetilde{\psi_\alpha}: \pi^{-1}(U_\alpha) \times F \to \hat{\pi}^{-1}(U_\alpha) \times G \ :\ (p,f) \mapsto (\widehat{\psi_\alpha}^{-1}(\pi(p), L(\pi_\alpha(p), f)), \pi_\alpha(p)),$$

so $\widetilde{\psi_\alpha}^{-1}: \hat{\pi}^{-1}(U_\alpha) \times G \to \pi^{-1}(U_\alpha) \times F \ :\ (b,g) \mapsto (\psi_\alpha^{-1}(\hat{\pi}(b), g), L(g^{-1}, \widehat{\pi_\alpha}(b))).$

The following diagram commutes for every $g \in G$:



Let $P(M,G)$ be a principal bundle and H a closed subgroup of G. In a natural way, G acts on G/H on the left and H acts on P on the right. So the associated bundle $P \times_G (G/H)$ and the quotient space P/H are well-defined and the following propositions hold ([6, p. 57]):

Proposition 2.19 *For every closed $H \leq G$ and any principal bundle $P(M,G)$, we can identify $P \times_G (G/H)$ with P/H by mapping every element $(p, gH)G$ into $R_g(p)H$. Thus P/H is a manifold and $P(M,G)$ is a principal bundle $P(P/H, H)$ over P/H with group H and canonical projection $\pi: P \to P/H$.*

Proposition 2.20 *The structure group G of $P(M,G)$ is reducible to a closed subgroup H iff $P/H = P \times_G (G/H)$ admits a cross-section $\sigma: M \to P/H$.*

Taking $H = \{e\}$ we get the following corollary (cf. [8, p. 36]):

Corollary 2.21 (Cross-section theorem) *A principal bundle $P(M,G)$ is equivalent to the product bundle $M \times G$ iff it admits a cross-section.*

Since associated bundles have the same transition functions, one deduces:

Corollary 2.22 *A bundle $B(M,F,G)$ is equivalent to the product bundle $M \times F$ iff the associated principal bundle $P(M,G)$ admits a section. The group of $B(M,F,G)$ is reducible to a closed subgroup H iff a section $\sigma: M \to P/H$ exists.*

Thus in order to decide whether a given bundle is trivial or not one can construct the associated principal bundle and look for sections there.

Proposition 2.23 *Every bundle $B(M,\mathbb{R}^m,G)$, $m \in \mathbb{N}_0$, over a paracompact base manifold M admits a section.*

Proof, using the axiom of choice: [6, pp. 58 – 59]. □

Every LIE group G that consists of a finite number of connected components — i. e., G/G_1 is finite, — is diffeomorphic to a direct product $K \times \mathbb{R}^m$, $m \in \mathbb{N}_0$, where K is a maximal compact subgroup of G, cf. HOCHSCHILD, [10, p. 180]. If G is compact then $m = 0$, if G is connected then K is connected, too. Now the following theorem is an immediate consequence of Corollary 2.22 and Proposition 2.23:

Theorem 2.24 *Let $B(M,F,G)$ be a bundle over a paracompact manifold M. If G/G_1 is finite then G is reducible to a maximal compact subgroup K.*

Corollary 2.25 *Every bundle $B(M,F,G)$ over a paracompact manifold M is equivalent to a trivial bundle if $G \cong \mathbb{R}^m$, $m \in \mathbb{N}_0$.*

As we have seen, triviality of a bundle $B(M,F,G)$ only depends on the triviality of the associated principal bundle $P(M,G)$, and we have found a criterion that depends on G. It is only natural to ask for another criterion that depends on the base manifold M. To this end, we recall the definition of a homotopy:

Definition 2.26 *Two (C^∞-)maps $f_i: M \to N$, $i = 1,2$, between manifolds M and N are said to be* homotopic*: $f_1 \sim f_2$, if a (C^∞-)map $F: M \times [0,1] \to N$, called* homotopy, *exists such that*

$$F(x,0) = f_1(x), \qquad F(x,1) = f_2(x) \qquad \textit{for all} \quad x \in M.$$

M and N are said to be of the same homotopy type, *if $f: M \to N$ and $g: N \to M$ exist with $g \circ f \sim \mathrm{id}_M$ and $f \circ g \sim \mathrm{id}_N$. M is called* contractible *if it is of the same homotopy type as a single point. In that case, id_M is homotopic to a constant map.*

Definition 2.27 *Be $B(N,F,G)$ a fiber bundle and $f\colon M \to N$ differentiable. Then the* pullback bundle *or* induced bundle *$(f^*B)(M,F,G)$ is defined by*

$$f^*B = \{(x,b) \in M \times B | \pi(b) = f(x)\} \subseteq M \times B$$

with induced projection $\mathrm{pr}_M | f^*B\colon f^*B \to M$ and fiber F. If $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in A}$ is a bundle atlas for B then $\{(f^{-1}(U_\alpha), \psi'_\alpha)\}_{\alpha \in A}$ is a bundle atlas for f^*B, where $\psi'_\alpha(x,b) := (x, \pi_\alpha(b))$ for all $x \in M$, $b \in B$.

Thus the following diagram commutes:

$$\begin{array}{ccc} f^*B & \xrightarrow{\mathrm{pr}_B} & B \\ \Big\downarrow{\scriptstyle \mathrm{pr}_M} & & \Big\downarrow{\scriptstyle \pi} \\ M & \xrightarrow{f} & N. \end{array}$$

Lemma 2.28 *Let $f\colon M \to N$ be differentiable and B and B' be bundles over N.*

1. *If B and B' are equivalent, resp., associated, then f^*B and f^*B' are equivalent, resp., associated.*

2. *If B is a principal bundle, so also is f^*B. $R'\colon G \times f^*B \to f^*B$ defined by $R'_g(x,b) := (x, R_g(b))$ is the induced free right action on f^*B.*

3. *If B, B' are vector bundles, resp., algebra bundles, so are f^*B and f^*B' and we have $f^*(B \oplus B') = f^*B \oplus f^*B'$, $f^*(B \otimes B') = f^*B \otimes f^*B'$, etc.*

4. *If $\sigma\colon N \to B$ is a section of B then $f^*\sigma = \sigma \circ f$ is a section of f^*B.*

5. *If $M = N$ and $f = \mathrm{id}_M$, then B and f^*B are equivalent.*

6. *If f is constant, then f^*B is equivalent to a trivial bundle.*

7. *If $g\colon P \to M$ is differentiable, then $(f \circ g)^*B = g^*f^*B$.*

E. g., if $B(M,F,G)$ with projection $\hat{\pi}\colon B \to M$ is associated with the principal bundle $P(M,G)$ then $\hat{\pi}^*P$ according to Lemma 2.28.2 is equivalent to the principal bundle from Lemma 2.18. Now the following theorem holds ([8, p. 53]):

Theorem 2.29 *Let $B(N,F,G)$ be a bundle and M a paracompact manifold. If $f_i\colon M \to N$, $i = 1,2$, are homotopic C^∞-maps then f_1^*B and f_2^*B are equivalent.*

Corollary 2.30 *Every bundle over a contractible, paracompact base manifold is equivalent to a trivial bundle.*

Proof: immediate from $\mathrm{id}_M \sim c$, with constant c, and Lemma 2.28.3 and 2.28.4. □

We close this section with the notion of the square of a bundle ([8, p. 49]).

Definition 2.31 *The* square of a bundle $B(M,F,G)$ *is defined to be the pullback bundle* $\pi^*B = \{(b,b') \in B \times B | \pi(b) = \pi(b')\}$ *with base* B*, fiber* F *and group* G*. The square of* B *admits a* natural cross-section $f: B \to \pi^*B$, $f(b) = (b,b)$.

If $B = P$ is a principal bundle then $\pi^*P \cong P \times G$ by the Cross-section theorem. The trivialization $\psi': \pi^*P \to P \times G$ is given by $\psi'^{-1}(p,g) = (p, R(g,p))$.

2.2 Connections on Principal Bundles

Every bundle chart for a fiber bundle B also induces a local trivialization of the tangent bundle of the given bundle: every tangent space splits into the direct product of a horizontal and a vertical subspace: $T_b(B) \cong H_b(B) \oplus V_b(B)$. Only the latter, consisting of all vectors tangential to the fiber, is given naturally and thus globally, as we have shown. Fixing global horizontal subspaces requires a new structure — a *connection* — on the bundle. Yet before we define connections on principal bundles, let us give the notion of fundamental vector fields.

Lemma 2.32 *If* R *means the right* LIE *group action on a principal bundle* $P(M,G)$ *and* $\mathfrak{g} = \mathbf{L}(G)$*, then* $(dR^p)_e: \mathfrak{g} \to V_p(P)$ *is a linear isomorphism for all* $p \in P$ *and every* $X \in \mathfrak{g}$ *induces a vector field* $\mathcal{R}_X \in v\mathcal{D}^1(P)$ *by* $(\mathcal{R}_X)_p := (dR^p)_e(X)$. $\mathcal{R}: \mathfrak{g} \to \mathcal{D}^1(P)$ *is an injective* LIE *algebra homomorphism with*

$$\begin{aligned}
[\mathcal{R}_X, \mathcal{R}_Y] &= \mathcal{R}_{[X,Y]}, \quad (R_{g^{-1}})_*\mathcal{R}_X = \mathcal{R}_{\mathrm{Ad}(g)X}, \quad \text{for all } g \in G, \quad X,Y \in \mathfrak{g}, \\
[\mathcal{R}_X, \mathcal{Y}] &= \lim_{t\to 0} \frac{1}{t}\{\mathcal{Y} - ((R_{e^{tX}})_*\mathcal{Y})\} \quad \text{for all } \mathcal{Y} \in \mathcal{D}^1(P), \quad X \in \mathfrak{g}. \\
(\mathcal{R}_X)_p(f) &= \lim_{t\to 0} \frac{1}{t}\{f(R(e^{tX},p)) - f(p)\} \quad \text{for all } f \in C^\infty(P),\ p \in P,\ X \in \mathfrak{g}.
\end{aligned}$$

$\mathcal{R}$ *induces a isomorphism* $\mathcal{R}': C^\infty(P,\mathfrak{g}) \to v\mathcal{D}^1(P)$ *of* $C^\infty(P)$*-modules; for every basis* $\{E_i\}_{i=1,\dots,\dim\mathfrak{g}}$ *for* $\mathfrak{g}$, $\{\mathcal{R}_{E_i}\}_{i=1,\dots,\dim\mathfrak{g}}$ *is a basis for the free* $C^\infty(P)$*-module* $v\mathcal{D}^1(P)$.

Proof. Since $\pi \circ R^p = \pi(p): G \to M$ is constant for all $p \in P$, one has $d\pi \circ dR^p = 0$. So $(dR^p)_e$ maps into $V_p(P)$. Anything else follows from Lemmas 1.91 and 1.94: just observe that for every $p \in P$ a (EUCLIDEAN) open neighborhood W exists such that $v\mathcal{D}^1(W)$ and $C^\infty(W)\mathcal{R}(\mathfrak{g})|_W \subseteq v\mathcal{D}^1(W)$ are both free modules of the same rank. Thus they are equal and we get $v\mathcal{D}^1(P) = C^\infty(P)\mathcal{R}(\mathfrak{g}) = \mathcal{R}'(C^\infty(P) \otimes \mathfrak{g})$. □

Definition 2.33 $\mathcal{R}_X$ *is called the* fundamental vector field corresponding to $X \in \mathfrak{g}$.

Definition 2.34 *A* connection Γ *on* $P(M,G)$ *associates with every* $p \in P$ *a horizontal subspace* $H_p(P) < T_p(P)$ *such that*

1. $T_p(P) = V_p(P) \oplus H_p(P)$ *with pointwise projections* $v_p: T_p(P) \to V_p(P)$ *and* $h_p: T_p(P) \to H_p(P)$;

2. vertical *and* horizontal projections v, h of vector fields *exist with:*

$$v: \mathcal{D}^1(P) \to v\mathcal{D}^1(P) = v(\mathcal{D}^1(P)) \subseteq \mathcal{D}^1(P) \; : \; X \mapsto vX, (vX)_p := v_p X_p$$
$$h: \mathcal{D}^1(P) \to h\mathcal{D}^1(P) := h(\mathcal{D}^1(P)) \subseteq \mathcal{D}^1(P) \; : \; X \mapsto hX, (hX)_p := h_p X_p;$$

3. $(R_g)_* H_p(P) = H_{R(g,p)}(P)$ *for all* $p \in P$, $g \in G$.

$\gamma(P(M,G))$ *denotes the set of all connections on* $P(M,G)$.

We also have $(R_g)_* V_p(P) = V_{R(g,p)}(P)$, so instead of 3. we could as well require that v, h commute with all $(R_g)_*$:

$$v \circ (R_g)_* = (R_g)_* \circ v, \qquad h \circ (R_g)_* = (R_g)_* \circ h, \qquad \text{for all} \quad g \in G.$$

Lemma 2.35 *For every* $X \in \mathfrak{g}$ *and all* $\mathcal{Y} \in h\mathcal{D}^1(P)$ *we have* $[\mathcal{R}_X, \mathcal{Y}] \in h\mathcal{D}^1(P)$. *If* $\mathcal{Y}$ *is invariant, resp.,* $\mathfrak{g}$*-invariant then* $[\mathcal{R}_X, \mathcal{Y}] = 0$.

Proof. By definition of a connection, $(R_{e^{tX}})_* \mathcal{Y} \in h\mathcal{D}^1(P)$ for all $\mathcal{Y} \in h\mathcal{D}^1(P)$ and all $t \in \mathbb{R}$. Thus $[\mathcal{R}_X, \mathcal{Y}] \in h\mathcal{D}^1(P)$ by Lemma 2.32. For the second statement, recall Definition 1.92 and $L_{\mathcal{R}_X} \mathcal{Y} = [\mathcal{R}_X, \mathcal{Y}]$ from Proposition 1.38.4 □

In the language of vector bundles, Definition 2.34 is equivalent to ([9, p. 276]):

Definition 2.36 *A connection* Γ *on a principal bundle is a vector subbundle* $H(P)$ *of* $T(P)$ *such that*

1. $H(P)$ *is complementary to the vertical bundle:* $T(P) = H(P) \oplus V(P)$,

2. $H(P)$ *is homogeneous:* $(R_g)_* H_p(P) = H_{R(g,p)}(P)$ *for all* $p \in P$, $g \in G$.

$H(P)$ *is called the* horizontal bundle, $h\mathcal{D}^1(P)$ *contains its sections, the* horizontal vector fields. *Thus the* $C^\infty(P)$*-module* $\mathcal{D}^1(P)$ *splits into* $\mathcal{D}^1(P) = h\mathcal{D}^1(P) \oplus v\mathcal{D}^1(P)$.

Definition 2.37 *Every connection* Γ *defines a* connection 1-form $\omega^\Gamma \in \mathcal{A}_1(P, \mathfrak{g})$ *by*

$$\omega^\Gamma(X)(p) = \omega^\Gamma(vX)(p) = (dR^p)^{-1}(v_p X_p) \qquad \textit{for all} \quad X \in \mathcal{D}^1(P).$$

ω^Γ is well-defined: we have $\omega^\Gamma = \mathcal{R}'^{-1} \circ v$. Obviously $\omega^\Gamma \circ h = 0$ and $(R^p)^* \omega^\Gamma = \Theta^L$ for all $p \in P$, since for $\mathcal{X} \in \mathcal{D}^1(G)$ and $g \in G$, $[(R^p)^* \omega^\Gamma(\mathcal{X})](g) = \omega^\Gamma_{R(g,p)}[(dR^p)_g \mathcal{X}_g] = \omega^\Gamma_{R(g,p)}[(dR^{R(g,p)})_e (d\lambda_{g^{-1}})_g \mathcal{X}_g)] = (d\lambda_{g^{-1}})_g \mathcal{X}_g = [\Theta^L(\mathcal{X})](g)$ by (38). Define

$$\mathcal{A}_\gamma(P(M,G)) := \left\{ \omega \in \mathcal{A}_1(P, \mathfrak{g}) \;\middle|\; \begin{array}{ll} \omega \circ \mathcal{R}' = \mathrm{id}_{C^\infty(P,\mathfrak{g})} & \text{and} \\ R_g^* \omega = \mathrm{Ad}(g^{-1})_* \omega & \text{for all } g \in G \end{array} \right\}.$$

Then one quickly verifies using Lemma 2.32 and the homogenity of $H(P)$:

Proposition 2.38 $(\Gamma \mapsto \omega^\Gamma)$: $\gamma(P(M,G)) \to \mathcal{A}_\gamma(P(M,G))$ *is bijective.*

For the inverse mapping, define Γ by its projections:

$$v := \mathcal{R}' \circ \omega, \qquad h := \mathrm{id}_{\mathcal{D}^1(P)} - \mathcal{R}' \circ \omega \qquad \text{for all} \quad \omega \in \mathcal{A}_\gamma(P(M,G)).$$

Then $v \circ v = v$ and $v \circ (R_g)_* = \mathcal{R}' \circ \mathrm{Ad}(g^{-1}) \circ \omega = (R_g)_* \circ \mathcal{R}' \circ \omega = (R_g)_* \circ v$. So every $\omega \in \mathcal{A}_\gamma(P(M,G))$ defines a connection Γ with $\omega(h\mathcal{D}^1(P)) = 0$, and then $\omega = \omega^\Gamma$. Proposition 1.97 yields for every $\omega \in \mathcal{A}_\gamma(P(M,G))$ and $X \in \mathfrak{g}$:

$$L_{\mathcal{R}_X}\omega = -\,\mathrm{ad}(X)_*\omega, \qquad \imath_{\mathcal{R}_X}\omega = X.$$

There are several ways how connections on principal bundles induce connections on other principal bundles. We only state (cf. [6, p. 81]):

Proposition 2.39 *Let $f: P'(M',G) \to P(M,G)$ be a G-equivariant mapping of principal bundles, i. e., $f \circ R'_g = R_g \circ f$ for all $g \in G$ (recall* Definition 1.89*), then $f^*\omega \in \mathcal{A}_\gamma(P'(M',G))$ for all $\omega \in \mathcal{A}_\gamma(P(M,G))$, thus every connection Γ on P induces a unique connection $\Gamma' = f^*\Gamma$ on P', such that f_* maps horizontal subspaces of Γ' into horizontal subspaces of Γ. In particular:*

1. *If $f: M' \to M$ is differentiable, then every connection on $P(M,G)$ induces a connection on the pullback bundle $f^*P(M'G)$.*

2. *Let U be open in M and $i: \pi^{-1}(U) \to P(M,G)$ denote the embedding. Then $i^*\omega \in \mathcal{A}_\gamma(\pi^{-1}(U))$ for all $\omega \in \mathcal{A}_\gamma(P(M,G))$. Thus every connection Γ on P induces a connection $\Gamma|_U$ on $\pi^{-1}(U)$.*

3. *Let $f: P(M,G) \to P'(M',G)$ be a G-equivariant* diffeomorphism *of principal bundles, then every connection Γ on P induces a connection Γ^f on P' since*

$$\omega \in \mathcal{A}_\gamma(P(M,G)) \iff (f^{-1})^*\omega \in \mathcal{A}_\gamma(P'(M',G)).$$

Definition 2.40 *For any connection $\Gamma \in \gamma(P(M,G))$, we denote the set of all horizontal G-invariant vector fields by $\mathcal{D}^\Gamma(P(M,G)) := h\mathcal{D}^1(P)_{\mathrm{inv}}$, i. e.,*

$$\mathcal{D}^\Gamma(P(M,G)) := \{\mathcal{Y} \in \mathcal{D}^1(P) \mid \mathcal{Y} = h\mathcal{Y} \quad \textit{and} \quad (R_g)_*\mathcal{Y} = \mathcal{Y} \quad \textit{for all } g \in G\}.$$

Recall $[\mathcal{R}_X, \mathcal{Y}] = 0$ for all $\mathcal{Y} \in \mathcal{D}^\Gamma(P(M,G))$ and $X \in \mathfrak{g}$ from Lemma 2.35. $\mathcal{D}^\Gamma(P)$ is a $C^\infty(M)$-module, where scalar multiplication with $f \in C^\infty(M)$ is understood to be multiplication with π^*f, since $R_g^* \circ \pi^* = \pi^*$ for all $g \in G$. This module is isomorphic to $\mathcal{D}^1(M)$, as the following proposition shows (cf. [6, p. 65]):

Proposition 2.41 $\mathbb{L}: \mathcal{D}^1(M) \to \mathcal{D}^\Gamma(P(M,G))$, *where $\mathbb{L}\mathcal{X} \in h\mathcal{D}^1(P)$ is uniquely defined by $d\pi_p(\mathbb{L}\mathcal{X})_p = \mathcal{X}_{\pi(p)}$ for all $p \in P$, is an isomorphism of $C^\infty(M)$-modules. $\mathbb{L}[\mathcal{X},\mathcal{Y}] = h[\mathbb{L}\mathcal{X}, \mathbb{L}\mathcal{Y}]$ for all $\mathcal{X},\mathcal{Y} \in \mathcal{D}^1(M)$. We call $\mathbb{L}\mathcal{X}$ the* (horizontal) lift *of $\mathcal{X}$, with inverse morphism π_* and $\mathbb{L}_p: T_{\pi(p)}(M) \to H_p(P)$ denotes the local inverse of the differential $d\pi_p$.*

Thus every connection Γ defines a $C^\infty(M)$-isomorphism $\mathbb{L}^\Gamma\colon \mathcal{D}^1(M) \to h\mathcal{D}^1(P)_{\text{inv}}$. Remember that $\gamma(P(M,G))$ and $\mathcal{A}_\gamma(P(M,G))$ are in bijective correspondence by Proposition 2.38: does such a bijection exist for connections and lifts, too? To be precise: does every $\mathbb{L} \in \mathrm{Hom}_{C^\infty(M)}(\mathcal{D}^1(M), \mathcal{D}^1(P)_{\text{inv}})$ with $\pi_* \circ \mathbb{L} = \mathrm{id}_{\mathcal{D}^1(M)}$ uniquely define a connection on P such that $\mathbb{L}$ maps onto $h\mathcal{D}^1(P)_{\text{inv}}$? This is indeed true: for all $p \in P$, $\mathbb{L}$ defines horizontal subspaces $H_p(P) := (\mathbb{L}T_{\pi(p)}(M))_p$ complementary to $V_p(P)$ and homogeneous with regard to $(R_g)_*$. Thus the horizontal projection is given by $h := [\mathbb{L}\pi_*]$, where $[\mathbb{L}\pi_*]\colon \mathcal{D}^1(P) \to \mathcal{D}^1(P)$ is defined by $[\mathbb{L}\pi_*]_p := \mathbb{L}_p d\pi_p$. Obviously h is pointwise well-defined and commutes with all $(R_g)_*$. We only have to look for differentiability. But this holds, because it holds locally on every bundle chart where we can trivialize our bundle. This proves:

Proposition 2.42 *The mapping that assigns a lift to every connection Γ on P:*

$$(\Gamma \mapsto \mathbb{L}^\Gamma)\colon \gamma(P(M,G)) \to \{\mathbb{L} \in \mathrm{Hom}_{C^\infty(M)}(\mathcal{D}^1(M), \mathcal{D}^1(P)_{\text{inv}}) | \pi_* \circ \mathbb{L} = \mathrm{id}_{\mathcal{D}^1(M)}\}$$

is bijective. For the inverse mapping, $h := [\mathbb{L}\pi_*]$, and the connection 1-form ω^Γ is

$$\omega^\Gamma = \mathcal{R}'^{-1} \circ v = \mathcal{R}'^{-1}(\mathrm{id}_{\mathcal{D}^1(P)} - [\mathbb{L}\pi_*]) = \mathcal{R}'^{-1} - \mathcal{R}'^{-1}[\mathbb{L}\pi_*].$$

Definition 2.43 *For any connection $\Gamma \in \gamma(P(M,G))$ and any $\omega_s \in \mathcal{A}_s(P,V)$, $s > 0$, where V is a vector space, we define horizontal and vertical projections $\omega_s h$, resp., $\omega_s v \in \mathcal{A}_s(P,V)$ by*

$$\begin{aligned} \omega_s h(\mathcal{X}^1,\ldots,\mathcal{X}^s) &:= \omega_s(h\mathcal{X}^1,\ldots,h\mathcal{X}^s), &&\text{for all } \mathcal{X}^i \in \mathcal{D}^1(P),\\ \omega_s v(\mathcal{X}^1,\ldots,\mathcal{X}^s) &:= \omega_s(v\mathcal{X}^1,\ldots,v\mathcal{X}^s), &&\text{for all } \mathcal{X}^i \in \mathcal{D}^1(P). \end{aligned}$$

$\mathcal{A}(P,V)h \subseteq \mathcal{A}(P,V)$ and $\mathcal{A}(P,V)v \subseteq \mathcal{A}(P,V)$ (with $\mathcal{A}_0(P,V)h := \mathcal{A}_0(P,V)v := \mathcal{A}_0(P,V) = C^\infty(P,V)$) denote the $C^\infty(P)$-submodules of $\mathcal{A}(P,V)$ that contain these horizontal, resp., vertical V-valued forms.

The following lemma justifies our previous Definition 1.92 of horizontal forms:

Lemma 2.44 *$\omega \in \mathcal{A}(P,V)$ is horizontal iff $\imath_{\mathcal{R}_X}\omega = 0$ for all $X \in \mathfrak{g}$.*

Proof. Since $h\mathcal{R}_X = 0$, one implication is obvious. So suppose $\omega \in \mathcal{A}_s(P,V)$, $s > 0$ (for $\omega \in \mathcal{A}_0(P,V)$ there is nothing to prove), and $\imath_{\mathcal{R}_X}\omega = 0$ for all $X \in \mathfrak{g}$. Then for $p \in P$, $\mathcal{X}^i \in \mathcal{D}^1(P)$ and any ω^Γ: $\omega_p h_p(\ldots,\mathcal{X}^i_p,\ldots) = \omega_p(\ldots,\mathcal{X}^i_p - v_p\mathcal{X}^i_p,\ldots) = \omega_p(\ldots,\mathcal{X}^i_p - (\mathcal{R}_{\omega^\Gamma_p(\mathcal{X}^i_p)})_p,\ldots) = \omega_p(\ldots,\mathcal{X}^i_p,\ldots)$ by multilinearity of ω_p. □

Lemma 2.45 *If $\Gamma \in \gamma(P(M,G))$ then $\mathcal{A}_1(P,V) = \mathcal{A}_1(P,V)h \oplus \mathcal{A}_1(P,V)v$ and*

1. *the projections of forms commute with $\wedge_\phi$: for $\alpha \in \mathcal{A}(P) \otimes V$, $\beta \in \mathcal{A}(P) \otimes W$*

$$(\alpha \wedge_\phi \beta)h = \alpha h \wedge_\phi \beta h, \qquad (\alpha \wedge_\phi \beta)v = \alpha v \wedge_\phi \beta v;$$

2. *h and v commute with* $\bullet$, $\blacktriangleleft$ *and* $\blacktriangleright$: *e. g. for* $\chi \in \mathcal{A}(P, \mathrm{Hom}(\mathcal{T}(W), Z))$, $\phi_r^q \in \mathcal{A}_r(P) \otimes \mathrm{Hom}(\bigotimes^q V, W)$

$$(\chi \bullet \phi_r^q)h = \chi h \bullet \phi_r^q h, \qquad (\chi \bullet \phi_r^q)v = \chi v \bullet \phi_r^q v;$$

3. *h and v commute with the right action on P:*

$$R_g^* \circ h = h \circ R_g^*, \qquad R_g^* \circ v = v \circ R_g^*, \qquad \text{for all } g \in G, \text{ and thus}$$
$$L_{\mathcal{R}_X} \circ h = h \circ L_{\mathcal{R}_X}, \qquad L_{\mathcal{R}_X} \circ v = v \circ L_{\mathcal{R}_X}, \qquad \text{for all } X \in \mathfrak{g}.$$

Definition 2.46 *Let* $P(M,G)$ *be a principal bundle and* $L: G \times V \to V$ *a (left) representation of G on a vector space V. Then a* pseudotensorial form of type (L,V) *is a V-valued form* $\omega \in \mathcal{A}(P) \otimes V$ *such that*

$$R_g^*\omega = (L_{g^{-1}})_*\omega \qquad \text{for all} \quad g \in G.$$

If ω *is horizontal, it is called a* tensorial form of type (L,V). *Let* $\mathcal{A}^P(P,L,V)$ *and* $\mathcal{A}^T(P,L,V)$ *denote the sets of pseudotensorial, resp., tensorial forms of type* (L,V). *For* $V = \mathfrak{g}$, *we put* $\mathcal{A}^P(P,\mathfrak{g}) := \mathcal{A}^P(P,\mathrm{Ad},\mathfrak{g})$ *and* $\mathcal{A}^T(P,\mathfrak{g}) := \mathcal{A}^T(P,\mathrm{Ad},\mathfrak{g})$.

$\mathcal{A}^P(P,L,V)$ and $\mathcal{A}^T(P,L,V)$ are $C^\infty(M)$-modules in the above sense. If L_0 is the trivial representation of G on V, then a tensorial form of type (L_0,V) is just a pullback $\pi^*\varphi$ with $\varphi \in \mathcal{A}(M) \otimes V$. Let $E(M,V,G)$ be the vector bundle associated with P with left action L. A tensorial r-form φ of type (L,V) may be regarded as an alternating $C^\infty(M)$-linear map $\bar{\varphi}: \mathcal{D}^r(M) \to \Gamma E$ uniquely defined by

$$\bar{\varphi}(\mathcal{X}^1,\ldots,\mathcal{X}^r) \circ \pi = \tilde{\pi} \circ (\mathrm{id}_P, \varphi(\mathbf{L}\mathcal{X}^1,\ldots,\mathbf{L}\mathcal{X}^r)). \tag{58}$$

In particular, a tensorial 0-form of type (L,V), i. e., map $f: P \to V$ with $f(R_g(p)) = L_{g^{-1}}(f(p))$, can be identified with a cross-section $\bar{f}: M \to E$, cf. [6, pp. 75 – 76].

Recall that in the sense of Section 1.5, pseudotensorial forms of type (L,V) are exactly the G-equivariant forms in $\mathcal{A}(P)\otimes V$ with regard to R and L; tensorial forms are those where in addition, $\iota_{\mathcal{R}_X}\omega = 0$ for all $X \in \mathfrak{g}$. For the induced representation l according to Note 1.95, we have

$$L_{\mathcal{R}_X}\omega = -(l_X)_*\omega \qquad \text{for all} \quad X \in \mathfrak{g}$$

and all pseudotensorial forms ω, cf. Propositions 1.93 and 1.97.

Lemma 2.47 1. *The definitions of* $\mathcal{A}(P,V)h$, $\mathcal{A}^P(P,L,V)$ *and* $\mathcal{A}^T(P,L,V)$ *are independent of* $\Gamma \in \gamma(P(M,G))$;

2. $\mathcal{A}_\gamma(P(M,G)) \subseteq \mathcal{A}_1^P(P,\mathfrak{g})$;

3. $\mathcal{A}^T(P,L,V) = \mathcal{A}^P(P,L,V)h$;

4. $d(\mathcal{A}_r^P(P,L,V)) \subseteq \mathcal{A}_{r+1}^P(P,L,V)$.

Lemma 2.48 *$\mathcal{A}^P(P,L,V) = \mathcal{A}^T(P,L,V)$ for all (L,V) iff $\mathfrak{g} = \{0\}$, i. e., iff G is discrete; in that case $\mathcal{A}(P)_{\text{inv}} \otimes V \cong \mathcal{A}(M) \otimes V = \mathcal{A}(P/G) \otimes V$.*

Proof. For $0 \neq X \in \mathfrak{g}$, we have $0 \neq \mathcal{R}_X \in \mathcal{D}^1(P)$. Take its dual $\rho_X \in \mathcal{A}_1(P)$ (cf. Lemma 1.4). Then $\omega := \rho_X \otimes X \in \mathcal{A}_1(P,\mathfrak{g})$ with $\omega h = 0$. On the other hand, if $\mathfrak{g} = \{0\}$ then every $\omega \in \mathcal{A}(P,V)$ is horizontal according to Lemma 2.44 and Lemma 2.47.3 applies. Finally $\mathcal{A}(P)_{\text{inv}} \otimes V = \mathcal{A}^P(P,L_0,V) = \mathcal{A}^T(P,L_0,V) \cong \mathcal{A}(M) \otimes V$. □

Definition 2.49 *For any connection $\Gamma \in \gamma(P(M,G))$ and vector space V, the* exterior covariant differentiation *$d^\Gamma : \mathcal{A}(P)\otimes V \to \mathcal{A}(P)h\otimes V$ is defined by $d^\Gamma \varphi := (d\varphi)h$.*

Lemmas 1.25 and 2.45 and Corollary 1.56 prove:

Lemma 2.50 *For any connection $\Gamma \in \gamma(P(M,G))$ and $p,q,r,s \in \mathbb{N}_0$, we have*

1. *$d^\Gamma \circ R_g^* = R_g^* \circ d^\Gamma$ for all $g \in G$, thus $d^\Gamma(\mathcal{A}_r^P(P,L,V)) \subseteq \mathcal{A}_{r+1}^T(P,L,V)$;*
2. *$d^\Gamma \circ \pi^* = \pi^* \circ d$ (with d on $\mathcal{A}(M) \otimes V$);*
3. *for all $\alpha_r \in \mathcal{A}_r(P) \otimes V$, $\beta \in \mathcal{A}(P) \otimes W$ and bilinear $\phi: V \times W \to Z$,*

$$d^\Gamma(\alpha_r \wedge_\phi \beta) = (d^\Gamma \alpha_r) \wedge_\phi \beta h + (-1)^r \alpha h \wedge_\phi (d^\Gamma \beta),$$

   analogous statements hold for $\wedge$, $\wedge_V$, etc.;

4. *for all $\chi_r^s \in \mathcal{A}_r(P) \otimes \text{Hom}(\bigotimes^s W, Z)$ and $\phi_p^q \in \mathcal{A}_p(M) \otimes \text{Hom}(\bigotimes^q V, W)$,*

$$d^\Gamma(\chi_r^s \bullet \phi_p^q) = (d^\Gamma \chi)_{r+1}^s \bullet \phi_p^q h + \sum_{j=0}^{s-1} (-1)^{r+jp} [(\chi_r^{j;s-j} h \blacktriangleleft \phi_p^q h)^{1;s-j-1} \blacktriangleleft (d^\Gamma \phi_p^q)] \blacktriangleright \phi_p^q h.$$

Corollary 2.51 *If a bilinear $\phi: V\times V \to V$, resp., the induced linear $\phi': V\otimes V \to V$ is G-equivariant in the sense that for a left representation $L: G\times V \to V$ and all $g \in G$, $v, w \in V$, $\phi(L(g,v), L(g,w)) = \phi'(L(g, v \otimes w)) = L(g, \phi'(v \otimes W)) = L(g, \phi(v,w))$ holds, then d^Γ is a skew-derivation of $\mathcal{A}^T(P,L,V)$ with regard to $\wedge_\phi$ of degree 1. Examples are $\mathcal{A}(P,\mathfrak{g})$ with regard to $\wedge_\mathfrak{g}$ and $\mathcal{A}(P,L_0,V)$ with regard to any $\wedge_V$.*

Lemma 1.90 yields:

Lemma 2.52 *Let $L^*: \text{Hom}(\mathcal{T}(W),V) \to \text{Hom}(\mathcal{T}(W),V)$ be the representation that is induced by a left representation $L: G \times W \to W$, i. e., $(L^*)_g := (L_g)^*$ for all $g \in G$. Then*

$$\bullet: \mathcal{A}^P(P,L^*,\text{Hom}(\mathcal{T}(W),V)) \times \mathcal{A}_r^P(P,L,W) \to \mathcal{A}^P(P,L_0,V) \text{ for all } r \in \mathbb{N}_0 \text{ and}$$
$$\bullet: \mathcal{A}^T(P,L^*,\text{Hom}(\mathcal{T}(W),V)) \times \mathcal{A}_r^T(P,L,W) \to \mathcal{A}^T(P,L_0,V) = \pi^*\mathcal{A}(M) \otimes V.$$

Definition 2.53 *$\Omega^\Gamma := d^\Gamma \omega^\Gamma \in \mathcal{A}_2^T(P,\mathfrak{g})$ is called* curvature 2-form *for Γ.*

Lemma 2.54 *Let $f: P'(M',G) \to P(M,G)$ be a G-equivariant mapping of principal bundles, let $\Gamma \in \gamma(P(M,G))$ and $f^*\Gamma$ be the induced connection on P'. Then*

1. *$f^*\varphi \in \mathcal{A}^P(P',L,V)$ for all $\varphi \in \mathcal{A}^P(P,L,V)$,*
2. *$f^*\varphi \in \mathcal{A}^T(P',L,V)$ for all $\varphi \in \mathcal{A}^T(P,L,V)$,*
3. *$f^*(d^\Gamma\varphi) = d^{f^*\Gamma}(f^*\varphi)$ for all $\varphi \in \mathcal{A}^P(P) \otimes V$, thus $\Omega^{f^*\Gamma} = f^*\Omega^\Gamma$.*

Analogous statements hold for $\Gamma|_U$ and Γ^f from Proposition 2.39.

Proposition 2.39 and Lemma 2.54 show that any connection Γ on a principal bundle induces connections $(\Gamma|_{\pi^{-1}(U_\alpha)})^{\psi_\alpha}$ on $U_\alpha \times G$ for every bundle chart (U_α, ψ_α). Thus a closer look on connections on trivial principal bundles is worth-while.

Lemma 2.55 *Let $\omega^\Gamma \in \mathcal{A}_\gamma(M \times G)$ be a connection 1-form on the trivial bundle, $x \in M$, $g,h \in G$, $Y \in \mathfrak{g}$ and $(\mathcal{X}_x^{(i)}, \mathcal{Y}_g^{(i)}) \in T_{(x,g)}(M \times G) \cong T_x(M) \oplus T_g(G)$. Then*

1. *$(\mathcal{R}_Y)_{(x,g)} = (0, d\lambda_g(Y))$, $\quad dR_h(\mathcal{X}_x, \mathcal{Y}_g) = (\mathcal{X}_x, d\rho_h(\mathcal{Y}_g))$,*
2. *$\omega^\Gamma_{(x,g)}(0, \mathcal{Y}_g) = d\lambda_{g^{-1}}(\mathcal{Y}_g)$, $\quad \omega^\Gamma_{(x,gh)}(\mathcal{X}_x, d\rho_h(\mathcal{Y}_g)) = \mathrm{Ad}(h^{-1})[\omega^\Gamma_{(x,g)}(\mathcal{X}_x, \mathcal{Y}_g)]$,*
3. *$(\forall \Omega \in \mathcal{A}^T(M \times G, \mathfrak{g}))\ \Omega_{(x,g)}(\ldots, (\mathcal{X}_x^i, \mathcal{Y}_g^i), \ldots) = \mathrm{Ad}(g^{-1})[\Omega_{(x,e)}(\ldots, (\mathcal{X}_x, 0), \ldots)]$,*
4. *$(\forall \varphi \in \mathcal{A}^T(M \times G, L, V))\ \varphi_{(x,g)}(\ldots, (\mathcal{X}_x^i, \mathcal{Y}_g^i), \ldots) = L_{g^{-1}}[\varphi_{(x,e)}(\ldots, (\mathcal{X}_x, 0), \ldots)]$.*

Proof. $dR^{(x,e)}Y = (0,Y)$ and the definition of a principal bundle yield *1.*, while *2.*, *3.* and *4.* follow from the properties of connection 1-forms and tensorial forms. □

Recall the notation of (local) sections $\sigma_{\alpha,e}: U_\alpha \to \pi^{-1}(U_\alpha)$, $x \mapsto \psi_\alpha^{-1}(x,e)$. For a trivial bundle $M \times G$ we have $\omega_{(x,e)}(\ldots, (\mathcal{X}_x^i, 0), \ldots) = (\sigma_e^*\omega)_x(\ldots, \mathcal{X}_x^i, \ldots)$ for any $\omega \in \mathcal{A}(M \times G, V)$. If ω^Γ is a connection 1-form then Lemma 2.55.2 yields

$$\begin{aligned}\omega^\Gamma_{(x,g)}(\mathcal{X}_x, \mathcal{Y}_g) &= \mathrm{Ad}(g^{-1})[\omega^\Gamma_{(x,e)}(\mathcal{X}_x, 0) + \omega^\Gamma_{(x,e)}(0, d\rho_{g^{-1}}(\mathcal{Y}_g))] \\ &= \mathrm{Ad}(g^{-1})[(\sigma_e^*\omega^\Gamma)_x(\mathcal{X}_x)] + d\lambda_{g^{-1}}(\mathcal{Y}_g).\end{aligned}$$

Thus $\sigma_e^*\omega^\Gamma$ determines ω^Γ completely (analogously for tensorial forms) and we get:

Proposition 2.56 *Let $(x,g) \in M \times G$, $\mathcal{X}_x^{(i)} \in T_x(M)$ and $\mathcal{Y}_g^{(i)} \in T_g(G)$.*

1. *$\sigma_e^*: \mathcal{A}_\gamma(M \times G) \to \mathcal{A}_1(M, \mathfrak{g})$ is bijective and for all $\omega \in \mathcal{A}_\gamma(M \times G)$*

$$\begin{aligned}\omega_{(x,g)}(\mathcal{X}_x, \mathcal{Y}_g) &= \mathrm{Ad}(g^{-1})[(\sigma_e^*\omega)_x(\mathcal{X}_x)] + d\lambda_{g^{-1}}(\mathcal{Y}_g), \\ \text{i. e.} \qquad \omega &= (\mathrm{Ad} \circ \eta \circ \mathrm{pr}_G) \bullet (\mathrm{pr}_M^* \sigma_e^*\omega) + \mathrm{pr}_G^* \Theta^L;\end{aligned}$$

2. *$\sigma_e^*: \mathcal{A}^T(M \times G, \mathfrak{g}) \to \mathcal{A}(M, \mathfrak{g})$ is bijective and for all $\Omega \in \mathcal{A}_\gamma^T(M \times G, \mathfrak{g})$*

$$\begin{aligned}\Omega_{(x,g)}(\ldots, (\mathcal{X}_x^i, \mathcal{Y}_g^i), \ldots) &= \mathrm{Ad}(g^{-1})[(\sigma_e^*\Omega)_x(\ldots, \mathcal{X}_x^i, \ldots)], \\ \text{i. e.} \qquad \Omega &= (\mathrm{Ad} \circ \eta \circ \mathrm{pr}_G) \bullet (\mathrm{pr}_M^* \sigma_e^*\Omega);\end{aligned}$$

3. $\sigma_e^*: \mathcal{A}^T(M\times G, L, V) \to \mathcal{A}(M)\otimes V$ *is bijective and for all* $\varphi \in \mathcal{A}^T(M\times G, L, V)$

$$\begin{aligned} \varphi_{(x,g)}(\ldots,(\mathcal{X}_x^i,\mathcal{Y}_g^i),\ldots) &= L(g^{-1},[(\sigma_e^*\varphi)_x(\ldots,\mathcal{X}_x^i,\ldots)]), \\ \text{i. e.} \qquad \varphi &= (L\circ\eta\circ \mathrm{pr}_G)\bullet(\mathrm{pr}_M^*\sigma_e^*\varphi). \end{aligned}$$

(Note that we have identified $L: G\times V\to V$ *and* $L: G\to \mathrm{Gl}(V)$.*)*

Definition 2.57 *The* canonical flat connection *on the trivial bundle* $M\times G$ *is the connection* Γ *with* $\omega^\Gamma = \mathrm{pr}_G^*\Theta^L$. *A connection* Γ *on any principal bundle* $P(M,G)$ *is called* flat, *if for every* $x\in M$ *a bundle chart* (U_α,ψ_α) *with* $x\in U_\alpha$ *exists such that* $\omega^\Gamma|_{\pi^{-1}(U_\alpha)} = \psi_\alpha^*\,\mathrm{pr}_G^*\Theta^L = \pi_\alpha^*\Theta^L$.

Theorem 2.58 *Let* Γ *be a connection on* $P(M,G)$ *and* $l: \mathfrak{g}\times V\to V$ *be the bilinear mapping induced by* $L: G\times V\to V$ *according to* Note 1.95. *Then for* $m\in\mathbb{N}$

$$(\forall\varphi\in\mathcal{A}^T(P,L,V)) \qquad d^\Gamma\varphi = d\varphi+\omega^\Gamma\wedge_l\varphi, \tag{59}$$

$$(\forall\varphi\in\mathcal{A}^T(P,L,V)) \quad (d^\Gamma)^{2m}\varphi = \underbrace{\Omega^\Gamma\wedge_l(\Omega^\Gamma\wedge_l\cdots\wedge_l(\Omega^\Gamma}_{m}\wedge_l\varphi)\cdots), \tag{60}$$

$$(\forall\varphi\in\mathcal{A}^P(P,L,V)) \quad (d^\Gamma)^{2m+1}\varphi = \underbrace{\Omega^\Gamma\wedge_l(\Omega^\Gamma\wedge_l\cdots\wedge_l(\Omega^\Gamma}_{m}\wedge_l d^\Gamma\varphi)\cdots), \tag{61}$$

$$(\forall\varphi\in\mathcal{A}^P(P,L,V)) \quad (d^\Gamma)^{2m}\varphi = \underbrace{\Omega^\Gamma\wedge_l(\Omega^\Gamma\wedge_l\cdots\wedge_l(\Omega^\Gamma}_{m-1}\wedge_l(d^\Gamma)^2\varphi)\cdots). \tag{62}$$

Proof. To prove (59), we show for all $\varphi\in\mathcal{A}_r^T(P,L,V)$, $p\in P$ and $X^i\in T_p(P)$ that

$$(d^\Gamma\varphi)_p(X^1,\ldots,X^{r+1}) = d\varphi_p(X^1,\ldots,X^{r+1}) + (\omega^\Gamma\wedge_l\varphi)_p(X^1,\ldots,X^{r+1}).$$

3 cases have to be distinguished: (i) All X^i are horizontal. This is trivial. (ii) At least two $X^i, X^j, i\neq j$, are vertical. This is trivial, too, since all terms are zero. (For $d\varphi_p$ use Definition 1.17 with fundamental vector fields $\mathcal{R}_{\omega_p^\Gamma X^i}, \mathcal{R}_{\omega_p^\Gamma X^j}$ and observe that $[\mathcal{R}_{\omega_p^\Gamma X^i}, \mathcal{R}_{\omega_p^\Gamma X^j}] = \mathcal{R}_{[\omega_p^\Gamma X^i,\omega_p^\Gamma X^j]}\in V_p(P)$.) So only (iii) remains, where $X^i\in H_p(P)$, $i=1,\ldots,r$ and $X^{r+1}\in V_p(P)$. Let $\mathcal{X}^i\in\mathcal{D}^\Gamma(P(M,G))$ with $\mathcal{X}_p^i = X^i$, $i=1,\ldots,r$ and $A=\omega_p^\Gamma X^{r+1}\in\mathfrak{g}$, such that $(\mathcal{R}_A)_p = X^{r+1}$. The first term is zero, thus by Definition 1.17 and because $[\mathcal{R}_A,\mathcal{X}^i]=0$ by Lemma 2.35 we have to prove $(\mathcal{R}_A)_p(\varphi(\mathcal{X}^1,\ldots,\mathcal{X}^r)) = -l(A,\varphi_p(X^1,\ldots,X^r))$. Lemma 2.32 yields

$$\begin{aligned} (\mathcal{R}_A)_p(\varphi(\mathcal{X}^1,\ldots,\mathcal{X}^r)) &= \lim_{t\to 0}\tfrac{1}{t}\{[\varphi(\mathcal{X}^1,\ldots,\mathcal{X}^r)](R(e^{tX},p)) - [\varphi(\mathcal{X}^1,\ldots,\mathcal{X}^r)](p)\} \\ &= \lim_{t\to 0}\tfrac{1}{t}\{[(R_{e^{tA}}^*\varphi)(\mathcal{X}^1,\ldots,\mathcal{X}^r)](p) - \varphi_p(X^1,\ldots,X^r)\} \\ &= \lim_{t\to 0}\tfrac{1}{t}\{L(e^{-tA},\varphi_p(X^1,\ldots,X^r)) - \varphi_p(X^1,\ldots,X^r)\}. \end{aligned}$$

Since l was suppose to be induced by L, (43) proves our claim.

For (60) and (61) for $\varphi\in\mathcal{A}_r^T(P,L,V)$, observe $(d^\Gamma)^2\varphi = d^\Gamma\omega^\Gamma\wedge_l\varphi h - \omega^\Gamma h\wedge_l d^\Gamma\varphi = \Omega^\Gamma\wedge_l\varphi$. Now the equations follow by induction because $(d^\Gamma)^i\varphi\in\mathcal{A}^T(P,L,V)$ again. Finally the equations for $\varphi\in\mathcal{A}^P(P,L,V)$ result from $d^\Gamma\varphi\in\mathcal{A}^T(P,L,V)$. □

Theorem 2.59 (Cartan's structure equation and Bianchi's identity) *Let Γ be a connection on $P(M,G)$ and $m \in \mathbb{N}$, then the following equalities hold:*

$$\begin{aligned}
\text{structure equation:} \quad \Omega^\Gamma &= d\omega^\Gamma + \tfrac{1}{2}\omega^\Gamma \wedge_{\mathfrak{g}} \omega^\Gamma; \\
\text{BIANCHI'S identity:} \quad d^\Gamma \Omega^\Gamma &= d\Omega^\Gamma + \omega^\Gamma \wedge_{\mathfrak{g}} \Omega^\Gamma = 0; \\
\textit{for all } \varphi \in \mathcal{A}^T(P,\mathfrak{g}): \quad d^\Gamma \varphi &= d\varphi + \omega^\Gamma \wedge_{\mathfrak{g}} \varphi, \\
(d^\Gamma)^{2m}\varphi = \underbrace{\Omega^\Gamma \wedge_{\mathfrak{g}} (\cdots (\Omega^\Gamma}_{m} \wedge_{\mathfrak{g}} \varphi)\cdots), \quad (d^\Gamma)^{2m+1}\varphi &= \underbrace{\Omega^\Gamma \wedge_{\mathfrak{g}} (\cdots (\Omega^\Gamma}_{m} \wedge_{\mathfrak{g}} d^\Gamma \varphi)\cdots).
\end{aligned}$$

Proof: analogous to the proof of Theorem 2.58, cf. [6, pp. 77 – 79]. Nevertheless observe that the last equations are just a corollary to Theorem 2.58. □

Suppose Γ is the canonical flat connection on $M \times G$. Then the structure equation yields $\Omega^\Gamma = \mathrm{pr}_G^*(\Theta^L + \frac{1}{2}\Theta^L \wedge_{\mathfrak{g}} \Theta^L)$. Since $(\mathrm{pr}_G \circ \sigma_e)^* = 0$, Proposition 2.56.2 yields that $\Omega^\Gamma = 0$ and thus $\Theta^L + \frac{1}{2}\Theta^L \wedge_{\mathfrak{g}} \Theta^L = 0$. So the MAURER-CARTAN identities are just a corollary to CARTAN'S structure equation.

The curvature 2-form vanishes not only for the canonical flat connection. Indeed, we have the following theorem (cf. [6, pp. 92 – 93]):

Theorem 2.60 *A connection Γ in $P(M,G)$ is flat iff its curvature 2-form vanishes identically. If in addition M is paracompact and simply connected, then P is isomorphic to the trivial bundle and Γ is isomorphic to the canonical flat connection.*

For a connection Γ on any principal bundle we define for every bundle chart

$$\mathrm{A}^\alpha := \sigma_{\alpha,e}^*(\omega^\Gamma|_{\pi^{-1}(U_\alpha)}) \in \mathcal{A}_1(U_\alpha,\mathfrak{g}), \quad \mathrm{F}^\alpha := \sigma_{\alpha,e}^*(\Omega^\Gamma|_{\pi^{-1}(U_\alpha)}) \in \mathcal{A}_2(U_\alpha,\mathfrak{g}). \tag{63}$$

Then by Proposition 2.56, the collection of A^α and F^α determines ω^Γ and Ω^Γ:

$$\begin{aligned}
\omega^\Gamma|_{\pi^{-1}(U_\alpha)} &= (\mathrm{Ad}\circ\eta\circ\pi_\alpha)\bullet(\pi^*\mathrm{A}^\alpha) + \pi_\alpha^*\Theta^L, &(64)\\
\Omega^\Gamma|_{\pi^{-1}(U_\alpha)} &= (\mathrm{Ad}\circ\eta\circ\pi_\alpha)\bullet(\pi^*\mathrm{F}^\alpha). &(65)
\end{aligned}$$

Theorem 2.61 *Let $\omega^\Gamma \in \mathcal{A}_\gamma(P(M,G))$ and $\{(U_\alpha,\psi_\alpha)\}_{\alpha\in A}$ be a bundle atlas for P, then for all $\alpha,\beta \in A$ with $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$:*

$$\begin{aligned}
\mathrm{F}^\alpha &= d\mathrm{A}^\alpha + \frac{1}{2}\mathrm{A}^\alpha \wedge_{\mathfrak{g}} \mathrm{A}^\alpha, \qquad d\mathrm{F}^\alpha = -\mathrm{A}^\alpha \wedge_{\mathfrak{g}} \mathrm{F}^\alpha &(66)\\
\mathrm{A}^\alpha|_{U_{\alpha\beta}} &= (\mathrm{Ad}\circ g_{\alpha\beta})\bullet\mathrm{A}^\beta|_{U_{\alpha\beta}} + g_{\beta\alpha}^*\Theta^L = (\mathrm{Ad}\circ g_{\alpha\beta})\bullet(\mathrm{A}^\beta|_{U_{\alpha\beta}} - g_{\alpha\beta}^*\Theta^L); &(67)\\
\mathrm{F}^\alpha|_{U_{\alpha\beta}} &= (\mathrm{Ad}\circ g_{\alpha\beta})\bullet\mathrm{F}^\beta|_{U_{\alpha\beta}}. &(68)
\end{aligned}$$

Vice versa, if for a bundle atlas $\{(U_\alpha,\psi_\alpha)\}_{\alpha\in A}$ on the principal bundle $P(M,G)$ a family $\{\mathrm{A}^\alpha \in \mathcal{A}_1(U_\alpha,\mathfrak{g})\}_{\alpha\in A}$ is given such that (67) holds, then there exists one unique $\omega^\Gamma \in \mathcal{A}_\gamma(P(M,G))$ such that $\mathrm{A}^\alpha = \sigma_{\alpha,e}^*(\omega^\Gamma|_{\pi^{-1}(U_\alpha)})$ for all $\alpha \in A$.

Proof. (66) follows from Theorem 2.59 (observe that conversely by Lemma 1.74, (66) yields the structure equation and BIANCHI'S identity); (67) is a consequence of Corollary 1.105 with $K = \mathrm{id}_{\mathfrak{g}}$, $f = \sigma_{\beta,e}|_{U_{\alpha\beta}}$ and $g = g_{\beta\alpha}$, since $(R^p)^*\omega^\Gamma = \Theta^L$ and $R_g^*\omega^\Gamma = \mathrm{Ad}(g^{-1})\circ\omega^\Gamma$ for all $p \in P$ and all $g \in G$. Finally (68) can be deduced from (65) and the fact that $g_{\beta\alpha}(x) = \pi_\beta(p)\cdot\pi_\alpha(p)^{-1}$ for all $p \in \pi^{-1}(\{x\})$. □

Note 2.62 It might seem that $\sigma_{\alpha,e}$ is not the most general choice for the definition of pullbacks A^α, resp., F^α on the base manifold. Indeed, one can develop generalized relations analogous to the upper equations of Theorem 1.70 for local sections σ_{α,h_α} with $h_\alpha \in G$. Yet this is not necessary if we equip P with a maximal bundle atlas, since then every section σ_{α,h_α} can be viewed as a section $\sigma_{\alpha',e}: U_{\alpha'} \to \pi^{-1}(U_{\alpha'})$, where $U_{\alpha'} = U_\alpha$ and $\pi_{\alpha'} = \rho_{h_\alpha^{-1}} \circ \pi_\alpha$, cf. [1, p. 53].

Note 2.63 The notation of A^α and F^α is adapted to the physics literature. There the A^α are called *gauge potentials* (resp., gauge potential 1-forms) and the F^α are named *gauge fields* (resp., gauge field 2-forms). Theorem 2.61 tells us how local gauge potentials and fields transform into each other whenever they define a global connection. For this, Theorem 2.61 is a fundamental result for all field theories in theoretical physics, e. g. for electromagnetism and YANG-MILLS theories, and the equations of motion of this field theory are contained in (66).

Theorem 2.61 is a first result in the direction how only locally defined forms have to patch in order to build up a global form on the bundle. If M is paracompact, we can prove a further result in this direction (cf. also Proposition 2.114 in Section 2.5):

Definition 2.64 *Let Γ be a connection on a principal fiber bundle $P(M,G)$ over a paracompact base manifold M. Let $\{\rho_\gamma\}_{\gamma\in A}$ denote a partition of unity subordinate to $\mathfrak{U}$. For all $\alpha \in A$, we define $\mathrm{C}^\alpha \in \mathcal{A}_1(U_\alpha, \mathfrak{g})$ by*

$$\mathrm{C}^\alpha := \mathrm{A}^\alpha - \sum_{\gamma\in A} \rho_\gamma(g^*_{\gamma\alpha}\Theta^L). \tag{69}$$

Although not mentioned explicitely, every statement on $\{\mathrm{C}^\alpha\}_{\alpha\in A}$ in the sequel will imply that M is assumed to be paracompact. Theorem 2.61 yields:

Corollary 2.65 *Let $\Gamma \in \gamma(P(M,G))$, where P is a principal bundle over paracompact M; $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in A}$ be a bundle atlas for P. Then for all $\alpha, \beta \in A$ with $U_{\alpha\beta} \neq \emptyset$*

$$\mathrm{C}^\alpha|_{U_{\alpha\beta}} = (\mathrm{Ad} \circ g_{\alpha\beta}) \bullet \mathrm{C}^\beta|_{U_{\alpha\beta}}. \tag{70}$$

As a consequence, $|\mathcal{A}_1^T(P,\mathfrak{g})| = |\mathcal{A}_\gamma(P(M,G))| = |\gamma(P(M,G))|$.

Proof. From (55) and Theorem 1.70 we conclude that

$$\begin{aligned}\sum_{\gamma\in A} \rho_\gamma(g^*_{\gamma\alpha}\Theta^L) &= \sum_{\gamma\in A} \rho_\gamma[(g_{\gamma\beta}\cdot g_{\beta\alpha})^*\Theta^L] = \sum_{\gamma\in A} \rho_\gamma[(\mathrm{Ad}\circ g_{\alpha\beta}) \bullet g^*_{\gamma\beta}\Theta^L + g^*_{\beta\alpha}\Theta^L] \\ &= (\mathrm{Ad}\circ g_{\alpha\beta}) \bullet [\sum_{\gamma\in A} \rho_\gamma(g^*_{\gamma\beta}\Theta^L)] + g^*_{\beta\alpha}\Theta^L(\sum_{\gamma\in A}\rho_\gamma).\end{aligned}$$

Now $(\sum_{\gamma\in A}\rho_\gamma) = 1$ and (67) yield (70). Via (69), every family $\{\mathrm{A}^\alpha \in \mathcal{A}_1(U_\alpha,\mathfrak{g})\}_{\alpha\in A}$ that obeys (67), defines a family $\{\mathrm{C}^\alpha \in \mathcal{A}_1(U_\alpha,\mathfrak{g})\}_{\alpha\in A}$ that obeys (70), and vice versa. By Proposition 2.56 such a family $\{\mathrm{C}^\alpha \in \mathcal{A}_1(U_\alpha,\mathfrak{g})\}_{\alpha\in A}$ uniquely defines a 1-form $\gamma \in \mathcal{A}_1^T(P,\mathfrak{g})$ with $\mathrm{C}^\alpha = \sigma^*_{\alpha,e}\gamma$ and $\gamma|_{\pi^{-1}(U_\alpha)} = (\mathrm{Ad}\circ\eta\circ\pi_\alpha)\bullet(\pi^*\mathrm{C}^\alpha)$. Combined with Proposition 2.38, this yields the last statement. □

Since $0 \in \mathcal{A}^T(P,\mathfrak{g})$, we obtain the following important result:

Theorem 2.66 (Existence theorem for connections) *If $P(M,G)$ is a principal bundle over a paracompact manifold M then P admits a connection.*

2.3 Connections on Associated Bundles

Every connection on a principal bundle induces connections on all associated bundles. In the literature ([6, pp. 87 – 88], [9, p. 290]) we find the following definition:

Definition 2.67 *Every connection Γ on a principal bundle $P(M,G)$ induces splittings $T(B) = H(B) \oplus V(B)$ on any associated bundle $B(M,F,G) = P \times_G F$. Let $\widetilde{\pi}: P \times F \to B$ be the natural projection then $H(B) := \widetilde{\pi}_*(H(P) \times \{0\})$. Since h and v on $\mathcal{D}^1(P)$ commute with all $(R_g)_*$, they induce horizontal and vertical projections $\widehat{h}: \mathcal{D}^1(B) \to \widehat{h}\mathcal{D}^1(B)$, resp., $\widehat{v}: \mathcal{D}^1(B) \to \widehat{v}\mathcal{D}^1(B)$ for any associated bundle B.*

Yet from this approach the projections of the vector fields cannot easily be read off. So we choose a slightly different approach in order to get formulae for $\widehat{h}$ and $\widehat{v}$. The next observation on the natural connection Γ^{nat} on trivial bundles is quite trivial:

Lemma 2.68 *We have natural lifts $\mathbf{L}_h^{\mathrm{nat}}, \mathbf{L}_v^{\mathrm{nat}}: \mathcal{D}^1(P) \to \mathcal{D}^1(P \times F)$ on the product manifold $P \times F$ with $(\mathrm{pr}_P)_* \circ \mathbf{L}_h^{\mathrm{nat}} = \mathrm{id}_{\mathcal{D}^1(P)}$ and $(\mathrm{pr}_F)_* \circ \mathbf{L}_v^{\mathrm{nat}} = \mathrm{id}_{\mathcal{D}^1(F)}$, which are injective homomorphisms of $C^\infty(P)$-modules, resp., $C^\infty(F)$-modules and* LIE *algebras and obey $(\bar{R}_g)_* \circ \mathbf{L}_h^{\mathrm{nat}} = \mathbf{L}_h^{\mathrm{nat}} \circ (R_g)_*$ and $(\bar{R}_g)_* \circ \mathbf{L}_v^{\mathrm{nat}} = \mathbf{L}_v^{\mathrm{nat}} \circ (L_{g^{-1}})_*$. If $i_f: P \to P \times F$ and $i_p: F \to P \times F$ defined by $i_f(p) = i_p(f) = (p,f)$, denote the natural injections then $(\mathbf{L}_h^{\mathrm{nat}}\mathcal{X})_{(p,f)} = (di_f)_p \mathcal{X}_p$ and $(\mathbf{L}_v^{\mathrm{nat}}\mathcal{Y})_{(p,f)} = (di_p)_f \mathcal{Y}_f$ for all $p \in P$, $f \in F$, $\mathcal{X} \in \mathcal{D}^1(P)$ and $\mathcal{D}^1(F)$.*

We also have natural projections of vector fields $h^{\mathrm{nat}}, v^{\mathrm{nat}}: \mathcal{D}^1(P \times F) \to \mathcal{D}^1(P \times F)$ with $\mathcal{D}^1(P \times F) = h^{\mathrm{nat}}\mathcal{D}^1(P \times F) \oplus v^{\mathrm{nat}}\mathcal{D}^1(P \times F)$ as a $C^\infty(P \times F)$-module and $h^{\mathrm{nat}} \circ \mathbf{L}_h^{\mathrm{nat}} = \mathbf{L}_h^{\mathrm{nat}}$, $v^{\mathrm{nat}} \circ \mathbf{L}_h^{\mathrm{nat}} = 0$, resp., $h^{\mathrm{nat}} \circ \mathbf{L}_v^{\mathrm{nat}} = 0$, $v^{\mathrm{nat}} \circ \mathbf{L}_v^{\mathrm{nat}} = \mathbf{L}_v^{\mathrm{nat}}$.

Since $\mathrm{pr}_P \circ \bar{R}_g = R_g \circ \mathrm{pr}_P$ and $\mathrm{pr}_F \circ \bar{R}_g = L_{g^{-1}} \circ \mathrm{pr}_F$ for all $g \in G$, we have

$$
\begin{aligned}
h^{\mathrm{nat}} \circ (\bar{R}_g)_* &= (\bar{R}_g)_* \circ h^{\mathrm{nat}} = (\bar{R}_g)_* \circ h^{\mathrm{nat}}, \quad h^{\mathrm{nat}} \circ \widetilde{\mathcal{R}}' \circ (\mathrm{pr}_P)^* = \mathbf{L}_h^{\mathrm{nat}} \circ \mathcal{R}', \\
v^{\mathrm{nat}} \circ (\bar{R}_g)_* &= (\bar{L}_{g^{-1}})_* \circ v^{\mathrm{nat}} = (\bar{R}_g)_* \circ v^{\mathrm{nat}}, \quad v^{\mathrm{nat}} \circ \widetilde{\mathcal{R}}' \circ (\mathrm{pr}_F)^* = -\mathbf{L}_v^{\mathrm{nat}} \circ \mathcal{L}',
\end{aligned}
$$

where $\bar{R}$ and $\bar{L}$ denote the actions on $P \times F$ naturally induced by R and L:

$$
\begin{aligned}
\bar{R}: G \times P \times F \to G \times F, &\qquad \bar{R}(g,p,f) = (R(g,p),f), \\
\bar{L}: G \times P \times F \to G \times F, &\qquad \bar{L}(g,p,f) = (p, L(g,f)).
\end{aligned}
$$

h and v induce projections h' and v' on $h^{\mathrm{nat}}\mathcal{D}^1(P \times F)$ such that $h' \circ \mathbf{L}_h^{\mathrm{nat}} = \mathbf{L}_h^{\mathrm{nat}} \circ h$, $v' \circ \mathbf{L}_h^{\mathrm{nat}} = \mathbf{L}_h^{\mathrm{nat}} \circ v$. Also a $C^\infty(P \times F)$-linear extension of ω^Γ on $h^{\mathrm{nat}}\mathcal{D}^1(P \times F)$ exists, which we denote by $\bar{\omega}^\Gamma$. Then $v' = \bar{\mathcal{R}}' \circ \bar{\omega}^\Gamma$ and $\bar{\mathcal{R}}' = h^{\mathrm{nat}}\widetilde{\mathcal{R}}'$. Note that the splitting of $T(P \times F)$ into $H(P \times F) = H(P) \times \{0\}$ and $V(P \times F) = V(p) \times \{0\} \oplus \{0\} \times T(F)$ corresponds to projections $h_{P \times F} := h' \circ h^{\mathrm{nat}}$ and $v_{P \times F} = \mathrm{id}_{\mathcal{D}^1(P \times F)} - h' \circ h^{\mathrm{nat}}$ with

$$
h_{P \times F} \circ (\bar{R}_g)_* = (\bar{R}_g)_* \circ h_{P \times F}, \qquad v_{P \times F} \circ (\bar{R}_g)_* = (\bar{R}_g)_* \circ v_{P \times F}.
$$

Yet these are not the only projections given on $P \times F$. Recall that $P \times F$ is a principal bundle over B (Lemma 2.18) equivalent to $\widehat{\pi}^* B$. Now every connection Γ

on P induces a connection $\widetilde{\Gamma} = \mathrm{pr}_P^* \Gamma$ on $P \times F$ according to Proposition 2.39 since pr_P is G-equivariant. We have $\widetilde{\omega}^{\widetilde{\Gamma}} = \mathrm{pr}_P^* \omega^\Gamma = \widetilde{\omega}^\Gamma \circ h^{\mathrm{nat}} \in \mathcal{A}_*((P \times F)(B,G))$ with

$$\widetilde{\mathcal{R}}_g^* \widetilde{\omega}^{\widetilde{\Gamma}} = \mathrm{Ad}(g^{-1})_* \widetilde{\omega}^{\widetilde{\Gamma}}, \qquad \widetilde{\omega}^{\widetilde{\Gamma}} \circ \widetilde{\mathcal{R}}' = \mathrm{id}_{C^\infty(P\times F,\mathfrak{g})}.$$

$\widetilde{\Gamma}$ defines projections and lifts on $(P\times F)(B,G)$, let us denote them by $\widetilde{h}$, $\widetilde{v}$, $\widetilde{\mathbb{L}}$. Then $\widetilde{v} := \widetilde{\mathcal{R}}' \circ \widetilde{\omega}^{\widetilde{\Gamma}} = \widetilde{\mathcal{R}}' \circ \widetilde{\omega}^\Gamma \circ h^{\mathrm{nat}} = \widetilde{\mathcal{R}}' \mathcal{R}'^{-1} \circ v' \circ h^{\mathrm{nat}}$ and $\widetilde{h} = \mathrm{id}_{\mathcal{D}^1(P\times F)} - \widetilde{\mathcal{R}}' \mathcal{R}'^{-1} \circ v' \circ h^{\mathrm{nat}}$. Thus $\widetilde{h} \circ \mathbb{L}_v^{\mathrm{nat}} = \mathbb{L}_v^{\mathrm{nat}}$ and $\widetilde{v} \circ \mathbb{L}_v^{\mathrm{nat}} = 0$. As for any connection on a principal bundle, we have

$$\widetilde{h} \circ (\widetilde{R}_g)_* = (\widetilde{R}_g)_* \circ \widetilde{h}, \qquad \widetilde{v} \circ (\widetilde{R}_g)_* = (\widetilde{R}_g)_* \circ \widetilde{v}.$$

Lemma 2.69 *Let $\Gamma \in \gamma(P(M,G))$, then the various projections on $\mathcal{D}^1(P\times F)$ obey*

$$\begin{array}{lll}
h_{P\times F} \circ \widetilde{v} = \widetilde{v} \circ h_{P\times F} = 0, & h_{P\times F} \circ \widetilde{h} = \widetilde{h} \circ h_{P\times F} = h_{P\times F}, & \\
v_{P\times F} \circ \widetilde{v} = \widetilde{v} \circ v_{P\times F} = \widetilde{v}, & v_{P\times F} \circ \widetilde{h} = \widetilde{h} \circ v_{P\times F} = \widetilde{h} - h_{P\times F} = v^{\mathrm{nat}} \circ \widetilde{h}, & \\
h^{\mathrm{nat}} \circ \widetilde{v} = v' \circ h^{\mathrm{nat}}, & \widetilde{v} \circ h^{\mathrm{nat}} = \widetilde{v}, & \widetilde{v} \circ h' \circ h^{\mathrm{nat}} = 0, \\
h^{\mathrm{nat}} \circ \widetilde{h} = h' \circ h^{\mathrm{nat}}, & \widetilde{h} \circ h^{\mathrm{nat}} = h^{\mathrm{nat}} - \widetilde{v}, & \widetilde{h} \circ h^{\mathrm{nat}} \circ \widetilde{h} = h^{\mathrm{nat}} \circ \widetilde{h}, \\
v^{\mathrm{nat}} \circ \widetilde{v} = \widetilde{v} - v' \circ h^{\mathrm{nat}}, & \widetilde{h} \circ v^{\mathrm{nat}} = v^{\mathrm{nat}}, & \widetilde{v} \circ v^{\mathrm{nat}} = 0.
\end{array}$$

By Lemma 2.69, h^{nat}, $h_{P\times F}$ and $v_{P\times F}$ also act on $\mathcal{D}^{\widetilde{\Gamma}}(P\times F)$ and

$$h^{\mathrm{nat}}|_{\mathcal{D}^{\widetilde{\Gamma}}(P\times F)} = h_{P\times F}|_{\mathcal{D}^{\widetilde{\Gamma}}(P\times F)} = \mathrm{id}_{\mathcal{D}^{\widetilde{\Gamma}}(P\times F)} - v_{P\times F}|_{\mathcal{D}^{\widetilde{\Gamma}}(P\times F)}.$$

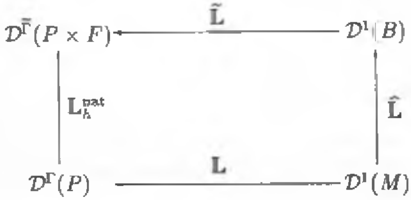
But $\widetilde{\mathbb{L}}: \mathcal{D}^1(B) \to \mathcal{D}^{\widetilde{\Gamma}}(P\times F)$ is a $C^\infty(B)$-module isomorphism according to Proposition 2.41, with inverse morphism $\widetilde{\pi}_*$. This defines the projections $\widehat{h}$, $\widehat{v}$ on $\mathcal{D}^1(B)$

$$\widehat{h} = \widetilde{\pi}_* h_{P\times F} \widetilde{\mathbb{L}} = \widetilde{\pi}_* h^{\mathrm{nat}} \widetilde{\mathbb{L}},\ \widehat{v} = \widetilde{\pi}_* v_{P\times F} \widetilde{\mathbb{L}} = \widetilde{\pi}_* v^{\mathrm{nat}} \widetilde{\mathbb{L}}, \quad \text{so } \mathcal{D}^1(B) = \widehat{h}\mathcal{D}^1(B) \oplus \widehat{v}\mathcal{D}^1(B).$$

Finally note that $\widetilde{h}\mathbb{L}_h^{\mathrm{nat}}\mathbb{L} = \widetilde{h}\mathbb{L}_h^{\mathrm{nat}} h \mathbb{L} = \widetilde{h} h' h^{\mathrm{nat}} \mathbb{L}_h^{\mathrm{nat}} \mathbb{L} = h' h^{\mathrm{nat}} \mathbb{L}_h^{\mathrm{nat}} \mathbb{L} = \mathbb{L}_h^{\mathrm{nat}} \mathbb{L}$ by Lemma 2.69 and $(\widetilde{R}_g)_* \mathbb{L}_h^{\mathrm{nat}} \mathbb{L} = \mathbb{L}_h^{\mathrm{nat}} (R_g)_* \mathbb{L} = \mathbb{L}_h^{\mathrm{nat}} \mathbb{L}$, so $\mathbb{L}_h^{\mathrm{nat}} \mathbb{L}: \mathcal{D}^1(M) \to \mathcal{D}^{\widetilde{\Gamma}}(P\times F)$ and the horizontal lift $\widehat{\mathbb{L}}: \mathcal{D}^1(M) \to \mathcal{D}^1(B)$ is well-defined by

$$\widehat{\mathbb{L}} := \widetilde{\pi}_* \circ \mathbb{L}_h^{\mathrm{nat}} \circ \mathbb{L}, \qquad \text{i. e.} \quad \widetilde{\mathbb{L}} \circ \widehat{\mathbb{L}} = \mathbb{L}_h^{\mathrm{nat}} \circ \mathbb{L}.$$

This is illustrated by the following commutative diagram:



$\widehat{h}\widehat{\mathbf{L}} = \widetilde{\pi}_* h^{\mathrm{nat}} \mathbf{L}_h^{\mathrm{nat}} \mathbf{L} = \widehat{\mathbf{L}}$ proves that $\widehat{\mathbf{L}}$ maps into $\widehat{h}\mathcal{D}^1(B)$, so $\widehat{h}_b = \widehat{\mathbf{L}}_b \circ d\widehat{\pi}_b$. Also

$$\begin{aligned}\widehat{h}[\widehat{\mathbf{L}}\mathcal{X}, \widehat{\mathbf{L}}\mathcal{Y}] &= \widetilde{\pi}_* h^{\mathrm{nat}} \widetilde{h}[\widetilde{\mathbf{L}}\widehat{\mathbf{L}}\mathcal{X}, \widetilde{\mathbf{L}}\widehat{\mathbf{L}}\mathcal{Y}] = \widetilde{\pi}_* h' h^{\mathrm{nat}}[\mathbf{L}_h^{\mathrm{nat}}\mathbf{L}\mathcal{X}, \mathbf{L}_h^{\mathrm{nat}}\mathbf{L}\mathcal{Y}] = \widetilde{\pi}_* h' \mathbf{L}_h^{\mathrm{nat}}[\mathbf{L}\mathcal{X}, \mathbf{L}\mathcal{Y}] \\ &= \widetilde{\pi}_* \mathbf{L}_h^{\mathrm{nat}} h[\mathbf{L}\mathcal{X}, \mathbf{L}\mathcal{Y}] = \widetilde{\pi}_* \mathbf{L}_h^{\mathrm{nat}} \mathbf{L}[\mathcal{X}, \mathcal{Y}] = \widehat{\mathbf{L}}[\mathcal{X}, \mathcal{Y}].\end{aligned}$$

We have thus proved the following analogue to Proposition 2.41:

Proposition 2.70 *The horizontal lift $\widehat{\mathbf{L}}: \mathcal{D}^1(M) \to \widehat{h}\mathcal{D}^1(B)$ is an injective homomorphism of $C^\infty(M)$-modules with $\widehat{\pi}_* \circ \widehat{\mathbf{L}} = \mathrm{id}_{\mathcal{D}^1(M)}$ and $\widehat{h}[\widehat{\mathbf{L}}\mathcal{X}, \widehat{\mathbf{L}}\mathcal{Y}] = \widehat{\mathbf{L}}[\mathcal{X}, \mathcal{Y}]$ for all $\mathcal{X}$, $\mathcal{Y} \in \mathcal{D}^1(M)$. $\widehat{\mathbf{L}}$ is uniquely defined by $\widetilde{\mathbf{L}}\widehat{\mathbf{L}} = \mathbf{L}_h^{\mathrm{nat}}\mathbf{L}: \mathcal{D}^1(M) \to h^{\mathrm{nat}}\mathcal{D}^{\widetilde{\Gamma}}(P \times F)$.*

Now what happens if $B = P$? One would expect that $\widehat{h} = h$ and $\widehat{\mathbf{L}} = \mathbf{L}$, and this is indeed true. We have the following commutative diagram:

$$\begin{array}{ccc} P \times G & \xrightarrow{\widetilde{\pi} = R \circ \tau_{PG}} & P \\ \Big\downarrow{\scriptstyle \mathrm{pr}_P} & & \Big\downarrow{\scriptstyle \pi} \\ P & \xrightarrow{\pi} & M \end{array}$$

On the left, $(P \times G)(P, G)$ is a trivial principal bundle with projection pr_P and right action $\widetilde{\rho} = \mathrm{id} \times \rho$. It is the trivialization of the square of P, which is the bundle on the top of the diagram. We can identify $\widetilde{\pi}$ and $R \circ \tau_{PG}$, where $\tau_{PG}: P \times G \to G \times P$ is the natural morphism exchanging P and G. Thus $d\widetilde{\pi}_{(p,g)}(\mathcal{P}_g, \mathcal{X}_g) = dR_g \mathcal{P}_p + dR^p \mathcal{X}_g$. We will prove $\widehat{\mathbf{L}} = \mathbf{L}$, then Proposition 2.42 yields that both connections Γ and $\widehat{\Gamma}$ on P coincide. For every $\mathcal{X} \in \mathcal{D}^1(M)$ and all $p \in P$ we have

$$(\widetilde{\pi}_* \mathbf{L}_h^{\mathrm{nat}} \mathbf{L}\mathcal{X})_p = d\widetilde{\pi}_{(R(g,p), g^{-1})}((\mathbf{L}\mathcal{X})_{R(g,p)}, 0_{g^{-1}}) = dR_{g^{-1}}(\mathbf{L}\mathcal{X})_{R(g,p)} = (\mathbf{L}\mathcal{X})_p,$$

since $(R_{g^{-1}})_* \mathbf{L}\mathcal{X} = \mathbf{L}\mathcal{X}$ for all $g \in G$. Thus $\widehat{\mathbf{L}} = \mathbf{L}$.

The following lemma in the spirit of Proposition 2.39 is quite obvious:

Lemma 2.71 *Let $\widehat{\Gamma}$ be a connection on $B(M, F, G)$ induced by Γ on the associated principal bundle $P(M, G)$. Every embedding $i: U \to M$ and every fiber preserving diffeomorphism of bundles $f: B(M, F, G) \to B'(M', F', G)$ induce connections $\widehat{\Gamma}|_U$ on $\widehat{\pi}^{-1}(U)$, resp., $\widehat{\Gamma}^f$ on B'. For every bundle chart (U_α, ψ_α) the induced connection $(\widehat{\Gamma}|_{U_\alpha})^{\psi_\alpha}$ on $U_\alpha \times F$ coincides with the connection induced by $(\Gamma|_{U_\alpha})^{\psi_\alpha}$ on $U_\alpha \times G$.*

Projections of forms on associated bundles are defined as in Definition 2.43:

Definition 2.72 *For any connection $\Gamma \in \gamma(P(M, G))$ and any $\omega_s \in \mathcal{A}_s(B, V)$, $s > 0$, where B is an associated bundle $B(M, F, G) = P \times_G F$ and V is a vector space, we define horizontal and vertical projections $\omega_s\widehat{h}$, resp., $\omega_s\widehat{v} \in \mathcal{A}_s(B, V)$ by*

$$\begin{aligned} \omega_s\widehat{h}(\mathcal{X}^1, \ldots, \mathcal{X}^s) &:= \omega_s(\widehat{h}\mathcal{X}^1, \ldots, \widehat{h}\mathcal{X}^s), \qquad \textit{for all} \quad \mathcal{X}^i \in \mathcal{D}^1(B), \\ \omega_s\widehat{v}(\mathcal{X}^1, \ldots, \mathcal{X}^s) &:= \omega_s(\widehat{v}\mathcal{X}^1, \ldots, \widehat{v}\mathcal{X}^s), \qquad \textit{for all} \quad \mathcal{X}^i \in \mathcal{D}^1(B). \end{aligned}$$

$\mathcal{A}(B, V)\widehat{h} \subseteq \mathcal{A}(B, V)$ and $\mathcal{A}(B, V)\widehat{v} \subseteq \mathcal{A}(B, V)$ (with $\mathcal{A}_0(B, V)\widehat{h} := \mathcal{A}_0(B, V)\widehat{v} := \mathcal{A}_0(B, V) = C^\infty(B, V)$) denote the $C^\infty(B)$-submodules of $\mathcal{A}(B, V)$ that contain these horizontal, resp., vertical V-valued forms.

Lemma 2.73 *If $\Gamma \in \gamma(P(M,G))$ and $B = P \times_G F$ is an associated bundle then*

1. $\mathcal{A}_1(B,V) = \mathcal{A}_1(B,V)\widehat{h} \oplus \mathcal{A}_1(B,V)\widehat{v}$ *and*
2. *the projections of forms commute with $\wedge_\phi$: for $\alpha \in \mathcal{A}(B)\otimes V$, $\beta \in \mathcal{A}(B)\otimes W$*

$$(\alpha \wedge_\phi \beta)\widehat{h} = \alpha\widehat{h} \wedge_\phi \beta\widehat{h}, \qquad (\alpha \wedge_\phi \beta)\widehat{v} = \alpha\widehat{v} \wedge_\phi \beta\widehat{v};$$

3. $\widehat{h}$ *and* $\widehat{v}$ *commute with $\bullet$, $\blacktriangleleft$ and $\blacktriangleright$: e. g. for $\chi \in \mathcal{A}(B, \mathrm{Hom}(\mathcal{T}(W),Z))$, $\phi^q_r \in \mathcal{A}_r(B) \otimes \mathrm{Hom}(\bigotimes^q V, W)$*

$$(\chi \bullet \phi^q_r)\widehat{h} = \chi\widehat{h} \bullet \phi^q_r\widehat{h}, \qquad (\chi \bullet \phi^q_r)\widehat{v} = \chi\widehat{v} \bullet \phi^q_r\widehat{v}.$$

The theory of fiber bundles is very often concerned with the problem how to "lift" or "extend" something that is defined on the base M, resp., the fiber F to the bundle $B(M,F,G)$. "Something" can mean a vector field, a differential form or, as we will see in the next chapter, a cohomology class. For the trivial bundle $M \times F$ with $G = \{e\}$ we can solve this problem using the natural projections pr_P and pr_F, resp., the natural injections i_f and i_x for $f \in F$ and $x \in M$. For arbitrary bundles only one global projection $\widetilde{\pi}$ is given naturally and we only have "global" (with regard to F) injections $i_{\alpha,x}$ on every bundle chart. These enable us to define a vertical bundle $V(B)$ and a global horizontal lift of differential forms $\widetilde{\pi}^*: \mathcal{A}(M,V) \to \mathcal{A}(B,V)$. We have seen that it requires a connection as an additional structure to define $H(B)$ and horizontal lifts of vector fields on M onto the bundle.

Now we will be concerned with the "dual problem" to extend forms on the fiber to the bundle. Locally we can achieve this using the pullbacks $\widetilde{\pi}^*_\alpha$ of the local projections onto the fiber, but normally for $\phi \in \mathcal{A}(F,V)$, $\{\widetilde{\pi}^*_\alpha\phi \in \mathcal{A}(\pi^{-1}(U_\alpha),V)\}_{\alpha\in A}$ will not define a global form since in general $(\widetilde{\pi}^*_\beta\phi|_{\pi^{-1}(U_{\alpha\beta})}) \neq (\widetilde{\pi}^*_\alpha\phi|_{\pi^{-1}(U_{\alpha\beta})})$. In order to investigate how a given connection will define global forms, we can compute $T^*_{\beta\alpha}$ and evaluate the projections of fields and forms locally. Let us postpone this access to the problem to Section 2.5. For now, we will again take the detour over $P \times F$ in order to derive global expressions for the extended forms.

But before note the following: in contrast to M we have an additional structure on F even for "trivial bundles", namely the effective left action $L: G \times F \to F$. Recall that even if $G \neq \{e\}$, we call a bundle B trivial, if we can find a (pre-)atlas for B such that all $g_{\alpha\beta} = e$ for all $U_{\alpha\beta} \neq \emptyset$. On the other hand, we equip B with a maximal atlas, cf. Note 2.62. Now observe that even for such a trivial bundle, the injections $i_{\alpha,x}$ define a global vertical vector field $i_*\mathcal{Y} \in v\mathcal{D}^1(M \times F)$ by $(i_*\mathcal{Y})_{(x,f)} := (di_{\alpha,x})_f\mathcal{Y}_f$ (if and) only if $\mathcal{Y} \in \mathcal{D}^1(F)$ is invariant under L. This is due to the following lemma:

Lemma 2.74 *$\mathcal{Y} \in \mathcal{D}^1(F)$ defines a vertical vector field $i_*\mathcal{Y} = \widetilde{\pi}_*\mathrm{L}^{\mathrm{nat}}_v\mathcal{Y} \in \mathcal{D}^1(B)$, such that locally $(i_*\mathcal{Y})_{\psi_\alpha^{-1}(x,f)} = (d\psi_\alpha^{-1})_{(x,f)}(0_x,\mathcal{Y}_f)$ on $\pi^{-1}(U_\alpha)$, iff $\mathcal{Y}$ is invariant.*

Proof. We already saw that $\widetilde{\pi}_*\mathrm{L}^{\mathrm{nat}}_v\mathcal{Y}$ defines a section of $\widetilde{\pi}^*T(B)$. A section of $\widetilde{\pi}^*T(B)$ is a section of $T(B)$ iff it is invariant under all $\widetilde{R}^*_g$. But this is the case iff $\mathbb{L}_v^{\mathrm{nat}}\mathcal{Y} = (\tilde{R}_{g^{-1}})_*\mathbb{L}_v^{\mathrm{nat}}\mathcal{Y} = \mathbb{L}_v^{\mathrm{nat}}(L_g)_*\mathcal{Y}$ for all $g \in G$. Since $\mathbb{L}_v^{\mathrm{nat}}$ is injective and $\hat{h}\tilde{\pi}_*\mathbb{L}_v^{\mathrm{nat}}\mathcal{Y} = \tilde{\pi}_* h^{\mathrm{nat}}\mathbb{L}_v^{\mathrm{nat}}\mathcal{Y} = 0$, this yields our assertion. That $(i_*\mathcal{Y})_{\psi_\alpha^{-1}(x,f)} = (d\psi_\alpha^{-1})_{(x,f)}(0_x, \mathcal{Y}_f)$ holds for all $x \in U_\alpha$ and $f \in F$, now follows from verticality and (57): $d\hat{\pi}_\alpha\, d\tilde{\pi}(\mathbb{L}_v^{\mathrm{nat}})_{(p,f)}\mathcal{Y}_f = dL^f\, d\pi_\alpha\, d\,\mathrm{pr}_P(\mathbb{L}_v^{\mathrm{nat}})_{(p,f)}\mathcal{Y}_f + dL_{\pi_\alpha(p)}\mathcal{Y}_f = \mathcal{Y}_{L(\pi_\alpha(p),f)}$. □

So the situation for M and F is not totally dual but involves L, and it is no surprise that, given a connection, we can only extend *invariant* forms $\phi \in \mathcal{A}(F,V)$ naturally onto the bundle. To see this, we observe that the only canonical way, how a differential form $\phi \in \mathcal{A}(F,V)$ acts on vector fields $\mathcal{Y}^i \in \mathcal{D}^1(B)$ is via

$$(\mathrm{pr}_F^*\,\phi)(\ldots, \tilde{\mathbb{L}}\mathcal{Y}^i, \ldots) = \tilde{f} \in C^\infty(P \times F, V).$$

This defines a form on B if and only if we find $f \in C^\infty(B,V)$ for any $\mathcal{Y}^i \in \mathcal{D}^1(B)$, such that $\tilde{f} = f \circ \tilde{\pi}$. We note that the resulting form will be vertical since

$$(\mathrm{pr}_F)_*\tilde{\mathbb{L}}\hat{v}\mathcal{Y}^i = (\mathrm{pr}_F)_*\tilde{\mathbb{L}}\tilde{\pi}_* v^{\mathrm{nat}}\tilde{\mathbb{L}}\mathcal{Y}^i = (\mathrm{pr}_F)_* v^{\mathrm{nat}}\tilde{\mathbb{L}}\mathcal{Y}^i = (\mathrm{pr}_F)_*\tilde{\mathbb{L}}\mathcal{Y}^i.$$

Proposition 2.75 *$\phi \in \mathcal{A}(F,V)$ defines a vertical V-valued form on $B(M,F,G)$ iff ϕ is invariant under all L_g^*. For such a ϕ and all $\mathcal{Y}^i \in \mathcal{D}^1(B)$ then there exists $f \in C^\infty(B,V)$ with*

$$(\mathrm{pr}_F^*\,\phi)(\ldots, \tilde{\mathbb{L}}\mathcal{Y}^i, \ldots) = f \circ \tilde{\pi}.$$

Proof. According to the previous discussion, ϕ defines a form on B if and only if $(\mathrm{pr}_F^*\,\phi)(\ldots, \tilde{\mathbb{L}}\mathcal{Y}^i, \ldots) \in C^\infty(P \times F, V)$ is invariant under all $\tilde{R}_g^*$, i. e., if and only if $\tilde{R}_g^*[(\mathrm{pr}_F^*\,\phi)(\ldots, \tilde{\mathbb{L}}\mathcal{Y}^i, \ldots)] = (\tilde{R}_g^*\,\mathrm{pr}_F^*\,\phi)(\ldots, (\tilde{R}_{g^{-1}})_*\tilde{\mathbb{L}}\mathcal{Y}^i, \ldots) = (\mathrm{pr}_F^*\,L_{g^{-1}}^*\phi)(\ldots, \tilde{\mathbb{L}}\mathcal{Y}^i, \ldots)$ for all $g \in G$ and $\mathcal{Y}^i \in \mathcal{D}^1(B)$. Obviously this relation holds if $\phi \in \mathcal{A}(F,V)$ is invariant. So let us assume, that ϕ is not invariant. Then we find $g \in G$, $f \in F$ and $\mathcal{X}^i \in \mathcal{D}^1(F)$ such that $(L_g^*\phi)_f(\ldots, \mathcal{X}_f^i, \ldots) = \phi_{L(g,f)}(\ldots, dL_g\mathcal{X}_f^i, \ldots) \neq \phi_f(\ldots, \mathcal{X}_f^i, \ldots)$. Since only $\mathcal{X}_f^i$ are involved, we may assume that all $\mathcal{X}^i$ are invariant and thus define $\tilde{\pi}_*\mathbb{L}_v^{\mathrm{nat}}\mathcal{X}^i \in \mathcal{D}^1(B)$ by Lemma 2.74. For these vector fields on B we compute $\tilde{\mathbb{L}}\tilde{\pi}_*\mathbb{L}_v^{\mathrm{nat}}\mathcal{X}^i = \tilde{h}\mathbb{L}_v^{\mathrm{nat}}\mathcal{X}^i = \tilde{h}v^{\mathrm{nat}}\mathbb{L}_v^{\mathrm{nat}}\mathcal{X}^i = v^{\mathrm{nat}}\mathbb{L}_v^{\mathrm{nat}}\mathcal{X}^i = \mathbb{L}_v^{\mathrm{nat}}\mathcal{X}^i$ and thus $(\tilde{R}_{g^{-1}}^*\,\mathrm{pr}_F^*\,\phi)(\ldots, \tilde{\mathbb{L}}\tilde{\pi}_*\mathbb{L}_v^{\mathrm{nat}}\mathcal{X}^i, \ldots)(p,f) = (L_g^*\phi)_f(\ldots, \mathcal{X}_f^i, \ldots) \neq \phi_f(\ldots, \mathcal{X}_f^i, \ldots) = (\mathrm{pr}_F^*\,\phi)(\ldots, \tilde{\mathbb{L}}\tilde{\pi}_*\mathbb{L}_v^{\mathrm{nat}}\mathcal{X}^i, \ldots)(p,f)$. So $(\mathrm{pr}_F^*\,\phi)(\ldots, \tilde{\mathbb{L}}\tilde{\pi}_*\mathbb{L}_v^{\mathrm{nat}}\mathcal{X}^i, \ldots)$ is not invariant under all $\tilde{R}_g^*$. Verticality was already proved above. □

Similar arguments hold for $\phi \in \mathcal{A}(P,V)$ acting on $\mathcal{Y}^i \in \mathcal{D}^1(B)$ via

$$(\mathrm{pr}_P^*\,\phi)(\ldots, \tilde{\mathbb{L}}\mathcal{Y}^i, \ldots) \in C^\infty(P \times F, V).$$

The resulting form will be horizontal because $(\mathrm{pr}_P)_*\tilde{\mathbb{L}}\hat{h} = (\mathrm{pr}_P)_*\tilde{\mathbb{L}}$. Moreover, only ϕh is of interest: $(\mathrm{pr}_P)_*\tilde{\mathbb{L}} = (\mathrm{pr}_P)_* h^{\mathrm{nat}}\tilde{h}\tilde{\mathbb{L}} = (\mathrm{pr}_P)_* h' h^{\mathrm{nat}}\tilde{\mathbb{L}} = h'(\mathrm{pr}_P)_*\tilde{\mathbb{L}}$, thus

$$(\mathrm{pr}_P^*\,\phi)(\ldots, \tilde{\mathbb{L}}\mathcal{Y}^i, \ldots) = (\mathrm{pr}_P^*\,\phi h)(\ldots, \tilde{\mathbb{L}}\mathcal{Y}^i, \ldots).$$

Proposition 2.76 *$\phi \in \mathcal{A}(P,V)$ defines a horizontal V-valued form on $B(M,F,G)$ iff $\phi h = \pi^*\varphi$, $\varphi \in \mathcal{A}(M,V)$. For such a ϕ and all $\mathcal{Y}^i \in \mathcal{D}^1(B)$ we then have*

$$(\mathrm{pr}_P^*\,\phi)(\ldots, \tilde{\mathbb{L}}\mathcal{Y}^i, \ldots) = (\tilde{\pi}^*\varphi)(\ldots, \mathcal{Y}^i, \ldots) \circ \tilde{\pi}.$$

Proof. We already saw that only ϕh matters and that the resulting form is horizontal. Now $\phi h = \pi^*\varphi$ iff $R_g^*(\phi h) = \phi h$ for all $g \in G$, and analogously to the previous proof we can show that this suffices to define a form on B. But then

$$(\mathrm{pr}_P^* \phi h)(\ldots, \tilde{\mathbb{L}}\mathcal{Y}^i, \ldots) = (\tilde{\pi}^*\bar{\pi}^*\varphi)(\ldots, \tilde{\mathbb{L}}\mathcal{Y}^i, \ldots) = (\tilde{\pi}^*\varphi)(\ldots, \tilde{\mathbb{L}}\mathcal{Y}^i, \ldots) \circ \tilde{\pi}.$$

On the other hand, if there exists $g \in G$ with $R_g^*\phi h \neq \phi h$, we can find invariant vector fields in $\mathcal{D}^\Gamma(P)$, i. e. $\mathcal{X}^i \in \mathcal{D}^1(M)$, such that $\phi h(\ldots, \mathbb{L}\mathcal{X}^i, \ldots) \circ R_g \neq \phi h(\ldots, \mathbb{L}\mathcal{X}^i, \ldots)$. So $(\mathrm{pr}_P^* \phi)(\ldots, \tilde{\mathbb{L}}\hat{\mathbb{L}}\mathcal{X}^i, \ldots) \circ \tilde{R}_g = \phi h(\ldots, \mathbb{L}\mathcal{X}^i, \ldots) \circ \mathrm{pr}_P \circ \tilde{R}_g = \phi h(\ldots, \mathbb{L}\mathcal{X}^i, \ldots) \circ R_g \circ \mathrm{pr}_P \neq \phi h(\ldots, \mathbb{L}\mathcal{X}^i, \ldots) \circ \mathrm{pr}_P = (\mathrm{pr}_P^* \phi)(\ldots, \tilde{\mathbb{L}}\hat{\mathbb{L}}\mathcal{X}^i, \ldots)$. Thus $(\mathrm{pr}_P^* \phi)(\ldots, \tilde{\mathbb{L}}\hat{\mathbb{L}}\mathcal{X}^i, \ldots)$ does not define $f \in C^\infty(B, V)$. □

As a simple example that only the horizontal part of $\phi \in \mathcal{A}(P, V)$ counts and needs to be invariant, we compute

$$(\mathrm{pr}_P^* \omega^\Gamma)(\tilde{\mathbb{L}}\mathcal{Y}) = \tilde{\omega}^\Gamma(\tilde{\mathbb{L}}\mathcal{Y}) = \widetilde{\mathcal{R}}'^{-1} \circ \tilde{v}\tilde{\mathbb{L}}\mathcal{Y} = 0. \tag{71}$$

Theorem 2.77 *If $\chi \in \mathcal{A}(F, \mathrm{Hom}(\mathcal{T}(\mathfrak{g}), V))_{\mathrm{equiv}}$ and $\phi \in \mathcal{A}_r^P(P, \mathfrak{g}) = \mathcal{A}_r(P, \mathfrak{g})_{\mathrm{equiv}}$, $r \in \mathbb{N}_0$, then $(\mathrm{pr}_F^* \chi) \bullet (\mathrm{pr}_P^* \phi) \in \mathcal{A}(P \times F, V)$ defines a V-valued form on B: for all vector fields $\mathcal{Y}^i \in \mathcal{D}^1(B)$ then there exists $f \in C^\infty(B, V)$ such that*

$$[(\mathrm{pr}_F^* \chi) \bullet (\mathrm{pr}_P^* \phi)](\ldots, \tilde{\mathbb{L}}\mathcal{Y}^i, \ldots) = [(\mathrm{pr}_F^* \chi) \bullet (\mathrm{pr}_P^* \phi h)](\ldots, \tilde{\mathbb{L}}\mathcal{Y}^i, \ldots) = f \circ \tilde{\pi}.$$

$(\mathrm{pr}_F^* \chi)$ defines the vertical and $(\mathrm{pr}_P^* \phi)$ defines the horizontal part of the form.

Proof. Analogously to the previous proofs, we must show that for any $\mathcal{Y} \in \mathcal{D}^1(B)$, $[(\mathrm{pr}_F^* \chi) \bullet (\mathrm{pr}_P^* \phi)](\ldots, \tilde{\mathbb{L}}\mathcal{Y}^i, \ldots) \in C^\infty(P \times F, V)$ is invariant. Again this means that $(\mathrm{pr}_F^* \chi) \bullet (\mathrm{pr}_P^* \phi) \in \mathcal{A}(P \times F, V)$ is invariant. $\mathrm{pr}_F \circ L_{g^{-1}} = \tilde{R}_g \circ \mathrm{pr}_F$ and $\mathrm{pr}_P \circ R_g = \tilde{R}_g \circ \mathrm{pr}_P$ yield that $\mathrm{pr}_F^* \chi$ and $\mathrm{pr}_P^* \phi$ are G-equivariant. Now Lemma 1.90 applies. □

All of these results are just special cases of the following theorem. If we replace $\mathfrak{g}$ by any vector space W with a left representation L', we may prove in total analogy for pseudotensorial forms of type (L', W) on P:

Theorem 2.78 *Let V, W be vector spaces, $L': G \times W \to W$ a left representation and $\phi \in \mathcal{A}_r^P(P, L', W)$, $r \in \mathbb{N}_0$. If $\chi \in \mathcal{A}(F, \mathrm{Hom}(\mathcal{T}(W), V))$ obeys $L_g^* \chi = ((L'_{g^{-1}})^*)_* \chi$ for all $g \in G$, then $(\mathrm{pr}_F^* \chi) \bullet (\mathrm{pr}_P^* \phi) \in \mathcal{A}(P \times F, V)$ defines a V-valued form on B: for all vector fields $\mathcal{Y}^i \in \mathcal{D}^1(B)$ then there exists $f \in C^\infty(B, V)$ such that*

$$[(\mathrm{pr}_F^* \chi) \bullet (\mathrm{pr}_P^* \phi)](\ldots, \tilde{\mathbb{L}}\mathcal{Y}^i, \ldots) = [(\mathrm{pr}_F^* \chi) \bullet (\mathrm{pr}_P^* \phi h)](\ldots, \tilde{\mathbb{L}}\mathcal{Y}^i, \ldots) = f \circ \tilde{\pi}.$$

$(\mathrm{pr}_F^* \chi)$ defines the vertical and $(\mathrm{pr}_P^* \phi)$ defines the horizontal part of the form.

Note that — since $P \times F$ is a principal bundle over B — Theorem 2.78 also is a consequence of Lemma 2.52 (to be exact: for $\chi \in \mathcal{A}(F) \otimes \mathrm{Hom}(\mathcal{T}(W), V) \subseteq \mathcal{A}(F, \mathrm{Hom}(\mathcal{T}(W), V))$, but Lemma 2.52 may be generalized). The conditions on ϕ and χ mean $\mathrm{pr}_P^* \phi \in \mathcal{A}_r^P(P \times F, L', W)$ and $\mathrm{pr}_F^* \chi \in \mathcal{A}^P(P \times F, (L')^*, \mathrm{Hom}(\mathcal{T}(W), V))$ and then $[(\mathrm{pr}_F^* \chi) \bullet (\mathrm{pr}_P^* \phi)]\tilde{h} \in \mathcal{A}^T(P \times F, L_0, V) = \tilde{\pi}^* \mathcal{A}(B) \otimes V$.

We are also interested in the exterior derivative of these forms $\varphi \in \mathcal{A}(B) \otimes V$ generated e. g. by $\phi \in \mathcal{A}(F) \otimes V$, and how far $d\varphi$ differs from the form generated by $d\phi$. Since d commutes with $\tilde{\pi}^*$, we can look at the forms $\tilde{\pi}^*\varphi \in \mathcal{A}^T(P \times F, L_0, V)$, and from (59) we know that $d(\tilde{\pi}^*\varphi) = d^\Gamma(\tilde{\pi}^*\varphi)$. Thus if $\phi \in \mathcal{A}^P_r(P, L', W)$ obeys $d^\Gamma\phi = 0$ (e. g. for Ω^Γ), we deduce from Lemma 2.50.4 $\tilde{\pi}^* d\varphi = d[(\mathrm{pr}_F^*\chi) \bullet (\mathrm{pr}_P^*\phi)]\tilde{h} = d^\Gamma[(\mathrm{pr}_F^*\chi)\tilde{h}] \bullet (\mathrm{pr}_P^*\phi)$. We will show in Section 2.5 that

$$\begin{aligned} d[(\mathrm{pr}_F^*\chi) \bullet (\mathrm{pr}_P^*\Omega^\Gamma)]\tilde{h} &= [(\mathrm{pr}_F^* d\chi) \bullet (\mathrm{pr}_P^*\Omega^\Gamma)]\tilde{h} + [(\mathrm{pr}_F^*(L_\bullet\chi)) \bullet (\mathrm{pr}_P^*\Omega^\Gamma)]\tilde{h}, \\ &= [(\mathrm{pr}_F^* d\chi) \bullet (\mathrm{pr}_P^*\Omega^\Gamma)]\tilde{h} + [(\mathrm{pr}_F^*(L_\bullet^\vee\chi)) \bullet (\mathrm{pr}_P^*\Omega^\Gamma)]\tilde{h}, \\ \text{resp.,} \qquad d(\mathrm{pr}_F^*\phi)\tilde{h} &= (\mathrm{pr}_F^* d\phi)\tilde{h} + (\mathrm{pr}_F^*(L_\bullet\varphi) \bullet (\mathrm{pr}_P^*\Omega^\Gamma))\tilde{h}, \end{aligned}$$

for all G-equivariant $\chi \in \mathcal{A}(F) \otimes \mathrm{Hom}(\mathcal{T}(W), V)$, resp., invariant $\phi \in \mathcal{A}(F) \otimes V$ (confer Theorem 2.120).

Note 2.79 We again consider the case $B = P$. Now $\mathcal{Y} \in \mathcal{D}^1(G)$ in Lemma 2.74 is invariant iff $\mathcal{Y}_g = d\lambda_g(X)$ for all $g \in G$ and $X \in \mathfrak{g}$. But then $(i_*\mathcal{Y})_{\psi_\alpha^{-1}(x,g)} = (d\psi_\alpha^{-1})_{(x,g)}(0_x, d\lambda_g(X)) = (\mathcal{R}_X)_{\psi_\alpha^{-1}(x,g)}$, so the vector field generated by $\mathcal{Y} = \mathcal{L}_X \in \mathcal{D}^1_L(G)$ is the fundamental vector field $\mathcal{R}_X$. Recall that the connection 1-form ω^Γ and the left canonical 1-form $\Theta^L \in \mathcal{A}^L_1(G)$ are connected via $(R^p)^*\omega^\Gamma = \Theta^L$ for all $p \in P$. According to Proposition 2.75, Θ^L defines a vertical $\mathfrak{g}$-valued 1-form "$\Theta^L v$" on P. Since $\Theta^L v$ is vertical, we may compute it by evaluating $(\Theta^L v)(\mathcal{R}_X)$. Now $(\mathrm{pr}_G^*\Theta^L)(\tilde{\mathrm{L}}\mathcal{R}_X) = (\mathrm{pr}_G^*\Theta^L)(\tilde{\mathrm{L}}i_*\mathcal{L}_X) = (\mathrm{pr}_G^*\Theta^L)(\mathrm{L}_v^{\mathrm{nat}}\mathcal{L}_X) = \Theta^L(\mathcal{L}_X) = X$. Thus $\Theta^L v = \omega^\Gamma$. Finally we can recover $\Omega^\Gamma \in \mathcal{A}^P_2(P, \mathfrak{g})$ using Theorem 2.77 with $\chi := \mathrm{Ad}\circ\eta \in C^\infty(G, \mathrm{Hom}(\mathfrak{g}, \mathfrak{g}))_{\mathrm{equiv}}$ since $\mathrm{pr}_G^*(\mathrm{Ad}\circ\eta) \bullet (\mathrm{pr}_P^*\Omega^\Gamma) = \tilde{\pi}^*\Omega^\Gamma$, cf. (65) and Corollary 2.118 below.

Given a connection on a bundle $B(M, F, G)$ and a (C^∞-)curve $\tau: [0,1] \to M$, there exists a unique horizontal lift $\hat{\tau}^\bullet: [0,1] \to B$ for every $b \in \tilde{\pi}^{-1}(\{\tau(0)\})$ such that $\hat{\tau}^\bullet(0) = b$, $\tilde{\pi} \circ \hat{\tau}^\bullet = \tau$ and $(d\hat{\tau}^\bullet)_r = \hat{\mathrm{L}}_{\hat{\tau}^\bullet(r)} \circ (d\tau)_r: \mathbb{R} \to H_{\hat{\tau}^\bullet(r)}(B)$ for all $r \in [0,1]$ ([6, p. 88]). Then $\hat{\tau}^\bullet(1) \in \tilde{\pi}^{-1}(\{\tau(1)\})$. By varying $b \in \tilde{\pi}^{-1}(\{\tau(0)\})$ we obtain a bijection $\hat{\tau}_0^1: \tilde{\pi}^{-1}(\{\tau(0)\}) \to \tilde{\pi}^{-1}(\{\tau(1)\})$, the so-called *parallel displacement of the fiber* $\tilde{\pi}^{-1}(\{\tau(0)\})$ *along the curve* τ. Its inverse is $\hat{\tau}_1^0 = \hat{\rho}_0^1$, where $\rho(r) := \tau(1 - r)$. For principal bundles we have $\hat{\tau}_r^s \circ R_g = R_g \circ \hat{\tau}_r^s$ for all $g \in G$, $r, s \in \mathbb{R}$ ([6, p. 70]).

Lemma 2.80 *If $B = P \times F$ is associated with $P(M, G)$ and $\tau^\bullet: [0,1] \to P$ is a horizontal lift of a curve $\tau: [0,1] \to M$, then for all $f \in F$, $\hat{\tau}^\bullet = \tilde{\pi} \circ i_f \circ \tau^\bullet: [0,1] \to B$ is the unique horizontal lift to B with $\hat{\tau}^\bullet(0) = \tilde{\pi}(\tau^\bullet(0), f)$.*

Proof. $d\hat{\tau}^\bullet_r = d\tilde{\pi} \circ (di_f) \circ \mathrm{L}_{\tau^\bullet(r)} \circ d\tau_r = d\tilde{\pi} \circ \mathrm{L}^{\mathrm{nat}}_{(\tau^\bullet(r), f)} \circ \mathrm{L}_{\tau^\bullet(r)} \circ d\tau_r = \hat{\mathrm{L}}_{\tilde{\pi}(\tau^\bullet(r), f)} \circ d\tau_r$ and $\tilde{\pi} \circ \hat{\tau}^\bullet = \tilde{\pi} \circ \tilde{\pi} \circ i_f \circ \tau^\bullet = \pi \circ \mathrm{pr}_P \circ i_f \circ \tau^\bullet = \pi \circ \tau^\bullet = \tau$ is obvious. □

Let $\sigma: M \to B$ be a section of B. By Lemma 2.28.4, $\sigma^*\mathcal{Y} \in \Gamma\sigma^*T(B)$ is a section of the pullback bundle $\sigma^*T(B)$ for every $\mathcal{Y} \in \mathcal{D}^1(B) = \Gamma T(B)$. We also observe that for all $\mathcal{X} \in \mathcal{D}^1(M)$, — although $\sigma_*\mathcal{X} \notin \mathcal{D}^1(B)$ — $\sigma_*\mathcal{X} \in \Gamma\sigma_*T(B)$ is well-defined by $\sigma_*\mathcal{X}(x) = d\sigma_x\mathcal{X}_x$ for all $x \in M$. $\sigma_*: \mathcal{D}^1(M) \to \Gamma\sigma^*T(B)$ is a natural injective $C^\infty(M)$-module homomorphism with $\pi_*\sigma_* = \mathrm{id}_{\mathcal{D}^1(M)}$. If Γ is a connection on B then $\sigma^*T(B) = \sigma^*H(B) \oplus \sigma^*V(B)$ by Lemma 2.28.3, thus we can decompose every $\sigma_*\mathcal{X}$ into a horizontal and a vertical part.

Definition 2.81 *A section $\sigma: M \to B$ is said to be* parallel *with respect to a given connection on $B(M, F, G)$ if $\sigma_* = \sigma^* \circ \widehat{\mathbf{L}}: \mathcal{D}^1(M) \to \Gamma\sigma^* H(B)$, resp., $d\sigma_x = \widehat{\mathbf{L}}_{\sigma(x)}$: for any curve $\tau: [0,1] \to M$ the parallel displacement of $\sigma(\tau(0))$ along τ gives $\sigma(\tau(1))$.*

For the trivial bundle $P \times F$ it is obvious that for every $f \in F$ the natural injection i_f is parallel with respect to Γ^{nat} on $P \times F$.

If E is a vector bundle over M, every connection Γ on E defines covariant derivatives of sections $\sigma: M \to E$ in the following way: we already saw that σ naturally induces $\sigma_*\mathcal{X} \in \Gamma\sigma^* T(E)$ for every $\mathcal{X} \in \mathcal{D}^1(M)$. By projection onto the vertical bundle we get $v(\sigma_*\mathcal{X}) \in \Gamma\sigma^* V(E)$. Now since $E(M, \mathbb{R}^n, G)$ is a vector bundle, we can identify the fiber $\mathbb{R}^n$ and its tangential space and $(\sigma^* V(E))(M, \mathbb{R}^n, G) \cong E(M, \mathbb{R}^n, G)$. Thus $v(\sigma_*\mathcal{X})$ defines a section $\nabla_{\mathcal{X}}\sigma \in \Gamma E$.

Definition 2.82 *If $E(M, \mathbb{R}^n, G)$ with $G < \mathrm{Gl}(\mathbb{R}^n)$ is a vector bundle, $\sigma: M \to E$ a section and $\mathcal{X} \in \mathcal{D}^1(M)$, then the section $\nabla_{\mathcal{X}}\sigma: M \to E$ is called the* covariant derivative of σ in the direction of X *with respect to the given connection Γ.*

Lemma 2.83 *$\sigma \in \Gamma E$ is parallel with respect to Γ iff $\nabla_{\mathcal{X}}\sigma = 0$ for all $\mathcal{X} \in \mathcal{D}^1(M)$.*

Proof. By definition, σ is parallel iff $\sigma_*\mathcal{X} \in \Gamma\sigma^* H(E)$ for all $\mathcal{X} \in \mathcal{D}^1(M)$, thus iff $v(\sigma_*\mathcal{X}) = 0$ for all $\mathcal{X} \in \mathcal{D}^1(M)$. □

The covariant derivative $\nabla_{\mathcal{X}}\sigma$ can be visualized locally in the following way (cf. [6, p. 114]): Let $\tau: [0,1] \to M$ be any (parametrized) curve with $\tau(0) = x$ and $\dot\tau(0) = d\tau_0(\frac{\partial}{\partial t}) = \mathcal{X}_x$. Then $(\nabla_{\mathcal{X}}\sigma)(x) = \nabla_{\mathcal{X}_x}\sigma = \nabla_{\dot\tau(0)}\sigma$, where

$$\nabla_{\dot\tau(t)}\sigma := \lim_{h\to 0} \frac{1}{h}[\bar\tau^t_{t+h}(\sigma(\tau(t+h))) - \sigma(\tau(t))].$$

(Recall that $\bar\tau^t_{t+h}: \bar\pi^{-1}(\{\tau(t+h)\}) \to \bar\pi^{-1}(\{\tau(t)\})$ denotes the parallel displacement of the fiber.) Again it becomes appearant that σ is parallel if $\nabla_{\dot\tau(t)}\sigma = 0$ — and thus $\bar\tau^t_{t+h}(\sigma(\tau(t+h))) = \sigma(\tau(t))$ — for all curves τ and $t \in [0,1]$.

Remember that Γ is a $C^\infty(M)$-module by Lemma 2.7.

Proposition 2.84 *$\nabla: \mathcal{D}^1(M) \times \Gamma E \to \Gamma E$, $\nabla(\mathcal{X}, \sigma) := \nabla_{\mathcal{X}}\sigma$ is $C^\infty(M)$-linear in its first argument and $\mathbb{R}$-linear in its second argument. For all $\mathcal{X}, \mathcal{Y} \in \mathcal{D}^1(M)$, all sections $\sigma, \sigma' \in \Gamma E$ and all $f, g \in C^\infty(M)$ we have*

$$\nabla_{(f\mathcal{X}+g\mathcal{Y})}\sigma = f\nabla_{\mathcal{X}}\sigma + g\nabla_{\mathcal{X}}\sigma \tag{72}$$

$$\nabla_{\mathcal{X}}(\sigma + \sigma') = \nabla_{\mathcal{X}}\sigma + \nabla_{\mathcal{X}}\sigma' \tag{73}$$

$$\nabla_{\mathcal{X}}(f\sigma) = f\nabla_{\mathcal{X}}\sigma + (\mathcal{X}f)\sigma \tag{74}$$

Proof. (72), (73) are clear. $\bar\tau^t_{t+h}[f(\tau(t+h))\sigma(\tau(t+h))] = f(\tau(t+h))\bar\tau^t_{t+h}[\sigma(\tau(t+h))]$ yields $\nabla_{\dot\tau(t)}(f\sigma) = f(\tau(t))\nabla_{\dot\tau(t)}\sigma + [\dot\tau(t)](f)\sigma(\tau(t))$, and this yields (74). □

We already saw in (58) that any section σ of $E(M, V, G)$ can be identified with a tensorial 0-form $f: P(M, G) \to V$ of type (L, V). Now covariant differentiation corresponds to Lie differentiation on the following sense (cf. [6, p. 116]):

Proposition 2.85 *If $\sigma: M \to E$ is a cross-section and $f: P(M,G) \to V$ is the corresponding function of type (L,V) defined by $\sigma \circ \pi = \tilde{\pi} \circ (\mathrm{id}_P, f)$ according to* (58)*, then $L_{\mathbf{L}\mathcal{X}} f$ is the function of type (L,V) that corresponds to $\nabla_{\mathcal{X}}\sigma$, i. e.*

$$(\nabla_{\mathcal{X}}\sigma) \circ \pi = \tilde{\pi} \circ (\mathrm{id}_P, L_{\mathbf{L}\mathcal{X}} f) = \tilde{\pi} \circ (\mathrm{id}_P, \mathbf{L}\mathcal{X}(f)) \qquad \textit{for all} \quad \mathcal{X} \in \mathcal{D}^1(M).$$

Proof. $\sigma \circ \pi = \tilde{\pi} \circ (\mathrm{id}_P, f)$ yields $\sigma_* \circ \pi_* = \tilde{\pi}_* \circ (\mathbf{L}_h^{\mathrm{nat}} + \mathbf{L}_v^{\mathrm{nat}} \circ f_*)$, thus

$$\begin{aligned} \hat{v}\sigma_* &= \tilde{\pi}_* v^{\mathrm{nat}} \tilde{\mathbf{L}} \sigma_* = \tilde{\pi}_* v^{\mathrm{nat}} \tilde{h} (\mathbf{L}_h^{\mathrm{nat}} \mathbf{L} + \mathbf{L}_v^{\mathrm{nat}} f_* \mathbf{L}) \\ &= \tilde{\pi}_* (v^{\mathrm{nat}} \tilde{\mathbf{L}} \hat{\mathbf{L}} + \mathbf{L}_v^{\mathrm{nat}} f_* \mathbf{L}) = \tilde{\pi}_* (v^{\mathrm{nat}} h^{\mathrm{nat}} \tilde{\mathbf{L}} \hat{\mathbf{L}} + \mathbf{L}_v^{\mathrm{nat}} f_* \mathbf{L}) = \tilde{\pi}_* \mathbf{L}_v^{\mathrm{nat}} f_* \mathbf{L}. \end{aligned}$$

This yields $(\nabla_{\mathcal{X}}\sigma) \circ \pi = (\hat{v}\sigma_* \mathcal{X}) \circ \pi = (\tilde{\pi}_* \mathbf{L}_v^{\mathrm{nat}} f_* \mathbf{L}\mathcal{X}) \circ \pi = \tilde{\pi} \circ (\mathrm{id}_P, \mathbf{L}\mathcal{X}(f))$. □

2.4 Linear Connections

Throughout this section, M will be of dimension $\dim M = n$, so $\mathbb{R}^n$ is the standard fiber of $T(M)$ and $G = \mathrm{Gl}(\mathbb{R}^n)$ acts on $\mathbb{R}^n$ by (matrix) multiplication $\cdot$ (instead of L). Recall that the bundle associated with $T(M)$ is the bundle of linear frames $L(M)$.

Definition 2.86 *A* linear connection *of a manifold M is a connection on $L(M)$.*

Every tensor field $K \in \mathcal{D}_s^r(M)$ is a section in the vector bundle $\bigotimes_s^r T(M)$ — and every V-valued $\omega \in \mathcal{D}_s(M) \otimes V$ is a section in $\bigotimes_s T(M) \otimes (M \times V)$. A linear connection defines covariant derivatives $\nabla_{\mathcal{X}} K$ and $\nabla_{\mathcal{X}}\omega$ for all $\mathcal{X} \in \mathcal{D}^1(M)$. Similar to the properties of LIE differentiation (cf. Proposition 1.38) we obtain for the covariant differentiation from Propositions 2.84 and 2.85 ([6, p. 132]):

Proposition 2.87 *The covariant differentiation $\nabla: \mathcal{D}^1(M) \times \mathcal{D}(M) \to \mathcal{D}(M)$ defined by a linear connection of M satisfies:*

1. *$\nabla_{(f\mathcal{X}+g\mathcal{Y})} K = f\nabla_{\mathcal{X}} K + g\nabla_{\mathcal{X}} K$ for all $f, g \in C^\infty(M)$, $\mathcal{X}, \mathcal{Y} \in \mathcal{D}^1(M)$;*
2. *$\nabla_{\mathcal{X}}$ is a type preserving derivation of $\mathcal{D}(M)$ commuting with contractions;*
3. *$\nabla_{\mathcal{X}} f = \mathcal{X}(f)$ for all $f \in C^\infty(M)$, $\mathcal{X} \in \mathcal{D}^1(M)$;*
4. *$\nabla_{\mathcal{X}}(fK) = f\nabla_{\mathcal{X}} K + (\mathcal{X} f) K$ for all $f \in C^\infty(M)$, $\mathcal{X} \in \mathcal{D}^1(M)$, $K \in \mathcal{D}(M)$.*

Analogous to Proposition 1.41, we have for a linear connection (cf. [6, p. 124]):

Proposition 2.88 *Let M be a manifold with a linear connection. Every derivation D of $\mathcal{D}(M)$ into the mixed tensor algebra $\mathcal{T}_*^*(T_x(M))$ at $x \in M$ with respect to the restriction map $|_{\{x\}}: \mathcal{D}(M) \to \mathcal{T}_*^*(T_x(M))$, that preserves type and commutes with contractions can be uniquely decomposed into*

$$D = \nabla_X + S \circ |_{\{x\}},$$

where $X \in T_x(M)$ and $S \in \mathrm{End}(T_x(M))$ (cf. Corollary 1.21*).*

Observe that, in contrast to LIE differentiation $L_{\mathcal{X}}$ with respect to a *vector field* $\mathcal{X}$, covariant differentiation $\nabla_{\mathcal{X}}$ is defined even for a *vector at a point* $x \in M$.

Definition 2.89 *For a linear connection* Γ *of a manifold* M *we define the* covariant differential $\nabla^{\Gamma}: \mathcal{D}(M) \to \mathcal{D}(M), \mathcal{D}^r_s(M) \to \mathcal{D}^r_{s+1}(M)$ *for* $K \in \mathcal{D}^r_s(M)$ *by*

$$(\nabla^{\Gamma} K)(\mathcal{X}^1, \ldots, \mathcal{X}^s; \mathcal{X}) := (\nabla_{\mathcal{X}} K)(\mathcal{X}^1, \ldots, \mathcal{X}^s) \qquad \text{for all} \quad \mathcal{X}, \mathcal{X}^i \in \mathcal{D}^1(M).$$

(Recall $\mathcal{D}^r_s(M) \cong \mathrm{Hom}(\mathcal{D}^s(M), \mathcal{D}^r(M))$*.) Restriction to the fibers defines local covariant differentials* $\nabla^{\Gamma}_x: \mathcal{T}^*_*(T_x(M)) \to \mathcal{T}^*_*(T_x(M)), \mathcal{T}^r_s(T_x(M)) \to \mathcal{T}^r_{s+1}(T_x(M))$.

Similar to (17), the following proposition holds ([6, pp. 124 – 125]):

Proposition 2.90 *If* $K \in \mathcal{D}^r_s(M)$ *and* $\mathcal{X}, \mathcal{X}^i, \mathcal{Y} \in \mathcal{D}^1(M)$ *then*

$$\begin{aligned}
(\nabla^{\Gamma} K)(\mathcal{X}^1, \ldots, \mathcal{X}^s; \mathcal{X}) &= \nabla_{\mathcal{X}}(K(\mathcal{X}^1, \ldots, \mathcal{X}^s)) - \sum_{i=1}^{s} K(\mathcal{X}^1, \ldots, \nabla_{\mathcal{X}} \mathcal{X}^i, \ldots, \mathcal{X}^s); \\
((\nabla^{\Gamma})^2 K)(\mathcal{X}^1, \ldots; \mathcal{X}; \mathcal{Y}) &= [\nabla_{\mathcal{Y}}(\nabla_{\mathcal{X}} K) - \nabla_{\nabla_{\mathcal{X}} \mathcal{Y}} K](\mathcal{X}^1, \ldots, \mathcal{X}^s).
\end{aligned}$$

As an immediate consequence of Lemma 2.83, we have

Lemma 2.91 *A tensor field* K *on* M *is parallel with respect to* Γ *iff* $\nabla^{\Gamma} K = 0$.

By Propositions 1.41 and 2.87.3, the operation of $\nabla_{\mathcal{X}}$ on $\mathcal{D}(M)$ is completely determined by its operation on $\mathcal{D}^1(M)$. We know that (72), (73) and (74) of Proposition 2.84 (with $\sigma := \mathcal{P} \in \mathcal{D}^1(M)$) hold for any covariant differentiation defined by a linear connection. For the reverse we have [6, p. 143]:

Theorem 2.92 *Any map* $\nabla: \mathcal{D}^1(M) \times \mathcal{D}^1(M) \to \mathcal{D}^1(M)$, $(\mathcal{X}, \mathcal{Y}) \mapsto \nabla_{\mathcal{X}} \mathcal{Y}$, *satisfying* (72), (73) *and* (74) *for* $E := T(M)$, *uniquely defines a linear connection* Γ *such that* $\nabla_{\mathcal{X}} \mathcal{Y}$ *is the covariant derivative of* $\mathcal{Y}$ *in the direction of* $\mathcal{X}$ *with respect to* Γ.

Definition 2.93 *The* canonical 1-form $\theta \in \mathcal{A}_1(L(M), \mathbb{R}^n)$ on the frame bundle *or* solder 1-form *is uniquely defined by*

$$\tilde{\pi}(p, \theta(\mathcal{Y}_p)) = \pi_* \mathcal{Y}_p \qquad \text{for all} \quad \mathcal{Y} \in \mathcal{D}^1(L(M)),\ p \in L(M)$$

with the projections $\pi: L(M) \to M$ *and* $\tilde{\pi}: L(M) \times \mathbb{R}^n \to T(M)$.

Obviously θ is horizontal and $\tilde{\pi} \circ (\mathrm{id}_{L(M)}, \theta(\mathbb{L}\mathcal{X})) = \mathcal{X}$ for all $\mathcal{X} \in \mathcal{D}^1(M)$ and lifts $\mathbb{L}: \mathcal{D}^1(M) \to \mathcal{D}^{\Gamma}(L(M))$ (θ is independent of Γ). Comparison with (58) yields:

Lemma 2.94 *The canonical 1-form* θ *on* $L(M)$ *is the unique tensorial 1-form of type* $(\mathrm{Gl}(\mathbb{R}^n), \mathbb{R}^n)$ *that corresponds to* $\mathrm{id}_{\mathcal{D}^1(M)}: \mathcal{D}^1(M) \to \Gamma T(M)$.

Definition 2.95 *For any linear connection of M we define $\mathcal{P}: \mathbb{R}^n \to h\mathcal{D}^1(L(M))$ by*

$$\mathcal{P}_p(v) := [\mathcal{P}(v)]_p := \mathbb{L}_p(\tilde{\pi}(p,v)) \qquad \text{for all} \quad v \in \mathbb{R}^n,\ p \in L(M).$$

$\mathcal{P}(v) \in h\mathcal{D}^1(L(M))$ *is called the* standard horizontal vector field corresponding to $v \in \mathbb{R}^n$. *Unlike the fundamental (vertical) vector fields, the standard horizontal vector fields depend on the choice of the connection.*

Definition 2.96 *A* geodesic *is a parametrized curve* $\tau:]a,b[\to M$, *where* $-\infty \leq a < b \leq \infty$, *such that the* tangent vector field $\mathcal{X} \in \mathcal{D}^1(\tau(]a,b[))$ *along the curve defined by* $\mathcal{X}_{\tau(t)} := \dot{\tau}(t)$ *is parallel along* τ: $\nabla_{\mathcal{X}}\mathcal{X} = 0$, *resp.*, $\dot{\tau}(s) = \tilde{\tau}_t^s(\tau(t))$ *for all* $t, s \in]a,b[$.

Note 2.97 Geodesics and standard horizontal vector fields are closely related. Geodesics are exactly the projections onto M of integral curves of standard horizontal vector fields. This proves that a unique geodesic exists for any initial point $x_0 \in M$ and tangent vector $X_0 \in T_{x_0}(M)$ [6, p. 139].

Lemma 2.98 *1. All $\mathcal{P}_p: \mathbb{R}^n \to T_p(L(M))$ are injective linear mappings: thus $\mathcal{P}(v)$ never vanishes for $v \neq 0$;*

2. $\theta(\mathcal{P}(v)) = v$ for the canonical 1-form θ on $L(M)$ and all $v \in \mathbb{R}^n$;

3. $\mathcal{P}$ is equivariant: $(R_g)_*\mathcal{P}(v) = \mathcal{P}(g^{-1} \cdot v)$ for all $g \in \mathrm{Gl}(\mathbb{R}^n)$ and $v \in \mathbb{R}^n$.

Proof. *1.*, *2.* are obvious, $(dR_g)_p\mathbb{L}_p(\tilde{\pi}(p,v)) = \mathbb{L}_{R(g,p)}(\tilde{\pi}(R_g(p), g^{-1} \cdot v))$ yields *3.* □

The conditions $\theta \circ \mathcal{P} = \mathrm{id}_{\mathbb{R}^n}$ and $\omega^\Gamma \circ \mathcal{P} = 0$ determine $\mathcal{P}: \mathbb{R}^n \to \mathcal{D}^1(L(M))$ completely. The situation is analogous to Lemma 1.94 and Lemma 2.32: the induced $\mathcal{P}': C^\infty(L(M), \mathbb{R}^n) \to h\mathcal{D}^1(L(M))$ is a $\mathrm{Gl}(\mathbb{R}^n)$-equivariant isomorphism of $C^\infty(L(M))$-modules, for every basis $\{e_i\}_{i=1,\ldots,n}$ for $\mathbb{R}^n$, $\{\mathcal{P}(e_i)\}_{i=1,\ldots,n}$ is a basis for the free $C^\infty(L(M))$-module $h\mathcal{D}^1(L(M))$. This proves (cf. [6, p. 122]):

Proposition 2.99 *For any connection on $L(M)(M, \mathrm{Gl}(\mathbb{R}^n))$, the n^2+n vector fields in $\{\mathcal{P}(e_i), \mathcal{R}_{E_k^j}\}_{i,j,k=1,\ldots,n}$, where $\{e_i\}_{i=1,\ldots,n}$ is a basis for $\mathbb{R}^n$ and $\{E_k^j\}_{j,k=1,\ldots,n}$ is a basis for $\mathrm{gl}(\mathbb{R}^n)$, form a basis for the free $C^\infty(L(M))$-module $\mathcal{D}^1(L(M))$.*

Lemma 2.100 *For $X \in \mathrm{gl}(\mathbb{R}^n)$, $v \in \mathbb{R}^n$ and $Xv = X \cdot v \in \mathbb{R}^n$ we have*

$$[\mathcal{R}_X, \mathcal{P}(v)] = \mathcal{P}(Xv).$$

Proof. Since all $\mathcal{P}_p$ are linear by Lemma 2.98.1, we obtain

$$[\mathcal{R}_X, \mathcal{P}(v)] = \lim_{t\to 0} \tfrac{1}{t}\{\mathcal{P}(v) - \mathcal{P}(e^{-tX} \cdot v)\} = \mathcal{P}(\lim_{t\to 0} \tfrac{1}{t}\{v - e^{-tX} \cdot v\})$$

from Lemma 2.32 and Lemma 2.98.3. But $\lim_{t\to 0} \frac{1}{t}\{v - e^{-tX} \cdot v\} = Xv$. □

Definition 2.101 $\Theta^\Gamma := d^\Gamma\theta$ *is called* torsion 2-form *of the linear connection* Γ.

Lemma 2.102 *Let $\mathcal{P}(v_1)$ and $\mathcal{P}(v_2)$ be standard horizontal vector fields on $L(M)$.*

1. *If the torsion form Θ^Γ vanishes then $[\mathcal{P}(v_1), \mathcal{P}(v_2)]$ is vertical.*
2. *If the curvature form Ω^Γ vanishes then $[\mathcal{P}(v_1), \mathcal{P}(v_2)]$ is horizontal.*

Proof. Since $\theta(\mathcal{P}(v_i)) = v_i$ are constant, $\theta([\mathcal{P}(v_1), \mathcal{P}(v_2)]) = -2d\theta(\mathcal{P}(v_1), \mathcal{P}(v_2)) = -2\Theta^\Gamma(\mathcal{P}(v_1), \mathcal{P}(v_2)) = 0$. Thus $[\mathcal{P}(v_1), \mathcal{P}(v_2)]$ is a vertical vector field. Analogously, using $\omega^\Gamma(\mathcal{P}(v_i)) = 0$, one proves the second claim. □

As a corollary to Theorem 2.58 and Theorem 2.59 we get:

Theorem 2.103 (Structure equations and Bianchi's identities)
Let Γ be a linear connection of M, then the following equalities hold:

$$\begin{aligned} &\text{structure equations:} & \Omega^\Gamma &= d\omega^\Gamma + \tfrac{1}{2}\omega^\Gamma \wedge_{\mathfrak{g}} \omega^\Gamma, & \Theta^\Gamma &= d\theta + \omega^\Gamma \wedge_l \theta; \\ &\text{BIANCHI'S identities:} & d^\Gamma\Omega^\Gamma &= d\Omega^\Gamma + \omega^\Gamma \wedge_{\mathfrak{g}} \Omega^\Gamma = 0, & d^\Gamma\Theta^\Gamma &= \Omega^\Gamma \wedge_l \theta. \end{aligned}$$

Definition 2.104 *For every linear connection Γ of a manifold M with torsion 2-form $\Theta^\Gamma \in \mathcal{A}_2^T(L(M), \mathbb{R}^n)$ and curvature 2-form $\Omega^\Gamma \in \mathcal{A}_2^T(L(M), \mathrm{gl}(\mathbb{R}^n))$ we define the* torsion (tensor field) $\mathrm{T} \in \mathcal{D}_2^1(M)$ *and the* curvature (tensor field) $\mathrm{R} \in \mathcal{D}_3^1(M)$ *by*

$$\begin{aligned} \mathrm{T}(\mathcal{X}, \mathcal{Y}) \circ \pi &= \tilde{\pi} \circ (\mathrm{id}_{L(M)}, 2\Theta^\Gamma(\mathbb{L}\mathcal{X}, \mathbb{L}\mathcal{Y})) \in \mathbf{\Gamma}\pi^*T(M) & &\textit{for all } \mathcal{X}, \mathcal{Y} \in \mathcal{D}^1(M), \\ \mathrm{R}(\mathcal{X}, \mathcal{Y}) \circ \pi &= \tilde{\tilde{\pi}} \circ (\mathrm{id}_{L(M)}, 2\Omega^\Gamma(\mathbb{L}\mathcal{X}, \mathbb{L}\mathcal{Y})) \in \mathbf{\Gamma}\pi^*\mathrm{End}(T(M)) & &\textit{for all } \mathcal{X}, \mathcal{Y} \in \mathcal{D}^1(M), \end{aligned}$$

with projections $\tilde{\pi}: L(M) \times \mathbb{R}^n \to T(M)$ and $\tilde{\tilde{\pi}}: L(M) \times \mathrm{gl}(\mathbb{R}^n) \to \mathrm{End}(T(M))$.

Thus $\frac{1}{2}\mathrm{T}$ and $\frac{1}{2}\mathrm{R}$ are the alternating $C^\infty(M)$-linear maps $\bar{\Theta}^\Gamma: \mathcal{D}^2(M) \to \mathbf{\Gamma}T(M)$ and $\bar{\Omega}^\Gamma: \mathcal{D}^2(M) \to \mathbf{\Gamma}\mathrm{End}(T(M))$ according to (58) and thus for all $\mathcal{X}, \mathcal{Y} \in \mathcal{D}^1(M)$

$$\mathrm{T}(\mathcal{X}, \mathcal{Y}) = -\mathrm{T}(\mathcal{Y}, \mathcal{X}) \in \mathcal{D}^1(M), \qquad \mathrm{R}(\mathcal{X}, \mathcal{Y}) = -\mathrm{R}(\mathcal{Y}, \mathcal{X}) \in \mathcal{D}_1^1(M).$$

We can also express T and R in terms of covariant differentiation and reformulate BIANCHI'S identities (cf. [6, p. 133 – 135]):

Theorem 2.105 *For any linear connection of M and all $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \mathcal{D}^1(M)$,*

$$\mathrm{T}(\mathcal{X}, \mathcal{Y}) = \nabla_\mathcal{X}\mathcal{Y} - \nabla_\mathcal{Y}\mathcal{X} - [\mathcal{X}, \mathcal{Y}], \tag{75}$$

$$\mathrm{R}(\mathcal{X}, \mathcal{Y})\mathcal{Z} = [\nabla_\mathcal{X}, \nabla_\mathcal{Y}]\mathcal{Z} - \nabla_{[\mathcal{X}, \mathcal{Y}]}\mathcal{Z}. \tag{76}$$

If $\mathfrak{S}$ denotes the symmetrization in $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ then BIANCHI'S *identities take the form*

$$\begin{aligned} \mathfrak{S}\{\mathrm{R}(\mathcal{X}, \mathcal{Y})\mathcal{Z} - \mathrm{T}(\mathrm{T}(\mathcal{X}, \mathcal{Y}), \mathcal{Z}) - (\nabla_\mathcal{X}\mathrm{T})(\mathcal{Y}, \mathcal{Z})\} &= 0, \\ \mathfrak{S}\{(\nabla_\mathcal{X}\mathrm{R})(\mathcal{Y}, \mathcal{Z}) + \mathrm{R}(\mathrm{T}(\mathcal{X}, \mathcal{Y}), \mathcal{Z})\} &= 0. \end{aligned}$$

In particular, if the torsion vanishes then $\mathfrak{S}\{\mathrm{R}(\mathcal{X}, \mathcal{Y})\mathcal{Z}\} = 0$, $\mathfrak{S}\{(\nabla_\mathcal{X}\mathrm{R})(\mathcal{Y}, \mathcal{Z})\} = 0$.

Recall the alternation $A: \mathcal{D}_\bullet(M) \otimes V \to \mathcal{A}(M) \otimes V$. We state (cf. [6, p. 149]):

Proposition 2.106 *If the torsion vanishes then* $d\omega = A(\nabla\omega)$ *for all* $\omega \in \mathcal{A}(M)\otimes V$.

The most important example for a linear connection is the LEVI-CIVITA connection on pseudo-Riemannian manifolds. Recall:

Definition 2.107 (M, g) *is called a* pseudo-Riemannian manifold *if* M *is a manifold and the so-called* pseudo-Riemannian metric $\mathrm{g} \in \mathfrak{S}\mathcal{D}_2(M)$ *is nondegenerate for all* $x \in M$. *If in addition* g *is positiv definite, we call* (M, g) *a* Riemannian manifold *and* g *a* Riemannian metric.

Theorem 2.108 *Every pseudo-Riemannian manifold* (M, g) *admits a unique linear connection* Γ *of* M *such that*

1. *the torsion vanishes:* $\Theta^\Gamma = 0$, *resp.*, $\mathrm{T} = 0$, *and*
2. g *is parallel with respect to* Γ: $\nabla^\Gamma \mathrm{g} = 0$.

Γ *is called* (pseudo-)Riemannian connection *or* LEVI-CIVITA connection.

Proof: cf. [6, p. 160]. Existence: define $\nabla_\mathcal{X}\mathcal{Y}$ for all $\mathcal{X}, \mathcal{Y} \in \mathcal{D}^1(M)$ by requiring

$$\begin{aligned} 2\mathrm{g}(\nabla_\mathcal{X}\mathcal{Y}, \mathcal{Z}) &= \mathcal{X}(\mathrm{g}(\mathcal{Y}, \mathcal{Z})) + \mathcal{Y}(\mathrm{g}(\mathcal{X}, \mathcal{Z})) - \mathcal{Z}(\mathrm{g}(\mathcal{X}, \mathcal{Y})) \\ &\quad + \mathrm{g}([\mathcal{X}, \mathcal{Y}], \mathcal{Z})) + \mathrm{g}([\mathcal{Z}, \mathcal{X}], \mathcal{Y})) + \mathrm{g}([\mathcal{Z}, \mathcal{Y}], \mathcal{X})) \end{aligned} \tag{77}$$

for all $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \mathcal{D}^1(M)$ (g is nondegenerate!). One checks that Γ is well-defined since the conditions of Theorem 2.92 are satisfied, that T vanishes and g is parallel.

On the other hand, one easily verifies that $\nabla_\mathcal{X}\mathrm{g} = 0$ and $\nabla_\mathcal{X}\mathcal{Y} - \nabla_\mathcal{Y}\mathcal{X} - [\mathcal{X}, \mathcal{Y}] = 0$ yield (77), which proves uniqueness of Γ. □

A few remarks on the local behavior of linear connections: Local evaluation of Γ on a chart U of the manifold with local coordinates $\{x^i\}_{i=1,\dots,n}$ and vector fields $\{\partial_i = \frac{\partial}{\partial x^i}\}_{i=1,\dots,n}$ as a basis for $\mathcal{D}^1(U)$, defines CHRISTOFFEL'S *symbols* Γ^k_{ij} by

$$\nabla_{\partial_i}\partial_j = \sum_{k=1}^n \Gamma^k_{ij}\partial_k \qquad \text{with} \quad \Gamma^k_{ij} \in C^\infty(U). \tag{78}$$

CHRISTOFFEL'S symbols do *not* define a tensor field. If we define the components T^k_{ij} of the torsion tensor by $\mathrm{T}(\partial_i, \partial_j) = \sum_{k=1}^n \mathrm{T}^k_{ij}\partial_k$, then (75) yields $\mathrm{T}^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}$. Geodesics $\vec{x}\colon]a, b[\to U$ are subject to the system of differential equations

$$\ddot{x}^k + \sum_{i,j=1}^n \Gamma^k_{ij}\dot{x}^i\dot{x}^j, \qquad k = 1, \dots, n,$$

as evaluation of $\nabla_{\dot{\vec{x}}}\dot{\vec{x}}$ using (78) and $\ddot{x}^k = \sum_{i=1}^n \dot{x}^i\partial_i(\dot{x}^k)$ shows.

For the LEVI-CIVITA connection on a pseudo-Riemannian manifold with $\mathrm{g}|_U = \sum_{i,j=1}^n \mathrm{g}_{ij}\, dx^i dx^j$ and $\mathrm{g}_{ij} = \mathrm{g}_{ji}$, we have

$$\sum_{l=1}^n \mathrm{g}_{lk}\Gamma^l_{ij} = \frac{1}{2}(\partial_i \mathrm{g}_{jk} + \partial_j \mathrm{g}_{ik} - \partial_k \mathrm{g}_{ij}), \qquad \text{so} \quad \Gamma^k_{ij} = \Gamma^k_{ji} \qquad \text{and} \quad \mathrm{T}^k_{ij} = 0.$$

Note 2.109 On pseudo-Riemannian manifolds, additional important linear mappings of forms — besides their exterior differentiation d — exist: for $0 \le p \le n$ we have the HODGE *star operator* $*: \mathcal{A}_p(M) \to \mathcal{A}_{n-p}(M)$ (on oriented manifolds): if $dV \in \mathcal{A}_n(M)$ denotes the volume form on M and $\langle\langle , \rangle\rangle$ denotes the scalar product of forms induced by g then $*$ is uniquely defined by $\alpha \wedge (*\beta) := \langle\langle \alpha, \beta \rangle\rangle dV$ for all $\alpha, \beta \in \mathcal{A}_p(M)$. Then $*1 = dV$, $*dV = \operatorname{sgn}(g) \cdot 1$ and $**\alpha = (-1)^{(n-p)p} \operatorname{sgn}(g)\alpha$.

The *co-differentiation* $\boldsymbol{\delta}: \mathcal{A}_p(M) \to \mathcal{A}_{p-1}(M)$ on pseudo-Riemannian manifolds with $\boldsymbol{\delta}^2 = 0$ is given by $\boldsymbol{\delta}\alpha := -(-1)^{n(p-1)} \operatorname{sgn}(g) * d * \alpha$ and is well-defined even if M is not orientable. Finally the LAPLACE-BELTRAMI *operator* $\boldsymbol{\Delta}: \mathcal{A}_p(M) \to \mathcal{A}_p(M)$, is defined by $\boldsymbol{\Delta} := (d + \boldsymbol{\delta})^2 = d\boldsymbol{\delta} + \boldsymbol{\delta} d$, cf. [1], [9].

2.5 Local Evaluation of Connections

Since we will be concerned with fiber bundles in general from now on, we will distinguish between π and $\hat{\pi}$, h and $\hat{h}$, L and $\hat{\mathrm{L}}$, etc., only where necessary, but use $\pi: M \to B$, etc., for convenience.

For many applications of fiber bundles, that involve numerical calculations, it is necessary to have coordinate functions for the bundle. Yet in most cases it is very difficult, if not impossible, to find global coordinates for a bundle. Especially in the case of Theorem 2.5 the bundle is given only by its bundle charts and their transition functions. Thus we are left with coordinate functions that are defined only locally on every bundle chart and we have to conclude every global property from the local ones and their interplay.

This illustrates the need for the local computations in this section. The situation is analogous to the situation for manifolds, where we have to decide from the transformation laws for functions, vector fields and tensor fields, whether a given set of locally defined fields or forms defines a global field, resp., form. For bundles we will have to compute the change of bundle charts to decide whether a set of fields or forms given for the local trivializations $U_\alpha \times F$ defines a global field or form on the bundle B.

Also, it will be one of our aims in this section to give local representations for the generated V-valued forms on B in Proposition 2.75 and Theorems 2.77 and 2.78. For this purpose we need to evaluate the local connections on $U_\alpha \times F$ that are induced by Γ due to Lemma 2.71 and thus to compute the local projections of fields and forms.

We start our local evaluations by computing the change of bundle charts. Definition 2.1 yields for all $x \in U_{\alpha\beta}$, $f^\alpha \in F$ that $T_{\beta\alpha}: U_{\alpha\beta} \times F \to U_{\alpha\beta} \times F$ is given by:

$(x, f^\beta) := T_{\beta\alpha}(x, f^\alpha) = (x, \pi_\beta|_{\pi^{-1}(\{x\})}(\pi_\alpha^{-1}|_{\pi^{-1}(\{x\})}(f^\alpha))) = (x, L(g_{\beta\alpha}(x), f^\alpha))$, thus

$$T_{\beta\alpha} = (\operatorname{pr}_{U_{\alpha\beta}}, L \circ (g_{\beta\alpha} \circ \operatorname{pr}_{U_{\alpha\beta}}, \operatorname{pr}_F)) = \bar{L} \circ (g_{\beta\alpha} \circ \operatorname{pr}_{U_{\alpha\beta}}, \operatorname{id}_{U_{\alpha\beta} \times F}), \tag{79}$$

with the induced action $\bar{L}$ on $U_{\alpha\beta} \times F$ from Lemma 2.68 ($P := U_{\alpha\beta}$). This yields:

Lemma 2.110 *Let $\sigma: M \to B$ be a section and define $s^\alpha := \pi_\alpha \circ \sigma|_{\pi^{-1}(U_\alpha)}: U_\alpha \to F$, i. e., $\psi_\alpha(\sigma(x)) = (x, s^\alpha(x))$ for all $x \in U_\alpha$. Then*

$$s^\beta|_{\pi^{-1}(U_{\alpha\beta})} = L \circ (g_{\beta\alpha}, s^\alpha|_{\pi^{-1}(U_{\alpha\beta})}). \tag{80}$$

Vice versa, if for a bundle atlas $\{(U_\alpha,\psi_\alpha)\}_{\alpha\in A}$ on the fiber bundle $B(M,F,G)$ a family $\{s^\alpha: U_\alpha \to F\}_{\alpha\in A}$ is given such that (80) holds, then there exists one unique section $\sigma: M \to B$ such that $s^\alpha = \pi_\alpha \circ \sigma|_{\pi^{-1}(U_\alpha)}$ for all $\alpha \in A$.

(13) in Lemma 1.30 yields that $(dT_{\beta\alpha})_{(x,f)} = d\bar{L}_{g_{\beta\alpha}(x)} + d\bar{L}^{(x,f)}\, dg_{\beta\alpha}\, d\,\mathrm{pr}_{U_{\alpha\beta}}$, resp.:

Lemma 2.111 *For $x \in U_{\alpha\beta} \neq \emptyset$ and $f \in F$ let $(X,F) \in T_x(M) \oplus T_f(F)$. Then*

$$(dT_{\beta\alpha})_{(x,f)}(X,F) = (X, dL_{g_{\beta\alpha}(x)}(F) + dL^f\, dg_{\beta\alpha}(X)).$$

Thus if $\mathcal{Y} \in \mathcal{D}^1(B)$ with $[(\psi_\alpha)_*(\mathcal{Y}|_{\pi^{-1}(U_\alpha)})]_{(x,f^\alpha)} =: (X^\alpha_{(x,f^\alpha)}, F^\alpha_{(x,f^\alpha)})$, then

$$X^\beta_{(x,f^\beta)} = X^\alpha_{(x,f^\alpha)} \quad \text{and} \quad F^\beta_{(x,f^\beta)} = dL_{g_{\beta\alpha}(x)}(F^\alpha_{(x,f^\alpha)}) + dL^{f^\alpha} dg_{\beta\alpha}(X^\alpha_{(x,f^\alpha)}) \tag{81}$$

for all $x \in U_{\alpha\beta} \neq \emptyset$, $f^\alpha \in F$ and $f^\beta = L(g_{\beta\alpha}(x), f^\alpha) \in F$.

(81) corresponds to the following transformation rule for 1-forms $\omega \in \mathcal{A}_1(B,V)$ with $[(\psi_\alpha^{-1})^*\omega|_{\pi^{-1}(U_\alpha)}](x,f^\alpha) = \mu^\alpha_{(x,f^\alpha)} + \phi^\alpha_{(x,f^\alpha)} \in \mathrm{Hom}(T_x(M),V) \oplus \mathrm{Hom}(T_{f^\alpha}(F),V)$:

$$\begin{aligned} \phi^\alpha_{(x,f^\alpha)} &= L^*_{g_{\beta\alpha}(x)}\phi^\beta_{(x,f^\beta)} \qquad \text{for all} \quad x \in U_{\alpha\beta} \qquad \text{and} \\ \mu^\alpha_{(x,f^\alpha)} &= \mu^\beta_{(x,f^\beta)} + g^*_{\beta\alpha}(L^{f^\alpha})^*\phi^\beta_{(x,f^\beta)} \;=\; \mu^\beta_{(x,f^\beta)} - ((L^{f^\beta})^*\phi^\beta_{(x,f^\beta)}) \circ (g^*_{\alpha\beta}\Theta^L_G)_x, \end{aligned}$$

cf. (14). In the general case, (81) yields $(T^*_{\beta\alpha}\omega^\beta)_{(x,f^\alpha)}(\ldots,(X^\alpha,F^\alpha)^i_{(x,f^\alpha)},\ldots) =$ $\omega^\beta_{(x,f^\beta)}(\ldots,(X^\alpha, dL_{g_{\beta\alpha}(x)}(F^\alpha) + dL^{f^\alpha} dg_{\beta\alpha}(X^\alpha)^i_{(x,f^\beta)},\ldots)$ for all $\omega^\beta \in \mathcal{A}(U_{\alpha\beta} \times F, V)$.

In order to get handier expressions independent of (x,f), we need to specialize. Suppose $\bar{L}^*_g\omega = (L'_g)_*\omega$ for all $g \in G$ with a representation $L': G \to \mathrm{Gl}(V)$. Then we may apply Theorem 1.104 on (79) and from (54) we get:

Proposition 2.112 *If L' is a representation of G on V and $\omega^\beta_n \in \mathcal{A}_n(U_{\alpha\beta} \times F, V)$ obeys $\bar{L}^*_g\omega^\beta_n = (L'_g)_*\omega^\beta_n$ for all $g \in G$, then*

$$T^*_{\beta\alpha}\omega^\beta_n = [(L' \circ g_{\beta\alpha} \circ \mathrm{pr}_{U_{\alpha\beta}}) \bullet \omega^\beta_n] \,\textcircled{\scriptsize D}\, (g_{\beta\alpha} \circ \mathrm{pr}_{U_{\alpha\beta}})^*\Theta^L_G.$$

Corollary 2.113 *If $\chi \in \mathcal{A}_n(F, \mathrm{Hom}(\mathcal{T}(\mathfrak{g}),V))_{\mathrm{equiv}}$ then*

$$T^*_{\beta\alpha}(\mathrm{pr}^*_F\,\chi) = [(\mathrm{Ad} \circ g_{\beta\alpha} \circ \mathrm{pr}_{U_{\alpha\beta}}) \bullet (\mathrm{pr}^*_F\,\chi)] \,\textcircled{\scriptsize D}\, (g_{\beta\alpha} \circ \mathrm{pr}_{U_{\alpha\beta}})^*\Theta^L_G.$$

If $\phi \in \mathcal{A}_n(F,V)_{\mathrm{inv}}$ then $T^*_{\beta\alpha}(\mathrm{pr}^*_F\,\phi) = (\mathrm{pr}^*_F\,\phi) \,\textcircled{\scriptsize D}\, (g_{\beta\alpha} \circ \mathrm{pr}_{U_{\alpha\beta}})^*\Theta^L_G$.

For $\mu \in \mathcal{A}(M,V)$ we obviously have $T^*_{\beta\alpha}((\mathrm{pr}_{U_{\alpha\beta}})^*\mu) = (\mathrm{pr}_{U_{\alpha\beta}})^*\mu$.

For a tensorial form $\varphi \in \mathcal{A}^T(P,L,V)$ on a principal bundle $P(M,G)$, we define analogously to (63) for every bundle chart

$$\mathrm{P}^\alpha := \sigma^*_{\alpha,e}(\varphi|_{\pi^{-1}(U_\alpha)}) \in \mathcal{A}(U_\alpha, V). \tag{82}$$

Then Proposition 2.56.3 yields that the collection of P^α determines φ completely:

$$\varphi|_{\pi^{-1}(U_\alpha)} = (L \circ \eta \circ \pi_\alpha) \bullet (\pi^*\mathrm{P}^\alpha), \tag{83}$$

and by (59) and Lemma 1.96 we get for the exterior covariant derivative

$$d^\Gamma\varphi|_{\pi^{-1}(U_\alpha)} = (L \circ \eta \circ \pi_\alpha) \bullet [\pi^*(d\mathrm{P}^\alpha + \mathrm{A}^\alpha \wedge_l \mathrm{P}^\alpha)].$$

Similar to Theorem 2.61 we now derive from $\eta \circ \pi_\beta \circ \sigma_{\alpha,e} = g_{\alpha\beta}$:

Proposition 2.114 *Let $\varphi \in \mathcal{A}^T(P(M,G),L,V)$ and $\{(U_\alpha,\psi_\alpha)\}_{\alpha\in A}$ be a bundle atlas for P, then for all $\alpha,\beta \in A$ with $U_{\alpha\beta} \neq \emptyset$:*

$$\mathrm{P}^\alpha|_{U_{\alpha\beta}} = (L \circ g_{\alpha\beta}) \bullet \mathrm{P}^\beta|_{U_{\alpha\beta}}. \tag{84}$$

Vice versa, if for a bundle atlas $\{(U_\alpha,\psi_\alpha)\}_{\alpha\in A}$ on the principal bundle $P(M,G)$ a family $\{\mathrm{P}^\alpha \in \mathcal{A}(U_\alpha,V)\}_{\alpha\in A}$ is given such that (84) holds, then there exists one unique $\varphi \in \mathcal{A}^T(P,L,V)$ such that $\mathrm{P}^\alpha = \sigma^*_{\alpha,e}(\varphi|_{\pi^{-1}(U_\alpha)})$ for all $\alpha \in A$.

Proposition 2.114 should be compared to Lemma 2.110: $\{\mathrm{P}^\alpha\}_{\alpha\in A}$ defines an alternating $C^\infty(M)$-linear map $\bar\varphi: \mathcal{D}^r(M) \to \Gamma E(M,V,G)$ by $\pi_\alpha \circ \bar\varphi|_{\pi^{-1}(U_\alpha)} = \mathrm{P}^\alpha$ for all $\alpha \in A$ and $\bar\varphi$ is exactly the map associated with φ according to (58).

Further note that (67) can be deduced from the transformation rule for 1-forms above. For principal bundles it reads for $x \in U_{\alpha\beta}$ and $g_\beta = g_{\beta\alpha}\cdot g_\alpha \in G$:

$$\phi^\alpha_{(x,g_\alpha)} = \lambda^*_{g_{\beta\alpha}(x)}\phi^\beta_{(x,g_\beta)}, \qquad \mu^\alpha_{(x,g_\alpha)} = \mu^\beta_{(x,g_\beta)} + g^*_{\beta\alpha}(\rho_{g_\alpha})^*\phi^\beta_{(x,g_\beta)}. \tag{85}$$

For $\omega = \omega^\Gamma$, (64) yields $\mu^\alpha_{(x,g_\alpha)} = \mathrm{Ad}(g_\alpha^{-1})(\mathrm{A}^\alpha_x)$ and $\phi^\alpha_{(x,g_\alpha)} = \Theta^L_{g_\alpha} = \lambda^*_{g_{\beta\alpha}(x)}\Theta^L_{g_\beta}$. Now with $g_\alpha = e$ and thus $g_\beta = g_{\beta\alpha}$, (85) yields

$$\mathrm{A}^\alpha_x = \mu^\alpha_{(x,e)} = \mu^\beta_{(x,g_{\beta\alpha})} + g^*_{\beta\alpha}(\rho_e)^*\phi^\beta_{(x,g_{\beta\alpha})} = \mathrm{Ad}(g_{\alpha\beta})(\mathrm{A}^\beta_x) + g^*_{\beta\alpha}\Theta^L_{g_{\beta\alpha}}.$$

The local evaluation of ω^Γ takes us to the computation of the local projections. $v = \mathcal{R}' \circ \omega^\Gamma$ and $\omega^\alpha_{(x,g)}(X,G) = \mathrm{Ad}(g^{-1})\mathrm{A}^\alpha_x(X) + d\lambda_{g^{-1}}(G)$ for all $x \in U_\alpha$, $g \in G$ and $(X,G) \in T_x(U_\alpha) \oplus T_g(G)$ induce on every $U_\alpha \times G$ projections

$$v^\alpha_{(x,g)}(X,G) = (0,(d\rho_g)_e\mathrm{A}^\alpha_x(X) + G), \qquad h^\alpha_{(x,g)}(X,G) = (X,-(d\rho_g)_e\mathrm{A}^\alpha_x(X)).$$

The horizontal lifts $\mathbb{L}^\alpha: \mathcal{D}^1(U_\alpha) \to \mathcal{D}^1(U_\alpha \times G)$ are thus given by

$$\mathbb{L}^\alpha_{(x,g)}(X) = (X,-(d\rho_g)_e\mathrm{A}^\alpha_x(X)). \tag{86}$$

In order to compute v^α for associated bundles, we first need the connection on $P \times F$ for our construction in Section 2.3. By Definition 2.17,

$$(d\tilde{R}^{(p,f)})_e(Y) = ((dR^p)_e(Y), -(dL^f)_e(Y)) \quad \text{for all} \quad p \in P,\ f \in F \quad \text{and} \quad Y \in \mathfrak{g},$$

$$\text{thus} \quad (d\tilde{R}^{(x,g,f)})^\alpha_e(Y) = (0,(d\lambda_g)_e(Y), -(dL^f)_e(Y)) \in T_x(U_\alpha) \oplus T_g(G) \oplus T_f(F).$$

With $\omega^\alpha_{(x,g)}$ from above, $\tilde{v}^\alpha_{(x,g,f)}(X,F,G) = (d\tilde{R}^{(x,g,f)})^\alpha_e\omega^\alpha_{(x,g)}(X,G)$ yields

$$\begin{aligned} \tilde{v}^\alpha_{(x,g,f)}(X,F,G) &= (0,(d\rho_g)_e\mathrm{A}^\alpha_x(X) + G, -(dL^f)_e[\mathrm{Ad}(g^{-1})\mathrm{A}^\alpha_x(X) + d\lambda_{g^{-1}}(G)]), \\ \tilde{h}^\alpha_{(x,g,f)}(X,F,G) &= (X,-(d\rho_g)_e\mathrm{A}^\alpha_x(X), +(dL^f)_e[\mathrm{Ad}(g^{-1})\mathrm{A}^\alpha_x(X) + d\lambda_{g^{-1}}(G)] + F). \end{aligned}$$

A little computation then shows using $d\tilde{\pi}(X,G,F) = (X,(dL^f)_gG + (dL_g)_fF)$

$$\tilde{\mathbb{L}}^\alpha_{(x,g,L(g^{-1},f))}(X,F) = (X,-(d\rho_g)_e\mathrm{A}^\alpha_x(X), +(dL_{g^{-1}})_f[(dL^f)_e\mathrm{A}^\alpha_x(X) + F]). \tag{87}$$

Now we obtain from $\hat{v} = \tilde{\pi}v^{\mathrm{nat}}\tilde{\mathbb{L}}$ the following lemma (omitting "$\sim$"):

Lemma 2.115 *Every connection* Γ *on an associated bundle* $B = P(M, G) \times_G F$, *that is defined by a collection of* $\mathrm{A}^\alpha \in \mathcal{A}_1(U_\alpha, \mathfrak{g})$ *according to* Theorem 2.61, *induces the following projections for all* $x \in U_\alpha$, $f \in F$ *and* $(X, F) \in T_x(U_\alpha) \oplus T_f(F)$:

$$v^\alpha_{(x,f)}(X, F) = (0, (dL^f)_e \mathrm{A}^\alpha_x(X) + F), \quad h^\alpha_{(x,f)}(X, F) = (X, -(dL^f)_e \mathrm{A}^\alpha_x(X)). \quad (88)$$

The horizontal lifts $\mathrm{L}^\alpha: \mathcal{D}^1(U_\alpha) \to \mathcal{D}^1(U_\alpha \times F)$ *are thus given by*

$$\mathrm{L}^\alpha_{(x,f)}(X) = (X, -(dL^f)_e \mathrm{A}^\alpha_x(X)).$$

Observe that for $B = P$, we indeed recover the original connection. Our result is no less than surprising since replacing $d\rho_g$ by dL^f is the only canonical way to generalize a connection on $U_\alpha \times G$ to associated connections on $U_\alpha \times F$.

Let us note in passing a formula for the local covariant derivatives of sections of vector bundles. With the notation of Lemma 2.110 we obtain from Lemma 2.115 $(d\psi_\alpha)_{\sigma(x)}[v(\sigma_* \mathcal{X})]_{\sigma(x)} = v^\alpha_{(x, s^\alpha(x))}(\mathcal{X}_x, ds^\alpha_x \mathcal{X}_x) = (0, \mathcal{X}_x(s^\alpha) + (dL^{s^\alpha})_e \mathrm{A}^\alpha_x(\mathcal{X}_x))$ for any $\mathcal{X} \in \mathcal{D}^1(M)$, and with l from (43), Definition 2.82 yields:

$$\nabla^\alpha_{\mathcal{X}} s^\alpha := \pi_\alpha \circ (\nabla_{\mathcal{X}} \sigma|_{\pi^{-1}(U_\alpha)}) = \mathcal{X}|_{U_\alpha}(s^\alpha) + l \circ (\mathrm{A}^\alpha(\mathcal{X}|_{U_\alpha}), s^\alpha). \quad (89)$$

One easily checks that the $\nabla^\alpha_{\mathcal{X}} s^\alpha$ transform according to (80) and thus these local covariant derivatives define a global unique section $\nabla_{\mathcal{X}} \sigma$ by Lemma 2.110.

Finally we compute the local projections of forms. Lemma 2.115 yields

$$(\omega^\alpha v^\alpha)_{(x,f)}(\ldots, (X^i, F^i), \ldots) = \omega^\alpha_{(x,f)}(\ldots, (0, (dL^f)_e \mathrm{A}^\alpha_x(X^i) + F^i), \ldots)$$

for all $\omega^\alpha \in \mathcal{A}(U_\alpha \times F, V)$ and $(X^i, F^i) \in T_x(U_\alpha) \oplus T_f(F)$. For 1-forms $\omega^\alpha = \mu^\alpha + \phi^\alpha$ with $\mu^\alpha_{(x,f)}: T_x(M) \to V$ and $\phi^\alpha_{(x,f)}: T_f(F) \to V$ as above, this yields

$$(\mu^\alpha v^\alpha)_{(x,f)} = 0, \qquad (\phi^\alpha v^\alpha)_{(x,f)} = \phi^\alpha_{(x,f)} + ((L^f)^* \phi^\alpha_{(x,f)}) \circ \mathrm{A}^\alpha_x.$$

Naturally, $(\mathrm{pr}^*_{U_\alpha} \mu) v^\alpha = 0$ holds for any $\mu \in \mathcal{A}(U_\alpha, V)$. One also easily proves:

Lemma 2.116 *If* $\phi \in \mathcal{A}_n(F, V)$ *then on every local trivialization* $U_\alpha \times F$:

$$(\mathrm{pr}^*_F \phi) v^\alpha = (\mathrm{pr}^*_F \phi) \odot (\mathrm{pr}^*_{U_\alpha} \mathrm{A}^\alpha).$$

Thus for all $x \in U_\alpha$, $i^*_{\alpha,x}[(\mathrm{pr}^*_F \phi) v^\alpha] = \phi$: *restriction to the fibers reproduces* ϕ.

Now we can evaluate Propositions 2.75, 2.76 and Theorems 2.77 and 2.78 on the bundle charts. For $\phi \in \mathcal{A}^p(U_\alpha \times G, L', W)$ one derives using (86) and (87) that

$$\begin{aligned} ((\mathrm{pr}_{U_\alpha \times G})^* \phi)(\ldots, \tilde{\mathrm{L}}^\alpha_{(x,g,L(g^{-1},f))}(X^i, F^i), \ldots) &= \phi_{(x,g)}(\ldots, \mathrm{L}^\alpha_{(x,g)}(X^i), \ldots) \\ &= (\phi h)_{(x,g)}(\ldots, \mathrm{L}^\alpha_{(x,g)}(X^i), \ldots). \end{aligned}$$

Since we already proved invariance under $\tilde{R}^*_g$, we may restrict ourselves to $g = e$. If we define $\mathrm{P}^\alpha = \sigma^*_{\alpha,e} \phi h \in \mathcal{A}(U_\alpha, W)$ as in (82), then (83) yields

$$(\phi h)_{(x,g)}(\ldots, \mathrm{L}^\alpha_{(x,g)}(X^i), \ldots) = \mathrm{P}^\alpha(\ldots, X^i, \ldots).$$

So the horizontal part ($\mathrm{pr}_P^* \phi$) of the form in Theorem 2.78 is locally just ($\widetilde{\mathrm{pr}}_{U_\alpha}^* \mathrm{P}^\alpha$), resp., ($\tilde{\pi}^* \mathrm{P}^\alpha$).

Analogously for the vertical part ($\mathrm{pr}_F^* \chi$), again (87) and (88) yield that it is locally given by $(\widetilde{\mathrm{pr}}_F^* \chi) v^\alpha$, resp., $(\hat{\pi}_\alpha \chi) v^\alpha$. So our results take the following form (again omitting "$\sim$" for convenience):

Theorem 2.117 *Let Γ be a connection on a principal fiber bundle $P(M,G)$ with associated bundle $B(M,F,G)$, V, W any vector spaces and $L' \colon G \times W \to W$ a left representation. Let v^α denote the local vertical projections of V-valued forms induced by Γ on $U_\alpha \times F$, resp., $\pi^{-1}(U_\alpha)$ for all $\alpha \in A$. Then for any family $\{\mathrm{P}^\alpha \in \mathcal{A}(U_\alpha, W)\}_{\alpha \in A}$ with $\mathrm{P}^\alpha|_{U_{\alpha\beta}} = (L' \circ g_{\alpha\beta}) \bullet \mathrm{P}^\beta|_{U_{\alpha\beta}}$ for all $\alpha, \beta \in A$ with $U_{\alpha\beta} \neq \emptyset$ (such that this family defines a pseudotensorial form of type (L', W) according to* Proposition 2.114*) and any $\chi \in \mathcal{A}(F, \mathrm{Hom}(\mathcal{T}(W), V))$ that obeys $L_g^* \chi = ((L'_{g^{-1}})^*)_* \chi$ for all $g \in G$,*

$$\begin{aligned} T_{\beta\alpha}^* \{[(\mathrm{pr}_F^* \chi) v^\beta] \bullet ((\mathrm{pr}_{U_\beta})^* \mathrm{P}^\beta)\} &= [(\mathrm{pr}_F^* \chi) v^\alpha] \bullet ((\mathrm{pr}_{U_\alpha})^* \mathrm{P}^\alpha), \quad \textit{resp.,} \\ [(\pi_\beta^* \chi) v^\beta] \bullet (\pi^* \mathrm{P}^\beta) &= [(\pi_\alpha^* \chi) v^\alpha] \bullet (\pi^* \mathrm{P}^\alpha), \end{aligned}$$

for all $\alpha, \beta \in A$ with $U_{\alpha\beta} \neq \emptyset$, where we omitted the restriction onto $U_{\alpha\beta}$. Thus $\{[(\pi_\alpha^* \chi) v^\alpha] \bullet (\pi^* \mathrm{P}^\alpha) \in \mathcal{A}(\pi^{-1}(U_\alpha), V)\}_{\alpha \in A}$ defines a global form "$\chi v \bullet \mathrm{P}$" on B.

Corollary 2.118 *For any G-equivariant $\chi \in \mathcal{A}(F, \mathrm{Hom}(\mathcal{T}(\mathfrak{g}), V))$ and $\alpha, \beta \in A$*

$$\begin{aligned} [(\pi_\beta^* \chi) v^\beta] \bullet (\pi^* \mathrm{F}^\beta)\} &= [(\pi_\alpha^* \chi) v^\alpha] \bullet (\pi^* \mathrm{F}^\alpha), \\ [(\pi_\beta^* \chi) v^\beta] \bullet (\pi^* \mathrm{C}^\beta)\} &= [(\pi_\alpha^* \chi) v^\alpha] \bullet (\pi^* \mathrm{C}^\alpha), \end{aligned}$$

where we omitted the restriction onto $U_{\alpha\beta} \neq \emptyset$. Thus $\{[(\pi_\alpha^* \chi) v^\alpha] \bullet (\pi^* \mathrm{F}^\alpha)\}_{\alpha \in A}$ and $\{[(\pi_\alpha^* \chi) v^\alpha] \bullet (\pi^* \mathrm{C}^\alpha)\}_{\alpha \in A}$ define global forms "$\chi v \bullet \mathrm{F}$" and "$\chi v \bullet \mathrm{C}$" on B.

Corollary 2.119 *If $\phi \in \mathcal{A}(F, V)$ is invariant then $\{(\mathrm{pr}_F^* \phi) v^\alpha \in \mathcal{A}(U_\alpha \times F, V)\}_{\alpha \in A}$, resp., $\{(\pi_\alpha^* \phi) v^\alpha \in \mathcal{A}(\pi^{-1}(U_\alpha), V)\}_{\alpha \in A}$ defines a global form $\phi v \in \mathcal{A}(B, V)$. If ϕ is invariant and locally vertical, then $\{\pi_\alpha^* \phi\}_{\alpha \in A}$ is global.*

The opposite is not true in general, as the case of a trivial bundle with Lie group $G \neq \{e\}$ shows, where every invariant $\phi \in \mathcal{A}(F, V)$ defines a global but not necessarily vertical form $\pi_\alpha^* \phi$ on the bundle (all $g_{\beta\alpha}^* \Theta_G^L$ in Corollary 2.113 vanish). Nevertheless, the canonically generated form due to Proposition 2.75 is always vertical.

Finally, from Lemma 2.116 and Corollaries 1.114, 1.116 and 1.118 we obtain:

Theorem 2.120 *Let Γ be a connection on a principal fiber bundle $P(M,G)$ and let $B(M,F,G)$ be an associated bundle, V any vector space, $\chi_n^s \in \mathcal{A}_n(F) \otimes \mathrm{Hom}(\bigotimes^s \mathfrak{g}, V)$ be G-equivariant and $\phi_n \in \mathcal{A}_n(F) \otimes V$ be invariant under G. Then*

$$\begin{aligned} d(\chi_n^s v \bullet \mathrm{F}) &= [(d\chi_n^s) v]_{n+1}^s \bullet \mathrm{F} \;+\; [(L_\bullet \chi_n^s) v]_{n-1}^{s+1} \bullet \mathrm{F}, \\ &= [(d\chi_n^s) v]_{n+1}^s \bullet \mathrm{F} \;+\; [(L_\bullet^\vee \chi_n^s) v]_{n-1}^{s+1} \bullet \mathrm{F}, \\ d(\phi_n v) &= (d\phi_n) v \;+\; [(L_\bullet \phi_n) v]_{n-1}^1 \bullet \mathrm{F}. \end{aligned}$$

2.6 Bundles with Abelian Structure Group

As already stated in Lemma 2.68, the left action on the fiber $L: G \times F \to F$ naturally induces a left action on the product manifold $\bar{L}: G \times P \times F \to P \times F$, that is trivial in the factor P. Thus, besides $\widetilde{\mathcal{R}}'$, we also have a G-equivariant (with respect to $\bar{L}''$ and $\bar{L}_*$) $C^\infty(P \times F)$-module homomorphism $\bar{\mathcal{L}}': C^\infty(P \times F, \mathfrak{g}) \to \mathcal{D}^1(P \times F)$ with $(\bar{L}_g)_* \widetilde{\mathcal{R}}' = \widetilde{\mathcal{R}}' \bar{L}^*_{g^{-1}}$ and $(\bar{R}_g)_* \bar{\mathcal{L}}' = \bar{\mathcal{L}}' \bar{R}^*_{g^{-1}}$. In addition, $\mathrm{pr}_P \circ \bar{L}_g = \mathrm{pr}_P$ yields

$$\begin{aligned} (\bar{L}_g)_* \circ \tilde{v} &= \tilde{v} \circ (\bar{L}_g)_* = \tilde{v} \qquad & (\bar{L}_g)_* \circ \tilde{h} &= \tilde{h} \circ (\bar{L}_g)_* = (\bar{L}_g)_* - \tilde{v}, \\ (\bar{L}_g)_* \circ v^{\mathrm{nat}} &= v^{\mathrm{nat}} \circ (\bar{L}_g)_*, & (\bar{L}_g)_* \circ h^{\mathrm{nat}} &= h^{\mathrm{nat}} \circ (\bar{L}_g)_*. \end{aligned}$$

$\mathrm{pr}_P \circ \bar{L}^{(p,f)} = p$ yields $h^{\mathrm{nat}} \bar{\mathcal{L}}' = 0$, thus $\bar{\mathcal{L}}': C^\infty(P \times F, \mathfrak{g}) \to v^{\mathrm{nat}} \mathcal{D}^1(P \times F)$.

Now $\bar{L}$ defines an action on the quotient manifold $P \times_G F$ iff $\bar{L}_{h^{-1}} \circ \bar{R}_g \circ \bar{L}_h \in \bar{R}_G$ for all $g, h \in G$, where $\bar{R}_G := \{\bar{R}_g \in \mathrm{Diff}(P \times F)\}_{g \in G}$. Thus $\bar{L}_G < N_{\mathrm{Diff}(P \times F)}(\bar{R}_G)$: $\bar{L}_G$ needs to be a subgroup of the normalizer of $\bar{R}_G$ in $\mathrm{Diff}(P \times F)$. Even if G is abelian and $\bar{R}$ acts freely, this does not hold automatically, as the example of the action of $\mathbb{Z}_4$ on $\mathbb{R}^3 \backslash \{$"axes"$\}$ by $\frac{\pi}{2}$-rotations around different axes shows.

In our case $(\bar{L}_{h^{-1}} \circ \bar{R}_g \circ \bar{L}_h)(p, f) = \bar{R}_g(p, L_{gh^{-1}g^{-1}h}(f))$, thus

$$\hat{L} \text{ defines an action } \hat{L}: G \times B \to B \quad \Longleftrightarrow \quad L_{G'} = \{\mathrm{id}_F\},$$

where G' means the commutator subgroup in G. This is equivalent to the requirement that G acts effectively only through its largest abelian factor group G/G'. Since we require G to act effectively itself, this means G is abelian.

Note 2.121 According to the structure theorem for abelian LIE groups [7, p. 228], a connected LIE group G is abelian iff it is isomorphic to $\mathfrak{g}/\ker\exp$, thus iff G is isomorphic to $\mathbb{R}^m \times (\mathbb{S}^1)^n = \mathbb{R}^m \times (\mathbb{R}^n/\mathbb{N}^n)$ where $m, n \in \mathbb{N}_0$. Thus for any abelian LIE group we will write the group operation additively, with neutral element 0, and we will identify all tangent spaces $T_g(G)$ with $T_0(G)$ in a natural way, such that $d\lambda_g = d\rho_g: T_h(G) \to T_{h+g}(G)$ becomes the identity morphism for all $g, h \in G$.

In that case, $\hat{L}_g \circ \tilde{\pi} = \tilde{\pi} \circ \bar{L}_g$ and $\hat{\pi} \circ \hat{L}_g = \hat{\pi}$ (and thus $\hat{\pi} \circ \hat{L}^b = \hat{\pi}(b)$), because

$$\hat{\pi} \circ \hat{L}_g \circ \tilde{\pi} = \hat{\pi} \circ \tilde{\pi} \circ \bar{L}_g = \pi \circ \mathrm{pr}_P \circ \bar{L}_g = \pi \circ \mathrm{pr}_P = \hat{\pi} \circ \tilde{\pi}$$

and $\tilde{\pi}$ is surjective. Since $(\bar{L}_g)_*$ commutes with $\tilde{h}$ and (for abelian G) commutes with $(\bar{R}_g)_*$, it defines an action on $\mathcal{D}^{\tilde{\Gamma}}(P \times F)$, i. e., $(\bar{L}_g)_* \tilde{\mathbb{L}} = \tilde{\mathbb{L}}(\hat{L}_g)_*$. This proves

$$(\hat{L}_g)_* \circ \hat{v} = \hat{v} \circ (\hat{L}_g)_*, \qquad (\hat{L}_g)_* \circ \hat{h} = \hat{h} \circ (\hat{L}_g)_*,$$

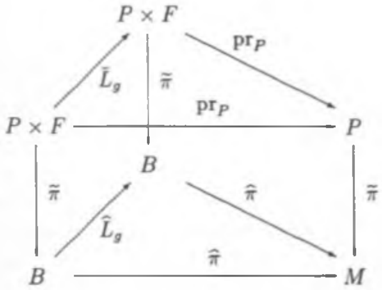
because $(\hat{L}_g)_* \hat{h} = (\hat{L}_g)_* \tilde{\pi}_* h^{\mathrm{nat}} \tilde{\mathbb{L}} = \tilde{\pi}_* (\bar{L}_g)_* h^{\mathrm{nat}} \tilde{\mathbb{L}} = \tilde{\pi}_* h^{\mathrm{nat}} (\bar{L}_g)_* \tilde{\mathbb{L}} = \hat{h}(\hat{L}_g)_*$. Finally $(\hat{L}_g)_* \hat{\mathbb{L}} = \tilde{\pi}_* (\bar{L}_g)_* \mathbb{L}^{\mathrm{nat}}_h \mathbb{L} = \hat{\mathbb{L}}$ and the horizontal lifts $\hat{\mathbb{L}}$ are $\hat{L}$-invariant. $\hat{h}\hat{\mathcal{L}}' = 0$, because $\hat{\mathcal{L}}': C^\infty(B, \mathfrak{g}) \to \hat{v}\mathcal{D}^1(B)$, since $\hat{\pi}_* \circ d\hat{L}^b = 0$ and $V_b(B)$ is the kernel of $d\hat{\pi}_b$. It is quite obvious that $\hat{L}$ coincides with the following locally defined action:

Lemma 2.122 *For abelian G, we have a left action $\widehat{L}$ of G on the whole bundle:*

$$\widehat{L}(g,b) := \psi_\alpha^{-1}(\widehat{\pi}(b), L(g, \widehat{\pi}_\alpha(b))) \quad \textit{for all } b \in B, g \in G, \quad \textit{where} \quad \widehat{\pi}(b) \in U_\alpha,$$

is then well-defined and fiber preserving: $\widehat{\pi}(\widehat{L}(g,b)) = \widehat{\pi}(b)$.

We thus get another diagram that commutes for every $g \in G$:



Note 2.123 Suppose $f: B(M,F,G) \to B'(M',F',G)$ is a fiber preserving bundle diffeomorphism between two bundles with left actions L, resp., L' of the abelian LIE group G and $\widehat{\Gamma}$ is a connection on B induced by Γ on $P(M,G)$, such that $(\widehat{L}_g)_* \circ \widehat{h} = \widehat{h} \circ (\widehat{L}_g)_*$ for all $g \in G$. By Lemma 2.71, $\widehat{\Gamma}$ induces a connection $\Gamma' = \widehat{\Gamma}^f$ on B'. For this new connection, h', v' and $(\widehat{L}'_g)_*$ need not commute on $\mathcal{D}^1(B')$. As an example, take $f = \mathrm{id}: M \times \mathbf{R} \to M \times \mathbf{R}$ and actions $L, L': \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ with $L(r,s) = e^r s$ and $L'(r,s) = r+s$. Then $h'(\mathcal{X}_x, \mathcal{Y}_s) = (\mathcal{X}_x, -sA_x\mathcal{X}_x)$ with $A \in \mathcal{A}_1(M)$ and $(\widehat{L}'_r)_* h'(\mathcal{X}_x, \mathcal{Y}_s) = (\mathcal{X}_x, -sA_x\mathcal{X}_x)$, while $h'(\widehat{L}'_r)_*(\mathcal{X}_x, \mathcal{Y}_s) = (\mathcal{X}_x, -rsA_x\mathcal{X}_x)$.

Analogous to Proposition 2.39, it is sufficient for commutativity of h', v' and $\widehat{L}'_g$ that f is G-equivariant. In fact, if B and B' are associated bundles over M and f is G-eqivariant and induces the identity on M, then $\widehat{\Gamma}$ and Γ induce the same connection $\widehat{\Gamma}'$ on $B'(M,F',G)$.

For abelian G, the adjoint action on $\mathfrak{g}$ is trivial, which makes life easier in most cases. Let us specialize our results: the discussion following Definition 2.46 shows:

Lemma 2.124 *If G is abelian then $\pi^*: \mathcal{A}(M,\mathfrak{g}) \to \mathcal{A}^T(P,\mathfrak{g})$ is an isomorphism of $C^\infty(M)$-moduls and* GRASSMANN *algebras, commuting with exterior differentiation.*

From Theorem 2.58 and Theorem 2.59 we immediately get

Theorem 2.125 *If G is abelian and $\omega^\Gamma \in \mathcal{A}_\gamma(P(M,G))$ then we have:*

structure equation for abelian G:	$\Omega^\Gamma = d^\Gamma\omega^\Gamma$	$=$	$d\omega^\Gamma$;	
BIANCHI identity for abelian G:	$d^\Gamma\Omega^\Gamma = d\Omega^\Gamma$	$=$	0;	
for all $\varphi \in \mathcal{A}^T(P,L,V)$: $d^\Gamma\varphi = d\varphi$,	$(d^\Gamma)^m\varphi$	$=$	0,	$m \geq 2$;
for all $\alpha \in \mathcal{A}^P(P,L,V)$:	$(d^\Gamma)^m\alpha$	$=$	0,	$m \geq 3$.

Theorem 2.61 and Corollary 2.65 yield (we have $g_{\beta\alpha}^*\Theta^L = dg_{\beta\alpha}$, cf. Note 2.121):

Theorem 2.126 *Let G be abelian, $\omega^\Gamma \in \mathcal{A}_\gamma(P(M,G))$ and $\{(U_\alpha,\psi_\alpha)\}_{\alpha\in A}$ a bundle atlas for P, then for all $\alpha,\beta \in A$ with $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$ and for all $x \in U_{\alpha\beta}$:*

$$\mathrm{F}^\alpha = d\mathrm{A}^\alpha, \qquad d\mathrm{F}^\alpha = 0; \tag{90}$$

$$\mathrm{A}^\alpha|_{U_{\alpha\beta}} = \mathrm{A}^\beta|_{U_{\alpha\beta}} + dg_{\beta\alpha} = \mathrm{A}^\beta|_{U_{\alpha\beta}} - dg_{\alpha\beta}; \tag{91}$$

$$\mathrm{F}^\alpha|_{U_{\alpha\beta}} = \mathrm{F}^\beta|_{U_{\alpha\beta}}. \tag{92}$$

$$\mathrm{C}^\alpha|_{U_{\alpha\beta}} = \mathrm{C}^\beta|_{U_{\alpha\beta}}. \tag{93}$$

Vice versa, if for a bundle atlas $\{(U_\alpha,\psi_\alpha)\}_{\alpha\in A}$ on the principal bundle $P(M,G)$ with abelian G a family $\{\mathrm{A}^\alpha \in \mathcal{A}_1(U_\alpha,\mathfrak{g})\}_{\alpha\in A}$ is given such that (91) holds, then there exists one unique $\omega^\Gamma \in \mathcal{A}_\gamma(P(M,G))$ such that $\mathrm{A}^\alpha = \sigma_{\alpha,e}^*(\omega^\Gamma|_{\pi^{-1}(U_\alpha)})$ for all $\alpha \in A$.

Thus for abelian G, the collection of F^α defines a *global* 2-form $\mathrm{F} \in \mathcal{A}_2(M,\mathfrak{g})$; if M is paracompact then the collection of C^α defines a *global* 1-form $\mathrm{C} \in \mathcal{A}_1(M,\mathfrak{g})$.

Finally let us treat the one-dimensional case, $\mathfrak{g} \cong \mathbb{R}$. So $G = D \times G_1$ with a discrete abelian subgroup D and $G_1 \cong \mathbb{S}^1$ or $G_1 \cong \mathbb{R}$. Recall from Corollary 2.25 that if G is connected, nontrivial bundles only exist for $G \cong \mathbb{S}^1$, e. g. for the electromagnetic gauge group $G_{\mathrm{em}} \cong \mathrm{U}_1 \cong \mathbb{S}^1$.

So suppose $\mathfrak{g} = E\mathbb{R}$, then the antisymmetry of differential forms yields that $L_\bullet^2\phi = 0$ for all $\phi \in \mathcal{A}_n(F,V)$. Thus Lemma 2.116 reads $(\mathrm{pr}_F^*\phi)v^\alpha = (\mathrm{pr}_F^*\phi) - (-1)^n[\mathrm{pr}_F^*(L_\bullet\phi)] \bullet (\mathrm{pr}_{U_{\alpha\beta}}^*\mathrm{A}^\alpha) = (\mathrm{pr}_F^*\phi) + \frac{1}{E}(\mathrm{pr}_{U_{\alpha\beta}}^*\mathrm{A}^\alpha) \wedge (\mathrm{pr}_F^*\imath_{\mathcal{L}_E}\phi)$. Analogously, Corollary 2.113 takes the form $T_{\beta\alpha}^*(\mathrm{pr}_F^*\phi) = (\mathrm{pr}_F^*\phi) + \frac{1}{E}(\mathrm{pr}_{U_{\alpha\beta}}^*dg_{\beta\alpha}) \wedge (\mathrm{pr}_F^*\imath_{\mathcal{L}_E}\phi)$ if $\phi \in \mathcal{A}(F,V)_{\mathrm{inv}}$. In that case, since $L_\bullet(L_\bullet\phi) = 0$ by (51), $\imath_{\mathcal{L}_E}\phi$ is vertical and global (it is invariant because Ad is trivial). Also recall that $d\imath_{\mathcal{L}_E}\phi + \imath_{\mathcal{L}_E}d\phi = L_{\mathcal{L}_E}\phi = 0$ if ϕ is invariant. Thus Corollary 2.119 and Theorem 2.120 prove:

Theorem 2.127 *Let Γ be a connection on $P(M,G)$ with abelian G, $\mathfrak{g} = E\mathbb{R} \cong \mathbb{R}$, $B(M,F,G)$ an associated bundle and V any vector space. For any $\phi \in \mathcal{A}_n(F,V)$ with $L_g^*\phi = \phi$ for all $g \in G$ define $\nu \in \mathcal{A}_{n-1}(F,V)$ by $\nu = \imath_{\mathcal{L}_E}\phi$, i. e.*

$$\nu_f(\mathcal{Y}_f^1,\ldots,\mathcal{Y}_f^{n-1}) := n\cdot\phi_f(dL^f(E),\mathcal{Y}_f^1,\ldots,\mathcal{Y}_f^{n-1}) \quad \text{for all } f \in F,\ \mathcal{Y}^i \in \mathcal{D}^1(F).$$

For any $U_\alpha \in \mathfrak{U}$ denote $\phi^\alpha := \pi_\alpha^*\phi$, $\nu^\alpha := \pi_\alpha^*\nu$. Then on all $U_{\alpha\beta} \neq \emptyset$

$$\phi^\alpha = \phi^\beta + \frac{1}{E}\pi^*dg_{\alpha\beta}\wedge\nu^\beta, \quad \phi^\alpha v = \phi^\alpha + \frac{1}{E}\pi^*\mathrm{A}^\alpha\wedge\nu^\alpha = \phi^\beta + \frac{1}{E}\pi^*\mathrm{A}^\beta\wedge\nu^\beta = \phi^\beta v,$$
$$\nu^\alpha = \nu^\alpha v = \nu^\beta = \nu^\beta v.$$

Thus ϕv and ν define global vertical invariant V-valued forms on B. The same holds for $(d\phi)v$ since $d\phi$ is also invariant, and we have

$$d(\phi v) = (d\phi)v + \frac{1}{E}\pi^*\mathrm{F}\wedge\nu, \quad \text{where} \quad (d\phi^\alpha)v = d\phi^\alpha - \frac{1}{E}\pi^*\mathrm{A}^\alpha\wedge d\nu^\alpha.$$

Note that $\mathfrak{g} \cong \mathbb{R}$ alone does not imply that G is abelian. $G = \mathbb{S}^1 \rtimes \mathbb{Z}_2$ with $(r,g)\cdot(r',e) = (r - r', g)$ for $r, r' \in \mathbb{S}^1$ and $g \neq e \in \mathbb{Z}_2$, is a simple counterexample, where $\mathrm{Ad}((0,g)) = -\mathrm{id}_{\mathfrak{g}}$, and thus ν in Theorem 2.127 would not be invariant and global for this Lie group G.

Chapter 3

Combining Cohomologies of Complexes with Connections

In this chapter we will introduce several cohomologies, not only the well known DE RHAM cohomology but also LIE algebra cohomology, the (trivial) ČECH cohomology and the combination of the latter with the DE RHAM cohomology of the ČECH-DE RHAM double complex. Another example for a cohomology is the integer valued so-called singular cohomology. For the purpose of covering them all we will introduce cohomology and the underlying differential complex in a broader version.

3.1 Complexes and Double Complexes

We start this section with the basic definitions and conclusions, cf. BOTT, TU, [11].

Definition 3.1 *A* (differential) complex $C = \bigoplus_{i\in\mathbb{Z}} C^i$ *with a* differential operator D *is a direct sum of modules (resp., merely abelian groups)* C^i, $i \in \mathbb{Z}$ *with homomorphisms* $D_i\colon C^i \to C^{i+1}$, *where* $D_{i+1} \circ D_i = 0$, *resp.,* $D^2 = 0$*:*

$$\cdots \longrightarrow C^{i-1} \xrightarrow{D_{i-1}} C^i \xrightarrow{D_i} C^{i+1} \xrightarrow{D_{i+1}} \cdots$$

For any complex with a smaller set of indices, e. g. $\{C^i\}_{i\in\mathbb{N}_0}$, *one can add an infinite number of copies* $C^i := C^0$ *for* $i < 0$, *combined with the zero map on* C^0, *to get a differential complex in the above sense.*

A chain map $f\colon A \to B$ *between two differential complexes* A, B *is a homomorphism that commutes with the differential operators of* A *and* B*:* $f \circ D_A = D_B \circ f$.

Definition 3.2 *An element* c *in a differential complex* C *is said to be* closed, *if* $Dc = 0$. c *is said to be* exact, *if there exist* $a \in C$, *such that* $Da = c$. *Since* $D^2 = 0$, *every exact element is closed:* $\operatorname{im} D_{i-1} \subseteq \ker D_i$. *Now the* cohomology *of the complex* C *is defined to be the direct sum of modules* $H^\bullet(C) := \bigoplus_{i\in\mathbb{Z}} H^i(C)$ *with abelian* cohomology groups

$$H^i(C) := \ker D_i / \operatorname{im} D_{i-1}.$$

Its elements are denoted by $[c] = c + D(C) \in H^\bullet(C)$, *where* $c \in C$.

Note 3.3 If the differential operators "descent", i. e., $D_i: C^i \to C^{i-1}$, we speak of the *homology* $H_\bullet(C)$ of the complex C. If the C^i are merely abelian groups, one also speaks of a *differential graded group* C and calls D a *(co-)boundary operator*, cf. SPANIER, [12]. A *chain complex* is a differential complex in which the differential operator is of degree -1 and a *co-chain complex* is a complex in which D is of degree $+1$. The elements of C are called *(co-)chains*, closed elements are also called *(co-)cycles* and exact elements are called *(co-)boundaries*.

Lemma 3.4 *Every chain map* $f: A \to B$ *induces a homomorphism of cohomologies* $[f]: H^\bullet(A) \to H^\bullet(B)$ *by* $[f][a] := [f(a)]$.

Proof. Since f commutes with D_A, it maps (co-)cycles onto (co-)cycles and maps (co-)boundaries onto (co-)boundaries. □

Definition 3.5 *Let* $f: A \to B$ *and* $g: B \to A$ *be two chain maps with* $f \circ g = \mathrm{id}_B$. *If a homomorphism* $K: A \to A$, $A^i \to A^{i-1}$ *obeys*

$$g \circ f - \mathrm{id}_A = \pm(D_A K \pm K D_A), \tag{94}$$

then K *is called a* homotopy operator *for* f *and* g *and* (94) *is called a* homotopy identity.

Lemma 3.6 *If* K *is a homotopy operator for* f *and* g *then* $H^\bullet(A) \cong H^\bullet(B)$.

Proof. On the one hand, $[f] \circ [g] = [\mathrm{id}_B] = \mathrm{id}_{H^\bullet(B)}$, on the other hand for any combination of signs, $\pm(D_A K \pm K D_A)$ maps closed elements of A onto exact elements of A. This proves $[g] \circ [f] = [\mathrm{id}_A] = \mathrm{id}_{H^\bullet(A)}$ and so $[f]$ and $[g]$ are inverse isomorphisms. □

Thus $L = \pm DK \pm KD$ yields $[L] = 0$. For the reverse we have:

Lemma 3.7 *Suppose* $L: A \to A$ *is a chain map with* $[L] = 0$ *and every module* A^i *decomposes into* $A^i = \ker D_i \oplus B^i$. *Then a homomorphism* $K: A \to A$, $A^i \to A^{i-1}$ *exists such that the homotopy identity* $L = DK + KD$ *holds. E. g.,* K *is given by*

$$K|_{\ker D_i} := D_{i-1}^{-1} \circ L|_{\ker D_i}, \qquad K|_{B^i} := 0,$$

where $D_i^{-1}: \mathrm{im}\, D_i \to B^i \cong \mathrm{im}\, D_i$ *for all* $i \in \mathbb{Z}$.

Proof. $[L] = 0$ means $L(\ker D_i) \subseteq \mathrm{im}\, D_{i-1}$. Since $D|_{B^i}$ is an isomorphism, K is well-defined. Let $a_i = a_i' + b_i \in A^i$ with $a_i' \in \ker D_i$ and $b_i \in B^i$. Then

$$(D_{i-1}K + KD_i)(a_i' + b_i) = D_{i-1}K(a_i') + KD_i(b_i) = L(a_i') + D_i^{-1}LD_i(b_i) = L(a_i' + b_i),$$

beause L is a chain map. □

Definition 3.8 *A sequence of abelian groups* A_i *with homomorphisms* $f_i\colon A_i \to A_{i+1}$,

$$\cdots \longrightarrow A_{i-1} \xrightarrow{f_{i-1}} A_i \xrightarrow{f_i} A_{i+1} \xrightarrow{f_{i+1}} \cdots,$$

is said to be exact at A_i *if* $\ker f_i = \operatorname{im} f_{i-1}$. *For an* exact sequence, $\ker f_i = \operatorname{im} f_{i-1}$ *for all* i. *A* short exact sequence *is an exact sequence of the form*

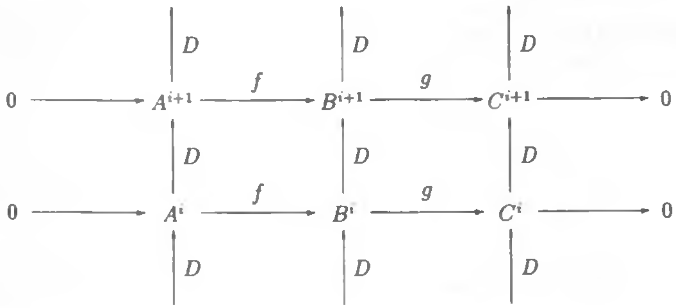
$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \qquad (95)$$

Lemma 3.9 *Any differential complex* $\{C^i\}_{i\in\mathbb{N}_0}$ *can be turned into a complex* $C = \bigoplus_{i\in\mathbb{Z}} C^i$ *by putting* $(C^{-1}, D_{-1}) := (\ker D_0, j)$ *and* $(C^i, D_i) := (\{0\}, 0)$ *for* $i < -1$, *where* $j\colon \ker D_0 \to C^0$ *denotes the injection. The resulting sequence*

$$\cdots \xrightarrow{0} C^{-2} \xrightarrow{0} C^{-1} \xrightarrow{j} C^0 \xrightarrow{D_0} C^1 \xrightarrow{D_1} \cdots$$

is then exact at all C^i *for* $i \leq 0$.

Figure 3.1: Commutative diagram for the exact sequence of differential complexes



The following proposition is an important tool in cohomology theory.

Proposition 3.10 *Any short exact sequence* (95) *of differential complexes, in which the homomorphisms* f, g *are chain maps, produces a long exact sequence of cohomology groups*

$$\cdots H^i(A) \xrightarrow{[f]} H^i(B) \xrightarrow{[g]} H^i(C) \xrightarrow{[D]} H^{i+1}(A) \xrightarrow{[f]} H^{i+1}(B) \xrightarrow{[g]} H^{i+1}(C) \xrightarrow{[D]} \cdots.$$

In this sequence $[f]$ *and* $[g]$ *are the naturally induced homomorphisms and the* connecting homomorphism $[D]$ *is obtained as follows (cf. the commutative diagram in Figure 3.1): For any closed* $c \in C^i$, *there exists* $b \in B^i$ *with* $g(b) = c$ *since* g *is surjective. Next* $g(Db) = Dc = 0$, *thus* $Db = f(a)$ *with* $a \in A^{i+1}$, *because of the exactness of the short sequence at* B. *Now* $[D]$ *is well-defined by* $[D][c] := [a]$.

Proof. First of all, since $f(Da) = Df(a) = DDb = 0$ and f is injective, $Da = 0$. To prove that $[D]$ is well-defined, we must show that $[D]DC = DA$. Thus let $c \in C^{i-1}$. Then we find $b, b' \in B$ with $c = g(b)$, $Dc = g(b')$ and thus $g(b' - Db) = 0$. Exactness of (95) at B yields that there exist $a, a' \in A$ with $b' = Db + f(a)$ and $Db' = f(a')$ (recall $g(Db') = D^2c = 0$). On the other hand, $Db' = Df(a) = f(Da)$, so $a' = Da$ since f is injective. This proves $[D][Dc] = [Da] = [0]$. Now we check that the resulting sequence is exact:

1. Exactness at $H^i(A)$: $[f][D][c] = [f][a] = [Db] = [0]$, so $\mathrm{im}[D] \subseteq \ker[f]$. Let $[f][a] = [f(a)] = [0]$. Then we find $b \in B$ with $f(a) = Db$. We put $c := g(b)$ and find $Dc = Dg(b) = g(f(a)) = 0$, thus $[a] = [D][c]$ and $\ker[f] \subseteq \mathrm{im}[D]$.

2. Exactness at $H^i(B)$: $[g][f][a] = [g(f(a))] = [0]$ proves $\mathrm{im}[f] \subseteq \ker[g]$. Let $[g][b] = [0] = DC$. Then we find $c \in C$ with $g(b) = Dc$ and $b' \in B$ with $g(b') = c$. This yields $g(b - Db') = 0$ and we find $a \in A$ such that $f(a) = b - Db'$. Thus $b = db' + f(a) \in [f(a)] = [f][a]$ and $\ker[g] \subseteq \mathrm{im}[f]$.

3. Exactness at $H^i(C)$: $[D][g][b] = [D][g(b)] = [0]$ since $Db = 0$, so $\mathrm{im}[g] \subseteq \ker[D]$. Let $[D][c] = [0]$. Then $c = g(b)$ with $Db = f(0) = 0$. Thus $[c] = [g][b]$ and $\ker[D] \subseteq \mathrm{im}[g]$. □

Definition 3.11 *By a* subcomplex C' *we mean a submodule* $C' \subseteq C$*, such that* $DC' \subseteq C'$*. A* filtration *of* C *is a sequence of subcomplexes* C_i

$$C = C_0 \supseteq C_1 \supseteq C_2 \supseteq \cdots.$$

Then C *becomes a* filtered complex *with* associated graded complex

$$GC := \bigoplus_{p=0}^{\infty} C_p/C_{p+1}.$$

We define a filtration for negative indices by putting $C_p := C$ *for* $p < 0$*. The module*

$$A := \bigoplus_{p \in \mathbb{Z}} C_p$$

is a complex with differential operator D*, too, and if* $i\colon A \to A$ *denotes the inclusion* $C_{p+1} \to C_p$*,* $p \in \mathbb{Z}$*, then the quotient of this map*

$$B := A/i(A) = \bigoplus_{p=0}^{\infty} C_p/C_{p+1}$$

is nothing but the graded complex GC *associated with* C*, equipped with the differential operator induced by* D*.*

The combination of two complexes with index sets $\mathbb{N}_0$ and commuting differential operators results in a double complex, where one operator acts horizontally and the other acts vertically:

Definition 3.12 *A* double complex *or* doubly graded complex $C^{\bullet,\bullet} := \bigoplus_{p,q\in\mathbb{N}_0} C^{p,q}$ *is the direct sum of modules $C^{p,q}$, $p, q \in \mathbb{N}_0$, for which commuting differential operators $\delta_{(p)}: C^{p,q} \to C^{p+1,q}$ and $d_{(q)}: C^{p,q} \to C^{p,q+1}$ exist. We can turn any double complex into a singly graded complex C by summing along the antidiagonal lines*

$$C^n := \bigoplus_{p+q=n} C^{p,q}$$

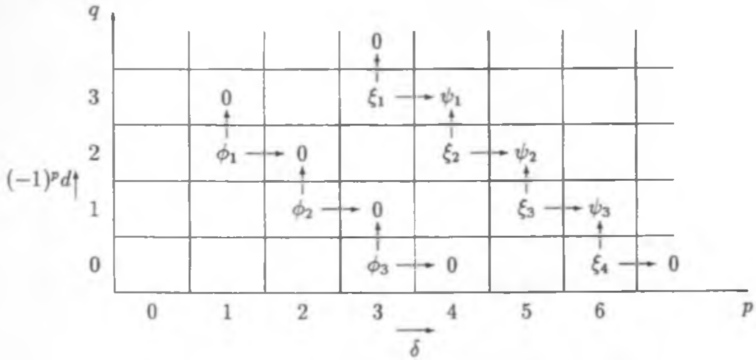
and introducing a new differential operator $D_{(n)}: C^n \to C^{n+1}$ by

$$D := D' + D'', \qquad D' := \delta, \qquad D'' := (-1)^p d \text{ on } C^{p,q}.$$

The cohomology $H_D^\bullet(C)$ is called the total cohomology *of the double complex.*

Note that the alternating sign guaranties that $D^2 = \delta^2 + \delta d - d\delta + d^2 = 0$, so D is indeed the base of a cohomology $H_D^\bullet(C)$. E. g., a D-closed element $\Phi \in C$, $D\Phi = 0$ looks like in Figure 3.2: $\Phi = \phi_1 + \phi_2 + \phi_3$ with $d\phi_1 = 0$, $\delta\phi_1 + D''\phi_2 = \delta\phi_1 + d\phi_2 = 0$, $\delta\phi_2 + D''\phi_3 = \delta\phi_2 - d\phi_3 = 0$ and $\delta\phi_3 = 0$.

Figure 3.2: D-closed and D-exact elements in a double complex



Analogously, $\Psi = \psi_1 + \psi_2 + \psi_3$ is a D-exact element of C, if there exists a cochain $\Xi = \xi_1 + \xi_2 + \xi_3 + \xi_4$ with $D\Xi = \Psi$, i. e., $0 = d\xi_1$, $\psi_1 = \delta\xi_1 + d\xi_2$, $\psi_2 = \delta\xi_2 - d\xi_3$, $\psi_3 = \delta\xi_3 + d\xi_3$ and $0 = \delta\xi_4$.

In view of Definition 3.1, we could turn $C^{\bullet,\bullet}$ into a double complex with index sets $\mathbb{Z}$ by putting $C^{p,q} := C^{p,0}$ for $q < 0$, resp., $C^{p,q} := C^{0,q}$ for $p < 0$, combined with zero maps δ_p and d_q for $p, q < 0$. Nevertheless it is more useful to enlarge $C^{\bullet,\bullet}$ analogously to Lemma 3.9, cf. Lemma 3.13 below.

For the double complex $C^{\bullet,\bullet} := \bigoplus_{p,q\in\mathbb{N}_0} C^{p,q}$, the sequence

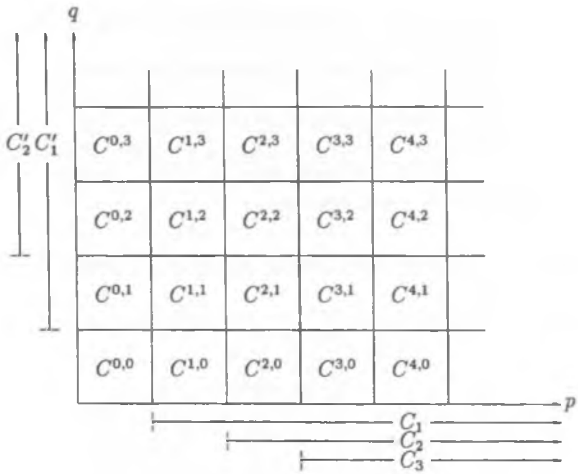
$$C_p := \bigoplus_{i\geq p} \bigoplus_{q\geq 0} C^{i,q}, \qquad p \in \mathbb{Z}$$

is a filtration of C along the columns of C with associated graded complex

$$B = \bigoplus_{p\in\mathbb{Z}} C_p/C_{p+1} = \bigoplus_{p\in\mathbb{Z}} \left[\left(\bigoplus_{q\geq 0} C^{p,q} \right) + C_{p+1} \right]. \tag{96}$$

We recognize the differential operator on B induced by D is just $(-1)^p d$, since $\delta: C_p \to C_{p+1}$ is zero on B.

Figure 3.3: Two filtrations of a double complex $C^{\bullet,\bullet}$



We can as well introduce a filtration of C along its rows:

$$C'_q := \bigoplus_{j\geq q}\bigoplus_{p\geq 0} C^{p,j}, \qquad q \in \mathbb{Z}, \qquad \text{thus}$$

$$B' = \bigoplus_{q\in\mathbb{Z}} C'_q/C'_{q+1} = \bigoplus_{q\in\mathbb{Z}} \left[\left(\bigoplus_{p\geq 0} C^{p,q} \right) + C'_{q+1} \right], \tag{97}$$

then δ is the differential operator on B' that is induced by D. Figure 3.3 illustrates these two filtrations of the double complex C.

Every double complex can be naturally augmented by an extra column and an extra row: every row of $C^{\bullet,\bullet}$ can be augmented on the left by injecting the kernel of $\delta_0: C^{0,\bullet} \to C^{1,\bullet}$. Then by definition the resulting sequence

$$0 \longrightarrow \ker \delta_0 \xrightarrow{i_\delta} C^{0,\bullet} \xrightarrow{\delta_0} C^{1,\bullet} \xrightarrow{\delta_1} \cdots$$

is exact at $\ker\delta_0$ and $C^{0,\bullet}$. Obviously $\ker\delta_0 = H^0_\delta(C^{\bullet,\bullet})$. We can also augment every column of $C^{\bullet,\bullet}$ at the bottom by injecting $\ker D''_0 = \ker d_0$ and obtain a sequence

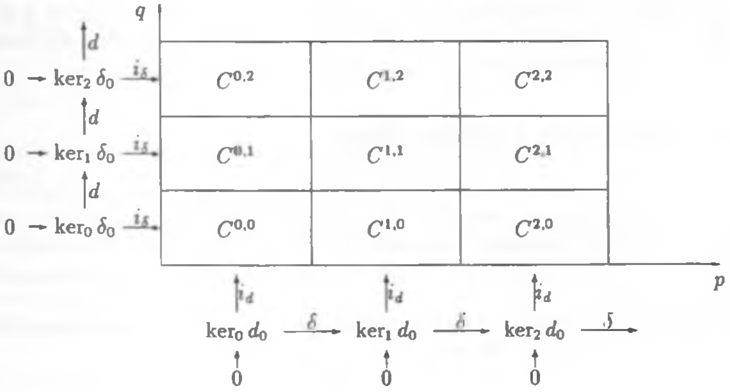
$$0 \longrightarrow \ker d_0 \xrightarrow{i_d} C^{\bullet,0} \xrightarrow{D''_0} C^{\bullet,1} \xrightarrow{D''_1} \cdots,$$

that is exact at $\ker d_0$ and $C^{\bullet,0}$, cf. Figure 3.4. Since δ and d commute, we have $\delta_0 d_q c_q = d_q \delta_0 c_q = 0$ for $c_q \in \ker_q \delta_0$. Thus $\ker \delta_0$ becomes a differential complex with restricted operator d, and i_δ becomes a chain map because $i_\delta \circ d = d \circ i_\delta = D \circ i_\delta$ by exactness at $C^{0,\bullet}$. Analogous conclusions for $\ker d_0$ prove:

Lemma 3.13 *The additional column* $\ker \delta_0 = H^0_\delta(C^{\bullet,\bullet})$ *and row* $\ker d_0 = H^0_d(C^{\bullet,\bullet})$ *of an augmented double complex* $C^{\bullet,\bullet}$ *are single complexes with operators* d*, resp.,* δ*. The inclusions* i_δ *and* i_d *are chain maps with respect to* D *and induce morphisms*

$$[i_\delta]\colon H^\bullet_d(\ker \delta_0) \to H^\bullet_D(C), \qquad \textit{resp.,} \qquad [i_d]\colon H^\bullet_\delta(\ker d_0) \to H^\bullet_D(C).$$

Figure 3.4: The augmented double complex



Analogously to Lemma 3.9, we have turned $C^{\bullet,\bullet}$ into a double complex $\bigoplus_{p,q\in\mathbb{Z}} C^{p,q}$

$$\text{with} \quad (C^{p,q}, \delta_p, d_q) := \begin{cases} (\ker \delta_0, i_\delta, d_q) & \text{for} \quad p = -1, q \geq 0, \\ (\ker d_0, \delta_p, i_d) & \text{for} \quad q = -1, p \geq 0, \\ (\ker \delta_0 \cap \ker d_0, i_\delta, i_d) & \text{for} \quad p = -1, q = -1, \\ (\{0\}, 0, 0) & \text{for} \quad p < -1 \text{ or } q < -1. \end{cases}$$

An important observation on the relationship between $H^\bullet_d(\ker \delta_0)$, $H^\bullet_\delta(\ker d_0)$ and $H^\bullet_D(C)$ is the following (cf. [11, p. 97]): If all rows of an augmented double complex are exact then $[i_\delta]$ is an isomorphism, and vice versa for the columns of $C^{\bullet,\bullet}$ and $[i_d]$. Moreover, we can prove:

Proposition 3.14 *If the rows of an augmented double complex are exact at* $C^{p,q}$ *for all* p, q *with* $n - 1 \leq p + q \leq n$*, then*

$$[i_\delta]\colon H^n_d(\ker \delta_0) \to H^n_D(C)$$

is an isomorphism. If the columns of an augmented double complex are exact at $C^{p,q}$ for all p, q with $n-1 \leq p+q \leq n$, then

$$[i_d]: H^n_\delta(\ker d_0) \to H^n_D(C)$$

is an isomorphism.

Proof. $[i_\delta]$ is surjective: Let $a = \sum_{i=0}^{n} a_i$ with $a_i \in C^{i,n-i}$ and $Da = 0$. Thus $\delta_n a_n = 0$. By δ-exactness we find $c_{n-1} \in C^{n-1,0}$ with $\delta_{n-1}c_{n-1} = a_n$. Now $a^{(1)} := a - Dc_{n-1} \in [a] \in H^n_D(C)$ is a representative of $[a]$ with lowest component removed. By induction we proceed to a representative $a^{(n-1)} \in [a]$ with $a^{(n-1)} \in C^{0,n}$. Now $Da^{(n-1)} = 0$ yields $a^{(n-1)} \in \ker_n \delta_0$ and $[i_\delta][a^{(n-1)}]_d = [a^{(n-1)}]_D = [a]$.

$[i_\delta]$ is injective: Suppose $[i_\delta][a]_d = [0]_D$ for $a \in \ker_n \delta_0$. Then $a = Db$ with $b = \sum_{i=0}^{n-1} b_i$, $b_i \in C^{i,n-1-i}$. Thus $\delta_{n-1}b_{n-1} = 0$, and as before we can shorten b by subtracting D-co-boundaries to obtain $b^{(n-2)} \in C^{0,n-1}$ with $Db^{(n-2)} = a$, i. e., $d_{n-1}b^{(n-2)} = a$ and $\delta_0 b^{(n-2)} = 0$. Thus $b^{(n-2)} \in \ker_{n-1}\delta_0$ and $[a]_d = [0]_d \in H^n_d(\ker \delta_0)$.

Analogous arguments hold for $[i_d]$ using the exactness of the columns. □

3.2 De Rham Cohomology

The most important example for a cohomology with regard to our purposes is the de Rham cohomology of a manifold. Let us assume from now on that M is paracompact. We already stated in Proposition 1.18.3 that the exterior differentiation of forms is a differential operator. Thus the Grassmann algebra $\mathcal{A}(M)$ is a complex:

Definition 3.15 *The (real-valued)* de Rham cohomology *of a n-dimensional manifold M is defined to be the $\mathbb{R}$-vector space*

$$H^\bullet(M) := H^\bullet_d(\mathcal{A}(M)) = \bigoplus_{p=0}^{\infty} H^p(M), \quad \textit{where} \quad H^p(M) = \ker d_p / \operatorname{im} d_{p-1}.$$

Analogously, for every vector space V, $\mathcal{A}(M) \otimes V$ is the differential complex for the V-valued de Rham *cohomology $H^\bullet(M) \otimes V$. Especially $H^\bullet(M, \mathbb{C}) = H^\bullet(M) \otimes \mathbb{C}$ denotes the complex-valued* de Rham *cohomology.*

Obviously $H^p(M) = \{0\}$ for $p > n$, since then $\mathcal{A}_p(M) = \{0\}$. The dimensions of the vector spaces are known as Betti numbers $b_p(M) := \dim_{\mathbb{R}} H^p(M)$; *the* Euler characteristic $\chi(M)$ *denotes their alternating sum:*

$$\chi(M) := \sum_{p=0}^{\infty}(-1)^p b_p(M) = \sum_{p=0}^{n} b_p(M).$$

Since d commutes with pullbacks (cf. Lemma 1.29.2) we obtain:

Lemma 3.16 *Every C^∞-map $f: M \to N$ induces a chain map $f^*: \mathcal{A}(N) \to \mathcal{A}(M)$ which in turn induces $[f^*]: H^\bullet(N) \to H^\bullet(M)$.*

Corollary 3.17 *If $\sigma: M \to B$ is a section of a bundle B, then π^* is injective and σ^* is surjective. Thus $[\pi^*]: H^\bullet(M) \to H^\bullet(B)$ is an injective homomorphism, while $[\sigma^*]: H^\bullet(B) \to H^\bullet(M)$ is a surjective homomorphism.*

Proof. By definition of a section, $\pi \circ \sigma = \mathrm{id}_M$. Thus $\sigma^* \circ \pi^* = \mathrm{id}_M^* = \mathrm{id}_{\mathcal{A}(M)}$ and $[\sigma^*] \circ [\pi^*] = \mathrm{id}_{H^\bullet(M)}$. □

From the homotopy identity in Proposition 1.39 we obtain immediately:

Proposition 3.18 *For every $\mathcal{X} \in \mathcal{D}^1(M)$, $[L_\mathcal{X}]: H^\bullet(M) \to H^\bullet(M)$ is the zero map. Every derivation of $\mathcal{A}(M)$ of degree 0 that commutes with d, induces 0 on $H^\bullet(M)$.*

We will prove that $H^\bullet(M \times \mathbb{R}) \cong H^\bullet(M)$ for any manifold M. Consider the maps $\mathrm{pr}_M: M \times \mathbb{R} \to M$ and $\imath_r: M \to M \times \mathbb{R}$ for any $r \in \mathbb{R}$. Since $\imath_r$ is a section, $\mathrm{pr}_M \circ \imath_r = \mathrm{id}_M$ proves $\imath_r^* \circ \mathrm{pr}_M^* = \mathrm{id}_{\mathcal{A}(M)}$, but obviously $\mathrm{pr}_M^* \circ \imath_r^* \neq \mathrm{id}_{\mathcal{A}(M\times\mathbb{R})}$. Yet if we find a homotopy operator for $\imath_r^*$ and pr_M^*, our result will follow from Lemma 3.6.

For this purpose, let φ denote the one-parameter group of diffeomorphisms φ_t of $M \times \mathbb{R}$ with $\varphi_t(x, t') := (x, t' + t)$, let $\mathcal{T} \in \mathcal{D}^1(M \times \mathbb{R})$ denote the induced vector field and dt the corresponding 1-form.

Definition 3.19 *We define the integral operator $\int_r: \mathcal{D}_\bullet(M \times \mathbb{R}) \to \mathcal{D}_\bullet(M \times \mathbb{R})$ of degree 0 for $r \in \mathbb{R}$ and $\omega \in \mathcal{D}_\bullet(M \times \mathbb{R})$ pointwise by*

$$\Big(\int_r \omega\Big)_{(x,t)} := \int_r^t [(\varphi_{t'-t})^*\omega]_{(x,t)} dt' = \int_r^t \omega_{(x,t')} dt' \qquad \text{for all} \quad x \in M,\ t, t' \in \mathbb{R}.$$

The last identity holds under natural identification of the tangent spaces $T_{(x,t)}(M \times \mathbb{R})$ and $T_{(x,t')}(M \times \mathbb{R})$. Linear extension defines $\int_r$ on $\mathcal{D}_\bullet(M \times \mathbb{R}) \otimes V$.

E. g., by evaluation on every chart $U_\alpha \times \mathbb{R}$ of $M \times \mathbb{R}$ one proves:

Lemma 3.20 *For every $\omega \in \mathcal{A}(M) \otimes V$ and $r \in \mathbb{R}$ we have*

$$\begin{aligned}
[(d\int_r - \int_r d)\omega]_{(x,t)} &= [dt \wedge (\mathrm{pr}_M^* \imath_r^* \omega)]_{(x,t)} = (dt \wedge \omega)_{(x,r)} \\
L_\mathcal{T} \int_r \omega &= \omega, \qquad \textit{whereas} \qquad [\int_r L_\mathcal{T}\omega]_{(x,t)} = \omega_{(x,t)} - \omega_{(x,r)}, \\
\int_r \imath_\mathcal{T}\omega &= \imath_\mathcal{T} \int_r \omega.
\end{aligned}$$

Proposition 3.21 *$K_r := \int_r \circ \imath_\mathcal{T}$ is a homotopy operator for $\imath_r^*$ and pr_M^* for all $r \in \mathbb{R}$:*

$$dK_r + K_r d = \mathrm{id}_{\mathcal{A}(M)} - \mathrm{pr}_M^* \circ \imath_r^*.$$

Thus $[\imath_r^*] = [\mathrm{pr}_M^*]^{-1}$ and we have $H^\bullet(M) \cong H^\bullet(M \times \mathbb{R})$ for every manifold M.

Theorem 3.22 (Homotopy axiom for the de Rham cohomology) *Homotopic maps induce the same map in cohomology.*

Proof: cf. [11, p. 35]. Let f_0, $f_1: M \to N$ be two homotopic maps. By Definition 2.26, we find a map $F: M \times \mathbb{R} \to N$ such that $f_j = F \circ i_j$, $j = 0, 1$ (we put $F(x,t) = f_1(x)$ for $t > 1$ and $F(x,t) = f_0(x)$ for $t < 0$). Due to Proposition 3.21, $[i_0^*] = [i_1^*]$, and thus $[f_0^*] = [i_0^*] \circ [F^*] = [i_1^*] \circ [F^*] = [f_1^*]$. □

Corollary 3.23 *Two manifolds of the same homotopy type have the same* DE RHAM *cohomology.*

Since the differential of a constant map $c: M \to N$ is zero and thus $c^*\omega = 0$ for all $\omega \in \mathcal{A}_p(N)$ with $p \geq 1$, we get:

Corollary 3.24 *For any contractible manifold M, $H^\bullet(M) = H^0(M) = \mathbb{R}$.*

Here we used the fact that for any manifold M,

$$H^0(M) = \{f \in C^\infty(M) \mid f \text{ locally constant}\} \cong \mathbb{R}^i, \tag{98}$$

where i is the number of components of M,cf. Corollary 3.28 below. This proves:

Corollary 3.25 (Poincaré lemma)

$$\textit{For all} \quad n \geq 0, p > 0: \qquad H^0(\mathbb{R}^n) \cong \mathbb{R}, \qquad H^p(\mathbb{R}^n) = \{0\}.$$

An important tool for the computation of the DE RHAM cohomology is the MAYER-VIETORIS sequence. It allows one to compute the cohomology of the union of two open sets $U, V \subseteq M$.

Definition 3.26 *For $M = U \cup V$ with open U, V, the* MAYER-VIETORIS sequence *reads*

$$0 \longrightarrow \mathcal{A}(M) \xrightarrow{(|_U, |_V)} \mathcal{A}(U) \oplus \mathcal{A}(V) \xrightarrow{\delta} \mathcal{A}(U \cap V) \longrightarrow 0, \tag{99}$$

where $|_U$ and $|_V$ are the restriction of forms and δ is the difference of the restricted forms, i. e., $\delta(\alpha, \beta) := \beta|_{U \cap V} - \alpha|_{U \cap V}$.

Proposition 3.27 *The* MAYER-VIETORIS *sequence is exact and thus induces a long exact* MAYER-VIETORIS *sequence in cohomology:*

$$\cdots \longrightarrow H^i(M) \longrightarrow H^i(U) \oplus H^i(V) \longrightarrow H^i(U \cap V) \longrightarrow H^{i+1}(M) \longrightarrow \cdots.$$

Proof: straightforward, cf. [11, p. 22] and Proposition 3.10. Proposition 3.27 also is a corollary to Theorem 3.70 below. □

Thus if $U \cap V = \emptyset$, $H^i(M) \cong H^i(U) \oplus H^i(V)$ for all $i \in \mathbb{N}_0$. This proves

Corollary 3.28 *If $M = \dot{\bigcup}_{i \in I} M_i$ with M_i open in M, then $H^\bullet(M) \cong \prod_{i \in I} H^\bullet(M_i)$.*

As a second example, we compute $H^\bullet(\mathbb{S}^n)$ for $n \geq 1$ ($H^\bullet(\mathbb{S}^0) = H^0(\mathbb{S}^0) \cong \mathbb{R}^2$ due to (98)). Let $U_1, U_2 \cong \mathbb{R}^n$ cover the northern, resp., southern hemisphere such that $U_1 \cap U_2 \cong \mathbb{S}^{n-1} \times \mathbb{R}$, where $\mathbb{S}^{n-1}$ is the equator. Thus $H^\bullet(U_i) = H^0(U_i) \cong \mathbb{R}$ by the POINCARÉ lemma, while $H^\bullet(U_1 \cap U_2) \cong H^\bullet(\mathbb{S}^{n-1})$ by Proposition 3.21. For $\mathbb{S}^1$ the induced exact sequence reads

$$0 \longrightarrow H^0(\mathbb{S}^1) \longrightarrow \mathbb{R} \oplus \mathbb{R} \xrightarrow{[\delta]} \mathbb{R} \oplus \mathbb{R} \longrightarrow H^1(\mathbb{S}^1) \longrightarrow 0 \longrightarrow \cdots.$$

Now $[\delta]((r,s)) = (s-r, s-r)$, thus $\mathrm{im}[\delta] \cong \mathbb{R}$. This proves $H^0(\mathbb{S}^1) = \ker[\delta] \cong \mathbb{R}$ (we already knew that from (98)) and $H^1(\mathbb{S}^1) = \mathrm{coker}[\delta] \cong \mathbb{R}$. All higher cohomology groups vanish. For $\mathbb{S}^n$, $n > 1$, the sequence reads

$$\begin{aligned} 0 \longrightarrow H^0(\mathbb{S}^n) \longrightarrow \mathbb{R} \oplus \mathbb{R} &\xrightarrow{[\delta]} H^0(\mathbb{S}^{n-1}) \longrightarrow H^1(\mathbb{S}^n) \longrightarrow 0 \longrightarrow \cdots \\ \cdots \longrightarrow 0 &\xrightarrow{[\delta]} H^{p-1}(\mathbb{S}^{n-1}) \longrightarrow H^p(\mathbb{S}^n) \longrightarrow 0 \longrightarrow \cdots \end{aligned}$$

for $p \geq 2$. $H^0(\mathbb{S}^n) \cong H^0(\mathbb{S}^{n-1}) \cong \mathbb{R}$ yields $H^1(\mathbb{S}^n) = 0$, and from $H^p(\mathbb{S}^n) \cong H^{p-1}(\mathbb{S}^{n-1})$ we obtain by induction:

Lemma 3.29 (De Rham cohomology of the spheres)

$$\begin{array}{lll} \textit{For all} \quad p \neq 0: & H^0(\mathbb{S}^0) \cong H^n(\mathbb{S}^0) \cong \mathbb{R}^2, & H^p(\mathbb{S}^0) = \{0\}; \\ \textit{for all} \quad n > 0, 0 \neq p \neq n: & H^0(\mathbb{S}^n) \cong H^n(\mathbb{S}^n) \cong \mathbb{R}, & H^p(\mathbb{S}^n) = \{0\}. \end{array}$$

Obviously, the long MAYER-VIETORIS sequence is quite efficient if the covering sets U, V and $U \cap V$ are diffeomorphic to $\mathbb{R}^n$. This leads to the following definition:

Definition 3.30 *An open cover $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$ of an n-dimensional manifold M is called a* good cover *if all finite intersections $U_{\alpha_0 \cdots \alpha_p} = U_{\alpha_0} \cap U_{\alpha_1} \cap \cdots \cap U_{\alpha_p}$, $p \in \mathbb{N}_0$ are diffeomorphic to $\mathbb{R}^n$. If the set of indices A is finite, $\mathfrak{U}$ is called a* finite good cover.

The following two propositions on good covers hold, cf. [11, pp. 42 – 44]:

Proposition 3.31 *Every paracompact manifold M has a good cover, if M is compact it has a finite good cover.*

Proposition 3.32 *If a manifold M has a finite good cover then its* DE RHAM *cohomology is finite dimensional.*

Proof. One proceeds by induction on the cardinality p of the finite good cover. The case $p = 1$ follows from the POINCARÉ lemma. If M is covered by $p+1$ open sets $U_0, \ldots, U_p$, then $M = U_0 \cup V$ where $V := \bigcup_{k=1}^p U_k$. Obviously U_0, V and $U_0 \cap V = \bigcup_{k=1}^p U_{0k}$ have finite good covers of cardinality $\leq p$. By induction $H^\bullet(U_0)$, $H^\bullet(V)$ and $H^\bullet(U_0 \cap V)$ are finite dimensional. But now the MAYER-VIETORIS sequence yields that $H^\bullet(U \cap V)$ is finite dimensional, too. □

We state some more general results on the DE RHAM cohomology from [11], [5] and SPIVAK, [13, p. 8-48].

Theorem 3.33 (Poincaré duality) *If M is an n-dimensional, compact and orientable manifold, then*

$$H^p(M) \cong H^{n-p}(M).$$

Theorem 3.34 *Let M be a n-dimensional paracompact connected manifold. If*

$$\begin{aligned} M \text{ compact, orientable} &\implies H^n(M) \cong \mathbb{R}, \\ M \text{ compact, non-orientable} &\implies H^n(M) = \{0\}, \\ M \text{ non-compact} &\implies H^n(M) = \{0\}. \end{aligned}$$

Theorem 3.35 (Künneth formula for the de Rham cohomology) *If the manifolds M, N are paracompact and $H^\bullet(M)$ or $H^\bullet(N)$ is finite dimensional, then*

$$H^\bullet(M \times N) \cong H^\bullet(M) \otimes H^\bullet(N) \quad i.\ e. \quad H^p(M \times N) \cong \bigoplus_{q+r=p} [H^q(M) \otimes H^r(N)].$$

The KÜNNETH formula is a consequence of the fact that we can extend all forms on M, resp., N to $M \times N$ by using pr_M^* and pr_N^*. Since d commutes with pullbacks, $\mathrm{pr}_M^* \omega \wedge \mathrm{pr}_N^* \alpha$ is closed iff $\omega \in \mathcal{A}_p(M)$ and $\alpha \in \mathcal{A}_q(N)$ are closed, and it is exact if in addition ω or α is exact. Since for bundles we only have one projection, it is no wonder that the KÜNNETH formula does not hold in general. The computation of the cohomology of a bundle is much more complicated and involves spectral sequences. We will postpone this to Section 3.5. Nevertheless the product relation for the EULER characteristics that can be deduced from the KÜNNETH formula, also holds for fiber bundles $B(M, F, G)$, cf. [11, p. 182]:

$$\chi(M \times N) = \chi(M)\chi(N) \qquad \text{and} \qquad \chi(B) = \chi(M)\chi(F). \tag{100}$$

Let $C_c^\infty(M)$ denote the algebra of all C^∞-maps on M with *compact* support. $C_c^\infty(M)$ is a $C^\infty(M)$-module. Then we may define $\mathcal{A}(M)_c$ as exterior algebra of all forms with compact support: $\mathcal{A}(M)_c = C_c^\infty(M) \otimes_{\mathbb{R}} \mathcal{A}(M)$. Like $\mathcal{A}(M)$, also $\mathcal{A}(M)_c$ is a complex and defines the so-called *compactly supported cohomology*, resp., *compact cohomology* $H_c^\bullet(M)$ analogously to the DE RHAM cohomology. For compact manifolds M, both $H_c^\bullet(M)$ and $H^\bullet(M)$ obviously coincide.

Although $H_c^\bullet(M)$ and $H^\bullet(M)$ are defined similary, they differ significantly on non-compact manifolds. In general, pullbacks $f^*: \mathcal{A}(N) \to \mathcal{A}(M)$ do not map $\mathcal{A}(N)_c$ onto $\mathcal{A}(M)_c$. On the other hand, every inclusion $\jmath: U \to M$ defines a push-out $\jmath_*: \mathcal{A}(U)_c \to \mathcal{A}(M)_c$ by extending compactly supported forms on U by zero to compactly supported forms on M. As a consequence, we get a short exact MAYER-VIETORIS sequence in the opposite direction (cf. [11, p. 26]),

$$0 \longleftarrow \mathcal{A}(M)_c \xleftarrow{\ s\ } \mathcal{A}(U)_c \oplus \mathcal{A}(V)_c \xleftarrow{\ i\ } \mathcal{A}(U \cap V)_c \longleftarrow 0,$$

where $i(\omega) = (-\jmath_*\omega, +\jmath_*\omega)$ and $s = (\jmath_U)_* + (\jmath_V)_*$. Thus the induced long exact sequence is also reversed. One has isomorphisms $H_c^p(M \times \mathbb{R}) \cong H_c^{p-1}(M)$ and a POINCARÉ lemma $H_c^n(\mathbb{R}^n) \cong \mathbb{R}$, $H_c^p(\mathbb{R}^n) = \{0\}$, $p \neq n$, which illustrates that $H_c^\bullet(M)$ is not invariant under homotopy equivalence, cf. [11, p. 39]. Since the spheres are compact, $H_c^\bullet(\mathbb{S}^n) = H^\bullet(\mathbb{S}^n)$. Similary to $H^\bullet(M)$, if M has a finite good cover, $H_c^\bullet(M)$ is finite dimensional. We also have a KÜNNETH formula for the compact cohomology ([5, I p. 211]):

$$H_c^\bullet(M \times N) \cong H_c^\bullet(M) \otimes H_c^\bullet(N) \quad \text{i. e.} \quad H_c^p(M \times N) \cong \bigoplus_{q+r=p} [H_c^q(M) \otimes H_c^r(N)].$$

The compact cohomology is of interest, since for arbitrary orientable n-dimensional paracompact manifolds M, the POINCARÉ duality reads

$$H^p(M) \cong [H_c^{n-p}(M)]^*,$$

(here * denotes the dual) even if $H^\bullet(M)$ is not finite dimensional (cf. [5, I pp. 14,198]). Note that $H_c^p(M) \cong [H^{n-p}(M)]^*$ does not hold in general: If M consists of countably many components, $M = \dot{\bigcup}_{i=1}^\infty M_i$, then $H^p(M)$ is the direct product $H^p(M) = \prod_{i=1}^\infty H^p(M_i)$, but $H_c^p(M)$ is the direct sum $H_c^p(M) = \bigoplus_{i=1}^\infty H_c^p(M_i)$

To conclude this section, suppose a LIE group action S is given on a manifold P. Since d and S_g^* commute on $\mathcal{A}(P) \otimes V$ for all $g \in G$, all S_g^* are chain maps on $\mathcal{A}(P) \otimes V$ and we have:

Definition 3.36 *$\mathcal{A}(P)_{\mathrm{inv}} \otimes V$ and $\mathcal{A}(P)_{\mathfrak{g}-\mathrm{inv}} \otimes V = \mathcal{A}(P)_{G_1-\mathrm{inv}} \otimes V$ are differential complexes and define the* $(G$-$)$invariant cohomology $H_{\mathrm{inv}}^\bullet(P) \otimes V$, *resp.*, $\mathfrak{g}$-invariant cohomology $H_{\mathfrak{g}-\mathrm{inv}}^\bullet(P) \otimes V$.

Analogously, for any representation S' of G on V, the (G-)equivariant, resp., $\mathfrak{g}$-equivariant forms with regard to S and S' constitute a differential complex and define the equivariant cohomology $H_{\mathrm{equiv}}^\bullet(P) \otimes V$, *resp.*, $\mathfrak{g}$-equivariant cohomology $H_{\mathfrak{g}-\mathrm{equiv}}^\bullet(P) \otimes V$. *Examples are $H_{\mathrm{equiv}}^\bullet(P, \mathfrak{g})$ and $H_{\mathrm{equiv}}^\bullet(P) \otimes \mathrm{Hom}(\mathcal{T}(\mathfrak{g}), V)$.*

For connected G, obviously $H_{\mathrm{inv}}^\bullet(P) \otimes V = H_{\mathfrak{g}-\mathrm{inv}}^\bullet(P) \otimes V$ and $H_{\mathrm{equiv}}^\bullet(P) \otimes V = H_{\mathfrak{g}-\mathrm{equiv}}^\bullet(P) \otimes V$. Since the inclusions $\imath: \mathcal{A}(P)_{\mathfrak{g}-\mathrm{inv}} \to \mathcal{A}(P)$, etc., are chain maps, we have natural homomorphisms

$$[\imath]_{\mathfrak{g}-\mathrm{inv}}: H_{\mathfrak{g}-\mathrm{inv}}^k(P) \otimes V \to H^k(P) \otimes V, \quad \text{etc.,} \qquad \text{for all} \quad k \in \mathbb{N}_0,$$

but in contrast to $\imath$, these homomorphisms need not be injective, as the example $G = \mathbb{R}$ acting on itself by translations $L_t(x) = x + t$ shows: the 1-form dx is invariant and generates $H_{\mathrm{inv}}^1(\mathbb{R}) \cong \mathbb{R}$, but $H^1(\mathbb{R}) = \{0\}$, since $dx = d\,\mathrm{id}_{\mathbb{R}}$ with $\mathrm{id}_{\mathbb{R}} \notin C^\infty(\mathbb{R})_{\mathrm{inv}}$.

If G is compact with HAAR measure μ, then the projections $p: \mathcal{A}(P) \to \mathcal{A}(P)_{\mathfrak{g}-\mathrm{inv}}$ onto ($\mathfrak{g}$-)invariant forms, resp., onto ($\mathfrak{g}$-)equivariant forms analogous to (29), (40) and (41) defined by integration over G_1, resp., G, are chain maps and thus define surjective homomorphisms

$$[p]_{\mathfrak{g}-\mathrm{inv}}: H^k(P) \otimes V \to H_{\mathfrak{g}-\mathrm{inv}}^k(P) \otimes V, \quad \text{etc.,} \qquad \text{for all} \quad k \in \mathbb{N}_0.$$

Also from $p \circ \imath = \mathrm{id}$ on $\mathcal{A}(P)_{\mathrm{inv}}$, resp., $\mathcal{A}(P)_{\mathfrak{g}-\mathrm{inv}}$, etc., and thus $[p] \circ [\imath] = \mathrm{id}$, we conclude that the induced homomorphisms $[\imath]$ are all injective if G is compact.

For every $g \in G_1$, S_g is homotopic to $S_e = \mathrm{id}_P$: if $\tau: [0,1] \to G_1$ is an arc connecting $\tau(0) = e$ and $\tau(1) = g$, then $F := S \circ (\tau \times \mathrm{id}_P): [0,1] \times P \to P$ is a homotopy connecting id_P and S_g. By Theorem 3.22, $[S_g^*] = \mathrm{id}_{H^\bullet(P)}$. So $[\imath]_{\mathfrak{g}-\mathrm{inv}}[p]_{\mathfrak{g}-\mathrm{inv}}[\omega] = [\omega]$ for all $\omega \in \mathcal{A}(P)$. We have proved:

Proposition 3.37 *For any* LIE *group action $S: G \times P \to P$, where G is compact, $[\imath]_{\mathfrak{g}-\mathrm{inv}}$ and $[p]_{\mathfrak{g}-\mathrm{inv}}$ are inverse isomorphisms and thus for all vector spaces V*

$$H^k(P) \otimes V \cong H^k_{\mathfrak{g}-\mathrm{inv}}(P) \otimes V \qquad \text{for all} \quad k \in \mathbb{N}_0.$$

The morphisms $[\imath]_{\mathrm{inv}}: H^\bullet_{\mathrm{inv}}(P)\otimes V \to H^\bullet(P)\otimes V$, $[\imath]_{\mathrm{equiv}}: H^\bullet_{\mathrm{equiv}}(P)\otimes V \to H^\bullet(P)\otimes V$ and $[\imath]_{\mathfrak{g}-\mathrm{equiv}}: H^\bullet_{\mathfrak{g}-\mathrm{equiv}}(P) \otimes V \to H^\bullet(P) \otimes V$ are injective.

If in addition G is connected, this yields $H^\bullet(P) \otimes V \cong H^\bullet_{\mathrm{inv}}(P) \otimes V$.

For $P = G$ we will use $H^\bullet_L(G)$ for the invariant cohomology with respect to the left multiplication, i. e., the cohomology of the differential complex $\mathcal{A}^L(G)$. Analogously, $H^\bullet_R(G)$ and $H^\bullet_I(G)$ will denote the cohomologies of $\mathcal{A}^R(G)$, resp., $\mathcal{A}^I(G)$. Proposition 3.37 yields (cf. [5, II p. 163]):

Theorem 3.38 *If G is a compact connected* LIE *group then*

$$\mathcal{A}^I(G) = H^\bullet_I(G) \cong H^\bullet_L(G) \cong H^\bullet_R(G) \cong H^\bullet(G).$$

Proof. Corollary 1.64 yields that every bi-invariant form is closed. Thus $\mathcal{A}^I(G) = H^\bullet_I(G)$. All other isomorphisms are immediate consequences of Proposition 3.37 with regard to the various actions: For the bi-invariant forms, note that these are exactly the forms that are invariant under $L: G \times G \to \mathrm{Gl}(G)$, where $L_{(a,b)}(g) = agb^{-1}$ (and if G is compact and connected, then $G \times G$ is so, too). □

3.3 LIE Algebra Cohomology

As another example for a cohomology of a differential complex, we will treat LIE algebra cohomology, as in [5] and [7]. Suppose $\mathfrak{g}$ is a $\mathbb{K}$-LIE algebra (for $\mathbb{K} = \mathbb{R}, \mathbb{C}$) and $l: \mathfrak{g} \to \mathrm{gl}(V)$ is a (left) representation of $\mathfrak{g}$ on a $\mathbb{K}$-vector space V. Recall $\mathrm{Alt}(\mathfrak{g}, V) = \bigoplus_{p=0}^{\infty} \mathrm{Alt}_p(\mathfrak{g}, V)$ from Definition 1.5: $\mathrm{Alt}_p(\mathfrak{g}, V)$ is the vector space of alternating p-linear maps from $\mathfrak{g}^p$ to V. $\mathrm{Alt}(\mathfrak{g}, V)$ becomes a complex C_l with the following differential operator $\mathbf{d}^l = (\mathbf{d}^l_p: C^p_l \to C^{p+1}_l)_{p\in\mathbb{N}_0}$: for $c \in C^p_l$ and $X_i \in \mathfrak{g}$,

$$\begin{aligned} \mathbf{d}^l_p c(X_1, \ldots, X_{p+1}) \;:=\; & \sum_{i=1}^{p+1} (-1)^{i+1} l(X_i)(c(X_1, \ldots, \widehat{X_i}, \ldots, X_{p+1})) \\ & + \sum_{i=1}^{p} \sum_{j=i+1}^{p+1} (-1)^i c(X_1, \ldots, \widehat{X_i}, \ldots, X_{j-1}, [X_i, X_j], X_{j+1}, \ldots, X_{p+1}). \end{aligned}$$

Our definition of $\mathbf{d}^l$ differs slightly from the definitions in [5] and [7], where analogously to (9) the second term reads

$$+\sum_{i=1}^{p} \sum_{j=i+1}^{p+1} (-1)^{i+j} c([X_i, X_j], X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_{p+1}).$$

Obviously both definitions coincide on C_l. Nevertheless with our definition not only $\mathrm{Alt}(\mathfrak{g}, V)$ becomes a differential complex, but also $\mathrm{Hom}(\mathcal{T}(\mathfrak{g}), V)$ becomes a complex $\overline{C_l}$ with subcomplex C_l. Indeed we can prove:

Proposition 3.39 *For any representation* $l: \mathfrak{g} \to \mathrm{gl}(V)$ *of* $\mathfrak{g}$, $\mathbf{d}^l_{p+1} \circ \mathbf{d}^l_p = 0$ *on* $\overline{C}_l$.

Definition 3.40 $H^p_l(\mathfrak{g}, V) := H^p_{\mathbf{d}}(C_l)$ *is called the* p-th (CHEVALLEY) cohomology space of $\mathfrak{g}$ *with values in* V *with regard to* l. *We put* $H^p_l(\mathfrak{g}) := H^p_l(\mathfrak{g}, \mathbb{K})$.

Denote the trivial representation by $o: \mathfrak{g} \to \mathrm{gl}(V)$. *Then* $b_p(\mathfrak{g}) := \dim_{\mathbb{K}} H^p_o(\mathfrak{g})$ *is called p-th* BETTI number of $\mathfrak{g}$.

Analogously, $\overline{H}^p_l(\mathfrak{g}, V) := H^p_{\mathbf{d}}(\overline{C}_l)$, $\overline{H}^p_l(\mathfrak{g}) := \overline{H}^p_l(\mathfrak{g}, \mathbb{K})$ *and* $\overline{b}_p(\mathfrak{g}) := \dim_{\mathbb{K}} \overline{H}^p_o(\mathfrak{g})$.

We will mainly be concerned with $H^p_l(\mathfrak{g}, V)$ for $p \leq 2$. Evaluation of $\mathbf{d}_p$ for these cases yields for $X, Y, Z \in \mathfrak{g}$:

$$\begin{aligned}
(\mathbf{d}^l_0 c)(X) &= l(X)c \qquad \text{for all} \quad c \in \overline{C}^0_l = V,\\
(\mathbf{d}^l_1 c)(X,Y) &= l(X)c(Y) - l(Y)c(X) - c([X,Y]) \quad \text{for all } c \in \overline{C}^1_l = \mathrm{Hom}(\mathfrak{g}, V),\\
(\mathbf{d}^l_2 c)(X,Y,Z) &= l(X)c(Y,Z) - l(Y)c(X,Z) + l(Z)c(X,Y)\\
&\quad -c([X,Y],Z) + c(X,[Y,Z]) - c(Y,[X,Z]) \quad \text{for all } c \in \overline{C}^2_l.
\end{aligned}$$

Definition 3.41 *We define* $\mathrm{Sym}(\mathfrak{g}, V)_{\mathfrak{g}\text{-inv}}$ *analogously to* $\mathrm{Alt}(\mathfrak{g}, V)_{\mathfrak{g}\text{-inv}}$. *Then* $\kappa_{\mathfrak{g}} \in \mathrm{Sym}_2(\mathfrak{g}, \mathbb{K})_{\mathfrak{g}\text{-inv}}$, *where* $\kappa_{\mathfrak{g}}$ *denotes the* KILLING form *of* $\mathfrak{g}$:

$$\kappa_{\mathfrak{g}}(X,Y) = \mathrm{Tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)).$$

Recall that a LIE algebra $\mathfrak{s}$ is called *simple*, if it is not abelian and its only ideals are $\{0\}$ and $\mathfrak{s}$. A LIE algebra is *semisimple*, if it is the direct sum of simple LIE algebras. Thus if $[\mathfrak{g}, \mathfrak{g}]$ denotes the commutator ideal in $\mathfrak{g}$, we have $[\mathfrak{s}, \mathfrak{s}] = \mathfrak{s}$ for semisimple LIE algebras. Semisimple LIE algebras have non-degenerate KILLING forms.

$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{s}$ is called *reductive*, if $\mathfrak{a}$ is abelian and $\mathfrak{s}$ is semisimple. As an important example, $\mathrm{gl}(\mathbb{C}^n) = Z(\mathrm{gl}(\mathbb{C}^n)) \oplus \mathrm{sl}(\mathbb{C}^n)$ is reductive. A real LIE algebra is called *compact*, if a (negative or positive) definite $\mathfrak{g}$-invariant scalar product s exists on $\mathfrak{g}$, i. e., a definite $s \in \mathrm{Sym}_2(\mathfrak{g}, \mathbb{R})_{\mathfrak{g}\text{-inv}}$. Compact LIE algebras are reductive, compact LIE groups have compact LIE algebras, cf. [7].

Lemma 3.42 *1.* $H^0_o(\mathfrak{g}, V) = \overline{H}^0_o(\mathfrak{g}, V) = V$.

2. $H^1_o(\mathfrak{g}, V) = \overline{H}^1_o(\mathfrak{g}, V) = [\mathfrak{g}, \mathfrak{g}]^\perp = \{c \in \mathrm{Hom}(\mathfrak{g}, V) | c([\mathfrak{g}, \mathfrak{g}]) = \{0\} \leq V\}$, *thus* $\mathbf{d}^o_1$ *is injective and* $H^1_o(\mathfrak{g}, V) = \{0\}$ *for all* LIE *algebras* $\mathfrak{g}$ *with* $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, *e. g. semisimple* LIE *algebras.*

3. If $\mathfrak{a}$ *is abelian, then* $H^p_o(\mathfrak{a}, V) = \mathrm{Alt}_p(\mathfrak{a}, V)$ *and* $\overline{H}^p_o(\mathfrak{a}, V) = \mathrm{Hom}(\otimes^p \mathfrak{a}, V)$ *and thus* $b_p(\mathfrak{a}) = \binom{\dim \mathfrak{a}}{p}$ *and* $\overline{b}_p(\mathfrak{a}) = (\dim \mathfrak{a})^p$ *for all* $p \in \mathbb{N}_0$.

Proof. $\mathbf{d}^o_0 = 0$ yields *1.* and proves that $H^1_o(\mathfrak{g}, V) = \ker \mathbf{d}^o_1 / \mathrm{im}\, \mathbf{d}^o_0 = \ker \mathbf{d}^o_1$. $(\mathbf{d}^o_1 c)(X,Y) = c([X,Y])$ yields *2.*, and *3.* follows from $\mathbf{d}^o = 0$ for abelian $\mathfrak{g}$. □

Definition 3.43 *Let* $l: \mathfrak{g} \to \mathrm{gl}(V)$ *be a representation of a* $\mathbb{K}$-LIE *algebra* $\mathfrak{g}$ *on a vector space* V. *Then* V *is a* $\mathfrak{g}$-module *with* $X \cdot v := l(X)v$ *for all* $X \in \mathfrak{g}$, $v \in V$.

1. *A subspace $V' \leq V$ is called a* $\mathfrak{g}$-submodule *if $X \cdot v$ for all $X \in \mathfrak{g}$, $v \in V$.*
2. *V is called* simple *if $\{0\}$ and V are its only submodules.*
3. *V is called* semisimple *if it is the direct sum of simple submodules.*

Thus a LIE algebra $\mathfrak{s}$ is simple iff it is not abelian and it is a simple $\mathfrak{s}$-module with respect to the adjoint representation; $\mathfrak{s}$ is semisimple iff it is semisimple as a $\mathfrak{s}$-module and $\mathfrak{s}$ does not act trivially on any submodule.

For semisimple LIE algebras, we state the following results:

Theorem 3.44 (Weyl's theorem) *If V is finite dimensional and $l: \mathfrak{s} \to \mathrm{gl}(V)$ is a representation of a semisimple $\mathbb{K}$-LIE algebra $\mathfrak{s}$, then the $\mathfrak{s}$-module V is semisimple.*

Proof: cf. [7, p. 149]; in fact, it involves WHITEHEAD'S first lemma below. □

Theorem 3.45 *Let $\mathfrak{s}$ be a semisimple $\mathbb{K}$-LIE algebra and $l: \mathfrak{s} \to \mathrm{gl}(V)$ a representation of $\mathfrak{s}$ on a finite dimensional vector space $V = V_1 \oplus \cdots \oplus V_n$ with representations $l_i: \mathfrak{s} \to \mathrm{gl}(V_i)$ on the simple $\mathfrak{s}$-modules V_i. Then the following results hold:*

1. *$H^0_l(\mathfrak{s}, V) = V_{i_1} \oplus \cdots \oplus V_{i_m}$, where V_{i_j} are those (one-dimensional) submodules with $l_{i_j} = 0$ for $j = 1, \ldots, m$, thus $b_0(\mathfrak{s}) = 1$;*
2. WHITEHEAD'S first lemma*: $H^1_l(\mathfrak{s}, V) = \{0\}$, thus $b_0(\mathfrak{s}) = 0$;*
3. WHITEHEAD'S second lemma*: $H^2_l(\mathfrak{s}, V) = \{0\}$, thus $b_0(\mathfrak{s}) = 0$.*

Proof: $\ker l_i \triangleleft V_i$ yields *1.*; for WHITEHEAD'S lemmas see [7, pp. 160 – 161]. □

Let us determine how the LIE algebra cohomlogy is related to the invariant cohomology of the corresponding LIE group. We know from Lemma 1.62 and Proposition 1.79 that the differential complexes $\mathcal{A}^S(G, V)$ are isomorphic to $\mathrm{Alt}(\mathfrak{g}, V)$ with induced differential operator d^S. Observe that d^S and $\mathbf{d}^\circ$ differ only by constants, thus they induce the same cohomology and we obtain that both

$$[\psi^S]: H^\bullet_\circ(\mathfrak{g}, V) \to H^\bullet_S(G, V), \quad S = L, R,$$

are isomorphisms. Recall Proposition 1.81: $\omega \in \mathcal{A}^S(G,V)_{\mathfrak{g}-\mathrm{inv}}$ yields $d\omega = 0$ since by (36), d^S is zero on $\mathrm{Alt}(\mathfrak{g}, V)_{\mathfrak{g}-\mathrm{inv}} = \mathrm{Alt}(\mathfrak{g}, V)_{G_1-\mathrm{inv}}$. Thus we may write $H^\bullet_\circ(\mathfrak{g}, V)_{\mathrm{inv}} = \mathrm{Alt}(\mathfrak{g}, V)_{\mathrm{inv}}$ and $H^\bullet_\circ(\mathfrak{g}, V)_{\mathfrak{g}-\mathrm{inv}} = \mathrm{Alt}(\mathfrak{g}, V)_{\mathfrak{g}-\mathrm{inv}}$. Now Theorem 3.38 proves:

Theorem 3.46 1. *For any (real)* LIE *group G, $H^\bullet_\circ(\mathfrak{g}) \cong H^\bullet_L(G) \cong H^\bullet_R(G)$.*

2. *If G is connected, then $\mathrm{Alt}(\mathfrak{g}, \mathbb{R})_{\mathfrak{g}-\mathrm{inv}} = \mathrm{Alt}(\mathfrak{g}, \mathbb{R})_{\mathrm{inv}} = H^\bullet_\circ(\mathfrak{g})_{\mathrm{inv}}$.*
3. *For compact connected G, $\mathrm{Alt}(\mathfrak{g}, \mathbb{R})_{\mathrm{inv}} = H^\bullet_\circ(\mathfrak{g}) \cong \mathcal{A}^I(G) \cong H^\bullet_S(G) \cong H^\bullet(G)$.*

We want to generalize Theorem 3.45 (for trivial representations) to reductive LIE algebras, according to [5, III].

For the direct sum $\mathfrak{g} \oplus \mathfrak{h}$ of two LIE algebras we have a natural isomorphism $\mathrm{Alt}(\mathfrak{g}, \mathbb{K}) \otimes \mathrm{Alt}(\mathfrak{h}, \mathbb{K}) \to \mathrm{Alt}(\mathfrak{g} \oplus \mathfrak{h}, \mathbb{K})$, $c \otimes d \mapsto c \wedge d$. Identifying these algebras we get for the operators of Proposition 1.79 for $X \in \mathfrak{g}, Y \in \mathfrak{h}$:

$$\begin{aligned} L^S_{X \oplus Y}(b \otimes c) &= L^S_X b \otimes c + b \otimes L^S_Y c, && \text{for all } b \in \mathrm{Alt}(\mathfrak{g}, \mathbb{K}), c \in \mathrm{Alt}(\mathfrak{h}, \mathbb{K}), \\ d^S(b \otimes c) &= d^S b \otimes c + (-1)^p b \otimes d^S c, && \text{for all } b \in \mathrm{Alt}_p(\mathfrak{g}, \mathbb{K}), c \in \mathrm{Alt}(\mathfrak{h}, \mathbb{K}). \end{aligned}$$

These relations are the main ingredients in the proof of (cf. [5, III p. 183]):

Proposition 3.47 (Künneth formulae for the Lie algebra cohomology)

$$H^*_o(\mathfrak{g} \oplus \mathfrak{h}) \cong H^*_o(\mathfrak{g}) \otimes H^*_o(\mathfrak{h}), \tag{101}$$

$$\mathrm{Alt}(\mathfrak{g} \oplus \mathfrak{h}, \mathbb{K})_{\mathfrak{g}\oplus\mathfrak{h}-\mathrm{inv}} = \mathrm{Alt}(\mathfrak{g}, \mathbb{K})_{\mathfrak{g}-\mathrm{inv}} \otimes \mathrm{Alt}(\mathfrak{h}, \mathbb{K})_{\mathfrak{h}-\mathrm{inv}}, \quad i.\ e. \tag{102}$$

$$H^*_o(\mathfrak{g} \oplus \mathfrak{h})_{\mathfrak{g}\oplus\mathfrak{h}-\mathrm{inv}} = H^*_o(\mathfrak{g})_{\mathfrak{g}-\mathrm{inv}} \otimes H^*_o(\mathfrak{h})_{\mathfrak{h}-\mathrm{inv}}. \tag{103}$$

For any reductive LIE algebra $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{s}$ and any finite dimensional vector space V, $\mathrm{Alt}(\mathfrak{g}, V)$ and $\mathrm{Hom}(\mathfrak{g}, V)$ are semisimple $\mathfrak{g}$-modules with respect to the representations L^L ([5, III, p. 188]). In particular, $\mathrm{Alt}(\mathfrak{g}, V)$ decomposes into

$$\mathrm{Alt}(\mathfrak{g}, V) = \mathrm{Alt}(\mathfrak{g}, V)_{\mathfrak{g}-\mathrm{inv}} \oplus L^S_{\mathfrak{g}}(\mathrm{Alt}(\mathfrak{g}, V)) \tag{104}$$

and we obtain projections $q\colon \mathrm{Alt}(\mathfrak{g}, V) \to \mathrm{Alt}(\mathfrak{g}, V)_{\mathfrak{g}-\mathrm{inv}}$. Moreover, we get:

Theorem 3.48 *If $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{s}$ is a reductive* LIE *algebra and V is finite dimensional, then the natural injection $i\colon \mathrm{Alt}(\mathfrak{g}, V)_{\mathfrak{g}-\mathrm{inv}} \to \mathrm{Alt}(\mathfrak{g}, V)$ induces an isomorphism*

$$[i]\colon \mathrm{Alt}(\mathfrak{g}, V)_{\mathfrak{g}-\mathrm{inv}} \to H^*_o(\mathfrak{g}, V), c \mapsto c + \mathrm{im}\, \mathbf{d}^o.$$

Proof: cf. [5, III p. 189]: $\ker \mathbf{d}^o = \mathrm{Alt}(\mathfrak{g}, V)_{\mathfrak{g}-\mathrm{inv}} \oplus \mathrm{im}\, \mathbf{d}^o$ and $\mathrm{im}\, \mathbf{d}^o = L^S_{\mathfrak{g}}(\ker \mathbf{d}^o)$. □

Proposition 3.49 *We have a linear map $\rho\colon \mathrm{Sym}_2(\mathfrak{g}, V)_{\mathfrak{g}-\mathrm{inv}} \to \mathrm{Alt}_3(\mathfrak{g}, V)_{\mathfrak{g}-\mathrm{inv}}$ defined by*

$$\rho(s)(X, Y, Z) := -\mathbf{d}^o_2 s(X, Y, Z) = s([X, Y], Z)$$

If $H^1_o(\mathfrak{g}) = H^2_o(\mathfrak{g}) = \{0\}$ then ρ is a linear isomorphism.

Proof: cf. [5, III p. 181]. □

Thus if $\mathfrak{s}$ is simple, the KILLING form $\kappa_{\mathfrak{s}}$ defines a non-zero element $\rho(\kappa_{\mathfrak{s}}) \in \mathrm{Alt}_3(\mathfrak{s}, \mathbb{K})_{\mathfrak{s}-\mathrm{inv}}$, namely

$$\rho(\kappa_{\mathfrak{s}})(X, Y, Z) = \mathrm{Tr}(\mathrm{ad}([X, Y]) \circ \mathrm{ad}(Z)).$$

So $b_3(\mathfrak{s}) \geq 1$ and the KUNNETH formula implies $b_3(\mathfrak{s}) \geq m$ for a semisimple LIE algebra $\mathfrak{s} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_m$. On the other hand for simple $\mathfrak{s}$ suppose $s \in \mathrm{Sym}_2(\mathfrak{s}, \mathbb{K})_{\mathfrak{s}-\mathrm{inv}}$. Since $\kappa_{\mathfrak{s}}$ is non-degenerate, we may define $\psi \in \mathrm{End}_{\mathfrak{s}-\mathrm{mod}}(\mathfrak{s})$ by

$$s(X, Y) = \kappa_{\mathfrak{s}}(\psi X, Y) \qquad \text{for all} \quad X, Y \in \mathfrak{s}.$$

In fact, $\mathfrak{s}$-invariance of s and $\kappa_\mathfrak{s}$ yields $\psi \circ \operatorname{ad} X = \operatorname{ad} X \circ \psi$ for all $X \in \mathfrak{s}$. Thus for every eigenvalue $\lambda \in \mathbb{K}$ of ψ, $\{0\} \neq \ker(\psi - \lambda \operatorname{id}) \triangleleft \mathfrak{s}$. This implies $\ker(\psi - \lambda \operatorname{id}) = \mathfrak{s}$, so $\psi = \lambda \operatorname{id}$ and $s = \lambda\kappa_\mathfrak{s}$, which in turn yields $b_3(\mathfrak{s}) = 1$.

For $\mathbb{K} = \mathbb{C}$, the condition $\lambda \in \mathbb{K}$ is automatically fulfilled. For $\mathbb{K} = \mathbb{R}$, observe that ψ is self-adjoint to $\kappa_\mathfrak{s}$. If $\kappa_\mathfrak{s}$ is negative definite, it defines a scalar product and then ψ has only real eigenvalues. (A positive definite $\kappa_\mathfrak{s}$ would mean $\{0\} = [\mathfrak{s}, \mathfrak{s}] = \mathfrak{s}$, cf. [7, p. 256].) But $\kappa_\mathfrak{s}$ is negativ definite iff $\mathfrak{s}$ is compact. We have proved:

Proposition 3.50 *For every semisimple $\mathfrak{s} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_m$ with simple $\mathfrak{s}_i$, $b_3(\mathfrak{s}) \geq m$. If $\mathbb{K} = \mathbb{C}$ or $\mathfrak{s}$ is compact, then $b_3(\mathfrak{s}) = m$.*

Definition 3.51 *For a reductive* LIE *algebra $\mathfrak{g}$, let $(d\mu_e)^*\colon \operatorname{Alt}(\mathfrak{g}, \mathbb{K}) \to \operatorname{Alt}(\mathfrak{g}\oplus\mathfrak{g}, \mathbb{K})$, where $[(d\mu_e)^*(K)](\ldots,(X^i,Y^i),\ldots) := K(\ldots,X^i+Y^i,\ldots)$, denotes the pullback of $d\mu_e\colon \mathfrak{g}\oplus\mathfrak{g} \to \mathfrak{g}, (X,Y) \mapsto X+Y$ (cf.* Definition 1.26*). If $i\colon \operatorname{Alt}(\mathfrak{g}, V)_{\mathfrak{g}-\mathrm{inv}} \to \operatorname{Alt}(\mathfrak{g}, V)$ and $q\colon \operatorname{Alt}(\mathfrak{g}\oplus\mathfrak{g}, \mathbb{K}) \to \operatorname{Alt}(\mathfrak{g}, \mathbb{K})_{\mathfrak{g}-\mathrm{inv}} \otimes \operatorname{Alt}(\mathfrak{g}, \mathbb{K})_{\mathfrak{g}-\mathrm{inv}}$ are defined by* (102) *and* (104),

$$\gamma_\mathfrak{g} := q \circ (d\mu_e)^* \circ i\colon \operatorname{Alt}(\mathfrak{g}, V)_{\mathfrak{g}-\mathrm{inv}} \to \operatorname{Alt}(\mathfrak{g}, \mathbb{K})_{\mathfrak{g}-\mathrm{inv}} \otimes \operatorname{Alt}(\mathfrak{g}, \mathbb{K})_{\mathfrak{g}-\mathrm{inv}}$$

is called co-multiplication map for $\mathfrak{g}$. *Let* $\operatorname{Alt}^+(\mathfrak{g}, V) := \bigoplus_{p=1}^{\infty} \operatorname{Alt}_p(\mathfrak{g}, V)$.

For $k \in \mathbb{K} = \operatorname{Alt}_0(\mathfrak{g}, \mathbb{K})_{\mathfrak{g}-\mathrm{inv}}$ obviously $\gamma_\mathfrak{g}(k) = k = 1 \otimes k = k \otimes 1$. On $\operatorname{Alt}^+(\mathfrak{g}, \mathbb{K})_{\mathfrak{g}-\mathrm{inv}}$, the algebra homomorphism $\gamma_\mathfrak{g}$ takes the following form:

Lemma 3.52 *Let $\mathfrak{g}$ be reductive. For all $K \in \operatorname{Alt}^+(\mathfrak{g}, \mathbb{K})_{\mathfrak{g}-\mathrm{inv}}$,*

$$\gamma_\mathfrak{g}(K) = K \otimes 1 + 1 \otimes K + K', \qquad K' \in \operatorname{Alt}^+(\mathfrak{g}, \mathbb{K})_{\mathfrak{g}-\mathrm{inv}} \otimes \operatorname{Alt}^+(\mathfrak{g}, \mathbb{K})_{\mathfrak{g}-\mathrm{inv}}.$$

Proof: (cf. [5, III pp. 193,201].) Write $\gamma_\mathfrak{g}(K) = K_1 \otimes 1 + 1 \otimes K_2 + K'$ with $K_1, K_2 \in \operatorname{Alt}(\mathfrak{g}, \mathbb{K})_{\mathfrak{g}-\mathrm{inv}}$. Then for all $X^i \in \mathfrak{g}$, $K_1(\ldots, X^i, \ldots) = \gamma_\mathfrak{g}(K)(\ldots, (X^i, 0), \ldots) = K(\ldots, X^i, \ldots)$. Thus $K_1 = K$ and analogously $K_2 = K$. □

Definition 3.53 *Let $\mathfrak{g}$ be a reductive* LIE *algebra. $K \in \operatorname{Alt}^+(\mathfrak{g}, \mathbb{K})_{\mathfrak{g}-\mathrm{inv}}$ is called* primitive *if*

$$\gamma_\mathfrak{g}(K) = K \otimes 1 + 1 \otimes K.$$

The primitive elements form a graded subspace $P_\mathfrak{g} = \bigoplus_j P_\mathfrak{g}^j$ of $\operatorname{Alt}(\mathfrak{g}, \mathbb{K})_{\mathfrak{g}-\mathrm{inv}}$, *called the* primitive subspace, $r := \dim P_\mathfrak{g}$ *is called the* rank *of* $\mathfrak{g}$.

Lemma 3.54

1. *$K \wedge K = 0$ for all $K \in P_\mathfrak{g}$.*
2. *The homogeneous primitive elements of* $\operatorname{Alt}(\mathfrak{g}, \mathbb{K})_{\mathfrak{g}-\mathrm{inv}}$ *have odd degree.*
3. *If $K_1, \ldots, K_p$ are linearly independent homogeneous primitive elements, then $K_1 \wedge \cdots \wedge K_p \neq 0$.*

Proof: cf. [5, III pp. 201,202]. *1.* is a consequence of *2.* □

Note that the exterior product $K_1 \wedge K_2$ of two primitive elements is not primitive since $\gamma_\mathfrak{g}(K_1 \wedge K_2) = (K_1 \wedge K_2) \otimes 1 + 1 \otimes (K_1 \wedge K_2) + K_1 \otimes K_2 - K_2 \otimes K_1$. Nevertheless Lemmas 1.9 and 3.54.1 yield that the inclusion map $h\colon P_\mathfrak{g} \to \operatorname{Alt}(\mathfrak{g}, \mathbb{K})_{\mathfrak{g}-\mathrm{inv}}$ extends to a unique algebra homomorphism $h_\wedge\colon \bigwedge P_\mathfrak{g} \to \operatorname{Alt}(\mathfrak{g}, \mathbb{K})_{\mathfrak{g}-\mathrm{inv}}$ of degree 0, if $\bigwedge P_\mathfrak{g}$ is given the gradation induced from that of $P_\mathfrak{g}$.

Theorem 3.55 *For a reductive n-dimensional* LIE *algebra* $\mathfrak{g}$,

$$h_\wedge : \bigwedge P_\mathfrak{g} \to \mathrm{Alt}(\mathfrak{g}, \mathbb{K})_{\mathfrak{g}-\mathrm{inv}}$$

is an isomorphism of graded algebras. Thus $\mathrm{Alt}(\mathfrak{g}, \mathbb{K})_{\mathfrak{g}-\mathrm{inv}}$ *and* $H_o^\bullet(\mathfrak{g})$ *are exterior algebras over graded subspaces with odd gradation. If* $r = \dim P_\mathfrak{g}$ *is the rank of* $\mathfrak{g}$ *and* g_j, $j = 1, \ldots, r$ *are the odd degrees of the homogeneous elements in* $P_\mathfrak{g}$, *then*

$$\sum_{j=1}^{r} g_j = n, \qquad \chi(\mathfrak{g}) = \sum_{p=0}^{n} (-1)^p b_p(\mathfrak{g}) = 0 \qquad \text{and} \qquad \sum_{p=0}^{n} b_p(\mathfrak{g}) = 2^r \qquad \text{if } n > 0.$$

For $n = 0$, *obviously* $\chi(\mathfrak{g}) = b_0(\mathfrak{g}) = 1$, *since* $b_p(\mathfrak{g}) = 0$ *for all* $p > 0$.

Proof: cf. [5, III pp. 202,203]. □

Theorem 3.56 *Let* $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{s}$ *(where* $\mathfrak{a}$ *is abelian and* $\mathfrak{s} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_m$ *with simple* $\mathfrak{s}_i$*) be a reductive* LIE *algebra over* $\mathbb{C}$ *or let* $\mathfrak{g}$ *be compact. If* $a = \dim \mathfrak{a}$, *then*

$$b_0(\mathfrak{g}) = 1, \quad b_1(\mathfrak{g}) = a, \quad b_2(\mathfrak{g}) = \binom{a}{2}, \quad b_3(\mathfrak{g}) = \binom{a}{3} + m, \quad b_4(\mathfrak{g}) = \binom{a}{4} + ma.$$

Thus for any real compact connected LIE *group* $G = (\mathbb{S}^1)^a \times G_1 \times \cdots \times G_m$ *with simple* G_i,

$$H^0(G) \cong \mathbb{R}, \quad H^1(G) \cong \mathbb{R}^a, \quad H^2(G) \cong \mathbb{R}^{\binom{a}{2}}, \quad H^3(G) \cong \mathbb{R}^{\binom{a}{3}+m}, \quad H^4(G) \cong \mathbb{R}^{\binom{a}{4}+ma}.$$

Proof: The first statement follows from Lemma 3.42.3, Theorems 3.45 and 3.55 and the KUNNETH formula. For the second statement, Theorem 3.46.3 applies. □

Examples are the classical LIE groups: the special orthogonal groups SO_{n+2}, the special unitary groups SU_{n+1} and the symplectic groups Sp_n are all compact connected semisimple for $n > 0$ and thus have trivial $H^1(G)$, $H^2(G)$ and $H^4(G)$. Nevertheless note that this result only holds for the DE RHAM cohomology, but in general not for the integer valued singular cohomology $H^\bullet(G, \mathbb{Z})$. [2] contains tables for the singular cohomology of the classical LIE groups, which illustrate that torsion elements may well appear in $H^2(G, \mathbb{Z})$. In fact we have $H^2(\mathrm{SO}_{n+2}, \mathbb{Z}) \cong \mathbb{Z}_2$.

Our results prove that the primitive subspaces are the keys to the DE RHAM cohomology of any LIE group G. According to Corollary 3.28, $H^\bullet(G) \cong \prod_{i\in I} H^\bullet(G_1)$, where $|I|$ is the number of components of G. Now G_1 is diffeomorphic to $K \times \mathbb{R}^m$, where $K = \exp \mathfrak{k}$ is the maximal compact connected subgroup of G_1, resp., G. Thus $H^\bullet(G_1) \cong H^\bullet(K)$ by the KÜNNETH formula and the POINCARÉ lemma. Since $\mathfrak{k}$ is reductive, Theorem 3.46, resp., Theorem 3.48 proves that $H^\bullet(G_1) \cong \mathrm{Alt}(\mathfrak{k}, \mathbb{R})_{\mathfrak{k}-\mathrm{inv}}$. Finally Theorem 3.55 shows:

Corollary 3.57 *If* $K = \exp \mathfrak{k}$ *denotes the maximal compact connected subgroup of a real* LIE *group* G *then* $H^\bullet(G) \cong \prod_{i\in I}(\bigwedge P_\mathfrak{k})$. *If* $K = \{1\}$, $\chi(G) = |I|$, *otherwise* $\chi(G) = 0$.

Corollary 3.58 *For any principal bundle $P(M,G)$ with $K = \{1\}$, $\chi(P) = |I| \cdot \chi(M)$, otherwise $\chi(P) = 0$.*

Proof: immediate by (100). □

So let us compute $P_{\mathfrak{g}}$ for the classical LIE groups. For $G < \mathrm{Gl}(\mathbb{C}^n)$ recall the bi-invariant $\omega_k^G \in \mathcal{A}^l(G,\mathbb{C})$ from (32). By Corollary 1.84, resp., Proposition 1.81 all ω_k^G are closed, but due to Lemma 1.76, $\omega_k^G \neq 0$ only for odd k. Let $\Phi_{2l-1}^G := (\omega_{2l-1}^G)_e \in \mathrm{Alt}_{2l-1}(\mathfrak{g},\mathbb{C})_{\mathrm{inv}}$. Then for all $X_i \in \mathfrak{g} < \mathrm{gl}(\mathbb{C}^n)$

$$\Phi_{2l-1}^G(X_1,\ldots,X_{2l-1}) = \frac{1}{(2l-1)!}\sum_{\rho\in S^{2l-1}}(-1)^\rho\,\mathrm{Tr}(X_{\rho(1)}\circ\cdots\circ X_{\rho(2l-1)}),\quad 1\le l\le n.$$

In addition we define the so-called *skew Pfaffian* Sf for $\mathfrak{g} = \mathrm{so}_{2m}$. If $\langle,\rangle$ denotes the inner product in $\mathbb{R}^n$ then

$$X\in \mathrm{so}_n \quad \text{iff} \quad \langle Xx,y\rangle = -\langle x,Xy\rangle \quad \text{for all } x,y\in\mathbb{R}^n.$$

$\langle,\rangle$ extends to an inner product in all spaces $\bigwedge^p\mathbb{R}^n$. Let $\beta\colon \bigwedge^2\mathbb{R}^n \to \mathrm{so}_n$ be the canonical isomorphism defined by

$$[\beta(x\wedge y)](z) := \langle x,z\rangle y - \langle y,z\rangle x \qquad \text{for all}\quad x,y,z\in\mathbb{R}^n.$$

Its inverse $\alpha\colon \mathrm{so}_n \to \bigwedge^2\mathbb{R}^n$ is given by

$$\langle\alpha(X), x\wedge y\rangle = \langle Xx,y\rangle \qquad \text{for all}\quad X\in\mathrm{so}_n,\ x,y\in\mathbb{R}^n.$$

Finally let $E\in\bigwedge^n\mathbb{R}^n$ denote the unique unit vector in $\bigwedge^n\mathbb{R}^n$ which represents the orientation. Then for $n = 2m$ the *skew Pfaffian* $\mathrm{Sf}_{2m-1}\in\mathrm{Alt}_{2m-1}(\mathrm{so}_{2m},\mathbb{R})_{\mathrm{so-inv}}$ is given by $\mathrm{Sf}_{2m-1}(X_1,\ldots,X_{2m-1}) =$

$$\frac{1}{(2m-1)!}\sum_{\rho\in S^{2m-1}}(-1)^\rho\langle E,\alpha(X_{\rho(1)})\wedge\alpha([X_{\rho(2)},X_{\rho(3)}])\wedge\cdots\wedge\alpha([X_{\rho(2m-2)},X_{\rho(2m-1)}])\rangle$$

for all $X_i\in\mathrm{so}_{2m}$. Now the following theorem holds (recall from Proposition 3.50 that a reductive $\mathfrak{g}$ is simple if $b_1(\mathfrak{g}) = 0$ and $b_3(\mathfrak{g}) = 1$), cf. [5, pp. 253 - 269]:

Theorem 3.59 *1. The elements $\Phi_{2l-1}^{\mathrm{Gl}(\mathbb{K}^n)}$, $1\le l\le n$ form a basis for $P_{\mathrm{gl}(\mathbb{K}^n)}$. In particular, $\mathrm{gl}(\mathbb{K}^n)$ has rank n.*

2. The elements $\Phi_{2l-1}^{\mathrm{Sl}(\mathbb{K}^n)}$, $2\le l\le n$ form a basis for $P_{\mathrm{sl}(\mathbb{K}^n)}$. In particular, $\mathrm{sl}(\mathbb{K}^n)$ has rank $n-1$ and is simple for $n>0$.

3. The elements $\Phi_{4l-1}^{\mathrm{SO}_{2m+1}}$, $1\le l\le m$ form a basis for $P_{\mathrm{so}_{2m+1}}$. In particular, so_{2m+1} has rank m and is simple for $m>0$.

4. The elements $\Phi_{4l-1}^{\mathrm{SO}_{2m}}$, $1\le l\le m-1$ and Sf_{2m+1} form a basis for $P_{\mathrm{so}_{2m}}$. In particular, so_{2m} has rank m and is simple for $m>2$.

5. *The elements* $\Phi^{\mathrm{Sp}_n}_{4l-1}$, $1 \le l \le n$ *form a basis for* P_{sp_n}. *In particular,* sp_n *has rank* n *and is simple for* $n > 0$.

6. *The elements* $i^l\Phi^{\mathrm{U}_n}_{2l-1}$, $1 \le l \le n$ *form a basis for* P_{u_n}. *In particular,* u_n *has rank* n.

7. *The elements* $i^l\Phi^{\mathrm{SU}_n}_{2l-1}$, $2 \le l \le n$ *form a basis for* P_{su_n}. *In particular,* su_n *has rank* $n-1$ *and is simple for* $n > 1$.

In particular, Theorem 3.59 yields that the DE RHAM cohomology of the real LIE groups Gl_n, Sl_n, SO_n, Sp_n, U_n and SU_n are isomorphic to exterior algebras over certain ω_{2l-1}, resp., $i^l\omega_{2l-1}$, and eventually $\psi^I(\mathrm{Sf}) \in \mathcal{A}^I(\mathrm{SO}_{2m})$.

Finally, given a LIE group action $S\colon G \times P \to P$, we will combine the $\mathfrak{g}$-invariant cohomology on P with the LIE algebra cohomology of $\mathfrak{g}$. To this purpose, we form the double complex

$$C^{\bullet,\bullet} := \mathcal{A}(P) \otimes \mathrm{Hom}(\mathcal{T}(\mathfrak{g}), V) = \bigoplus_{p,q\in\mathbb{N}_0} \mathcal{A}_q(P) \otimes \mathrm{Hom}(\otimes^p \mathfrak{g}, V).$$

$(-1)^p d_{(q)}\colon C^{p,q} \to C^{p,q+1}$ is the vertical operator, and for the horizontal operator we have $\mathbf{d}^l_{(q)}\colon C^{p,q} \to C^{p+1,q}$. For the representation $l\colon \mathfrak{g} \to \mathrm{gl}(\mathcal{A}(P) \otimes V)$ several choices are possible. E. g., one can take the trivial representation o. Then $\mathbf{d}^o$ and d obviously commute.

Instead we choose l defined by $l(X) := \mathrm{sgn}(S) L_{S_X}$ and define $\delta := \mathrm{sgn}(S)\mathbf{d}^L$. Since LIE differentiation and exterior differentiation commute, δ and d commute on the double complex and define an operator D on the associated single complex.

As in Section 3.1 we augment this double complex by an extra column $\ker \delta_0$ and an extra row $\ker d_0$. Let $P = \dot{\bigcup}_{i\in I} P_i$ with components P_i, then one easily verifies

$$\ker \delta_0 = \mathcal{A}(P)_{\mathfrak{g}-\mathrm{inv}} \otimes V, \qquad H^\bullet_d(\ker \delta_0) = H^\bullet_{\mathfrak{g}-\mathrm{inv}}(P) \otimes V,$$
$$\ker d_0 \cong \prod_{i\in I} \mathrm{Hom}(\mathcal{T}(\mathfrak{g}), V), \qquad H^\bullet_\delta(\ker d_0) \cong \prod_{i\in I} \widetilde{H}^\bullet_o(\mathfrak{g}, V).$$

Moreover, for the various subcomplexes we obtain:

Lemma 3.60 *For a* LIE *group action* S *of* G *on* P, *let* $P = \dot{\bigcup}_{i\in I} P_i$ *and* $P/G = \dot{\bigcup}_{j\in J}(P/G)_j$ *with components* P_i, *resp.,* $(P/G)_j$, *where* $|I| = |J| \cdot |G/G_1|$.

1. $A^{\bullet,\bullet} := \mathcal{A}(P) \otimes \mathrm{Alt}(\mathfrak{g}, V)$ *is a subcomplex of* $C^{\bullet,\bullet}$ *with*

$$\ker \delta_0 = \mathcal{A}(P)_{\mathfrak{g}-\mathrm{inv}} \otimes V, \qquad H^\bullet_d(\ker \delta_0) = H^\bullet_{\mathfrak{g}-\mathrm{inv}}(P) \otimes V,$$
$$\ker d_0 \cong \prod_{i\in I} \mathrm{Alt}(\mathfrak{g}, V), \qquad H^\bullet_\delta(\ker d_0) \cong \prod_{i\in I} H^\bullet_o(\mathfrak{g}, V).$$

2. $A^{\bullet,\bullet}_{\mathfrak{g}-\mathrm{inv}} := \mathcal{A}(P)_{\mathfrak{g}-\mathrm{inv}} \otimes \mathrm{Alt}(\mathfrak{g}, V)$ *and* $A^{\bullet,\bullet}_{\mathfrak{g}-\mathrm{equiv}} := \mathcal{A}(P)_{\mathfrak{g}-\mathrm{equiv}} \otimes \mathrm{Alt}(\mathfrak{g}, V)$ *are subcomplexes of* $A^{\bullet,\bullet}$ *with* $\delta = \mathrm{sgn}(S)\mathbf{d}^o$, *resp.,* $\delta = -\mathrm{sgn}(S)\mathbf{d}^o$ *and*

$$\ker \delta_0 = \mathcal{A}(P)_{\mathfrak{g}-\mathrm{inv}} \otimes V, \qquad H^\bullet_d(\ker \delta_0) = H^\bullet_{\mathfrak{g}-\mathrm{inv}}(P) \otimes V,$$
$$\ker d_0 \cong \prod_{i\in I} \mathrm{Alt}(\mathfrak{g}, V), \qquad H^\bullet_\delta(\ker d_0) \cong \prod_{i\in I} H^\bullet_o(\mathfrak{g}, V).$$

3. $A^{\bullet,\bullet}_{\mathrm{inv}} := \mathcal{A}(P)_{\mathrm{inv}} \otimes \mathrm{Alt}(\mathfrak{g}, V)$ *and* $A^{\bullet,\bullet}_{\mathrm{equiv}} := \mathcal{A}(P)_{\mathrm{equiv}} \otimes \mathrm{Alt}(\mathfrak{g}, V)$ *are subcomplexes of* $A^{\bullet,\bullet}_{\mathfrak{g}-\mathrm{inv}}$*, resp.,* $A^{\bullet,\bullet}_{\mathfrak{g}-\mathrm{equiv}}$ *with* $\delta = \pm\,\mathrm{sgn}(S)\mathbf{d}^o$ *and*

$$\ker\delta_0 = \mathcal{A}(P)_{\mathrm{inv}} \otimes V, \qquad H^\bullet_d(\ker\delta_0) = H^\bullet_{\mathrm{inv}}(P) \otimes V,$$
$$\ker d_0 \cong \prod_{j\in J} \mathrm{Alt}(\mathfrak{g}, V), \qquad H^\bullet_\delta(\ker d_0) \cong \prod_{j\in J} H^\bullet_o(\mathfrak{g}, V).$$

Corollary 3.61 *If all components P_i of P are diffeomorphic to $\mathbb{R}^n$ then*

$$H^\bullet_D(C^{\bullet,\bullet}) \cong \prod_{i\in I} \overline{H}^\bullet_o(\mathfrak{g}, V), \quad H^\bullet_D(A^{\bullet,\bullet}) \cong \prod_{i\in I} H^\bullet_o(\mathfrak{g}, V).$$

For compact G, also $H^\bullet_D(A^{\bullet,\bullet}_{\mathfrak{g}-\mathrm{inv}}) \cong \prod_{i\in I} H^\bullet_o(\mathfrak{g}, V) \cong H^\bullet_D(A^{\bullet,\bullet})$*. For semisimple G,*

$$H^i_D(A^{\bullet,\bullet}_{\mathfrak{g}-\mathrm{inv}}) \cong H^i_D(A^{\bullet,\bullet}_{\mathfrak{g}-\mathrm{equiv}}) \;\cong\; H^i_{\mathfrak{g}-\mathrm{inv}}(P) \otimes V, \qquad i \le 2,$$
$$H^i_D(A^{\bullet,\bullet}_{\mathrm{inv}}) \cong H^i_D(A^{\bullet,\bullet}_{\mathrm{equiv}}) \;\cong\; H^i_{\mathrm{inv}}(P) \otimes V, \qquad i \le 2.$$

Proof. For the first statement use Proposition 3.14 and the POINCARÉ lemma; for compact G, Proposition 3.37 applies. For the last statements use Theorem 3.45 and again Proposition 3.14. □

Lemma 3.62 *For all* $\omega_n \in \mathcal{A}_n(P) \otimes V = \mathcal{A}_n(P) \otimes \mathrm{Alt}_0(\mathfrak{g}, V)$ *and* $i \le n+1$*,*

$$S^i_\bullet d\omega_n - (-1)^i d S^i_\bullet \omega_n = \delta_{i-1} S^{i-1}_\bullet \omega_n.$$

Proof. This is an immediate consequence of Lemma 1.106. □

Definition 3.63 *We define the homomorphism* $\mathbf{S}\colon \mathcal{A}(P) \otimes V \to \mathcal{A}(P) \otimes \mathrm{Alt}(\mathfrak{g}, V)$ *by* $\mathbf{S}\omega_n := \sum_{i=0}^n S^i_\bullet \omega_n$ *for all* $\omega_n \in \mathcal{A}_n(P) \otimes V$.

The homomorphism $S^\bullet_\bullet\colon \mathcal{A}(P) \otimes V \to \mathrm{Alt}(\mathfrak{g}, V)$ *is given by* $S^\bullet_\bullet\omega := \sum_{n=0}^\infty S^n_\bullet \omega_n$ *for all* $\omega = \sum_{n=0}^\infty \omega_n$ *with* $\omega_n \in \mathcal{A}_n(P) \otimes V$.

Let $p_0\colon \mathcal{A}(P) \otimes \mathrm{Alt}(\mathfrak{g}, V) \to \mathcal{A}(P) \otimes V$ denote the canonical projection onto $\mathcal{A}(P) \otimes \mathrm{Alt}_0(\mathfrak{g}, V)$. Since $p_0 \circ D = d \circ p_0$, p_0 is a chain map. Obviously $p_0 \circ \mathbf{S} = \mathrm{id}_{\mathcal{A}(P)\otimes V}$, thus if $\mathbf{S}$ is a chain map, we obtain $[p_0]\circ[\mathbf{S}] = \mathrm{id}_{H^\bullet(P)\otimes V}$ and $[\mathbf{S}]$ is injective. Indeed we find:

Proposition 3.64 *1.* $\mathbf{S}$ *is a chain map and induces an injective homomorphism*

$$[\mathbf{S}]\colon H^\bullet(P) \otimes V \to H^\bullet_D(\mathcal{A}(P) \otimes \mathrm{Alt}(\mathfrak{g}, V)).$$

2. $S^\bullet_\bullet$ *is a chain map and thus induces a homomorphism*

$$[S^\bullet_\bullet]\colon H^\bullet(P) \otimes V \to H^\bullet_L(\mathfrak{g}, V).$$

Proof. By Lemma 3.62 we have $D(\mathbf{S}\omega) = \sum_{i=0}^{n} D(S_\bullet^i\omega_n) = \sum_{i=0}^{n}[\delta_i S_\bullet^i\omega_n + (-1)^i dS_\bullet^i\omega_n] = \sum_{i=0}^{n}[S_\bullet^{i+1}d\omega_n + (-1)^i dS_\bullet^{i+1}\omega_n + (-1)^i dS_\bullet^i\omega_n] = \sum_{i=0}^{n}(S_\bullet^{i+1}d\omega_n) + (-1)^n dS_\bullet^{n+1}\omega_n + d\omega_n = \mathbf{S}(d\omega_n)$ since $S_\bullet^{n+1}\omega_n = 0$. *2.* follows from Lemma 3.62 for $i = n+1$. □

We may also restrict $\mathbf{S}$ to $\mathcal{A}(P)_{\mathfrak{g}-\mathrm{inv}} \otimes V$. Then by (50), $\operatorname{im}\mathbf{S} \subseteq A^{\bullet,\bullet}_{\mathfrak{g}-\mathrm{equiv}}$ and $\mathbf{S}$ induces an homomorphism $[\mathbf{S}]: H^\bullet(P)_{\mathfrak{g}-\mathrm{inv}} \otimes V \to H_D^\bullet(\mathcal{A}(P)_{\mathfrak{g}-\mathrm{equiv}} \otimes \mathrm{Alt}(\mathfrak{g}, V))$. Since also $i: \mathcal{A}(P)_{\mathfrak{g}-\mathrm{inv}} \otimes V \to \mathcal{A}(P)_{\mathfrak{g}-\mathrm{equiv}} \otimes \mathrm{Alt}(\mathfrak{g}, V)$ is a chain map, we obtain a chain map $\mathbf{S} - i$ (with $p_0 \circ (\mathbf{S} - i) = 0$) and an induced homomorphism

$$[\mathbf{S}] - [i]: H^\bullet(P)_{\mathfrak{g}-\mathrm{inv}} \otimes V \to H_D^\bullet(\mathcal{A}(P)_{\mathfrak{g}-\mathrm{equiv}} \otimes \mathrm{Alt}(\mathfrak{g}, V)).$$

Theorem 3.65 *If* $\mathfrak{g}$ *is semisimple and* $\omega \in \mathcal{A}_2(P)_{\mathfrak{g}-\mathrm{inv}} \otimes V$ *is closed, there exists a unique* $\chi \in \mathcal{A}_0(P)_{\mathfrak{g}-\mathrm{equiv}} \otimes \mathrm{Alt}_1(\mathfrak{g}, V)$, *such that*

$$d\chi = -S_\bullet\omega \qquad \text{and} \qquad \delta\chi = S_\bullet^2\omega.$$

Proof. By Lemma 3.62 $\delta S_\bullet^2\omega = 0$. Since $H_o^2(\mathfrak{g}, V) = 0$, we find $\chi \in \mathcal{A}_0(P)_{\mathfrak{g}-\mathrm{equiv}} \otimes \mathrm{Alt}_1(\mathfrak{g}, V)$ with $\delta\chi = S_\bullet^2\omega$. Lemma 3.42 yields that δ_1 is injective, so χ is unique. On the other hand we know from $D\mathbf{S}\omega = 0$ that $-\delta S_\bullet\omega = dS_\bullet^2\omega = d\delta\chi = \delta d\chi$. Thus $d\chi + S_\bullet\omega \in \ker\delta_1$. But δ_1 is injective. □

3.4 The Čech-de Rham Complex

As further examples for differential complexes we will discuss the Čech complex and the resulting Čech-de Rham double complex. Let M be any (paracompact) manifold with a cover $\mathfrak{U} = \{U_\alpha\}_{\alpha\in A}$, where A is countable and ordered. Recall that we have defined $U_{\alpha_0\cdots\alpha_p} := U_{\alpha_0} \cap U_{\alpha_1} \cap \cdots \cap U_{\alpha_p}$. Let $\prod_{\alpha_0} U_{\alpha_0}$, $\prod_{\alpha_0<\alpha_1} U_{\alpha_0\alpha_1}$ and $\prod_{\alpha_0<\cdots<\alpha_p} U_{\alpha_0\cdots\alpha_p}$ denote the (disjoint) direct products of these open sets.

$$C^0 := \prod_{\alpha_0} \mathcal{A}(U_{\alpha_0}), \quad C^1 := \prod_{\alpha_0<\alpha_1} \mathcal{A}(U_{\alpha_0\alpha_1}) \quad \text{and} \quad C^p := \prod_{\alpha_0<\cdots<\alpha_p} \mathcal{A}(U_{\alpha_0\cdots\alpha_p})$$

mean the products of the Grassmann algebras of these open sets. We denote the components of the elements $\omega \in C^p$ by $\omega_{\alpha_0\cdots\alpha_p} \in \mathcal{A}(U_{\alpha_0\cdots\alpha_p})$. We will also allow indices α_i in arbitrary order (even with repetitions) subject to the convention that the forms $\omega_{\alpha_0\cdots\alpha_p}$ are totally antisymmetrical with respect to these indices.

Definition 3.66 *The* Čech complex $C^\bullet(\mathfrak{U}, \mathcal{A}) := \bigoplus_{p\in\mathbb{N}_0} C^p$ *is equipped with the following differential operator* $\delta: C^p \to C^{p+1}$, $\omega \mapsto \delta\omega$:

$$(\delta\omega)_{\alpha_0\cdots\alpha_{p+1}} := \sum_{j=0}^{p+1}(-1)^j \omega_{\alpha_0\cdots\hat{\alpha}_j\cdots\alpha_{p+1}}|_{U_{\alpha_0\cdots\alpha_{p+1}}}.$$

Lemma 3.67 $\delta^2 = 0$.

Proof. $(\delta^2\omega)_{\alpha_0\cdots\alpha_{p+2}} = \sum_{i=0}^{p+1}(-1)^i(\delta\omega)_{\alpha_0\cdots\widehat{\alpha_i}\cdots\alpha_{p+2}} = \sum_{j<i}(-1)^i(-1)^j\omega_{\alpha_0\cdots\widehat{\alpha_j}\cdots\widehat{\alpha_i}\cdots\alpha_{p+2}} + \sum_{i<j}(-1)^i(-1)^{j-1}\omega_{\alpha_0\cdots\widehat{\alpha_i}\cdots\widehat{\alpha_j}\cdots\alpha_{p+2}} = 0$, where we have omitted the restrictions. □

δ is the generalization of δ in (99) to the case of countably many open sets. Obviously $\ker\delta_0 \cong \mathcal{A}(M)$. If we identify $\ker\delta_0$ and $\mathcal{A}(M)$ then the injection j in Lemma 3.9 is just the restriction of forms $r\colon \mathcal{A}(M) \to C^0$, $\omega \mapsto (\ldots,\omega|_{U_\alpha},\ldots)$. Let $C^\bullet_{\mathrm{aug}}(\mathfrak{U},\mathcal{A})$ denote this augmented Čech complex.

Definition 3.68 *Let M be a paracompact manifold with a cover $\mathfrak{U} = \{U_\alpha\}_{\alpha\in A}$ and let $\{\rho_\alpha\}_{\alpha\in A}$ denote a partition of unity subordinate to $\mathfrak{U}$. Then we define a homotopy operator $K\colon C^\bullet_{\mathrm{aug}}(\mathfrak{U},\mathcal{A}) \to C^\bullet_{\mathrm{aug}}(\mathfrak{U},\mathcal{A})C^p(\mathfrak{U},\mathcal{A}) \to C^{p-1}(\mathfrak{U},\mathcal{A})$, $\omega \mapsto K\omega$ with*

$$(K\omega)_{\alpha_0\cdots\alpha_{p-1}} := \sum_{\alpha\in A} \rho_\alpha\,\omega_{\alpha\alpha_0\cdots\alpha_{p-1}}. \tag{105}$$

Lemma 3.69 *K obeys the homotopy identity $K\delta + \delta K = \mathrm{id}_{C^\bullet_{\mathrm{aug}}(\mathfrak{U},\mathcal{A})}$.*

Proof. $(K\delta\omega + \delta K\omega)_{\alpha_0\cdots\alpha_p} = \sum_{\alpha\in A}\rho_\alpha\,(\delta\omega)_{\alpha\alpha_0\cdots\alpha_p} + \sum_{i=0}^{p}(-1)^i(K\omega)_{\alpha_0\cdots\widehat{\alpha_i}\cdots\alpha_p} = (\sum_\alpha\rho_\alpha)\omega_{\alpha\alpha_0\cdots\alpha_p} + \sum_{i,\alpha}(-1)^{i+1}\rho_\alpha\,\omega_{\alpha\alpha_0\cdots\widehat{\alpha_i}\cdots\alpha_p} + \sum_{i,\alpha}(-1)^i\rho_\alpha\,\omega_{\alpha\alpha_0\cdots\widehat{\alpha_i}\cdots\alpha_p} = \omega_{\alpha_0\cdots\alpha_p}$. □

Just as in Proposition 3.27, we obtain that the cohomology of the Čech complex consists only of $H^0_\delta(C^\bullet(\mathfrak{U},\mathcal{A})) = \ker\delta_0 \cong \mathcal{A}(M)$, resp., $H^\bullet_\delta(C^\bullet_{\mathrm{aug}}(\mathfrak{U},\mathcal{A})) = 0$:

Theorem 3.70 (Generalized Mayer-Vietoris sequence) *Let $r\colon \mathcal{A}(M) \to C^0$ denote the restriction of forms, $\omega \mapsto (\ldots,\omega|_{U_\alpha},\ldots)$. Then the sequence*

$$0 \longrightarrow \mathcal{A}(M) \xrightarrow{\ r\ } C^0 \xrightarrow{\ \delta\ } C^1 \xrightarrow{\ \delta\ } C^2 \xrightarrow{\ \delta\ } \cdots$$

is exact and thus $H^0_\delta(C^\bullet(\mathfrak{U},\mathcal{A})) = \ker\delta_0 \cong \mathcal{A}(M)$ and $H^i_\delta(C^\bullet(\mathfrak{U},\mathcal{A})) = 0$ for $i > 0$.

Proof. Lemma 3.69 yields that $\mathrm{id}_{H^\bullet_\delta(C^\bullet_{\mathrm{aug}}(\mathfrak{U},\mathcal{A}))} = [\mathrm{id}_{C^\bullet_{\mathrm{aug}}(\mathfrak{U},\mathcal{A})}] = 0$. □

If we also consider the exterior differentiation of forms, then we obtain the Čech-de Rham complex:

Definition 3.71 *The* Čech-de Rham complex *is the double complex*

$$C^{\bullet,\bullet}(\mathfrak{U},\mathcal{A}) := \bigoplus_{p,q\in\mathbb{N}_0} C^p(\mathfrak{U},\mathcal{A}_q) \qquad \text{with} \quad C^p(\mathfrak{U},\mathcal{A}_q) := \prod_{\alpha_0<\cdots<\alpha_p} \mathcal{A}_q(U_{\alpha_0\cdots\alpha_p}),$$

where δ is the horizontal operator and d is the vertical operator.

We obtain the associated singly graded complex $C(\mathfrak{U},\mathcal{A})$ with differential operator $D = \delta + (-1)^p d$ by summation along the antidiagonal lines:

$$C(\mathfrak{U},\mathcal{A})^n := \bigoplus_{p+q=n} C^p(\mathfrak{U},\mathcal{A}_q).$$

Theorem 3.72 (Generalized Mayer-Vietoris principle) *For any manifold M with a countable cover $\mathfrak{U}$, the restriction map $r\colon \mathcal{A}(M) \to C(\mathfrak{U},\mathcal{A})$ is a chain map and induces an isomorphism of cohomologies:*

$$[r]\colon H^\bullet(M) \to H^\bullet_D(C(\mathfrak{U},\mathcal{A})),\, H^n(M) \to H^n_D(C(\mathfrak{U},\mathcal{A})).$$

Proof. This is a consequence of Proposition 3.14 and Theorem 3.70. □

The inverse map $[r]^{-1}$ is less intuitive. We need a chain map $f: C(\mathfrak{U}, \mathcal{A}) \to \mathcal{A}(M), C(\mathfrak{U}, \mathcal{A})^n \to \mathcal{A}_n(M)$, that tells us how to "collate" together the components of a ČECH-DE RHAM co-chain into a global form on M. Such a f with $[f] = [r]^{-1}$ is given by the following formula, cf. [11, p. 102]:

Theorem 3.73 (Collating formula) *Let K be the homotopy operator defined in (105). For $\alpha = \sum_{i=0}^n \alpha_i \in C(\mathfrak{U}, \mathcal{A})^n$ with $\alpha_i \in C^i(\mathfrak{U}, \mathcal{A}_{n-i})$ and $D\alpha = \beta = \sum_{i=0}^{n+1} \beta_i$ with $\beta_i \in C^i(\mathfrak{U}, \mathcal{A}_{n+1-i})$,*

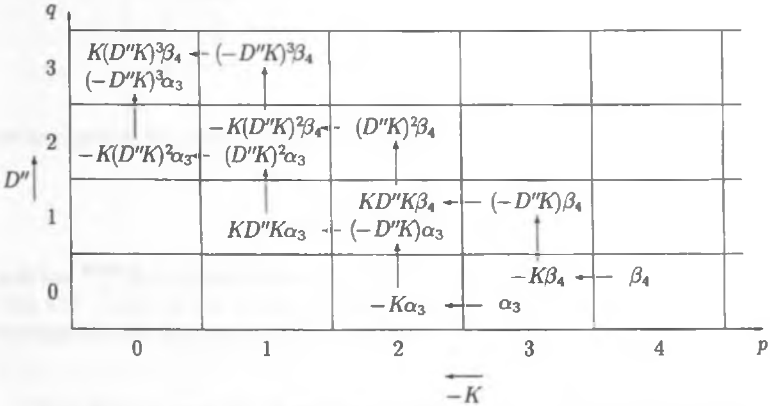
$$f(\alpha) := \sum_{i=0}^{n}(-D''K)^i\alpha_i - \sum_{i=1}^{n+1} K(-D''K)^{i-1}\beta_i \in C^0(\mathfrak{U}, \mathcal{A}_n)$$

is a global form on M (resp., the restriction of such a form to the open sets U_α). $f \circ r = id_{\mathcal{A}(M)}$ and $\mathrm{id}_{C(\mathfrak{U},\mathcal{A})} - r \circ f = DL + LD$, where the homotopy operator $L: C(\mathfrak{U}, \mathcal{A}) \to C(\mathfrak{U}, \mathcal{A})$, $C(\mathfrak{U}, \mathcal{A})^n \to C(\mathfrak{U}, \mathcal{A})^{n-1}$, $\alpha \mapsto \sum_{i=0}^{n-1}(L\alpha)_i$ is given by

$$(L\alpha)_i := \sum_{j=i+1}^{n} K(-D''K)^{j-i-1}\alpha_j \in C^i(\mathfrak{U}, \mathcal{A}_{n-i-1}).$$

Figure 3.5 illustrates how the components α_3 and β_4 of α, resp., $\beta = D\alpha$ are sent to $C^0(\mathfrak{U}, \mathcal{A}_n)$. In order to obtain a global form on M, all components α_i and β_i have to be treated in this manner.

Figure 3.5: Illustration of the Collating formula



Let us now enlarge the ČECH-DE RHAM complex as in Lemma 3.13. By Theorem 3.70, we have $\ker \delta_0 \cong \mathcal{A}(M)$. For the additional row, $\ker d_0$ obviously consists of all locally constant maps on the sets $U_{\alpha_0 \cdots \alpha_p}$. We denote the complex $\ker d_0$ by $C(\mathfrak{U}, \mathbf{R})$, cf. Figure 3.6.

Definition 3.74 *The cohomology* $H^\bullet(\mathfrak{U},\mathbb{R}) := H^\bullet_\delta(C(\mathfrak{U},\mathbb{R}))$ *of the differential complex* $C(\mathfrak{U},\mathbb{R}) := \ker d_0$ *is called* ČECH cohomology of the cover $\mathfrak{U}$.

The ČECH cohomology of a cover $\mathfrak{U}$ is a purely combinatorial object. Note that the argument for the exactness of the generalized MAYER-VIETORIS sequence breaks down for the complex $C(\mathfrak{U},\mathbb{R})$, because the elements of $C(\mathfrak{U},\mathbb{R})$ are locally constant functions so that partitions of unity are not applicable and K in (105) is not defined for $C(\mathfrak{U},\mathbb{R})$.

Figure 3.6: The augmented ČECH-DE RHAM complex

$$\begin{array}{lllll}
q & & & & \\
0 \to A_3(M) \xrightarrow{r} & \prod_{\alpha_0} A_3(U_{\alpha_0}) & \prod_{\alpha_0<\alpha_1} A_3(U_{\alpha_0\alpha_1}) & \prod_{\alpha_0<\alpha_1<\alpha_2} A_3(U_{\alpha_0\alpha_1\alpha_2}) & \\
0 \to A_2(M) \xrightarrow{r} & \prod_{\alpha_0} A_2(U_{\alpha_0}) & \prod_{\alpha_0<\alpha_1} A_2(U_{\alpha_0\alpha_1}) & \prod_{\alpha_0<\alpha_1<\alpha_2} A_2(U_{\alpha_0\alpha_1\alpha_2}) & \\
0 \to A_1(M) \xrightarrow{r} & \prod_{\alpha_0} A_1(U_{\alpha_0}) & \prod_{\alpha_0<\alpha_1} A_1(U_{\alpha_0\alpha_1}) & \prod_{\alpha_0<\alpha_1<\alpha_2} A_1(U_{\alpha_0\alpha_1\alpha_2}) & \\
0 \to A_0(M) \xrightarrow{r} & \prod_{\alpha_0} A_0(U_{\alpha_0}) & \prod_{\alpha_0<\alpha_1} A_0(U_{\alpha_0\alpha_1}) & \prod_{\alpha_0<\alpha_1<\alpha_2} A_0(U_{\alpha_0\alpha_1\alpha_2}) & \\
 & i\uparrow & i\uparrow & i\uparrow & p \\
 & C^0(\mathfrak{U},\mathbb{R}) \xrightarrow{\delta} & C^1(\mathfrak{U},\mathbb{R}) \xrightarrow{\delta} & C^2(\mathfrak{U},\mathbb{R}) \xrightarrow{\delta} & \\
 & \uparrow & \uparrow & \uparrow & \\
 & 0 & 0 & 0 &
\end{array}$$

Theorem 3.75 *If* $\mathfrak{U}$ *is a* good *cover of* M*, then the* DE RHAM *cohomology and the* ČECH *cohomology of* $\mathfrak{U}$ *are isomorphic:*

$$H^\bullet(M) \cong H^\bullet(\mathfrak{U},\mathbb{R}).$$

Proof. If $\mathfrak{U}$ is a good cover, then all sets $U_{\alpha_0\cdots\alpha_p}$ are diffeomorphic to $\mathbb{R}^{\dim M}$ and thus the columns of the augmented ČECH-DE RHAM complex are all exact. The rows are exact by Theorem 3.70. Now Proposition 3.14 yields that both cohomologies are isomorphic to $H^\bullet_D(C(\mathfrak{U},\mathcal{A}))$. □

Corollary 3.76 *The* ČECH *cohomologies* $H^\bullet(\mathfrak{U},\mathbb{R})$ *are the same for all good covers* $\mathfrak{U}$ *of a manifold.*

As a consequence, one can compute the DE RHAM cohomology of a manifold M using purely combinatorical considerations by computing the ČECH cohomology of a good cover of M, cf. [11] and [2].

3.5 Spectral Sequences of Double Complexes

Definition 3.77 *A* spectral sequence *is a sequence of complexes* $\{E_r\}_{r\in\mathbb{N}_0}$ *with differential operators* D_r*, where every* E_r *is the cohomology of its predecessor:*

$$E_{r+1} = H^*_{D_r}(E_r).$$

If E_R *becomes* stationary, *i. e.,* $E_r = E_{r+1}$ *for all* $r \geq R$*, we denote* E_R *by* E_∞ *and say that the spectral sequence* converges *to some filtered complex* H *if* $E_\infty \cong GH$.

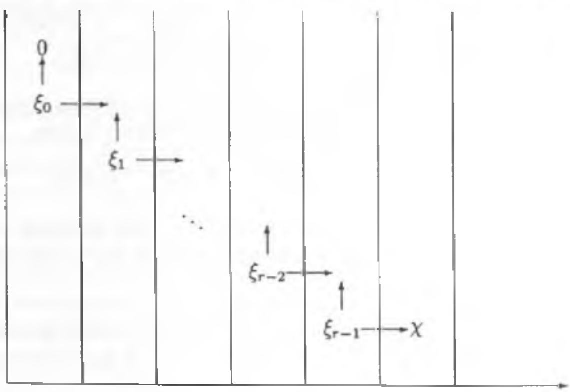
We obtain the spectral sequence of a double complex $C^{\bullet,\bullet}$ *by putting* $E_0 = B = GC$ *from* (96) *and defining* D_r *to be the differential operator induced by* D *on* E_r*, so* $E_{r+1} = H^*_D(E_r)$. *We say that an element* $\beta \in C^{\bullet,\bullet}$ lives *to* E_r, $r > 0$, *if it represents a cohomology class* $[\beta]_r \in E_r$*, resp. equivalently, if* β *is* D*-closed in* $E_0, E_1, \ldots, E_{r-1}$*:* $D_i[\beta]_i = 0$, $i = 0, \ldots, r-1$.

Lemma 3.78 $\beta \in C^{\bullet,\bullet}$ *lives to* E_r, $r > 0$, *iff* β *is* d*-closed and we have a* "zig-zag" $\Xi = \xi_0 + \xi_1 + \ldots + \xi_{r-1}$ *of elements* $\xi_i \in C^{\bullet,\bullet}$ *with* $\xi_0 := \beta$ *and (cf.* Figure 3.7*)*

$$D'\xi_i = \delta\xi_i = -D''\xi_{i+1}, \qquad i = 0, \ldots, r-2.$$

Now $D_r[\beta]_r = [\delta\xi_{r-1}]_r = [\chi]_r$*, so* D_r *is given by* δ *at the end of the zig-zag.*

Figure 3.7: Illustration of the Differential operator D_r: $D_r[\xi_0]_r = [\delta\xi_{r-1}]_r = [\chi]_r$



Thus like $C^{\bullet,\bullet}$ every E_r is a double complex, too: $E_r = \bigoplus_{p,q\in\mathbb{N}_0} E_r^{p,q}$ and D_r shifts the bidegrees by $(r, -r+1)$: $D_r\colon E_r^{p,q} \to E_r^{p+r,q-r+1}$. Obviously

$$E_1 = H_d(C) \qquad \text{and} \qquad E_2 = H_\delta(H_d(C)).$$

The spectral sequence of a double complex computes the total cohomology of the double complex. We have, cf. [11, p. 165]:

Theorem 3.79 *Given a double complex* $C^{\bullet,\bullet} = \bigoplus_{p,q\in\mathbb{N}_0} C^{p,q}$ *there is a spectral sequence* $\{E_r, D_r\}_{r\in\mathbb{N}_0}$ *converging to the total cohomology* $H_D^\bullet(C)$ *such that each* $E_r = \bigoplus_{p,q\in\mathbb{N}_0} E_r^{p,q}$ *has a bigrading with:*

$$D_r\colon E_r^{p,q} \to E_r^{p+r,q-r+1}.$$

The first terms of this sequence are

$$E_0^{p,q} = B^{p,q}, \qquad E_1^{p,q} = H_d^{p,q}(C) \qquad \text{and} \qquad E_2^{p,q} = H_\delta^{p,q}(H_d(C)).$$

Furthermore, the associated graded complex of the total cohomology is given by

$$GH_D^n(C) = \bigoplus_{p+q=n} E_\infty^{p,q}(C).$$

Naturally, we can also use the filtration given by (97). Then we obtain a second spectral sequence $\{E'_r, D'_r\}_{r\in\mathbb{N}_0}$ that converges to $H_D^\bullet(C)$ with

$$(E'_0)^{p,q} = (B')^{p,q}, \qquad (E'_1)^{p,q} = H_\delta^{p,q}(C), \qquad (E'_2)^{p,q} = H_d^{p,q}(H_\delta(C))$$

$$\text{and} \qquad D'_r\colon (E'_r)^{p,q} \to (E'_r)^{p-r+1,q+r}.$$

Let us compute these two sequences for the ČECH-DE-RHAM complex $C(\mathfrak{U}, \mathcal{A})$. For the second sequence, Theorem 3.70 yields

$$(E'_1)^{p,q} = \begin{cases} \mathcal{A}_q(M) & \text{if } p = 0 \\ 0 & \text{else,} \end{cases} \qquad \text{thus} \qquad (E'_2)^{p,q} = \begin{cases} H^q(M) & \text{if } p = 0 \\ 0 & \text{else.} \end{cases}$$

Since E'_2 consists only of one column, we obtain $D_2 = 0$. Thus E'_2 becomes stationary and the analogon of Theorem 3.79 for the second sequence proves

$$H^n(M) \cong H_D^n(C(\mathfrak{U}, \mathcal{A})) \qquad \text{for all} \qquad n \in \mathbb{N}_0,$$

which gives an alternative proof of the MAYER-VIETORIS principle 3.72.

On the other hand, if $\mathfrak{U}$ is a good cover and thus all $U_{\alpha_0\cdots\alpha_p}$ are contractible, we obtain for the first sequence:

$$E_1^{p,q} = \begin{cases} C^p(\mathfrak{U}, \mathbb{R}) & \text{if } q = 0 \\ 0 & \text{else,} \end{cases} \qquad \text{thus} \qquad E_2^{p,q} = \begin{cases} H^p(\mathfrak{U}, \mathbb{R}) & \text{if } q = 0 \\ 0 & \text{else.} \end{cases}$$

Again $D_2 = 0$ and $D_r = 0$ for all $r > 2$, since E_2 consists of only one row. Thus $E_2 = E_\infty$ becomes stationary and Theorem 3.79 proves (cf. Theorem 3.75):

$$H^n(\mathfrak{U}, \mathbb{R}) \cong H_D^n(C(\mathfrak{U}, \mathcal{A})) \cong H^n(M) \qquad \text{for all} \qquad n \in \mathbb{N}_0.$$

Now we are prepared for the definition of the *spectral sequence of a fiber bundle.* For a bundle $B(M, F, G)$, if $\mathfrak{U}$ is a good cover of M, $\pi^{-1}\mathfrak{U}$ is a cover of B and for all $U_{\alpha_0\cdots\alpha_p}$ we have $\pi^{-1}(U_{\alpha_0\cdots\alpha_p}) \cong \mathbb{R}^n \times F$, so $H^q(\pi^{-1}(U_{\alpha_0\cdots\alpha_p})) \cong H^q(F)$ by the

POINCARÉ lemma. We form the following double complex, the so-called ČECH-DE RHAM complex for the bundle B:

$$C^{\bullet,\bullet}(\pi^{-1}\mathfrak{U},\mathcal{A}) = \bigoplus_{p,q\in\mathbb{N}_0} C^p(\pi^{-1}\mathfrak{U},\mathcal{A}_q) \quad \text{with} \quad C^p(\pi^{-1}\mathfrak{U},\mathcal{A}_q) = \prod_{\alpha_0<\cdots<\alpha_p} \mathcal{A}_q(\pi^{-1}(U_{\alpha_0\cdots\alpha_p})).$$

Theorem 3.72 yields that $H_D^\bullet(C(\pi^{-1}\mathfrak{U},\mathcal{A})) \cong H^\bullet(B)$. According to Theorem 3.79, there is a spectral sequence converging to $H_D^\bullet(C(\pi^{-1}\mathfrak{U},\mathcal{A}))$ with E_1 term given by

$$E_1^{p,q} = H_d^{p,q}(C(\pi^{-1}\mathfrak{U},\mathcal{A})) = \prod_{\alpha_0<\cdots<\alpha_p} H^q(\pi^{-1}(U_{\alpha_0\cdots\alpha_p})) \cong \prod_{\alpha_0<\cdots<\alpha_p} H^q(F).$$

Recall that the projection π induces a homomorphism $[\pi^*]: H^\bullet(M) \to H^\bullet(B)$. Thus one would expect that not only $H^\bullet(F)$ but also $H^\bullet(M)$ entered into this spectral sequence for $H^\bullet(B)$. Indeed, if $H^\bullet(F)$ is finitely generated and in addition M simply connected or $B \cong M \times F$, then one can prove that for the E_2 term, $E_2^{p,q} = H_\delta^{p,q}(H_d(C(\pi^{-1}\mathfrak{U},\mathcal{A}))) \cong H^p(M) \otimes H^q(F)$ holds, cf. [11, p. 170]. This proves

Theorem 3.80 (Leray's theorem for the de Rham cohomology) *Suppose $B(M,F,G)$ is a fiber bundle and $\mathfrak{U} = \{U_\alpha\}_{\alpha\in A}$ is a good cover of M then there is a spectral sequence converging to $H^\bullet(B)$ with E_1 term*

$$E_1^{p,q} = \prod_{\alpha_0<\cdots<\alpha_p} H^q(\pi^{-1}(U_{\alpha_0\cdots\alpha_p})) \cong \prod_{\alpha_0<\cdots<\alpha_p} H^q(F).$$

If $H^\bullet(F)$ is finitely generated and in addition M simply connected or $B \cong M \times F$, then

$$E_2^{p,q} = H^p(M, H^q(F)) \cong H^p(M) \otimes H^q(F).$$

Thus it is possible to compute the cohomology of a bundle from the cohomology of the fiber, whenever a good cover of the base is given. Further note that LERAY's theorem proves the KÜNNETH formula, because all forms in E_2 are closed global forms on $M \times F$ for which $d = \delta = 0$ and thus $D_2 = 0$. So E_2 becomes stationary, which proves Theorem 3.35.

Recall once more that any closed form on the base can be extended to the bundle B and defines a cohomology class of B. Using the spectral sequence for B, we are now able to answer the analogous question, which closed forms on the fiber can be extended to B and which of these extended forms are closed, too, and thus define a cohomology class of B.

Definition 3.81 *A closed differential form $\phi_q \in \mathcal{A}_q(F)$, resp., its cohomology class $[\phi_q] \in H^q(F) \hookrightarrow E_1^{0,q}$ is called* transgressive *if it lives to the E_{q+1} term of the spectral sequence in* Theorem 3.80. *We call a transgressive form* 0-transgressive *if $D_{q+1}[\phi_q]_{q+1} = 0$, i. e., if it lives to E_{q+2}. Denote the set of transgressive, resp., 0-transgressive forms by $\mathcal{A}(F)_{\text{trans}}$, resp., $\mathcal{A}(F)_{0-\text{trans}}$.*

Transgressive forms on connected fibers represent forms on the bundle because of the following theorem [11, p. 247]:

Theorem 3.82 *Let $B(M, F, G)$ be a bundle with connected fiber F. Then a differential form $\phi \in A_q(F)$ is transgressive iff it is the restriction of a $\psi \in \mathcal{A}_q(B)$ with $d\psi = \pi^*\tau$ for some $\tau \in \mathcal{A}_{q+1}(M)$.*

Note that if ϕ_q and ϕ' are transgressive with connected fiber,

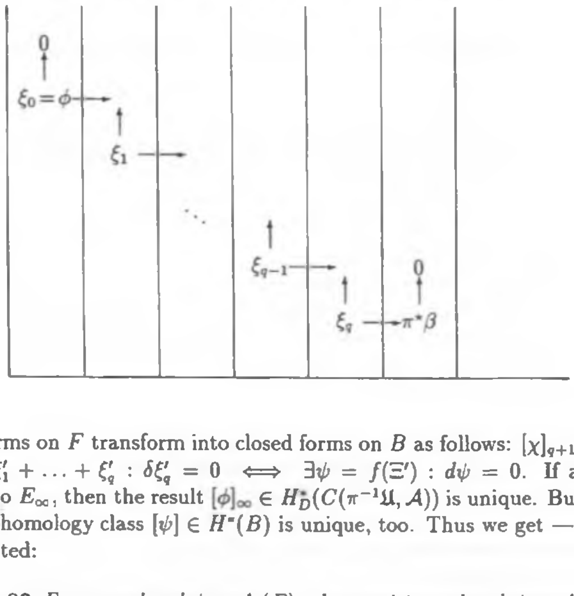
$$d(\psi_q \wedge \psi') = \pi^*\tau \wedge \psi' + (-1)^q \psi_q \wedge \pi^*\tau' \notin \pi^*(\mathcal{A}(M))$$

in general. Thus $\mathcal{A}(F)_{\text{trans}}$ is only a vector space but not a $\mathbb{R}$-subalgebra of $\mathcal{A}(F)$.

We find ψ and τ using the Collating formula: first we take a zig-zag $\Xi = \xi_0 + \ldots + \xi_q$ according to Lemma 3.78 (where $q = r - 1$). Now $d\chi = d\delta\xi_q = \delta d\xi_q = \pm\delta\delta\xi_{q-1} = 0$, thus the components of χ are locally constant on $\pi^{-1}(U_{\alpha_0\cdots\alpha_p}) \cong U_{\alpha_0\cdots\alpha_p} \times F$, and because F is connected we have $\chi = \pi^*\beta$ with $\beta \in C^{q+1}(\mathfrak{U}, \mathcal{A}_0)$, cf. Figure 3.8. Now Theorem 3.73 yields

$$\begin{aligned} \psi &:= f(\Xi) = \sum_{i=0}^{q}(-D''K)^i\xi_i - K(-D''K)^q\pi^*\beta \in \mathcal{A}_q(B) \quad \text{and} \qquad (106)\\ d\psi &= (-D''K)^{q+1}\pi^*\beta = \pi^*\tau, \quad \text{where} \quad \tau = (-D''K)^{q+1}\beta \in \mathcal{A}_{q+1}(M). \end{aligned}$$

Figure 3.8: Zigzag of a transgressive form



Closed forms on F transform into closed forms on B as follows: $[\chi]_{q+1} = 0 \iff \exists\Xi' = \xi_0 + \xi_1' + \ldots + \xi_q' : \delta\xi_q' = 0 \iff \exists\psi = f(\Xi') : d\psi = 0$. If any closed form ϕ lives to E_∞, then the result $[\phi]_\infty \in H_D^*(C(\pi^{-1}\mathfrak{U}, \mathcal{A}))$ is unique. But then the DE RHAM cohomology class $[\psi] \in H^*(B)$ is unique, too. Thus we get — even if F is not connected:

Corollary 3.83 *For any closed $\phi \in \mathcal{A}_q(F)$, there exists a closed $\psi \in \mathcal{A}_q(B)$ such that ϕ is the restriction of ψ iff ϕ is 0-transgressive. In that case $[\psi]$ is unique.*

As a consequence, $\mathcal{A}(F)_{0-\text{trans}}$ is a $\mathbb{R}$-subalgebra of $\mathcal{A}(F)$.

Let us apply spectral sequences to the special unitary groups SU_n. To this purpose, we note that SU_n acts on the unit sphere $\mathbb{S}^{2n-1}$ in $\mathbb{C}^n$ by multiplication. For $\vec{e}_n$, the stabilizer subgroup is SU_{n-1}, thus SU_n is a principal bundle $P(\mathbb{S}^{2n-1}, SU_{n-1})$ for $n \geq 2$ (apply Proposition 2.19 with $P := \{x\} \times G \cong G$). We want to prove that $H^{\bullet}(SU_n)$ is isomorphic to the exterior $\mathbb{R}$-algebra over the volume forms dV_{2l-1} of $\mathbb{S}^{2l-1}$, $2 \leq l \leq n$, cf. Theorem 3.59.7.

For $SU_2 \cong \mathbb{S}^3$ there is nothing to prove. Suppose the statement is true for SU_{n-1}. We compute the spectral sequence $\{E_r\}_{r \in \mathbb{N}_0}$ for $P(\mathbb{S}^{2n-1}, SU_{n-1})$. Due to LERAY'S theorem, $E_2^{p,q} \cong H^p(\mathbb{S}^{2n-1}) \otimes H^q(SU_{n-1})$, since $\mathbb{S}^{2n-1}$ is simply connected and $H^{\bullet}(SU_{n-1})$ is finitely generated. Thus E_2 consists only of two nontrivial columns, namely for $p = 0$ and $p = 2n-1$. Thus $D_2(dV_{2l-1}) = 0$ for all $l < n$. (We identify dV_{2l-1} on $\mathbb{S}^{2l-1}$ and the generated forms on SU_{n-1} for convenience.) Analogously $D_r(dV_{2l-1}) = 0$ for all $r \geq 2$. Thus all dV_{2l-1} are 0-transgressive and define cohomology classes in $H^{2l-1}(SU_n)$. Since $\mathcal{A}(SU_{n-1})_{0-\text{trans}}$ is a $\mathbb{R}$-subalgebra of $\mathcal{A}(SU_{n-1})$, the same holds for all products of the forms dV_{2l-1} for $l < n$. Finally dV_{2n-1} represents a cohomology class in $H^{2n-1}(SU_n)$, because it is a form on the base. Since E_2 consists only of elements that represent products of dV_{2l-1}, $l \leq n$, the desired result follows by induction.

3.6 Cohomology and Connection on Bundles

In the previous section we have developed spectral sequences in order to compute the cohomology of a fiber bundle from the cohomologies of the base and the fiber. We found that only transgressive forms on the fiber can be extended to B and only 0-transgressive forms $\phi_n \in \mathcal{A}_n(F)$ define cohomology classes in $H^n(B)$. Nevertheless recall from the definition of the homotopy operator K in (106) — confer (105) — that for any such ϕ_n, the Collating formula generates a form ψ_n on B that depends on a partition of unity given on the paracompact manifold M.

Now suppose a connection is given on B. Especially for applications in physics one would like to obtain forms that are adapted to this connection and do not depend on an arbitrary physically meaningless partition of unity. So the question not only is whether ϕ_n defines a cohomology class $[\psi_n] \in H^n(B)$, but whether we can find a representative $\phi_n^A \in \mathcal{A}_n(B)$ for this class $[\psi_n] = [\phi_n^A]$, such that ϕ_n^A is adapted to the given connection.

At this point we should say what we exactly mean by "adapted to a given connection." According to Proposition 2.75, any invariant $\phi_n \in \mathcal{A}_n(F)$ can be naturally extended to B and defines a vertical form there. Yet this form $\phi_n v \in \mathcal{A}_n(B)$ will not be closed in general: if $d\phi_n = 0$ we know that $d(\phi_n v) = (L_{\bullet}\phi_n)v \bullet \mathrm{F}$ from Theorem 2.120, where $L_{\bullet}\phi_n \in \mathcal{A}_{n-1}(F) \otimes \mathrm{Hom}(\mathfrak{g}, \mathbb{R})$ is equivariant.

Thus we are led to equivariant forms $\chi \in \mathcal{A}(F, \mathrm{Hom}(\mathcal{T}(\mathfrak{g}), \mathbb{R})_{\text{equiv}}$. By Theorem 2.77, χ can be extended to B, whenever a pseudotensorial $\psi \in \mathcal{A}_r^P(P, \mathfrak{g})$ is given. Then $(\mathrm{pr}_F^* \chi) \bullet (\mathrm{pr}_P^* \psi) \in \mathcal{A}(P \times F)$ defines a form "$\chi v \bullet \psi$"$\in \mathcal{A}(B)$, where the vertical part is given by $(\mathrm{pr}_F^* \chi)$ and the horizontal part is given by $(\mathrm{pr}_P^* \psi)$.

The only pseudotensorial forms $\psi \in \mathcal{A}^P(P, \mathfrak{g})$ given naturally by a connection Γ are ω^Γ and Ω^Γ. Yet recall from (71) that ω^Γ produces only zero. This justifies our following definition (as in Corollary 2.118, we will use the notation $\chi v \bullet \mathrm{F}$ instead of $\chi v \bullet \Omega^\Gamma$):

Definition 3.84 *Let Γ be a connection on $P(M,G)$ and $B = P \times_G F$. A differential form $\phi^{\mathrm{A}} \in \mathcal{A}(B,V)$ is called* adapted to Γ *if $\chi^i \in \mathcal{A}(F, \mathrm{Hom}(\mathcal{T}(\mathfrak{g}), V))_{\mathrm{equiv}}$ are given such that*

$$\phi^{\mathrm{A}} = \sum\nolimits_i \chi^i v \bullet \mathrm{F}.$$

For convenience we further define (recall $[r] := \max_{z \in \mathbb{Z}}\{z \leq r\}$ for all $r \in \mathbb{R}$):

Definition 3.85 *Let $L: G \times F \to F$ be a left* LIE *group action. An invariant closed differential form $\phi_n \in \mathcal{A}_n(F)_{\mathrm{inv}} \otimes V$ will be called* G-transgressive *if equivariant differential forms $\chi^i \in \mathcal{A}_{n-2i}(F)_{\mathrm{equiv}} \otimes \mathrm{Sym}_i(\mathfrak{g}, V)$ exist for $0 \leq i \leq [n/2]$ with*

$$\chi^0 = \phi_n, \quad -L^\vee_\bullet \chi^i = d\chi^{i+1} \quad \textit{for all} \quad 0 \leq i \leq [n/2]-1 \quad \textit{and} \quad L^\vee_\bullet \chi^{[n/2]} = 0. \quad (107)$$

Denote the set of all G-transgressive forms on F by $\mathcal{A}(F)_{G-\mathrm{trans}} \otimes V$.

Using the fact that d and $L^\vee_\bullet$ are skew derivations of $\mathcal{A}(F)_{\mathrm{equiv}} \otimes \mathrm{Sym}(\mathfrak{g}, V)$ of degree 1, resp., -1 (cf. Lemma 1.119), one proves:

Proposition 3.86 *$\mathcal{A}(F)_{G-\mathrm{trans}} \otimes V$ is a $\mathbb{R}$-subalgebra of $\mathcal{A}(F) \otimes V$, whenever $\wedge_V$ is defined. If ϕ_m and ψ_n are G-transgressive and $\chi^i \in \mathcal{A}_{m-2i}(F)_{\mathrm{equiv}} \otimes \mathrm{Sym}_i(\mathfrak{g}, V)$, resp., $\xi^j \in \mathcal{A}_{n-2j}(F)_{\mathrm{equiv}} \otimes \mathrm{Sym}_j(\mathfrak{g}, V)$ are the differential forms given by (107) for ϕ_m, resp., ψ_n, then*

$$\zeta^k := \sum_{i+j=k} \chi^i \wedge_V \xi^j \in \mathcal{A}_{m+n-2k}(F)_{\mathrm{equiv}} \otimes \mathrm{Sym}_k(\mathfrak{g}, V), \qquad 0 \leq k \leq [m/2] + [n/2],$$

(and $\zeta^{[(m+n)/2]} := 0$ if m and n are odd) are the corresponding forms for $\phi_m \wedge_V \psi_n$.

Now we are ready for the following theorem:

Theorem 3.87 *Let Γ be a connection on a principal bundle $P(M,G)$ and $B = P \times_G F$ an associated bundle with left* LIE *group action $L: G \times F \to F$. Let V denote any vector space. If $\phi_n \in \mathcal{A}_n(F)_{\mathrm{inv}} \otimes V$ is G-transgressive and the equivariant forms $\chi^i_{n-2i} \in \mathcal{A}_{n-2i}(F)_{\mathrm{equiv}} \otimes \mathrm{Sym}_i(\mathfrak{g}, V)$ are given by (107), then*

$$\phi^{\mathrm{A}}_n := \sum_{i=0}^{[n/2]} (\chi^i_{n-2i} v) \bullet \mathrm{F} \in \mathcal{A}_n(B) \otimes V$$

is closed and adapted to Γ. Its restriction to the fibers is ϕ_n, i. e. for any $x \in M$, $i^*_{\alpha,x} \phi^{\mathrm{A}}_n = \phi_n$.

Proof. ϕ_n^A is obviously adapted to Γ. Furthermore Theorem 2.120 yields:

$$\begin{aligned} d\phi_n^A &= \sum_{i=0}^{[n/2]} (d\chi_{n-2i}^i)v \bullet F + (L_\bullet^\vee \chi_{n-2i}^i)v \bullet F \\ &= (d\phi_n)v + \sum_{i=0}^{[n/2]-1} (d\chi_{n-2i-2}^{i+1} + L_\bullet^\vee \chi_{n-2i}^i)v \bullet F + (L_\bullet^\vee \chi_{n-2[n/2]}^{[n/2]})v \bullet F = 0, \end{aligned}$$

since ϕ_n is G-transgressive. Finally, since $i_{\alpha,x}^*\pi^*F^\alpha = 0$ for all $x \in U_\alpha$ (resp., $i_p^* \mathrm{pr}_P^* \Omega^\Gamma = 0$ for all $p \in P$), $i_{\alpha,x}^*\phi_n^A = i_{\alpha,x}^*(\chi^0 v) = i_{\alpha,x}^*(\phi_n v)$. By Lemma 2.116, $i_{\alpha,x}^*(\phi_n v) = \phi_n$. □

Note that the property of being G-transgressive only depends on L, G and F. G-transgressive forms define DE RHAM cohomology classes on all fiber bundles where L is the action of the structure group G on the fiber F. In particular, this condition is independent of the base M and of the question whether the bundle is trivial or not. Indeed we have:

Corollary 3.88 *Let $L: G \times F \to F$ be a left* LIE *group action. If $\phi_n \in \mathcal{A}_n(F)$ is G-transgressive, it is 0-transgressive for any bundle $B(M,F,G)$ that comes along with L. Thus ϕ_n defines a unique cohomology class $[\phi_n^A] \in H^n(B)$ with $[i_{\alpha,x}^*][\phi_n^A] = [\phi_n] \in H^n(F)$, independently of the (paracompact) base M and the transition functions $g_{\alpha\beta}$.*

Proof. By Theorem 2.66, we find a connection Γ on $P(M,G)$. Thus ϕ_n^A is well-defined and Theorem 3.87 applies. Uniqueness follows from Corollary 3.83. □

Corollary 3.89 *If Γ and Γ' are two connections on $P(M,G)$ and $\phi \in \mathcal{A}(F)$ is G-transgressive then there exists $\psi \in \mathcal{A}(B)$ such that the forms ϕ^A and $\phi^{A'}$ obey:*

$$\phi^A - \phi^{A'} = d\psi \qquad \textit{with} \qquad d(i_{\alpha,x}^*\psi) = 0 \qquad \textit{for all} \quad x \in U_\alpha.$$

Let us compute the cases where $n = 0$, 1 or 2.

$d\phi_0 = 0$ means that $\phi_0 \in C^\infty(F)$ is locally constant. Obviously $L_\bullet^\vee \phi_0 = 0$. So every closed G-invariant $\phi_0 \in C^\infty(F)$ is G-transgressive. Since ϕ_0 is invariant, it is global and vertical. Thus $(\phi_0^A)^\alpha = \pi_\alpha^*\phi_0$ and $[i_{\alpha,x}^*][\phi_0^A] = [\phi_0]$. This proves:

Corollary 3.90 *For any $x \in U_\alpha$, $[i_{\alpha,x}^*]: H^0(B(M,F,G)) \to H_{\mathrm{inv}}^0(F)$ is surjective.*

(Note that this also implies $H_{\mathrm{inv}}^0(F) \leq H^0(F)$, if we put $B := \{x\} \times F$, but this is nothing new.)

For $n = 1$ and $\phi_1 \in \mathcal{A}(F)_{\mathrm{inv}}$, Lemma 3.62 yields that $d\phi_1 = 0$ implies $d_1^\circ L_\bullet \phi_1 = 0$, i. e. for all $f \in F$, $[L_\bullet \phi_1(f)] \in [\mathfrak{g},\mathfrak{g}]^\perp$ by Lemma 3.42. Thus for a semisimple LIE algebra $\mathfrak{g}$, $L_\bullet \phi_1 = 0$. As a consequence for any bundle $B(M,F,G)$ that comes along with L, $\{\pi_\alpha^*\phi_1\}_{\alpha\in A}$ defines a global vertical form on B. We have proved:

Corollary 3.91 *If L is a* LIE *group action of a semisimple* LIE *group G on F, then every closed invariant 1-form $\phi_1 \in \mathcal{A}_1(F)_{\mathrm{inv}}$ is G-transgressive and defines a unique cohomology class $[\phi_1 v] = [\{\pi_\alpha^*\phi_1\}_{\alpha\in A}] \in H^1(B)$ for any bundle $B(M,F,G)$ that comes along with L. Thus for any $x \in U_\alpha$, $[i_{\alpha,x}^*]: H^1(B(M,F,G)) \to H_{\mathrm{inv}}^1(F)$ is surjective.*

To show that the condition "G semisimple" is necessary, take $G = \mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$ acting on itself by left multiplication. Then $\mathfrak{g} = \mathbb{R}$ and the (left) canonical 1-form Θ^L is invariant. Since $\mathbb{S}^1$ is abelian, $d\Theta^L = \Theta^L \wedge_{\mathfrak{g}} \Theta^L = 0$. Θ^L is the volume form on $\mathbb{S}^1$ and generates $H^1_{\text{inv}}(\mathbb{S}^1) \cong H^1(\mathbb{S}^1) \cong \mathbb{R}$, cf. Proposition 3.37. Yet $(L_\bullet \Theta^L)(X) = \Theta^L(\mathcal{L}_X) = X$ for all $X \in \mathbb{R}$. Thus $L_\bullet \Theta^L = \text{id}_{\mathbb{R}}$ and Θ^L is *not* $\mathbb{S}^1$-transgressive.

Recall the principal bundles $P_m(\mathbb{S}^2, \mathbb{S}^1)$ from (56). For their DE RHAM cohomology one obtains from the spectral sequence for P_m with $m \neq 0$, cf. [2, p. 72]:

$$H^0(P_m) \cong \mathbb{R}, \qquad H^1(P_m) = 0, \qquad H^2(P_m) = 0, \qquad H^3(P_m) \cong \mathbb{R}.$$

So $[i^*_{\alpha,x}]: H^1(P_m) \to H^1_{\text{inv}}(G)$ is never surjective. Moreover, recall from Note 2.79, that $\Theta^L v = \omega^\Gamma$, even for $m = 0$. Since $d\omega^\Gamma = d^\Gamma \omega^\Gamma = \Omega^\Gamma$, our canonical construction does not produce closed forms on P_m, in general.

Finally we consider the case $n = 2$ for semisimple LIE groups. Using Theorem 3.65 we obtain that every closed invariant 2-form on F is G-transgressive. Thus we have:

Corollary 3.92 *If L is a* LIE *group action of a semisimple* LIE *group G on F, then every closed invariant 2-form $\phi_2 \in \mathcal{A}_2(F)_{\text{inv}}$ is G-transgressive and defines a unique cohomology class $[\phi_2^{\mathsf{A}}] \in H^2(B)$ for any bundle $B(M, F, G)$ that comes along with L. If $\chi_0^1 \in C^\infty(F)_{\text{equiv}} \otimes \text{Hom}(\mathfrak{g}, \mathbb{R})$ is the unique map with $d\chi_0^1 = -L_\bullet \phi_2$ and $\delta\chi_0^1 = L_\bullet^2 \phi_2$ according to* Theorem 3.65, *then ϕ_2^{A} is given by*

$$\phi_2^{\mathsf{A}} = \phi_2 v + (\chi_0^1 v) \bullet \mathrm{F} \in \mathcal{A}_2(B).$$

Thus for any $x \in U_\alpha$, $[i^*_{\alpha,x}]: H^2(B(M, F, G)) \to H^2_{\text{inv}}(F)$ is surjective.

In view of Proposition 3.37 we thus have proved:

Theorem 3.93 *If L is a* LIE *group action of a semisimple* LIE *group G on F, then every closed invariant $\phi_n \in \mathcal{A}_n(F)_{\text{inv}}$, $n \leq 2$, is G-transgressive and defines a unique cohomology class $[\phi_n^{\mathsf{A}}] \in H^n(B)$ for any bundle $B(M, F, G)$ that comes along with L. For any $x \in U_\alpha$, $[i^*_{\alpha,x}]: H^n(B(M, F, G)) \to H^n_{\text{inv}}(F)$ is surjective.*

If in addition, G is compact and connected then $H^n_{\text{inv}}(F) \cong H^n(F)$, thus for every bundle $B(M, F, G)$, $H^n(B)$ contains a subgroup isomorphic to $H^n(F)$ for $n \leq 2$.

In the following section we will show that the closed invariant 3-form ω_3 on SU_m is not SU_m-transgressive if we define L to be left multiplication. Thus Theorem 3.93 does not hold for $n = 3$.

In Corollary 3.88 we have proved that any G-transgressive form is 0-transgressive for all bundles with fiber and left action L. We presume that the reverse is also true for compact LIE groups: if a cohomology class in $H^n_{\text{inv}}(F)$, resp., the corresponding closed invariant form ϕ_n generates closed forms $\psi_n \in \mathcal{A}_n(B)$ and thus cohomology classes in $H^n(B)$ for all bundles B with fiber F and left action L such that ϕ_n is the restriction of ψ_n to the fibers, then ϕ_n is necessarily G-transgressive. Yet we are not able to prove this conjecture at the moment. Our conjecture is based on the following observation. If $\phi_n \in \mathcal{A}_n(F)_{\text{inv}}$ then we obtain for $(\delta\phi_n)_{\alpha\beta} = (\pi_\beta^*\phi_n - \pi_\alpha^*\phi_n)|_{U_{\alpha\beta}}$ in the ČECH-DE RHAM double complex using Corollary 2.113:

$$(\delta\phi_n)_{\alpha\beta} = \sum_{i=1}^{n} \frac{(-1)^{i(n-i)}}{i!}[\mathrm{pr}_F^*(L_\bullet^i\phi_n)] \bullet (g_{\beta\alpha} \circ \mathrm{pr}_{U_{\alpha\beta}})^*\Theta^L.$$

We concentrate on the term where $i = 1$. If ϕ_n lives to E_2 for any bundle that comes along with L, we can find differential forms $\xi_{\alpha\beta} \in \mathcal{A}(U_{\alpha\beta} \times F)$ such that

$$d\xi_{\alpha\beta} \in (-1)^n(L_\bullet\phi_n) \bullet (g_{\beta\alpha}^*\Theta^L) + \sum_{i=2}^{n} C^\infty(U_{\alpha\beta} \times F) \cdot (\mathcal{A}_{n-i}(F) \wedge \mathcal{A}_i(U_{\alpha\beta})),$$

where we have omitted the pullbacks pr_F^* and $\mathrm{pr}_{U_{\alpha\beta}}^*$. Since this is supposed to hold for any transition functions $g_{\alpha\beta}$ it looks as if this condition requires the existence of a $\chi^1 \in \mathcal{A}_{n-2}(F, \mathrm{Hom}(\mathfrak{g}, \mathbb{R}))$ with $d\chi^1 = -L_\bullet\phi_n$. In that case the forms $\xi_{\alpha\beta}$ are given by $\xi_{\alpha\beta} = (-1)^n\chi^1 \bullet (g_{\beta\alpha}^*\Theta^L)$. Analogously by concentrating on those forms where the factor in $\mathcal{A}(F)$ has highest degree, one should be able to prove the existence of all forms χ^i in (107).

Yet for $G = \mathbb{R}$ the conjecture is false. Define $L: \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}^k$ by $L(r, \vec{v}) = \vec{v} + r\vec{z}$ with $\vec{z} \in \mathbb{R}^k$. Then all forms ϕ_n with constant coefficients are closed and invariant. Because every bundle with structure group $\mathbb{R}$ is trivial, every ϕ_n defines a closed form $\mathrm{pr}_{\mathbb{R}^k}^*\phi_n$ on the bundle. But $L_\bullet\phi_1 \neq 0$ in general, e. g., for $\phi_1 \in \mathcal{A}_1(\mathbb{R}^k)$ defined by $\phi_1(\vec{v})(\vec{x}) := \langle\vec{v}, \vec{z}\rangle$ for all $\vec{x} \in \mathbb{R}^k$ and $\vec{v} \in T_{\vec{x}}(\mathbb{R}^k)$, where $L_\bullet\phi_1(\vec{x}) = \mathrm{id}_{\mathbb{R}}$. Thus ϕ_1 is not G-transgressive.

According to Corollary 2.25, every bundle with $G \cong \mathbb{R}^m$ is equivalent to the trivial bundle and thus our condition is automatically satisfied. For this reason, we can only expect to prove the reverse of Corollary 3.88 for compact LIE groups.

Finally let us derive the analogon to Theorem 3.87 for one-dimensional abelian LIE groups G (cf. Theorem 2.127):

Lemma 3.94 *If G is abelian with $\mathfrak{g} = E\mathbb{R}$, then $\phi_n \in \mathcal{A}(F)_{\text{inv}} \otimes V$ is G-transgressive iff $\chi^i \in \mathcal{A}_{n-2i}(F)_{\text{inv}} \otimes V$ exist for $0 \leq i \leq [n/2]$ such that with $\nu^i := \iota_{\mathcal{L}_E}\chi^i$ the following equations hold:*

$$\chi^0 = \phi_n, \quad -\nu^i = d\chi^{i+1} \quad \textit{for all} \quad 0 \leq i \leq [n/2] - 1 \quad \textit{and} \quad \nu^{[n/2]} = 0. \tag{108}$$

Theorem 3.95 *Let Γ be a connection on a principal bundle $P(M, G)$, where G is abelian with $\mathfrak{g} = E\mathbb{R}$, and let $B = P \times_G F$ be any associated bundle with left LIE group action $L: G \times F \to F$. If $\phi_n \in \mathcal{A}_n(F)_{\text{inv}} \otimes V$ is G-transgressive and $\chi_{n-2i}^i \in \mathcal{A}_{n-2i}(F)_{\text{inv}} \otimes V$ are given by (108), then with $\bar{\mathrm{F}} := \frac{1}{E}\pi^*\mathrm{F} \in \mathcal{A}_2(B)$,*

$$\phi_n^{\mathrm{A}} = \sum_{i=0}^{[n/2]} (\chi_{n-2i}^i v) \wedge \underbrace{\bar{\mathrm{F}} \wedge \cdots \wedge \bar{\mathrm{F}}}_{i} = \sum_{i=0}^{[n/2]} \underbrace{\bar{\mathrm{F}} \wedge \cdots \wedge \bar{\mathrm{F}}}_{i} \wedge (\chi_{n-2i}^i v) \in \mathcal{A}_n(B) \otimes V$$

is closed and adapted to Γ. Its restriction to the fibers is ϕ_n, i. e. for any $x \in U_\alpha$, $i_{\alpha,x}^*\phi_n^{\mathrm{A}} = \phi_n$.

3.7 Skyrmion Bundle and Yang-Mills Theories

As an application for the presented ideas, that combine the cohomology of a fiber bundle with connections on the bundle, we present the skyrmion bundle in theoretical nuclear physics, as discussed in detail in [2] and [14]. To this purpose, let us first introduce the ungauged SKYRME model, that treats the purely hadronic case.

The SKYRME model[15] in theoretical nuclear physics is an effective field theory modelled to describe the low energy limit of quantum chromodynamics (QCD) and related to QCD by its underlying "chiral" symmetry, cf. below. Let N_F denote the number of flavors in QCD and let X_a, $1 \leq a \leq N_F^2 - 1$ denote generators of su_{N_F}, i. e., $X_a = -(X_a)^\dagger \in \mathbb{C}^{N_F \times N_F}$ and $\mathrm{Tr}(X_a) = 0$. Then defined by

$$U = \exp(\sum_a \pi^a X_a),$$

the meson fields $\pi^a \in C^\infty(M)$ generate differentiable functions $U: M \to \mathrm{SU}_{N_F}$ from space-time M to the special unitary group SU_{N_F}. The vacuum is represented by the unit matrix $\mathbb{1} \in \mathrm{SU}_{N_F}$. Requiring $\pi^a(r) \to 0$ and thus $U(r) \to \mathbb{1}$ for $r \to \infty$ one can compactify EUCLIDIAN space $\mathbb{R}^3$, resp., space-time $\mathbb{R}^4$, so that the meson fields constitute functions

$$U: \mathbb{R}_{(t)} \times \mathbf{S}^3 \to \mathrm{SU}_{N_F}, \quad \text{resp.,} \quad U: \mathbf{S}^4 \to \mathrm{SU}_{N_F}.$$

The SKYRME model requires the knowledge of the homotopy groups of SU_m, which we have not introduced so far. For $n \in \mathbb{N}_0$, let $\pi_n(\mathrm{SU}_m)$ denote the n-th homotopy group of SU_m. Its elements are the equivalence classes of homotopic maps from $\mathbf{S}^n$ to SU_m. Homotopy groups are topological invariants. They are abelian for $n \geq 2$. BOTT's periodicity theorem yields that

$$\pi_{2n}(\mathrm{SU}_m) = \pi_{2n}(\mathrm{U}_m) = 0, \ \pi_{2n+1}(\mathrm{SU}_m) \cong \pi_{2n+1}(\mathrm{U}_m) \cong \mathbb{Z}, \quad m > n \in \mathbb{N}. \tag{109}$$

[16] exhibits explicit representatives for the elements of $\pi_{2n+1}(\mathrm{SU}_m)$, $m > n < 3$.

Recall the left and right invariant currents $L, R \in \mathcal{A}_1(\mathrm{SU}_n, \mathrm{End}(\mathbb{C}^n))$ and the differential forms λ_k^Q, ρ_k^Q and ω_k from Section 1.4 (cf. (32)). For coordinates x^μ, $0 \leq \mu \leq 3$, we have $L = \sum_{\mu=0}^3 L_\mu dx^\mu$ with $L_\mu := U^{-1}\partial_\mu U$ (and analogously $R_\mu := \partial_\mu U\, U^{-1}$). The meson fields obey the field equations derived as EULER-LAGRANGE equations from a lagrangian $\mathcal{L}(U, dU)$ by variation of the action integral $\Gamma(U) = \int_{\mathbf{S}^4} \mathcal{L}\, dV$. The latter splits into two parts: the nonanomalous action

$$\Gamma_{NA}(U) = \int_{\mathbf{S}^4} \left(-\frac{f_\pi^2}{4} \sum_{\mu=0}^{3} \mathrm{Tr}(L_\mu\, L^\mu) + \frac{1}{32a^2} \sum_{\mu,\nu=0}^{3} \mathrm{Tr}([L_\mu, L_\nu][L^\mu, L^\nu]) \right) dV, \tag{110}$$

where f_π is the pion decay constant and a^{-2} a coupling constant, and the WESS-ZUMINO term [17] (N_C is the number of colors in QCD)

$$\Gamma_{WZ}(U) = \frac{iN_C}{240\pi^2} \int_{D^5} (U')^* \omega_5, \tag{111}$$

that describes the anomalous processes of QCD. Now $\mathcal{A}_5(\mathrm{SU}_2, \mathbb{C}) = 0$, so the WESS-ZUMINO term only contributes to the total action for $N_F \geq 3$. In that case one uses $\pi_4(\mathrm{SU}_{N_F}) = 0$ and extends U to a differentiable map $U' : D^5 \to \mathrm{SU}_3$ from a five-dimensional disk D^5 whose boundary ∂D^5 is space-time $\mathbb{S}^4$.

The action is invariant under all chiral transformations $U \mapsto g_L U g_R^{-1}$ with $g_L, g_R \in \mathrm{SU}_{N_F}$. This symmetry is spontaneously broken: the vacuum state is only invariant under diagonal SU_{N_F} transformations $U \mapsto VUV^{-1}$. (One can add further chiral invariant terms of fourth order to the nonanomalous lagrangian, cf. [14])

$$\frac{1}{32f^2} \sum_{\mu,\nu=0}^{3} \mathrm{Tr}(\{L_\mu, L_\nu\}\{L^\mu, L^\nu\}) + \frac{1}{32g^2} \sum_{\mu,\nu=0}^{3} \mathrm{Tr}(\partial_\mu L_\nu\, \partial^\mu L^\nu), \tag{112}$$

with coupling constants f^2 and g^2 and anticommutator braces $\{,\}$, or — in order to take the finite pion mass M_π into account — a mass term, breaking the axial symmetry

$$\frac{f_\pi^2 M_\pi^2}{2} \mathrm{Tr}(U - \mathbb{1}), \qquad \text{resp.,} \qquad \frac{f_\pi^2 M_\pi^2}{2(m_u + m_d)} \mathrm{Tr}(M_q(U + U^\dagger - 2 \cdot \mathbb{1}))$$

for $N_F = 2$, resp., 3, where $M_q = \mathrm{diag}(m_u, m_d, m_s)$ is the quark mass matrix, and m_u, m_d, m_s denote the masses of up, down, and strange quarks, respectively.

Baryons appear as topological soliton solutions — as "skyrmions" — of the meson fields. (Topological soliton solutions mean solutions of the field equations that carry nontrivial topological invariants.) The number B of baryons described by a given mesonic field configuration U can be computed by an integration over the space manifold:

$$B(U) = \int_{\mathbb{S}^3} -\frac{1}{24\pi^2} U^* \omega_3 \tag{113}$$

Compactification of space-time is crucial for the existence of nontrivial soliton solutions. Normally there is no guarantee that the integral in (113) is an integer, but for spheres we have the following theorem (cf. BOTT, SEELEY [18, p. 237]):

Theorem 3.96 *For every continuous map $U : \mathbb{S}^{2n-1} \to \mathrm{U}_m$ the integral*

$$n(U) = \int_{\mathbb{S}^{2n-1}} \left(\frac{i}{2\pi}\right)^n \frac{(n-1)!}{(2n-1)!} U^* \omega_{2n-1}$$

is an integer. The assignment $[U] \mapsto n(U) : \pi_{2n-1}(\mathrm{U}_m) \to \mathbb{Z}$ is an isomorphism for $m \geq n$.

Recall from Theorem 3.59 that $i^n \omega_{2n-1}$ are the generators of the real valued DE RHAM cohomology of SU_n. By Theorem 3.96 we are able to identify the normalized forms $\left(\frac{i}{2\pi}\right)^n \frac{(n-1)!}{(2n-1)!} \omega_{2n-1}$ with the generators of the integer valued cohomology $H^*(\mathrm{U}_m, \mathbb{Z})$, resp., $H^*(\mathrm{SU}_m, \mathbb{Z})$. At any time t the meson fields form C^∞-functions $U(t, \cdot) : \mathbb{S}^3 \to \mathrm{SU}_{N_F}$ and thus represent elements of the homotopy groups $\pi_3(\mathrm{U}_{N_F}) \cong \mathbb{Z}$ for $N_F \geq 2$. Although these fields need not be constant in time, continuity forces them to change only within their equivalence class of homotopic functions. Thus the integer characterizing the homotopy class is a topological invariant, the "topological charge", that can be interpreted as the number of baryons and be computed by (113).

The vacuum map represents the zero element, and so $B(U \equiv 1\!\!1) = 0$. For proton and neutron we have $B = 1$, for their antiparticles $B = -1$. Annihilation of proton and antiproton corresponds to the "addition" of their maps within the homotopy group and generates a mesonic field of topological charge $B = 0$.

We only note that the topological quantization of the coupling constant $\lambda = \frac{iN_C}{240\pi^2}$ in (111) is also a consequence of Theorem 3.96, and of the requirement that for any extension U' the result has to be unique.

So much for the ungauged SKYRME model. Now we want to treat interactions with electromagnetic fields. We already stated in the previous chapter (cf. Note 2.63) that the electromagnetic gauge potentials $\mathrm{A}^\alpha = \sum_{\mu=0}^{3} \mathrm{A}^\alpha_\mu dx^\mu$ and the gauge fields $\mathrm{F}^\alpha = \frac{1}{2}\sum_{\mu,\nu=0}^{3} \mathrm{F}^\alpha_{\mu\nu} dx^\mu \wedge dx^\nu$ can conveniently be described by a so-called MAXWELL connection on a principal bundle $P(M, G_{\mathrm{em}})$, where $G_{\mathrm{em}} = 2g_D \cdot \mathbb{S}^1 \cong \mathrm{U}_1$ is the electromagnetic gauge group. e and g_D denote the electric, resp., magnetic unit charge, we have $2eg_D = 1$. The forms A^α and F^α determine the connection 1-form ω^Γ, resp., the curvature 2-form Ω^Γ according to (64) and (65). Recall that G_{em} is the only possible choice for a connected LIE group that allows for the existence of nontrivial bundles and, on the other hand, guaranties that the F^α define a global real valued form (cf. the discussion that followed Theorem 2.126).

If we are interested in the special case of a single magnetic monopole that rests in the origin of the space manifold such that $M \cong \mathbb{R}_{(t)} \times \mathbb{R}^+_{(r)} \times \mathbb{S}^2$, then we obtain a countable number of nonequivalent principal bundles, characterized by the magnetic charge $m \in \mathbb{Z}$ of the monopole:

$$P_m(\mathbb{R}_{(t)} \times \mathbb{R}^+_{(r)} \times \mathbb{S}^2, G_{\mathrm{em}}) \cong P_m(\mathbb{S}^2, G_{\mathrm{em}}) \times \mathbb{R}_{(t)} \times \mathbb{R}^+_{(r)}, \qquad m \in \mathbb{Z},$$

where $P_m(\mathbb{S}^2, G_{\mathrm{em}}) \cong P_m(\mathbb{S}^2, \mathbb{S}^1)$ is the only topologically interesting part. In fact, the principal bundles $P_m(\mathbb{S}^2, \mathbb{S}^1)$ are the bundles we listed in (56).

The electromagnetic gauge field F (sometimes also called FARADAY 2-form F) is connected with the electric and the magnetic field in the following way, cf. EGUCHI ET AL. [19], NASH, SEN [20] or ABRAHAM ET AL. [21]: Recall from Note 2.109 that on the pseudo-Riemannian manifold M (equipped with the LORENTZIAN metric of signature $(+---)$) we have the HODGE star operator $*\colon \mathcal{A}_p(M) \to \mathcal{A}_{4-p}(M)$ and the co-differentiation $\boldsymbol{\delta}\colon \mathcal{A}_p(M) \to \mathcal{A}_{p-1}(M)$, which is a differential operator on $\mathcal{A}(M)$. If we define

$$\begin{aligned}
\text{electric 1-form E,} \quad \mathrm{E}|_U &= \sum\nolimits_{\mu=0}^{3} \mathrm{E}_\mu dx^\mu, \mathrm{E}_\mu := (0, \vec{E}),\\
\text{magnetic 1-form B,} \quad \mathrm{B}|_U &= \sum\nolimits_{\mu=0}^{3} \mathrm{B}_\mu dx^\mu, \mathrm{B}_\mu := (0, \vec{B}),\\
\text{source 1-form J,} \quad \mathrm{J}|_U &= \sum\nolimits_{\mu=0}^{3} \mathrm{J}_\mu dx^\mu, \mathrm{J}_\mu := (-\rho, \vec{j}),
\end{aligned}$$

then $\mathrm{F} = \frac{1}{2}[(\mathrm{E} \wedge dt - dt \wedge \mathrm{E}) + *(\mathrm{B} \wedge dt - dt \wedge \mathrm{B})]$ and MAXWELL'S equations

$$\begin{aligned} \vec{\nabla} \times \vec{E} + \tfrac{\partial}{\partial t}\vec{B} &= 0, & \vec{\nabla} \times \vec{B} - \tfrac{\partial}{\partial t}\vec{E} &= 4\pi\vec{j}, \\ \vec{\nabla} \cdot \vec{B} &= 0, & \vec{\nabla} \cdot \vec{E} &= 4\pi\rho, \end{aligned}$$

simply read $d\mathrm{F} = 0$ and $\boldsymbol{\delta}\mathrm{F} = -4\pi\mathrm{J}$. The continuity equation $\vec{\nabla} \cdot \vec{j} + \frac{\partial}{\partial t}\rho = 0$ reads $\boldsymbol{\delta}\mathrm{J} = 0$ and is a consequence of $\boldsymbol{\delta}^2 = 0$.

Now we construct a bundle $B(M, \mathrm{SU}_{N_F}, G_{\mathrm{em}})$ associated with $P(M, G_{\mathrm{em}})$ in order to treat interactions between electromagnetic fields and meson fields: meson fields are considered as global sections in this associated "skyrmion bundle" B. The left action of G_{em} on SU_{N_F} is given by the inner automorphisms

$$L(g, U) \ := \ e^{-iegQ} U e^{+iegQ},$$

which do not affect the vacuum state being diagonal symmetry operations. Q is the $N_F \times N_F$-matrix containing the quark charges in units of e (again $N_F = 2$, resp., 3)

$$Q = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & -\frac{1}{3} \end{pmatrix}, \qquad \text{resp.,} \qquad Q = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}.$$

From a physical point of view it is obvious that any coupling between baryons and electromagnetic fields has to involve these charges. From a mathematical point of view we observe that Q's eigenvalues $\lambda_i \in \mathbb{R}$ obey the conditions $\lambda_i - \lambda_j \in \mathbb{Z}$ and $\gcd\{\lambda_i - \lambda_j\} = 1$, which guarantee that the action is well-defined and effective. Under a change of bundle charts we have

$$U^\alpha(x) = L(g_{\alpha\beta}(x), U^\beta(x)) = e^{-ieg_{\alpha\beta}(x)Q}\, U^\beta(x)\, e^{+ieg_{\alpha\beta}(x)Q}.$$

So not only vacuum $U \equiv \mathbb{1}$ is a global section but every $U(x) = e^{i\chi(x)Q}$ with a differentiable map $\chi: M \to \mathbf{S}^1$. Observe that if we include SU_{N_F} into $\mathbb{C}^{N_F \times N_F}$, then L also defines a representation of G_{em} on the vector space $\mathbb{C}^{N_F \times N_F}$. For the induced representation $l: 2g_D\mathbb{R} \to \mathbb{C}^{N_F \times N_F}$ according to (43) we obtain

$$l(X, U) = \mathcal{L}_X(U) = -ieX\,[Q, U] \qquad \text{for all} \quad X \in 2g_D\mathbb{R},\ U \in \mathbb{C}^{N_F \times N_F}. \tag{114}$$

Evaluation of our results in Section 2.5 (e. g., confer (89) and Lemma 2.116) yields

$$\begin{aligned} dU^\alpha &= l(g_{\alpha\beta}, dU^\beta - ie\, dg_{\alpha\beta}\,[Q, U^\beta]), \\ (dU^\alpha)h &= ie\,\mathrm{A}^\alpha\,[Q, U^\alpha], \quad (dU^\alpha)v = dU^\alpha - ie\,\mathrm{A}^\alpha\,[Q, U^\alpha] \quad \text{and} \\ \nabla^\alpha_\mu U^\alpha &= \partial_\mu U^\alpha - ie\mathrm{A}^\alpha_\mu\,[Q, U]. \end{aligned}$$

Moreover, since the forms ρ^Q_k, λ^Q_k and ω_k are invariant, we obtain the following lemma from Theorem 2.127:

Lemma 3.97 *$\rho_l^Q v$, $\lambda_l^Q v$, $\omega_{2l+1}v$, $\rho_1^Q + \lambda_1^Q$ and $\rho_{2l}^Q - \lambda_{2l}^Q$ for $l \in \mathbb{N}_0$ are global vertical forms on B, and we have:*

$$\begin{aligned}
(\rho_{2l}^Q - \lambda_{2l}^Q)^\alpha &= (\rho_{2l}^Q - \lambda_{2l}^Q)^\beta = (\rho_{2l}^Q - \lambda_{2l}^Q)^\alpha v,\\
(\rho_1^Q + \lambda_1^Q)^\alpha &= (\rho_1^Q + \lambda_1^Q)^\beta = (\rho_1^Q + \lambda_1^Q)^\alpha v,\\
(\rho_3^Q + \lambda_3^Q)^\alpha &= (\rho_3^Q + \lambda_3^Q)^\beta - 2ie\, dg_{\alpha\beta} \wedge \mathrm{Tr}\left[Q^2(R^2 - L^2) + Q\, dU^\dagger \wedge Q\, dU\right]^\beta,\\
(\rho_3^Q + \lambda_3^Q)^\alpha v &= (\rho_3^Q + \lambda_3^Q)^\alpha - 2ie\, \mathrm{A}^\alpha \wedge \mathrm{Tr}\left[Q^2(R^2 - L^2) + Q\, dU^\dagger \wedge Q\, dU\right]^\alpha,\\
(\rho_{2l+1}^Q + \lambda_{2l+1}^Q)^\alpha &= (\rho_{2l+1}^Q + \lambda_{2l+1}^Q)^\beta - 2ie\, dg_{\alpha\beta} \wedge \sum\nolimits_{j=1}^{l} \mathrm{Tr}(QUL^{2j-1}QL^{2l-2j+1}U^\dagger)^\beta\\
&\quad -ie\, dg_{\alpha\beta} \wedge \sum\nolimits_{j=0}^{l} \mathrm{Tr}(QR^{2j}QR^{2l-2j} - QL^{2j}QL^{2l-2j})^\beta,\\
\omega_{2l+1}^\alpha &= \omega_{2l+1}^\beta - (2l+1)ie\, dg_{\alpha\beta} \wedge (\rho_{2l}^Q - \lambda_{2l}^Q)^\beta,\\
\omega_{2l+1}^\alpha v &= \omega_{2l+1}^\alpha - (2l+1)ie\, \mathrm{A}^\alpha \wedge (\rho_{2l}^Q - \lambda_{2l}^Q)^\alpha.
\end{aligned}$$

For calculations we need the action integral and the topological charge. Both consist of forms on B, whose pullbacks by the mesonic sections $U: M \to B$ are integrated over space-time, resp., the space manifold only. For the nonanomalous action, our task is easy: we replace the partial derivatives by covariant derivatives. Defining $\tilde{L}_\mu^\alpha := (U^\alpha)^\dagger \nabla_\mu^\alpha U^\alpha$, we get for the lagrangian from (110) and (112):

$$\begin{aligned}
\mathcal{L}_{NA}(\mathrm{A}) &= -\frac{f_\pi^2}{4}\sum_{\mu=0}^{3} \mathrm{Tr}(\tilde{L}_\mu \tilde{L}^\mu) + \frac{1}{32a^2}\sum_{\mu,\nu=0}^{3} \mathrm{Tr}([\tilde{L}_\mu, \tilde{L}_\nu][\tilde{L}^\mu, \tilde{L}^\nu])\\
&\quad + \frac{1}{32f^2}\sum_{\mu,\nu=0}^{3} \mathrm{Tr}(\{\tilde{L}_\mu, \tilde{L}_\nu\}\{\tilde{L}^\mu, \tilde{L}^\nu\}) + \frac{1}{32g^2}\sum_{\mu,\nu=0}^{3} \mathrm{Tr}(\nabla_\mu \tilde{L}_\nu\, \nabla^\mu \tilde{L}^\nu),
\end{aligned}$$

where we omitted the index α since covariant derivation yields $\mathcal{L}_{NA}^\alpha = \mathcal{L}_{NA}^\beta \in C^\infty(B)$. A mass term may also be included ($[M_g, Q] = 0$). Combined with the pullback of the volume form $\pi^* dV \in \mathcal{A}_4(B)h$ we get

$$\Gamma_{NA}(U, \mathrm{A}) = \int_M U^*(\mathcal{L}_{NA}(\mathrm{A})\, \pi^* dV) = \int_M \mathcal{L}_{NA}(U, \mathrm{A})\, dV. \tag{115}$$

For the anomalous action and the topological charge, the old difficulty arises that we have to extend the forms ω_3 and ω_5 to the bundle. Several approaches "by trial and error" have been made to "generalize" ω_3 and ω_5, cf. CALLAN, WITTEN [22], KAYMAKCALAN ET AL. [23] or PAK, ROSSI [24]. In terms of the language we are using, we would like to obtain differential forms ω_3^{A} and ω_5^{A} that are adapted to the MAXWELL connection. Thus we will examine whether the forms ω_3 and ω_5 are G_{em}-transgressive.

This is indeed the case. According to Lemma 3.94 we have to find $\chi_{n-2i}^i \in \mathcal{A}_n(\mathrm{SU}_{N_F}, \mathbb{C})$ and $\nu_{n-2i-1}^i = \imath_{\mathcal{L}_{2g_D}} \chi_{n-2i}^i$ that obey (108) for $\phi = \omega_3$, resp., $\phi = \omega_5$. From Lemma 3.97 we conclude that for $\phi = \omega_{2l+1}$, we have $\nu_{2l}^0 = -(2l+1)i(\rho_{2l}^Q - \lambda_{2l}^Q)$. Now (37) yields that $\rho_{2l}^Q - \lambda_{2l}^Q = d(\rho_{2l-1}^Q + \lambda_{2l-1}^Q)$, so $\chi_{2n-1}^1 = (2l+1)i(\rho_{2l-1}^Q + \lambda_{2l-1}^Q)$. For ω_3 we are already done, since χ_1^1 is global and vertical due to Lemma 3.97: $\nu_0^1 = 0$.

For χ_3^1, again Lemma 3.97 yields $\nu_2^2 = -10i^2 \operatorname{Tr}\left[Q^2(R^2 - L^2) + Q\,dU^\dagger \wedge Q\,dU\right]$. One easily verifies that

$$\chi_1^2 = 10i^2(\rho_1^{Q^2} + \lambda_1^{Q^2}) + 5i^2 \operatorname{Tr}(Q\,dU\,QU^\dagger - QUQ\,dU^\dagger) + ri^2 d\operatorname{Tr}(QU^\dagger QU), \quad r \in \mathbb{R},$$

is an admissible choice and that $\nu_0^2 = 0$, thus χ_1^2 is global and vertical. For physical reasons (parity invariance, cf. [23]), we put $r = 0$. We thus obtain from Theorem 3.95:

Theorem 3.98 *ω_3 and ω_5 are G_{em}-transgressive and generate cohomology groups isomorphic to $\mathbb{R}$ for any skyrmion bundle. Representatives for the generated cohomology groups, that are adapted to the* MAXWELL *connection, are*

$$\begin{aligned}
\omega_3^{\mathrm{A}} &= \omega_3 v + ie\,\mathrm{F} \wedge \chi_1^1 v = [\omega_3^\alpha - 3ie\,\mathrm{A}^\alpha \wedge (\rho_2^Q - \lambda_2^Q)] + 3ie\,\mathrm{F} \wedge (\rho_1^Q + \lambda_1^Q),\\
\omega_5^{\mathrm{A}} &= \omega_5 v + ie\,\mathrm{F} \wedge \chi_3^1 v + (ie)^2 \mathrm{F} \wedge \mathrm{F} \wedge \chi_1^2 v = [\omega_5^\alpha - 5ie\,\mathrm{A}^\alpha \wedge (\rho_4^Q - \lambda_4^Q)]\\
&\quad +5ie\,\mathrm{F} \wedge \{(\rho_3^Q + \lambda_3^Q)^\alpha - 2ie\,\mathrm{A}^\alpha \wedge \operatorname{Tr}[Q^2(R^2 - L^2) + Q\,dU^\dagger \wedge Q\,dU]^\alpha\}\\
&\quad +5(ie)^2\,\mathrm{F} \wedge \mathrm{F} \wedge [2(\rho_1^{Q^2} + \lambda_1^{Q^2})^\alpha + \operatorname{Tr}(Q\,dU\,QU^\dagger - QUQ\,dU^\dagger)^\alpha].
\end{aligned}$$

Analogous to (113), the integral over ω_3^{A} computes the number of baryons in the skyrmion bundle:

$$B^{\mathrm{A}}(U) = \int_{S^3} -\frac{1}{24\pi^2} U^* \omega_3^{\mathrm{A}},$$

whereas the integral over ω_5^{A} is the WESS-ZUMINO term for the skyrmion bundle

$$\Gamma_{WZ}(U, \mathrm{A}) = \frac{iN_C}{240\pi^2} \int_{D^5} (U')^* \omega_5^{\mathrm{A}},$$

completing $\Gamma(U, \mathrm{A}) = \Gamma_{NA}(U, \mathrm{A}) + \Gamma_{WZ}(U, \mathrm{A})$ with $\Gamma_{NA}(U, \mathrm{A})$ from (115). Note that Theorem 3.96 does not apply any more for the skyrmion bundle, so there is no guarantee that $B^{\mathrm{A}}(U)$ is an integer nor that it is conserved. This allows for the treatment of baryon number violating processes within the skyrmion bundle, such as the monopole induced proton decay, where the topological charge may vanish through the monopole singularities of the manifold, cf. [2], [22] and CHEMTOB [25].

Finally let us compute the cohomology for the bundles $B_m(M, \mathrm{SU}_{N_F}, G_{\mathrm{em}}) \cong B_m(\mathrm{S}^2, \mathrm{SU}_{N_F}, G_{\mathrm{em}}) \times \mathbb{R}_{(t)} \times \mathbb{R}^+_{(r)}$ as an application of spectral sequences. We have $H^\bullet(B_m(M, \mathrm{SU}_{N_F}, G_{\mathrm{em}})) \cong H^\bullet(B_m(\mathrm{S}^2, \mathrm{SU}_{N_F}, G_{\mathrm{em}}))$, and since S^2 is simply connected, LERAY's theorem yields

$$E_2^{p,q} \cong H^p(\mathrm{S}^2) \otimes H^q(\mathrm{SU}_{N_F}) = \begin{cases} H^q(\mathrm{SU}_{N_F}) & \text{for } p = 0, 2,\\ 0 & \text{otherwise.} \end{cases}$$

We want to prove $E_2 = E_\infty$. Because E_2 only consists of the two columns we merely have to show $D_2 = 0$. Thus let us compute the zig-zag for the generators ω_{2l+1} of $H^\bullet(SU_{N_F})$. Using the local trivializations we inject ω_{2l+1} into $C^0(\pi^{-1}\mathfrak{U}, \mathcal{A}_{2l+1})$, so ξ_0 in Figure 3.7 is given by $(\xi_0)_\alpha = \omega_{2l+1}^\alpha$. Now for $U_{\alpha\beta} \neq \emptyset$ by Lemma 3.97

$$\begin{aligned}
(\delta\xi_0)_{\alpha\beta} &= (\omega_{2l+1}^\beta - \omega_{2l+1}^\alpha)|_{U_{\alpha\beta}} = (2l+1)ie\,dg_{\alpha\beta} \wedge (\rho_{2l}^Q - \lambda_{2l}^Q),\\
&= d[(2l+1)ie\,dg_{\beta\alpha} \wedge (\rho_{2l-1}^Q + \lambda_{2l-1}^Q)^{\alpha/\beta}] = (d\xi_1)_{\alpha\beta}, \quad \text{where}\\
(\xi_1)_{\alpha\beta} &:= ie\,dg_{\beta\alpha} \wedge (\chi_{2l-1}^1)^{\alpha/\beta} := (2l+1)ie\,dg_{\beta\alpha} \wedge (\rho_{2l-1}^Q + \lambda_{2l-1}^Q)^{\alpha/\beta}.
\end{aligned}$$

Here α/β indicates that one may use both trivializations. Using Lemma 3.97 again and $dg_{\gamma\alpha} = dg_{\gamma\beta} + dg_{\beta\alpha}$ on $U_{\alpha\beta\gamma} \neq \emptyset$ we have

$$\begin{aligned}(\delta\xi_1)_{\alpha\beta\gamma} &= [(\xi_1)_{\beta\gamma} - (\xi_1)_{\alpha\gamma} + (\xi_1)_{\alpha\beta}]|_{U_{\alpha\beta\gamma}} = ie\, dg_{\gamma\beta} \wedge [(\chi^1_{2l-1})^\beta - (\chi^1_{2l-1})^\alpha] \\ &= (2l+1)(ie)^2\, dg_{\gamma\beta} \wedge dg_{\alpha\beta} \wedge \Big[\sum_{j=0}^{l-1} \mathrm{Tr}(QR^{2j}QR^{2l-2j-2} - QL^{2j}QL^{2l-2j-2}) \\ &\qquad + 2\sum_{j=1}^{l-1} \mathrm{Tr}(QUL^{2j-1}QL^{2l-2j-1}U^\dagger)\Big]^{\alpha/\beta/\gamma} = (-d\xi_2)_{\alpha\beta\gamma} \quad \text{with}\end{aligned}$$

$$(\xi_2)_{\alpha\beta\gamma} = (ie)^2\, dg_{\gamma\beta} \wedge dg_{\beta\alpha} \wedge (\chi^2_{2l-3})^{\alpha/\beta/\gamma},$$

and for $\chi^2_{2l-3} \in \mathcal{A}_{2l-3}(\mathrm{SU}_{N_F})$ we may take using Corollary 1.84:

$$\begin{aligned}\chi^2_{2l-3} = (2l+1)\Big[&2(\rho^{Q^2}_{2l-3} + \lambda^{Q^2}_{2l-3}) + \sum_{j=1}^{l-2} \mathrm{Tr}(QR^{2j-1}QR^{2l-2j-2} + QL^{2j-1}QL^{2l-2j-2}) \\ &+ \sum_{j=1}^{l-1} \mathrm{Tr}(QUL^{2j-1}QL^{2l-2j-2}U^\dagger + QUL^{2j-2}QL^{2l-2j-1}U^\dagger)\Big].\end{aligned}$$

We terminate at this point. $D_2[\omega_{2l+1}]_2 = [\delta\xi_1]_2$, so whenever $\delta\xi_1 = 0$, ω_{2n-1} lives to E_∞. In any event this is the case if for our cover $dg_{\gamma\beta} \wedge dg_{\beta\alpha} = 0$ for all combinations of α, β and γ. E. g., this holds for the special case of a single monopole, where we only have two nontrivial transition functions $g_{+-} = -g_{-+}$. We have found:

Theorem 3.99 *The cohomology of the skyrmion bundle $B_m(M, \mathrm{SU}_{N_F}, G_{\mathrm{em}})$ is independent of the monopole charge mg_D, but isomorphic to the cohomology of $M \times \mathrm{SU}_{N_F}$:*

$$H^k(B_m(M, \mathrm{SU}_{N_F}, G_{\mathrm{em}})) \cong \bigoplus_{p+q=k} H^p(M) \otimes H^q(\mathrm{SU}_{N_F}), \qquad k \in \mathbb{N}_0.$$

The same holds for all skyrmion bundles of manifolds, where a good cover $\mathfrak{U} = \{U_\alpha\}_{\alpha\in A}$ exists such that $dg_{\gamma\beta} \wedge dg_{\beta\alpha} = 0$ for all $\alpha, \beta, \gamma \in A$.

Applications to non-abelian YANG-MILLS theories are also possible. E. g., instead of $G \cong \mathbb{S}^1$ and $F = \mathrm{SU}_n$ take $G = \mathrm{U}^L_n \times \mathrm{U}^R_n$ and $F = \mathrm{U}_n$ with $L_{(g_L, g_R)}(U) = g_L U g_R^{-1}$. As a generalization of (114) we have for all $(X_L, X_R) \in \mathfrak{u}^L_n \oplus \mathfrak{u}^R_n$:

$$l((X_L, X_R), U) = \mathcal{L}_{(X_L, X_R)}(U) = X_L U - U X_R \qquad \text{for all} \quad (X_L, X_R) \in \mathfrak{u}^L_n \oplus \mathfrak{u}^R_n.$$

Let $\mathrm{A}^\alpha = (\mathrm{A}^\alpha_L, \mathrm{A}^\alpha_R)$ and $\mathrm{F}^\alpha = (\mathrm{F}^\alpha_L, \mathrm{F}^\alpha_R) \in \mathcal{A}(U_\alpha, \mathfrak{u}^L_n \oplus \mathfrak{u}^R_n)$ define the connection Γ on $P(M, G)$. Then the covariant differentiation is given by $\nabla_\mu U = \partial_\mu U + \mathrm{A}_{L,\mu} U - U \mathrm{A}_{R,\mu}$ and analogously $dUv = dU + \mathrm{A}_L U - U\mathrm{A}_R$, so

$$Lv = L + U^\dagger \mathrm{A}_L U - \mathrm{A}_R, \quad Rv = R + \mathrm{A}_L - U\mathrm{A}_R U^\dagger \quad \text{and} \quad \omega_1 v = \omega_1 + \mathrm{Tr}(\mathrm{A}_L - \mathrm{A}_R)$$

since $L_\bullet \omega_1 = \mathrm{Tr}(\pi_L - \pi_R)$ with the projections $\pi^{L/R}\colon \mathfrak{g} = \mathfrak{u}^L_n \oplus \mathfrak{u}^R_n \to \mathfrak{u}^{L/R}_n$. Thus for any LIE subgroup $H < G$, the closed invariant form ω_1 is H-transgressive iff $\mathrm{Tr}(X_L - X_R) = 0$ for all $(X_L, X_R) \in \mathfrak{h}$. E. g., we could choose a subgroup of the diagonal $D_n = \mathrm{U}^L_n \times \mathrm{U}^L_n$ in G such that $g_L = g_R$ for all $(g_L, g_R) \in D_n$. (Note that this is the case for the skyrmion bundle.) Or we could choose $H = \mathrm{SU}^L_n \times \mathrm{SU}^R_n$, resp., a subgroup of H. Since $\mathrm{SU}_n^L \times \mathrm{SU}_n^R$ is semisimple for $n > 2$, the form ω_1 is then necessarily H-transgressive by Theorem 3.93.

For ω_3 we obtain $L_\bullet\omega_3 = 3\,\mathrm{Tr}(R^2\pi^L - L^2\pi^R)$, thus

$$\chi_1^1 := -3\,\mathrm{Tr}(R\pi^L + L\pi^R) \in \mathcal{A}_1(\mathrm{U}_n, \mathrm{Hom}(\mathfrak{g}, \mathbb{C}))$$

obeys $d\chi_1^1 = -L_\bullet\omega_3$ due to Corollary 1.84. Omitting the symmetrization $\vee$, we compute $L_\bullet^\vee\chi_1^1 = 3\,\mathrm{Tr}(\pi^R\pi^R - \pi^L\pi^L)$, i. e.,

$$(L_\bullet^\vee\chi_1^1)((X_L, X_R), (Y_L, Y_R)) = 3\,\mathrm{Tr}(X_R Y_R - X_L Y_L) \neq 0.$$

Thus ω_3 is not G-transgressive. In fact, let $\tilde{\chi}_1^1 \in \mathcal{A}_1(U_n, \mathrm{Hom}(\mathfrak{g}, \mathbb{C}))_{\mathrm{equiv}}$ with $d\tilde{\chi}_1^1 = -L_\bullet\omega_3$. Then $\xi_1^1 := \tilde{\chi}_1^1 - \chi_1^1 \in \mathcal{A}_1(U_n, \mathrm{Hom}(\mathfrak{g}, \mathbb{C}))_{\mathrm{equiv}}$ with $d\xi_1^1 = 0$. Since $H^1(\mathrm{SU}_n) = 0$, we find $\xi_0^1 \in C^\infty(U_n, \mathrm{Hom}(\mathfrak{g}, \mathbb{C}))$ with $d\xi_0^1 = \xi_1^1$. In fact, we may choose ξ_0^1 equivariant, because SU_n is compact, cf. (40). But then for all $X, Y \in \mathfrak{g}$,

$$\begin{aligned} L_\bullet^\vee\xi_1^1(X,Y) &= (\imath_{\mathcal{L}_X} d\xi_0^1)(Y) + (\imath_{\mathcal{L}_Y} d\xi_0^1)(X) = (L_{\mathcal{L}_X}\xi_0^1)(Y) + (L_{\mathcal{L}_Y}\xi_0^1)(X) \\ &= \mathcal{L}_X(\xi_0^1)(Y) + \mathcal{L}_Y(\xi_0^1)(X) = \xi_0^1([Y,X]) + \xi_0^1([X,Y]) = 0. \end{aligned}$$

Thus $(L_\bullet^\vee\tilde{\chi}_1^1) = (L_\bullet^\vee\chi_1^1) \neq 0$. Since ω_3 is not G-transgressive, the generated form

$$\omega_3^A = \omega_3 v + \chi_1^1 v \bullet \mathrm{F} \in \mathcal{A}_3(B(M, \mathrm{U}_n, G), \mathbb{C})$$

is not closed in general: $d\omega_3^A = (L_\bullet^\vee\chi_1^1)v \bullet \mathrm{F} = (L_\bullet^\vee\chi_1^1) \bullet \mathrm{F}$. Yet if we again restrict L to a subgroup $H < G$ with generators $X^\sigma = (X_L^\sigma, X_R^\sigma)$, $\sigma \in I$, such that $\mathrm{Tr}(X_L^\sigma X_L^\tau) = \mathrm{Tr}(X_R^\sigma X_R^\tau)$ for all $\sigma, \tau \in I$, then $L_\bullet^\vee\chi_1^1 = 0$ and ω_3 is H-transgressive. Note that this condition holds for any subgroup of the diagonal D_n and thus for the skyrmion bundle.

Finally, some cumbersome calculations show that the voluminous expressions for the anomalous action $\Gamma_{WZ}(U, A_L, A_R)$ in [23, (4.18)], resp., [26, (24)] are equal to the integral over

$$\omega_5^A = \omega_5 v + \chi_3^1 v \bullet \mathrm{F} + \chi_1^2 v \bullet \mathrm{F} \in \mathcal{A}_5(B(M, \mathrm{U}_n, G), \mathbb{C}),$$

where the differential forms $\chi_{5-2l}^l \in \mathcal{A}_{5-2l}(\mathrm{U}_n, \mathrm{Sym}_l(\mathfrak{g}, \mathbb{C}))_{\mathrm{equiv}}$ are given by:

$$\begin{aligned} \chi_3^1 &:= -5\,\mathrm{Tr}(R^3\pi^L + L^3\pi^R), \quad \text{i. e.,} \quad \chi_3^1 \bullet \mathrm{F} = -5\,\mathrm{Tr}(R^3 F_L + L^3 F_R) \quad \text{and} \\ \chi_1^2 &:= 10\,\mathrm{Tr}(R\pi^L\pi^L + L\pi^R\pi^R) + 5\,\mathrm{Tr}(dU\pi^R U^\dagger \pi^L - d(U^\dagger)\pi^L U \pi^R). \end{aligned}$$

Analogously to the skyrmion case, one may add a term

$$r\,[d\,\mathrm{Tr}(\pi^L U \pi^R U^\dagger)v] \bullet \mathrm{F} = r\,d\,\mathrm{Tr}(F_L U F_R U^\dagger), \quad r \in \mathbb{C},$$

or exclude it by parity invariance, cf. [23]. Also in this case, ω_5 is not G-transgressive: we obtain $L_\bullet^\vee\chi_1^2 = 10\,\mathrm{Tr}(\pi^L\pi^L\pi^L - \pi^R\pi^R\pi^R)$, thus again ω_5 is H-transgressive for any subgroup $H \leq D$.

Nevertheless note that $d\omega_5^A = (L_\bullet^\vee\chi_1^2) \bullet \mathrm{F}$ consists of a 6-form on the base. Thus as long as we stick to space-time M — or even a five-dimensional extension — this form vanishes and ω_5^A is in fact closed. The same holds for ω_3^A: although it might not be closed on space-time M, ω_3^A is closed, of course, when restricted to three-dimensional space.

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List of Symbols

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