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Generating Functions of $\pi_{2n-1}(SU_n)$

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In many questions concerning homotopy groups of LIE groups G, especially of the unitary groups U_m and SU_m, it suffices to know the mere group structure of $\pi_n(G)$. For this purpose one can consult tables. For example, it is known that

$$(1)\quad \pi_{2n}(SU_m) = \pi_{2n}(U_m) = 0,\ \pi_{2n+1}(SU_m) \cong \pi_{2n+1}(U_m) \cong \mathbb{Z}, \quad m > n \in \mathbb{N}$$

by BOTT's periodicy theorem [1]. However, often we have to know representatives $U: S^n \to G$ for the generators of these homotopy groups.

One example for this situation is the SKYRME model [5] in theoretical nuclear physics, a chiral invariant effective field theory describing the low energy limit of the quantum chromodynamics (QCD). By compactification of euclidian space $\mathbb{R}^3$, resp., of space-time $\mathbb{R}^4$, the meson fields are differentiable functions $\hat{U}: \mathbb{R}_{(t)} \times S^3 \to SU_{N_F}$, resp., $U: S^4 \to SU_{N_F}$, N_F being the number of flavors in the QCD ($N_F = 2$, resp., $N_F = 3$). In this model nucleons appear as topological soliton solutions of these field configurations. The number of nucleons described by a certain meson field $\hat{U}$ can be computed by integration of the pullback

$$(2)\qquad \frac{1}{24\pi^2}\,\hat{U}^\star\omega_3 \;=\; \frac{1}{24\pi^2}\,\mathrm{Tr}\,(L\wedge L\wedge L)$$

over the space manifold, with $L := \hat{U}^\dagger\, d\hat{U}$, where $\wedge$ is the wedge product of differential forms and ω_3 is the generator of the DE-RHAM cohomology $H_3(SU_m) \cong H_3(U_m) \cong \mathbb{R}$ for $m \geq 2$.

The meson fields obey the field equations derived as EULER-LAGRANGE equations from a lagrangian $\mathcal{L}(U, dU)$ by variation of the action integral $\int_{S^4} \mathcal{L}\; dV$. Let $\epsilon^{\mu\nu\rho\sigma}$ denote the totally antisymmetrical LEVI-CIVITA symbol, $L_\mu = U^\dagger\,\partial_\mu U$ and λ a coupling constant. Then for $N_F = 3$ the field equations involve an additional term

$$(3)\qquad \lambda\,\epsilon^{\mu\nu\rho\sigma}\,L_\mu L_\nu L_\rho L_\sigma,$$

that describes anomalous processes of the QCD. (In (3) we have used the EINSTEIN summation convention.) Unfortunately, it is impossible to build up the global corresponding term in the lagrangian from which (3) could be derived by variation. Instead by using $\pi_4(SU_3) = 0$ from (1) one argues that U can be extended to a differentiable function $U': D^5 \to SU_3$ from a five-dimensional disc D^5 whose boundary ∂D^5 is space-time S^4 [7]. Now the corresponding term for (3), the so-called WESS-ZUMINO term [6], is $\lambda \int_{D^5}(U')^\star\omega_5$, with ω_5 being the generator of $H_5(SU_m) \cong \mathbb{R}$ for $m \geq 3$. Using STOKES' theorem we can perform the integration along space-time which leads — at least locally — to (3).

For any possible extension U' the result has to be unique. This is equivalent to the requirement that

$$\lambda \int_{S^5} (\tilde{U})^\star \omega_5 = 2\pi z, \qquad z \in \mathbb{Z},$$

where S^5 is the 5-sphere which one obtains by gluing any two 5-cells $D^5_{(1)}$ and $D^5_{(2)}$ at space-time $S^4 = \partial D^5_{(1)} = \partial D^5_{(2)}$ together, and where we have defined $\tilde{U} = U'_{(1)} \cup U'_{(2)}: S^5 \to SU_3$ as the corresponding extension to this 5-sphere. This forces λ to be set equal to $\frac{1}{240\pi^2}$ by the following index theorem (cf. BOTT, SEELEY [2]). The factor $\frac{1}{24\pi^2}$ in (2) can also be deduced from this conclusion. Recall $L_\mu = U^\dagger \partial_\mu U$.

Theorem 1. *For every map $U: S^{2n-1} \to U_m$ the integral*

$$\int_{S^{2n-1}} \left(\frac{i}{2\pi}\right)^n \frac{(n-1)!}{(2n-1)!} U^\star \omega_{2n-1} =$$

$$\int_{S^{2n-1}} \left(\frac{i}{2\pi}\right)^n \frac{(n-1)!}{(2n-1)!} \epsilon^{\mu_1 \mu_2 \cdots \mu_{2n-1}} \operatorname{Tr}(L_{\mu_1} L_{\mu_2} \cdots L_{\mu_{2n-1}})\, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{2n-1}$$

is an integer $n(U)$. The assignment $[U] \mapsto n(U): \pi_{2n-1}(U_m) \to \mathbb{Z}$ is an isomorphism for $m \geq n$.

We have seen that in the case of the SKYRME model, explicit representatives $\hat{U}(t,\cdot): S^3 \to SU_{N_F}$ and $\tilde{U}: S^3 \to SU_{N_F}$ for the generators of $\pi_3(SU_{N_F})$ and $\pi_5(SU_{N_F})$ have physical significance. Thus it is worthwhile to look for such explicit representatives. This is the task of the following article.

For $\pi_3(SU_2)$ there is the so-called Hedgehog Ansatz [5] where the field equations can be transformed into a differential equation for the radial part of this ansatz. Unfortunately, this is not transferable to $\pi_5(SU_m)$, let alone $\pi_{2n-1}(SU_m)$. In order to achieve such an extension we take the more mathematical point of view and do not demand our representatives to obey certain physical field equations. A first result is the following: having found a generator U of $\pi_{2n-1}(SU_n)$ one also has a generator $j \circ U$ of $\pi_{2n-1}(SU_m)$ for $m > n$ through the inclusion

$$j: SU_n \to SU_m, \quad U \mapsto \begin{pmatrix} U & 0 \\ 0 & \mathbb{1}_{m-n} \end{pmatrix},$$

because of $U^\star \omega_k = (j \circ U)^\star \omega_k$. On the other hand one obtains a generator $i \circ U$ of $\pi_{2n-1}(U_n)$ (and thereby of $\pi_{2n-1}(U_m)$ for $m \geq n$) via the inclusion $i: SU_n \to U_n$.

So the main problem is to find representatives for $\pi_{2n-1}(SU_n)$. By looking at the LIE algebra of U_n and the use of the exponential map we make the following ansatz for a function of a $(2n-1)$-dimensional disc $D^{2n-1}_{(1)}$ into U_n: let H denote the hermitian operator

$$
\begin{pmatrix}
x_0 & z_1 & 0 & & & \\
\overline{z_1} & -x_0 & z_2 & 0 & & \\
0 & \overline{z_2} & x_0 & z_3 & \ddots & \\
 & 0 & \overline{z_3} & -x_0 & \ddots & 0 \\
 & & \ddots & \ddots & \ddots & z_{n-1} \\
 & & & 0 & \overline{z_{n-1}} & \pm x_0
\end{pmatrix}, \tag{4}
$$

where $z_j := x_{2j-1} + ix_{2j}$ for $j = 1, \ldots, n-1$. Let $\mathbf{x} = (x_0, x_1, \ldots, x_{2n-2})$ and define $U_1 \colon D^{2n-1}_{(1)} \to U_n$ by $U_1(\mathbf{x}) = \exp(i\pi H(\mathbf{x}))$. To obtain a representative of a generator of $\pi_{2n-1}(U_n)$, resp., of $\pi_{2n-1}(SU_n)$ one has to construct a second function $\widetilde{U_1}$ of a second disc $D^{2n-1}_{(2)}$ (northern and southern hemisphere) so that $U_1 \cup \widetilde{U_1}$ is a continous function of $D^{2n-1}_{(1)} \cup D^{2n-1}_{(2)} \cong S^{2n-1}$, well defined on the equator $\partial D^{2n-1}_{(1)} = \partial D^{2n-1}_{(2)} \cong S^{2n-2}$. In order to get a generator we must make sure that this "gluing process" is not trivial: if we were so careless as to choose $\widetilde{U_1}$ so that $U_1 \cup \widetilde{U_1}$ is symmetric about the equator, then we would obtain a candidate for the zero element of $\pi_{2n-1}(U_n)$ instead of a generator. In this paper we shall carry out this program for $n = 1, 2, 3$.

In [3] LUNDELL has proven an iteration for the construction of representatives for generators of $\pi_{2n-1}(SU_n)$. This iteration even leads to functions $U \colon S^{2n-1} \to SU_n$ directly, one doesn't have to look for fitting second functions on the northern hemisperes. But unfortunately, as he himself admits, "the actual formulae are too complicated for reasonable calculation". They do not inherit any symmetries between the matrix elements — like the ones built up by (4) — that allow for the calculation of the integral in Theorem 1. So this iteration is of more theoretical interest, whereas the representatives presented here could be of practical use whenever the problem of finding functions for the northern hemispheres is solved for $n \geq 4$.

Using CLIFFORD algebras LUNDELL and TOSA constructed representatives for generators of the stabe homotopy groups of SO, SU and Sp [4]. In the case of SU their formalism leads to functions $U \colon S^{2n+1} \to SU_{2^n}$, so $\pi_{2n-1}(SU_n), n \geq 3$ isn't covered either.

A Generator of $\pi_1(U_1)$

For the sake of illustration and completeness we begin by discussing the simplest case. The isomorphism $U_1 \to S^1$ yields a representative for the generator of $\pi_1(U_1)$. We also obtain this representative by using our scheme in (4). In this case we set:

$$H = (x_0), \qquad U_1(\mathbf{x}) = U_1(x_0) = \exp(i\pi x_0).$$

Here we have $U_1(-1) = U_1(1) = -1$. Therefore we can map D^1 onto S^1 by identifying 1 and -1 (and so we define our second function from $D^1_{(2)}$ to S^1 by $\widetilde{U_1}(\mathbf{x}) = -1 = \text{const}$). The mapping $U \colon S^1 \to U_1$ we obtain is a homeomorphism and thus generates $\pi(U_1)$. This is confirmed by our invoking Theorem 1: Because of $(U_1)^\star \omega_1 = \operatorname{Tr}[\exp(-i\pi x_0) i\pi \exp(i\pi x_0)]\, dx_0 = i\pi\, dx_0$, integration gives

$$\int_{-1}^{+1} \left(\frac{i}{2\pi}\right)^1 \frac{0!}{1!}\, i\pi\, dx_0 = -1.$$

Note. Representatives for the other elements of $\pi_1(U_1)$ are obtained by expanding the domain for U_1 to be $n \cdot D^1 = [-n, n]$. Because of $U_1(-n) = U_1(n) = (-1)^n$, we can again identify n and $-n$ and thereby transform $n \cdot D^1$ into S^1. Integration leads to

$$\int_{-n}^{+n} \left(\frac{i}{2\pi}\right)^1 \frac{0!}{1!}\, i\pi\, dx_0 = -n.$$

If we keep $D^1 = [-1, 1]$ as domain, $U_n = \exp(in\pi x_0)$, resp., $U_{-n} = \exp(-in\pi x_0)$ is a representative for the n-th element of $\pi_1(U_1)$.

A Generator of $\pi_3(SU_2)$

Here we have $SU_2 \cong S^3$. Under this identification the identity on S^3 is again a representative for the generator of $\pi_3(SU_2)$. We are led to it through our scheme defined by (4) (remember $z_1 = x_1 + ix_2$):

$$H = \begin{pmatrix} x_0 & z_1 \\ \overline{z_1} & -x_0 \end{pmatrix}, \qquad U_1(\mathbf{x}) = U_1(x_0, x_1, x_2) = \exp(i\pi H(\mathbf{x})).$$

Evaluating the exponential map and using $R^2 := x_0^2 + x_1^2 + x_2^2$, we obtain

$$U_1(\mathbf{x}) = \begin{pmatrix} \cos\pi R + i\frac{x_0}{R}\sin\pi R & i\frac{z_1}{R}\sin\pi R \\ i\frac{\overline{z_1}}{R}\sin\pi R & \cos\pi R - i\frac{x_0}{R}\sin\pi R \end{pmatrix}.$$

Setting $y_0 := \cos\pi R$, $y_1 := \frac{x_1}{R}\sin\pi R$, $y_2 := \frac{x_2}{R}\sin\pi R$, and $y_3 := \frac{x_0}{R}\sin\pi R$, we get $\sum_{j=0}^3 y_j^2 = 1$ and realize the isomorphism $\chi\colon SU(2) \to S^3$ as follows:

$$\begin{pmatrix} y_0 + iy_3 & -y_2 + iy_1 \\ y_2 + iy_1 & y_0 - iy_3 \end{pmatrix} \mapsto (y_0, y_1, y_2, y_3) \in S^3.$$

In particular, $R = 1$ yields

$$U_1(\mathbf{x}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

and therefore, similarly to the previous case, we can transform our function $U_1\colon D^3 \to SU_2$ into a continous mapping $U\colon S^3 \to SU_2$ by collapsing all points $\mathbf{x} \in D^3$ with $R = 1$ into one single point ∞, the "North Pole".

For the evaluation of the integral in Theorem 1 we use the three independent parameters x_0, r_1, ϕ_1 defined by $z_1 = r_1 e^{i\phi_1}$. We thus rather compute $\psi^\star(U_1)^\star\omega_3$ instead of $U^\star\omega_3$, where $\psi\colon \mathbf{R} \times [0,1] \times [-\pi, +\pi] \to D^3$ is defined by $\psi(x_0, r_1, \phi_1) = (x_0, r_1\cos\phi_1, r_1\sin\phi_1)$. By cyclic permutation under the trace we get

$$\psi^\star(U_1)^\star\omega_3 = -3 \cdot \mathrm{Tr}\{L_{x_0}[L_{\phi_1}, L_{r_1}]\}\, dx_0 \wedge dr_1 \wedge d\phi_1.$$

Next we compute the L_μ's, using the abbrevations $c := \cos\pi R, s := \sin\pi R$:

$$L_{x_0} = i\begin{pmatrix} \frac{\pi x_0^2}{R^2} + \frac{r_1^2 sc}{R^3} & z_1\left[\frac{\pi x_0}{R^2} - \frac{x_0 sc}{R^3} + i\frac{s^2}{R^2}\right] \\ \overline{z_1}\left[\frac{\pi x_0}{R^2} - \frac{x_0 sc}{R^3} - i\frac{s^2}{R^2}\right] & -\frac{\pi x_0^2}{R^2} - \frac{r_1^2 sc}{R^3}\end{pmatrix}, \tag{5}$$

$$L_{\phi_1} = i\begin{pmatrix} -\frac{r_1^2 s^2}{R^2} & z_1\left[\frac{x_0 s^2}{R^2} + i\frac{sc}{R}\right] \\ \overline{z_1}\left[\frac{x_0 s^2}{R^2} - i\frac{sc}{R}\right] & \frac{r_1^2 s^2}{R^2}\end{pmatrix}, \tag{6}$$

$$L_{r_1} = ir_1\begin{pmatrix} \frac{\pi x_0}{R^2} - \frac{x_0 sc}{R^3} & z_1\left[\frac{\pi}{R^2} + \frac{x_0^2 sc}{r_1^2 R^3} - i\frac{x_0 s^2}{r_1^2 R^2}\right] \\ \overline{z_1}\left[\frac{\pi}{R^2} + \frac{x_0^2 sc}{r_1^2 R^3} + i\frac{x_0 s^2}{r_1^2 R^2}\right] & -\frac{\pi x_0}{R^2} + \frac{x_0 sc}{R^3}\end{pmatrix}. \tag{7}$$

This yields

$$[L_{\phi_1}, L_{r_1}] = \frac{2ir_1}{R^2}\begin{pmatrix} -\frac{\pi r_1^2 sc}{R} - \frac{x_0^2 s^2}{R^2} & z_1\left[\frac{\pi x_0 sc}{R} - \frac{x_0 s^2}{R^2} - i\pi s^2\right] \\ \overline{z_1}\left[\frac{\pi x_0 sc}{R} - \frac{x_0 s^2}{R^2} + i\pi s^2\right] & \frac{\pi r_1^2 sc}{R} + \frac{x_0^2 s^2}{R^2}\end{pmatrix}$$

and $\mathrm{Tr}\{L_{x_0}[L_{\phi_1}, L_{r_1}]\} = \frac{4\pi r_1}{R^2}\sin^2\pi R$, from which we deduce

$$\psi^\star(U_1)^\star\omega_3 = -\frac{12\pi r_1}{R^2}\sin^2\pi R\, dx_0 \wedge dr_1 \wedge d\phi_1,$$

$$\text{resp.,} \qquad U^\star\omega_3 = -\frac{12\pi}{R^2}\sin^2\pi R\, dx_0 \wedge dx_1 \wedge dx_2.$$

By the transformation rule for integrals we obtain for the integral in theorem 1

$$I_1 = \int_{S^3} -\frac{1}{24\pi^2}U^\star\omega_3 = \int_{-1}^{+1} dx_0 \int_0^{\sqrt{1-x_0^2}} dr_1 \int_0^{2\pi} d\phi_1 \frac{r_1}{2\pi R^2}\sin^2\pi R = \int_0^{+1} dx_0 \int_0^{\sqrt{1-x_0^2}} dr_1 \frac{2r_1}{R^2}\sin^2\pi R$$

(the integrand is even in x_0). We choose new variables R, r_1^2 , observe $dR\wedge d(r_1^2) = \frac{2x_0 r_1}{R}\, dx_0 \wedge dr_1$, and finally get

$$I_1 = \int_0^{+1} dR \int_0^{R^2} d(r_1^2)\frac{\sin^2\pi R}{R\sqrt{R^2 - r_1^2}} = \int_0^{+1} 2\sin^2\pi R\, dR = 1.$$

This confirmes that U is a representative for the generator of $\pi_3(SU_2)$.

Note. As for $\pi_1(U_1)$, we obtain representatives for all other elements of $\pi_3(SU_2)$ by expanding our domain to the ball of radius $R = n$. For $\mathbf{x}$ with $\|\mathbf{x}\| = R = n$ we have

$$U_1(\mathbf{x}) = \begin{pmatrix} (-1)^n & 0 \\ 0 & (-1)^n \end{pmatrix}.$$

For this reason even a mapping from $n \cdot D^3$ can be transformed into a continous mapping from S^3, resp., $n \cdot S^3$, into S^3, which yields

$$I_n = \int_0^n 2\sin^2\pi R\, dR = n.$$

If we want to keep our domain D^3, we simply replace $\mathbf{x}$ by $n\mathbf{x}$ and obtain $U_n\colon D^3 \to SU_2$:

$$U_n(\mathbf{x}) = \begin{pmatrix} \cos n\pi R + i\frac{x_0}{R}\sin n\pi R & i\frac{z_1}{R}\sin n\pi R \\ i\frac{\overline{z_1}}{R}\sin n\pi R & \cos n\pi R - i\frac{x_0}{R}\sin n\pi R \end{pmatrix}$$

as representative for the n-th element of $\pi_3(SU_2)$. In order to get the inverse elements we replace x_0 by $-x_0$, then L_{x_0} changes into $-L_{x_0}$ and $U^\star\omega_3$ changes into $-U^\star\omega_3$. We obtain $U_{-n}\colon D^3 \to SU_2$:

$$U_{-n}(\mathbf{x}) = \begin{pmatrix} \cos n\pi R - i\frac{x_0}{R}\sin n\pi R & i\frac{z_1}{R}\sin n\pi R \\ i\frac{\overline{z_1}}{R}\sin n\pi R & \cos n\pi R + i\frac{x_0}{R}\sin n\pi R \end{pmatrix}.$$

A Generator of $\pi_5(SU_3)$

There is no isomorphism between SU_3 and a sphere and for the first time we will have to make use of the gluing process described in the introduction. Two mappings $U_1\colon D^5_{(1)} \to SU_3$ and $\widetilde{U_1}\colon D^5_{(2)} \to SU_3$ that coincide on the boundaries $\partial D^5_{(1)} = \partial D^5_{(2)} = S^4$, are transformed into a well defined continous function $U = U_1 \cup \widetilde{U_1}\colon S^5 \to SU_3$. In analogy with (4) we have

$$H = \begin{pmatrix} x_0 & z_1 & 0 \\ \overline{z_1} & -x_0 & z_2 \\ 0 & \overline{z_2} & x_0 \end{pmatrix}, \quad \left\{ \begin{matrix} z_1 = x_1 + ix_2 = r_1 e^{i\phi_1} \\ z_2 = x_3 + ix_4 = r_2 e^{i\phi_2} \end{matrix} \right\}, \quad \sum_{i=0}^{4} x_i^2 = x_0^2 + r^2 = R^2,$$

and a mapping $U_1'\colon D^5 \to U_3$ defined by $U_1'(\mathbf{x}) = \exp(i\pi H)(\mathbf{x})$.

$$\det U_1' = \exp(i\pi \operatorname{Tr} H) = \exp(i\pi x_0),$$

so we have $U_1'(D^5) \not\subseteq SU_3$. Using the diagonalisation of H we compute

$$U_1'(\mathbf{x}) = \begin{pmatrix} \frac{r_1^2}{r^2}(c + i\frac{x_0}{R}s) + \frac{r_2^2}{r^2}e^{i\pi x_0} & i\frac{z_1}{R}s & \frac{z_1 z_2}{r^2}(c + i\frac{x_0}{R}s - e^{i\pi x_0}) \\ i\frac{\overline{z_1}}{R}s & c - i\frac{x_0}{R}s & i\frac{z_2}{R}s \\ \frac{\overline{z_1 z_2}}{r^2}(c + i\frac{x_0}{R}s - e^{i\pi x_0}) & i\frac{\overline{z_2}}{R}s & \frac{r_2^2}{r^2}(c + i\frac{x_0}{R}s) + \frac{r_1^2}{r^2}e^{i\pi x_0} \end{pmatrix},$$

where we again used $c = \cos\pi R$ and $s = \sin\pi R$ for convenience. In order to obtain $U_1\colon D^5 \to SU_3$, we multiply every matrix $U_1'(\mathbf{x})$ by a matrix $T(\mathbf{x})$ of determinant $\det T(\mathbf{x}) = \exp(-i\pi x_0)$, preserving a convenient degree of symmetry between its elements. Thus we choose

$$(8) \qquad U_1(\mathbf{x}) = T(\mathbf{x}) \cdot U_1'(\mathbf{x}) \qquad \text{with} \qquad T(\mathbf{x}) = \begin{pmatrix} e^{-i\frac{\pi}{2}x_0} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\frac{\pi}{2}x_0} \end{pmatrix}.$$

Using $\pi^\pm := c \pm i\frac{x_0}{R}s$ for further convenience we obtain

$$U_1(\mathbf{x}) = \begin{pmatrix} \frac{r_1^2}{r^2}\pi^+ e^{-i\frac{\pi}{2}x_0} + \frac{r_2^2}{r^2}e^{+i\frac{\pi}{2}x_0} & i\frac{z_1}{R}se^{-i\frac{\pi}{2}x_0} & \frac{z_1 z_2}{r^2}(\pi^+ e^{-i\frac{\pi}{2}x_0} - e^{+i\frac{\pi}{2}x_0}) \\ i\frac{\overline{z_1}}{R}s & \pi^- & i\frac{z_2}{R}s \\ \frac{\overline{z_1 z_2}}{r^2}(\pi^+ e^{-i\frac{\pi}{2}x_0} - e^{+i\frac{\pi}{2}x_0}) & i\frac{\overline{z_2}}{R}se^{-i\frac{\pi}{2}x_0} & \frac{r_2^2}{r^2}\pi^+ e^{-i\frac{\pi}{2}x_0} + \frac{r_1^2}{r^2}e^{+i\frac{\pi}{2}x_0} \end{pmatrix}$$

as the function on the southern hemisphere of S^5. On the equator ($R = 1$) it turns out to be

$$U_1(\mathbf{x}) = \begin{pmatrix} -\frac{r_1^2}{r^2}e^{-i\frac{\pi}{2}x_0} + \frac{r_2^2}{r^2}e^{+i\frac{\pi}{2}x_0} & 0 & -\frac{z_1 z_2}{r^2}(e^{+i\frac{\pi}{2}x_0} + e^{-i\frac{\pi}{2}x_0}) \\ 0 & -1 & 0 \\ \frac{\overline{z_1 z_2}}{r^2}(e^{+i\frac{\pi}{2}x_0} + e^{-i\frac{\pi}{2}x_0}) & 0 & -\frac{r_2^2}{r^2}e^{-i\frac{\pi}{2}x_0} + \frac{r_1^2}{r^2}e^{+i\frac{\pi}{2}x_0} \end{pmatrix}.$$

Obviously this is *not* constant like in the previous cases, so it is impossible to contract the boundary into one single point, but we have to look for a nontrivial mapping $\widetilde{U_1}$ on the northern hemisphere, that coincides with U_1 on the equator. There are two possibilities:

$$\widetilde{U_1}(\mathbf{x}) = \begin{pmatrix} -\frac{r_2^2}{r^2}\pi^- e^{+i\frac{\pi}{2}x_0} - \frac{r_1^2}{r^2}e^{-i\frac{\pi}{2}x_0} & \pm i\frac{z_2}{R}se^{+i\frac{\pi}{2}x_0} & \frac{z_1 z_2}{r^2}(\pi^- e^{+i\frac{\pi}{2}x_0} - e^{-i\frac{\pi}{2}x_0}) \\ \mp i\frac{\overline{z_2}}{R}s & \pi^+ & \pm i\frac{z_1}{R}s \\ \frac{\overline{z_1 z_2}}{r^2}(\pi^- e^{+i\frac{\pi}{2}x_0} - e^{-i\frac{\pi}{2}x_0}) & \mp i\frac{\overline{z_1}}{R}se^{+i\frac{\pi}{2}x_0} & -\frac{r_1^2}{r^2}\pi^- e^{+i\frac{\pi}{2}x_0} - \frac{r_2^2}{r^2}e^{-i\frac{\pi}{2}x_0} \end{pmatrix}.$$

To secure the property of being unitary we have to choose either the upper or the lower signs. Once the choice has been made, it propagates to all products, its derivations and inverses, and so — by forming the trace at the end of the computation of $(\widetilde{U_1})^\star\omega_5$ — does not influence the value of this pullback. In the following we choose the upper signs.

Again we will use polar coordinates for the evaluation of our 5-form: let $K = \mathbf{R} \times \mathbf{R}_0^+ \times \mathbf{R}_0^+ \times [-\pi, +\pi] \times [-\pi, +\pi]$ and define $\psi: K \to \mathbf{R}^5$ by $\psi(x_0, r_1, r_2, \phi_1, \phi_2) = (x_0, r_1\cos\phi_1, r_1\sin\phi_1, r_2\cos\phi_2, r_2\sin\phi_2)$, resp., restrict K to $\psi^{-1}(D^5)$. By cyclic permutation under the trace we then obtain

$$\psi^\star(U_1)^\star\omega_5 = 5\cdot\mathrm{Tr}\{L\cdot L_{x_0}\}\, dx_0\wedge dr_1\wedge dr_2\wedge d\phi_1\wedge d\phi_2 \tag{9}$$

with the hermitian matrix

$$\begin{aligned} L = &+[L_{\phi_1}, L_{\phi_2}][L_{r_1}, L_{r_2}] - [L_{\phi_1}, L_{r_1}][L_{\phi_2}, L_{r_2}] + [L_{\phi_1}, L_{r_2}][L_{\phi_2}, L_{r_1}] \\ &+[L_{r_1}, L_{r_2}][L_{\phi_1}, L_{\phi_2}] - [L_{\phi_2}, L_{r_2}][L_{\phi_1}, L_{r_1}] + [L_{\phi_2}, L_{r_1}][L_{\phi_1}, L_{r_2}] \end{aligned} \tag{10}$$

($L^\dagger = L$ is a consequence of $[L_\mu, L_\nu]^\dagger = -[L_\mu, L_\nu]$, which itself follows from $L_\mu^\dagger = -L_\mu$, cf. (5) to (7)).

The computation of $\psi^\star(U_1)^\star\omega_5$ is straightforward but long and tedious. We have collected the main steps in the appendix. We end up with (14):

$$\begin{aligned} \psi^\star(U_1)^\star\omega_5 = 30i\pi\frac{r_1 r_2}{R^3}\Bigg[&-\pi\sin^2\pi R(\sin\pi R\cos\pi x_0 - \frac{x_0}{R}\cos\pi R\sin\pi x_0) \\ &+ 2\pi\sin\pi R(\cos\pi x_0 - \cos\pi R) + (2\frac{R^2}{r^2}+2)\frac{\sin^2\pi R}{R}(1-\cos\pi R\cos\pi x_0) \\ &- (2\frac{R^2}{r^2}+1)\frac{x_0}{R^2}\sin^3\pi R\sin\pi x_0\Bigg] dx_0\wedge dr_1\wedge dr_2\wedge d\phi_1\wedge d\phi_2. \end{aligned}$$

For the mapping on the northern hemisphere, it turns out that — cf. (15) —

$$(\widetilde{U_1})^\star\omega_5 = \overline{(U_1)^\star\omega_5} = -(U_1)^\star\omega_5.$$

Fortunately, the negative sign compensates the factor (-1), that arises as a consequence of the opposite orientation of the northern hemisphere. So both integrals yield the same value:

$$\begin{aligned} I_1 &= \int\limits_{S^5,\text{"south"}} -\frac{i}{480\pi^3}(U_1)^\star\omega_5 + \int\limits_{S^5,\text{"north"}} -\frac{i}{480\pi^3}(\widetilde{U_1})^\star\omega_5 \\ &= 2\int\limits_{S^5,\text{"south"}} -\frac{i}{480\pi^3}(U_1)^\star\omega_5 = 2\int\limits_{\psi^{-1}(D^5)} -\frac{i}{480\pi^3}\psi^\star(U_1)^\star\omega_5. \end{aligned}$$

Because $\psi^\star(U_1)^\star\omega_5$ is even in x_0, we integrate twice over positive values of x_0, the integration over ϕ_1 and ϕ_2 just yields the factor $4\pi^2$. Using new variables R, x_0 and r_1^2 and observing $dR \wedge dx_0 \wedge dr_1^2 = 2\frac{r_1 r_2}{R} dx_0 \wedge dr_1 \wedge dr_2$, we obtain

$$\begin{aligned} I_1 = &\int_0^1 \frac{dR}{R^2}\int_0^R dx_0 \int_0^{R^2-x_0^2}\Big[-\pi\sin^2\pi R(\sin\pi R\cos\pi x_0 - \frac{x_0}{R}\cos\pi R\sin\pi x_0) \\ &\quad + 2\pi\sin\pi R(\cos\pi x_0 - \cos\pi R) - (2\frac{R^2}{r^2}+1)\frac{x_0}{R^2}\sin^3\pi R\sin\pi x_0 \\ &\quad + (2\frac{R^2}{r^2}+2)\frac{\sin^2\pi R}{R}(1-\cos\pi R\cos\pi x_0)\Big]\, dr_1^2 \\ = &\int_0^1 \frac{dR}{R^2}\int_0^R \Big[-\pi(R^2-x_0^2)\sin^2\pi R(\sin\pi R\cos\pi x_0 - \frac{x_0}{R}\cos\pi R\sin\pi x_0) \\ &\quad + 2\pi(R^2-x_0^2)\sin\pi R(\cos\pi x_0 - \cos\pi R) - (3R^2-x_0^2)\frac{x_0}{R^2}\sin^3\pi R\sin\pi x_0 \\ &\quad + (4R^2-2x_0^2)\frac{\sin^2\pi R}{R}(1-\cos\pi R\cos\pi x_0)\Big]\, dx_0. \end{aligned}$$

Partial integration yields:

$$\begin{aligned} I_1 &= \int_0^1 \Big[+2\sin^2\pi R - \frac{2}{3}\pi R\sin\pi R\cos\pi R - \frac{1}{3}\sin^2\pi R \\ &\qquad -2\frac{\sin\pi R\cos\pi R}{\pi R} + \frac{\sin^2\pi R}{\pi^2 R^2} + 4\frac{\sin^3\pi R\cos\pi R}{\pi^3 R^3} - 3\frac{\sin^4\pi R}{\pi^4 R^4}\Big]\, dR \\ &= \int_0^1 2\sin^2\pi R\, dR + \Big[-\frac{1}{3}R\sin^2\pi R - \frac{\sin^2\pi R}{\pi^2 R} + \frac{\sin^4\pi R}{\pi^4 R^3}\Big]_{R=0}^{R=1} \\ &= 1 + 0 = 1. \end{aligned}$$

This finally proves that our mapping constructed from U_1 and $\widetilde{U_1}$ represents the generator of $\pi_5(SU_3)$.

Representatives for further Elements of $\pi_5(SU_3)$

Having found a representative U for the generator $[U]$ of $\pi_5(SU_3)$, we could use standard techniques to construct representatives for the powers $[U]^n$, notably, since SU_3 is a group. But neither of these is practical for an explicit numerical representation of a V_n with $[V_n] = [U]^n$. Fortunately, there is a simple technique due to the fact that we can expand the domains for U_1 and $\widetilde{U_1}$. They not only can be glued together at $R = 1$, but as well at $R = 2n + 1$, yet not at $R = 2n$ ($n \in \mathbb{N}_0$).

$$I_{2n+1} = \int_0^{2n+1} 2\sin^2 \pi R \, dR = 2n + 1,$$

we thus easily obtain further representatives for all odd products of $[U]$. For even products U_1 has to be combined with another function $\widehat{U_1}: 2n \cdot D^5 \to SU_3$. Choosing

$$\widehat{U_1}(\mathbf{x}) = \begin{pmatrix} \frac{r_2^2}{r^2}\pi^- e^{+i\frac{\pi}{2}x_0} + \frac{r_1^2}{r^2}e^{-i\frac{\pi}{2}x_0} & i\frac{z_2}{R}s e^{+i\frac{\pi}{2}x_0} & -\frac{z_1 z_2}{r^2}(\pi^- e^{+i\frac{\pi}{2}x_0} - e^{-i\frac{\pi}{2}x_0}) \\ i\frac{\overline{z_2}}{R}s & \pi^+ & i\frac{z_1}{R}s \\ -\frac{\overline{z_1 z_2}}{r^2}(\pi^- e^{+i\frac{\pi}{2}x_0} - e^{-i\frac{\pi}{2}x_0}) & i\frac{\overline{z_1}}{R}s e^{+i\frac{\pi}{2}x_0} & \frac{r_1^2}{r^2}\pi^- e^{+i\frac{\pi}{2}x_0} + \frac{r_2^2}{r^2}e^{-i\frac{\pi}{2}x_0} \end{pmatrix}$$

for points on the northern hemisphere, we recognize that $\widehat{U_1}$ can be glued together with U_1 at $R = 2n$, since

$$\widehat{U_1}(\mathbf{x}) = \begin{pmatrix} \frac{r_1^2}{r^2}e^{-i\frac{\pi}{2}x_0} + \frac{r_2^2}{r^2}e^{+i\frac{\pi}{2}x_0} & 0 & -\frac{z_1 z_2}{r^2}(e^{+i\frac{\pi}{2}x_0} - e^{-i\frac{\pi}{2}x_0}) \\ 0 & 1 & 0 \\ \frac{\overline{z_1 z_2}}{r^2}(e^{+i\frac{\pi}{2}x_0} - e^{-i\frac{\pi}{2}x_0}) & 0 & \frac{r_2^2}{r^2}e^{-i\frac{\pi}{2}x_0} + \frac{r_1^2}{r^2}e^{+i\frac{\pi}{2}x_0} \end{pmatrix} = U_1(\mathbf{x})$$

for all points $\mathbf{x}$ with $\|\mathbf{x}\| = R = 2n$. Because of $\widehat{U_1}(\mathbf{x} = 0) = \mathbb{1}_3 = U_1(\mathbf{x} = 0)$, both North Pole and South Pole of S^5 are mapped onto the base point of SU_3. Using the fact that the L_μ are invariant under left multiplications, we have

$$\widehat{U_1}(\mathbf{x}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \cdot \widetilde{U_1}(\mathbf{x}) \quad \Longrightarrow \quad \widehat{L_\mu}(\mathbf{x}) = \widetilde{L_\mu}(\mathbf{x}).$$

This yields $(\widehat{U_1})^\star \omega_5 = (\widetilde{U_1})^\star \omega_5$ and thus:

$$\begin{aligned} I_{2n} &= \int\limits_{2n\cdot S^5,\text{“south”}} -\frac{i}{480\pi^3}(U_1)^\star \omega_5 + \int\limits_{2n\cdot S^5,\text{“north”}} -\frac{i}{480\pi^3}(\widehat{U_1})^\star \omega_5 \\ &= 2 \int\limits_{2n\cdot S^5,\text{“south”}} -\frac{i}{480\pi^3}(U_1)^\star \omega_5 = \int_0^{2n} 2\sin^2 \pi R \, dR = 2n. \end{aligned}$$

In order to obtain representatives for the corresponding inverse elements of $\pi_5(SU_3)$ we replace x_0 by $-x_0$, or define U_1 to be the mapping of the northern hemisphere and $\widetilde{U_1}$, resp., $\widehat{U_1}$ to be the mapping of the southern hemisphere of S^5. If we replace $\mathbf{x}$ by $2n\mathbf{x}$, resp., by $(2n + 1)\mathbf{x}$, we can keep D^5 instead of $2n \cdot D^5$, resp., $(2n + 1) \cdot D^5$ as domain.

Representatives for Elements of $\pi_5(U_3)$

As already mentioned in the introduction, $U: S^5 \to SU_3$ constructed above also is a representative for the generator of $\pi_5(U_3)$ via the inclusion $i: SU_3 \to U_3$. Alternatively, we can also use the function $U_1': D^5 \to U_3$ that we had obtained by (4) directly, to build up representatives for all the elements of $\pi_5(U_3)$, as a short computation will show.

For the mappings on the northern hemisphere we define:

$$\widetilde{U_1'}(\mathbf{x}) = \begin{pmatrix} -\frac{r_2^2}{r^2}\pi^- e^{i\pi x_0} - \frac{r_1^2}{r^2} & i\frac{z_2}{R} s e^{i\frac{\pi}{2}x_0} & \frac{z_1 z_2}{r^2}(\pi^- e^{i\pi x_0} - 1) \\ -i\frac{\overline{z_2}}{R} s e^{i\frac{\pi}{2}x_0} & \pi^+ & i\frac{z_1}{R} s e^{i\frac{\pi}{2}x_0} \\ \frac{\overline{z_1 z_2}}{r^2}(\pi^- e^{i\pi x_0} - 1) & -i\frac{\overline{z_1}}{R} s e^{i\frac{\pi}{2}x_0} & -\frac{r_1^2}{r^2}\pi^- e^{i\pi x_0} - \frac{r_2^2}{r^2} \end{pmatrix}, \quad \text{resp.,}$$

$$\widehat{U_1'}(\mathbf{x}) = \begin{pmatrix} \frac{r_2^2}{r^2}\pi^- e^{i\pi x_0} + \frac{r_1^2}{r^2} & i\frac{z_2}{R} s e^{i\frac{\pi}{2}x_0} & \frac{z_1 z_2}{r^2}(-\pi^- e^{i\pi x_0} + 1) \\ i\frac{\overline{z_2}}{R} s e^{i\frac{\pi}{2}x_0} & \pi^+ & -i\frac{z_1}{R} s e^{i\frac{\pi}{2}x_0} \\ \frac{\overline{z_1 z_2}}{r^2}(-\pi^- e^{i\pi x_0} + 1) & -i\frac{\overline{z_1}}{R} s e^{i\frac{\pi}{2}x_0} & \frac{r_1^2}{r^2}\pi^- e^{i\pi x_0} + \frac{r_2^2}{r^2} \end{pmatrix};$$

these can be glued together with U_1' at $R = 2n+1$, resp., $R = 2n$, because

$$\widetilde{U_1'}(\mathbf{x}) = \begin{pmatrix} -\frac{r_1^2}{r^2} + \frac{r_2^2}{r^2} e^{i\pi x_0} & 0 & -\frac{z_1 z_2}{r^2}(1 + e^{i\pi x_0}) \\ 0 & -1 & 0 \\ \frac{\overline{z_1 z_2}}{r^2}(1 + e^{i\pi x_0}) & 0 & -\frac{r_2^2}{r^2} + \frac{r_1^2}{r^2} e^{i\pi x_0} \end{pmatrix} = U_1'(\mathbf{x})$$

for all $\mathbf{x}$ with $\|\mathbf{x}\| = R = 2n+1$ and

$$\widehat{U_1'}(\mathbf{x}) = \begin{pmatrix} \frac{r_1^2}{r^2} + \frac{r_2^2}{r^2} e^{i\pi x_0} & 0 & \frac{z_1 z_2}{r^2}(1 - e^{i\pi x_0}) \\ 0 & 1 & 0 \\ \frac{\overline{z_1 z_2}}{r^2}(1 - e^{i\pi x_0}) & 0 & \frac{r_2^2}{r^2} + \frac{r_1^2}{r^2} e^{i\pi x_0} \end{pmatrix} = U_1'(\mathbf{x})$$

for all $\mathbf{x}$ with $\|\mathbf{x}\| = 2n$. Recalling $T(\mathbf{x})$ from (8) we get

$$U_1'(\mathbf{x}) = T^{-1}(\mathbf{x}) \cdot U_1(\mathbf{x}), \quad \widetilde{U_1'}(\mathbf{x}) = \widetilde{U_1}(\mathbf{x}) \cdot T^{-1}(\mathbf{x}), \quad \widehat{U_1'}(\mathbf{x}) = \widehat{U_1}(\mathbf{x}) \cdot T^{-1}(\mathbf{x}).$$

$T(\mathbf{x})$ only depends on x_0, so the matrices that occur in our calculation of $(U_1)^\star \omega_5$ (conf. (9)), only change in the following manner (we omit the argument $\mathbf{x}$ for convenience):

$$\begin{aligned} L'_{x_0} &= L_{x_0} + i\frac{\pi}{2}\, U_1^\dagger E U_1, \\ \widetilde{L'_{x_0}} &= T \cdot \widetilde{L_{x_0}} \cdot T^{-1} + i\frac{\pi}{2} E, \\ \widehat{L'_{x_0}} &= T \cdot \widehat{L_{x_0}} \cdot T^{-1} + i\frac{\pi}{2} E, \\ L' &= L, \qquad \widetilde{L'} = T \cdot \widetilde{L} \cdot T^{-1}, \qquad \widehat{L'} = T \cdot \widehat{L} \cdot T^{-1}, \end{aligned}$$

where we have defined $E := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. We easily deduce

$$\mathrm{Tr}\{L' \cdot L'_{x_0}\} = \mathrm{Tr}\{L \cdot L_{x_0}\} + i\frac{\pi}{2}\,\mathrm{Tr}\{L \cdot U_1^\dagger E U_1\} \quad \text{and}$$

$$\mathrm{Tr}\{\widetilde{L'} \cdot \widetilde{L'_{x_0}}\} = \mathrm{Tr}\{\widehat{L'} \cdot \widehat{L'_{x_0}}\} = -\,\mathrm{Tr}\{L \cdot L_{x_0}\} + i\frac{\pi}{2}\,\mathrm{Tr}\{L \cdot E\}.$$

For the total integral we get $I_n' = I_n + \Delta$ with

$$\Delta = \int_0^n \frac{dR}{R^2} \int_{-R}^{R} dx_0 \int_0^{R^2 - x_0^2} dr_1^2 \frac{1}{48} \frac{R^3}{r_1 r_2}\, \mathrm{Tr}\{L \cdot (U_1^\dagger E U_1 - E)\},$$

and for $M := U_1^\dagger E U_1 - E$ we compute

$$M = \begin{pmatrix} -\frac{r_1^2 s^2}{R^2} & z_1(x_0\frac{s^2}{R^2} + i\frac{sc}{R}) & -z_1 z_2 \frac{s^2}{R^2} \\ \overline{z_1}(x_0\frac{s^2}{R^2} - i\frac{sc}{R}) & \frac{r^2 s^2}{R^2} & z_2(x_0\frac{s^2}{R^2} - i\frac{sc}{R}) \\ -\overline{z_1 z_2}\frac{s^2}{R^2} & \overline{z_2}(x_0\frac{s^2}{R^2} + i\frac{sc}{R}) & -\frac{r_2^2 s^2}{R^2} \end{pmatrix}.$$

Using (11) to (13) from the appendix we obtain

$$\begin{aligned} \mathrm{Tr}\{L\cdot M\} &= \frac{s^2}{R^2}S + 2\frac{sc}{R}\Big(r_1^2\,\mathrm{Im}\{\frac{L_{12}}{z_1}\} - r_2^2\,\mathrm{Im}\{\frac{L_{23}}{z_2}\}\Big) \\ &= \frac{r_1 r_2}{R^3}\left[-24\pi s^3 c(1-cc_x-\frac{x_0}{R}ss_x) + 24\pi s^3 c(1-cc_x-\frac{x_0}{R}ss_x)\right] = 0. \end{aligned}$$

This yields $\Delta = 0$, and thus — as expected —

$$I_n' = n.$$

This result once again confirms that our ansatz (4) *directly* leads to representatives for generators of $\pi_{2n-1}(SU_n)$, resp., $\pi_{2n-1}(U_n)$, depending on n being even or odd — at least for the lower dimensions examinated here.

Appendix

In order to compute $\psi^\star(U_1)^\star\omega_5$ we first calculate the antihermitian L_μ's. Throughout all computations we will use the following abbrevations for convenience and clarity:

$$\begin{aligned} c &= \cos\pi R, & s &= \sin\pi R, \\ \pi^+ &= \cos\pi R + i\frac{x_0}{R}\sin\pi R, & \pi^- &= \cos\pi R - i\frac{x_0}{R}\sin\pi R, \\ c_x &= \cos\pi x_0, & s_x &= \sin\pi x_0, \\ e^+ &= \exp(+i\pi x_0), & e^- &= \exp(-i\pi x_0). \end{aligned}$$

Remember $K = \mathbf{R}\times\mathbf{R}_0^+\times\mathbf{R}_0^+\times[-\pi,+\pi]\times[-\pi,+\pi]$ as domain for the polar coordinate function ψ and let $\mathbf{v} := (x_0, r_1, r_2, \phi_1, \phi_2) \in K$. Define the linear involution $\Lambda\colon K \to K$ by $\mathbf{w} = \Lambda(\mathbf{v}) = (x_0, r_2, r_1, -\phi_2, -\phi_1)$. So $U_1(\psi(\mathbf{w}))$ is the matrix we obtain from $U_1(\psi(\mathbf{v}))$ by replacing (z_1, z_2) by $(\overline{z_2}, \overline{z_1})$, resp., (r_1, ϕ_1) by $(r_2, -\phi_2)$, and vice versa.

Let A^P denote the matrix A "rotated by 180°", so that A_{11} becomes A_{33}, A_{12} becomes A_{32}, A_{13} becomes A_{31}, and so on. Obviously this operation commutes with the hermitian conjugation and the derivation of A. We have $(AB)^P = A^P B^P$ and $\mathrm{Tr}\{A^P\} = \mathrm{Tr}\{A\}$. Because of $U_1^P\circ\psi = U_1\circ\psi\circ\Lambda$ we obtain $\frac{\partial}{\partial x_0}(U_1^P\circ\psi) = \frac{\partial}{\partial x_0}(U_1\circ\psi\circ\Lambda) = \frac{\partial}{\partial x_0}(U_1\circ\psi)\circ\Lambda$, $\frac{\partial}{\partial r_1}(U_1^P\circ\psi) = \frac{\partial}{\partial r_1}(U_1\circ\psi\circ\Lambda) = \frac{\partial U_1}{\partial r_2}(U_1\circ\psi)\circ\Lambda$ and $\frac{\partial}{\partial\phi_1}(U_1^P\circ\psi) = \frac{\partial}{\partial\phi_1}(U_1\circ\psi\circ\Lambda) = -\frac{\partial U_1}{\partial\phi_2}(U_1\circ\psi)\circ\Lambda$. We thus have an additional symmetry between the elements of the antihermitian L_μ's (here $L_\mu = (U_1\circ\psi)^\dagger\frac{\partial}{\partial\mu}(U_1\circ\psi)\colon K \to M_3(\mathbf{C})$ for $\mu = x_0, r_1, r_2, \phi_1, \phi_2$):

$$\begin{aligned} &L_{x_0}(\mathbf{v}) = L_{x_0}^P(\mathbf{w}), \\ &L_{\phi_1}(\mathbf{v}) = -L_{\phi_2}^P(\mathbf{w}), \quad L_{\phi_2}(\mathbf{v}) = -L_{\phi_1}^P(\mathbf{w}), \\ &L_{r_1}(\mathbf{v}) = +L_{r_2}^P(\mathbf{w}), \quad L_{r_2}(\mathbf{v}) = +L_{r_1}^P(\mathbf{w}), \end{aligned}$$

which makes life a bit easier. We obtain

$$
\begin{aligned}
L_{x_0}(\mathbf{v}) &= \frac{i\pi}{2R^2}\cdot \\
&\begin{pmatrix}
R^2+r_1^2(\frac{2sc}{\pi R}-1-c^2) & z_1\left[-x_0(\frac{2sc}{\pi R}-1-c^2)+i(\frac{2s^2}{\pi}-Rsc)\right] & z_1 z_2(\frac{2sc}{\pi R}-1-c^2) \\
\overline{z_1}\left[-x_0(\frac{2sc}{\pi R}-1-c^2)-i(\frac{2s^2}{\pi}-Rsc)\right] & -2R^2-r^2(\frac{2sc}{\pi R}-1-c^2) & z_2\left[-x_0(\frac{2sc}{\pi R}-1-c^2)-i(\frac{2s^2}{\pi}-Rsc)\right] \\
\overline{z_1 z_2}(\frac{2sc}{\pi R}-1-c^2) & \overline{z_2}\left[-x_0(\frac{2sc}{\pi R}-1-c^2)+i(\frac{2s^2}{\pi}-Rsc)\right] & R^2+r_2^2(\frac{2sc}{\pi R}-1-c^2)
\end{pmatrix}, \\
L_{\phi_1}(\mathbf{v}) &= i\cdot \\
&\begin{pmatrix}
-\frac{r_1^2 s^2}{R^2}-\frac{r_1^2 r_2^2}{r^4}|\pi^+-e^+|^2 & +iz_1\frac{s}{R}\left(\frac{r_1^2}{r^2}\pi^-+\frac{r_2^2}{r^2}e^-\right) & +\frac{z_1 z_2}{r^4}(r_1^2\pi^-+r_2^2 e^-)(\pi^+-e^+) \\
-i\overline{z_1}\frac{s}{R}\left(\frac{r_1^2}{r^2}\pi^++\frac{r_2^2}{r^2}e^+\right) & +\frac{r_1^2 s^2}{R^2} & -iz_2\frac{s}{R}\frac{r_1^2}{r^2}(\pi^+-e^+) \\
+\frac{\overline{z_1 z_2}}{r^4}(r_1^2\pi^++r_2^2 e^+)(\pi^--e^-) & +i\overline{z_2}\frac{s}{R}\frac{r_1^2}{r^2}(\pi^--e^-) & +\frac{r_1^2 r_2^2}{r^4}|\pi^+-e^+|^2
\end{pmatrix}, \\
L_{\phi_2}(\mathbf{v}) &= i\cdot \\
&\begin{pmatrix}
-\frac{r_1^2 r_2^2}{r^4}|\pi^+-e^+|^2 & -iz_1\frac{s}{R}\frac{r_2^2}{r^2}(\pi^--e^-) & -\frac{z_1 z_2}{r^4}(r_2^2\pi^++r_1^2 e^+)(\pi^--e^-) \\
+i\overline{z_1}\frac{s}{R}\frac{r_2^2}{r^2}(\pi^+-e^+) & -\frac{r_2^2 s^2}{R^2} & +iz_2\frac{s}{R}\left(\frac{r_2^2}{r^2}\pi^++\frac{r_1^2}{r^2}e^+\right) \\
-\frac{\overline{z_1 z_2}}{r^4}(r_2^2\pi^-+r_1^2 e^-)(\pi^+-e^+) & -i\overline{z_2}\frac{s}{R}\left(\frac{r_2^2}{r^2}\pi^-+\frac{r_1^2}{r^2}e^-\right) & +\frac{r_2^2 s^2}{R^2}+\frac{r_1^2 r_2^2}{r^4}|\pi^+-e^+|^2
\end{pmatrix}, \\
L_{r_1}(\mathbf{v}) &= ir_1\cdot \\
&\begin{pmatrix}
\frac{x_0 r_1^2}{r^2}(\frac{\pi}{R^2}-\frac{sc}{R^3})+i\frac{r_2^2}{r^4}(\pi^-e^+-\pi^+e^-) & z_1\left[(\frac{\pi}{R^2}-\frac{sc}{R^3})+\frac{s(r_1^2\pi^-+r_2^2e^-)}{r_1^2 r^2 R}\right] & z_1 z_2\left[\frac{x_0}{r^2}(\frac{\pi}{R^2}-\frac{sc}{R^3})-\frac{r^2-r_1^2\pi^-e^+-r_2^2\pi^+e^-}{ir_1^2 r^4}\right] \\
\overline{z_1}\left[(\frac{\pi}{R^2}-\frac{sc}{R^3})+\frac{s(r_1^2\pi^++r_2^2e^+)}{r_1^2 r^2 R}\right] & -x_0(\frac{\pi}{R^2}-\frac{sc}{R^3}) & z_2\left[(\frac{\pi}{R^2}-\frac{sc}{R^3})+\frac{s(\pi^+-e^+)}{r^2 R}\right] \\
\overline{z_1 z_2}\left[\frac{x_0}{r^2}(\frac{\pi}{R^2}-\frac{sc}{R^3})+\frac{r^2-r_1^2\pi^-e^+-r_2^2\pi^+e^-}{ir_1^2 r^4}\right] & \overline{z_2}\left[(\frac{\pi}{R^2}-\frac{sc}{R^3})+\frac{s(\pi^--e^-)}{r^2 R}\right] & \frac{x_0 r_2^2}{r^2}(\frac{\pi}{R^2}-\frac{sc}{R^3})-i\frac{r_2^2}{r^4}(\pi^-e^+-\pi^+e^-)
\end{pmatrix}, \\
L_{r_2}(\mathbf{v}) &= ir_2\cdot \\
&\begin{pmatrix}
\frac{x_0 r_1^2}{r^2}(\frac{\pi}{R^2}-\frac{sc}{R^3})-i\frac{r_1^2}{r^4}(\pi^-e^+-\pi^+e^-) & z_1\left[(\frac{\pi}{R^2}-\frac{sc}{R^3})+\frac{s(\pi^--e^-)}{r^2 R}\right] & z_1 z_2\left[\frac{x_0}{r^2}(\frac{\pi}{R^2}-\frac{sc}{R^3})+\frac{r^2-r_1^2\pi^-e^+-r_2^2\pi^+e^-}{ir_2^2 r^4}\right] \\
\overline{z_1}\left[(\frac{\pi}{R^2}-\frac{sc}{R^3})+\frac{s(\pi^+-e^+)}{r^2 R}\right] & -x_0(\frac{\pi}{R^2}-\frac{sc}{R^3}) & z_2\left[(\frac{\pi}{R^2}-\frac{sc}{R^3})+\frac{s(r_2^2\pi^++r_1^2e^+)}{r_2^2 r^2 R}\right] \\
\overline{z_1 z_2}\left[\frac{x_0}{r^2}(\frac{\pi}{R^2}-\frac{sc}{R^3})-\frac{r^2-r_1^2\pi^-e^+-r_2^2\pi^+e^-}{ir_2^2 r^4}\right] & \overline{z_2}\left[(\frac{\pi}{R^2}-\frac{sc}{R^3})+\frac{s(r_2^2\pi^-+r_1^2e^-)}{r_2^2 r^2 R}\right] & \frac{x_0 r_2^2}{r^2}(\frac{\pi}{R^2}-\frac{sc}{R^3})+i\frac{r_1^2}{r^4}(\pi^-e^+-\pi^+e^-)
\end{pmatrix}.
\end{aligned}
$$

For the antihermitian $[L_\mu, L_\nu]$ we have the following additional symmetries:

$$
\begin{aligned}
&[L_{\phi_1}, L_{\phi_2}](\mathbf{v}) = -[L_{\phi_1}, L_{\phi_2}]^P(\mathbf{w}), \quad [L_{r_1}, L_{r_2}](\mathbf{v}) = -[L_{r_1}, L_{r_2}]^P(\mathbf{w}), \\
&[L_{\phi_1}, L_{r_1}](\mathbf{v}) = -[L_{\phi_2}, L_{r_2}]^P(\mathbf{w}), \quad [L_{\phi_2}, L_{r_1}](\mathbf{v}) = -[L_{\phi_1}, L_{r_2}]^P(\mathbf{w}), \\
&[L_{\phi_1}, L_{r_2}](\mathbf{v}) = -[L_{\phi_2}, L_{r_1}]^P(\mathbf{w}), \quad [L_{\phi_2}, L_{r_2}](\mathbf{v}) = -[L_{\phi_1}, L_{r_1}]^P(\mathbf{w}),
\end{aligned}
$$

so that $[L_{\phi_1}, L_{r_2}](\mathbf{v})$ and $[L_{\phi_2}, L_{r_2}](\mathbf{v})$ do not need to be computed. For the others we obtain

$$[L_{\phi_1}, L_{\phi_2}](\mathbf{v}) = \begin{pmatrix} 0 & -iz_1\frac{s}{R}\frac{r_2^2}{r^2}(\pi^- - e^-) & z_1 z_2 \frac{s^2}{R^2} \\ -i\overline{z_1}\frac{s}{R}\frac{r_2^2}{r^2}(\pi^+ - e^+) & 0 & iz_2\frac{s}{R}\frac{r_1^2}{r^2}(\pi^+ - e^+) \\ -\overline{z_1 z_2}\frac{s^2}{R^2} & i\overline{z_2}\frac{s}{R}\frac{r_1^2}{r^2}(\pi^- - e^-) & 0 \end{pmatrix},$$

$$[L_{r_1}, L_{r_2}](\mathbf{v}) = \frac{r_1 r_2}{r^2 R^2} \cdot$$

$$\begin{pmatrix} \begin{array}{c} \pi Rs(e^+ - e^-) \\ +i\pi x_0(ce^+ + ce^- - 2) \\ +ix_0\frac{s}{R}(2c - e^+ - e^-) \end{array} & \frac{iz_1}{r_1^2}\left\{\begin{array}{c} \pi r^2(ce^- - 1) \\ +x_0^2\frac{s}{R}(e^- - c) \\ +ix_0 s^2 \end{array}\right\} & \frac{z_1 z_2}{r_1^2 r_2^2}\left\{\begin{array}{c} -\pi Rs(r_1^2 e^+ + r_2^2 e^-) \\ +i\pi x_0 r_2^2(ce^- - 1) \\ -i\pi x_0 r_1^2(ce^+ - 1) \\ +ix_0 r_1^2\frac{s}{R}(e^+ - c) \\ -ix_0 r_2^2\frac{s}{R}(e^- - c) \end{array}\right\} \\ \frac{i\overline{z_1}}{r_1^2}\left\{\begin{array}{c} \pi r^2(ce^+ - 1) \\ +x_0^2\frac{s}{R}(e^+ - c) \\ -ix_0 s^2 \end{array}\right\} & 0 & -\frac{iz_2}{r_2^2}\left\{\begin{array}{c} \pi r^2(ce^+ - 1) \\ +x_0^2\frac{s}{R}(e^+ - c) \\ -ix_0 s^2 \end{array}\right\} \\ \frac{\overline{z_1 z_2}}{r_1^2 r_2^2}\left\{\begin{array}{c} \pi Rs(r_1^2 e^- + r_2^2 e^+) \\ +i\pi x_0 r_2^2(ce^+ - 1) \\ -i\pi x_0 r_1^2(ce^- - 1) \\ +ix_0 r_1^2\frac{s}{R}(e^- - c) \\ -ix_0 r_2^2\frac{s}{R}(e^+ - c) \end{array}\right\} & -\frac{i\overline{z_2}}{r_2^2}\left\{\begin{array}{c} \pi r^2(ce^- - 1) \\ +x_0^2\frac{s}{R}(e^- - c) \\ +ix_0 s^2 \end{array}\right\} & \begin{array}{c} -\pi Rs(e^+ - e^-) \\ -i\pi x_0(ce^+ + ce^- - 2) \\ -ix_0\frac{s}{R}(2c - e^+ - e^-) \end{array} \end{pmatrix},$$

$$[L_{\phi_1}, L_{r_1}](\mathbf{v}) = r_1 \cdot$$

$$\begin{pmatrix} \begin{array}{c} \frac{2i\pi x_0 r_1^2 r_2^2 csx}{r^4R^2} - \frac{2i\pi r_1^4 sc}{r^2R^3} \\ -\frac{2i\pi r_1^2 r_2^2 scx}{r^4R} - \frac{2ix_0 r_1^2 r_2^2 ssx}{r^4R^3} \\ -\frac{2ir_2^2(r_2^2 - r_1^2)}{r^6}|\pi^+ - e^+|^2 \\ +\frac{2ir_1^2 r_2^2 s^2}{r^4R^2} + \frac{2ir_1^4 s^2}{r^2R^4} - \frac{2is^2}{R^2} \end{array} & z_1\left\{\begin{array}{c} \frac{2\pi r_1^2 s^2}{r^2R^2} + \frac{r_2^2 s}{r^2R^3}(e^- - c) \\ -\frac{\pi r_2^2}{r^2R^2}(ce^- - 1) \\ +\frac{2ix_0 r_2^2 s}{r^2R^4}(\pi Rc - s) \\ +\frac{2r_2^2 s}{r^4R}(e^- - \pi^-) \end{array}\right\} & z_1 z_2\left\{\begin{array}{c} -\frac{x_0}{r^4R^3}(r_1^2e^+ + r_2^2e^-)(\pi Rc - s) \\ +\frac{i\pi s}{r^4R}(r_1^2e^+ - r_2^2e^-) \\ +\frac{\pi x_0}{r^2R^2} - \frac{2i\pi r_1^2 sc}{r^2R^3} \\ +\frac{is^2}{R^4} + \frac{4ir_2^2}{r^6}|\pi^+ - e^+|^2 \\ -\frac{x_0 sc}{r^2R^3} - \frac{ix_0^2(r_1^2 - r_2^2)s^2}{r^4R^4} \end{array}\right\} \\ \overline{z_1}\left\{\begin{array}{c} -\frac{2\pi r_1^2 s^2}{r^2R^2} - \frac{r_2^2 s}{r^2R^3}(e^+ - c) \\ +\frac{\pi r_2^2}{r^2R^2}(ce^+ - 1) \\ +\frac{2ix_0 r_2^2 s}{r^2R^4}(\pi Rc - s) \\ -\frac{2r_2^2 s}{r^4R}(e^+ - \pi^+) \end{array}\right\} & \frac{2is}{R^4}[r_1^2(\pi Rc - s) + R^2 s] & z_2\left\{\begin{array}{c} -\frac{2\pi r_1^2 s^2}{r^2R^2} + \frac{r_1^2 s}{r^2R^3}(e^+ - c) \\ -\frac{\pi r_1^2}{r^2R^2}(ce^+ - 1) \\ +\frac{2ix_0 r_2^2 s}{r^2R^4}(\pi Rc - s) \\ -\frac{2r_2^2 s}{r^4R}(e^+ - \pi^+) \end{array}\right\} \\ \overline{z_1 z_2}\left\{\begin{array}{c} \frac{x_0}{r^4R^3}(r_1^2e^- + r_2^2e^+)(\pi Rc - s) \\ +\frac{i\pi s}{r^4R}(r_1^2e^- - r_2^2e^+) \\ -\frac{\pi x_0}{r^2R^2} - \frac{2i\pi r_1^2 sc}{r^2R^3} \\ +\frac{is^2}{R^4} + \frac{4ir_2^2}{r^6}|\pi^+ - e^+|^2 \\ +\frac{x_0 sc}{r^2R^3} - \frac{ix_0^2(r_1^2 - r_2^2)s^2}{r^4R^4} \end{array}\right\} & \overline{z_2}\left\{\begin{array}{c} \frac{2\pi r_1^2 s^2}{r^2R^2} - \frac{r_1^2 s}{r^2R^3}(e^- - c) \\ +\frac{\pi r_1^2}{r^2R^2}(ce^- - 1) \\ +\frac{2ix_0 r_2^2 s}{r^2R^4}(\pi Rc - s) \\ +\frac{2r_2^2 s}{r^4R}(e^- - \pi^-) \end{array}\right\} & \begin{array}{c} -\frac{2i\pi x_0 r_1^2 r_2^2 csx}{r^4R^2} - \frac{2i\pi r_1^2 r_2^2 sc}{r^2R^3} \\ +\frac{2i\pi r_1^2 r_2^2 scx}{r^4R} + \frac{2ix_0 r_1^2 r_2^2 ssx}{r^4R^3} \\ +\frac{2ir_2^2(r_2^2 - r_1^2)}{r^6}|\pi^+ - e^+|^2 \\ -2ir_1^2 r_2^2 s^2 r^4 R^2 + \frac{2ir_1^2 r_2^2 s^2}{r^2R^4} \end{array} \end{pmatrix},$$

$$
[L_{\phi_2}, L_{r_1}](\mathbf{v}) \quad = \quad r_1 \cdot
$$

$$
\begin{pmatrix}
\begin{array}{c}
\frac{2i\pi x_0 r_1^2 r_2^2 c s_x}{r^4 R^2} + \frac{2i\pi r_1^2 r_2^2 s c}{r^2 R^3} \\
-\frac{2i\pi r_1^2 r_2^2 s c_x}{r^4 R} - \frac{2i x_0 r_1^2 r_2^2 s s_x}{r^4 R^3} \\
-\frac{2i r_2^2 (r_2^2 - r_1^2)}{r^6} |\pi^+ - e^+|^2 \\
+\frac{2i r_1^2 r_2^2 s^2}{r^4 R^2} - \frac{2i r_1^2 r_2^2 s^2}{r^2 R^4}
\end{array}
&
z_1 \left\{ \begin{array}{c}
-\frac{2\pi r_2^2 s^2}{r^2 R^2} + \frac{r_2^2 s}{r^2 R^3}(e^- - c) \\
-\frac{\pi r_2^2}{r^2 R^2}(c e^- - 1) \\
-\frac{2i x_0 r_2^2 s}{r^2 R^4}(\pi R c - s) \\
+\frac{r_2^2 (r_1^2 - r_2^2) s}{r_1^2 r^4 R}(e^- - \pi^-)
\end{array} \right\}
&
z_1 z_2 \left\{ \begin{array}{c}
-\frac{x_0}{r^4 R^3}(r_1^2 e^+ + r_2^2 e^-)(\pi R c - s) \\
+\frac{i\pi s}{r^4 R}(r_1^2 e^+ - r_2^2 e^-) \\
+\frac{\pi x_0}{r^2 R^2} - \frac{2i\pi r_1^2 s c}{r^2 R^3} \\
+\frac{i r_2^2 s^2}{r_1^2 r^2 R^2} + \frac{4 i r_2^2}{r^6} |\pi^+ - e^+|^2 \\
-\frac{x_0 s c}{r^2 R^3} + \frac{2i x_0^2 r_2^2 s^2}{r^4 R^4}
\end{array} \right\}
\\
\overline{z_1} \left\{ \begin{array}{c}
\frac{2\pi r_2^2 s^2}{r^2 R^2} - \frac{r_2^2 s}{r^2 R^3}(e^+ - c) \\
+\frac{\pi r_2^2}{r^2 R^2}(c e^+ - 1) \\
-\frac{2i x_0 r_2^2 s}{r^2 R^4}(\pi R c - s) \\
-\frac{r_2^2 (r_1^2 - r_2^2) s}{r_1^2 r^4 R}(e^+ - \pi^+)
\end{array} \right\}
&
-\frac{2is}{R^4}[r_2^2(\pi R c - s)]
&
z_2 \left\{ \begin{array}{c}
\frac{2\pi r_2^2 s^2}{r^2 R^2} + \frac{r_1^2 s}{r^2 R^3}(e^+ - c) \\
-\frac{\pi r_1^2}{r^2 R^2}(c e^+ - 1) \\
-\frac{2i x_0 r_2^2 s}{r^2 R^4}(\pi R c - s) \\
-\frac{(r_1^2 + 3 r_2^2) s}{r^4 R}(e^+ - \pi^+)
\end{array} \right\}
\\
\overline{z_1 z_2} \left\{ \begin{array}{c}
\frac{x_0}{r^4 R^3}(r_1^2 e^- + r_2^2 e^+)(\pi R c - s) \\
+\frac{i\pi s}{r^4 R}(r_1^2 e^- - r_2^2 e^+) \\
-\frac{\pi x_0}{r^2 R^2} + \frac{2i\pi r_2^2 s c}{r^2 R^3} \\
+\frac{i r_2^2 s^2}{r_1^2 r^2 R^2} + \frac{4 i r_2^2}{r^6} |\pi^+ - e^+|^2 \\
+\frac{x_0 s c}{r^2 R^3} + \frac{2i x_0^2 r_2^2 s^2}{r^4 R^4}
\end{array} \right\}
&
\overline{z_2} \left\{ \begin{array}{c}
-\frac{2\pi r_2^2 s^2}{r^2 R^2} - \frac{r_1^2 s}{r^2 R^3}(e^- - c) \\
+\frac{\pi r_1^2}{r^2 R^2}(c e^- - 1) \\
-\frac{2i x_0 r_2^2 s}{r^2 R^4}(\pi R c - s) \\
+\frac{(r_1^2 + 3 r_2^2) s}{r^4 R}(e^- - \pi^-)
\end{array} \right\}
&
\begin{array}{c}
-\frac{2i\pi x_0 r_1^2 r_2^2 c s_x}{r^4 R^2} + \frac{2i\pi r_2^4 s c}{r^2 R^3} \\
+\frac{2i\pi r_1^2 r_2^2 s c_x}{r^4 R} + \frac{2i x_0 r_1^2 r_2^2 s s_x}{r^4 R^3} \\
+\frac{2i r_2^2 (r_2^2 - r_1^2)}{r^6} |\pi^+ - e^+|^2 \\
-\frac{2i r_1^2 r_2^2 s^2}{r^4 R^2} + \frac{2i r_1^2 r_2^2 s^2}{r^2 R^4} - \frac{2i r_2^2 s^2}{R^4}
\end{array}
\end{pmatrix}.
$$

We further have

$$
\begin{aligned}
[L_{\phi_1}, L_{\phi_2}][L_{r_1}, L_{r_2}](\mathbf{v}) &= ([L_{\phi_1}, L_{\phi_2}][L_{r_1}, L_{r_2}])^P(\mathbf{w}), \\
[L_{r_1}, L_{r_2}][L_{\phi_1}, L_{\phi_2}](\mathbf{v}) &= ([L_{r_1}, L_{r_2}][L_{\phi_1}, L_{\phi_2}])^P(\mathbf{w}), \\
[L_{\phi_1}, L_{r_1}][L_{\phi_2}, L_{r_2}](\mathbf{v}) &= ([L_{\phi_1}, L_{r_1}][L_{\phi_2}, L_{r_2}])^P(\mathbf{w}), \\
[L_{\phi_2}, L_{r_2}][L_{\phi_1}, L_{r_1}](\mathbf{v}) &= ([L_{\phi_2}, L_{r_2}][L_{\phi_1}, L_{r_1}])^P(\mathbf{w}), \\
[L_{\phi_2}, L_{r_1}][L_{\phi_1}, L_{r_2}](\mathbf{v}) &= ([L_{\phi_2}, L_{r_1}][L_{\phi_1}, L_{r_2}])^P(\mathbf{w}), \\
[L_{\phi_1}, L_{r_2}][L_{\phi_2}, L_{r_1}](\mathbf{v}) &= ([L_{\phi_1}, L_{r_2}][L_{\phi_2}, L_{r_1}])^P(\mathbf{w}),
\end{aligned}
$$

from which we deduce for L defined by (10)

$$
L(\mathbf{v}) = L^P(\mathbf{w}),\ L \cdot L_{x_0}(\mathbf{v}) = (L \cdot L_{x_0})^P(\mathbf{w}) \text{ and } \mathrm{Tr}\{L \cdot L_{x_0}\}(\mathbf{v}) = \mathrm{Tr}\{L \cdot L_{x_0}\}(\mathbf{w}).
$$

Since $dx_0 \wedge dr_1 \wedge dr_2 \wedge d\phi_1 \wedge d\phi_2 = dx_0 \wedge dr_2 \wedge dr_1 \wedge d(-\phi_2) \wedge d(-\phi_1)$, we have $\psi^\star(U_1)^\star \omega_5(\mathbf{v}) = \psi^\star(U_1)^\star \omega_5(\mathbf{w})$ by (9). Even if we make good use of these symmetries there is still some work left over to compute L. We finally obtain

$$
L(\mathbf{v}) \quad = \quad \frac{r_1 r_2}{R^3} \cdot
$$

$$
\begin{pmatrix}
\begin{array}{c}
\frac{12 r_1^2 s^2}{r^2}(\pi s c_x - \pi \frac{x_0}{R} c s_x + \frac{x_0}{R^2} s s_x) \\
+8s(2 - 3\frac{r_2^2}{r^2}) [\pi(c - c_x) \\
+\frac{x_0^2 s}{r^2 R}(1 - c c_x) - \frac{x_0}{r^2} s^2 s_x]
\end{array}
&
z_1 \left\{ \begin{array}{c}
\frac{12 s}{r^2}(1 - c c_x - \frac{x_0}{R} s s_x) \cdot \\
\cdot(-\pi x_0 c + i\pi R s + \frac{x_0}{R} s)
\end{array} \right\}
&
z_1 z_2 \left\{ \begin{array}{c}
\frac{12 s^2}{r^2}(\pi s c_x - \pi \frac{x_0}{R} c s_x + \frac{x_0}{R^2} s s_x) \\
+24 \frac{s}{r^2} [\pi(c - c_x) \\
+\frac{x_0^2 s}{r^2 R}(1 - c c_x - \frac{x_0}{R} s s_x)]
\end{array} \right\}
\\
\overline{z_1} \left\{ \begin{array}{c}
\frac{12 s}{r^2}(1 - c c_x - \frac{x_0}{R} s s_x) \cdot \\
\cdot(-\pi x_0 c - i\pi R s + \frac{x_0}{R} s)
\end{array} \right\}
&
\begin{array}{c}
-12 s^2(\pi s c_x - \pi \frac{x_0}{R} c s_x + \frac{x_0}{R^2} s s_x) \\
-8s [\pi(c - c_x) \\
+\frac{x_0^2 s}{r^2 R}(1 - c c_x) - \frac{x_0}{r^2} s^2 s_x]
\end{array}
&
z_2 \left\{ \begin{array}{c}
\frac{12 s}{r^2}(1 - c c_x - \frac{x_0}{R} s s_x) \cdot \\
\cdot(-\pi x_0 c - i\pi R s + \frac{x_0}{R} s)
\end{array} \right\}
\\
\overline{z_1 z_2} \left\{ \begin{array}{c}
\frac{12 s^2}{r^2}(\pi s c_x - \pi \frac{x_0}{R} c s_x + \frac{x_0}{R^2} s s_x) \\
+24 \frac{s}{r^2} [\pi(c - c_x) \\
+\frac{x_0^2 s}{r^2 R}(1 - c c_x - \frac{x_0}{R} s s_x)]
\end{array} \right\}
&
\overline{z_2} \left\{ \begin{array}{c}
\frac{12 s}{r^2}(1 - c c_x - \frac{x_0}{R} s s_r) \cdot \\
\cdot(-\pi x_0 c + i\pi R s + \frac{x_0}{R} s)
\end{array} \right\}
&
\begin{array}{c}
\frac{12 r_2^2 s^2}{r^2}(\pi s c_x - \pi \frac{x_0}{R} c s_x + \frac{x_0}{R^2} s s_x) \\
+8s(2 - 3\frac{r_1^2}{r^2}) [\pi(c - c_x) \\
+\frac{x_0^2 s}{r^2 R}(1 - c c_x) - \frac{x_0}{r^2} s^2 s_x]
\end{array}
\end{pmatrix}.
$$

Using

$$\mathrm{Tr}\{L\cdot L_{x_0}\}=\sum_{ij}L_{ij}(L_{x_0})_{ji}=\sum_{i}L_{ii}(L_{x_0})_{ii}+2i\sum_{i<j}\mathrm{Im}\{L_{ij}(L_{x_0})_{ji}\}$$

we have (omitting the argument $\mathbf{v}$)

$$\begin{aligned}\mathrm{Tr}\{L\cdot L_{x_0}\}=&i(\frac{2s^2}{R^2}-\frac{\pi sc}{R})\Big(r_1^2\,\mathrm{Im}\{\frac{L_{12}}{z_1}\}-r_2^2\,\mathrm{Im}\{\frac{L_{23}}{z_2}\}\Big)\\&+i\frac{\pi}{2}(L_{11}-2L_{22}+L_{33})+i\Big(\frac{\pi(1+c^2)}{2R^2}-\frac{sc}{R^3}\Big)S\\ \text{with}\qquad S=&\Big[-r_1^2L_{11}+r^2L_{22}-r_2^2L_{33}-2r_1^2r_2^2\,\mathrm{Re}\{\frac{L_{13}}{z_1z_2}\}\\&+2x_0\Big(r_1^2\,\mathrm{Re}\{\frac{L_{12}}{z_1}\}+r_2^2\,\mathrm{Re}\{\frac{L_{23}}{z_2}\}\Big)\Big].\end{aligned}\tag{11}$$

We obtain

$$r_1^2\,\mathrm{Im}\{\frac{L_{12}}{z_1}\}-r_2^2\,\mathrm{Im}\{\frac{L_{23}}{z_2}\}=\frac{r_1r_2}{R^3}12\pi Rs^2(1-cc_x-\frac{x_0}{R}ss_x),\tag{12}$$

$$r_1^2\,\mathrm{Re}\{\frac{L_{12}}{z_1}\}+r_2^2\,\mathrm{Re}\{\frac{L_{23}}{z_2}\}=\frac{r_1r_2}{R^3}12x_0s(\frac{s}{R}-\pi c)(1-cc_x-\frac{x_0}{R}ss_x),$$

$$\begin{aligned}L_{11}-2L_{22}+L_{33}=&\frac{r_1r_2}{R^3}\Big\{36s^2\Big[\pi(sc_x-\frac{x_0}{R}cs_x)+\frac{x_0}{R^2}ss_x\Big]\\&+24s\Big[\pi(c-c_x)+\frac{x_0^2s}{r^2R}(1-cc_x)-\frac{x_0}{r^2}s^2s_x\Big]\Big\},\\ -2r_1^2r_2^2\,\mathrm{Re}\{\frac{L_{13}}{z_1z_2}\}=&\frac{r_1r_2}{R^3}\frac{r_1^2r_2^2}{r^2}\Big\{-24s^2\Big[\pi(sc_x-\frac{x_0}{R}cs_x)-\frac{x_0}{R^2}ss_x\Big]\\&-48s\Big[\pi(c-c_x)+\frac{x_0^2s}{r^2R}(1-cc_x-\frac{x_0}{R}ss_x)\Big]\Big\},\\ -r_1^2L_{11}+r^2L_{22}-r_2^2L_{33}=&\frac{r_1r_2}{R^3}r^2\Big\{-24s^2(1-\frac{r_1^2r_2^2}{r^4})\Big[\pi(sc_x-\frac{x_0}{R}cs_x)+\frac{x_0}{R^2}ss_x\Big]\\&-24s\frac{r_1^4+r_2^4}{r^4}\Big[\pi(c-c_x)+\frac{x_0^2s}{r^2R}(1-cc_x)-\frac{x_0}{r^2}s^2s_x\Big]\Big\},\end{aligned}$$

$$\text{thus:}\qquad S=-\frac{r_1r_2}{R^3}24\pi R^2sc(1-cc_x-\frac{x_0}{R}ss_x)\tag{13}$$

$$\begin{aligned}\text{and}\qquad \mathrm{Tr}\{L\cdot L_{x_0}\}=&i\frac{r_1r_2}{R^3}\Big\{12\pi^2s(c_x-c)-6\pi^2s^2(sc_x-\frac{x_0}{R}cs_x)\\&-6\pi(2\frac{R^2}{r^2}+1)\frac{x_0}{R^2}s^3s_x+6\pi(2\frac{R^2}{r^2}+2)\frac{s^2}{R}(1-cc_x)\Big\}.\end{aligned}$$

It is finally done. For the desired 5-form on the southern hemisphere of the S^5 we end up with

$$\begin{aligned}\psi^\star(U_1)^\star\omega_5=&30i\pi\frac{r_1r_2}{R^3}\Big[-\pi\sin^2\pi R(\sin\pi R\cos\pi x_0-\frac{x_0}{R}\cos\pi R\sin\pi x_0)\\&+2\pi\sin\pi R(\cos\pi x_0-\cos\pi R)+(2\frac{R^2}{r^2}+2)\frac{\sin^2\pi R}{R}(1-\cos\pi R\cos\pi x_0)\\&-(2\frac{R^2}{r^2}+1)\frac{x_0}{R^2}\sin^3\pi R\sin\pi x_0\Big]dx_0\wedge dr_1\wedge dr_2\wedge d\phi_1\wedge d\phi_2.\end{aligned}\tag{14}$$

After this preliminary work it is quite easy now to compute $(\widetilde{U_1})^\star\omega_5$ on the northern hemisphere. We rewrite $\widetilde{U_1}$ as

$$\widetilde{U_1} = F\overline{U_1}^P F^P = \left(F^P\overline{U_1}F\right)^P,$$

where we have defined $F := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. This yields

$$\widetilde{U_1}^\dagger = \left(F\overline{U_1}^\dagger F^P\right)^P, \qquad \frac{\partial\widetilde{U_1}}{\partial x_i} = \left(F^P\overline{\frac{\partial U_1}{\partial x_i}}F\right)^P,$$

because complex conjugation and differentiation along real variables commute. We thus have

$$\begin{aligned} \widetilde{L_{x_i}} &= (F\overline{L_{x_i}}F)^P, \\ \mathrm{Tr}\{\widetilde{L}\cdot\widetilde{L_{x_0}}\} &= \mathrm{Tr}\{(F\overline{L\cdot L_{x_0}}F)^P\} = \mathrm{Tr}\{F\overline{L\cdot L_{x_0}}F\} \\ &= \mathrm{Tr}\{\overline{L\cdot L_{x_0}}\} = \overline{\mathrm{Tr}\{L\cdot L_{x_0}\}} \end{aligned}$$

and immediately obtain

$$(\widetilde{U_1})^\star\omega_5 = \overline{(U_1)^\star\omega_5} = -(U_1)^\star\omega_5. \tag{15}$$

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