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Normal bundle and Almgren's geometric inequality for singular varieties of bounded mean curvature

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In this paper we deal with a class of varieties of bounded mean curvature in the viscosity sense that has the remarkable property to contain the blow up sets of all sequences of varifolds whose mean curvatures are uniformly bounded and whose boundaries are uniformly bounded on compact sets. We investigate the second-order properties of these varieties, obtaining results that are new also in the varifold's setting. In particular we prove that the generalized normal bundle of these varieties satisfies a natural Lusin (N) condition, a property that allows to prove a Coarea-type formula for their generalized Gauss map. Then we use this formula to extend a sharp geometric inequality of Almgren and the associated soap bubble theorem. As a consequence of the geometric inequality we obtain sufficient conditions to conclude that the area-blow-up set is empty for sequences of varifolds whose first variation is controlled.

Keywords: Area blow-up set, varifolds, generalized second fundamental form, generalized Gauss map, Almgren's geometric inequality, soap bubbles.

Mathematics Subject Classification: 49Q20, 49Q10, 53A07, 53C24, 35D40

1. Introduction

Overview

In this paper, we deal with the following class of singular varieties.

Definition 1.1. (see [31, 2.1][a]) Suppose $1 \leq m < n$ are integers, Ω is an open subset of $\mathbf{R}^n$, Γ is relatively closed in Ω and $h \geq 0$. We say that Γ is an (m, h) *subset of* Ω provided it has the following property: if $x \in \Gamma$ and f is a $\mathscr{C}^2$ function in a neighbourhood of x such that $f|\Gamma$ has a local maximum at x and $\nabla f(x) \neq 0$, then

$$\operatorname{trace}_m \mathrm{D}^2 f(x) \leq h|\nabla f(x)|,$$

where $\operatorname{trace}_m \mathrm{D}^2 f(x)$ is the sum of the lowest m eigenvalues of $\mathrm{D}^2 f(x)$.

Similar notions also appear in the theory of viscosity solutions of PDE's; see [5], [23] and [24]. A reason to study the class of (m, h) sets is certainly given by the fact, proved in [31, 2.6], that the area blow-up set of a sequence of smooth submanifolds (or varifolds) belongs to this class, provided that their mean curvatures are uniformly bounded by h and their boundaries are uniformly bounded on compact sets. In particular, the support of an m dimensional varifold V such that $\|\delta V\| \leq h\|V\|$ is an (m, h) set; see [31, 2.8][b]. In [31] several theorems for smooth varieties are extended to (m, h) sets (including a constancy theorem, barrier principles and halfspace theorems).

In the first part of this paper we use the normal bundle and the second fundamental form introduced in [20] for all closed sets, to prove the coarea formula for the Gauss map for (m, h) sets (cf. Theorem 1.3). On the other hand, this formula might easily fail to hold outside the class of (m, h) sets; for example there exist convex hypersurfaces of class $\mathcal{C}^1$ for which it is not true (see [20, 6.3]).

Since for varifolds of bounded mean curvature the trace of the aforementioned second fundamental form agrees with the distributional mean curvature (cf. Corollary 1.4), the coarea formula in Theorem 1.3 gives also a meaningful and new result in the theory of varifolds.

Then we use Theorem 1.3 to prove a sharp geometric inequality and a soap bubble theorem (cf. Theorem 1.5) for all compact (m, h) sets. These results extend a well known result that Almgren proved for varifolds of bounded mean curvature in [2, Theorem 1].

Finally, Theorem 1.5 is used to derive sufficient conditions to conclude that the area blow up set of certain sequences of varifolds is empty; see Corollaries 4.4 and 4.5.

[a]This definition is equivalent to [31, 2.1] by [31, 8.1].
[b]In this paper we adopt the terminology in [2, Appendix C] for varifolds; in particular note that the variation function $\mathbf{h}(V, \cdot)$ (i.e. generalized mean curvature of V) differs from the one adopted in Allard's paper [1, 4.2] by a sign.

Outline of the paper

In this section we give more details on the results and the methods of this paper.

Firstly, we recall certain definitions from [20] (see also [29] and [11]). If $A \subseteq \mathbf{R}^n$ is closed we define the *generalized unit normal bundle of* A as

$$N(A) = (A \times \mathbf{R}^n) \cap \{(a, u) : |u| = 1,\ \boldsymbol{\delta}_A(a + su) = s \text{ for some } s > 0\}$$

(here $\boldsymbol{\delta}_A$ is the distance function from A). It might be interesting to notice that $N(A)$ is the natural geometric version of the second-order super-differential of a function used in [30] to study second-order pointwise differentiability properties of viscosity solutions of elliptic PDE's. The set $N(A)$ is always a countably $n-1$ rectifiable subset of $\mathbf{R}^n \times \mathbf{R}^n$ (in the sense of [7, 3.2.14]) and an appropriate notion of second fundamental form

$$Q_A(a, u) : T_A(a, u) \times T_A(a, u) \to \mathbf{R},$$

where $T_A(a, u)$ is a linear subspace of $\mathbf{R}^n$, exists at $\mathscr{H}^{n-1}$ almost all $(a, u) \in N(A)$ (see 2.4). For an arbitrary closed set A the dimension of T_A may vary from point to point. One of the key facts proved in this paper (see 3.8) shows that if A is an (m, h) subset of $\mathbf{R}^n$ then the normal bundle $N(A)$ satisfies the following remarkable Lusin (N) condition, provided that A is a countable union of sets of finite $\mathscr{H}^m$ measure (this proviso is unnecessary if $m = n - 1$, see 3.9).

Definition 1.2. Suppose $A \subseteq \mathbf{R}^n$ is a closed set, $\Omega \subseteq \mathbf{R}^n$ is an open set and $1 \le m < n$ is an integer. We say that $N(A)$ satisfies the m *dimensional Lusin (N) condition in* Ω if and only if the following property holds:

$$\mathscr{H}^{n-1}(N(A) \cap \{(a, u) : a \in Z\}) = 0$$

for every $Z \subseteq A \cap \Omega$ with $\mathscr{H}^m(A^{(m)} \cap Z) = 0$. Here $A^{(m)}$ is the set of points where A can be touched by a ball from $n - m$ linearly independent directions, (see 2.5).

It follows from a recent result of Schneider [28] that a typical (in the sense of Baire category) compact convex hypersurface in $\mathbf{R}^n$ is of class $\mathcal{C}^1$ but its unit normal bundle does not satisfy the $n-1$ dimensional Lusin (N) condition. Remarkably this condition implies that the first m principal curvatures of an (m, h) set are finite; this is in sharp contrast with the typical behavior of a convex surface; see [28] and [20, 6.3]. This good curvature-behavior is the key to extend the Coarea formula for the generalized Gauss map. If M is an m dimensional $\mathcal{C}^2$ submanifold of $\mathbf{R}^n$ without boundary, $N(M)$ is the unit normal bundle and Q_M is the second fundamental form then the area of the generalized Gauss map of M can be expressed in terms of the curvature of M in the following way: if B is an $\mathscr{H}^{n-1}$ measurable subset of

$N(M)$ then

$$\int_{\mathbf{S}^{n-1}} \mathscr{H}^0\{a : (a,u) \in B\}\, d\mathscr{H}^{n-1}u = \int_M \int_{\{\eta:(z,\eta)\in B\}} |\operatorname{discr} Q_M(z,\zeta)| d\mathscr{H}^{n-m-1}(\zeta)\, d\mathscr{H}^m(z), \qquad (1)$$

where $\operatorname{discr} Q_M(z,\zeta)$ is the discriminant of the symmetric bilinear form $Q_M(z,\zeta)$, see [7, 1.7.10]. This formula follows combining the two coarea formulas (see [7, 3.2.22]) given by the projections maps from $N(M)$ into M and from $N(M)$ into $\mathbf{S}^{n-1}$. Thanks to the Lusin (N) condition, this argument works unchanged for (m,h) sets. Summarizing, we state the first main result of the paper.

Theorem 1.3 (cf. 3.3, 3.8 and 3.9). *Suppose* $1 \le m \le n-1$, $0 \le h < \infty$, Γ *is an* (m,h) *subset of* $\mathbf{R}^n$. *If* $m \le n-2$ *we further assume that* Γ *is a countable union of sets with finite* $\mathscr{H}^m$ *measure.*

Then $N(\Gamma)$ *satisfies the* m *dimensional Lusin (N) condition and*

$$\int_{\mathbf{S}^{n-1}} \mathscr{H}^0\{a : (a,u) \in B\}\, d\mathscr{H}^{n-1}u = \int_\Gamma \int_{\{\eta:(z,\eta)\in B\}} |\operatorname{discr} Q_\Gamma| d\mathscr{H}^{n-m-1} d\mathscr{H}^m z.$$

whenever $B \subseteq N(\Gamma)$ *is* $\mathscr{H}^{n-1}$ *measurable. Moreover,*

$$\dim T_\Gamma(z,\eta) = m \quad \text{and} \quad \operatorname{trace} Q_\Gamma(z,\eta) \le h$$

for $\mathscr{H}^{n-1}$ *a.e.* $(z,\eta) \in N(\Gamma)$.

Theorem 1.3 clearly shows that Q_Γ and $\operatorname{trace} Q_\Gamma$ provide natural notions of second fundamental form and mean curvature for (m,h) sets. In case of *integral varifolds* we also prove the agreement of the trace of the second fundamental form with their distributional mean curvature. The restriction to *integral* varifolds is technical and only due to the fact that we rely on the locality theorem of Schätzle [26, 4.2], which is currently not available for non-integral varifolds.

Corollary 1.4 (cf. 3.11). *Suppose* $1 \le m \le n-1$, $V \in \mathbf{V}_m(\mathbf{R}^n)$ *is an integral varifold such that* $\|\delta V\| \le c\|V\|$ *for some* $0 \le c < \infty$. *Then*

$$\operatorname{trace} Q_{\operatorname{spt}\|V\|}(z,\eta) = \mathbf{h}(V,z) \bullet \eta \quad \text{for } \mathscr{H}^{n-1} \text{ a.e. } (z,\eta) \in N(\operatorname{spt}\|V\|).$$

Thanks to 1.4, the coarea formula in 1.3 turns out to be a natural result in the theory of varifolds. We recall that the coarea formula for the Gauss map for certain codimension-one integral varifolds has been announced in [32]. Mentioning other contributions to the study of generalized notions of curvature in varifolds theory, we recall [12]–[14] in the setting of curvature varifolds and [25],[26],[16], [22] for the study of the second-order rectifiability properties.

The other main contribution of this paper is the following result.

Theorem 1.5. *If* $1 \le m \le n-1$, $h > 0$ *and* Γ *is a non-empty compact* (m, h) *subset of* $\mathbf{R}^n$ *then*

$$\mathscr{H}^m(\Gamma) \ge \left(\frac{m}{h}\right)^m \mathscr{H}^m(\mathbf{S}^m).$$

Moreover if the equality holds and $\Gamma = \operatorname{spt}(\mathscr{H}^m \llcorner \Gamma)$ *then there exists an* $m+1$ *dimensional plane* T *and* $a \in \mathbf{R}^n$ *such that*

$$\Gamma = \partial \mathbf{B}(a, m/h) \cap T.$$

As already mentioned, if Γ is the support of a rectifiable varifold V with a uniform lower bound on the density such that $\|\delta V\| \le h\|V\|$ then this theorem is contained in [2, Theorem 1]. Our proof generalizes Almgren's method to (m, h) sets and some parts originate from [15], where Almgren's techniques have been revisited to be linked to the newer concept of second-order rectifiability. If Γ is the smooth boundary of an open set, then sharp stability estimates for Almgrens geometric inequality are obtained in [13]. We now briefly describe the main steps of the proof. Firstly we can suitably rescale Γ to have $m = h$. For the inequality case we use compactness of Γ to see that for each $\eta \in \mathbf{S}^{n-1}$ there exists an $(n-1)$ dimensional plane π perpendicular to η such that Γ lies on one side of π and touches π at least in one point. This can be precisely stated saying that the projection onto $\mathbf{S}^{n-1}$ of the contact set

$$C = (\Gamma \times \mathbf{S}^{n-1}) \cap \{(z, \eta) : (w - z) \bullet \eta \le 0 \text{ for every } w \in \Gamma\} \subseteq N(\Gamma)$$

equals $\mathbf{S}^{n-1}$. Then the estimate $\operatorname{trace} Q_\Gamma \le m$ in 1.3 and the more elementary fact that Q_Γ has a sign when restricted on C, allows to obtain $\mathscr{H}^m(\Gamma) \ge \mathscr{H}^m(\mathbf{S}^m)$. We remark that we obtain this inequality directly working on the projection of the contact set C of Γ, combining the Coarea formula 1.3 and the Barrier principle of White [31, 7.1]. Moving to the proof of the equality case, we first combine [31, 3.2] with the Strong Barrier principle in [31, 7.3] to conclude that at each point of Γ its tangent cone is the unique supporting hyperplane of the convex hull of Γ. This implies that Γ actually coincides with the boundary of its convex hull and it is a $\mathcal{C}^1$ hypersurface. Now the last step of the proof consists in applying the barrier principle [31, 7.1] in combination with a result of Federer [7, 3.1.23] to prove that Γ is a $\mathcal{C}^{1,1}$ hypersurface. The idea of this last step comes from [15]. Once we know that Γ is a $\mathcal{C}^{1,1}$ hypersurface, we conclude that Γ is a sphere by a direct computation. In the aforementioned last step of the proof, Almgren used instead the more sophisticated Allard's regularity theory for varifolds to show that Γ is a smooth submanifold.

2. Preliminaries

As a general rule, the notation and the terminology used without comments agree with [7, pp. 669–676]. For varifolds our terminology is based on [2, Appendix C].

The symbol $\sim$ denotes the difference between two sets and the operator discr is the discriminant [7, 1.7.10]. The symbols $\mathbf{U}(a,r)$ and $\mathbf{B}(a,r)$ denote the open and closed ball with center a and radius r [7, 2.8.1]; $\mathbf{S}^m$ is the m-dimensional unit sphere in $\mathbf{R}^{m+1}$ [7, 3.2.13]; $\mathscr{L}^m$ and $\mathscr{H}^m$ are the m-dimensional Lebesgue and Hausdorff measure [7, 2.10.2]; $\mathbf{G}(m,k)$ is the Grassmann manifold of all k-dimensional subspaces in $\mathbf{R}^m$ [7, 1.6.2]. Given a measure μ, we denote by $\boldsymbol{\Theta}^m(\mu,\cdot)$ the m-dimensional density of μ [7, 2.10.19]. Moreover, given a function f, we denote by $\operatorname{dmn} f$, $\operatorname{im} f$ and ∇f the domain, the image and the gradient of f. The closure and the boundary in $\mathbf{R}^n$ of a set A are denoted by $\overline{A}$ and ∂A and, if $\lambda > 0$ and $x \in \mathbf{R}^n$ then $\lambda(A-x) = \{\lambda(y-x) : y \in A\}$. The symbols $\operatorname{Tan}(A,a)$ and $\operatorname{Nor}(A,a)$ denote the tangent and the normal cone of A at a [7, 3.1.21]. The symbol $\bullet$ denotes the standard inner product of $\mathbf{R}^n$. If T is a linear subspace of $\mathbf{R}^n$, then $T_\natural : \mathbf{R}^n \to \mathbf{R}^n$ is the orthogonal projection onto T and $T^\perp = \mathbf{R}^n \cap \{v : v \bullet u = 0 \text{ for } u \in T\}$. If X and Y are sets and $Z \subseteq X \times Y$ we define

$$Z \,|\, S = Z \cap \{(x,y) : x \in S\} \quad \text{for } S \subseteq X,$$
$$Z(x) = Y \cap \{y : (x,y) \in Z\} \quad \text{for } x \in X.$$

The maps $\mathbf{p}, \mathbf{q} : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ are

$$\mathbf{p}(x,v) = x, \quad \mathbf{q}(x,v) = v.$$

If $A \subseteq \mathbf{R}^n$ and $m \geq 1$ is an integer, we say that A *is countably* $(\mathscr{H}^m, m)$ *rectifiable of class* 2 if A can be $\mathscr{H}^m$ almost covered by the union of countably many m-dimensional submanifolds of class 2 of $\mathbf{R}^n$; we omit the prefix "countably" when $\mathscr{H}^m(A) < \infty$. If X and Y are metric spaces and $f : X \to Y$ is a function such that f and f^{-1} are Lipschitzian functions, then we say that f is a *bi-Lipschitzian homeomorphism*.

Approximate second fundamental form

In this paper, we employ weak notions of second fundamental form and mean curvature that can be naturally associated to each set $A \subseteq \mathbf{R}^n$ at those points $a \in \mathbf{R}^n$, where A is approximately differentiable of order 2 in the sense of [21]. In order to keep this preliminary section relatively short we directly refer to [20, 2.7–2.11], where relevant definitions and remarks about the theory developed in [21] are summarized. On the basis of [20, 2.7–2.8] we can introduce the following definitions.

Definition 2.1. The *approximate second fundamental form of* A *at* a is

$$\operatorname{ap} \mathbf{b}_A(a) = \operatorname{ap} \mathrm{D}^2 A(a) \,|\, \operatorname{ap} \operatorname{Tan}(A,a) \times \operatorname{ap} \operatorname{Tan}(A,a)$$

and the associated *approximate mean curvature of* A *at* a is

$$\operatorname{ap} \mathbf{h}_A(a) = \operatorname{trace}(\operatorname{ap} \mathbf{b}_A(a)).$$

If A is an m-dimensional submanifold of class 2 then these notions agree with the classical notions from differential geometry, see [20, 2.9].

Curvature for arbitrary closed sets

Besides the concept of approximate second fundamental form, in this paper, we make use of a more general notion of second fundamental form introduced in [20] that can be associated to arbitrary closed sets. The theory of curvature for arbitrary closed sets has been developed in [11, 20, 29] and here we summarize those concepts that are relevant for our purpose in this paper.

Suppose A is a closed subset of $\mathbf{R}^n$.

2.2 (cf. [20, 2.12, 3.1]). The *distance function to* A is denoted by $\boldsymbol{\delta}_A$ and $S(A,r) = \{x : \boldsymbol{\delta}_A(x) = r\}$. It follows from [20, 2.13] that if $r > 0$ and $K \subseteq \mathbf{R}^n$ is compact then $\mathscr{H}^{n-1}(S(A,r) \cap K) < \infty$ and $S(A,r)$ is countably $(\mathscr{H}^{n-1}, n-1)$ rectifiable of class 2.

If U is the set of all $x \in \mathbf{R}^n$ such that there exists a unique $a \in A$ with $|x-a| = \boldsymbol{\delta}_A(x)$, we define the *nearest point projection onto* A as the map $\boldsymbol{\xi}_A$ characterized by the requirement

$$|x - \boldsymbol{\xi}_A(x)| = \boldsymbol{\delta}_A(x) \quad \text{for } x \in U.$$

Let $U(A) = \operatorname{dmn} \boldsymbol{\xi}_A \sim A$. The functions $\boldsymbol{\nu}_A$ and $\boldsymbol{\psi}_A$ are defined by

$$\boldsymbol{\nu}_A(z) = \boldsymbol{\delta}_A(z)^{-1}(z - \boldsymbol{\xi}_A(z)) \quad \text{and} \quad \boldsymbol{\psi}_A(z) = (\boldsymbol{\xi}_A(z), \boldsymbol{\nu}_A(z)),$$

whenever $z \in U(A)$.

2.3 (cf. [20, 3.6, 3.13]). We define the Borel function $\rho(A,\cdot)$ setting

$$\rho(A,x) = \sup\{t : \boldsymbol{\delta}_A(\boldsymbol{\xi}_A(x) + t(x - \boldsymbol{\xi}_A(x))) = t\boldsymbol{\delta}_A(x)\} \quad \text{for } x \in U(A),$$

and we say that $x \in U(A)$ is a *regular point of* $\boldsymbol{\xi}_A$ provided that $\boldsymbol{\xi}_A$ is approximately differentiable at x with symmetric approximate differential and $\operatorname{ap}\lim_{y \to x} \rho(A,y) = \rho(A,x) > 1$ (see [20, 2.4, 2.5] for the definition of approximate limit and approximate differentiability). The set of regular points of $\boldsymbol{\xi}_A$ is denoted by $R(A)$.

2.4 (cf. [20, 4.1, 4.4, 4.7, 4.9]). *The generalized unit normal bundle of* A is defined as

$$N(A) = (A \times \mathbf{S}^{n-1}) \cap \{(a,u) : \boldsymbol{\delta}_A(a + su) = s \text{ for some } s > 0\}$$

and $N(A,a) = \{v : (a,v) \in N(A)\}$ for $a \in A$. The set $N(A)$ is a countably $n-1$ rectifiable subset of $\operatorname{Nor}(A)$ (cf. [20, 4.2, 4.3])

Next, we define

$$R(N(A)) = \boldsymbol{\psi}_A[R(A)].$$

One may check (cf. [20, 4.5]) that $\mathscr{H}^{n-1}[N(A) \sim R(N(A))] = 0$. If $(a,u) \in R(N(A))$, $x \in R(A)$ and $\boldsymbol{\psi}_A(x) = (a,u)$ we introduce (cf. [20, 4.7]) the linear subspace

$$T_A(a,u) = \operatorname{ap} \mathrm{D}\, \boldsymbol{\xi}_A(x)[\mathbf{R}^n] \in \bigcup_{m=0}^{n+1} \mathbf{G}(n,m)$$

and we define the symmetric bilinear form $Q(a,u) : T_A(a,u) \times T_A(a,u) \to \mathbf{R}$ by the formula

$$Q_A(a,u)(\tau,\tau_1) = \tau \bullet \operatorname{ap} \mathrm{D}\, \boldsymbol{\nu}_A(x)(\sigma_1),$$

where $(\tau,\tau_1) \in T_A(a,u) \times T_A(a,u)$ and $\sigma_1 \in \operatorname{ap} \mathrm{D}\, \boldsymbol{\xi}_A(x)^{-1}[\tau_1]$. This is a well-posed definition (cf. [20, 4.6, 4.8]). The bilinear form $Q_A(a,u)$ is the second fundamental form of A at a in the direction u.

If $(a,u) \in R(N(A))$ *the principal curvatures of A at (a,u)* are the numbers

$$\kappa_{A,1}(a,u) \le \cdots \le \kappa_{A,n-1}(a,u),$$

defined so that $\kappa_{A,m+1}(a,u) = \infty$, $\kappa_{A,1}(a,u), \ldots, \kappa_{A,m}(a,u)$ are the eigenvalues of $Q_A(a,u)$ and $m = \dim T_A(a,u)$. Moreover,

$$\chi_{A,1}(x) \le \cdots \le \chi_{A,n-1}(x)$$

are the eigenvalues of $\operatorname{ap} \mathrm{D}\, \boldsymbol{\nu}_A(x)|\{v : v \bullet \boldsymbol{\nu}_A(x) = 0\}$ for $x \in R(A)$.

It follows from [20, 4.10] that if $r > 0$ and $x \in S(A,r) \cap R(A)$ then

$$\kappa_{A,i}(\boldsymbol{\psi}_A(x)) = \chi_{A,i}(x)(1 - r\chi_{A,i}(x))^{-1} \quad \text{for } i = 1, \ldots, n-1. \tag{2}$$

2.5 (cf. [20, 5.1, 5.2]). For each $a \in A$ we define the closed convex subset

$$\operatorname{Dis}(A,a) = \{v : |v| = \boldsymbol{\delta}_A(a+v)\}$$

and we notice that $N(A,a) = \{v/|v| : 0 \ne v \in \operatorname{Dis}(A,a)\}$. For every integer $0 \le m \le n$ we define *the mth stratum of A* by

$$A^{(m)} = A \cap \{a : \dim \operatorname{Dis}(A,a) = n - m\};$$

this is a Borel set which is countably m rectifiable and countably $(\mathscr{H}^m, m)$ rectifiable of class 2; see [18, 4.12].

The following assertion will be useful: *if $a \in A^{(m)}$ then*

$$\mathscr{H}^{n-m-1}(N(A,a) \cap V) > 0$$

whenever V is an open subset of $\mathbf{R}^n$ such that $V \cap N(A,a) \ne \varnothing$. In fact, noting that $\mathscr{H}^{n-m}(U \cap \operatorname{Dis}(A,a)) > 0$ whenever U is open and $U \cap \operatorname{Dis}(A,a) \ne \varnothing$, the assertion follows applying Coarea formula [7, 3.2.22(3)].

The relation between the two notions of second fundamental form defined in 2.1 and 2.4 is given by the following result, proved in [20, 6.2].

Theorem 2.6. *If $A \subseteq \mathbf{R}^n$ is a closed set, $1 \le m \le n-1$ and $S \subseteq A$ is $\mathscr{H}^m$ measurable and $(\mathscr{H}^m, m)$ rectifiable of class 2 then there exists $R \subseteq S$ such that $\mathscr{H}^m(S \sim R) = 0$,*

$$\operatorname{ap} \operatorname{Tan}(S,a) = T_A(a,u) \in \mathbf{G}(n,m), \quad \operatorname{ap} \mathbf{b}_S(a)(\tau,\upsilon) \bullet u = -Q_A(a,u)(\tau,\upsilon)$$

for every $\tau, \upsilon \in T_A(a,u)$ and for $\mathscr{H}^{n-1}$ a.e. $(a,u) \in N(A)|R$.

Remark 2.7. It is in general not possible to replace $N(A)|R$ with $N(A)|S$ in the conclusion, even if S is the boundary of a $\mathcal{C}^{1,\alpha}$ convex set A; see the example in [20, 6.3].

Level sets of the distance function. We conclude this preliminary section providing a structural result for the level sets of the distance function from an arbitrary closed set, which is sufficient for the purpose of this work. Other structural results are available, in particular we refer to [19] and references therein.

Theorem 2.8 (Gariepy–Pepe). *Suppose A is a closed subset of $\mathbf{R}^n$, $r > 0$, $x \in S(A,r)$, $\boldsymbol{\delta}_A$ is differentiable at x and $T = \{v : v \bullet \nabla \boldsymbol{\delta}_A(x) = 0\}$.*

Then there exists an open neighborhood V of x and a Lipschitzian function $f : T \to T^\perp$ such that f is differentiable at $T_\natural(x)$ with $\mathrm{D}\, f(T_\natural(x)) = 0$ and

$$V \cap S(A,r) = V \cap \{\chi + f(\chi) : \chi \in T\}.$$

Proof. The arguments in the proof of [9, Theorem 1] prove the statement with the exception of the differentiability properties of f, which can be easily deduced[c] noting that $\mathrm{Tan}(S(A,r),x) \subseteq T$. □

Lemma 2.9. *If $A \subseteq \mathbf{R}^n$ is a closed set then the following conclusion holds for $\mathscr{L}^1$ a.e. $r > 0$ and for $\mathscr{H}^{n-1}$ a.e. $x \in S(A,r)$:*

$$\mathrm{Tan}(S(A,r),x) = \mathrm{ap}\,\mathrm{Tan}(S(A,r),x) = \{v : v \bullet \boldsymbol{\nu}_A(x) = 0\}$$

and, if $T = \mathrm{Tan}(S(A,r),x)$, there exists an open neighborhood V of x and a Lipschitzian function $f : T \to T^\perp$ such that f is pointwise differentiable of order 2 at $T_\natural(x)$, $\mathrm{D}\, f(T_\natural(x)) = 0$,

$$\mathrm{D}^2 f(T_\natural(x))(u,v) \bullet \boldsymbol{\nu}_A(x) = -\mathrm{ap}\,\mathrm{D}\,\boldsymbol{\nu}_A(x)(u) \bullet v \quad \textit{for } u, v \in T$$

and $V \cap S(A,r) = V \cap \{\chi + f(\chi) : \chi \in T\}$.

Proof. Since $\boldsymbol{\delta}_A$ is differentiable at $\mathscr{L}^n$ a.e. $x \in \mathbf{R}^n$, it follows from [6, 4.8(3)] and Coarea formula that $\boldsymbol{\nu}_A(x) = \nabla \boldsymbol{\delta}_A(x)$ for $\mathscr{H}^{n-1}$ a.e. $x \in S(A,r)$ and for $\mathscr{L}^1$ a.e. $r > 0$. Henceforth, it follows from [17, 3.14] and 2.8 that for $\mathscr{L}^1$ a.e. $r > 0$ the level

[c]In fact the following statement follows from the definition of tangent cone (see [7, 3.1.21]). *If $T \in \mathbf{G}(n, n-1)$, $\alpha \in T$, $f : T \to T^\perp$ is continuous at α, $a = \alpha + f(\alpha)$, $A = \{\chi + f(\chi) : \chi \in T\}$ and $\mathrm{Tan}(A,a) \subseteq T$ then f is differentiable at α with $\mathrm{D}\, f(\alpha) = 0$.*

set $S(A,r)$ is pointwise differentiable of order 1 at $\mathscr{H}^{n-1}$ a.e. $x \in S(A,r)$ with

$$\operatorname{Tan}(S(A,r),x) = \{v : v \bullet \boldsymbol{\nu}_A(x) = 0\}.$$

Noting [20, 2.16], we can argue as in the first paragraph of [20, 3.12] to infer that for all $\mathscr{L}^1$ a.e. $r > 0$ and for $\mathscr{H}^{n-1}$ a.e. $x \in S(A,r)$ there exists $s > 0$ such that

$$\mathbf{U}(x + s\boldsymbol{\nu}_A(x), s) \cap S(A,r) = \varnothing;$$

therefore, since it is obvious that for every $x \in S(A,r)$ there exists $a \in A$ such that $|x-a| = r$ and $\mathbf{U}(a,r) \cap S(A,r) = \varnothing$, it follows that

$$\limsup_{t \to 0} t^{-2} \sup\{\boldsymbol{\delta}_{\operatorname{Tan}(S(A,r),x)}(z-x) : z \in \mathbf{U}(x,t) \cap S(A,r)\} < \infty$$

for $\mathscr{H}^{n-1}$ a.e. $x \in S(A,r)$ and for $\mathscr{L}^1$ a.e. $r > 0$. It follows that $S(A,r)$ is pointwise differentiable of order $(1,1)$ at $\mathscr{H}^{n-1}$ a.e. $x \in S(A,r)$ and for $\mathscr{L}^1$ a.e. $r > 0$ (see [17, 3.3]) and we employ [17, 5.7(3)] to conclude that $S(A,r)$ is pointwise differentiable of order 2 at $\mathscr{H}^{n-1}$ a.e. $x \in S(A,r)$ and for $\mathscr{L}^1$ a.e. $r > 0$. Now, the conclusion can be easily deduced with the help of 2.8, [17, 3.14, footnote of 3.12] and [20, 3.12]. □

3. Area Formula for the Spherical Image

We introduce now the key concept of Lusin (N) condition for the generalized unit normal bundle.

Definition 3.1. Suppose $A \subseteq \mathbf{R}^n$ is a closed set, $\Omega \subseteq \mathbf{R}^n$ is an open set and $1 \leq m < n$ is an integer. We say that $N(A)$ satisfies the *m-dimensional Lusin (N) condition in* Ω if and only if (see 2.4–2.5)

$$\mathscr{H}^{n-1}(N(A)|S) = 0 \quad \text{whenever } S \subseteq A \cap \Omega \text{ such that } \mathscr{H}^m(A^{(m)} \cap S) = 0.$$

Remark 3.2. If $N(A)$ satisfies the m-dimensional Lusin (N) condition in Ω then it follows from [20, 6.1; 18, 4.12] that

$$\dim T_A(a,u) = m \quad \text{for } \mathscr{H}^{n-1} \text{ a.e. } (a,u) \in N(A)|\Omega.$$

The following coarea-type formula is a consequence of the Lusin (N) condition.

Theorem 3.3. *Suppose $1 \leq m < n$ is an integer, $\Omega \subseteq \mathbf{R}^n$ is open, $A \subseteq \mathbf{R}^n$ is closed and $N(A)$ satisfies the m-dimensional Lusin (N) condition in Ω.*

Then for every $\mathscr{H}^{n-1}$ *measurable set* $B \subseteq N(A) \,|\, \Omega$,

$$\int_{\mathbf{S}^{n-1}} \mathscr{H}^0\{a : (a,u) \in B\} d\mathscr{H}^{n-1} u = \int_A \int_{B(z)} |\operatorname{discr} Q_A| d\mathscr{H}^{n-m-1} d\mathscr{H}^m z.$$

Proof. It follows from 3.2 that for $\mathscr{H}^{n-1}$ a.e. $(a,u) \in N(A)|\Omega$,

$$\kappa_{A,m+1}(a,u) = \infty \quad \text{and} \quad \operatorname{discr} Q_A(a,u) = \prod_{i=1}^{m} \kappa_{A,i}(a,u).$$

Therefore, we use [20, 4.11(3), 5.4] to compute

$$\begin{aligned}
&\int_{\mathbf{S}^{n-1}} \mathscr{H}^0\{a : (a,v) \in B\} d\mathscr{H}^{n-1} v \\
&\quad = \int_B \prod_{i=1}^{n-1} |\kappa_{A,i}(a,u)|(1+\kappa_{A,i}(a,u)^2)^{-1/2} d\mathscr{H}^{n-1}(a,u) \\
&\quad = \int_{B|A^{(m)}} |\operatorname{discr} Q_A(a,u)| \prod_{i=1}^{m} (1+\kappa_{A,i}(a,u)^2)^{-1/2} d\mathscr{H}^{n-1}(a,u) \\
&\quad = \int_{A^{(m)}} \int_{B(z)} |\operatorname{discr} Q_A| d\mathscr{H}^{n-m-1} d\mathscr{H}^m z,
\end{aligned}$$

whenever $B \subseteq N(A)|\Omega$ is $\mathscr{H}^{n-1}$ measurable. □

We point out a simple and useful generalization of the barrier principle in [31, 7.1].

Lemma 3.4. *Suppose* $1 \le m < n$ *are integers,* $T \in \mathbf{G}(n, n-1)$, $\eta \in T^\perp$, $f : T \to T^\perp$ *is pointwise differentiable of order* 2 *at* 0 *such that* $f(0) = 0$ *and* $\operatorname{D} f(0) = 0$, $h \ge 0$, Ω *is an open subset of* $\mathbf{R}^n$ *and* Γ *is an* (m,h) *subset of* Ω *such that* $0 \in \Gamma$ *and*

$$\Gamma \cap V \subseteq \{z : z \bullet \eta \le f(T_\natural(z)) \bullet \eta\}$$

for some open neighborhood V *of* 0. *Then, denoting by* $\chi_1 \ge \cdots \ge \chi_{n-1}$ *the eigenvalues of* $\operatorname{D}^2 f(0) \bullet \eta$, *it follows that*

$$\chi_1 + \cdots + \chi_m \ge -h.$$

Proof. Fix $\epsilon > 0$. We define

$$\begin{aligned}
\psi(\chi) &= \left(\frac{1}{2}\operatorname{D}^2 f(0)(\chi,\chi) \bullet \eta + \epsilon|\chi|^2\right)\eta \quad \text{for } \chi \in T, \\
M &= \{\chi + \psi(\chi) : \chi \in T\},
\end{aligned}$$

and we select $r > 0$ such that $f(\chi) \bullet \eta \le \psi(\chi) \bullet \eta$ for $\chi \in \mathbf{U}(0,r) \cap T$. By [31, 7.1], if $\kappa_1 \le \cdots \le \kappa_{n-1}$ are the principal curvatures at 0 of M with respect to the unit normal that points into $\{z : z \bullet \eta \leq \psi(T_\natural(z)) \bullet \eta\}$, then

$$\kappa_1 + \cdots + \kappa_m \leq h.$$

Since a standard and straightforward computation shows that $\kappa_i = -\chi_i - \epsilon$ for $i = 1, \ldots, n-1$, we obtain the conclusion letting $\epsilon \to 0$. □

Finally, the following immediate consequence of Federer's Coarea formula is needed.

Lemma 3.5. *Suppose $0 \leq \mu \leq m$ are integers, W is a $(\mathscr{H}^m, m)$ rectifiable and $\mathscr{H}^m$ measurable subset of $\mathbf{R}^n$, $S \subseteq \mathbf{R}^\nu$ is a countable union of sets with finite $\mathscr{H}^\mu$ measure and $f : W \to \mathbf{R}^\nu$ is a Lipschitzian map such that*

$$\mathscr{H}^m(W \cap \{w : \|\textstyle\bigwedge_\mu((\mathscr{H}^m \llcorner W, m)\mathrm{ap}\, \mathrm{D}\, f(w))\| = 0\}) = 0,$$

$$\mathscr{H}^\mu(S \cap \{z : \mathscr{H}^{m-\mu}(f^{-1}\{z\}) > 0\}) = 0.$$

Then $\mathscr{H}^m(f^{-1}[S]) = 0$.

Proof. First, we reduce the problem to the case $\mathscr{H}^\mu(S) < \infty$; then, by [7, 2.1.4, 2.10.26], to the case of a Borel subset S of $\mathbf{R}^\nu$. Now, the conclusion comes from the coarea formula in [8, p. 300]. □

Remark 3.6. If $m = \mu$ then the result is true even if we omit to assume that S is a countable union of sets with finite $\mathscr{H}^\mu$ measure, as one may check noting that $\{z : \mathscr{H}^0(f^{-1}\{z\}) > 0\} = f[W]$ and applying 3.5 with S replaced by $S \cap f[W]$.

In the proof of the next result it is convenient to introduce the following Borel sets (see [20, 3.8]).

Definition 3.7. If $A \subseteq \mathbf{R}^n$ is closed and $\lambda \geq 1$ we define (see 2.3)

$$A_\lambda = \{x : \rho(A, x) \geq \lambda\}.$$

We are now in the position to prove the main result of this section.

Theorem 3.8. *Suppose $1 \leq m \leq n-1$, Ω is an open subset of $\mathbf{R}^n$, $0 \leq h < \infty$, Γ is an (m, h) subset of Ω that is a countable union of sets with finite $\mathscr{H}^m$ measure and $A = \overline{\Gamma}$.*

Then $N(A)$ satisfies the m dimensional Lusin (N) condition in Ω; moreover

$$\dim T_A(z, \eta) = m \quad \text{and} \quad \operatorname{trace} Q_A(z, \eta) \leq h$$

for $\mathscr{H}^{n-1}$ a.e. $(z, \eta) \in N(A)|\Omega$.

Proof. We divide the proof in several claims. Fix $\tau > 2m$.

Claim 1. *If* $0 < r < \frac{m}{3(2m-1)h}$ *and* $x \in S(A,r) \cap R(A) \cap A_\tau \cap \boldsymbol{\xi}_A^{-1}(\Gamma)$ (*see* 2.3) *are such that* $\boldsymbol{\Theta}^{n-1}(\mathscr{H}^{n-1} \llcorner S(A,r) \sim A_\tau, x) = 0$ *and the conclusion of* 2.9 *holds, then*

$$\sum_{i=1}^{m} \chi_{A,i}(x) \le h \quad \textit{and} \quad \|{\textstyle\bigwedge_m}((\mathscr{H}^{n-1} \llcorner S(A,r), n-1)\operatorname{ap} \mathrm{D}\, \boldsymbol{\xi}_A(x))\| > 0.$$

Noting that $\boldsymbol{\xi}_A | A_{2m}$ is approximately differentiable at x, we employ [20, 3.10(3)(6)] and [7, 3.2.16] to conclude that

$$\chi_{A,j}(x) \ge -(2m-1)^{-1} r^{-1} \quad \text{for } i = 1, \dots, n-1, \tag{3}$$

$$\operatorname{ap} \mathrm{D}\, \boldsymbol{\xi}_A(x) | \operatorname{Tan}^{n-1}(\mathscr{H}^{n-1} \llcorner S(A,r), x) = (\mathscr{H}^{n-1} \llcorner S(A,r), n-1)\operatorname{ap} \mathrm{D}\, \boldsymbol{\xi}_A(x). \tag{4}$$

We assume $\boldsymbol{\xi}_A(x) = 0$ and we notice that $T_\natural(x) = 0$ and $\boldsymbol{\nu}_A(x) = r^{-1}x$. We choose f, V and T as in 2.9 and $0 < s < r/2$ such that $\mathbf{U}(x,s) \subseteq V$. Then we define $g(\zeta) = f(\zeta) - x$ for $\zeta \in T$,

$$U = T_\natural(\mathbf{U}(x,s) \cap \{\chi + f(\chi) : \chi \in T\}), \quad W = \{y - x : y \in T_\natural^{-1}(U) \cap \mathbf{U}(x,s)\}.$$

It follows that W is an open neighborhood of 0 and

$$W \cap A \subseteq \{z : z \bullet \boldsymbol{\nu}_A(x) \le g(T_\natural(z)) \bullet \boldsymbol{\nu}_A(x)\}. \tag{5}$$

If (5) did not hold then there would be $y \in \mathbf{U}(x,s) \cap T_\natural^{-1}[U]$ such that $y - x \in A$ and $y \bullet \boldsymbol{\nu}_A(x) > f(T_\natural(y)) \bullet \boldsymbol{\nu}_A(x)$; noting that

$$T_\natural(y) + f(T_\natural(y)) \in \mathbf{U}(x,s) \cap S(A,r), \quad |T_\natural(y) + f(T_\natural(y)) - y| < r,$$

we would conclude

$$|T_\natural(y) + f(T_\natural(y)) - (y - x)| = r - (y - f(T_\natural(y))) \bullet \boldsymbol{\nu}_A(x) < r$$

which is a contradiction. Since $-\chi_{A,1}(x), \dots, -\chi_{A,n-1}(x)$ are the eigenvalues of $\mathrm{D}^2 g(0) \bullet \boldsymbol{\nu}_A(x)$, we may apply 3.4 to infer that

$$\chi_{A,1}(x) + \cdots + \chi_{A,m}(x) \le h \tag{6}$$

and combining (3) and (6) it follows that

$$\chi_{A,j}(x) \le \frac{4m-3}{6m-3} r^{-1} < r^{-1} \quad \text{for } j = 1, \dots, m.$$

Since it follows by (4) and [20, 3.5] that $1 - r\chi_{A,i}(x)$ are the eigenvalues of $(\mathscr{H}^{n-1} \llcorner S(A,r), n-1)\operatorname{ap} \mathrm{D}\, \boldsymbol{\xi}_A(x)$ for $i = 1, \dots, n-1$, we get that

$$\|{\textstyle\bigwedge_m}((\mathscr{H}^{n-1} \llcorner S(A,r), n-1)\operatorname{ap} \mathrm{D}\, \boldsymbol{\xi}_A(x))\| \ge \prod_{i=1}^{m} (1 - \chi_{A,i}(x) r) > 0.$$

Claim 2. *For* $\mathscr{H}^{n-1}$ *a.e.* $x \in S(A,r) \cap A_\tau \cap \boldsymbol{\xi}_A^{-1}(\Gamma)$ *and for* $\mathscr{L}^1$ *a.e.* $0 < r < \frac{m}{3(2m-1)h}$ *the following inequalities hold*:

$$\sum_{i=1}^{m} \chi_{A,i}(x) \le h \quad \textit{and} \quad \|{\textstyle\bigwedge_m}((\mathscr{H}^{n-1} \llcorner S(A,r), n-1)\operatorname{ap} \mathrm{D}\, \boldsymbol{\xi}_A(x))\| > 0.$$

Notice that

$$\boldsymbol{\Theta}^{n-1}(\mathscr{H}^{n-1} \llcorner S(A,r) \sim A_\tau, x) = 0$$

for $\mathscr{H}^{n-1}$ a.e. $x \in S(A,r) \cap A_\tau$ and for every $r > 0$ by [20, 2.13(1)] and [7, 2.10.19(4)], and $\mathscr{H}^{n-1}(S(A,r) \sim R(A)) = 0$ for $\mathscr{L}^1$ a.e. $r > 0$ by [20, 3.15]. Then Claim 2 follows from 2.9 and Claim 1.

Claim 3. *$N(A)$ satisfies the m-dimensional Lusin (N) condition in Ω.*

Let $S \subseteq \Gamma$ such that $\mathscr{H}^m(S \cap A^{(m)}) = 0$. For $r > 0$ it follows from *[20, 3.16, 3.17(1), 4.3]* that $\boldsymbol{\psi}_A | A_\tau \cap S(A,r)$ is a bi-Lipschitzian homeomorphism and

$$\boldsymbol{\psi}_A(\boldsymbol{\xi}_A^{-1}(x) \cap A_\tau \cap S(A,r)) \subseteq N(A,x) \quad \text{for } x \in A;$$

then we apply [20, 5.2] to get

$$A \cap \{x : \mathscr{H}^{n-m-1}(\boldsymbol{\xi}_A^{-1}\{x\} \cap A_\tau \cap S(A,r)) > 0\} \subseteq \bigcup_{i=0}^{m} A^{(i)} \quad \text{for every } r > 0.$$

Since $\mathscr{H}^m(A^{(i)}) = 0$ for $i = 0, \ldots, m-1$ (see 2.5), it follows

$$\mathscr{H}^m\big(S \cap \{x : \mathscr{H}^{n-m-1}(\boldsymbol{\xi}_A^{-1}\{x\} \cap A_\tau \cap S(A,r)) > 0\}\big) = 0 \quad \text{for every } r > 0.$$

Noting Claim 2 and [20, 3.10(1)], we can apply 3.5 with W and f replaced by $S(A,r) \cap A_\tau \cap \boldsymbol{\xi}_A^{-1}(\Gamma)$ and $\boldsymbol{\xi}_A | S(A,r) \cap A_\tau \cap \boldsymbol{\xi}_A^{-1}(\Gamma)$ to infer that

$$\mathscr{H}^{n-1}(\boldsymbol{\xi}_A^{-1}(S) \cap S(A,r) \cap A_\tau) = 0 \quad \text{for } \mathscr{L}^1 \text{ a.e. } 0 < r < \frac{m}{3(2m-1)} h^{-1}.$$

We notice that $N(A)|S = \bigcup_{r>0} \boldsymbol{\psi}_A(S(A,r) \cap A_\tau \cap \boldsymbol{\xi}_A^{-1}(S))$ by [20, 4.3] and $\boldsymbol{\psi}_A(S(A,r) \cap A_\tau) \subseteq \boldsymbol{\psi}_A(S(A,s) \cap A_\tau)$ if $s < r$ by [20, 3.17(2)]. Henceforth, it follows that

$$\mathscr{H}^{n-1}(N(A) \,|\, S) = 0.$$

Claim 4. *For $\mathscr{H}^{n-1}$ a.e. $(z,\eta) \in N(A) \,|\, \Omega$,*

$$\dim T_A(z,\eta) = m \quad \textit{and} \quad \operatorname{trace} Q_A(z,\eta) \le h.$$

By Claim 3, Remark 3.2, Claim 2 and Eq. (2) in 2.4, it follows that

$$\dim T_A(z,\eta) = m \quad \text{for } \mathscr{H}^{n-1} \text{ a.e. } (z,\eta) \in N(A) \,|\, \Omega,$$

$$\sum_{l=1}^{m} \frac{\kappa_{A,l}(\boldsymbol{\psi}_A(x))}{1 + r\kappa_{A,l}(\boldsymbol{\psi}_A(x))} \le h \tag{7}$$

for $\mathscr{H}^{n-1}$ a.e. $x \in S(A,r) \cap A_\tau \cap \boldsymbol{\xi}_A^{-1}(\Gamma)$ and for $\mathscr{L}^1$ a.e. $0 < r < \frac{m}{3(2m-1)} h^{-1}$. We choose a positive sequence $r_i \to 0$ such that if M_i is the set of points $x \in$

$S(A, r_i) \cap A_\tau \cap \boldsymbol{\xi}_A^{-1}(\Gamma)$ satisfying (7) with r replaced by r_i, then

$$\mathscr{H}^{n-1}((S(A, r_i) \cap A_\tau \cap \boldsymbol{\xi}_A^{-1}(\Gamma)) \sim M_i) = 0 \quad \text{for every } i \geq 1.$$

It follows that

$$\operatorname{trace} Q_A(z, \eta) \leq h \quad \text{if } (z, \eta) \in \bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} \boldsymbol{\psi}_A(M_j)$$

and the inclusion

$$(N(A) \,|\, \Omega) \sim \bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} \boldsymbol{\psi}_A(M_j) \subseteq \bigcup_{i=1}^{\infty} (\boldsymbol{\psi}_A(S(A, r_i) \cap A_\tau \cap \boldsymbol{\xi}_A^{-1}(\Gamma)) \sim \boldsymbol{\psi}_A(M_i))$$

readily implies

$$\mathscr{H}^{n-1}\left((N(A)|\Omega) \sim \bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} \boldsymbol{\psi}_A(M_j)\right) = 0. \qquad \square$$

Remark 3.9. We assume in 3.8 that Γ is a countable union of sets of finite $\mathscr{H}^m$ measure only because this hypothesis ensures the applicability of 3.5 in the proof of Claim 3. Consequently, in view of 3.6, we have that if $m = n - 1$ the result is still true even if we omit the aforementioned hypothesis.

Corollary 3.10. *Suppose* $1 \leq m \leq n-1$, Ω *is an open subset of* $\mathbf{R}^n$, $0 \leq h < \infty$, Γ *is an* (m, h) *subset of* Ω *such that* $\mathscr{H}^m(\Gamma \cap K) < \infty$ *for every compact set* $K \subseteq \Omega$, $A = \overline{\Gamma}$ *and* $S = A^{(m)} \cap \Omega$. *Then*

$$\operatorname{ap}\operatorname{Tan}(S, z) = T_A(z, \eta) \in \mathbf{G}(n, m) \quad \textit{and} \quad \operatorname{ap}\mathbf{b}_S(z) \bullet \eta = -Q_A(z, \eta)$$

for $\mathscr{H}^{n-1}$ *a.e.* $(z, \eta) \in N(A)|\Omega$.

Proof. If $K \subseteq \Omega$ is compact then $S \cap K$ is $(\mathscr{H}^m, m)$ rectifiable of class 2 by [18, 4.12]. Henceforth the conclusion follows from 3.8(1) and 2.6. $\square$

Remark 3.11. Suppose $V \in \mathbf{V}_m(\Omega)$ is an *integral* varifold such that $\|\delta V\| \leq c\|V\|$ for some $0 \leq c < \infty$, $A = \overline{\operatorname{spt}\|V\|}$ and $S = A^{(m)} \cap \Omega$. Since $S \cap K$ is $(\mathscr{H}^m, m)$ rectifiable of class 2 by [18, 4.12] whenever $K \subseteq \Omega$ is compact, we use the locality theorem [26, 4.2] to conclude that

$$\operatorname{ap}\mathbf{h}_S(z) = -\mathbf{h}(V, z)$$

for $\mathscr{H}^m$ a.e. $z \in S$. It follows from 3.8(1) and 3.10 that

$$\operatorname{trace} Q_A(z, \eta) = \mathbf{h}(V, z) \bullet \eta \quad \text{for } \mathscr{H}^{n-1} \text{ a.e. } (z, \eta) \in N(A)|\Omega.$$

Here, we consider only *integral* varifolds because the locality theorem [26, 4.2] is not currently available for non-integral ones.

4. Almgren's Sharp Geometric Inequality

The following lemma will be useful in the proof of the rigidity theorem.

Lemma 4.1. *Let $1 \leq m \leq n$ be integers and let B be an m-dimensional submanifold of class 1 in $\mathbf{R}^n$. If $0 < \lambda < 1$ and $\varphi : B \to \mathbf{R}^n$ is a Lipschitzian map such that*

$$\|\mathrm{D}(\varphi - \mathbf{1}_B)(b)\| \leq \lambda \quad \text{for } \mathscr{H}^m \text{ a.e. } b \in B,$$

then for each $b \in B$ there exists an open neighborhood V of b such that $\varphi \mid V \cap B$ is a bi-Lipschitzian homeomorphism.

Proof. First, we prove the following claim. *If U is an open convex subset of $\mathbf{R}^m$, $0 \leq M < \infty$ and $g : U \to \mathbf{R}^n$ is a Lipschitzian map such that $\|\mathrm{D}\, g(x)\| \leq M$ for $\mathscr{L}^m$ a.e. $x \in U$, then* $\operatorname{Lip} g \leq M$. In fact, if $a \in U$ and $r > 0$ such that $\mathbf{U}(a, r) \subseteq U$ then Coarea formula [7, 3.2.22(3)] and the fundamental theorem of calculus [7, 2.9.20(1)] imply that for $\mathscr{H}^{n-1}$ a.e. $v \in \mathbf{S}^{n-1}$,

$$|g(a + tv) - g(a)| \leq Mt \quad \text{for } 0 < t < r;$$

since g is continuous,

$$\limsup_{x \to a} \frac{|g(x) - g(a)|}{|x - a|} \leq M \quad \text{for } a \in U$$

and $\operatorname{Lip} g \leq M$ by [7, 2.2.7].

Now, we fix $b \in B$ and $\sqrt{\lambda} < t < 1$, we define $A = \operatorname{Tan}(B, b) + b$ and we select $r > 0$ and diffeomorphism $f : \mathbf{R}^n \to \mathbf{R}^n$ of class 1 as in [7, 3.1.23]. In particular, we have that $f(\mathbf{U}(b, tr)) \cap A = f(\mathbf{U}(b, tr) \cap B)$ and

$$\|\mathrm{D}((\varphi - \mathbf{1}_B) \circ f^{-1})(x)\| \leq \lambda t^{-1} \quad \text{for } \mathscr{H}^m \text{ a.e. } x \in f(\mathbf{U}(b, tr) \cap B).$$

It follows that if U is a convex subset of $f(\mathbf{U}(b, tr) \cap B)$ such that $f(b) \in U$ and U is relatively open in A then

$$\operatorname{Lip}[((\varphi - \mathbf{1}_B) \circ f^{-1}) | U] \leq \lambda t^{-1}. \tag{8}$$

Therefore, one uses (8) and $\operatorname{Lip} f \leq t^{-1}$ to conclude

$$|\varphi(c) - \varphi(d)| \geq |c - d|(1 - \lambda t^{-2}) \quad \text{for } c, d \in f^{-1}(U).$$ □

Theorem 4.2. *If $1 \leq m \leq n - 1$, $h > 0$ and Γ is a nonempty compact (m, h) subset of $\mathbf{R}^n$ then*

$$\mathscr{H}^m(\Gamma) \geq \left(\frac{m}{h}\right)^m \mathscr{H}^m(\mathbf{S}^m).$$

Moreover, if the equality holds and $\Gamma = \operatorname{spt}(\mathscr{H}^m \llcorner \Gamma)$ then there exists an $(m+1)$-dimensional plane T and $a \in \mathbf{R}^n$ such that

$$\Gamma = \partial \mathbf{B}(a, m/h) \cap T.$$

Proof. We assume $\mathscr{H}^m(\Gamma) < \infty$. Since $\lambda\Gamma$ is an $(m, h/\lambda)$ set whenever $\lambda > 0$, we reduce the proof to the case $h = m$.

We define

$$C = (\Gamma \times \mathbf{R}^n) \cap \{(a, \eta) : \eta \bullet (w - a) \geq 0 \text{ for every } w \in \Gamma\}$$

and we notice that C is a closed subset of $\Gamma \times \mathbf{R}^n$, $C(a)$ is a closed convex cone[d] containing 0 for every $a \in \Gamma$ and, since Γ is compact, for every $\eta \in \mathbf{S}^{n-1}$ there exists $a \in \Gamma$ such that $\inf_{w \in \Gamma}(w \bullet \eta) = (a \bullet \eta)$; in other words,

$$\mathbf{q}(C \cap (\Gamma \times \mathbf{S}^{n-1})) = \mathbf{S}^{n-1}.$$

Moreover, we let $B = \{(a, -\eta) : (a, \eta) \in C, \ |\eta| = 1\}$ and we notice that

$$B \subseteq N(\Gamma) \quad \text{and} \quad \mathbf{q}[B] = \mathbf{S}^{n-1}. \tag{9}$$

We define $X \subseteq \Gamma^{(m)}$ as the set of $a \in \Gamma^{(m)}$ such that the following conditions are satisfied:

(i) $\Gamma^{(m)}$ is approximately differentiable of order 2 at a with

$$\operatorname{ap}\operatorname{Tan}(\Gamma^{(m)}, a) \in \mathbf{G}(n, m),$$

(ii) $\operatorname{ap}\mathbf{b}_{\Gamma^{(m)}}(a) \bullet \eta = -Q_\Gamma(a, \eta)$ for $\mathscr{H}^{n-m-1}$ a.e. $\eta \in N(\Gamma, a)$,

(iii) $\operatorname{trace} Q_\Gamma(a, \eta) \leq m$ for $\mathscr{H}^{n-m-1}$ a.e. $\eta \in N(\Gamma, a)$.

Since $\Gamma^{(m)}$ is $(\mathscr{H}^m, m)$ rectifiable of class 2 by [18, 4.12], it follows from [21, 3.23] that condition (i) is satisfied $\mathscr{H}^m$ almost everywhere on $\Gamma^{(m)}$. Moreover, noting 3.8 and 3.10, we infer that

$$\operatorname{ap}\mathbf{b}_{\Gamma^{(m)}}(a) \bullet \eta = -Q_\Gamma(a, \eta) \quad \text{and} \quad \operatorname{trace} Q_\Gamma(a, \eta) \leq m \tag{10}$$

for $\mathscr{H}^{n-1}$ a.e. $(a, \eta) \in N(\Gamma)$ and we apply [7, 2.10.25], with X and f replaced by $N(\Gamma)|\Gamma^{(m)}$ and $\mathbf{p}$, to conclude that (10) holds for $\mathscr{H}^m$ a.e. $a \in \Gamma^{(m)}$ and for $\mathscr{H}^{n-m-1}$ a.e. $\eta \in N(\Gamma, a)$. Henceforth,

$$\mathscr{H}^m(\Gamma^{(m)} \sim X) = 0. \tag{11}$$

Furthermore, we notice that if $a \in X$ then

$$\operatorname{ap}\mathbf{h}_{\Gamma^{(m)}}(a) \bullet \eta \geq -m \quad \text{for } \mathscr{H}^{n-m-1} \text{ a.e. } \eta \in N(\Gamma, a),$$

whence we readily infer from 2.5 that

$$\operatorname{ap}\mathbf{h}_{\Gamma^{(m)}}(a) \bullet \eta \geq -m \quad \text{for all } \eta \in N(\Gamma, a). \tag{12}$$

If $a \in X$ we define

$$g(a) = \boldsymbol{\xi}_{C(a)}(\operatorname{ap}\mathbf{h}_{\Gamma^{(m)}}(a))$$

(notice $C(a) \subseteq \operatorname{ap}\operatorname{Nor}(\Gamma^{(m)}, a) \in \mathbf{G}(n, n-m)$ and $\operatorname{ap}\mathbf{h}_{\Gamma^{(m)}}(a) \in \operatorname{ap}\operatorname{Nor}(\Gamma^{(m)}, a)$), we infer from [18, 3.9(3)] that $\operatorname{ap}\mathbf{h}_{\Gamma^{(m)}}(a) - g(a) \in \operatorname{Nor}(C(a), g(a))$ and

$$(\operatorname{ap}\mathbf{h}_{\Gamma^{(m)}}(a) - g(a)) \bullet (\eta - g(a)) \leq 0 \quad \text{for every } \eta \in C(a)$$

[d] A subset C of $\mathbf{R}^n$ is a cone if and only if $\lambda c \in C$ whenever $0 < \lambda < \infty$ and $c \in C$.

and, noting that $0 \in C(a)$ and $2g(a) \in C(a)$, we conclude that

$$(\mathrm{ap}\,\mathbf{h}_{\Gamma^{(m)}}(a) - g(a)) \bullet g(a) = 0,$$

$$(\mathrm{ap}\,\mathbf{h}_{\Gamma^{(m)}}(a) - g(a)) \bullet \eta \leq 0 \quad \text{for every } \eta \in C(a).$$

Since $-g(a)/|g(a)| \in N(\Gamma, a)$ when $g(a) \neq 0$ by (9), we obtain from (12) that

$$|g(a)| = \mathrm{ap}\,\mathbf{h}_{\Gamma^{(m)}}(a) \bullet (g(a)/|g(a)|) \leq m. \tag{13}$$

Moreover, it follows from [21, 4.12(3)],

$$\mathrm{ap}\,\mathbf{b}_{\Gamma^{(m)}}(a) \bullet \eta \geq 0 \quad \text{for } a \in X \quad \text{and} \quad \eta \in C(a),$$

whence we deduce

$$g(a) \bullet \eta \geq \mathrm{ap}\,\mathbf{h}_{\Gamma^{(m)}}(a) \bullet \eta \geq 0 \quad \text{for } a \in X \quad \text{and} \quad \eta \in C(a), \tag{14}$$

and, employing the classical inequality relating the arithmetic and geometric means of a family of non-negative numbers,[e]

$$0 \leq \mathrm{discr}(\mathrm{ap}\,\mathbf{b}_{\Gamma^{(m)}}(a) \bullet \eta) \leq m^{-m}(\mathrm{ap}\,\mathbf{h}_{\Gamma^{(m)}}(a) \bullet \eta)^m \leq m^{-m}(g(a) \bullet \eta)^m \tag{15}$$

for every $a \in X$ and $\eta \in C(a)$.

If $T \in \mathbf{G}(n, m)$ and $v \in T^\perp$ we define

$$D(T, v) = T^\perp \cap \{u : u \bullet v \geq 0\}.$$

We readily infer that there exists $0 < \gamma(n, m) < \infty$ such that

$$\gamma(n, m)|v|^m = \int_{D(T,v)\cap \mathbf{S}^{n-1}} (\eta \bullet v)^m d\mathscr{H}^{n-m-1}\eta$$

for every $T \in \mathbf{G}(n, m)$ and $v \in T^\perp$. It follows from (14),

$$C(a) \subseteq D(\mathrm{ap}\,\mathrm{Tan}(\Gamma^{(m)}, a), g(a)) \quad \text{for every } a \in X; \tag{16}$$

[e] If $a_1, \dots, a_m$ are non-negative real numbers,

$$a_1 a_2 \dots a_m \leq \left(\frac{a_1 + a_2 + \cdots + a_m}{m}\right)^m$$

with equality only if $a_1 = a_2 = \cdots = a_m$.

noting (9), (11), (15) and (13), we apply Theorem 3.3 to estimate

$$
\begin{aligned}
&\mathscr{H}^{n-1}(\mathbf{S}^{n-1})\\
&\quad\overset{\text{(I)}}{\le} \int_{\mathbf{S}^{n-1}} \mathscr{H}^0\{a : (a,\eta) \in B\} d\mathscr{H}^{n-1}\eta\\
&\quad= \int_\Gamma \int_{B(a)} |\operatorname{discr} Q_\Gamma(a,\eta)| d\mathscr{H}^{n-m-1}\eta \, d\mathscr{H}^m a\\
&\quad= \int_{\Gamma^{(m)}} \int_{C(a)\cap \mathbf{S}^{n-1}} \operatorname{discr}(\operatorname{ap} \mathbf{b}_{\Gamma^{(m)}}(a) \bullet \eta) d\mathscr{H}^{n-m-1}\eta \, d\mathscr{H}^m a\\
&\quad\overset{\text{(II)}}{\le} m^{-m} \int_{\Gamma^{(m)}} \int_{C(a)\cap \mathbf{S}^{n-1}} (g(a)\bullet\eta)^m d\mathscr{H}^{n-m-1}\eta \, d\mathscr{H}^m a\\
&\quad\overset{\text{(III)}}{\le} m^{-m} \int_{\Gamma^{(m)}} \int_{D(\operatorname{ap}\operatorname{Tan}(\Gamma^{(m)},a),g(a))\cap \mathbf{S}^{n-1}} (g(a)\bullet\eta)^m d\mathscr{H}^{n-m-1}\eta \, d\mathscr{H}^m a\\
&\quad= m^{-m}\gamma(n,m) \int_{\Gamma^{(m)}} |g(a)|^m d\mathscr{H}^m a\\
&\quad\overset{\text{(IV)}}{\le} \gamma(n,m)\mathscr{H}^m(\Gamma^{(m)})\\
&\quad\overset{\text{(V)}}{\le} \gamma(n,m)\mathscr{H}^m(\Gamma).
\end{aligned}
$$

Suppose $T \in \mathbf{G}(n, m+1)$ and $\Sigma = T \cap \mathbf{S}^{n-1}$. We observe that if $a \in \Sigma$ then

$$
\begin{gathered}
D(\operatorname{Tan}(\Sigma,a),-a) = \mathbf{R}^n \cap \{\eta : \eta \bullet (w-a) \ge 0 \text{ for every } w \in \Sigma\},\\
\mathbf{b}_\Sigma(a)(u,v) = -a(u\bullet v) \quad \text{for } u,v \in \operatorname{Tan}(\Sigma,a), \quad \mathbf{h}_\Sigma(a) = -ma,
\end{gathered}
$$

where $\mathbf{b}_\Sigma$ and $\mathbf{h}_\Sigma$ are the second fundamental form and mean curvature of Σ; moreover $\Sigma = \Sigma^{(m)}$ and

$$
\mathscr{H}^0\{a : \eta \in D(\operatorname{Tan}(\Sigma,a),-a)\} = 1 \quad \text{for every } \eta \in \mathbf{S}^{n-1} \sim T^\perp.
$$

Then we infer that inequalities (I)–(V) in the previous estimate are actually equalities when $\Gamma = \Sigma$, and we conclude that

$$
\mathscr{H}^{n-1}(\mathbf{S}^{n-1}) = \gamma(n,m)\mathscr{H}^m(\Sigma) = \gamma(n,m)\mathscr{H}^m(\mathbf{S}^m).
$$

From this equation we finally obtain that

$$
\mathscr{H}^m(\Gamma) \ge \mathscr{H}^m(\mathbf{S}^m)
$$

and the proof of the first part of the theorem is concluded.

We now assume $\mathscr{H}^m(\Gamma) = \mathscr{H}^m(\mathbf{S}^m)$ and $\operatorname{spt}(\mathscr{H}^m \llcorner \Gamma) = \Gamma$.

First, noting that inequalities (I)–(V) are equalities, we infer that

$$\mathscr{H}^m(\Gamma \sim \Gamma^{(m)}) = 0 \quad \text{(by (V))}$$

and the following equalities hold for $\mathscr{H}^m$ a.e. $a \in \Gamma$,

$$|g(a)| = m \quad \text{(by (IV) and (13))}, \quad \dim \operatorname{ap} \operatorname{Tan}(\Gamma, a) = m, \tag{17}$$

$$D(\operatorname{ap} \operatorname{Tan}(\Gamma, a), g(a)) \cap \mathbf{S}^{n-1} = C(a) \cap \mathbf{S}^{n-1} \quad \text{(by (III) and (16))}, \tag{18}$$

$$g(a) = \operatorname{ap} \mathbf{h}_\Gamma(a), \quad |\operatorname{ap} \mathbf{h}_\Gamma(a)| = m, \tag{19}$$

$$\operatorname{discr}(\operatorname{ap} \mathbf{b}_\Gamma(a) \bullet \eta) = m^{-m}(\operatorname{ap} \mathbf{h}_\Gamma(a) \bullet \eta)^m \quad \text{for } \eta \in C(a) \quad \text{(by (II) and (15))},$$

$$\operatorname{ap} \mathbf{b}_\Gamma(a)(u, v) = m^{-1}(u \bullet v) \operatorname{ap} \mathbf{h}_\Gamma(a) \quad \text{for } u, v \in \operatorname{ap} \operatorname{Tan}(\Gamma, a). \tag{20}$$

Let A be the convex hull of Γ and let B be the relative boundary of A. Note that $A - a \subseteq \{v : v \bullet \eta \geq 0\}$ for every $a \in \Gamma$ and $\eta \in C(a)$. Then it follows from (17) and (18) that

$$\dim\{u + \lambda g(a) : u \in \operatorname{ap} \operatorname{Tan}(\Gamma, a),\ \lambda \geq 0\} = m + 1,$$

$$A - a \subseteq \{u + \lambda g(a) : u \in \operatorname{ap} \operatorname{Tan}(\Gamma, a),\ \lambda \geq 0\}$$

for $\mathscr{H}^m$ a.e. $a \in \Gamma$, whence we deduce that $\dim A \leq m + 1$. If $\dim A = m$ then we could apply [20, 2.9] with M replaced by the relative interior of A to infer that $\operatorname{ap} \mathbf{h}_\Gamma(x) = 0$ for $\mathscr{H}^m$ a.e. $x \in \Gamma$. Since this contradicts (19) we have proved that $\dim A = m+1$. Then we notice that $\mathscr{H}^m(\Gamma \sim B) = 0$ and, since $\Gamma = \operatorname{spt}(\mathscr{H}^m \llcorner \Gamma)$, it follows that $\Gamma \subseteq B$.

At this point it is not restrictive to assume $m = n - 1$ in the sequel.

Now, we prove that *if $x \in \Gamma$ then* $\operatorname{Tan}(\Gamma, x)$ *is the unique supporting hyperplane of A at x*. We fix $x \in \Gamma$. By [27, 1.3.2] there exists a closed half space H of $\mathbf{R}^{m+1}$ such that $0 \in \partial H$ and $A - x \subseteq H$. By [3, Theorem 1.1.7] we choose a sequence λ_i converging to $+\infty$ and a closed set Z in $\mathbf{R}^{m+1}$ such that (see [3, 1.1.1])

$$\lambda_i(\Gamma - x) \to Z \quad \text{as } i \to \infty \text{ in the sense of Kuratowski.}$$

Then we notice that $0 \in Z \subseteq H$, Z is an $(m, 0)$ subset of $\mathbf{R}^{m+1}$ by [31, 1.6, 3.2] and $\partial H \subseteq Z$ by [31, 7.3]. Henceforth, we have the following inclusions

$$\partial H \subseteq Z \subseteq \operatorname{Tan}(\Gamma, x) \subseteq \operatorname{Tan}(\partial A, x) \subseteq \operatorname{Tan}(A, x) \subseteq H$$

and one may infer from [10, 5.7] that $\operatorname{Tan}(A, x) = H$ and $\operatorname{Tan}(\Gamma, x) = \partial H$.

Next, we check that

$$\partial A = \Gamma.$$

Let $x \in \Gamma$. Then there exist an m-dimensional plane T, an open neighborhood W of x and a convex Lipschitzian function $f : U \to T^\perp$ defined on a relatively open convex subset U of T containing $T_\natural(x)$, such that

$$W \cap \partial A = \{\chi + f(\chi) : \chi \in U\}.$$

Since $\operatorname{Lip} f < \infty$ it follows that $\operatorname{Tan}(\partial A, y) \cap T^\perp = \{0\}$ for $y \in W \cap \partial A$ and, since we have proved in the previous paragraph that $\operatorname{Tan}(\Gamma, y)$ is an m-dimensional plane for every $y \in \Gamma$, we employ [7, first paragraph p. 234] to conclude

$$T = T_\natural(\operatorname{Tan}(\Gamma, y)) = \operatorname{Tan}(T_\natural(W \cap \Gamma), T_\natural(y)) \quad \text{for every } y \in W \cap \Gamma.$$

Noting that $T_\natural(W \cap \Gamma)$ is relatively closed in U, we infer[f] that $T_\natural(W \cap \Gamma) = U$ and $W \cap \partial A = W \cap \Gamma$. Since x is arbitrarily chosen in Γ, it follows that $\partial A = \Gamma$.

We combine the assertions of the previous two paragraphs with [27, 2.2.4] to conclude that ∂A *is an m-dimensional submanifold of class* 1 *in* $\mathbf{R}^{m+1}$. Moreover, it is well known that $\operatorname{dmn} \boldsymbol{\xi}_A = \mathbf{R}^{m+1}$, $\operatorname{Lip} \boldsymbol{\xi}_A \leq 1$ (see [27, 1.2]) and $\{x : \boldsymbol{\delta}_A(x) < r\}$ is an open convex set whose boundary $S(A, r)$ is an m-dimensional submanifold of class $\mathcal{C}^{1,1}$ for $r > 0$ (see [6, 4.8]). Let $0 < r < m^{-1}$ and $\xi = \boldsymbol{\xi}_A | S(A, r)$. For $\mathscr{H}^m$ a.e. $x \in S(A, r)$ we apply the barrier principle 3.4, with T, η and f replaced by $\{v : v \bullet \boldsymbol{\nu}_A(x) = 0\}$, $\boldsymbol{\nu}_A(x)$ and a concave function whose graph corresponds to $S(A, r)$ in a neighborhood of x, to infer (see 2.4) that

$$\chi_{A,i}(x) \geq 0 \quad \text{for } i = 1, \ldots, m, \quad \sum_{i=1}^{m} \chi_{A,i}(x) \leq m,$$

and we combine these inequalities to conclude that $\chi_{A,i}(x) \leq m$ for $i = 1, \ldots, m$. Therefore, $\|\mathrm{D}(\xi - \mathbf{1}_{S(A,r)})(x)\| \leq mr < 1$ for $\mathscr{H}^m$ a.e. $x \in S(A, r)$ and, noting that ξ is univalent by [6, 4.8(12)], we apply 4.1 to conclude that the function $\xi^{-1} : \partial A \to S(A, r)$ is a locally Lipschtzian map and the unit normal vector field on ∂A,

$$\eta = \boldsymbol{\nu}_A \circ \xi^{-1},$$

is locally Lipschitzian. Combining [21, 3.25] with (19) and (20), we infer for $\mathscr{H}^m$ a.e. $x \in \Gamma$ and for $u, v \in \operatorname{Tan}(\Gamma, x)$ that

$$\begin{aligned} \mathrm{D}\,\eta(x)(u) \bullet v &= -\operatorname{ap} \mathbf{b}_\Gamma(x)(u, v) \bullet \eta(x) \\ &= -m^{-1}(\operatorname{ap} \mathbf{h}_\Gamma(x) \bullet \eta(x))(u \bullet v) \\ &= u \bullet v, \end{aligned}$$

whence we conclude that $\mathrm{D}(\eta - \mathbf{1}_\Gamma)(x) = 0$ for $\mathscr{H}^m$ a.e. $x \in \Gamma$. Therefore, there exists $a \in \mathbf{R}^{m+1}$ such that

$$\eta(z) = z - a \quad \text{for every } z \in \Gamma$$

[f] *Suppose $C \subseteq U \subseteq \mathbf{R}^n$, U is open and C is relatively closed in U. If* $\operatorname{Tan}(C, x) = \mathbf{R}^n$ *for every $x \in C$ then $C = U$.* In fact, if there was $y \in U \sim C$ and if $t = \sup\{s : \mathbf{U}(y, s) \cap C = \varnothing\}$ then $t > 0$, $\mathbf{U}(y, t) \cap C = \varnothing$, $\mathbf{B}(y, t) \cap C \neq \varnothing$ and $y - x \in \operatorname{Nor}(C, x)$ for every $x \in \mathbf{B}(y, t) \cap C$. This is clearly a contradiction.

and, since $|\eta(z)| = 1$ for $z \in \Gamma$, we conclude that

$$\Gamma = \partial \mathbf{B}(a, 1).$$

□

Remark 4.3. If V is a varifold as in [2, Theorem 1] and if we additionally assume that V is *integral* then Brakke perpendicularity theorem [4, 5.8] implies that $|\mathbf{h}(V, x)| \leq m$, whence we deduce by [31, 2.8] that spt $\|V\|$ is an (m, m) subset of $\mathbf{R}^n$.

Theorem 4.2 readily provides a sufficient condition to conclude that the area-blow up set is empty for certain sequences of m-dimensional varifolds whose mean curvature is uniformly bounded outside a set that is not too large.

Corollary 4.4. *Let V_i be a sequence of m-dimensional varifolds in $\mathbf{R}^n$ whose total variation $\|\delta V_i\|$ is a Radon measure and such that the following three conditions hold for some $0 < h < \infty$:*

(1) *the generalized boundaries of V_i are uniformly bounded on compacts sets*; i.e. *if μ_i is the singular part of $\|\delta V_i\|$ with respect to $\|V_i\|$ then*
$$\limsup_{i \to \infty} \mu_i(K) < \infty \quad \textit{for every compact set } K \subseteq \mathbf{R}^n;$$
(2) *there exists a compact set Γ such that $\mathscr{H}^m(\Gamma) < (m/h)^m \mathscr{H}^m(\mathbf{S}^m)$ and*
$$\limsup_{i \to \infty} \|V_i\|(K) < \infty \quad \textit{for every compact set } K \subseteq \mathbf{R}^n \sim \Gamma;$$
(3) $\limsup_{i \to \infty} \int_K (|\mathbf{h}(V_i, z)| - h)^+ \, d\|V_i\| z < \infty$ *whenever $K \subseteq \mathbf{R}^n$ is compact, where $t^+ = \sup\{t, 0\}$ for $t \in \mathbf{R}$.*

Then $\limsup_{i \to \infty} \|V_i\|(K) < \infty$ for every compact set $K \subseteq \mathbf{R}^n$.

Proof. If $Z = \{x : \limsup_{i \to \infty} \|V_i\|(\mathbf{B}(x, r)) = \infty$ for every $r > 0\}$ then Z is an (m, h) subset of $\mathbf{R}^{n+1}$ by [31, 2.6]. Since $Z \subseteq \Gamma$ and Z is compact, it follows from 4.2 that $Z = \varnothing$. □

Here is the limit-case $h = 0$.

Corollary 4.5. *Suppose V_i is a sequence of m-dimensional varifolds in $\mathbf{R}^n$ such that*

(1) $\limsup_{i \to \infty} \|\delta V_i\|(K) < \infty$ *whenever $K \subseteq \mathbf{R}^n$ is compact,*
(2) *there exists a compact set $\Gamma \subseteq \mathbf{R}^n$ such that $\mathscr{H}^m(\Gamma) < \infty$ and*
$$\limsup_{i \to \infty} \|V_i\|(K) < \infty \quad \textit{for every compact set } K \subseteq \mathbf{R}^n \sim \Gamma.$$

Then $\limsup_{i \to \infty} \|V_i\|(K) < \infty$ for every compact set $K \subseteq \mathbf{R}^n$.

Proof. Choose $h > 0$ small so that $\mathscr{H}^m(\Gamma) < (m/h)^m \mathscr{H}^m(\mathbf{S}^m)$ and apply 4.4. □

Remark 4.6. The reader may find useful to compare 4.4 and 4.5 with [31, 1.4].

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