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Multiobjective optimization under uncertainty: A multiobjective robust (relative) regret approach

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ABSTRACT

Consider a multiobjective decision problem with uncertainty in the objective functions, given as a set of scenarios. In the single-criterion case, robust optimization methodology helps to identify solutions which remain feasible and of good quality for all possible scenarios. A well-known alternative method in the single-objective case is to compare possible decisions under uncertainty with the optimal decision with the benefit of hindsight, i.e. to minimize the (possibly scaled) regret of not having chosen the optimal decision. In this contribution, we extend the concept of regret from the single-objective case to the multiobjective setting and introduce a proper definition of multivariate (robust) (relative) regret. In contrast to the few existing ideas that mix scalarization and optimization, we clearly separate the modelling of multiobjective (robust) regret from its numerical solution. Moreover, our approach is not limited to a finite uncertainty set or interval uncertainty and furthermore, computations or at least approximations remain tractable in several important special cases. We illustrate all approaches based on a biobjective shortest path problem under uncertainty.

1. Introduction

1.1. Motivation

The task of decision making under uncertainty appears in various fields. Quite often, the considered decision problem cannot be expressed by a standard formulation as an optimization problem with a single objective. Instead, multiple conflicting criteria have to be considered and thus a formulation as an *uncertain (or parametric) multiobjective optimization problem* is required[1] In the recent past, several promising approaches have emerged which allow to generalize the idea of a *robust counterpart* of an uncertain single-objective optimization problem to the multiobjective setup, see for instance the summary (Ide & Schöbel, 2016) or the extensive overview (Goberna, Jeyakumar, Li, & Vicente-Pérez, 2015). In Goberna et al. (2015, Table 1), an overview of different approaches to robust multiobjective optimization is provided; more recently, Dranichak and Wiecek (2019) and Talbi and Todosijevic (2020) add further concepts to the literature, while Schöbel and Zhou-Kangas (2020) provides (theoretical) comparisons between some of these approaches. More specifically, extending the concept of point-based minmax robust efficiency from the single-objective to the multiobjective case might either lead to multiobjective formulations or to set-based concepts. Subsequently, we have chosen to focus on the pointwise approach, cf. Ehrgott, Ide, and Schöbel (2014); Fliege and Werner (2014); Kuroiwa and Lee (2012), which relies on the simple and intuitive (albeit quite conservative) strategy to minimize the worst possible outcome among the possible scenarios (for each objective function), cf. Section 1.2.2. Quite related to Ehrgott et al. (2014); Fliege and Werner (2014); Kuroiwa and Lee (2012), Hassanzadeh, Nemati, and Sun (2013) considers the special case of budgeted uncertainty for function-wise box uncertainty for the coefficients of linear objective functions. In the framework of Ehrgott et al. (2014), Bokrantz and Fredriksson (2017) discusses necessary and sufficient conditions for robust efficiency in terms of scalarization functions. An alternative concept (light robustness) is proposed in Ide and Schöbel (2016) by extending the light robustness concept from the single-objective case. Similarly, the generalization of Hassanzadeh et al. (2013) of budgeted uncertainty introduced by Bertsimas and Sim (2003) to the multiobjective setup is developed further in Raith, Schmidt, Schöbel, and Thom (2018). The further alternative of multi-scenario

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[1] We assume a certain familiarity of the reader with the concepts of robust optimization and multiobjective optimization.

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efficiency is introduced by Botte and Schöbel (2019). Finally, Ide and Köbis (2014) and Ide, Köbis, Kuroiwa, Schöbel, and Tammer (2014) discuss the relation of uncertain multiobjective optimization problems to the field of set-valued optimization.

Although the concept of a robust counterpart to an uncertain single-objective optimization problem as in Ben-Tal, El Ghaoui, and Nemirovski (2009) is in our view most popular, an alternative is available based on the notion of *regret*. For example, Inuiguchi and Sakawa (1995); Kouvelis and Yu (1997) contain early treatments of regret from an optimization point of view and Hauser, Krishnamurthy, and Tütüncü (2013); Simões, McDonald, Williams, Fenn, and Hauser (2018); Takeda, Taguchi, and Tanaka (2010) provide interesting mathematical analyses as well as real world applications in the single-objective setting. Similar to Ben-Tal et al. (2009), this alternative is recommended whenever probabilities for each individual scenario are unknown, cf. Kouvelis and Yu (1997); in such situations it is perceived as a useful way for selecting decisions under uncertainty, cf. Kouvelis and Yu (1997, Section 6.1.1). Moreover, considering the worst case only is a relevant source of criticism to the classical worst-case approach, cf. Hauser et al. (2013). For instance[2], considering evaluations of investment managers, it becomes important to compete against the best competitor's performance and therefore regret is more suited in this setting, cf. Simões et al. (2018). In summary, it can be noted that on the one hand, robust regret has some modelling advantage over traditional robust optimization in several occasions, while on the other hand it might lead to computationally harder optimization problems. As we will illustrate in the following, several important special cases still remain computationally tractable, although computational burden of course increases in comparison to solving instances of an uncertain multiobjective optimization problem (i.e. $(RR^{(m)}(U))$, Section 2.1 compared to $(P^{(m)}(u))$), Section 1.2.2). Eventually, for a graphical illustration of the two approaches in the single criterion case, the reader is referred to Fig. 1.

In contrast to the single-objective case, no corresponding concept of regret is available for the case of an uncertain multiobjective optimization problem. The only existing approaches in this direction so far were given by Rivaz and Yaghoobi (2013, 2018); Rivaz, Yaghoobi, and Hladík (2016) and Xidonas, Mavrotas, Hassapis, and Zopounidis (2017). Still, none of these approaches actually considers a proper concept of regret in the multiobjective case. Instead, it is suggested to first scalarize the uncertain multiobjective optimization problems to uncertain single-criterion instances which are then handled within the known single-objective regret setting. In the following, we will close this gap and introduce a proper concept of *multivariate (relative) regret*. We will motivate the choice of this regret formulation and show that this indeed represents a generalization of the single-objective setting to the multiobjective case. We further analyze the structure of the corresponding multiobjective robust regret formulation. We especially compare the computational effort for solving such problems instead of solving standard robust multiobjective optimization problems in a variety of common special cases. Not surprisingly, our analysis yields similar results as the analysis in the single-objective case, cf. Hauser et al. (2013). Finally, we compare our approach to the related approaches by Drezner, Drezner, and Salhi (2006); Rivaz and Yaghoobi (2013, 2018); Rivaz et al. (2016); Xidonas et al. (2017) in more detail. In summary, our approach covers the following novel aspects of uncertain multiobjective optimization:

- For the first time, we introduce a consistent framework for regret optimization in an uncertain multiobjective optimization (MOP) setup. In particular, we clearly distinguish the modeling approach (i.e. the fact that an uncertain multiobjective optimization problem is cast as a multiobjective regret optimization problem) from its numerical solution (e.g. by scalarization, by specific genetic algorithms, by direct multisearch methods, etc.). As a side result, we obtain the novel insight that Chebyshev scalarization actually commutes with robustification, thus the order of scalarization and regret is not important in this case, see Proposition 2.4 for more details.
- In contrast to the few existing and rather scattered results in the literature on multiobjective regret, our approach is neither limited to linear objectives, nor to finite or interval uncertainty sets. Thus, all existing approaches are unified and generalized, besides the separation of regret modelling and scalarization.
- We provide a detailed view on the structure of the multiobjective regret optimization problem with a special focus on convex optimization problems. We also highlight, which instances can be solved with polynomial effort and how general formulations can be approached via (inner and outer) polytopal approximations. For this purpose, a thorough discussion concerning continuity with respect to the uncertainty set is carried out.

We believe that as scalarization is an auxiliary computational tool to solve a given multiobjective optimization problem and not a modeling paradigm as such, our setting seems to be a quite natural one. Furthermore, in our opinion, sticking to the multiobjective setting as we do, is more intuitive from a modeler's perspective. Finally, our approach now allows to put all existing approaches into more context, cf. Section 5, and also allows for easy interpretation.

1.2. Problem formulation

Before we introduce the general multiobjective setup, we briefly recall the main idea of (relative) regret in the single-objective case.

1.2.1. Uncertain single-objective optimization problems

Consider the following (family of) uncertain optimization problem(s)

$$\min_{x\in X} \; f(x,u) \tag{P(u)}$$

where $f: X \times U \to \mathbb{R}$ is some continuous function, $x \in X \subset \mathbb{R}^n$ represents the decision variables and $u \in U \subset \mathbb{R}^{n'}$ represents uncertain parameters. For simplicity of presentation, we assume that both X and U are compact. We would like to point out that this in particular covers the case of finite X. The *robust counterpart* to (P(u)) as in Ben-Tal et al. (2009) is given as

$$\min_{x\in X} \; F(x) \tag{RC(U)}$$

with $F(x) := \max_{u\in U} f(x,u)$. It is well-known, see e.g. Ben-Tal et al. (2009), that (RC(U)) can be solved efficiently, if the original uncertain optimization problem satisfies certain structural requirements, e.g. f has to be convex in x and F needs to be easily computable. However, it is also well-known that (RC(U)) leads to rather conservative solutions as it is focused on the worst case instance only. As a remedy to this problem, the alternative concept of (relative) regret can be applied, see e.g. Kouvelis and Yu (1997) for a more detailed discussion. In the single-objective setting, the *regret of a decision x in a scenario $u \in U$* is defined as

$$r(x,u) := f(x,u) - f^*(u), \quad \text{with } f^*(u) := \min_{x\in X} \; f(x,u),$$

while the *relative or scaled regret of a decision x in a scenario $u \in U$* can be represented as

$$s(x,u) := \frac{f(x,u) - f^*(u)}{f^*(u)} = \frac{f(x,u)}{f^*(u)} - 1 = \frac{1}{f^*(u)} r(x,u),$$

[2] Another example is the choice of a tour from some point A to a point B, which is usually compared with the optimal choice in the given scenario, not against the worst case.

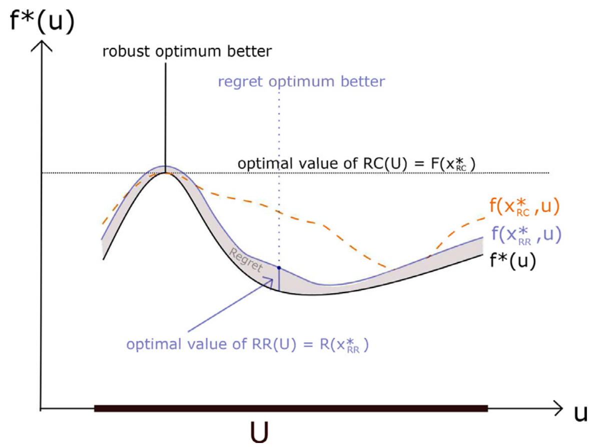

Fig. 1. Illustration of worst-case optimization versus regret optimization under the assumption $F(x^*_{\mathrm{RC}}) = \max_{u\in U} f^*(u)$.

if $f^*(u) > 0$ holds for all $u \in U$. It usually depends on the given application, whether regret or relative regret is considered by the decision maker.[3]

Based on the (relative) regret, the decision maker can now optimize with respect to regret. In uncertain environments, similar to the robust counterpart, a worst case (relative) regret is often deemed appropriate; hence we introduce

$$R(x,U) := \max_{u\in U} \; r(x,u),$$

and, analogously,

$$S(x,U) := \max_{u\in U} \; s(x,u).$$

This leads to the corresponding *robust (relative) regret counterparts* to (P(u))

$$\min_{x\in X} \; R(x,U), \qquad \text{(RR(U))}$$

and

$$\min_{x\in X} \; S(x,U). \qquad \text{(RS(U))}$$

The differences of the robust counterpart to the robust regret counterpart are illustrated in Fig. 1. As it can be observed, the worst case performance of the robust counterpart is (by definition) better than the worst case performance of the minimum regret decision. The latter, however, usually comes with a lower worst case opportunity loss. Accordingly, in Hauser et al. (2013), optimal solutions to these optimization problems are called robust (relative) regret solutions and in Kouvelis and Yu (1997), they are called robust (relative) deviation decisions. For interval uncertainty, this approach can also already be found in Inuiguchi and Sakawa (1995).

Remark 1.1. It needs to be mentioned that *stochastic programming* represents an important (quite related) approach to optimization under uncertainty. There, the uncertain parameter u is treated as a random variable and probabilistic criteria are applied instead of a worst case criterion. The same distinction can of course be made here, leading to a framework of stochastic regret, where instead *expected regret* or some *risk measure of regret* could be considered. For instance, in the related paper (Xu, Zhou, & Xu, 2020), total (and average) expected regret is considered, already in a multiobjective framework. The main difference to our approach, besides a rather different focus of their paper, is that the authors only consider a finite uncertainty set and that no worst-case regret is considered. As a thorough analysis of stochastic regret is beyond the scope of this contribution, we prefer to leave this for future research.

We have already mentioned that the robust counterpart can be solved efficiently in a variety of common setups; see Ben-Tal et al. (2009) for an overview and more details. We will demonstrate in the following that although the robust regret counterpart represents a somewhat more involved concept (due to the fact that the optimal value $f^*(u)$ appears in the formulation), it can still be solved with polynomial effort in common specific situations, see e.g. Hauser et al. (2013) for a detailed analysis of regret in the single-objective case. As we will see later in Section 4.2, this remains to be true for our generalization of robust (relative) regret to the multiobjective setup.

1.2.2. Uncertain multiobjective optimization problems

If instead of an uncertain single-objective optimization problem an uncertain multiobjective optimization problem is given, the considerations from Section 1.2.1 get somewhat more involved. As already mentioned, there have been successful approaches how to formulate a robust counterpart for an uncertain multiobjective optimization problem, while the corresponding concept for (relative) regret is still missing. For this purpose, let now $f: X\times U \to \mathbb{R}^m$ be an m-variate continuous objective function, representing potentially conflicting aims. The uncertain multiobjective optimization problem then reads as

$$\min_{x\in X}{}_{\mathbb{R}^m_{\geq}} \; f(x,u). \qquad \text{(P}^m\text{(u))}$$

Multiobjective optimization problems constitute a special case of vector optimization problems (see for example Jahn, 2004), where the optimization is carried out with respect to a general ordering cone K. In the case of multiobjective optimization problems, the corresponding ordering cone is always given by the specific cone $K = \mathbb{R}^m_{\geq}$. In contrast to the single-objective setup, minima of the image set $Y_{f(\cdot,u)} := \{f(x,u)\,|\,x\in X\}$ will not exist in general. Therefore, usually (weakly) non-dominated elements of $Y_{f(\cdot,u)}$, denoted by $Y^N_{f(\cdot,u)}$ and $Y^{wN}_{f(\cdot,u)}$ are sought. The corresponding pre-images are so-called (weakly) efficient solutions and denoted by $X^E_{f(\cdot,u)}$ and $X^{wE}_{f(\cdot,u)}$, cf. Ehrgott (2005), Jahn (2004). In our setup, a solution $\bar{x}\in X$ is called *efficient* (for fixed parameter u) if and only if there does not exist $x\in X$ with $f_i(x,u) \leq f_i(\bar{x},u)$ for every $i = 1,\dots,m$ and $f_j(x,u) < f_j(\bar{x},u)$ for at least one $j\in\{1,\dots,m\}$. A slightly weaker version of this concept is the following: a solution $\bar{x}\in X$ is called *weakly efficient* if and only if there does not exist $x\in X$ with $f_i(x,u) < f_i(\bar{x},u)$ for every $i = 1,\dots,m$.

As recently suggested by Ehrgott et al. (2014), Fliege and Werner (2014), Kuroiwa and Lee (2012), a reasonable, albeit rather conservative formulation for a multiobjective robust counterpart, taking into account all uncertainty, looks as follows:

$$\min_{x\in X}{}_{\mathbb{R}^m_{\geq}} \; F(x), \qquad \text{(RC}^m\text{(U))}$$

where, in complete analogy to the single-objective case, $F_i(x) := \max_{u\in U} f_i(x,u)$ for $i = 1,\dots,m$.

Remark 1.2. Let us point out that although formulation $(RC^{(m)}(U))$ seems quite straightforward, a thorough discussion of the pros and cons of such a formulation had been appropriate and necessary, see especially Ehrgott et al. (2014) and Fliege and Werner (2014). As emphasized there, it is not instantaneously clear how to interpret the term "$\max_{u\in U} f(x,u)$" for an m-variate f. Several interpretations, ranging from the straightforward idea of a kind of *anti-efficient frontier* with respect to the negative ordering cone to a

[3] If the goal is not to compare the decision to the optimal solution, but to some given benchmark, one can consider the so-called benchmark regret as introduced in Simões et al. (2018). In this case we only need to consider some benchmark performance $b(u)$ instead of the individual optimal solutions $f^*(u)$ in the regret formulation and require the same structural properties for $b(u)$ as for $f^*(u)$.

set-valued interpretation have appeared in the literature. For the standard cone, the *pointwise* formulation has shown to be quite successful in terms of interpretation, application and ease of computation.

In this paper, we will show that we can obtain a multiobjective robust (relative) regret formulation along the same lines as the multiobjective robust counterpart has been obtained, thus generalizing the concept of regret from the univariate to the multivariate setup. For this purpose, the remainder of the paper is organized as follows: In Section 2 we introduce and motivate the generalization of robust regret to the multivariate case. To solve these multiobjective optimization problems, scalarization techniques are usually employed, see e.g. Ehrgott (2005) for more details. We pay special attention to the separation of robustification (as a modelling tool) and scalarization (as a solution procedure). We specifically prove that weighted Chebyshev scalarization actually commutes with robustification in our context (Proposition 2.4). We then discuss numerical aspects of the multiobjective robust regret formulations in Section 4 and we show that these formulations can be solved efficiently (or at least be well-approximated) in a variety of common setups, especially under polytopal uncertainty. For this purpose, a thorough analysis of the continuity of the objective function with respect to the uncertainty set is provided in Section 3, as well as its consequences on approximations. As a main tool of approximation for general convex uncertainty sets we will consider (inner and outer) polytopal approximations in Section 3.2.3. We provide a detailed comparison to existing approaches in Section 5, before we apply the techniques to a specific illustrative example in Section 6. We close the paper by a brief summary and an outlook to future research directions.

2. Multiobjective robust (relative) regret

As already discussed, there is currently no extension of the robust (relative) regret approach for uncertain optimization problems to the multiobjective setting. The main question in this context is how to replace the former scalar term $f^*(u) \in \mathbb{R}$ in a multivariate regret formulation. It has to be noted that in the multiobjective setup, the optimal value is no longer a unique scalar value, but represented by the whole non-dominated set $Y^N_{f(\cdot,u)}$. From this observation, as outlined in Remark 1.2, it is not straightforward what quantity should be used to compare to the what is now multivariate $f(x,u) \in \mathbb{R}^m$ to obtain a meaningful notion of regret.

2.1. Extension of robust regret to the multiobjective setting

In the following, we argue that a meaningful choice is obtained, if the scalar value $f^*(u) \in \mathbb{R}$ is replaced by the corresponding *ideal point* $f^*(u) \in \mathbb{R}^m$ in the multiobjective setting, defined by

$$f_i^*(u) := \min_{x\in X} f_i(x,u), \quad \text{for } i = 1,\dots,m.$$

Alternatively[4], one could prefer a set valued formulation and work with $Y^N_{f(\cdot,u)}$ instead of $f^*(u)$. To see that the ideal point is indeed a reasonable choice which also leads to computationally tractable optimization problems, we have a closer look at the single-objective setup. In this setup we have:

$$\min_{x\in X} R(x,U) = \min_{x\in X} \max_{u\in U} r(x,u).$$

[4] Indeed, this also represents an interesting and promising approach, which, however, needs separate thorough discussion, see also Section 7.

Introducing a slack variable α for the objective function leads to

$$\min_{x\in X} R(x,U) = \min_{x\in X,\, \alpha\in\mathbb{R}} \alpha \quad \text{s.t.} \quad r(x,u) \le \alpha \quad \forall u \in U.$$

Note that the latter constitutes an optimization problem with (potentially) infinitely many constraints, depending on the cardinality of the uncertainty set U. We can continue the reformulation and obtain

$$\min_{x\in X} R(x,U) = \min_{x\in X,\, \alpha\in\mathbb{R}} \alpha \quad \text{s.t.} \quad f(x,u) \le \alpha + \min_{y\in X} f(y,u) \quad \forall u \in U.$$

This can be reformulated in an equivalent way as

$$\min_{x\in X} R(x,U) = \min_{x\in X,\, \alpha\in\mathbb{R}} \alpha \quad \text{s.t.} \quad f(x,u) \le \alpha + f(y,u) \quad \forall u \in U, \forall y \in X.$$

Now we make the important observation that this final reformulation with (potentially) infinitely many constraints can easily be generalized to multivariate functions f (together with a corresponding slack $\alpha \in \mathbb{R}^m$). Taking $\mathbb{R}^m_{\geq}$ as ordering cone, we obtain the multiobjective generalization

$$\min_{\mathbb{R}^m_{\geq}} \alpha \quad x\in X,\, \alpha\in\mathbb{R}^m$$
$$\text{s.t.} \quad f_i(x,u) \le \alpha_i + f_i(y,u) \quad \forall u \in U, \forall y \in X, \forall i = 1,\dots,m. \quad (1)$$

Remark 2.1. Considering general ordering cones $K \subset \mathbb{R}^m$ and replacing $\le$ with the cone inequality $\le_K$ yields optimization problems of type:

$$\min_{K} \alpha \quad x\in X, \alpha \in\mathbb{R}^m$$
$$\text{s.t.} \quad f(x,u) \le_K \alpha + f(y,u) \quad \forall u \in U, \forall y \in X.$$

Note that for general ordering cones, the subsequent reformulations in this Section 2.1 do not apply and different argumentation would be necessary.

As the constraint (1) can be interpreted as *rowwise uncertainty*, we can get rid of the (potentially) infinitely many constraints parametrized by $y \in X$, which yields

$$\min_{\mathbb{R}^m_{\geq}} \alpha \quad x\in X,\, \alpha\in\mathbb{R}^m$$
$$\text{s.t.} \quad f_i(x,u) \le \alpha_i + \min_{y\in X} f_i(y,u) \quad \forall u \in U, \forall i = 1,\dots,m.$$

By replacing $\min_{y\in X} f_i(y,u)$ by $f_i^*(u)$ we obtain

$$\min_{\mathbb{R}^m_{\geq}} \alpha \quad x\in X,\, \alpha\in\mathbb{R}^m$$
$$\text{s.t.} \quad f_i(x,u) \le \alpha_i + f_i^*(u) \quad \forall u \in U, \forall i = 1,\dots,m.$$

This can be equivalently reformulated to

$$\min_{\mathbb{R}^m_{\geq}} \alpha \quad x\in X,\, \alpha\in\mathbb{R}^m$$
$$\text{s.t.} \quad \max_{u\in U} f_i(x,u) - f_i^*(u) \le \alpha_i \quad \forall i = 1,\dots,m.$$

Eliminating the slack variable α, we finally arrive at the *multiobjective robust regret optimization problem*

$$\min_{x\in X}{}_{\mathbb{R}^m_{\geq}} R(x,U) \qquad (\mathrm{RR}^m(\mathrm{U}))$$

with

$$R_i(x,U) := \max_{u\in U} r_i(u,x) := \max_{u\in U} f_i(x,u) - f_i^*(u),$$

for $i = 1, \dots, m$. We immediately see that this indeed represents a generalization of the single-objective regret formulation to a multivariate setting. Quite obviously, the same arguments can be repeated to obtain the *multiobjective robust relative (or scaled) regret optimization problem*

$$\min_{x\in X}{}_{\mathbb{R}^m_{\geq}}\; S(x,U) \qquad (\mathrm{RS}^m(\mathrm{U}))$$

as long as $f^*(u) > 0$ for all $u \in U$. Therefore, from now on we assume throughout the rest of the paper that

$\forall u \in U:\; f^*(u) > 0.$

We would like to mention that both $(\mathrm{RR}^m(\mathrm{U}))$ and $(RS^{(m)}(U))$, like the robust counterpart (RC(U)), constitute classical multiobjective optimization problems. Their structure of course depends on the sets X and U, as well as on the specific structure of f in x and/or u.

Remark 2.2. Note that both problems $(RR^{(m)}(U))$ and $(RS^{(m)}(U))$ obviously remain invariant under scalar multiplication of f_i by some $\mu_i > 0$. However, while $R_i(x,U)$ itself remains invariant under additive shifts of f_i, $S_i(x,U)$ might change in a non-linear fashion, as already the corresponding worst-case $u \in U$ may change due to such shifts.

While in the definition of $s_i(x,u)$ in Kouvelis and Yu (1997) the normalization of $r_i(x,u)$ is by $1/f_i^*(u)$ (as we do here), the authors in Xidonas et al. (2017) prefer to apply a different normalization based on $f_i^{\times}(u) := \max_{x\in X} f_i(x,u)$:

$$\tilde{s}_i(x,u) := \frac{f_i(x,u) - f_i^*(u)}{f_i^{\times}(u) - f_i^*(u)},$$

which avoids this issue. Clearly, this kind of normalization is mainly (numerically) recommendable for finite X or for linear optimization problems in x as considered in Xidonas et al. (2017). To avoid potentially computationally hard maximization problems in x, it might be more advisable to use the normalization

$$\bar{s}_i(x,u) := \frac{f_i(x,u) - f_i^*(u)}{f_i^{N}(u) - f_i^*(u)},$$

based on the nadir point f^N, which might be easier to compute in some specific setups. Unfortunately, in general, both choices typically lead to numerically rather intractable optimization problems and are therefore not considered further in this paper.

2.2. An alternative motivation

By a closer inspection of the robust (relative) regret, it can be observed that there is a close relationship to the robust counterpart. Indeed, starting with the family of uncertain optimization problems (P(u)), we can shift (or scale and shift) each objective function to obtain the uncertain families

$$\min_{x\in X}\; f(x,u) - f^*(u), \qquad (\mathrm{P'(u)})$$

and

$$\min_{x\in X}\; \frac{f(x,u)}{f^*(u)} - 1, \qquad (\mathrm{P''(u)})$$

respectively. Note that these transformations do not change the set of optimal solutions of (P(u)), and hence (P'(u)) and (P"(u)) are equivalent to (P(u)). Now, it becomes obvious that the robust counterpart to (P"(u)) is exactly (RR(U)) and the robust counterpart to (P"(u)) coincides with (RS(U)). The same is of course true for the multiobjective case. We would like to emphasize that – in alternative to our motivation presented in Section 2.1 – we could have introduced the multiobjective robust regret concept via the observations made here in Section 2.2 instead. These observations especially imply that all discussions on robust counterparts to multiobjective optimization problems directly transfer to the case of multiobjective regret.

Remark 2.3. As argued here in Section 2.2, the regret formulation is closely linked to a robust counterpart formulation. Thus, applying the weighted sum scalarization (or the ε-constraint technique) first and then robustifying is not the same as robustifying in the multiobjective setting and then applying these scalarizations. A more detailed discussion of this issue can for instance be found in Fliege and Werner (2014). Hence, robustification and scalarization will not commute in general.

2.3. Chebyshev scalarization commutes with robustification

In contrast to the negative results from Fliege and Werner (2014), let us now provide a positive result concerning the ordering of robustification and scalarization: it turns out that in our setup weighted Chebyshev scalarization indeed commutes with robustification (for more details on Chebyshev scalarization, we refer to Ehrgott, 2005).

Proposition 2.4 (Chebyshev scalarization commutes with robustification)**.** *For $w \in \mathbb{R}^m_{\geq}$ it holds*

$$\max_{u\in U}\max_{1\leq i\leq m} w_i r_i(x,u) = \max_{1\leq i\leq m} w_i R_i(x,U)$$

and especially

$$\min_{x\in X}\max_{u\in U}\max_{1\leq i\leq m} w_i r_i(x,u) = \min_{x\in X}\max_{1\leq i\leq m} w_i R_i(x,U)$$

Proof. As can be easily seen, 0 represents the ideal point of both $r(x,u)$ and $s(x,u)$ for each fixed u. Thus, we have for the robustified scalarized regret:

$$\max_{u\in U}\max_{1\leq i\leq m} w_i r_i(x,u) = \max_{1\leq i\leq m}\max_{u_i\in U} w_i r_i(x,u_i) = \max_{1\leq i\leq m} w_i \max_{u_i\in U} r_i(x,u_i) = \max_{1\leq i\leq m} w_i R_i(x,U). \qquad \square$$

The right hand side in Proposition 2.4 coincides with the (Chebyshev-)scalarized robust regret using 0 as reference point for R, while the left hand side represents the robustified (Chebyshev-) scalarized regret. Thus Chebyshev scalarization commutes with robustification. The same arguments hold of course for S instead of R. We further note that this argumentation remains true for a general uncertain f instead of r or s, as long as 0 represents a reasonable reference point – thus extending the analysis on commutation given in Fliege and Werner (2014). The reason lying behind this surprising result is the connection of the weighted Chebyshev scalarization to the ε-constraint scalarization technique. As shown in Fliege and Werner (2014), the ε-constraint scalarization technique commutes with robustification of *generalized instances* of uncertain optimization problems, a property which has been used in the above proof when switching the order of maximization in u and i.

3. Problem properties

Before we consider potential solution approaches for both problems $(RR^{(m)}(U))$ and $(RS^{(m)}(U))$ in Section 4, let us first gain some further insight into the objective functions R and S, especially under some structural assumptions. We start by providing some results concerning continuity and monotonicity of R and S with respect to U. This will in turn yield results concerning the approximation of problems $(RR^{(m)}(U))$ and $(RS^{(m)}(U))$. Furthermore, recall that we assume f to be continuous in (x,u) throughout this paper. Finally, we would like to remark that the results obtained in this section can be straightforwardly generalized to the pointwise robust approach in uncertain multiobjective optimization, as considered for example in Ehrgott et al. (2014), Fliege and Werner (2014), Kuroiwa and Lee (2012).

3.1. Approximation with respect to U

We start with an obvious observation concerning monotonicity:

Proposition 3.1 (Monotonicity of R and S in U). *Let $U \subseteq V \subseteq \mathbb{R}^{n'}$. Then*

$$\forall x \in X: \quad R(x,U) \leq R(x,V) \quad \text{and} \quad S(x,U) \leq S(x,V). \tag{2}$$

As (2) is especially true for each $u \in U$, we get

$$\forall x \in X, \forall u \in U: \quad r(x,u) \leq \max_{v \in U} r(x,v) = R(x,U) \quad \text{and}$$
$$s(x,u) \leq \max_{v \in U} s(x,v) = S(x,U),$$

where the maximum should of course be understood componentwise. The consequence of this result is that the non-dominated set of the worst-case regret lies to the upper right of all non-dominated sets for all scenarios.

Due to continuity of f in u, we also immediately obtain the following continuity results, where $d_H(A,B)$ denotes the Hausdorff distance between two compact sets A and B.

Proposition 3.2 (Continuity of R and S in U)**.** *Let $(U_n)_{n \in \mathbb{N}}$ be a sequence of compact sets in $\mathbb{R}^{n'}$. Then $d_H(U_n, U) \to 0$ for $n \to \infty$ implies*

$$\forall x \in X: \quad \lim_{n \to \infty} R(x,U_n) = R(x,U) \quad \text{and} \quad \lim_{n \to \infty} S(x,U_n) = S(x,U).$$

Proof. This follows directly from (Bank, Guddat, Klatte, Kummer, & Tammer, 1983, Theorem 4.2.2,), where U plays the role of the parameter. □

Interestingly, under our assumptions, Propositions 3.1 and 3.2 are already sufficient for a uniform convergence (of the objective functions) of monotone outer and inner approximations to the original objective.

Corollary 3.3 (Uniform convergence of R and S)**.** *Let $(U_n)_{n \in \mathbb{N}}$ be a sequence of compact sets in $\mathbb{R}^{n'}$ with $U_n \subseteq U_{n+1}$ (or, alternatively, $U_{n+1} \subseteq U_n$) for all n. If $d_H(U_n, U) \to 0$ for $n \to \infty$, then both $R(\cdot, U_n)$ and $S(\cdot, U_n)$ converge uniformly on X to $R(\cdot, U)$ and $S(\cdot, U)$, respectively.*

Proof. This result follows directly from Dini's theorem (see Edwards, 1994, page 165) based on Propositions 3.1 and 3.2 and due to the compactness of X. □

Under the additional regularity assumption $Y^N_{R(\cdot,U)} = Y^{wN}_{R(\cdot,U)}$, we also get convergence of the corresponding non-dominated sets:

Theorem 3.4 (Convergence of the non-dominated set)**.** *Let $(U_n)_{n \in \mathbb{N}}$ be a sequence of compact sets in $\mathbb{R}^{n'}$ with $d_H(U_n, U) \to 0$ for $n \to \infty$. If $Y^N_{R(\cdot,U)} = Y^{wN}_{R(\cdot,U)}$, then*

$$\lim_{n \to \infty} d_H\left(Y^N_{R(\cdot,U_n)}, Y^N_{R(\cdot,U)}\right) = 0.$$

The same assumptions imply upper semi-continuity of the corresponding map $V \mapsto X^E_{R(\cdot,V)}$ at U. Analogous statements hold true for S.

Proof. This result follows from Theorems 3.1 and 3.2 in Tanino (1990). All prerequisites of these theorems are obviously satisfied, including $\mathbb{R}^m_{\geq}$-mini-completeness. The upper semi-continuity of $V \mapsto X^E_{R(\cdot,V)}$ follows from (Tanino, 1990, Theorem 3.3.). □

Unfortunately, for the corresponding efficient solutions one can only expect upper semi-continuity, as lower semi-continuity comes with much stronger requirements on R (or S), cf. Condition 3 of Theorem 3.4 in Tanino (1990). Interestingly, Theorem 3.4 yields that if the uncertainty set shrinks to a single scenario, the corresponding non-dominated set will converge to the non-dominated set of the corresponding single scenario.

From a more practical perspective, Proposition 3.1 already provides us with the insight that under monotone inner and outer approximations, one obtains non-dominated sets which *sandwich* the true non-dominated set:

Proposition 3.5 (Sandwiching the set of non-dominated points)**.** *Let $U^i \subseteq U \subseteq U^o$, and for R let further $y^N_R(U^o)$ denote the nadir point of the outer approximation based on U^o and $y^*_R(U^i)$ the ideal point of the inner approximation based on U^i. Further let $B_R(U^i, U^o) := \{y \in \mathbb{R}^m \mid y^*_R(U^i) \leq y \leq y^N_R(U^o)\}$ (and accordingly for S). Then we have:*

$$Y^N_{R(\cdot,U)} \subseteq \left(\left(Y^N_{R(\cdot,U^i)} + \mathbb{R}^m_{\geq}\right) \setminus \left(Y^N_{R(\cdot,U^o)} + \mathbb{R}^m_{>}\right)\right) \cap B_R(U^i, U^o), \quad \text{and}$$
$$Y^N_{S(\cdot,U)} \subseteq \left(\left(Y^N_{S(\cdot,U^i)} + \mathbb{R}^m_{\geq}\right) \setminus \left(Y^N_{S(\cdot,U^o)} + \mathbb{R}^m_{>}\right)\right) \cap B_S(U^i, U^o).$$

Proof. This directly follows from Proposition 3.1 and the definition of non-dominated points. □

3.2. Exploiting structure in the uncertainty

Let us now consider structural assumptions on the set U and the dependence of f on u. We will focus on the most important practical cases, which are

- U is finite,
- U is a (convex) polytope, or,
- U is a convex body.

The last setup especially covers the quite popular case of ellipsoidal uncertainty sets. The assumptions on U will be supplemented by further assumptions on $f(x,u)$, as e.g. linear uncertainty dependence, to obtain relevant structural results.

3.2.1. Finite uncertainty set

Let us assume that $p := |U| < \infty$ such that U is given as a finite set of scenarios, i.e. $U = \{u_1, \ldots, u_p\}$. In this case, it is possible to precompute all optimal values $f^*_i(u)$ for all $i = 1, \ldots, m$ and for all $u \in U$. Then

$$R_i(x, \{u_1, \ldots, u_p\}) = \max_{u \in \{u_1, \ldots, u_p\}} f_i(x,u) - f^*_i(u) = \max_{j \in \{1, \ldots, p\}} f_i(x,u_j) - f^*_i(u_j).$$

The analogous statement for $S(x,U)$ is true as well.

Remark 3.6. The precomputation of the ideal points $f^*(u)$ can be carried out with polynomial effort, if either X is finite, or (P(u)) is a convex optimization problem. More details on this are given in Section 4.1 for finite X and in Section 4.2 for continuous X.

3.2.2. Polytopal uncertainty set

In contrast to the previous Section 3.2.1, where finiteness of U was assumed, we now instead assume that the uncertainty set U is given as a convex polytope. As we will demonstrate next, this can be used to reduce the polytopal case to the previous setup of finite U, given that f has some additional structure in u. More precisely, we require that f is linear in u for all $x \in X$. We start with the following observation.

Proposition 3.7 (Polytopal uncertainty)**.** *Let U be a convex polytope and let $V(U)$ denote the finite set of its vertices. Further, let f be linear in u for all $x \in X$. Then*

$$\max_{u \in U} f_i(x,u) - f^*_i(u) = \max_{u \in V(U)} f_i(x,u) - f^*_i(u)$$

and

$$\max_{u\in U}\frac{f_i(x,u)-f_i^*(u)}{f_i^*(u)}=\max_{u\in V(U)}\frac{f_i(x,u)-f_i^*(u)}{f_i^*(u)}$$

holds for every $i\in\{1,\dots,m\}$.

Proof. Consider the first statement: As $u\mapsto f_i(x,u)$ is linear for each $x\in X$, the mapping $u\mapsto f_i^*(u)=\min_{x\in X} f_i(x,u)$ is concave. Thus, the function $g_i(u)=f_i(x,u)-f_i^*(u)$ is convex in u. Since convex functions that attain their maximum also attain it in one of the extreme points of the feasible domain (in our case in one of the vertices), the first statement of the proposition follows.

For the second statement, note that it is no longer true that the function $g_i(u)=\frac{f_i(x,u)-f_i^*(u)}{f_i^*(u)}$ is convex in u. Still, as the numerator is convex in u and the denominator is concave in u (and positive), it is straightforward to see that g_i is quasiconvex in u. Since quasiconvex functions that attain their maximum also attain it in one of the vertices of the feasible domain (see e.g. Greenberg & Pierskalla, 1971), the second statement of the proposition follows. Note that quasiconvexity of g_i was already observed in Takeda et al. (2010), Section 2.2. □

Thanks to Proposition 3.7 we can replace the uncertainty set U by its finite set of vertices $V(U)=\{u_1,\dots,u_p\}$ in case of linear uncertainty dependence:

Proposition 3.8 (Special case of polytopal uncertainty)**.** *Let U be a convex polytope and let $V(U)$ denote the finite set of its vertices. Further, let f be linear in u for all $x\in X$. Then*

$$R(x,U)=R(x,V(U))\quad\text{and}\quad S(x,U)=S(x,V(U))$$

Remark 3.9. For completeness, we would like to remark that more generally, it can be shown that if f is linear in u, then for convex uncertainty sets U one has

$$R(x,U)=R(x,\partial U)=R(x,\mathcal{E}(U)),$$

i.e. a convex uncertainty set can be replaced by its boundary ∂U or its set of extreme points $\mathcal{E}(U)$.

3.2.3. Convex uncertainty set

For convex uncertainty sets, more precisely for convex bodies, we will now provide some analysis for a lower and upper approximation based on inner and outer polytopal approximations of the uncertainty set. The main reason for this investigation is that under linear uncertainty dependence, due to Proposition 3.8, the robust regret functions R and S can be computed more easily if the uncertainty set is polytopal.

Remark 3.10. An excellent reference on polytopal approximation of convex bodies is given by the survey paper Bronstein (2008) and the many references therein. Among the numerous interesting results concerning polytopal approximations, it is important in our case that an inner and an outer ε-approximation[5] (with respect to the Hausdorff metric) can be obtained by polytopes with $\mathbf{O}(1/\varepsilon^{\frac{d-1}{2}})$ vertices in d dimensions, cf. Bronstein (2008), Section 4.1.

It should be emphasized that in our case the number of vertices of the polytopal approximation is the crucial quantity which impacts the numerical complexity, especially also for the outer approximation (and not the number of faces, nor some combinatorial complexity as e.g. recently considered by Arya, da Fonseca, & Mount, 2017).

[5] For arbitrary convex sets, it is still possible to obtain a polytopal approximation, see e.g. Bronšteĭn and Ivanov (1976), but with a slightly worse approximation rate. To the best of our knowledge, this just yields an arbitrary approximation, not necessarily an inner or outer approximation.

Remark 3.11. For the actual computation of a reasonable inner and outer polytopal approximation to a convex body, refer to Bronstein Bronstein (2008), Section 8 and the references mentioned there. For practical purposes, it is usually sufficient to sample random points from U (or ∂U under linear uncertainty dependence) from an almost arbitrary distribution for which the density is bounded away from zero. By known results on random approximations of convex bodies, the convex hull of the sampled points yields a reasonable inner approximation also in the Hausdorff sense, albeit with a (slightly) worse approximation rate. For more details on the exact statement and the exact asymptotic rates, see Dümbgen and Walther (1996), Corollary 1.

Theorem 3.12 (Inner and outer polytopal approximations)**.** *Let U_n^i and U_n^o be sequences of inner and outer polytopal approximations to a convex uncertainty set U, i.e. let*

$$U_1^i\subset U_2^i\subset\cdots\subset U\subset\cdots\subset U_2^o\subset U_1^o$$

and $\lim_{n\to\infty} d_H(U_n^i,U)=0=\lim_{n\to\infty} d_H(U_n^o,U)$. *Then, for all n*

$$\forall x\in X:\quad R(x,U_n^i)\le R(x,U_{n+1}^i)\le\cdots\le R(x,U)\le\cdots\le R(x,U_n^o)\le R(x,U_{n-1}^o)$$

and

$$\lim_{n\to\infty}\sup_{x\in X}||R(x,U)-R(x,U_n^i)||=0=\lim_{n\to\infty}\sup_{x\in X}||R(x,U)-R(x,U_n^o)||.$$

If furthermore $Y^N_{R(\cdot,U)}=Y^{wN}_{R(\cdot,U)}$, then

$$\lim_{n\to\infty} d_H\left(Y^N_{R(\cdot,U_n^i)},Y^N_{R(\cdot,U)}\right)=0=\lim_{n\to\infty} d_H\left(Y^N_{R(\cdot,U_n^o)},Y^N_{R(\cdot,U)}\right)$$

The same results hold true for S.

Proof. The monotonicity is straightforward. The uniform convergence follows directly from Corollary 3.3 and the convergence of the sequence of approximating non-dominated sets is due to Theorem 3.4. □

As a consequence, we note that in the setting of Theorem 3.12, the volume of the sandwiching sets

$$\left(\left(Y^N_{R(\cdot,U_n^i)}+\mathbb{R}^m_{\geq}\right)\setminus\left(Y^N_{R(\cdot,U_n^o)}+\mathbb{R}^m_{>}\right)\right)\cap B_R(U_n^i,U_n^o)$$

decreases to 0. Further, under the regularity condition $Y^N_{R(\cdot,U)}=Y^{wN}_{R(\cdot,U)}$, we have convergence of this covering set to the true efficient frontier and analogously for S.

3.3. Exploiting structure in x and (x,u) jointly

In this section, we are interested in exploiting structural results concerning the dependence of f in x for a continuous feasible set X. Let us start with an easy-to-see fact:

Lemma 3.13. *For some $i\in\{1,\dots,m\}$ let f_i be convex in x for all u in U. Then R_i is convex in x as well. Under the assumption* $\min_{u\in U} f_i^*(u)>0$, *S_i is convex in x as well.*

Proof. This follows directly from the well-known fact that the supremum of convex functions is again convex and that $r_i(x,u)$ and $s_i(x,u)$ are convex in x for all u. □

3.3.1. Lipschitz continuity of R and S

To improve Corollary 3.3 to a more quantitative result, let us now assume some additional local Lipschitz continuity[6] of f in

[6] The function $f:X\times U\to\mathbb{R}^m$ is locally Lipschitz continuous in (x,u) jointly if for all $(x,u)\in X\times U$ there exists a neighbourhood $V\subset X\times U$ of (x,u) and a constant $L>0$ such that for all $(y,v),(z,w)\in V:||f(y,v)-f(z,w)||_1\le L(||y-z||_1+||v-w||_1)$ holds. Here, for convenience, the norm is chosen to be the 1-norm on $X\times U$ as this can be represented as the sum of the individual 1-norms on X and U.

(x, u) jointly. With this we can obtain the following much more quantitative result.

Theorem 3.14 (Uniform Lipschitz bounds)**.** *Let f be locally Lipschitz with respect to (x, u) jointly and $U, V \subseteq \mathbb{R}^{n'}$. Then there exists a constant $K > 0$ such that uniformly for all $x \in X$:*

$$||R(x,U) - R(x,V)||_1 \le K d_H(U,V)$$

and

$$||S(x,U) - S(x,V))||_1 \le K d_H(U,V).$$

We first state Remark 3.15 and Corollary 3.16 before we proceed to the proof of Theorem 3.14 based on Lemma 3.17.

Remark 3.15. If f is linear in u for all $x \in X$, we have seen in Remark 3.9 that it is sufficient that the boundary ∂U of U is well-approximated by the vertex set of the approximating polytope. However, if no structure of f in u is available, then the complete uncertainty set has to be approximated.

Based on Theorem 3.14 we can now state an immediate corollary, which improves the results of Proposition 3.5.

Corollary 3.16 (Uniform Lipschitz sandwich)**.** *Let $U^i \subseteq U \subseteq U^o$ and let f be locally Lipschitz with respect to (x, u) jointly. Then there exists a constant $\tilde{K} > 0$ such that for all $x \in X$*

$$R(x,U^i) \le R(x,U) \le R(x,U^i) + \tilde{K} d_H(U,U^i) \quad \text{and}$$

$$R(x,U^o) - \tilde{K} d_H(U,U^o) \le R(x,U) \le R(x,U^o)$$

and thus

$$Y^N_{R(\cdot,U)} \subset \left(\left(Y^N_{R(\cdot,U^i)} + \mathbb{R}^m_{\geq}\right) \setminus \left(Y^N_{R(\cdot,U^i)} + \tilde{K} d_H(U,U^i) + \mathbb{R}^m_{>}\right)\right) \cap \ B_R(U^i,U^o),$$

as well as

$$Y^N_{R(\cdot,U)} \subset \left(\left(Y^N_{R(\cdot,U^o)} - \tilde{K} d_H(U,U^o) + \mathbb{R}^m_{\geq}\right) \setminus \left(Y^N_{R(\cdot,U^o)} + \mathbb{R}^m_{>}\right)\right) \cap \ B_R(U^i,U^o),$$

Analogous results hold for S.

To prove Theorem 3.14, we first establish a helpful auxiliary result on the global Lipschitz continuity of the (scaled) regret.

Lemma 3.17. *Let $f : X \times U \to \mathbb{R}^m$ be locally Lipschitz in (x, u) jointly. Then there exists a constant $\bar{L} > 0$ such that*

$$\forall x, y \in X,\ u, v \in U: \quad ||f(x,u) - f(y,v)||_1 \le \bar{L}\left(||x-y||_1 + ||u-v||_1\right)$$

and

$$\forall u, v \in U: \quad ||f^*(u) - f^*(v)||_1 \le m\bar{L}||u-v||_1.$$

Thus, r is globally Lipschitz continuous with constant $(m+1)\bar{L}$ and s is also globally Lipschitz continuous with some (usually different) constant $\tilde{L} > 0$.

Proof. Since f is locally Lipschitz on $X \times U$ and since X and U are compact, there exists an $\bar{L} > 0$ such that f is globally Lipschitz continuous on $X \times U$, i.e.

$$\forall x, y \in X,\ u, v \in U: \quad ||f(x,u) - f(y,v)||_1 \le \bar{L}\left(||x-y||_1 + ||u-v||_1\right)$$

which proves the first statement. For the second claim, consider the following inequality for f_i^*:

$$f_i^*(u) - f_i^*(v) = \min_{x\in X} f_i(x,u) - \min_{x\in X} f_i(x,v) \le \min_{x\in X} f_i(x,v) + \bar{L}||u-v||_1 - \min_{x\in X} f_i(x,v) = \bar{L}||u-v||_1.$$

Swapping the roles of u and v yields $|f_i^*(u) - f_i^*(v)| \le \bar{L}||u-v||_1$. Adding all components yields the second claim. The third claim follows directly from the definition of r as the difference of two globally Lipschitz continuous functions with Lipschitz constants $\bar{L}$ and $m\bar{L}$. The analogous statement for s is a bit more involved. For this purpose, we define the constants

$$A_i := \max_{u\in U} \frac{1}{f_i^*(u)}, \quad B_i := \max_{u\in U} f_i^*(u), \quad \text{and} \quad C_i := \max_{x\in X, u\in U} f_i(x,u).$$

Then

$$s_i(x,u) - s_i(y,v) = \frac{f_i(x,u)}{f_i^*(u)} - \frac{f_i(y,v)}{f_i^*(v)} = \frac{1}{f_i^*(u) f_i^*(v)}\left(f_i(x,u) f_i^*(v) - f_i(y,v) f_i^*(u)\right).$$

As for the term in brackets, we have

$$f_i(x,u) f_i^*(v) - f_i(y,v) f_i^*(u) = f_i(x,u) f_i^*(v) - f_i(x,u) f_i^*(u) + f_i(x,u) f_i^*(u) - f_i(y,v) f_i^*(u),$$

thus proper inspection yields that $A_i^2(B_i + mC_i)\bar{L}$ is a global Lipschitz constant for s_i. The claim then follows with $\tilde{L} := mA_i^2(B_i + mC_i)\bar{L}$. □

We now come to the (straightforward) proof of the theorem.

Proof of Theorem 3.14. We only prove the statement for the case $m = 1$, the generalization to arbitrary dimensions is straightforward. To show the statement, note that $R(x,U) = r(x, u^*(x))$ for some $u^*(x)$ which maximizes $r(x,u)$ in $u \in U$. We can now replace $u^*(x)$ by some $v^*(x) \in V$ with $||u^*(x) - v^*(x)||_1 \le \sqrt{n'} d_H(U,V)$ (the constant n' appears as the Hausdorff distance is typically defined via the 2-norm). Since $R(x,V) \ge r(x, v^*(x))$ and since r is globally Lipschitz continuous with Lipschitz constant $(m+1)\bar{L}$ thanks to Lemma 3.17, we get for $K = (m+1)\bar{L}\sqrt{n'}$ that:

$$R(x,U) - R(x,V) \le r(x,u^*(x)) - r(x,v^*(x)) \le (m+1)\bar{L}||u^*(x) - v^*(x)||_1 \le K d_H(U,V).$$

Swapping the roles of U and V shows the claim. □

3.3.2. Improving the inner polytopal approximation

Let us finally mention that it is possible to improve the inner polytopal approximation along the lines of Takeda et al. (2010). For specific special cases of f and U (see Takeda et al., 2010 for more details), this improved approach still remains computationally tractable. For this purpose, choose (for fixed i) $u_1, \ldots, u_K \in U$ together with some $x_k \in X$ such that $f_i(x_k, u_k) = f_i^*(u_k)$ for $k = 1, \ldots, K$. The main idea of Takeda et al. (2010) is to replace the approximation of U by an approximation of $f^*(u)$. We start by observing that

$$R_i(x, \{u_1, \ldots, u_K\}) = \max_{u\in\{u_1,\ldots,u_K\}} r_i(x,u) = \max_{k=1,\ldots,K} f_i(x,u_k) - f_i(x_k,u_k) \le \max_{k=1,\ldots,K} \max_{u\in U} f_i(x,u) - f_i(x_k,u) = \max_{u\in U} \max_{k=1,\ldots,K} f_i(x,u) - f_i(x_k,u)$$

and furthermore, as $f_i(x_k,u) \ge f_i^*(u)$,

$$\max_{u\in U} \max_{k=1,\ldots,K} f_i(x,u) - f_i(x_k,u) \le \max_{u\in U} \max_{k=1,\ldots,K} f_i(x,u) - f_i^*(u) = \max_{u\in U} r_i(x,u) = R(x,U).$$

In this way, for the function

$$\rho_i(x, \{u_1, \ldots, u_K\}, U) := \max_{u\in U} \max_{k=1,\ldots,K} f_i(x,u) - f_i(x_k,u),$$

motivated by the definition of (CRP) in Takeda et al. (2010), we have that

$$\forall x: \ R_i(x, \{u_1, \ldots, u_K\}) \le \rho_i(x, \{u_1, \ldots, u_K\}, U) \le R(x,U),$$

and thus ρ_i provides a better lower bound than the simple inner polytopal approximation. Although at first glance the definition of ρ_i does not seem to be numerically tractable (due to the appearance of $\max_{u\in U}$), it is shown in Takeda et al. (2010), Section 4, that for f convex-quadratic in x and linear in u, ρ_i can be efficiently computed. More precisely, for norm-constrained uncertainty, this approximation can be formulated as a multiobjective second order cone program, and for ellipsoidal U, it can be formulated as a multiobjective semidefinite program. For more details on this reformulation, we refer to the original paper Takeda et al. (2010).

4. Numerical tractability

In this section, we discuss the computational tractability of a few important special cases of problems $(RR^{(m)}(U))$ and $(RS^{(m)}(U))$, based on structural assumptions similar to Section 3. We start with some straightforward observations for finite X, before we discuss the case of continuous optimization problems in more detail.

4.1. Numerical tractability of finite optimization problems

In the following let $\chi := |X| < \infty$ and let $p = |U| < \infty$. Then easy observations yield the following results, where we provide the complexity in terms of the number of function evaluations of one component of $f(x, u)$ and assume that this is computationally more expensive than for instance comparisons, look-ups, etc.

- (C1): The computation of $\{f^*(u)\,|\,u \in U\}$ has complexity $\mathbf{O}(\chi mp)$: this holds as mp minimization problems in x have to be solved (plus the same number of comparisons).
- (C2): The computation of $\{R(x, U)\,|\,x \in X\}$ has complexity $\mathbf{O}(\chi mp)$: based on (C1), the remaining argumentation is analogous to (C1), with r replacing f.
- (C3): The computation of $Y^N_{R(\cdot,U)}$ based on $\{R(x, U)\,|\,x \in X\}$ needs at most χ^2 comparisons.

Summarizing and and keeping in mind the similarities between R and S, we obtain:

Proposition 4.1 (Complexity for finite X)**.** *For $\chi = |X| < \infty$ and $p = |U| < \infty$, the sets $Y^N_{R(\cdot,U)}$ and $Y^N_{S(\cdot,U)}$ can be computed with at most $\mathbf{O}(\chi mp)$ function evaluations and at most $\mathbf{O}(\chi mp + \chi^2)$ comparisons.*

This result shows in particular that the effort to compute reasonable approximations to $Y^N_{R(\cdot,U)}$ and $Y^N_{S(\cdot,U)}$ for general convex U grows linearly with the number of vertices of the polytopal approximation to U, which in turn grows polynomially in the approximation accuracy ε of the uncertainty set.

Remark 4.2. For simplicity, in this Section 4.1, we focus on complexity considerations in $|X|$ only, as it is well-known that for some multiobjective combinatorial optimization problems (like multiobjective shortest path, see for example Serafini, 1987), the cardinality of the non-dominated set can be almost as large as $|X|$; more precisely, for some combinatorial multiobjective optimization problems with $X = \{0, 1\}^n$ one has that $|Y^N|$ is exponential in n. Thus, these kind of optimization problems need individual+/ different investigations which we leave for future investigations. For some recent related reference, let us refer to the PhD thesis (Bökler, 2018).

4.2. Numerical tractability of continuous optimization problems

Let us now focus on numerically tractable continuous optimization problems, i.e. problems which can be solved numerically within polynomial time up to a certain precision. To be able to apply classical complexity results, in this Section 4.2 we make the usual assumptions (e.g. similar to Jarre, 1992) that

- (A1): $X = \{x \in \mathbb{R}^n\,|\,g_i(x) \le 0, i = 1, \dots, M\}$ for some $M \in \mathbb{N}$ with g_i convex and sufficiently smooth[7] for $i = 1, \dots, M$.
- (A2): $\exists \bar{x} \in \mathbb{R}^n$ with $g_i(\bar{x}) < 0$ for $i = 1, \dots, M$ (Slater condition).
- (A3): f_i is convex and sufficiently smooth for $i = 1, \dots, m$ for all $u \in U$.
- (A4): All function values, gradients and Hessian matrices can be computed with an effort of at most $\mathbf{O}(n^2)$ and efforts for comparisons, look-ups, etc. can be neglected.

Based on these assumptions, we can state the classical complexity result for (P(u)), where we especially emphasize the mild dependence of the complexity on the number of constraints.

Theorem 4.3 (Complexity of IPMs, Jarre, 1992)**.** *Under assumptions (A1) to (A4), each instance of (P(u)) can be solved to ε-optimality in polynomial time with an effort of at most $\mathbf{O}(\sqrt{M}(n + M)n^2 \log(1/\varepsilon))$.*

Proof. In Jarre (1992), an interior-point method (IPM) is presented which needs $\mathbf{O}(\sqrt{M}\log(1/\varepsilon))$ iterations to obtain an ε-optimal feasible point. The effort needed in each iteration (typically for solving a Newton system) is at most $\mathbf{O}((n + M)n^2)$, see e.g. Jarre (1989), p. ~71. □

Theorem 4.3 immediately yields for $p = |U| < \infty$ that the precomputation of $f^*_i(u_j)$ up to an accuracy ε_{f^*} for $i = 1, \dots, m$, $j = 1, \dots, p$, comes with an effort of at most $[\mathbf{O}(mp\sqrt{M}(M + n)n^2 \log(1/\varepsilon_{f^*}))]$ iterations. Of course, warm-start ideas might be exploited to reduce this complexity further.

These considerations can now easily be extended from (P(u)) to convex multiobjective programming, i.e. to $(P^m(u))$ as well as to $(RR^{(m)}(U))$ and $(RS^{(m)}(U))$. We here follow the ideas from the pioneering work (Fliege, 2006) and the very recent analysis (Bergou, Diouane, & Kungurtsev, 2020), who (implicitly) work with a solution concept, which we formalize here:

Definition 4.4. Let $\Lambda := \{\lambda \in \mathbb{R}^m_{\geq}\,|\,\sum_{i=1}^m \lambda_i = 1\}$, $\delta > 0$, and $\Lambda_\delta \subset \Lambda$ such that $|\Lambda_\delta| < \infty$ and $\sup_{\lambda\in\Lambda} \inf_{\lambda_\delta\in\Lambda_\delta} ||\lambda - \lambda_\delta||_\infty < \delta/2$. Then a set $Y_{\varepsilon,\delta} \subset f(X)$ is called an (ε, δ)*-solution* for

$$\min_{x\in X}{}_{\mathbb{R}^m_{\geq}}\ f(x) \tag{3}$$

if it satisfies the two conditions

$$(i)\ \forall y \in Y_{\varepsilon,\delta} :\ \exists \lambda_\delta \in \Lambda_\delta :\ \lambda_\delta^\top y \le \min_{x\in X} \lambda_\delta^\top f(x) + \varepsilon, \quad \text{and}$$

$$(ii)\ \forall \lambda_\delta \in \Lambda_\delta :\ \exists y \in Y_{\varepsilon,\delta} :\ \lambda_\delta^\top y \le \min_{x\in X} \lambda_\delta^\top f(x) + \varepsilon.$$

If in (3) all f_i, $i = 1, \dots, m$ are strongly convex then one can easily show that

$$\lim_{\varepsilon,\delta\to 0} Y_{\varepsilon,\delta} = cl(Y^N_f) \tag{4}$$

in a Painlevé-Kuratowski sense, cf. Rockafellar and Wets (1998, Definition 4.1). Without strong convexity, only

$$\limsup_{\varepsilon,\delta\to 0} Y_{\varepsilon,\delta} \subseteq cl(Y^N_f) \tag{5}$$

holds. Due to (5), Fliege (2006) additionally considers a related quadratic scalarization, for which (4) already holds in the general convex case (but which comes with a slightly worse polynomial complexity).

Obviously, an (ε, δ)-solution for (3) with cardinality $\mathbf{O}(1/\delta^{m-1})$ can be obtained by the following construction:

- Choose a grid Λ_δ with cardinality $\mathbf{O}(1/\delta^{m-1})$ within the set of scalarization parameters Λ as in Bergou et al. (2020).

[7] We say that a function is sufficiently smooth if it is twice continuously differentiable and if its Hessian satisfies the relative Lipschitz condition (1.2) in Jarre (1992).

- Solve the corresponding weighted-sum scalarized instance of (3) for each scalarization parameter in Λ_δ up to accuracy ε.

Based on this concept of an (ε, δ)-solution, it is possible to obtain the following complexity result for a finite uncertainty set U, which yields a polynomial complexity in n, p, $1/\delta$ and $1/\varepsilon$, but an exponential complexity in m:

Proposition 4.5 (Complexity for finite U)**.** *Under assumptions (A1) to (A4), for $p = |U| < \infty$, the sets $Y^N_{R(\cdot,U)}$ and $Y^N_{S(\cdot,U)}$ can be approximated by an (ε, δ)-solution in the sense of (4) and (5) with a computational effort of at most*

$$\mathbf{O}\Big(\sqrt{M+mp}(n+m+M+pm)(n+m)^2\log(1/\varepsilon)/\delta^{m-1} + mp\sqrt{M}(M+n)n^2\log(1/\varepsilon)\Big). \quad (6)$$

Proof. Consider the weighted sum scalarization for $(RR^{(m)}(U))$ for finite U and $\lambda \in \Lambda_\delta$:

$$\min_{x\in X,\, \alpha\in\mathbb{R}^m} \lambda^\top \alpha \quad \text{s.t.} \quad f_i(x,u_j) - f_i^*(u_j) \le \alpha_i \quad \forall i = 1,\dots,m,\ j = 1,\dots,p.$$

Due to Theorem 4.3, each of the $\mathbf{O}(1/\delta^{m-1})$ many instances can be solved up to $\varepsilon/2$-accuracy with an effort of at most $\mathbf{O}\Big(\sqrt{M+mp}(n+m+M+pm)(n+m)^2\log(1/\varepsilon)\Big)$. Taking into account that $||\lambda||_1 \le 1$ and that each $f_i^*(u_j)$ can be precomputed up to $\varepsilon/2$-accuracy with effort $\mathbf{O}(\sqrt{M}(M+n)n^2\log(1/\varepsilon))$ yields a total effort of

$$\mathbf{O}\Big(\sqrt{M+mp}(n+m+M+pm)(n+m)^2\log(1/\varepsilon)/\delta^{m-1} + mp\sqrt{M}(M+n)n^2\log(1/\varepsilon)\Big).$$

□

Remark 4.6. We would like to mention that Fliege (2006) and Bergou et al. (2020) have successfully exploited warm-start ideas to significantly reduce the complexity (6) to $\mathbf{O}(\log(1/\varepsilon) + \log(\log(1/\varepsilon))/\delta^{m-1})$ (for $p = 1$, m and n fixed). We expect that similar ideas could be applied to the precomputation of $f_i^*(u_j)$ such that the above complexity might be further reduced.

Based on Proposition 4.5 and the results from Sections 3.2 and 3.3, we can now approximate $Y^N_{R(\cdot,U)}$ or $Y^N_{S(\cdot,U)}$ sufficiently well in polynomial time in the corresponding frameworks of Sections 3.2 and 3.3.

4.3. Alternative computational approaches for continuous optimization problems

Note that besides the approach via polytopal inner and outer approximation, there are alternative approaches to solve for $Y^N_{R(\cdot,U)}$ or $Y^N_{S(\cdot,U)}$ in case of general convex U. As these are not within the focus of this contribution, we prefer to just outline these approaches rather briefly and leave a detailed analysis (especially compared to the complexity of the polytopal approximation) for future research.

4.3.1. Semi-infinite formulation

For example, let us return to the equivalent formulations of $(RR^{(m)}(U))$ or $(RS^{(m)}(U))$, where the slack variable α has been introduced, i.e. we write $(RR^{(m)}(U))$ as

$$\min_{x\in X,\, \alpha\in\mathbb{R}^m}{}_{\mathbb{R}^m_{\geq}} \ \alpha \quad \text{s.t.} \quad \alpha_i \ge r_i(x,u) \quad \text{for } i = 1,\dots,m,\ \forall u \in U,$$

and analogously for $(RS^{(m)}(U))$. Both formulations constitute semi-infinite multiobjective optimization problems. Unfortunately, to the best of our knowledge no specific numerical algorithm for such a type of problem is available, although it has been already analyzed from a theoretical perspective, see for instance (Chuong & Kim, 2014; Guerra-Vásquez & Rückmann, 2015) and further references therein. Of course, after scalarization, the semi-infinite multiobjective optimization problem becomes a (parametric) standard convex semi-infinite optimization problem (SIP) which is then open to a variety of existing methods for convex SIPs. For a survey on semi-infinite programming, we refer to Stein (2012) or the more detailed book (Goberna & López, 2001). More specifically, some numerical experience is reported in Auslender, Ferrer, Goberna, and López (2015), while modern versions of an exchange method and a cutting surface method are covered in Mehrotra and Papp (2014) and Okuno, Hayashi, Yamashita, and Gomoto (2016). For the non-smooth case, more details are given in Pang, Lv, and Wang (2016). Further results on the (typically linear) rate of convergence of a cutting surface method can for example be found in Mehrotra and Papp (2014), while Still (2001) provides convergence rates for discretization methods. For a very recent improved result, we finally refer to the working paper (Seidel & Küfer, 2020).

4.3.2. Improved inner polytopal approximation

As a second alternative, let us mention the approach considered in Takeda et al. (2010): Similar to the idea of selecting a random inner polytopal approximation, it is suggested there that points $u_1, \dots, u_K$ are selected (in an optimal manner / randomly) in U. Based on these, the lower approximation ρ_i is used to approximate R_i from below. For the single-objective case, a convergence analysis is provided in Takeda et al. (2010), Section 3, which shows that the optimal value converges in probability. The corresponding analysis can be extended to our setup, straightforwardly for scalarized instances, and also to the multiobjective case due to the fact that the approximation ρ_i is sandwiched between the inner polytopal approximation and the true objective function. As mentioned, to keep focus of presentation, we prefer to leave rigorous mathematical statements for future work.

5. Connection to existing approaches

As already mentioned, other authors (Drezner et al., 2006; Rivaz & Yaghoobi, 2013; Xidonas et al., 2017) have already studied robust regret approaches to solve uncertain multiobjective optimization problems. However, all these approaches are based on the main idea to first scalarize the uncertain multiobjective optimization problem by some scalarization technique and then to apply the single-objective robust regret approach. In our view, the main drawback of proceeding in this order is that it mixes the computational technique scalarization with the modelling paradigm multiobjective optimization. Instead, we favor a clear separation between problem modelling and problem solution, including a transparent definition of what we understand as a solution. In the following, we discuss in more detail these existing approaches.

5.1. Connection to Drezner et al. (2006)

The approach by Drezner et al. in Drezner et al. (2006) can be seen as a first step into multiobjective regret, introducing (deterministic) *relative multiobjective regret* for the first time. To be more precise,

- Drezner et al. do not consider uncertainty, or, to fit within our setting, they assume $U = \{\bar{u}\}$ to be a singleton; and,

- similiar to our approach, the authors work with the ideal point based relative regret as

$$s_i(x,\bar{u}) = \frac{f_i(x,\bar{u}) - f_i^*(\bar{u})}{f_i^*(\bar{u})},$$

for each objective function $f_1(x,\bar{u}),\dots,f_m(x,\bar{u})$ individually.
- By applying a specific Chebyshev scalarization with weight vector w equal to the all-ones vector and using the ideal point 0 (which is indeed the ideal point for s in the given setup, whereas 0 becomes a utopian point in the uncertain setup), the authors finally suggest to consider the single-objective robust optimization problem

$$\min_{x\in X} \max_{i=1,\dots,m} s_i(x,\bar{u}).$$

- The authors do not consider different weight vectors and thus obtain only a single weakly efficient point.[8]

The insight that this approach indeed yields a weakly efficient solution is of course only possible within our setup and thus not discussed by the authors.

5.2. Connection to Xidonas et al. (2017)

An approach, which is in our opinion closer to ours than the one of Drezner et al. (2006) is due to Xidonas et al. (2017). It can be seen as a second step, adding uncertainty to the approach by Drezner et al. However, instead of considering general uncertain setups, in Xidonas et al. (2017) the authors focus on the special case where the uncertainty set is a finite set of scenarios $U=\{u_1,\dots,u_p\}$; this setup is later extended to ellipsoidal uncertainty in Li and Wang (2020). Furthermore, our understanding of the authors' framework is that they also focus solely on linear multiobjective optimization problems. Xidonas et al. apply the weighted sum scalarization technique prior to the regret formulation in order to work with the single-criterion regret. To be more precise,

- the authors consider the same approach as in Drezner et al. (2006), except for the scalarization technique. For each i, let $\tilde{s}_i(x,u)$ again denote the individual regret as introduced in Remark 2.2. Then the corresponding scalarized single-objective function reads as

$$\sum_{i=1}^m \lambda_i \tilde{s}_i(x,u).$$

Applying the classical robust counterpart to this objective function, they finally introduce the *robust scalarized relative regret* optimization problem:

$$\min_{x\in X} \max_{u\in U} \sum_{i=1}^m \lambda_i \tilde{s}_i(x,u).$$

- To obtain different *optimal* solutions of the single-objective formulation, they suggest to vary the scalarization parameter λ.
- As the authors do not discuss any (robust regret) multiobjective formulation or solution concept thereof, they simply take the single-objective optimal solutions as solutions to the original question. This is in strong contrast to our approach which starts by introducing a corresponding multiobjective formulation of robust regret together with the usual concepts of (weakly) efficient solutions.

Without further strong assumptions on the uncertainty (like e.g. separability) it cannot be expected that any solution obtained by the authors' approach is (weakly) efficient in our setting.

[8] It needs to be mentioned that one particular weakly efficient solution was sufficient for the specific application in Drezner et al. (2006).

5.3. Connection to Rivaz and Yaghoobi (2013, 2018); Rivaz et al. (2016)

Finally, the following approach by Rivaz et al. in Rivaz and Yaghoobi (2013) is again motivated by a priori scalarization. For convenience, we have translated their maximization approach (with corresponding adjusted definition of regret) to our notation.

- The approach in Rivaz and Yaghoobi (2013) is similar to that of Drezner et al., but focuses on a linear multiobjective optimization problem with interval uncertainty for the coefficients of the objective functions. In Rivaz and Yaghoobi (2013), the authors aim to find *necessarily / possibly efficient solutions* of the linear multiobjective optimization problem under interval uncertainty.
- Translated to our notation, the authors consider the same definitions as we do, but instead of first setting up a multiobjective regret optimization problem, the authors directly start with a scalarized formulation:

$$\min_{x\in X} \max_{u\in U} \max_{1\le i\le m} r_i(x,u).$$

This represents the minimization of the robustified scalarized regret, where a Chebyshev scalarization with ideal point 0 (for $r(x,u)$) has been used together with the weight vector of all-ones.
- Since for the Chebyshev scalarization we can swap robustification and scalarization (see Proposition 2.4), this can be equivalently reformulated as

$$\min_{x\in X} \max_{1\le i\le m} \max_{u\in U} r_i(x,u). \tag{7}$$

The same observation was made by the authors in Rivaz and Yaghoobi (2013), however without explicitly noting that this indeed allows to swap the order of robustification and scalarization in general, as observed in Section 2.3.

In (7) the authors first robustify then scalarize by Chebyshev scalarization, however, without referring to a corresponding multiobjective regret optimization setup. Nevertheless, Problem (7) can be seen as another step towards introducing a kind of *scalarized robust regret* for the first time, of course limited to the special setting considered in Rivaz and Yaghoobi (2013).

The same uncertain multiobjective optimization problem as in Rivaz and Yaghoobi (2013) is considered in the two follow-up papers (Rivaz & Yaghoobi, 2018; Rivaz et al., 2016), with the same aim of identifying possibly or necessarily efficient solutions. Within this setup, the authors introduce weights in Rivaz et al. (2016) and suggest a robustified weighted Chebyshev scalarized[9] regret formulation. Based on Proposition 2.4, we can now recognize that their formulation is actually equivalent to a weighted Chebyshev scalarized multiobjective robust regret formulation. In their most recent paper (Rivaz & Yaghoobi, 2018), they start with a swapped order of scalarization and robustification and replace the robustified weighted Chebyshev scalarized regret by a weighted-sum scalarized robust regret (see their Eq. (10)), i.e. in our notation a weighted-sum scalarized multiobjective robust regret. In both Rivaz and Yaghoobi (2018); Rivaz et al. (2016), the authors provide sufficient conditions that optimal solutions to their single-objective optimization problems are indeed possibly / necessarily (weakly) efficient solutions. As the connection to a multiobjective robust regret formulation has not been recognized in Rivaz and Yaghoobi (2018); Rivaz et al. (2016), no statement concerning (weak) efficiency could be made.

[9] As reference point for the scalarization, some arbitrary image point is used instead of the ideal or a utopian point. Therefore, the authors also slightly modify the regret function.

Note that several of these results, more precisely, Rivaz et al. (2016), Theorems 3, 5 and 6, as well as Rivaz and Yaghoobi (2018), Theorems 4.1, 4.5 and 4.6 become straightforward corollaries in our setup. Connections are thus established between possibly / necessarily (weakly) efficient solutions and our multiobjective robust regret framework. Further, it becomes apparent that the mentioned theorems just differ in the scalarization techniques, thus allowing for a unification of these results. Finally, this also shows that their framework can be easily generalized to the non-linear setup.

6. An illustrative example

To illustrate the techniques and theoretical aspects covered in this paper, we consider an illustrative example motivated by the routing of airplanes subject to uncertain flight costs on the one hand and to uncertain weather conditions on the other hand. This setting was introduced in Kuhn and Raith (2010) and investigated numerically in Kuhn, Raith, Schmidt, and Schöbel (2016).

In this specific application, a detailed description of the route the plane will take needs to be determined a priori. The routing is driven by a trade-off between efficiency and risk, due to possible turbulence or hazardous weather conditions. Therefore the aircraft route guidance problem can be interpreted as a biobjective shortest path problem on a network representing a discretized airspace. More specifically, we consider a graph $G(V, E)$, with vertices $v \in V$ representing grid cells in a discretized airspace and edges $e \in E$ representing potential flight routes between these grid cells within the network. We highlight two vertices: the origin $s \in V$ (source) and the destination $t \in V$ (sink). For the uncertainty set $U \subset \mathbb{R}^{|E|}$ let us introduce the corresponding uncertain functions $c : E \times U \to \mathbb{R}_{\geq}$ for the costs and $w : E \times U \to \mathbb{R}_{\geq}$ for the uncertain *weather exposure* for all connections within the network, as introduced in Kuhn et al. (2016). Further, let $u \in U$ be some fixed scenario and let x denote the decision variable, i.e. some path from s to t. The total costs $c(x, u)$ and the weather exposure $w(x, u)$ for path x in scenario u are then given as

$$c(x,u) = \sum_{e\in x} c(e,u) \quad \text{and} \quad w(x,u) = \sum_{e\in x} w(e,u).$$

With X denoting the finite set of all paths from s to t, we obtain the uncertain biobjective shortest path problem:

$$\min_{\substack{\mathbb{R}^2_{\geq} \\ x\in X}} \begin{pmatrix} c(x,u) \\ w(x,u) \end{pmatrix}.$$

The multiobjective robust regret formulation now reads as follows, cf. $(RR^{(m)}(U))$:

$$\min_{\substack{\mathbb{R}^2_{\geq} \\ x\in X}} R(x) = \min_{\substack{\mathbb{R}^2_{\geq} \\ x\in X}} \begin{pmatrix} \max_{u\in U} c(x,u) - c^*(u) \\ \max_{u\in U} w(x,u) - w^*(u) \end{pmatrix}.$$

For illustration purposes, let us specify a toy instance of this problem, i.e. we consider the graph displayed in Fig. 2 together with its edge labels c and w, which are chosen to be linear in $u \in U \subseteq \mathbb{R}^2$.

6.1. Finite set of scenarios

First, we consider a finite set of scenarios, e.g. let $U = \{(-1,0), (1,0), (0,-1), (0,1)\}$. As a first step, we compute the optimal values $c^*(u) = \min_{x\in X} c(x,u)$ and $w^*(u) = \min_{x\in X} w(x,u)$ for every $u \in U$. The results as well as the corresponding optimal solutions are collected in Table 1. In our example, the set X consists of all 13 paths from s to t, denoted as

Table 1
Optimal solutions and optimal values for c and w for the example in Fig. 2, with U = {(-1, 0), (1, 0), (0,-1), (0, 1)}.

u	(−1, 0)	(1,0)	(0, −1)	(0,1)
$c^*(u)$	3	19	7	14
x^* for c	$s-f-c-d-t$	$s-b-t$	$s-f-c-d-t$	$s-f-c-t$
$w^*(u)$	5	8	3	10
x^* for w	$s-a-c-t$	$s-b-t$	$s-f-b-t$	$s-a-c-t$

$$\begin{aligned} X &= \{x_1, x_2, \dots, x_{13}\} \\ &:= \{(s-a-c-d-t), (s-a-c-t), (s-a-d-t), \\ &\quad (s-f-c-t), (s-f-c-d-t), (s-f-b-c-d-t), \\ &\quad (s-f-b-c-t), (s-f-b-t), (s-f-b-e-t), \\ &\quad (s-b-t), (s-b-e-t), (s-b-c-d-t), (s-b-c-t)\} \end{aligned}$$

For the worst case regret, we now compute $R_c(x) := \max_{u\in U}\{c(x,u) - c^*(u)\}$ as well as $R_w(x) = \max_{u\in U}\{w(x,u) - w^*(u)\}$ for every $x \in X$. The results are collected in Table 2. The set $\{(R_c(x), R_w(x)) \mid x \in X\}$ as well as the non-dominated points are illustrated in Fig. 3 and were obtained by direct comparison. We obtain $Y^N_{R(\cdot,U)} = \{(5,17), (7,13), (9,7), (10,5)\}$ and the corresponding efficient paths $x_3 = (s-a-d-t)$, $x_5 = (s-f-c-d-t)$, $x_4 = (s-f-c-t)$ and $x_{10} = (s-b-t)$.

6.2. Polytopal uncertainty set

As a second step, we now consider the polytopal uncertainty set $U = \{u \in \mathbb{R}^2 \mid \|u\|_1 \leq 1\}$. Since c and w are linear in u for every path, it is sufficient to analyse the vertices $\{(-1,0), (1,0), (0,-1), (0,1)\}$ due to Proposition 3.8. Thus, we again obtain the results in Table 1. Moreover, the values $R_c(x)$ and $R_w(x)$ for every $x \in X$ are exactly those collected in Table 2 while Fig. 3 again shows the resulting non-dominated points.

6.3. Convex uncertainty set and inner and outer polytopal approximation

In the third step, we consider a more general, i.e. convex, uncertainty set, e.g. $U := \{u \in \mathbb{R}^2 \mid \|u\|_2 \leq 1\}$. Accordingly, we choose the set $U^i := \{u \in \mathbb{R}^2 \mid \|u\|_1 \leq 1\}$ as a polyhedral inner approximation of U, and $U^o := \{u \in \mathbb{R}^2 \mid \|u\|_\infty \leq 1\}$ as a polyhedral outer approximation of U. For these specific approximations, the quality of the approximation is still rather low, as the computation of the corresponding Hausdorff distances shows:

$$d_H(U^i, U) = d_H(conv\{(-1,0), (1,0), (0,-1), (0,1)\}, \{u \mid \|u\|_2 \leq 1\}) \approx 0.29$$
$$d_H(U^o, U) = d_H(conv\{(-1,-1), (-1,1), (1,1), (1,-1)\}, \{u \mid \|u\|_2 \leq 1\}) \approx 0.41.$$

As can be seen in Table 2, the set of non-dominated points for the inner approximation reads as $Y^N_{R(\cdot,U^i)} = \{(5,17), (7,13), (9,7), (10,5)\}$. To obtain the set of non-dominated points for the outer approximation, we proceed analogously and obtain $c^*(u) = \min_{x\in X} c(x,u)$, $w^*(u) = \min_{x\in X} w(x,u)$ and the corresponding optimal solutions for every $u \in V(U^o) := \{(-1,-1), (-1,1), (1,1), (1,-1)\}$. The result can be found in Table 3. The values $R^o_c(x) := \max_{u\in U^o}\{c(x,u) - c^*(u)\}$ as well as $R^o_w(x) = \max_{u\in U^o}\{w(x,u) - w^*(u)\}$ for every $x \in X$ based on the outer approximation $U^o = \left\{u \in \mathbb{R}^2 \mid \|u\|_\infty \leq 1\right\}$ of U are collected in Table 4. We finally get $Y^N_{R(\cdot,U^o)} = \{(7,20), (11,12), (14,7)\}$ with corresponding efficient paths $x_3 = (s-a-d-t), x_8 = (s-f-b-t)$ and $x_4 = (s-f-c-t)$. The set $\{(R^o_c(x), R^o_w(x)) \mid x \in X\}$ as well as the non-dominated points are illustrated in Fig. 4. According to Proposition 3.5 which yields

$$Y^N_{R(\cdot,U)} \subset \left(\left(Y^N_{R(\cdot,U^i)} + \mathbb{R}^m_{\geq}\right) \setminus \left(Y^N_{R(\cdot,U^o)} + \mathbb{R}^m_{>}\right)\right) \cap B_R(U^i, U^o),$$

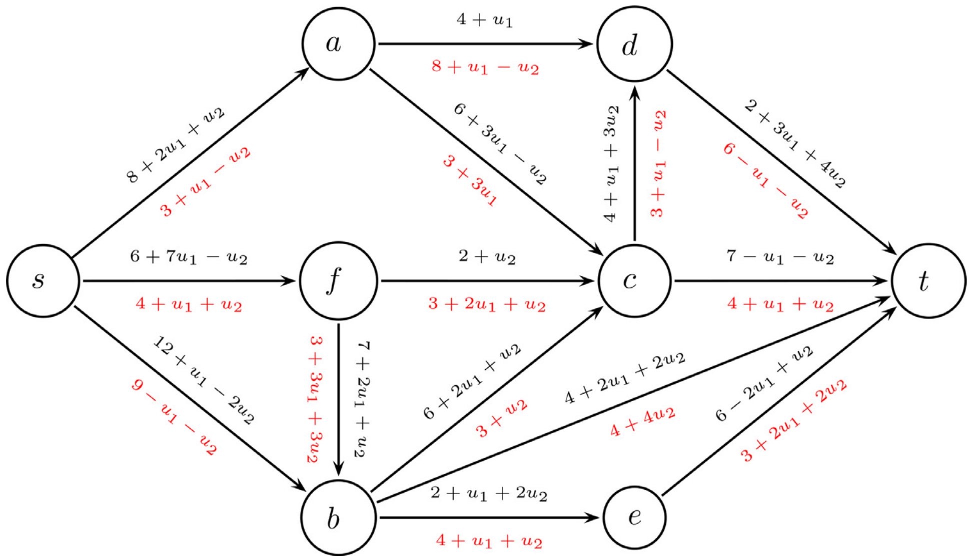

Fig. 2. Illustrative example graph for the biobjective shortest path problem. The cost function is displayed as an edge label in black and the weather exposure as an edge label in red. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Table 2
The values $R_c(x)$ and $R_w(x)$ for every $x \in X$, based on $U = \{(-1,0),(1,0),(0,-1),(0,1)\}$ and the optimal values in Table 1. Bold values denote non-dominated points.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}
$R_c(\cdot)$	13	15	**5**	**9**	**7**	21	19	9	11	**10**	17	16	20
$R_w(\cdot)$	15	7	**17**	**7**	**13**	15	11	9	13	**5**	7	16	8

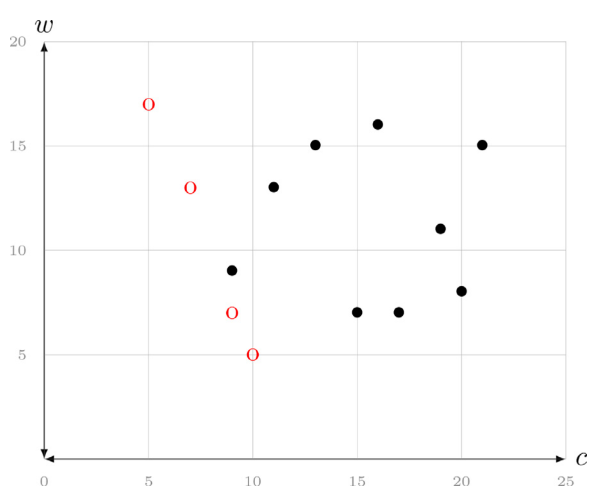

Fig. 3. The dominated points $\{(R_c(x), R_w(x)) \mid x \in X\}$ marked by black dots and the non-dominated points $Y^N_{R(\cdot,U)}$ illustrated by red circles. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Table 3
Optimal solutions and optimal values for the outer approximation $U^o = \{u \in \mathbb{R}^2 \mid \|u\|_\infty \le 1\}$.

u	$(-1,-1)$	$(-1,1)$	(1,1)	$(1,-1)$
$c^*(u)$	-4	8	19	15
x^* for c	$s-f-c-d-t$	$s-f-b-t$	$s-b-t$	$s-a-d-t$
$w^*(u)$	-1	5	11	5
x^* for w	$s-f-b-t$	$s-a-c-t$	$s-b-t$	$s-b-t$

we can determine a region where the true non-dominated set of the robust regret optimization problem has to be located. The corresponding visualization of this region can be found in Fig. 5. Successively improving the quality of the inner and outer approximation then enables us to identify the true non-dominated set: Since the assumptions in Theorem 3.12 are satisfied, the volume of the sandwiching set shrinks to 0. Moreover, since the mapping $x \mapsto R(x,U)$ is bijective, we can even identify the corresponding efficient paths. The true efficient paths and the corresponding non-dominated points eventually read as $[x_{10}, (12.63, 5.71)]$, $[x_4, (10.43, 7)]$, $[x_8, (9.24, 9.06)]$, $[x_5, (8.63, 13.06)]$, $[x_3, (5.38, 17.39)]$.

7. Conclusion and outlook

We have introduced a novel consistent framework for multiobjective robust regret, which can be seen as an extension of early approaches in Drezner et al. (2006), Rivaz and Yaghoobi (2013) and Xidonas et al. (2017). In contrast to these, our framework is not limited to linear objective functions and/or finite uncertainty sets, or linear interval uncertainty. Furthermore, we

Table 4
$R^o_k(x)$ for every $k \in \{c,w\}$ and $x \in X$ based on the outer approximation $U^o = \{u \in \mathbb{R}^2 \mid \|u\|_\infty \le 1\}$ of U. Bold values denote non-dominated points.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}
$R^o_c(\cdot)$	17	22	**7**	**14**	13	29	21	**11**	14	17	23	18	29
$R^o_w(\cdot)$	17	10	**20**	**7**	14	15	14	**12**	17	8	9	21	12

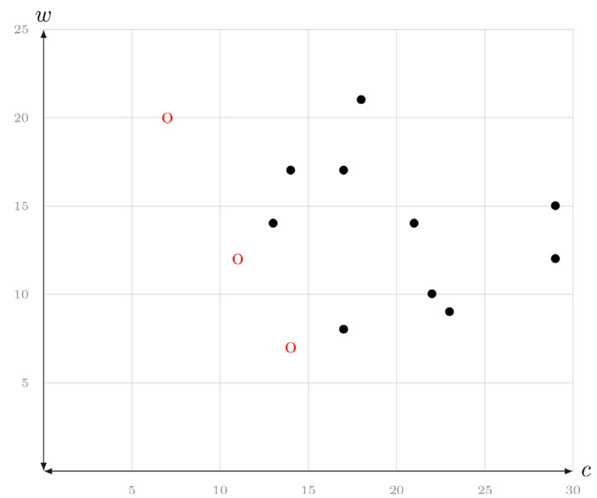

Fig. 4. The dominated points in $\{(R^o_c(x), R^o_w(x)) \mid x \in X\}$ marked by black dots and the corresponding non-dominated points $Y^N_{R(\cdot,U^o)}$ illustrated by red circles. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

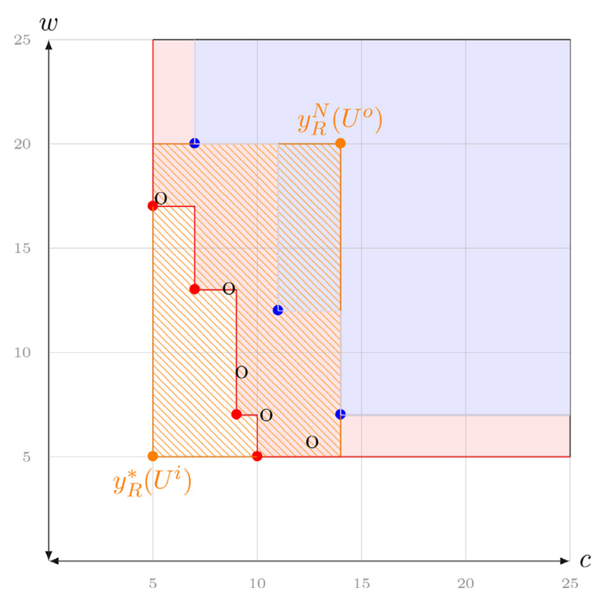

Fig. 5. Illustrating the set $Y^N_{R(\cdot,U)}$ (black circles) as a subset of the sandwiching set $\left(\left(Y^N_{R(\cdot,U^i)} + \mathbb{R}^m_{\geq}\right) \setminus \left(Y^N_{R(\cdot,U^o)} + \mathbb{R}^m_{\geq}\right)\right) \cap B_R(U^i, U^o)$.

consistently work in a multiobjective setting and introduce a multivariate (relative) regret based on a clear separation between problem modelling and problem solution, whereas the earlier mentioned approaches first scalarize the optimization problem to be able to apply the concept of single criterion robust regret. In addition, we observe that Chebyshev scalarization actually commutes with robustification in the context of this paper. For the multiobjective regret framework, we gain several interesting insights concerning continuity of the objective functions with respect to the uncertainty set; results which are also valid within the classical framework of (pointwise) robust multiobjective programming. We especially analyze the impact of the uncertainty on numerical tractability by investigating all common cases for the uncertainty. For approximations of the non-dominated set in the case of general convex uncertainty sets, we introduce inner and outer polytopal approximations. Finally, we show that the effort to compute reasonable approximations to the set of non-dominated points for general convex U grows linearly with the number of vertices of the polytopal approximation to U in the finite setting and can still be computed in polynomial time in the continuous setting.

Finally, we would like to suggest some further research opportunities. From a modeling perspective, first of all, alternatives to the pointwise approach for robust multiobjective optimization need to be considered, cf. Section 1.1 for a list of alternative concepts. Second, set valued concepts for $f^*(u)$ could be investigated, e.g. by replacing the ideal point $f^*(u)$ by the set $Y^N_{f(\cdot,u)}$. Third, our analysis is focused on multiobjective optimization, i.e. vector optimization with ordering cone $\mathbb{R}^m_{\geq}$. It is currently open how regret could be extended to a general vector optimization problem under uncertainty. Last but not least, the framework of multiobjective stochastic regret as for instance considered in Xu et al. (2020) also constitutes a promising field for future research.

From an algorithmic point of view, it currently remains open which approach is the most suitable one for general convex uncertainty. As promising alternatives to inner and outer polytopal approximations, we have mentioned semi-infinite programming and the improved inner approximating, which both deserve further investigations in our context.

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