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NUMERICAL HOMOGENIZATION OF H(CURL)-PROBLEMS*

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Abstract. If an elliptic differential operator associated with an **H**(curl)-problem involves rough (rapidly varying) coefficients, then solutions to the corresponding **H**(curl)-problem admit typically very low regularity, which leads to arbitrarily bad convergence rates for conventional numerical schemes. The goal of this paper is to show that the missing regularity can be compensated through a corrector operator. More precisely, we consider the lowest-order Nédélec finite element space and show the existence of a linear corrector operator with four central properties: it is computable, **H**(curl)-stable, and quasi-local and allows for a correction of coarse finite element functions so that first-order estimates (in terms of the coarse mesh size) in the **H**(curl) norm are obtained provided the right-hand side belongs to **H**(div). With these four properties, a practical application is to construct generalized finite element spaces which can be straightforwardly used in a Galerkin method. In particular, this characterizes a homogenized solution and a first-order corrector, including corresponding quantitative error estimates without the requirement of scale separation. The constructed generalized finite element method falls into the class of localized orthogonal decomposition methods, which have not been studied for **H**(curl)-problems so far.

Key words. multiscale method, wave propagation, Maxwell's equations, finite element method, a priori error estimates

AMS subject classifications. 35Q61, 65N12, 65N15, 65N30, 78M10

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1. Introduction.

Electromagnetic wave propagation plays an essential role in many physical applications, for instance, in the large field of wave optics. In recent years, multiscale and heterogeneous materials have been studied with great interest, e.g., in the context of photonic crystals [32]. These materials can exhibit unusual and astonishing (optical) properties, such as band gaps, perfect transmission, or negative refraction [36, 17, 35]. These problems are modeled by Maxwell's equations, which involve the curl-operator and the associated Sobolev space **H**(curl). Additionally, the coefficients in the problems are rapidly oscillating on a fine scale for the context of photonic crystals and metamaterials. The numerical simulation and approximation of the solution are then a challenging task for the following three reasons: 1. As with all multiscale problems, a direct treatment with standard methods is infeasible in many cases because it needs grids which resolve all discontinuities or oscillations of the material parameters. 2. Solutions to **H**(curl)-problems with discontinuous coefficients in Lipschitz domains can have arbitrarily low regularity; see [5, 14, 13]. Hence, standard

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methods (see, e.g., [38] for an overview) suffer from bad convergence rates and fine meshes are needed to have a tolerably small error. 3. Due to the large kernel of the curl-operator, we cannot expect that the L^2-norm is of a lower order compared to the full $\mathbf{H}(\mathrm{curl})$-norm (the energy norm). Thus, it is necessary to consider dual norms or the Helmholtz decomposition to obtain improved a priori error estimates.

In order to deal with the rapidly oscillating material parameters, we consider multiscale methods and thereby aim at a feasible numerical simulation. In general, these methods try to decompose the exact solution into a macroscopic contribution (without oscillations), which can be discretized on a coarse mesh, and a fine-scale contribution. Analytical homogenization for locally periodic $\mathbf{H}(\mathrm{curl})$-problems shows that there exists such a decomposition, where the macroscopic part is a good approximation in H^{-1} and an additional fine-scale corrector leads to a good approximation in L^2 and $\mathbf{H}(\mathrm{curl})$; cf. [48, 27, 49]. Based on these analytical results, multiscale methods are developed, e.g., the heterogeneous multiscale method in [27, 12] and asymptotic expansion methods in [9]. The question is now how far such considerations can be extended beyond the (locally) periodic case.

The main contribution of this paper is the numerical homogenization of $\mathbf{H}(\mathrm{curl})$-elliptic problems—beyond the periodic case and without assuming scale separation. The main findings can be summarized as follows. We show that the exact solution can indeed be decomposed into a coarse and a fine part, using a suitable interpolation operator. The coarse part gives an optimal approximation in a negative Sobolev norm, the best we can hope for in this situation, namely, the $\mathbf{H}(\mathrm{div})'$ norm. In order to obtain optimal L^2 and $\mathbf{H}(\mathrm{curl})$ approximations, we have to add a so-called fine-scale corrector or corrector Green's operator. This corrector shows exponential decay and can therefore be truncated to local patches of macroscopic elements, so that it can be computed efficiently.

This technique of numerical homogenization is known as localized orthogonal decomposition (LOD) and arose from the framework of the variational multiscale method, where we refer to [6, 30, 31, 37, 43] for historically important steps in this direction. A game theoretic interpretation of the methodology, using so-called gamblets, was recently given in [40] (see also [41]). Let us describe the contribution of this paper in the usual language of an LOD: The LOD framework decomposes the solution space into a coarse finite-dimensional space (spanned by standard finite element functions) and a fine-scale space, expressed as the kernel of a suitable interpolation/projection operator. A generalized finite element basis is constructed by adding *corrections* from the fine-scale space to the standard basis functions. These corrections are computed as solutions of a PDE on a fine grid, i.e., what we called the corrector Green's operator above. For all problem classes considered so far (see below), the corrections show an exponential decay, which justifies truncating their computation to patches of coarse elements. The LOD has been extensively studied in the context of Lagrange finite elements [37, 26, 28], where we particularly refer to the contributions written on wave phenomena [1, 7, 8, 24, 39, 44, 45]. Aside from Lagrange finite elements, an LOD application in Raviart–Thomas spaces was given in [25].

In this spirit, this contribution can be seen as an extension of periodic homogenization results to more general rapidly varying coefficients or as an application of the LOD framework to a new problem class, namely, $\mathbf{H}(\mathrm{curl})$-elliptic problems. We try to cover both views throughout this paper.

A crucial ingredient for numerical homogenization procedures in the spirit of LODs is the choice of a suitable interpolation operator. As we will see later, in our case we require it to be computable, $\mathbf{H}(\mathrm{curl})$-stable, and (quasi-)local and that it commutes with the curl-operator. Constructing an operator that enjoys such properties is a very delicate task and a lot of operators have been suggested—with different backgrounds and applications in mind. The nodal interpolation operator (see, e.g., [38, Theorem 5.41]) and the interpolation operators introduced in [15] are not well-defined on $\mathbf{H}(\mathrm{curl})$ and hence lack the required stability. Various (quasi-)interpolation operators are constructed as a composition of smoothing and some (nodal) interpolation, such as [10, 11, 16, 19, 46, 47]. For all of them, the kernel of the operator is practically hard or even impossible to compute and they fulfill only the projection *or* the locality property. Finally, we mention the interpolation operator of [20] which is local and a projection, but which, however, does not commute with the exterior derivative. A suitable candidate (and to the authors' best knowledge, the only one) that enjoys all required properties was proposed by Falk and Winther in [21].

This paper thereby also shows the applicability of the Falk–Winther operator. In this context, we mention two results, which may be of their own interest: a localized regular decomposition of the interpolation error (in the spirit of [47]) and the practicable implementation of the Falk–Winther operator as a matrix. The last point admits the efficient implementation of our numerical scheme and we refer to [18] for general considerations.

The paper is organized as follows. Section 2 introduces the general curl-curl-problem under consideration and briefly mentions its relation to Maxwell's equations. In section 3, we give a short motivation of our approach from two perspectives: periodic homogenization and the (ideal) LOD. Section 4 introduces the necessary notation for meshes, finite element spaces, and interpolation operators. We introduce the corrector Green's operator in section 5 and show its approximation properties. We localize the corrector operator in section 6 and present the main a priori error estimates. The proofs of the decay of the correctors are given in section 7. Details on the definition of the interpolation operator and its implementation are given in section 8.

The notation $a \lesssim b$ is used for $a \leq Cb$ with a constant C independent of the mesh size H and the oversampling parameter m. It will be used in (technical) proofs for simplicity and readability.

2. Model problem.

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded, contractible domain with polyhedral Lipschitz boundary. We consider the following so-called curl-curl-problem: Find $\mathbf{u} : \Omega \to \mathbb{C}^3$ such that

$$\begin{aligned} \mathrm{curl}(\mu\, \mathrm{curl}\, \mathbf{u}) + \kappa \mathbf{u} &= \mathbf{f} \quad \text{in } \Omega, \\ \mathbf{u} \times \mathbf{n} &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{2.1}$$

with the outer unit normal $\mathbf{n}$ of Ω. Exact assumptions on the parameters μ and κ and the right-hand-side $\mathbf{f}$ are given in Assumption 1 below, but we implicitly assume that the above problem is a multiscale problem, i.e., the coefficients μ and κ are rapidly varying on a very fine scale.

Such curl-curl-problems arise in various formulations and reductions of Maxwell's equations and we give a few examples. In all cases, our coefficient μ equals $\tilde{\mu}^{-1}$ with the magnetic permeability $\tilde{\mu}$, a material parameter. The right-hand-side $\mathbf{f}$ is related to (source) current densities. One possible example is Maxwell's equations in a linear conductive medium, subject to Ohm's law, together with the so-called time-harmonic ansatz $\hat{\boldsymbol{\psi}}(x, t) = \boldsymbol{\psi}(x) \exp(-i\omega t)$ for all fields. In this case, one obtains the above curl-curl-problem with $\mathbf{u} = \mathbf{E}$, the electric field, and $\kappa = i\omega\sigma - \omega^2\varepsilon$ related to the electric permittivity ε and the conductivity σ of the material. Another example is implicit time-step discretizations of eddy current simulations, where the above curl-curl-problem has to be solved in each time step. In that case $\mathbf{u}$ is the vector potential associated with the magnetic field and $\kappa \approx \sigma/\tau$, where τ is the time-step size. Coefficients with multiscale properties can, for instance, arise in the context of photonic crystals.

Before we define the variational problem associated with our general curl-curl-problem (2.1), we need to introduce some function spaces. In the following, bold face letters will indicate vector-valued quantities and all functions are complex-valued, unless explicitly mentioned. For any bounded subdomain $G \subset \Omega$, we define the space

$$\mathbf{H}(\mathrm{curl}, G) := \{\mathbf{v} \in L^2(G, \mathbb{C}^3) |\, \mathrm{curl}\, \mathbf{v} \in L^2(G, \mathbb{C}^3)\}$$

with the inner product $(\mathbf{v}, \mathbf{w})_{\mathbf{H}(\mathrm{curl},G)} := (\mathrm{curl}\, \mathbf{v}, \mathrm{curl}\, \mathbf{w})_{L^2(G)} + (\mathbf{v}, \mathbf{w})_{L^2(G)}$ with the complex L^2-inner product. We will omit the domain G if it is equal to the full domain Ω. The restriction of $\mathbf{H}(\mathrm{curl}, \Omega)$ to functions with a zero tangential trace is defined as

$$\mathbf{H}_0(\mathrm{curl}, \Omega) := \{\mathbf{v} \in \mathbf{H}(\mathrm{curl}, \Omega) |\; \mathbf{v} \times \mathbf{n}|_{\partial\Omega} = 0\}.$$

Similarly, we define the space

$$\mathbf{H}(\mathrm{div}, G) := \{\mathbf{v} \in L^2(G, \mathbb{C}^3) |\, \mathrm{div}\, \mathbf{v} \in L^2(G, \mathbb{C})\}$$

with corresponding inner product $(\cdot, \cdot)_{\mathbf{H}(\mathrm{div},G)}$. For more details we refer to [38].

We make the following assumptions on the data of our problem.

ASSUMPTION 1. *Let $\mathbf{f} \in \mathbf{H}(\mathrm{div}, \Omega)$ and $\mu \in L^\infty(\Omega, \mathbb{R}^{3\times3})$ and $\kappa \in L^\infty(\Omega, \mathbb{C}^{3\times3})$ be self-adjoint. For any open subset $G \subset \Omega$, we define the sesquilinear form $\mathcal{B}_G$: $\mathbf{H}(\mathrm{curl}, G) \times \mathbf{H}(\mathrm{curl}, G) \to \mathbb{C}$ as*

$$\mathcal{B}_G(\mathbf{v}, \boldsymbol{\psi}) := (\mu\, \mathrm{curl}\, \mathbf{v}, \mathrm{curl}\, \boldsymbol{\psi})_{L^2(G)} + (\kappa \mathbf{v}, \boldsymbol{\psi})_{L^2(G)} \tag{2.2}$$

and set $\mathcal{B} := \mathcal{B}_\Omega$. The form $\mathcal{B}_G$ is obviously continuous, i.e., there is $C_B > 0$ such that

$$|\mathcal{B}_G(\mathbf{v}, \boldsymbol{\psi})| \leq C_B \|\mathbf{v}\|_{\mathbf{H}(\mathrm{curl},G)} \|\boldsymbol{\psi}\|_{\mathbf{H}(\mathrm{curl},G)} \quad \textit{for all } \mathbf{v}, \boldsymbol{\psi} \in \mathbf{H}(\mathrm{curl}, G).$$

We furthermore assume that μ and κ are such that $\mathcal{B} : \mathbf{H}_0(\mathrm{curl}) \times \mathbf{H}_0(\mathrm{curl}) \to \mathbb{C}$ is $\mathbf{H}_0(\mathrm{curl})$-elliptic, i.e., there is $\alpha > 0$ such that

$$|\mathcal{B}(\mathbf{v}, \mathbf{v})| \geq \alpha \|\mathbf{v}\|^2_{\mathbf{H}(\mathrm{curl})} \quad \textit{for all } \mathbf{v} \in \mathbf{H}_0(\mathrm{curl}).$$

The assumption that $\mathcal{B}$ is self-adjoint is made for better readability because in this case the discretization is a Galerkin method instead of a Petrov–Galerkin method. It is not an essential restriction. We now give a precise definition of our model problem for this article. Let Assumption 1 be fulfilled. We look for $\mathbf{u} \in \mathbf{H}_0(\mathrm{curl}, \Omega)$ such that

$$\mathcal{B}(\mathbf{u}, \boldsymbol{\psi}) = (\mathbf{f}, \boldsymbol{\psi})_{L^2(\Omega)} \quad \text{for all } \boldsymbol{\psi} \in \mathbf{H}_0(\mathrm{curl}, \Omega). \tag{2.3}$$

Existence and uniqueness of a solution to (2.3) follow from the Lax–Milgram–Babuška theorem [4]. Assumption 1 is fulfilled in the following two important examples mentioned at the beginning: (i) a strictly positive real function in the identity term, i.e., $\kappa \in L^\infty(\Omega, \mathbb{R})$, as it occurs in the time-step discretization of eddy-current problems; and (ii) a complex κ with strictly negative real part and strictly positive imaginary part, as it occurs for time-harmonic Maxwell's equations in a conductive medium. Further possibilities of μ and κ yielding an $\mathbf{H}(\mathrm{curl})$-elliptic problem are described in [23].

Remark 2. The assumption of contractibility of Ω is required only to ensure the existence of local regular decompositions later used in the proof of Lemma 4. We note that this assumption can be relaxed by assuming that Ω is simply connected in certain local subdomains formed by unions of tetrahedra (i.e., in patches of the form $\mathrm{N}(\Omega_P)$, using the notation from Lemma 4).

3. Motivation of the approach. As explained in the introduction, our contribution can be viewed from two perspectives: (numerical) homogenization and the LOD framework, which of course are connected.

3.1. Motivation via homogenization. For the sake of the argument, let us consider model problem (2.1) for the case that the coefficients μ and κ are replaced by parametrized multiscale coefficients μ_δ and κ_δ, respectively. Here, $0 < \delta \ll 1$ is a small parameter that characterizes the roughness of the coefficient or respectively the speed of the variations, i.e., the smaller the δ, the faster the oscillations of μ_δ and κ_δ. If we discretize this model problem in the lowest-order Nédélec finite element space $\mathring{\mathcal{N}}(\mathcal{T}_H)$, we have the classical error estimate of the form

$$\inf_{\mathbf{v}_H\in\mathring{\mathcal{N}}(\mathcal{T}_H)} \|\mathbf{u}_\delta - \mathbf{v}_H\|_{\mathbf{H}(\mathrm{curl})} \leq CH\left(\|\mathbf{u}_\delta\|_{H^1(\Omega)} + \|\operatorname{curl}\mathbf{u}_\delta\|_{H^1(\Omega)}\right)$$

with the mesh size H. However, if the coefficients μ_δ and κ_δ are discontinuous the necessary regularity for this estimate is not available; see [13, 14, 5]. On the other hand, if μ_δ and κ_δ are sufficiently regular but δ small, then we face the blow-up with $\|\mathbf{u}_\delta\|_{H^1(\Omega)} + \|\operatorname{curl}\mathbf{u}_\delta\|_{H^1(\Omega)} \to \infty$ for $\delta \to 0$, which makes the estimate useless in practice, unless the mesh size H becomes very small to compensate for the blow-up. This does not change if we replace the $\mathbf{H}$(curl)-norm by the $L^2(\Omega)$-norm since both norms are equivalent in the kernel of the curl-operator, i.e., in the subspace of gradient functions.

To understand if there exist any meaningful approximations of $\mathbf{u}_\delta$ in $\mathring{\mathcal{N}}(\mathcal{T}_H)$ (even on coarse meshes), we make a short excursus to classical homogenization theory. For that we assume that the coefficients $\mu_\delta(x) = \mu(x/\delta)$ and $\kappa_\delta(x) = \kappa(x/\delta)$ are periodically oscillating with period δ. In this case it is known (cf. [12, 27, 49]) that the sequence of exact solutions $\mathbf{u}_\delta$ converges weakly in $\mathbf{H}_0$(curl) to a *homogenized* function $\mathbf{u}_0$. Since $\mathbf{u}_0 \in \mathbf{H}_0(\mathrm{curl})$ is δ-independent and slow, it can be well approximated in $\mathring{\mathcal{N}}(\mathcal{T}_H)$. Furthermore, there exists a *corrector* $\mathcal{K}_\delta(\mathbf{u}_0)$ such that

$$\mathbf{u}_\delta \approx (\mathrm{id} + \mathcal{K}_\delta)\mathbf{u}_0$$

is a good approximation in $\mathbf{H}$(curl), i.e., the error converges strongly to zero with

$$\|\mathbf{u}_\delta - (\mathbf{u}_0 + \mathcal{K}_\delta(\mathbf{u}_0))\|_{\mathbf{H}(\mathrm{curl})} \to 0 \qquad \text{for } \delta \to 0.$$

Here the nature of the corrector is revealed by two estimates. In fact, $\mathcal{K}_\delta(\mathbf{u}_0)$ admits a decomposition into a gradient part and a part with small amplitude (cf. [27, 48, 49]) such that

$$\mathcal{K}_\delta(\mathbf{u}_0) = \mathbf{z}_\delta + \nabla\theta_\delta$$

with

$$\delta^{-1}\|\mathbf{z}_\delta\|_{L^2(\Omega)} + \|\mathbf{z}_\delta\|_{\mathbf{H}(\mathrm{curl})} \leq C\|\mathbf{u}_0\|_{\mathbf{H}(\mathrm{curl})} \tag{3.1}$$

$$\text{and} \qquad \delta^{-1}\|\theta_\delta\|_{L^2(\Omega)} + \|\nabla\theta_\delta\|_{L^2(\Omega)} \leq C\|\mathbf{u}_0\|_{\mathbf{H}(\mathrm{curl})}, \tag{3.2}$$

where $C = C(\alpha, C_B)$ only depends on the constants appearing in Assumption 1. First, we immediately see that the estimates imply that $\mathcal{K}_\delta(\mathbf{u}_0)$ is $\mathbf{H}(\text{curl})$-stable in the sense that it holds that

$$\|\mathcal{K}_\delta(\mathbf{u}_0)\|_{\mathbf{H}(\text{curl})} \le C\|\mathbf{u}_0\|_{\mathbf{H}(\text{curl})}.$$

Second, and more interestingly, we see that alone from the above properties, we can conclude that $\mathbf{u}_0$ *must* be a good approximation of the exact solution in the space $H^{-1}(\Omega, \mathbb{C}^3) := H^1_0(\Omega, \mathbb{C}^3)'$. In fact, using (3.1) and (3.2) we have for any $\mathbf{v} \in H^1_0(\Omega, \mathbb{C}^3)$ with $\|\mathbf{v}\|_{H^1(\Omega)} = 1$ that

$$\left|\int_\Omega \mathcal{K}_\delta(\mathbf{u}_0)\cdot\mathbf{v}\right| = \left|\int_\Omega \mathbf{z}_\delta\cdot\mathbf{v} - \int_\Omega \theta_\delta\,(\nabla\cdot\mathbf{v})\right| \le \|\mathbf{z}_\delta\|_{L^2(\Omega)} + \|\theta_\delta\|_{L^2(\Omega)} \le C\delta\|\mathbf{u}_0\|_{\mathbf{H}(\text{curl})}.$$

Consequently we have strong convergence in $H^{-1}(\Omega)$ with

$$\|\mathbf{u}_\delta - \mathbf{u}_0\|_{H^{-1}(\Omega)} \le \|\mathbf{u}_\delta - (\mathbf{u}_0 + \mathcal{K}_\delta(\mathbf{u}_0))\|_{H^{-1}(\Omega)} + \|\mathcal{K}_\delta(\mathbf{u}_0)\|_{H^{-1}(\Omega)} \overset{\delta\to 0}{\longrightarrow} 0.$$

We conclude two things. First, even though the coarse space $\mathring{\mathcal{N}}(\mathcal{T}_H)$ does not contain good $\mathbf{H}(\text{curl})$- or L^2-approximations, it still contains meaningful approximations in $H^{-1}(\Omega)$. Second, the fact that the coarse part $\mathbf{u}_0$ is a good H^{-1}-approximation of $\mathbf{u}_\delta$ is an intrinsic conclusion from the properties of the correction $\mathcal{K}_\delta(\mathbf{u}_0)$. A refined analysis reveals that the numerical homogenization method presented here allows for estimates in the stronger $\mathbf{H}(\text{div})'$ norm.

In this paper we are concerned with the question of whether the above considerations can be transferred to a discrete setting beyond the assumption of periodicity. More precisely, defining a coarse level of resolution through the space $\mathring{\mathcal{N}}(\mathcal{T}_H)$, we ask if it is possible to find a coarse function $\mathbf{u}_H$ and an (efficiently computable) $\mathbf{H}(\text{curl})$-stable operator $\mathcal{K}$ such that

$$\|\mathbf{u}_\delta - \mathbf{u}_H\|_{H^{-1}(\Omega)} \le CH \qquad \text{and} \qquad \|\mathbf{u}_\delta - (\text{id} + \mathcal{K})\mathbf{u}_H\|_{\mathbf{H}(\text{curl})} \le CH, \tag{3.3}$$

with C being independent of the oscillations in terms of δ. A natural ansatz for the coarse part is $\mathbf{u}_H = \pi_H(\mathbf{u}_\delta)$ for a suitable projection $\pi_H : \mathbf{H}(\text{curl}) \to \mathring{\mathcal{N}}(\mathcal{T}_H)$. From the considerations above, it is desirable that the (interpolation) error $\mathbf{u}_\delta - \pi_H(\mathbf{u}_\delta)$ fulfills a discrete analogue to the estimates (3.1) and (3.2). Hence, we look for a projector π_H with the following property: there are $\mathbf{z} \in \mathbf{H}^1_0(\Omega)$ and $\theta \in H^1_0(\Omega)$ such that

$$\mathbf{v} - \pi_H\mathbf{v} = \mathbf{z} + \nabla\theta$$

and

$$\begin{aligned} H^{-1}\|\mathbf{z}\|_{L^2(\Omega)} + \|\nabla\mathbf{z}\|_{L^2(\Omega)} &\le C\|\operatorname{curl}\mathbf{v}\|_{L^2(\Omega)},\\ H^{-1}\|\theta\|_{L^2(\Omega)} + \|\nabla\theta\|_{L^2(\Omega)} &\le C\|\mathbf{v}\|_{\mathbf{H}(\text{curl})}. \end{aligned} \tag{3.4}$$

Note that the above properties are not fulfilled for, e.g., the L^2-projection.

We conclude this paragraph by summarizing that we want to have a projection π_H fulfilling (3.4). We can then define a coarse scale numerically through the space $\mathring{\mathcal{N}}(\mathcal{T}_H) = \text{im}(\pi_H)$. Moreover, the corrector $\mathcal{K}$ should be constructed such that it maps into the kernel of the projection operator, i.e., $\text{im}(\mathcal{K}) \subset \ker(\pi_H)$ in order to inherit the estimates (3.3).

3.2. Motivation via the localized orthogonal decomposition. The question of decomposing the solution space into a coarse and a fine part is also the key motivation for the LOD. The idea is to write $\mathbf{H}_0(\text{curl}) = \mathring{\mathcal{N}}(\mathcal{T}_H) \oplus \mathbf{W}$ with $\mathbf{W} = \ker\pi_H$, where $\pi_H : \mathbf{H}_0(\mathrm{curl}) \to \mathring{\mathcal{N}}(\mathcal{T}_H)$ is a suitable projection. One can define a correction operator $\mathcal{K} : \mathbf{H}_0(\mathrm{curl}) \to \mathbf{W}$ via

$$\mathcal{B}(\mathcal{K}\mathbf{v}, \mathbf{w}) = -\mathcal{B}(\mathbf{v}, \mathbf{w}) \qquad \text{for all} \quad \mathbf{w} \in \mathbf{W}. \tag{3.5}$$

The (ideal) LOD is now a Galerkin method over the space $(\mathrm{id} + \mathcal{K})\mathring{\mathcal{N}}(\mathcal{T}_H)$, i.e., we look for $\mathbf{u}_H \in \mathring{\mathcal{N}}(\mathcal{T}_H)$ such that

$$\mathcal{B}((\mathrm{id} + \mathcal{K})\mathbf{u}_H, (\mathrm{id} + \mathcal{K})\mathbf{v}_H) = (\mathbf{f}, (\mathrm{id} + \mathcal{K})\mathbf{v}_H) \qquad \text{for all} \quad \mathbf{v}_H \in \mathring{\mathcal{N}}(\mathcal{T}_H). \tag{3.6}$$

Problem (3.5), however, is global and therefore very costly to solve. In order to obtain a localized method the computation has to be truncated to patches $\mathrm{N}^m(T)$ of diameter approximately mH; see section 4 for a precise definition. The overall scheme can then be described as follows: Consider a basis $\{\mathbf{\Phi}_k | \, 1 \leq k \leq N\}$ of $\mathring{\mathcal{N}}(\mathcal{T}_H)$. For all $T \in \mathcal{T}_H$ with $T \subset \mathrm{supp}(\mathbf{\Phi}_k)$, we solve for $\mathcal{K}_{T,m}(\mathbf{\Phi}_k) \in \mathbf{W}(\mathrm{N}^m(T))$ with

$$\mathcal{B}_{\mathrm{N}^m(T)}(\mathcal{K}_{T,m}(\mathbf{\Phi}_k), \mathbf{w}) = -\mathcal{B}_T(\mathbf{\Phi}_k, \mathbf{w}) \qquad \text{for all } \mathbf{w} \in \mathbf{W}(\mathrm{N}^m(T)).$$

Defining the corrector operator $\mathcal{K}_m$ via

$$\mathcal{K}_m(\mathbf{\Phi}_k) = \sum_{\substack{T \in \mathcal{T}_H \\ T \subset \mathrm{supp}(\mathbf{\Phi}_k)}} \mathcal{K}_{T,m}(\mathbf{\Phi}_k),$$

one looks for the solution $\mathbf{u}_{H,m} \in \mathring{\mathcal{N}}(\mathcal{T}_H)$ of (3.6) with $\mathcal{K}$ replaced by $\mathcal{K}_m$; see section 6 for details.

In the LOD framework the following questions now have to be answered: (i) What approximation properties do $\mathbf{u}_H$ and $(\mathrm{id} + \mathcal{K})\mathbf{u}_H$ have? (ii) Can we truncate the computation in (3.5) to patches of elements without losing the approximation properties?

With a view to these two questions, let us briefly describe the main challenges for $\mathbf{H}(\mathrm{curl})$-problems in contrast to elliptic diffusion problems. Concerning (i), denoting $e = \mathbf{u} - (\mathrm{id} + \mathcal{K})\mathbf{u}_H$, where $\mathbf{u}$ solves (2.3), one observes $\pi_H e = 0$ and quickly comes to the estimate

$$\|e\|^2_{\mathbf{H}(\mathrm{curl})} \lesssim |(\mathbf{f}, e)| = |(\mathbf{f}, e - \pi_H e)|.$$

At this point, the approximation properties of π_H play a crucial role. For elliptic diffusion problems, there are several possible choices for I_H which fulfill

$$\|v - I_H v\|_{L^2(\Omega)} \lesssim H \|\nabla v\|_{L^2(\Omega)} \qquad \text{for all} \quad v \in H^1(\Omega).$$

Such an estimate (with the gradient replaced by the curl), however, cannot hold in $\mathbf{H}(\mathrm{curl})$ because of the large kernel of the curl-operator. Instead, one has to hope for estimates like (3.4) in order to deduce

$$|(\mathbf{f}, e - \pi_H e)| = |(\mathbf{f}, \mathbf{z} + \nabla \theta)| \leq |(\mathbf{f}, \mathbf{z})| + |(\mathrm{div}\, \mathbf{f}, \theta)| \lesssim H \|\mathbf{f}\|_{\mathbf{H}(\mathrm{div})} \|e\|_{\mathbf{H}(\mathrm{curl})},$$

where we also see the role of the assumption $\mathbf{f} \in \mathbf{H}(\mathrm{div})$. This difference between the gradient subspace and its complement also has to be considered when studying the exponential decay of $\mathcal{K}$ to answer (ii).

4. Mesh and interpolation operator. In this section we introduce the basic notation for establishing our coarse scale discretization and we will present a projection operator that fulfills the sufficient conditions derived in the previous section.

Let $\mathcal{T}_H$ be a regular partition of Ω into tetrahedra such that $\cup\mathcal{T}_H = \overline{\Omega}$ and any two distinct $T, T' \in \mathcal{T}_H$ either are disjoint or share a common vertex, edge, or face. We assume the partition $\mathcal{T}_H$ to be shape-regular and quasi-uniform. The global mesh size is defined as $H := \max\{\operatorname{diam}(T) | T \in \mathcal{T}_H\}$. $\mathcal{T}_H$ is a coarse mesh in the sense that it does not resolve the fine-scale oscillations of the parameters.

Given any (possibly even not connected) subdomain $G \subset \overline{\Omega}$ define its neighborhood via

$$\mathrm{N}(G) := \operatorname{int}(\cup\{T \in \mathcal{T}_H | T \cap \overline{G} \neq \emptyset\})$$

and for any $m \geq 2$ the patches

$$\mathrm{N}^1(G) := \mathrm{N}(G) \qquad \text{and} \qquad \mathrm{N}^m(G) := \mathrm{N}(\mathrm{N}^{m-1}(G)).$$

The shape regularity implies that there is a uniform bound $C_{\mathrm{ol},m}$ on the number of elements in the mth order patch

$$\max_{T \in \mathcal{T}_H} \operatorname{card}\{K \in \mathcal{T}_H | K \subset \overline{\mathrm{N}^m(T)}\} \leq C_{\mathrm{ol},m}$$

and the quasi-uniformity implies that $C_{\mathrm{ol},m}$ depends polynomially on m. We abbreviate $C_{\mathrm{ol}} := C_{\mathrm{ol},1}$.

The space of $\mathcal{T}_H$-piecewise affine and continuous functions is denoted by $\mathcal{S}^1(\mathcal{T}_H)$. We denote the lowest-order Nédélec finite element (cf. [38, section 5.5]) by

$$\mathring{\mathcal{N}}(\mathcal{T}_H) := \{\mathbf{v} \in \mathbf{H}_0(\mathrm{curl}) | \text{ for all}\, T \in \mathcal{T}_H : \mathbf{v}|_T(\mathbf{x}) = \mathbf{a}_T \times \mathbf{x} + \mathbf{b}_T \text{ with } \mathbf{a}_T, \mathbf{b}_T \in \mathbb{C}^3\}$$

and the space of Raviart–Thomas fields by

$$\mathring{\mathcal{RT}}(\mathcal{T}_H) := \{\mathbf{v} \in \mathbf{H}_0(\mathrm{div}) | \text{ for all } T \in \mathcal{T}_H : \mathbf{v}|_T(\mathbf{x}) = a_T\mathbf{x} + \mathbf{b}_T \text{ with } a_T \in \mathbb{C}, \mathbf{b}_T \in \mathbb{C}^3\}.$$

As motivated in section 3 we require an $\mathbf{H}(\mathrm{curl})$-stable interpolation operator $\pi_H^E : \mathbf{H}_0(\mathrm{curl}) \to \mathring{\mathcal{N}}(\mathcal{T}_H)$ that allows for a decomposition with the estimates such as (3.4). However, from the view point of numerical homogenization where corrector problems should be localized to small subdomains, we also need that π_H^E is local and (as we will see later) that it fits into a commuting diagram with other stable interpolation operators for lowest order $H^1(\Omega)$, $\mathbf{H}(\mathrm{div})$, and $L^2(\Omega)$ elements. As discussed in the introduction, the only known candidate is the Falk–Winther interpolation operator π_H^E [21]. We postpone a precise definition of π_H^E to section 8 and just summarize its most important properties in the following proposition.

PROPOSITION 3. *There exists a projection* $\pi_H^E : \mathbf{H}_0(\mathrm{curl}) \to \mathring{\mathcal{N}}(\mathcal{T}_H)$ *with the following local stability properties: For all* $\mathbf{v} \in \mathbf{H}_0(\mathrm{curl})$ *and all* $T \in \mathcal{T}_H$ *it holds that*

$$\|\pi_H^E(\mathbf{v})\|_{L^2(T)} \leq C_\pi \big(\|\mathbf{v}\|_{L^2(\mathrm{N}(T))} + H\|\operatorname{curl}\mathbf{v}\|_{L^2(\mathrm{N}(T))}\big), \tag{4.1}$$

$$\|\operatorname{curl}\pi_H^E(\mathbf{v})\|_{L^2(T)} \leq C_\pi \|\operatorname{curl}\mathbf{v}\|_{L^2(\mathrm{N}(T))}. \tag{4.2}$$

Furthermore, there exists a projection $\pi_H^F : \mathbf{H}_0(\mathrm{div}) \to \mathring{\mathcal{RT}}(\mathcal{T}_H)$ *to the Raviart–Thomas space such that the following commutation property holds*

$$\operatorname{curl}\pi_H^E(\mathbf{v}) = \pi_H^F(\operatorname{curl}\mathbf{v}).$$

A corresponding proof that is also valid (verbatim) in the case of homogeneous boundary values is found in [21].

As explained in the motivation in section 3, we also require that π_H^E allows for a regular decomposition in the sense of (3.4). In general, regular decompositions are an important tool for the study of $\mathbf{H}(\mathrm{curl})$-elliptic problems and involve that a vector field $\mathbf{v} \in \mathbf{H}_0(\mathrm{curl})$ is split—in a nonunique way—into a gradient and a (regular) remainder in $\mathbf{H}^1$; see [29, 42]. In contrast to the Helmholtz decomposition, this splitting is not orthogonal with respect to the L^2-inner product. If the function $\mathbf{v} \in \mathbf{H}_0(\mathrm{curl})$ is additionally known to be in the kernel of a suitable quasi interpolation, a modified decomposition can be derived that is localized and H-weighted. In particular, the weighting with H allows for estimates similar to the one stated in (3.4). The first proof of such a modified decomposition was given by Schöberl [47]. In the following we shall use his results and the locality of the Falk–Winther operator to recover a similar decomposition for the projection π_H^E. More precisely, we have the following lemma, which is crucial for our analysis.

LEMMA 4. *Let π_H^E denote the projection from Proposition 3. For any $\mathbf{v} \in \mathbf{H}_0(\mathrm{curl}, \Omega)$, there are $\mathbf{z} \in \mathbf{H}_0^1(\Omega)$ and $\theta \in H_0^1(\Omega)$ such that*

$$\mathbf{v} - \pi_H^E(\mathbf{v}) = \mathbf{z} + \nabla\theta$$

with the local bounds for every $T \in \mathcal{T}_H$

$$\begin{aligned} H^{-1}\|\mathbf{z}\|_{L^2(T)} + \|\nabla \mathbf{z}\|_{L^2(T)} &\le C_z \|\mathrm{curl}\,\mathbf{v}\|_{L^2(\mathrm{N}^3(T))}, \\ H^{-1}\|\theta\|_{L^2(T)} + \|\nabla\theta\|_{L^2(T)} &\le C_\theta\big(\|\mathbf{v}\|_{L^2(\mathrm{N}^3(T))} + H\|\mathrm{curl}\,\mathbf{v}\|_{L^2(\mathrm{N}^3(T))}\big), \end{aligned} \tag{4.3}$$

where $\nabla\mathbf{z}$ stands for the Jacobi matrix of $\mathbf{z}$. Here C_z and C_θ are generic constants that only depend on the regularity of the coarse mesh.

Observe that (4.3) implies the earlier formulated condition (3.4).

Proof. Let $\mathbf{v} \in \mathbf{H}_0(\mathrm{curl}, \Omega)$. Denote by $I_H^S : \mathbf{H}_0(\mathrm{curl}, \Omega) \to \mathring{\mathcal{N}}(\mathcal{T}_H)$ the quasi-interpolation operator introduced by Schöberl in [47]. It is shown in [47, Theorem 6] that there exists a decomposition

$$\mathbf{v} - I_H^S(\mathbf{v}) = \sum_{\substack{P \text{ vertex} \\ \text{of } \mathcal{T}_H}} \mathbf{v}_P, \tag{4.4}$$

where, for any vertex P, $\mathbf{v}_P \in \mathbf{H}_0(\mathrm{curl}, \Omega_P)$ and Ω_P is the support of the local hat function associated with P. Moreover, [47, Theorem 6] provides the stability estimates

$$\|\mathbf{v}_P\|_{L^2(\Omega_P)} \lesssim \|\mathbf{v}\|_{L^2(\mathrm{N}(\Omega_P))} \quad \text{and} \quad \|\mathrm{curl}\,\mathbf{v}_P\|_{L^2(\Omega_P)} \lesssim \|\mathrm{curl}\,\mathbf{v}\|_{L^2(\mathrm{N}(\Omega_P))} \tag{4.5}$$

for any vertex P. With these results we deduce, since π_H^E is a projection onto the finite element space, that

$$\mathbf{v} - \pi_H^E(\mathbf{v}) = \mathbf{v} - I_H^S(\mathbf{v}) - \pi_H^E(\mathbf{v} - I_H^S\mathbf{v}) = \sum_{\substack{P \text{ vertex} \\ \text{of } \mathcal{T}_H}} (\mathrm{id} - \pi_H^E)(\mathbf{v}_P).$$

Due to the locality of π_H^E, we have $(\mathrm{id} - \pi_H^E)(\mathbf{v}_P) \in \mathbf{H}_0(\mathrm{curl}, \mathrm{N}(\Omega_P))$. The local stability of π_H^E, (4.1) and (4.2), and the stability (4.5) imply

$$\begin{aligned} \|(\mathrm{id} - \pi_H^E)(\mathbf{v}_P)\|_{L^2(\mathrm{N}(\Omega_P))} &\lesssim \|\mathbf{v}\|_{L^2(\mathrm{N}(\Omega_P))} + H\|\mathrm{curl}\,\mathbf{v}\|_{L^2(\mathrm{N}(\Omega_P))}, \\ \|\mathrm{curl}(\mathrm{id} - \pi_H^E)(\mathbf{v}_P)\|_{L^2(\mathrm{N}(\Omega_P))} &\lesssim \|\mathrm{curl}\,\mathbf{v}\|_{L^2(\mathrm{N}(\Omega_P))}. \end{aligned}$$

We can now apply the regular splitting to $(\mathrm{id}-\pi_H^E)\mathbf{v}_P$ (cf. [42]), i.e., there are $\mathbf{z}_P \in \mathbf{H}^1_0(\mathrm{N}(\Omega_P))$, $\theta_P \in H^1_0(\mathrm{N}(\Omega_P))$ such that $(\mathrm{id}-\pi_H^E)\mathbf{v}_P = \mathbf{z}_P + \nabla\theta_P$ and with the estimates

$$\begin{aligned} H^{-1}\|\mathbf{z}_P\|_{L^2(\mathrm{N}(\Omega_P))} + \|\nabla \mathbf{z}_P\|_{L^2(\mathrm{N}(\Omega_P))} &\lesssim \|\operatorname{curl}((\mathrm{id}-\pi_H^E)(\mathbf{v}_P))\|_{L^2(\mathrm{N}(\Omega_P))},\\ H^{-1}\|\theta_P\|_{L^2(\mathrm{N}(\Omega_P))} + \|\nabla \theta_P\|_{L^2(\mathrm{N}(\Omega_P))} &\lesssim \|(\mathrm{id}-\pi_H^E)(\mathbf{v}_P)\|_{L^2(\mathrm{N}(\Omega_P))}. \end{aligned}$$

Set $\mathbf{z} = \sum_P \mathbf{z}_P$ and $\theta = \sum_P \theta_P$, which is a regular decomposition of $\mathbf{v} - \pi_H^E(\mathbf{v})$. The local estimate follows from the foregoing estimates for $\mathbf{v}_P$ and the decomposition (4.4) which yields

$$\begin{aligned} H^{-1}\|\mathbf{z}\|_{L^2(T)} + \|\nabla\mathbf{z}\|_{L^2(T)} &\le \sum_{\substack{P \text{ vertex}\\ \text{of } T}} \left(H^{-1}\|\mathbf{z}_P\|_{L^2(\Omega_P)} + \|\nabla\mathbf{z}_P\|_{L^2(\Omega_P)}\right)\\ &\lesssim \sum_{\substack{P \text{ vertex}\\ \text{of } T}} \|\operatorname{curl}(\mathrm{id}-\pi_H^E)(\mathbf{v}_P)\|_{L^2(\mathrm{N}(\Omega_P))} \lesssim \|\operatorname{curl}\mathbf{v}\|_{L^2(\mathrm{N}^3(T))}. \end{aligned}$$

The local estimate for θ follows analogously. ∎

5. The corrector Green's operator. In this section we introduce an ideal *corrector Green's operator* (also known as the fine-scale Green's operator in the context of the variational multiscale method; see [31]) that allows us to derive a decomposition of the exact solution into a coarse part (which is a good approximation in $H^{-1}(\Omega, \mathbb{C}^3)$) and two different corrector contributions. For simplicity, we let from now on $\mathcal{L} : \mathbf{H}_0(\mathrm{curl}) \to \mathbf{H}_0(\mathrm{curl})'$ denote the differential operator associated with the sesquilinear form $\mathcal{B}(\cdot,\cdot)$, i.e., $\mathcal{L}(v)(w) = \mathcal{B}(v, w)$.

Using the Falk-Winther interpolation operator π_H^E for the Nédélec elements, we split the space $\mathbf{H}_0(\mathrm{curl})$ into the finite, low-dimensional coarse space $\mathring{\mathcal{N}}(\mathcal{T}_H) = \mathrm{im}(\pi_H^E)$ and a corrector space given as the kernel of π_H^E, i.e., we set $\mathbf{W} := \ker(\pi_H^E) \subset \mathbf{H}_0(\mathrm{curl})$. This yields the direct sum splitting $\mathbf{H}_0(\mathrm{curl}) = \mathring{\mathcal{N}}(\mathcal{T}_H) \oplus \mathbf{W}$. Note that $\mathbf{W}$ is closed since it is the kernel of a continuous (i.e., $\mathbf{H}(\mathrm{curl})$-stable) operator. With this the ideal corrector Green's operator is defined as follows.

DEFINITION 5 (corrector Green's operator). *For $\mathbf{F} \in \mathbf{H}_0(\mathrm{curl})'$, we define the corrector Green's operator*

$$\mathcal{G} : \mathbf{H}_0(\mathrm{curl})' \to \mathbf{W} \qquad \text{by} \qquad \mathcal{B}(\mathcal{G}(\mathbf{F}), \mathbf{w}) = \mathbf{F}(\mathbf{w}) \qquad \text{for all } \mathbf{w} \in \mathbf{W}. \tag{5.1}$$

It is well defined by the Lax–Milgram–Babuška theorem, which is applicable since $\mathcal{B}(\cdot,\cdot)$ is $\mathbf{H}_0(\mathrm{curl})$-elliptic and since $\mathbf{W}$ is a closed subspace of $\mathbf{H}_0(\mathrm{curl})$.

Using the corrector Green's operator we obtain the following decomposition of the exact solution.

LEMMA 6 (ideal decomposition). *The exact solution $\mathbf{u} \in \mathbf{H}_0(\mathrm{curl})$ to (2.3) and $\mathbf{u}_H := \pi_H^E(\mathbf{u})$ admit the decomposition*

$$\mathbf{u} = \mathbf{u}_H - (\mathcal{G} \circ \mathcal{L})(\mathbf{u}_H) + \mathcal{G}(\mathbf{f}).$$

Proof. Since $\mathbf{H}_0(\mathrm{curl}) = \mathring{\mathcal{N}}(\mathcal{T}_H) \oplus \mathbf{W}$, we can write $\mathbf{u}$ uniquely as

$$\mathbf{u} = \pi_H^E(\mathbf{u}) + (\mathrm{id}-\pi_H^E)(\mathbf{u}) = \mathbf{u}_H + (\mathrm{id}-\pi_H^E)(\mathbf{u}),$$

where $(\mathrm{id}-\pi_H^E)(\mathbf{u}) \in \mathbf{W}$ by the projection property of π_H^E. Using the differential equation for test functions $\mathbf{w} \in \mathbf{W}$ yields that

$$\mathcal{B}((\mathrm{id} - \pi_H^E)(\mathbf{u}), \mathbf{w}) = -\mathcal{B}(\mathbf{u}_H, \mathbf{w}) + (\mathbf{f}, \mathbf{w})_{L^2(\Omega)} = -\mathcal{B}((\mathcal{G} \circ \mathcal{L})(\mathbf{u}_H), \mathbf{w}) + \mathcal{B}(\mathcal{G}(\mathbf{f}), \mathbf{w}).$$

Since this holds for all $\mathbf{w} \in \mathbf{W}$ and since $\mathcal{G}(\mathbf{f}) - (\mathcal{G} \circ \mathcal{L})(\mathbf{u}_H) \in \mathbf{W}$, we conclude that

$$(\mathrm{id} - \pi_H^E)(\mathbf{u}) = \mathcal{G}(\mathbf{f}) - (\mathcal{G} \circ \mathcal{L})(\mathbf{u}_H),$$

which finishes the proof. ∎

The corrector Green's operator has the following approximation and stability properties, which reveal that its contribution is always negligible in the $\mathbf{H}(\mathrm{div})'$-norm and negligible in the $\mathbf{H}(\mathrm{curl})$-norm if applied to a function in $\mathbf{H}(\mathrm{div})$.

LEMMA 7 (ideal corrector estimates). *Any $\mathbf{F} \in \mathbf{H}_0(\mathrm{curl})'$ satisfies*

$$H\|\mathcal{G}(\mathbf{F})\|_{\mathbf{H}(\mathrm{curl})} + \|\mathcal{G}(\mathbf{F})\|_{\mathbf{H}(\mathrm{div})'} \leq CH\alpha^{-1}\|\mathbf{F}\|_{\mathbf{H}_0(\mathrm{curl})'}. \tag{5.2}$$

If $\mathbf{F} = \mathbf{f} \in \mathbf{H}(\mathrm{div})$ we even have

$$H\|\mathcal{G}(\mathbf{f})\|_{\mathbf{H}(\mathrm{curl})} + \|\mathcal{G}(\mathbf{f})\|_{\mathbf{H}(\mathrm{div})'} \leq CH^2\alpha^{-1}\|\mathbf{f}\|_{\mathbf{H}(\mathrm{div})}. \tag{5.3}$$

Here, the constant C does only depend on the maximum number of neighbors of a coarse element and the generic constants appearing in Lemma 4.

Remark 8. We phrase all results in the $\mathbf{H}(\mathrm{div})'$ norm because we do not require more. Note, however, that all results are still valid if we replace the $\mathbf{H}(\mathrm{div})'$-norm by the $H^{-1}(\Omega, \mathbb{C}^3)$-norm, which is the norm we used in the motivation in section 3.

Proof. The stability estimate $\|\mathcal{G}(\mathbf{F})\|_{\mathbf{H}(\mathrm{curl})} \leq \alpha^{-1}\|\mathbf{F}\|_{\mathbf{H}_0(\mathrm{curl})'}$ is obvious. Next, with $\mathcal{G}(\mathbf{F}) \in \mathbf{W}$ and Lemma 4 we have

$$\begin{aligned} \|\mathcal{G}(\mathbf{F})\|_{\mathbf{H}(\mathrm{div})'} &= \sup_{\substack{\mathbf{v} \in \mathbf{H}(\mathrm{div}) \\ \|\mathbf{v}\|_{\mathbf{H}(\mathrm{div})}=1}} \left| \int_\Omega \mathbf{z} \cdot \mathbf{v} - \int_\Omega \theta (\nabla \cdot \mathbf{v}) \right| \\ &\leq (\|\mathbf{z}\|^2_{L^2(\Omega)} + \|\theta\|^2_{L^2(\Omega)})^{1/2} \leq CH\|\mathcal{G}(\mathbf{F})\|_{\mathbf{H}(\mathrm{curl})} \leq CH\alpha^{-1}\|\mathbf{F}\|_{\mathbf{H}_0(\mathrm{curl})'}, \end{aligned} \tag{5.4}$$

which proves (5.2). Note that this estimate exploited $\theta \in H^1_0(\Omega)$, which is why we do not require the function $\mathbf{v}$ to have a vanishing normal trace. Let us now consider the case that $\mathbf{F} = \mathbf{f} \in \mathbf{H}(\mathrm{div})$. The ellipticity, the relation (5.1), and (5.4) imply that

$$\alpha\|\mathcal{G}(\mathbf{f})\|^2_{\mathbf{H}(\mathrm{curl})} \leq \|\mathcal{G}(\mathbf{f})\|_{\mathbf{H}(\mathrm{div})'}\|\mathbf{f}\|_{\mathbf{H}(\mathrm{div})} \leq CH\|\mathcal{G}(\mathbf{f})\|_{\mathbf{H}(\mathrm{curl})}\|\mathbf{f}\|_{\mathbf{H}(\mathrm{div})}.$$

We conclude $\|\mathcal{G}(\mathbf{f})\|_{\mathbf{H}(\mathrm{curl})} \leq CH\alpha^{-1}\|\mathbf{f}\|_{\mathbf{H}(\mathrm{div})}$. Finally, we can use this estimate again in (5.4) to obtain

$$\|\mathcal{G}(\mathbf{f})\|_{\mathbf{H}(\mathrm{div})'} \leq CH\|\mathcal{G}(\mathbf{f})\|_{\mathbf{H}(\mathrm{curl})} \leq CH^2\alpha^{-1}\|\mathbf{f}\|_{\mathbf{H}(\mathrm{div})}.$$

This finishes the proof. ∎

An immediate conclusion of Lemmas 6 and 7 is the following.

CONCLUSION 9. *Let $\mathbf{u}$ denote the exact solution to (2.1) for $\mathbf{f} \in \mathbf{H}(\mathrm{div})$. Then with the coarse part $\mathbf{u}_H := \pi_H^E(\mathbf{u})$ and corrector operator $\mathcal{K} := -\mathcal{G} \circ \mathcal{L}$ it holds that*

$$\begin{aligned} H^{-1}\|\mathbf{u} - (\mathrm{id} + \mathcal{K})\mathbf{u}_H\|_{\mathbf{H}(\mathrm{div})'} + \|\mathbf{u} - (\mathrm{id} + \mathcal{K})\mathbf{u}_H\|_{\mathbf{H}(\mathrm{curl})} + \|\mathbf{u} - \mathbf{u}_H\|_{\mathbf{H}(\mathrm{div})'} \\ \leq CH\|\mathbf{f}\|_{\mathbf{H}(\mathrm{div})}. \end{aligned}$$

Here, C only depends on α, on the mesh regularity, and on the constants appearing in Lemma 4.

Proof. The estimates for $\mathbf{u} - (\mathrm{id} + \mathcal{K})\mathbf{u}_H = \mathcal{G}(\mathbf{f})$ directly follow from (5.3). For the estimate of $\mathbf{u} - \mathbf{u}_H = \mathcal{K}\mathbf{u}_H + \mathcal{G}\mathbf{f}$, observe that (5.2) and Proposition 3 imply

$$\|\mathcal{K}\mathbf{u}_H\|_{\mathbf{H}(\mathrm{div})'} \lesssim H\|\mathcal{L}\mathbf{u}_H\|_{\mathbf{H}_0(\mathrm{curl})'} \lesssim H\|\mathbf{u}_H\|_{\mathbf{H}(\mathrm{curl})} = H\|\pi_H^E\mathbf{u}\|_{\mathbf{H}(\mathrm{curl})} \lesssim H\|\mathbf{u}\|_{\mathbf{H}(\mathrm{curl})}.$$

Thus, the proof follows from the stability of the problem and the triangle inequality. ∎

It only remains to derive an equation that characterizes $(\mathrm{id} + \mathcal{K})\mathbf{u}_H$ as the unique solution of a variational problem. This is done in the following theorem.

THEOREM 10. *We consider the setting of Conclusion 9. Then $\mathbf{u}_H = \pi_H^E(\mathbf{u}) \in \mathring{\mathcal{N}}(\mathcal{T}_H)$ is characterized as the unique solution to*

$$\mathcal{B}(\,(\mathrm{id} + \mathcal{K})\mathbf{u}_H, (\mathrm{id} + \mathcal{K})\mathbf{v}_H\,) = (\mathbf{f}, (\mathrm{id} + \mathcal{K}\mathbf{v}_H)_{L^2(\Omega)} \qquad \text{for all } \mathbf{v}_H \in \mathring{\mathcal{N}}(\mathcal{T}_H). \tag{5.5}$$

The sesquilinear form $\mathcal{B}(\,(\mathrm{id} + \mathcal{K})\,\cdot\,, (\mathrm{id} + \mathcal{K})\,\cdot\,)$ is $\mathbf{H}(\mathrm{curl})$-elliptic on $\mathring{\mathcal{N}}(\mathcal{T}_H)$.

We mention that in the non-self-adjoint case, the correction operator for the test functions would be the adjoint $\mathcal{K}^*$.

Proof. Since Lemma 6 guarantees $\mathbf{u} = \mathbf{u}_H - (\mathcal{G} \circ \mathcal{L})(\mathbf{u}_H) + \mathcal{G}(\mathbf{f})$, the weak formulation (2.3) yields

$$\mathcal{B}(\mathbf{u}_H - (\mathcal{G} \circ \mathcal{L})(\mathbf{u}_H) + \mathcal{G}(\mathbf{f}), \mathbf{v}_H) = (\mathbf{f}, \mathbf{v}_H)_{L^2(\Omega)} \qquad \text{for all } \mathbf{v}_H \in \mathring{\mathcal{N}}(\mathcal{T}_H).$$

We observe that by definition of $\mathcal{G}$ we have

$$\mathcal{B}(\mathcal{G}(\mathbf{f}), \mathbf{v}_H) = (\mathbf{f}, (\mathcal{G} \circ \mathcal{L})\mathbf{v}_H)_{L^2(\Omega)}$$

and

$$\mathcal{B}(\mathbf{u}_H - (\mathcal{G} \circ \mathcal{L})(\mathbf{u}_H), (\mathcal{G} \circ \mathcal{L})\mathbf{v}_H) = 0.$$

Combining the three equations shows that $(\mathrm{id} + \mathcal{K})\mathbf{u}_H$ is a solution to (5.5). The uniqueness follows from the following norm equivalence:

$$\|\mathbf{u}_H\|_{\mathbf{H}(\mathrm{curl})} = \|\pi_H^E((\mathrm{id} + \mathcal{K})\mathbf{u}_H)\|_{\mathbf{H}(\mathrm{curl})} \leq C\|(\mathrm{id} + \mathcal{K})\mathbf{u}_H\|_{\mathbf{H}(\mathrm{curl})} \leq C\|\mathbf{u}_H\|_{\mathbf{H}(\mathrm{curl})}.$$

This is also the reason why the $\mathbf{H}(\mathrm{curl})$-ellipticity of $\mathcal{B}(\cdot, \cdot)$ implies the $\mathbf{H}(\mathrm{curl})$-ellipticity of $\mathcal{B}(\,(\mathrm{id} + \mathcal{K})\,\cdot\,, (\mathrm{id} + \mathcal{K})\,\cdot\,)$ on $\mathring{\mathcal{N}}(\mathcal{T}_H)$. ∎

Dropping the correction on the right-hand side of (5.5) still allows for a numerical homogenization result. However, not all estimates from Conclusion 9 can be recovered in this case, as the quadratic order convergence for $\|\mathbf{u} - (\mathrm{id} + \mathcal{K})\widetilde{\mathbf{u}}_H\|_{\mathbf{H}(\mathrm{div})'}$ is typically lost (at least in the asymptotic regime). In general, the following result is available.

CONCLUSION 11. *For $\mathbf{f} \in \mathbf{H}(\mathrm{div})$, let $\widetilde{\mathbf{u}}_H \in \mathring{\mathcal{N}}(\mathcal{T}_H)$ denote the unique solution to*

$$\mathcal{B}((\mathrm{id} + \mathcal{K})\widetilde{\mathbf{u}}_H, (\mathrm{id} + \mathcal{K})\mathbf{v}_H) = (\mathbf{f}, \mathbf{v}_H)_{L^2(\Omega)} \quad \text{for all} \quad \mathbf{v}_H \in \mathring{\mathcal{N}}(\mathcal{T}_H). \tag{5.6}$$

Then we have the error estimate

$$\|\mathbf{u} - (\mathrm{id} + \mathcal{K})\widetilde{\mathbf{u}}_H\|_{\mathbf{H}(\mathrm{curl})} + \|\mathbf{u} - \widetilde{\mathbf{u}}_H\|_{\mathbf{H}(\mathrm{div})'} \leq CH\|\mathbf{f}\|_{\mathbf{H}(\mathrm{div})}.$$

Proof. We estimate the error $\mathbf{u}_H - \widetilde{\mathbf{u}}_H$, where $\mathbf{u}_H$ solves (5.5). For any $\mathbf{v}_H \in \mathring{\mathcal{N}}(\mathcal{T}_H)$, we have that

$$\mathcal{B}((\mathrm{id}+\mathcal{K})(\mathbf{u}_H - \widetilde{\mathbf{u}}_H), (\mathrm{id}+\mathcal{K})\mathbf{v}_H) = (\mathbf{f}, \mathcal{K}\mathbf{v}_H)_{L^2(\Omega)}.$$

Hence, we conclude with the coercivity and continuity of $\mathcal{B}$ and Lemma 7 that

$$\begin{aligned}\|\mathbf{u}_H - \widetilde{\mathbf{u}}_H\|^2_{\mathbf{H}(\mathrm{curl})} &\lesssim \|(\mathrm{id}+\mathcal{K})(\mathbf{u}_H - \widetilde{\mathbf{u}}_H)\|^2_{\mathbf{H}(\mathrm{curl})} \lesssim \|\mathbf{f}\|_{\mathbf{H}(\mathrm{div})}\|\mathcal{K}(\mathbf{u}_H - \widetilde{\mathbf{u}}_H)\|_{\mathbf{H}(\mathrm{div})'}\\ &\lesssim H\|\mathbf{f}\|_{\mathbf{H}(\mathrm{div})}\|\mathcal{L}(\mathbf{u}_H - \widetilde{\mathbf{u}}_H)\|_{\mathbf{H}_0(\mathrm{curl})'} \lesssim H\|\mathbf{f}\|_{\mathbf{H}(\mathrm{div})}\|\mathbf{u}_H - \widetilde{\mathbf{u}}_H\|_{\mathbf{H}(\mathrm{curl})}.\end{aligned}$$

The estimate for $\|\mathbf{u}-\widetilde{\mathbf{u}}_H\|_{\mathbf{H}(\mathrm{div})'}$ follows with the triangle inequality and the properties of the corrector $\mathcal{K}$. ∎

The result from Conclusion 11 reflects the fact that in periodic homogenization, correctors typically do not appear on the right-hand side. However, as mentioned before, problem (5.6) has the disadvantage that its suffers from a slight loss in accuracy which is expected to cause reduced convergence rates for $\|\mathbf{u} - (\mathrm{id}+\mathcal{K})\widetilde{\mathbf{u}}_H\|_{\mathbf{H}(\mathrm{div})'}$.

Numerical homogenization. Let us summarize the most important findings and relate them to (numerical) homogenization. We defined a *homogenization scale* through the coarse finite element space $\mathring{\mathcal{N}}(\mathcal{T}_H)$. We proved that there exists a numerically homogenized function $\mathbf{u}_H \in \mathring{\mathcal{N}}(\mathcal{T}_H)$ which approximates the exact solution well in $\mathbf{H}(\mathrm{div})'$ with

$$\|\mathbf{u} - \mathbf{u}_H\|_{\mathbf{H}(\mathrm{div})'} \leq CH\|\mathbf{f}\|_{\mathbf{H}(\mathrm{div})}.$$

From the periodic homogenization theory (cf. section 3) we know that this is the best we can expect and that $\mathbf{u}_H$ is typically not a good L^2-approximation due to the large kernel of the curl-operator. Furthermore, we showed the existence of an $\mathbf{H}(\mathrm{curl})$-stable corrector operator $\mathcal{K} : \mathring{\mathcal{N}}(\mathcal{T}_H) \to \mathbf{W}$ that corrects the homogenized solution in such a way that the approximation is also accurate in $\mathbf{H}(\mathrm{curl})$ with

$$\|\mathbf{u} - (\mathrm{id}+\mathcal{K})\mathbf{u}_H\|_{\mathbf{H}(\mathrm{curl})} \leq CH\|\mathbf{f}\|_{\mathbf{H}(\mathrm{div})}.$$

Since $\mathcal{K} = -\mathcal{G} \circ \mathcal{L}$, we know that we can characterize $\mathcal{K}(\mathbf{v}_H) \in \mathbf{W}$ as the unique solution to the (ideal) corrector problem

$$\mathcal{B}(\mathcal{K}(\mathbf{v}_H), \mathbf{w}) = -\mathcal{B}(\mathbf{v}_H, \mathbf{w}) \qquad \text{for all } \mathbf{w} \in \mathbf{W}. \tag{5.7}$$

The above result shows that $(\mathrm{id}+\mathcal{K})\mathbf{u}_H$ approximates the analytical solution with linear rate without any assumptions on the regularity of the problem or the structure of the coefficients that define $\mathcal{B}(\cdot,\cdot)$. Also it does not assume that the mesh resolves the possible fine-scale features of the coefficient. On the other hand, the ideal corrector problem (5.7) is global, which significantly limits its practical usability in terms of real computations.

However, as we will see next, the corrector Green's function associated with problem (5.1) shows an exponential decay measured in units of H. This property will allow us to split the global corrector problem (5.7) into several smaller problems on subdomains, similar to how we encounter it in classical homogenization theory. We show the exponential decay of the corrector Green's function indirectly through the properties of its corresponding Green's operator $\mathcal{G}$. The localization is established in section 6, whereas we prove the decay in section 7.

6. Quasi-local numerical homogenization. In this section we describe how the ideal corrector $\mathcal{K}$ can be approximated by a sum of local correctors, without destroying the overall approximation order. This is of central importance for an efficient computability. Furthermore, it also reveals that the new corrector is a quasi-local operator, which is in line with homogenization theory.

We follow the standard procedure for the localization in the LOD, as displayed, for instance, in [1, 26, 37, 44], just to name a few. We start with quantifying the decay properties of the corrector Green's operator in subsection 6.1. In subsection 6.2 we apply the result to our numerical homogenization setting and state the error estimates for the "localized" corrector operator. We close with a few remarks on a fully discrete realization of the localized corrector operator in subsection 6.3.

6.1. Exponential decay and localized corrector. The property that $\mathcal{K}$ can be approximated by local correctors is directly linked to the decay properties of the Green's function associated with problem (5.1). These decay properties can be quantified explicitly by measuring distances between points in units of the coarse mesh size H. We have the following result, which states—loosely speaking—in which distance from the support of a source term $\mathbf{F}$ becomes the $\mathbf{H}(\mathrm{curl})$-norm of $\mathcal{G}(\mathbf{F})$ negligibly small. For that, recall the definition of the element patches from the beginning of section 4, where $\mathrm{N}^m(T)$ denotes the patch that consists of a coarse element $T \in \mathcal{T}_H$ and m layers of coarse elements around it. A proof of the following proposition is given in section 7.

PROPOSITION 12. *Let $T \in \mathcal{T}_H$ denote a coarse element and $m \in \mathbb{N}$ a number of layers. Furthermore, let $\mathbf{F}_T \in \mathbf{H}_0(\mathrm{curl})'$ denote a local source functional in the sense that $\mathbf{F}_T(\mathbf{v}) = 0$ for all $\mathbf{v} \in \mathbf{H}_0(\mathrm{curl})$ with $\mathrm{supp}(\mathbf{v}) \subset \Omega \setminus T$. Then there exists $0 < \tilde{\beta} < 1$, independent of H, T, m, and $\mathbf{F}_T$, such that*

$$\|\mathcal{G}(\mathbf{F}_T)\|_{\mathbf{H}(\mathrm{curl},\Omega\setminus \mathrm{N}^m(T))} \lesssim \tilde{\beta}^m \|\mathbf{F}_T\|_{\mathbf{H}_0(\mathrm{curl})'}. \tag{6.1}$$

In order to use this result to approximate $\mathcal{K}(\mathbf{v}_H) = -(\mathcal{G} \circ \mathcal{L})\mathbf{v}_H$ (which has a nonlocal argument), we introduce, for any $T \in \mathcal{T}_H$, localized differential operators $\mathcal{L}_T : \mathbf{H}(\mathrm{curl}, T) \to \mathbf{H}(\mathrm{curl}, \Omega)'$ with

$$\langle \mathcal{L}_T(\mathbf{u}), \mathbf{v}\rangle := \mathcal{B}_T(\mathbf{u}, \mathbf{v}),$$

where $\mathcal{B}_T(\cdot,\cdot)$ denotes the restriction of $\mathcal{B}(\cdot,\cdot)$ to the element T. By linearity of $\mathcal{G}$ we have that

$$\mathcal{G} \circ \mathcal{L} = \sum_{T\in\mathcal{T}_H} \mathcal{G} \circ \mathcal{L}_T$$

and consequently we can write

$$\mathcal{K}(\mathbf{v}_H) = \sum_{T\in\mathcal{T}_H} \mathcal{G}(\mathbf{F}_T) \qquad \text{with } \mathbf{F}_T := -\mathcal{L}_T(\mathbf{v}_H).$$

Obviously, $\mathcal{G}(\mathbf{F}_T)$ fits into the setting of Proposition 12. This suggests truncating the individual computations of $\mathcal{G}(\mathbf{F}_T)$ to a small patch $\mathrm{N}^m(T)$ and then collecting the results to construct a global approximation for the corrector. Typically, m is referred to as an *oversampling parameter*. The strategy is detailed in the following definition.

DEFINITION 13 (localized corrector approximation). *For an element $T \in \mathcal{T}_H$ we define the element patch $\Omega_T := \mathrm{N}^m(T)$ of order $m \in \mathbb{N}$. Let $\mathbf{F} \in \mathbf{H}_0(\mathrm{curl})'$ be* *the sum of local functionals with* $\mathbf{F} = \sum_{T\in\mathcal{T}_H} \mathbf{F}_T$*, where* $\mathbf{F}_T \in \mathbf{H}_0(\mathrm{curl})'$ *is as in Proposition* 12*. Furthermore, let* $\mathbf{W}(\Omega_T) \subset \mathbf{W}$ *denote the space of functions from* $\mathbf{W}$ *that vanish outside* Ω_T*, i.e.,*

$$\mathbf{W}(\Omega_T) = \{\mathbf{w} \in \mathbf{W} | \mathbf{w} = 0 \text{ outside } \Omega_T\}.$$

We call $\mathcal{G}_{T,m}(\mathbf{F}_T) \in \mathbf{W}(\Omega_T)$ *the* localized corrector *if it solves*

$$\mathcal{B}(\mathcal{G}_{T,m}(\mathbf{F}_T), \mathbf{w}) = \mathbf{F}_T(\mathbf{w}) \qquad \text{for all } \mathbf{w} \in \mathbf{W}(\Omega_T). \tag{6.2}$$

With this, the global corrector approximation is given by

$$\mathcal{G}_m(\mathbf{F}) := \sum_{T\in\mathcal{T}_H} \mathcal{G}_{T,m}(\mathbf{F}_T).$$

Observe that problem (6.2) is only formulated on the patch Ω_T and that it admits a unique solution by the Lax–Milgram–Babuška theorem.

Based on decay properties stated in Proposition 12, we can derive the following error estimate for the difference between the exact corrector $\mathcal{G}(\mathbf{F})$ and its approximation $\mathcal{G}_m(\mathbf{F})$ obtained by an mth level truncation. The proof of the following result is again postponed to section 7.

Theorem 14. *We consider the setting of Definition* 13 *with ideal Green's corrector* $\mathcal{G}(\mathbf{F})$ *and its* m*th level truncated approximation* $\mathcal{G}_m(\mathbf{F})$*. Then there exist constants* $C_d > 0$ *and* $0 < \beta < 1$ *(both independent of* H *and* m*) such that*

$$\|\mathcal{G}(\mathbf{F}) - \mathcal{G}_m(\mathbf{F})\|_{\mathbf{H}(\mathrm{curl})} \leq C_d \sqrt{C_{\mathrm{ol},m}}\, \beta^m \left(\sum_{T\in\mathcal{T}_H} \|\mathbf{F}_T\|^2_{\mathbf{H}_0(\mathrm{curl})'}\right)^{1/2} \tag{6.3}$$

and

$$\|\mathcal{G}(\mathbf{F}) - \mathcal{G}_m(\mathbf{F})\|_{\mathbf{H}(\mathrm{div})'} \leq C_d \sqrt{C_{\mathrm{ol},m}}\, \beta^m H \left(\sum_{T\in\mathcal{T}_H} \|\mathbf{F}_T\|^2_{\mathbf{H}_0(\mathrm{curl})'}\right)^{1/2}. \tag{6.4}$$

As a direct conclusion from Theorem 14 we obtain the main result of this paper that we present in the next subsection.

6.2. The quasi-local corrector and homogenization. Following the above motivation we split the ideal corrector $\mathcal{K}(\mathbf{v}_H) = -(\mathcal{G}\circ\mathcal{L})\mathbf{v}_H$ into a sum of quasi-local contributions of the form $\sum_{T\in\mathcal{T}_H}(\mathcal{G}\circ\mathcal{L}_T)\mathbf{v}_H$. Applying Theorem 14, we obtain the following result.

Conclusion 15. *Let* $\mathcal{K}_m := -\sum_{T\in\mathcal{T}_H}(\mathcal{G}_{T,m}\circ\mathcal{L}_T) : \mathring{\mathcal{N}}(\mathcal{T}_H) \to \mathbf{W}$ *denote the localized corrector operator obtained by truncation of* m*th order. Then it holds that*

$$\inf_{\mathbf{v}_H\in\mathring{\mathcal{N}}(\mathcal{T}_H)} \|\mathbf{u} - (\mathrm{id} + \mathcal{K}_m)\mathbf{v}_H\|_{\mathbf{H}(\mathrm{curl})} \leq C\left(H + \sqrt{C_{\mathrm{ol},m}}\beta^m\right)\|\mathbf{f}\|_{\mathbf{H}(\mathrm{div})}. \tag{6.5}$$

Note that even though the ideal corrector $\mathcal{K}$ is a nonlocal operator, we can approximate it by a quasi-local corrector $\mathcal{K}_m$. Here, the quasi locality is seen by the fact that if $\mathcal{K}$ is applied to a function $\mathbf{v}_H$ with local support, the image $\mathcal{K}(\mathbf{v}_H)$ will typically still have a global support in Ω. On the other hand, if $\mathcal{K}_m$ is applied to a locally supported function, the support will only increase by a layer with thickness of order mH.

Proof of Conclusion 15. With $\mathcal{K}_m = -\sum_{T\in\mathcal{T}_H}(\mathcal{G}_{T,m}\circ\mathcal{L}_T)$ we apply Conclusion 9 and Theorem 14 to obtain

$$\begin{aligned}\inf_{\mathbf{v}_H\in\mathring{\mathcal{N}}(\mathcal{T}_H)} &\|\mathbf{u}-(\mathrm{id}+\mathcal{K}_m)\mathbf{v}_H\|_{\mathbf{H}(\mathrm{curl})}\\ &\le \|\mathbf{u}-(\mathrm{id}+\mathcal{K})\mathbf{u}_H\|_{\mathbf{H}(\mathrm{curl})} + \|(\mathcal{K}-\mathcal{K}_m)\mathbf{u}_H\|_{\mathbf{H}(\mathrm{curl})}\\ &\le CH\|\mathbf{f}\|_{\mathbf{H}(\mathrm{div})} + C\sqrt{C_{\mathrm{ol},m}}\,\beta^m\left(\sum_{T\in\mathcal{T}_H}\|\mathcal{L}_T(\mathbf{u}_H)\|^2_{\mathbf{H}_0(\mathrm{curl})'}\right)^{1/2},\end{aligned}$$

where we observe with $\|\mathcal{L}_T(\mathbf{v}_H)\|_{\mathbf{H}_0(\mathrm{curl})'}\le C\|\mathbf{v}_H\|_{\mathbf{H}(\mathrm{curl},T)}$ that

$$\begin{aligned}\sum_{T\in\mathcal{T}_H}\|\mathcal{L}_T(\mathbf{u}_H)\|^2_{\mathbf{H}_0(\mathrm{curl})'} &\le C\|\mathbf{u}_H\|^2_{\mathbf{H}(\mathrm{curl})} = C\|\pi_H^E(\mathbf{u})\|^2_{\mathbf{H}(\mathrm{curl})}\\ &\le C\|\mathbf{u}\|^2_{\mathbf{H}(\mathrm{curl})}\le C\|\mathbf{f}\|^2_{\mathbf{H}(\mathrm{div})}.\end{aligned}$$

In the last line, the first inequality is due to the stability of π_H^E and the second inequality is the energy estimate for the original problem (2.3). ∎

Conclusion 15 has immediate implications from the computational point of view. First, we observe that $\mathcal{K}_m$ can be computed by solving local decoupled problems. Considering a basis $\{\mathbf{\Phi}_k|\ 1\le k\le N\}$ of $\mathring{\mathcal{N}}(\mathcal{T}_H)$, we require determining $\mathcal{K}_m(\mathbf{\Phi}_k)$. For that, we consider all $T\in\mathcal{T}_H$ with $T\subset\mathrm{supp}(\mathbf{\Phi}_k)$ and solve for $\mathcal{K}_{T,m}(\mathbf{\Phi}_k)\in\mathbf{W}(\mathrm{N}^m(T))$ with

$$\mathcal{B}_{\mathrm{N}^m(T)}(\mathcal{K}_{T,m}(\mathbf{\Phi}_k),\mathbf{w}) = -\mathcal{B}_T(\mathbf{\Phi}_k,\mathbf{w})\qquad\text{for all }\mathbf{w}\in\mathbf{W}(\mathrm{N}^m(T)).\tag{6.6}$$

The global corrector approximation is now given by

$$\mathcal{K}_m(\mathbf{\Phi}_k) = \sum_{\substack{T\in\mathcal{T}_H\\ T\subset\mathrm{supp}(\mathbf{\Phi}_k)}}\mathcal{K}_{T,m}(\mathbf{\Phi}_k),$$

as already presented in the motivation in section 3. Next, we observe that selecting the localization parameter m such that

$$m\gtrsim|\log H|/|\log\beta|,$$

we have with Conclusion 15 that

$$\inf_{\mathbf{v}_H\in\mathring{\mathcal{N}}(\mathcal{T}_H)}\|\mathbf{u}-(\mathrm{id}+\mathcal{K}_m)\mathbf{v}_H\|_{\mathbf{H}(\mathrm{curl})}\le CH\|\mathbf{f}\|_{\mathbf{H}(\mathrm{div})},\tag{6.7}$$

which is of the same order as for the ideal corrector $\mathcal{K}$. Note that the polynomial (in m) growth of $C_{\mathrm{ol},m}$ does only influence the constant hidden in $\gtrsim$ in the selection rule $m\gtrsim|\log H|$ and not (6.7). The choice $m\approx|\log H|$ is the standard condition for the oversampling parameter m in LOD-type methods. However, numerical experiments for other types of problems show that moderate sizes of m such as $m=1,2,3$ are often sufficient in practice; cf. [26, 44]. This indicates that there is hope for similar observations for $\mathbf{H}$(curl)-problems, though this still remains open for investigations.

Consequently, we can consider the Galerkin finite element method, where we seek $\mathbf{u}_{H,m}\in\mathring{\mathcal{N}}(\mathcal{T}_H)$ such that

$$\mathcal{B}((\mathrm{id}+\mathcal{K}_m)\mathbf{u}_{H,m},(\mathrm{id}+\mathcal{K}_m)\mathbf{v}_H) = (\mathbf{f},(\mathrm{id}+\mathcal{K}_m)\mathbf{v}_H)_{L^2(\Omega)}\qquad\text{for all }\mathbf{v}_H\in\mathring{\mathcal{N}}(\mathcal{T}_H).$$

Since a Galerkin method yields the $\mathbf{H}(\mathrm{curl})$-quasi-best approximation of $\mathbf{u}$ in the space $(\mathrm{id} + \mathcal{K}_m)\mathring{\mathcal{N}}(\mathcal{T}_H)$ we have with (6.7) that

$$\|\mathbf{u} - (\mathrm{id} + \mathcal{K}_m)\mathbf{u}_{H,m}\|_{\mathbf{H}(\mathrm{curl})} \le CH\|\mathbf{f}\|_{\mathbf{H}(\mathrm{div})}$$

and we have with (5.2), (6.4), and the $\mathbf{H}(\mathrm{curl})$-stability of π_H^E that

$$\|\mathbf{u} - \mathbf{u}_{H,m}\|_{\mathbf{H}(\mathrm{div})'} \le CH\|\mathbf{f}\|_{\mathbf{H}(\mathrm{div})}.$$

This result is a homogenization result in the sense that it yields a coarse function $\mathbf{u}_{H,m}$ that approximates the exact solution in $\mathbf{H}(\mathrm{div})'$. Furthermore, it yields an appropriate (quasi-local) corrector $\mathcal{K}_m(\mathbf{u}_{H,m})$ that is required for an accurate approximation in $\mathbf{H}(\mathrm{curl})$.

Finally, we note that the error estimate in $\mathbf{H}(\mathrm{curl})$ above can also be obtained for the Galerkin method without corrector $\mathcal{K}_m$ on the right-hand side; see Conclusion 11. Moreover, the assumption $\mathbf{f} \in \mathbf{H}(\mathrm{div})$ is essential to obtain a linear rate: If we only have $\mathbf{f} \in \mathbf{H}_0(\mathrm{curl})'$, the results of Conclusion 15 do not hold. As seen in Lemma 7, we lose a power of H for less regular right-hand sides.

Remark 16 (refined estimates). With a more careful proof, the constants in the estimate of Conclusion 15 can be specified as

$$\begin{aligned}&\inf_{\mathbf{v}_H \in \mathring{\mathcal{N}}(\mathcal{T}_H)} \|\mathbf{u} - (\mathrm{id} + \mathcal{K}_m)\mathbf{v}_H\|_{\mathbf{H}(\mathrm{curl})} \\ &\quad \le \alpha^{-1}(1+H)\left(H\max\{C_z, C_\theta\}\sqrt{C_{\mathrm{ol},3}} + C_d C_\pi C_B^2 \sqrt{C_{\mathrm{ol},m} C_{\mathrm{ol}}}\,\beta^m\right)\|\mathbf{f}\|_{\mathbf{H}(\mathrm{div})},\end{aligned}$$

where α and C_B are as in Assumption 1, C_d is the constant appearing in the decay estimate (6.3), C_π is as in Proposition 3, C_z and C_θ are from (4.3), and $C_{\mathrm{ol},m}$ is as detailed at the beginning of section 4. Note that if m is large enough so that $\mathrm{N}^m(T) = \Omega$ for all $T \in \mathcal{T}_H$, we have as a refinement of Conclusion 9 that

$$\inf_{\mathbf{v}_H \in \mathring{\mathcal{N}}(\mathcal{T}_H)} \|\mathbf{u} - (\mathrm{id} + \mathcal{K})\mathbf{v}_H\|_{\mathbf{H}(\mathrm{curl})} \le \alpha^{-1}(1+H)\big(H\max\{C_z, C_\theta\}\sqrt{C_{\mathrm{ol},3}}\big)\|\mathbf{f}\|_{\mathbf{H}(\mathrm{div})}.$$

6.3. A fully discrete localized multiscale method. The procedure described in the previous section is still not yet "ready to use" for a practical computation as the local corrector problems (6.6) involve the infinite-dimensional spaces $\mathbf{W}(\Omega_T)$. Hence, we require an additional fine-scale discretization of the corrector problems (just like the cell problems in periodic homogenization theory can typically not be solved analytically).

For a fully discrete formulation, we introduce a second shape-regular partition $\mathcal{T}_h$ of Ω into tetrahedra. This partition may be nonuniform and is assumed to be obtained from $\mathcal{T}_H$ by at least one global refinement. It is a fine discretization in the sense that $h < H$ and that $\mathcal{T}_h$ resolves all fine-scale features of the coefficients. Let $\mathring{\mathcal{N}}(\mathcal{T}_h) \subset \mathbf{H}_0(\mathrm{curl})$ denote the space of Nédélec elements with respect to the partition $\mathcal{T}_h$. We then introduce the space

$$\mathbf{W}_h(\Omega_T) := \mathbf{W}(\Omega_T) \cap \mathring{\mathcal{N}}(\mathcal{T}_h) = \{\mathbf{v}_h \in \mathring{\mathcal{N}}(\mathcal{T}_h) | \mathbf{v}_h = 0 \text{ outside } \Omega_T, \pi_H^E(\mathbf{v}_h) = 0\}$$

and discretize the corrector problem (6.6) with this new space. The corresponding correctors are denoted by $\mathcal{K}_{T,m,h}$ and $\mathcal{K}_{m,h}$. With this modification we can prove analogously to the error estimate (6.5) that it holds that

$$\inf_{\mathbf{v}_H\in\mathring{\mathcal{N}}(\mathcal{T}_H)} \|\mathbf{u}_h - (\mathrm{id}+\mathcal{K}_{m,h})\mathbf{v}_H\|_{\mathbf{H}(\mathrm{curl})} \leq C\left(H+\sqrt{C_{\mathrm{ol},m}}\tilde{\beta}^m\right)\|\mathbf{f}\|_{\mathbf{H}(\mathrm{div})}, \tag{6.8}$$

where $\mathbf{u}_h$ is the Galerkin approximation of $\mathbf{u}$ in the discrete fine space $\mathring{\mathcal{N}}(\mathcal{T}_h)$. If $\mathcal{T}_h$ is fine enough, we can assume that $\mathbf{u}_h$ is a good $\mathbf{H}$(curl)-approximation to the true solution $\mathbf{u}$. Consequently, it is justified to formulate a fully discrete (localized) multiscale method by seeking $\mathbf{u}_{H,h,m}\in\mathring{\mathcal{N}}(\mathcal{T}_H)$ such that

(6.9)
$$\mathcal{B}((\mathrm{id}+\mathcal{K}_{m,h})\mathbf{u}_{H,h,m},(\mathrm{id}+\mathcal{K}_{m,h})\mathbf{v}_H) = (\mathbf{f},(\mathrm{id}+\mathcal{K}_{m,h})\mathbf{v}_H)_{L^2(\Omega)} \text{ for all } \mathbf{v}_H\in\mathring{\mathcal{N}}(\mathcal{T}_H).$$

As before, we can conclude from (6.8) together with the choice $m\gtrsim|\log H|/|\log\beta|$ that it holds that

$$\|\mathbf{u}_h-(\mathrm{id}+\mathcal{K}_{m,h})\mathbf{u}_{H,h,m}\|_{\mathbf{H}(\mathrm{curl})}+\|\mathbf{u}_h-\mathbf{u}_{H,h,m}\|_{\mathbf{H}(\mathrm{div})'}\leq CH\|\mathbf{f}\|_{\mathbf{H}(\mathrm{div})}.$$

Thus, the additional fine-scale discretization does not affect the overall error estimates and we therefore concentrate in the proofs (for simplicity) on the semidiscrete case as detailed in subsections 6.1 and 6.2. Compared to the fully discrete case, only some small modifications are needed in the proofs for the decay of the correctors. These modifications are outlined at the end of section 7. Note that $\mathbf{u}_h$ is not needed in the practical implementation of the method.

7. Proof of the decay for the corrector Green's operator. In this section, we prove Proposition 12 and Theorem 14. Since the latter is based on the first result, we start with proving the exponential decay of the Green's function associated with $\mathcal{G}$. Recall that we quantified the decay indirectly through estimates of the form

$$\|\mathcal{G}(\mathbf{F}_T)\|_{\mathbf{H}(\mathrm{curl},\Omega\setminus\mathrm{N}^m(T))}\lesssim\tilde{\beta}^m\|\mathbf{F}_T\|_{\mathbf{H}_0(\mathrm{curl})'},$$

where $\mathbf{F}_T$ is a T-local functional and $0<\tilde{\beta}<1$. The proof techniques rely on the multiplication of a corrector function with a cut-off function and a Caccioppoli-type argument, as is the usual strategy for LOD methods; see, e.g., [37, 44]. Alternatively, the LOD has been recently reinterpreted in the form of an iterative method (additive subspace correction method) and a new technique for proving the exponential decay has been proposed; see [34, 33]. However, this modified approach would require a different localization strategy than the one that we chose in section 6.

Proof of Proposition 12. Let $\eta\in\mathcal{S}^1(\mathcal{T}_H)\subset H^1(\Omega)$ be a scalar-valued, piecewise linear and globally continuous cut-off function with

$$\eta=0 \quad\text{in}\quad \mathrm{N}^{m-6}(T), \qquad\qquad \eta=1\quad\text{in}\quad\Omega\setminus\mathrm{N}^{m-5}(T).$$

Denote $\mathcal{R}=\mathrm{supp}(\nabla\eta)$ and $\boldsymbol{\phi}:=\mathcal{G}(\mathbf{F}_T)\in\mathbf{W}$. In the following we use $\mathrm{N}^k(\mathcal{R})=\mathrm{N}^{m-5+k}(T)\setminus\mathrm{N}^{m-6-k}(T)$. Note that $\|\nabla\eta\|_{L^\infty(\mathcal{R})}\sim H^{-1}$. Furthermore, let $\boldsymbol{\phi}=\boldsymbol{\phi}-\pi_H^E\boldsymbol{\phi}=\mathbf{z}+\nabla\theta$ be the splitting from Lemma 4.

Set $\mathbf{w}:=(\mathrm{id}-\pi_H^E)(\eta\mathbf{z}+\nabla(\eta\theta))$ and note that (i) $\mathrm{curl}\,\mathbf{w}=\mathrm{curl}(\mathrm{id}-\pi_H^E)(\eta\mathbf{z})$, (ii) $\mathbf{w}\in\mathbf{W}$, and (iii) $\mathrm{supp}\,\mathbf{w}\subset\Omega\setminus T$. Property (i) holds because of $\mathrm{curl}\,\nabla=0$ and $\mathrm{curl}\,\pi_H^E\nabla v=\pi_H^F(\mathrm{curl}\,\nabla v)=0$ for all $v\in H_0^1(\Omega)$ due to the commuting property of π_H^E. Since $\pi_H^E\boldsymbol{\phi}=0$, $\eta=1$ in $\Omega\setminus\mathrm{N}^m(T)$ and because of the coercivity, we obtain that

$$\begin{aligned}\|\boldsymbol{\phi}\|^2_{\mathbf{H}(\mathrm{curl},\Omega\setminus\mathrm{N}^m(T))}&=\|(\mathrm{id}-\pi_H^E)(\mathbf{z}+\nabla\theta)\|^2_{\mathbf{H}(\mathrm{curl},\Omega\setminus\mathrm{N}^m(T))}\leq\|\mathbf{w}\|^2_{\mathbf{H}(\mathrm{curl},\Omega)}\\&\leq\alpha^{-1}|\mathcal{B}(\mathbf{w},\mathbf{w})|.\end{aligned}$$

Using the definition of the corrector Green's operator in (5.1) and the fact that $\mathbf{F}_T(\mathbf{w}) = 0$ due to $\operatorname{supp}\mathbf{w} \subset \Omega \setminus T$ yields $\mathcal{B}(\boldsymbol{\phi}, \mathbf{w}) = 0$. Using that $\operatorname{supp}\mathbf{w} \cap \operatorname{supp}(\boldsymbol{\phi} - \mathbf{w}) \subset \mathrm{N}(\mathcal{R})$, we obtain with the continuity of $\mathcal{B}$

$$\begin{aligned}\alpha\|\boldsymbol{\phi}\|^2_{\mathbf{H}(\mathrm{curl},\Omega\setminus\mathrm{N}^m(T))} &\leq |\mathcal{B}(\mathbf{w},\mathbf{w})| = |\mathcal{B}(\mathbf{w}-\boldsymbol{\phi},\mathbf{w})| \\ &\lesssim \|\mathbf{w}-\boldsymbol{\phi}\|_{\mathbf{H}(\mathrm{curl},\mathrm{N}(\mathcal{R}))}\|\mathbf{w}\|_{\mathbf{H}(\mathrm{curl},\mathrm{N}(\mathcal{R}))} \\ &\leq \|\mathbf{w}-\boldsymbol{\phi}\|_{\mathbf{H}(\mathrm{curl},\mathrm{N}(\mathcal{R}))}\big(\|\mathbf{w}-\boldsymbol{\phi}\|_{\mathbf{H}(\mathrm{curl},\mathrm{N}(\mathcal{R}))} + \|\boldsymbol{\phi}\|_{\mathbf{H}(\mathrm{curl},\mathrm{N}(\mathcal{R}))}\big).\end{aligned}$$

We now estimate $\boldsymbol{\phi} - \mathbf{w} = (\mathrm{id} - \pi_H^E)(\boldsymbol{\phi} - \eta\mathbf{z} - \nabla(\eta\theta))$. We deduce with the stability of π_H^E, (4.1) and (4.2), and Lemma 4

$$\begin{aligned}&\|\boldsymbol{\phi}-\mathbf{w}\|_{\mathbf{H}(\mathrm{curl},\mathrm{N}(\mathcal{R}))} \\ &\quad\lesssim \|\boldsymbol{\phi}-\eta\mathbf{z}-\nabla(\eta\theta)\|_{L^2(\mathrm{N}^2(\mathcal{R}))} + H\|\operatorname{curl}(\boldsymbol{\phi}-\eta\mathbf{z})\|_{L^2(\mathrm{N}^2(\mathcal{R}))} \\ &\quad\lesssim (\|\boldsymbol{\phi}\|_{L^2(\mathrm{N}^2(\mathcal{R}))} + H\|\operatorname{curl}\boldsymbol{\phi}\|_{L^2(\mathrm{N}^2(\mathcal{R}))} + \|\eta\mathbf{z}\|_{L^2(\mathrm{N}^2(\mathcal{R}))} \\ &\qquad + \|\nabla\eta\|_{L^\infty(\mathcal{R})}\|\theta\|_{L^2(\mathcal{R})} + \|\eta\|_{L^\infty(\mathrm{N}^2(\mathcal{R}))}\|\nabla\theta\|_{L^2(\mathrm{N}^{m-3}(T)\setminus\mathrm{N}^{m-6}(T))} \\ &\qquad + H\big(\|\nabla\eta\|_{L^\infty(\mathcal{R})}\|\mathbf{z}\|_{L^2(\mathcal{R})} + \|\eta\|_{L^\infty(\mathrm{N}^2(\mathcal{R}))}\|\operatorname{curl}\mathbf{z}\|_{L^2(\mathrm{N}^{m-3}(T)\setminus\mathrm{N}^{m-6}(T))}\big) \\ &\quad\lesssim \|\boldsymbol{\phi}\|_{L^2(\mathrm{N}^m(T)\setminus\mathrm{N}^{m-9}(T))} + H\|\operatorname{curl}\boldsymbol{\phi}\|_{L^2(\mathrm{N}^m(T)\setminus\mathrm{N}^{m-9}(T))}.\end{aligned}$$

All in all, this gives

$$\|\boldsymbol{\phi}\|^2_{\mathbf{H}(\mathrm{curl},\Omega\setminus\mathrm{N}^m(T))} \leq \tilde{C}\|\boldsymbol{\phi}\|^2_{\mathbf{H}(\mathrm{curl},\mathrm{N}^m(T)\setminus\mathrm{N}^{m-9}(T))}$$

for some $\tilde{C} > 0$. Moreover, it holds that

$$\|\boldsymbol{\phi}\|^2_{\mathbf{H}(\mathrm{curl},\Omega\setminus\mathrm{N}^m(T))} = \|\boldsymbol{\phi}\|^2_{\mathbf{H}(\mathrm{curl},\Omega\setminus\mathrm{N}^{m-9}(T))} - \|\boldsymbol{\phi}\|^2_{\mathbf{H}(\mathrm{curl},\mathrm{N}^m(T)\setminus\mathrm{N}^{m-9}(T))}.$$

Thus, we obtain finally with $\tilde{\beta}_{\mathrm{pre}} := (1 + \tilde{C}^{-1})^{-1} < 1$, a reiteration of the above argument, and Lemma 7 that

$$\|\boldsymbol{\phi}\|^2_{\mathbf{H}(\mathrm{curl},\Omega\setminus\mathrm{N}^m(T))} \lesssim \tilde{\beta}_{\mathrm{pre}}^{\lfloor m/9\rfloor}\|\boldsymbol{\phi}\|^2_{\mathbf{H}(\mathrm{curl})} \lesssim \tilde{\beta}_{\mathrm{pre}}^{\lfloor m/9\rfloor}\|\mathbf{F}_T\|^2_{\mathbf{H}_0(\mathrm{curl})'}.$$

Algebraic manipulations give the assertion. ∎

Proof of Theorem 14. We start by proving the local estimate

$$\|\mathcal{G}(\mathbf{F}_T) - \mathcal{G}_{T,m}(\mathbf{F}_T)\|_{\mathbf{H}(\mathrm{curl})} \leq C_1\tilde{\beta}^m\|\mathbf{F}_T\|_{\mathbf{H}_0(\mathrm{curl})'} \tag{7.1}$$

for some constant $C_1 > 0$ and $0 < \tilde{\beta} < 1$. Let $\eta \in \mathcal{S}^1(\mathcal{T}_H)$ be a piecewise linear and globally continuous cut-off function with

$$\eta = 0 \quad \text{in} \quad \Omega \setminus \mathrm{N}^{m-1}(T), \qquad\qquad \eta = 1 \quad \text{in} \quad \mathrm{N}^{m-2}(T).$$

Due to Céa's lemma we have

$$\|\mathcal{G}(\mathbf{F}_T) - \mathcal{G}_{T,m}(\mathbf{F}_T)\|_{\mathbf{H}(\mathrm{curl})} \lesssim \inf_{\mathbf{w}_{T,m}\in\mathbf{W}(\Omega_T)}\|\mathcal{G}(\mathbf{F}_T) - \mathbf{w}_{T,m}\|_{\mathbf{H}(\mathrm{curl})}.$$

We use the splitting of Lemma 4 and write $\mathcal{G}(\mathbf{F}_T) = (\mathrm{id} - \pi_H^E)(\mathcal{G}(\mathbf{F}_T)) = \mathbf{z} + \nabla\theta$. Then we choose $\mathbf{w}_{T,m} = (\mathrm{id} - \pi_H^E)(\eta\mathbf{z} + \nabla(\eta\theta)) \in \mathbf{W}(\Omega_T)$ and derive with the stability of π_H^E and (4.3)

$$\begin{aligned}\|\mathcal{G}(\mathbf{F}_T)-\mathcal{G}_{T,m}(\mathbf{F}_T)\|_{\mathbf{H}(\mathrm{curl})} &\lesssim \|(\mathrm{id}-\pi_H^E)(\mathcal{G}(\mathbf{F}_T)-\eta\mathbf{z}-\nabla(\eta\theta))\|_{\mathbf{H}(\mathrm{curl})}\\ &= \|(\mathrm{id}-\pi_H^E)((1-\eta)\mathbf{z}+\nabla((1-\eta)\theta))\|_{\mathbf{H}(\mathrm{curl})}\\ &\lesssim \|(1-\eta)\mathbf{z}\|_{L^2(\Omega\setminus\{\eta=1\})}+\|\nabla((1-\eta)\theta)\|_{L^2(\Omega\setminus\{\eta=1\})}\\ &\quad+(1+H)\|\operatorname{curl}((1-\eta)\mathbf{z})\|_{L^2(\Omega\setminus\{\eta=1\})}\\ &\lesssim (1+H)\,\|\mathcal{G}(\mathbf{F}_T)\|_{\mathbf{H}(\mathrm{curl},\mathrm{N}^3(\Omega\setminus\{\eta=1\}))}.\end{aligned}$$

Combination with Proposition 12 gives estimate (7.1).

To prove the main estimate of Theorem 14, i.e., estimate (6.3), we define, for a given simplex $T\in\mathcal{T}_H$, the piecewise linear, globally continuous cut-off function $\eta_T\in\mathcal{S}^1(\mathcal{T}_H)$ via

$$\eta_T=0 \quad \text{in} \quad \mathrm{N}^{m+1}(T), \qquad\qquad \eta_T=1 \quad \text{in} \quad \Omega\setminus\mathrm{N}^{m+2}(T).$$

Denote $\mathbf{w}:=(\mathcal{G}-\mathcal{G}_m)(\mathbf{F})=\sum_{T\in\mathcal{T}_H}\mathbf{w}_T$ with $\mathbf{w}_T:=(\mathcal{G}-\mathcal{G}_{T,m})(\mathbf{F}_T)$ and split $\mathbf{w}$ according to Lemma 4 as $\mathbf{w}=\mathbf{w}-\pi_H^E(\mathbf{w})=\mathbf{z}+\nabla\theta$. Due to the ellipticity of $\mathcal{B}$ and its sesquilinearity, we have

$$\alpha\|\mathbf{w}\|^2_{\mathbf{H}(\mathrm{curl})}\leq\Big|\sum_{T\in\mathcal{T}_H}\mathcal{B}(\mathbf{w}_T,\mathbf{w})\Big|\leq\sum_{T\in\mathcal{T}_H}|\mathcal{B}(\mathbf{w}_T,\mathbf{z}+\nabla\theta)|\leq\sum_{T\in\mathcal{T}_H}(A_T+B_T),$$

where, for any $T\in\mathcal{T}_H$, we abbreviate

$$A_T:=|\mathcal{B}(\mathbf{w}_T,(1-\eta_T)\mathbf{z}+\nabla((1-\eta_T)\theta))| \quad\text{and}\quad B_T:=|\mathcal{B}(\mathbf{w}_T,\eta_T\mathbf{z}+\nabla(\eta_T\theta))|.$$

For the term A_T, we derive by using the properties of the cut-off function and the regular decomposition (4.3)

$$\begin{aligned}A_T &\lesssim \|\mathbf{w}_T\|_{\mathbf{H}(\mathrm{curl})}\|(1-\eta_T)\mathbf{z}+\nabla((1-\eta_T)\theta)\|_{\mathbf{H}(\mathrm{curl},\{\eta_T\neq1\})}\\ &\leq \|\mathbf{w}_T\|_{\mathbf{H}(\mathrm{curl})}\,(1+H)\,\|\mathbf{w}\|_{\mathbf{H}(\mathrm{curl},\mathrm{N}^3(\{\eta_T\neq1\}))}.\end{aligned}$$

The term B_T can be split as

$$B_T\leq|\mathcal{B}(\mathbf{w}_T,(\mathrm{id}-\pi_H^E)(\eta_T\mathbf{z}+\nabla(\eta_T\theta)))|+|\mathcal{B}(\mathbf{w}_T,\pi_H^E(\eta_T\mathbf{z}+\nabla(\eta_T\theta)))|.$$

Denoting $\boldsymbol{\phi}:=(\mathrm{id}-\pi_H^E)(\eta_T\mathbf{z}+\nabla(\eta_T\theta))$, we observe $\boldsymbol{\phi}\in\mathbf{W}$ and $\operatorname{supp}\boldsymbol{\phi}\subset\Omega\setminus\mathrm{N}^m(T)$. Because $\boldsymbol{\phi}\in\mathbf{W}$ with support outside T, we have $\mathcal{B}(\mathcal{G}(\mathbf{F}_T),\boldsymbol{\phi})=\mathbf{F}_T(\boldsymbol{\phi})=0$. Since $\boldsymbol{\phi}$ has support outside $\mathrm{N}^m(T)=\Omega_T$, but $\mathcal{G}_{T,m}(\mathbf{F}_T)\in\mathbf{W}(\Omega_T)$, we also have $\mathcal{B}(\mathcal{G}_{T,m}(\mathbf{F}_T),\boldsymbol{\phi})=0$. All in all, this means $\mathcal{B}(\mathbf{w}_T,\boldsymbol{\phi})=0$. Using the stability of π_H^E (4.1), (4.2), and the regular decomposition (4.3), we obtain

$$\begin{aligned}B_T &\leq |\mathcal{B}(\mathbf{w}_T,\pi_H^E(\eta_T\mathbf{z}+\nabla(\eta_T\theta)))|\\ &\lesssim \|\mathbf{w}_T\|_{\mathbf{H}(\mathrm{curl})}\big(\|\eta_T\mathbf{z}+\nabla(\eta_T\theta)\|_{L^2(\mathrm{N}^2(\{\eta_T\neq1\}))}+(1+H)\|\operatorname{curl}(\eta_T\mathbf{z})\|_{L^2(\mathrm{N}^2(\{\eta_T\neq1\}))}\big)\\ &\lesssim \|\mathbf{w}_T\|_{\mathbf{H}(\mathrm{curl})}(1+H)\,\|\mathbf{w}\|_{\mathbf{H}(\mathrm{curl},\mathrm{N}^5(\{\eta_T\neq1\}))}.\end{aligned}$$

Combining the estimates for A_T and B_T and observing that $\{\eta_T\neq1\}=\mathrm{N}^{m+2}(T)$, we deduce

$$\begin{aligned}\alpha\|\mathbf{w}\|^2_{\mathbf{H}(\mathrm{curl})} &\lesssim \sum_{T\in\mathcal{T}_H}\|\mathbf{w}_T\|_{\mathbf{H}(\mathrm{curl})}\,\|\mathbf{w}\|_{\mathbf{H}(\mathrm{curl},\mathrm{N}^{m+7}(T))}\\ &\lesssim \sqrt{C_{\mathrm{ol},m}}\,\|\mathbf{w}\|_{\mathbf{H}(\mathrm{curl})}\sqrt{\sum_{T\in\mathcal{T}_H}\|\mathbf{w}_T\|^2_{\mathbf{H}(\mathrm{curl})}}.\end{aligned}$$

Combination with estimate (7.1) finishes the proof of (6.3). Finally, estimate (6.4) follows with

$$\|\mathbf{w}\|_{\mathbf{H}(\mathrm{div})'} \leq CH\|\mathbf{w}\|_{\mathbf{H}(\mathrm{curl})}. \qquad \square$$

Changes for the fully discrete localized method. Let us briefly consider the fully discrete setting described in subsection 6.3. Here we note that, up to a modification of the constants, Theorem 14 also holds for the difference $(\mathcal{G}_h - \mathcal{G}_{h,m})(\mathbf{F})$, where $\mathcal{G}_h(\mathbf{F})$ is the Galerkin approximation of $\mathcal{G}(\mathbf{F})$ in the discrete space $\mathbf{W}_h := \{\mathbf{v}_h \in \mathring{\mathcal{N}}(\mathcal{T}_h) | \pi_H^E(\mathbf{v}_h) = 0\}$ and where $\mathcal{G}_{h,m}(\mathbf{F})$ is defined analogously to $\mathcal{G}_m(\mathbf{F})$ but where $\mathbf{W}_h(\Omega_T) := \{\mathbf{w}_h \in \mathbf{W}_h | \, \mathbf{w}_h \equiv 0 \text{ in } \Omega \setminus \Omega_T\}$ replaces $\mathbf{W}(\Omega_T)$ in the local problems. Again, the central observation is a decay result similar to Proposition 12, but now for $\mathcal{G}_h(\mathbf{F}_T)$. A few modifications to the proof have to be made, though: The product of the cut-off function η and the regular decomposition $\mathbf{z} + \nabla\theta$ does not lie in $\mathring{\mathcal{N}}(\mathcal{T}_h)$. Therefore, an additional interpolation operator into $\mathring{\mathcal{N}}(\mathcal{T}_h)$ has to be applied. Here it is tempting to just use the nodal interpolation operator and its stability on piecewise polynomials, since $\eta\, \mathcal{G}_h(\mathbf{F}_T)$ is a piecewise (quadratic) polynomial, as done, for instance, for the Helmholtz equation in [44]. However, the regular decomposition employed is no longer piecewise polynomial and we hence have to use the Falk–Winther operator π_h^E onto the fine space $\mathring{\mathcal{N}}(\mathcal{T}_h)$ here. This means that we have to modify $\mathbf{w}$ to $\widetilde{\mathbf{w}} := (\mathrm{id} - \pi_H^E)\pi_h^E(\eta\mathbf{z} + \nabla(\eta\theta))$. Note that the additional interpolation operator π_h^E will enlarge the patches slightly, so that we should define η via

$$\eta = 0 \quad \text{in} \quad \mathrm{N}^{m-8}(T), \qquad\qquad \eta = 1 \quad \text{in} \quad \Omega \setminus \mathrm{N}^{m-7}(T).$$

With the same arguments as in the proof of Proposition 12, we can now deduce that

$$\alpha\|\boldsymbol{\phi}\|^2_{\mathbf{H}(\mathrm{curl},\Omega\setminus \mathrm{N}^m(T))} \leq |\mathcal{B}(\widetilde{\mathbf{w}}, \widetilde{\mathbf{w}})| = |\mathcal{B}(\widetilde{\mathbf{w}} - \boldsymbol{\phi}, \widetilde{\mathbf{w}})|.$$

Note that $\boldsymbol{\phi} - \widetilde{\mathbf{w}} = (\mathrm{id} - \pi_h^E)(\boldsymbol{\phi} - \eta\mathbf{z} - \nabla(\eta\theta)) + (\mathrm{id} - \pi_H^E)(\mathrm{id} - \pi_h^E)(\eta\mathbf{z} + \nabla(\eta\theta))$. The first term is the same as in the proof of Proposition 12. The second term can be estimated simply using the stability of π_h^E, the properties of η, and the regular decomposition (4.3).

8. Falk–Winther interpolation. This section briefly describes the construction of the bounded local cochain projection of [21] for the present case of $\mathbf{H}(\mathrm{curl})$-problems in three space dimensions. The two-dimensional case is thoroughly described in the gentle introductory paper [22]. After giving the definition of the operator, we describe how it can be represented as a matrix. This is important because the interpolation operator is part of the algorithm and not a mere theoretical tool and therefore required in a practical realization.

8.1. Definition of the operator. Let Δ_0 denote the set of vertices of $\mathcal{T}_H$ and let $\mathring{\Delta}_0 := \Delta_0 \cap \Omega$ denote the interior vertices. Let Δ_1 denote the set of edges and let $\mathring{\Delta}_1$ denote the interior edges, i.e., the elements of Δ_1 that are not a subset of $\partial\Omega$. The space $\mathring{\mathcal{N}}(\mathcal{T}_H)$ is spanned by the well-known edge-oriented basis $(\boldsymbol{\psi}_E)_{E\in\mathring{\Delta}_1}$ defined for any $E \in \mathring{\Delta}_1$ through the property

$$⨍_E \boldsymbol{\psi}_E \cdot \mathbf{t}_E \, ds = 1 \quad \text{and} \quad ⨍_{E'} \boldsymbol{\psi}_E \cdot \mathbf{t}_E \, ds = 0 \quad \text{for all } E' \in \mathring{\Delta}_1 \setminus \{E\}.$$

Here $\mathbf{t}_E$ denotes the unit tangent to the edge E with a globally fixed sign. Any vertex $z \in \Delta_0$ possesses a nodal patch (sometimes also called macroelement)

$$\omega_z := \operatorname{int}\Big(\bigcup\{T \in \mathcal{T}_H : z \in T\}\Big).$$

For any edge $E \in \Delta_1$ shared by two vertices $z_1, z_2 \in \Delta_0$ such that $E = \operatorname{conv}\{z_1, z_2\}$, the extended edge patch reads

$$\omega_E^{ext} := \omega_{z_1} \cup \omega_{z_2}.$$

The restriction of the mesh $\mathcal{T}_H$ to ω_E^{ext} is denoted by $\mathcal{T}_H(\omega_E^{ext})$. Let $\mathcal{S}^1(\mathcal{T}_H(\omega_E^{ext}))$ denote the (scalar-valued) first-order Lagrange finite element space with respect to $\mathcal{T}_H(\omega_E^{ext})$ and let $\mathcal{N}(\mathcal{T}_H(\omega_E^{ext}))$ denote the lowest-order Nédélec finite element space over $\mathcal{T}_H(\omega_E^{ext})$. The operator

$$Q_E^1 : \mathbf{H}(\operatorname{curl}, \omega_E^{ext}) \to \mathcal{N}(\mathcal{T}_H(\omega_E^{ext}))$$

is defined for any $\mathbf{u} \in \mathbf{H}(\operatorname{curl}, \omega_E^{ext})$ via

$$\begin{aligned} (\mathbf{u} - Q_E^1\mathbf{u}, \nabla\tau) &= 0 && \text{for all } \tau \in \mathcal{S}^1(\mathcal{T}_H(\omega_E^{ext})),\\ (\operatorname{curl}(\mathbf{u} - Q_E^1\mathbf{u}), \operatorname{curl}\mathbf{v}) &= 0 && \text{for all } \mathbf{v} \in \mathcal{N}(\mathcal{T}_H(\omega_E^{ext})). \end{aligned}$$

Given any vertex $y \in \Delta_0$, define the piecewise constant function z_y^0 by

$$z_y^0 = \begin{cases} (\operatorname{meas}(\omega_y))^{-1} & \text{in } \omega_y,\\ 0 & \text{in } \Omega \setminus \omega_y. \end{cases}$$

Given any edge $E \in \Delta_1$ shared by vertices $y_1, y_2 \in \Delta_0$ such that $E = \operatorname{conv}\{y_1, y_2\}$, define

$$(\delta z^0)_E := z_{y_2}^0 - z_{y_1}^0.$$

Let $E \in \Delta_1$ and denote by $\mathring{\mathcal{RT}}(\mathcal{T}_H(\omega_E^{ext}))$ the lowest-order Raviart–Thomas space with respect to $\mathcal{T}_H(\omega_E^{ext})$ with vanishing normal trace on the boundary $\partial(\omega_E^{ext})$. Let for any $E \in \Delta_1$ the field $\mathbf{z}_E^1 \in \mathring{\mathcal{RT}}(\mathcal{T}_H(\omega_E^{ext}))$ be defined by

$$\begin{aligned} \operatorname{div}\mathbf{z}_E^1 &= -(\delta z^0)_E,\\ (\mathbf{z}_E^1, \operatorname{curl}\boldsymbol{\tau}) &= 0 && \text{for all } \boldsymbol{\tau} \in \mathring{\mathcal{N}}(\mathcal{T}_H(\omega_E^{ext})), \end{aligned}$$

where $\mathring{\mathcal{N}}(\mathcal{T}_H(\omega_E^{ext}))$ denotes the Nédélec finite element functions over $\mathcal{T}_H(\omega_E^{ext})$ with vanishing tangential trace on the boundary $\partial(\omega_E^{ext})$. The operator $M^1 : L^2(\Omega; \mathbb{C}^3) \to \mathring{\mathcal{N}}(\mathcal{T}_H)$ maps any $\mathbf{u} \in L^2(\Omega; \mathbb{C}^3)$ to

$$M^1\mathbf{u} := \sum_{E \in \mathring{\Delta}_1} (\operatorname{length}(E))^{-1} \int_{\omega_E^{ext}} \mathbf{u} \cdot \mathbf{z}_E^1 \, dx \, \boldsymbol{\psi}_E.$$

Recall that the weights in the (modified) Clément interpolation of a function $v \in H_0^1$ are $(\operatorname{meas}(\omega_z))^{-1} \int_{\omega_z} v \, dx$ for all vertices z. The operator M^1 now generalizes this (local) averaging process of the Clément operator to the case of edge elements. M^1, however, is not a projection onto the edge elements yet.

The operator

$$Q_{y,-}^1 : \mathbf{H}(\operatorname{curl}, \omega_E^{ext}) \to \mathcal{S}^1(\mathcal{T}_H(\omega_E^{ext}))$$

is the solution operator of a local discrete Neumann problem. For any $\mathbf{u} \in \mathbf{H}(\operatorname{curl}, \omega_E^{ext})$, the function $Q_{y,-}^1\mathbf{u}$ solves

$$(\mathbf{u} - \nabla Q^1_{y,-}\mathbf{u}, \nabla v) = 0 \qquad \text{for all } v \in \mathcal{S}^1(\mathcal{T}_H(\omega_E^{ext})),$$
$$\int_{\omega_E^{ext}} Q^1_{y,-}\mathbf{u}\,dx = 0.$$

Define now the operator $S^1 : \mathbf{H}_0(\mathrm{curl},\Omega) \to \mathring{\mathcal{N}}(\mathcal{T}_H)$ via

$$S^1\mathbf{u} := M^1\mathbf{u} + \sum_{y\in\mathring{\Delta}_0} (Q^1_{y,-}\mathbf{u})(y)\nabla\lambda_y. \tag{8.1}$$

S^1 preserves the degrees of freedom of $\mathcal{N}(\mathcal{T}_H)$ for all gradient functions $\nabla\mathcal{S}(\mathcal{T}_H)$, which is a first step to the projection property. However, S^1 will not commute with the exterior derivative in general and hence needs to be further modified. The second sum on the right-hand side of (8.1) can be rewritten in terms of the basis functions $\boldsymbol{\psi}_E$. The inclusion $\nabla\mathring{\mathcal{S}}^1(\mathcal{T}_H) \subseteq \mathring{\mathcal{N}}(\mathcal{T}_H)$ follows from the principles of finite element exterior calculus [2, 3]. Given an interior vertex $z \in \mathring{\Delta}_0$, the expansion in terms of the basis $(\boldsymbol{\psi}_E)_{E\in\mathring{\Delta}_1}$ reads

$$\nabla\lambda_z = \sum_{E\in\mathring{\Delta}_1} \fint_E \nabla\lambda_z\cdot\mathbf{t}_E\,ds\,\boldsymbol{\psi}_E = \sum_{E\in\Delta_1(z)} \frac{\mathrm{sign}(\mathbf{t}_E\cdot\nabla\lambda_z)}{\mathrm{length}(E)}\boldsymbol{\psi}_E,$$

where $\Delta_1(z) \subseteq \mathring{\Delta}_1$ is the set of all edges that contain z. Thus, S^1 from (8.1) can be rewritten as

$$S^1\mathbf{u} := M^1\mathbf{u} + \sum_{E\in\mathring{\Delta}_1} (\mathrm{length}(E))^{-1}\big((Q^1_{y_2(E),-}\mathbf{u})(y_2(E)) - (Q^1_{y_1(E),-}\mathbf{u})(y_1(E))\big)\boldsymbol{\psi}_E, \tag{8.2}$$

where $y_1(E)$ and $y_2(E)$ denote the endpoints of E (with the orientation convention $\mathbf{t}_E = (y_2(E) - y_1(E))/\,\mathrm{length}(E)$). Finally, the Falk–Winther interpolation operator $\pi_H^E : \mathbf{H}_0(\mathrm{curl},\Omega) \to \mathring{\mathcal{N}}(\mathcal{T}_H)$ is defined as

$$\pi_H^E\mathbf{u} := S^1\mathbf{u} + \sum_{E\in\mathring{\Delta}_1} \fint_E \big((\mathrm{id} - S^1)Q^1_E\mathbf{u}\big)\cdot\mathbf{t}_E\,ds\,\boldsymbol{\psi}_E. \tag{8.3}$$

8.2. Algorithmic aspects. Given a mesh $\mathcal{T}_H$ and a refinement $\mathcal{T}_h$, the linear projection $\pi_H : \mathring{\mathcal{N}}(\mathcal{T}_h) \to \mathring{\mathcal{N}}(\mathcal{T}_H)$ can be represented by a matrix $\mathsf{P} \in \mathbb{C}^{\dim\mathring{\mathcal{N}}(\mathcal{T}_H)\times\dim\mathring{\mathcal{N}}(\mathcal{T}_h)}$. This subsection briefly sketches the assembling of that matrix. The procedure involves the solution of local discrete problems on the macroelements. It is important to note that these problems are of small size.

Given an interior edge $E \in \mathring{\Delta}_1^H$ of $\mathcal{T}_H$ and an interior edge $e \in \mathring{\Delta}_1^h$ of $\mathcal{T}_h$, the interpolation $\pi_H\boldsymbol{\psi}_e$ has an expansion

$$\pi_H\boldsymbol{\psi}_e = \sum_{E'\in\mathring{\Delta}_1^H} c_{E'}\boldsymbol{\psi}_{E'}$$

for coefficients $(c_{E'})_{E'\in\mathring{\Delta}_1^H}$. The coefficient c_E is zero whenever e is not contained in the closure of the extended edge patch $\overline{\omega}_E^{ext}$. The assembling can therefore be organized in a loop over all interior edges in $\mathring{\Delta}_1^H$. Given a global numbering of the edges in $\mathring{\Delta}_1^H$, each edge $E \in \mathring{\Delta}_1^H$ is equipped with a unique index $I_H(E) \in \{1,\dots,\mathrm{card}(\mathring{\Delta}_1^H)\}$. Similarly, the numbering of edges in $\mathring{\Delta}_1^h$ is denoted by I_h.

The matrix $\mathsf{P} = \mathsf{P}_1 + \mathsf{P}_2$ will be composed as the sum of matrices P_1, P_2 that represent the two summands on the right-hand side of (8.3). Those will be assembled in loops over the interior edges. Matrices P_1, P_2 are initialized as empty sparse matrices.

8.2.1. Operator P_1. for $E \in \mathring{\Delta}_1^H$ do. Let the interior edges in $\mathring{\Delta}_1^h$ that lie inside $\overline{\omega}_E^{ext}$ be denoted with $\{e_1, e_2, \ldots, e_N\}$ for some $N \in \mathbb{N}$. The entries $\mathsf{P}_1(I_H(E), [I_h(e_1) \ldots I_h(e_N)])$ of the matrix P_1 are now determined as follows. Compute $\mathbf{z}_E^1 \in \mathring{\mathcal{RT}}(\mathcal{T}_H(\omega_E^{ext}))$. The matrix $\mathsf{M}_E \in \mathbb{C}^{1\times N}$ defined via

$$\mathsf{M}_E := (\operatorname{length}(E))^{-1} \left[\int_{\omega_E^{ext}} \mathbf{z}_E^1 \cdot \boldsymbol{\psi}_{e_j}\, dx \right]_{j=1}^N$$

represents the map of the basis functions on the fine mesh to the coefficient of M^1 contributing to $\boldsymbol{\psi}_E$ on the coarse mesh. Denote by $\mathsf{A}_{y_j(E)}$ and $\mathsf{B}_{y_j(E)}$ $(j = 1, 2)$ the stiffness and right-hand-side matrix representing the system for the operator $Q_{y_j(E),-}$

$$\mathsf{A}_{y_j(E)} := \left[\int_{\omega_{y_j(E)}} \nabla\phi_y \cdot \nabla\phi_z\, dx \right]_{y,z\in\Delta_0(\mathcal{T}_H(\omega_{y_j(E)}))},$$

$$\mathsf{B}_{y_j(E)} := \left[\int_{\omega_{y_j(E)}} \nabla\phi_y \cdot \boldsymbol{\psi}_{e_j}\, dx \right]_{\substack{y\in\Delta_0(\mathcal{T}_H(\omega_{y_j(E)}))\\ j=1,\ldots,N}}.$$

After enhancing the system to $\tilde{\mathsf{A}}_{y_j(E)}$ and $\tilde{\mathsf{B}}_{y_j(E)}$ (with a Lagrange multiplier accounting for the mean constraint), it is uniquely solvable. Set $\tilde{\mathsf{Q}}_{y_j(E)} = \tilde{\mathsf{A}}_{y_j(E)}^{-1}\tilde{\mathsf{B}}_{y_j(E)}$ and extract the row corresponding to the vertex $y_j(E)$

$$\mathsf{Q}_j := (\operatorname{length}(E))^{-1}\tilde{\mathsf{Q}}_{y_j(E)}[y_j(E), :] \in \mathbb{C}^{1\times N}.$$

Set

$$\mathsf{P}_1(I_H(E), [I_h(e_1) \ldots I_h(e_N)]) = \mathsf{M}_E + \mathsf{Q}_1 - \mathsf{Q}_2.$$

end

8.2.2. Operator P_2. for $E \in \mathring{\Delta}_1^H$ do. Denote the matrices—where indices j, k run from 1 to $\operatorname{card}(\Delta_1(\mathcal{T}_H(\omega_E^{ext})))$, y through $\Delta_0(\mathcal{T}_H(\omega_E^{ext}))$, and $\ell = 1, \ldots, N$—

$$\mathsf{S}_E := \left[\int_{\omega_E^{ext}} \operatorname{curl}\boldsymbol{\psi}_{E_j} \cdot \operatorname{curl}\boldsymbol{\psi}_{E_k}\, dx \right]_{j,k} \quad \mathsf{T}_E := \left[\int_{\omega_E^{ext}} \boldsymbol{\psi}_{E_j} \cdot \nabla\lambda_y\, dx \right]_{j,y}$$

and

$$\mathsf{F}_E := \left[\int_{\omega_E^{ext}} \operatorname{curl}\boldsymbol{\psi}_{E_j} \cdot \operatorname{curl}\boldsymbol{\psi}_{e_\ell}\, dx \right]_{j,\ell} \quad \mathsf{G}_E := \left[\int_{\omega_E^{ext}} \boldsymbol{\psi}_{e_\ell} \cdot \nabla\lambda_y\, dx \right]_{y,\ell}.$$

Solve the saddle-point system

$$\begin{bmatrix} \mathsf{S} & \mathsf{T}^* \\ \mathsf{T} & 0 \end{bmatrix} \begin{bmatrix} \mathsf{U} \\ \mathsf{V} \end{bmatrix} = \begin{bmatrix} \mathsf{F} \\ \mathsf{G} \end{bmatrix}.$$

(This requires an additional one-dimensional gauge condition because the sum of the test functions $\sum_y \nabla\lambda_y$ equals zero.) Assemble the operator S^1 (locally) as described in the previous step and denote this matrix by P_1^{loc}. Compute $\mathsf{U} - \mathsf{P}_1^{loc}\mathsf{U}$ and extract the line X corresponding to the edge E

$$\mathsf{P}_2(I_H(E), [I_h(e_1) \ldots I_h(e_N)]) = \mathsf{X}.$$

end

We note that this representation of the Falk–Winther operator as a matrix is an essential step toward a practical implementation: Computations requiring test or ansatz functions in the kernel space $\mathbf{W}$ or its modifications can be written as saddle-point problems now; see [18]. As the rest of our construction follows the LOD framework, we refer to [18] for a discussion of an efficient implementation.

Conclusion. In this paper, we suggested a procedure for the numerical homogenization of $\mathbf{H}(\mathrm{curl})$-elliptic problems. The exact solution is decomposed into a coarse part, which is a good approximation in $\mathbf{H}(\mathrm{div})'$, and a corrector contribution by using the Falk–Winther interpolation operator. We showed that this decomposition gives an optimal order approximation in $\mathbf{H}(\mathrm{curl})$, independent of the regularity of the exact solution. Furthermore, the corrector operator can be localized to patches of macro elements, which allows for an efficient computation. This results in a generalized finite element method in the spirit of the LOD which utilizes the bounded local cochain projection of the Falk–Winther as part of the algorithm.

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