Randomly repeated measurements on quantum systems: correlations and topological invariants of the quantum evolution

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Abstract
Randomly repeated measurements during the evolution of a closed quantum system create a sequence of probabilities for the first detection of a certain quantum state. The related discrete monitored evolution for the return of the quantum system to its initial state is investigated. We found that the mean number of measurements (MNM) until the first detection is an integer, namely the dimensionality of the accessible Hilbert space. Moreover, the mean first detected return (FDR) time is equal to the average time step between successive measurements times the MNM. Thus, the mean FDR time scales linearly with the dimensionality of the accessible Hilbert space. The main goal of this work is to explain the quantization of the mean return time in terms of a quantized Berry phase.

Keywords: dynamical invariants, randomly repeated measurements, monitored quantum evolution

1. Introduction

The unitary evolution (UE) of a closed quantum system from the initial state $|\Psi\rangle$ to the state $|\Psi(\tau)\rangle$ on the time interval τ is defined by $|\Psi(\tau)\rangle = \exp(-iH\tau/\hbar)|\Psi\rangle$, where H is the Hamiltonian of the system. The result of this evolution is characterized by the overlap amplitude

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$\langle\psi|\Psi(\tau)\rangle$ with respect to a given state $|\psi\rangle$. Then $|\langle\psi|\Psi(\tau)\rangle|^2$ is the probability that the evolution has reached the state $|\psi\rangle$, which obviously depends on the time τ. In a single experiment the measurement time is fixed and, therefore, the UE allows us to detect the state $|\psi\rangle$ with probability $|\langle\psi|\Psi(\tau)\rangle|^2$ only once. The detection of this state at different times would require the repetition of the experiment, prepared in the same initial state $|\Psi\rangle$, for different values of τ [1]. An alternative approach, which we call 'monitored evolution (ME)' is to allow the system to evolve from the initial state $|\Psi\rangle$ for the time τ_1 and then measure whether or not the system is in the state $|\psi\rangle$. If the answer is 'yes', we stop the experiment, if the answer is 'no' we allow the system to evolve further after the measurement [2] and perform a second measurement at time $\tau_1+\tau_2$. This procedure is repeated for times $t_2=\tau_1+\tau_2, \ldots, t_k=\tau_1+\cdots+\tau_k$ until the measurement detects the state $|\psi\rangle$ for the first time. We consider cases in which the state is detected with probability one. This is called a recurrent measurement process [1]. If at time t_1 the outcome of the measurement is null this can be associated with a projection of the quantum system $(\mathbf{1}-|\psi\rangle\langle\psi|)|\Psi(t_1)\rangle$ with a subsequent normalization of the resulting state [3], and similarly for t_2 etc. And when the state is detected for the first time at time t_k, the amplitude of the corresponding state is ϕ_k and its probability is $|\phi_k|^2$ [4, 5].

For fixed time steps τ with $t_k=k\tau$ the repeated measurement approach, also known as the stroboscopic protocol, has been studied in great detail in references [1, 3–19]. Two cases have been distinguished, the return probability for $|\psi\rangle=|\Psi\rangle$ and the transition probability for $|\psi\rangle=|\Psi\rangle$. It was found that (i) the return and the transition probabilities differ qualitatively, (ii) the average first detected return (FDR) time is quantized and given by the winding number of the Laplace transform of the return amplitude [1], (iii) near degeneracies of the spectrum of the evolution operator, the fluctuations become very large and diverge at the degeneracies [3, 11, 18], and (iv) the average first detected transition time also diverges at the degeneracies [20].

Our main intention is to present in this article a study of the effect of independent and identically distributed random time steps $\{\tau_k\}$ on the ME. For this purpose we want to answer the following questions: (1) do we still obtain quantization of the number of attempts for the first successful measurement? (2) Is the average mean time for the first detection also quantized, as found for stroboscopic measurements? (3) How can dynamical quantization be related to topological invariants? (4) Do random time steps affect the divergent fluctuations near resonances? To answer these questions we will develop a theory of the first detection time under repeated random measurements. Our work is based on the ideas presented by Grünbaum *et al* [9] for fixed time steps and related to a work by Varbanov *et al* [21] on random time steps, who studied the conditions for the existence of non-detectable dark states. Random time measurements were discussed also for open quantum systems [22], while for closed quantum systems they were recently studied by us in terms of average return and transition probabilities [23]. The present work is an extension of the latter in which we explain in detail the origin of the quantization of the mean return time. For this purpose we compare the Berry phase of the return amplitude, averaged with respect to the distribution of the measurement times. It will be shown that the average Berry phase is equal to the average number of measurements for the FDR and equal to the dimensionality of the accessible Hilbert space. The average Berry phase is reminiscent of the quantized winding number in case of stroboscopic measurements [9].

The structure of this article is as follows. In section 2 the definitions of the relevant quantities for the first detection under repeated measurements are given. Then a short summary of the main results are presented in section 3. After a brief discussion of the ME for fixed time steps τ in section 4, we formulate the ME for random time steps in terms of random matrix products in sections 5–6.2.1. This includes a description of the averaging procedure (section 6) and the introduction of a generating function for the average return time and its higher moments (section 6.1). The quantization of the mean number of measurements (MNM) of the FDR by the dimensionality of the accessible Hilbert space is studied in sections 6.2 and 6.2.1. Finally, in section 7 the example of a symmetric two-level system is analyzed, followed by a discussion of all the results in section 8. Details of the calculations are presented in appendices A–F.

2. Return amplitude of the ME

The return probability as a function of time, with or without intermediate measurements, provides a measure of how big the accessible space is and how long it takes to return to the initial state. This is an important quantity for classical random walks [24, 25] and plays also an important role for characterizing localization in many-body quantum systems [26, 27]. We will investigate the return amplitude for the ME with measurements at time steps $\{\tau_k\}_{k=1,2,\ldots}$. First, we will have need to refer to the return amplitude of the UE

$$u_k = \langle\Psi|\mathrm{e}^{-\mathrm{i}H(\tau_1+\cdots+\tau_k)}|\Psi\rangle, \tag{1}$$

for the state $|\Psi\rangle$ when we measure only once after the time $t_k = \tau_1 + \cdots + \tau_k$, assuming that the UE is governed by the Hamiltonian H. Then we turn to the return amplitude for the ME [3, 4, 13]

$$\phi_k = \langle\Psi|\mathrm{e}^{-\mathrm{i}H\tau_k}(P\mathrm{e}^{-\mathrm{i}H\tau_{k-1}})\ldots(P\mathrm{e}^{-\mathrm{i}H\tau_1})|\Psi\rangle, \quad P = \mathbf{1} - |\Psi\rangle\langle\Psi|, \tag{2}$$

which is the major objective of our inquiry. This is the return amplitude at time t_k, provided that we also measure at times $t_1, t_2, \ldots, t_{k-1}$ but detect the quantum state $|\Psi\rangle$ for the first time at t_k with probability $|\phi_k|^2$. For the time of measurements we assume a distribution density $\mathcal{P}(\{\tau_k\})$ of independent and identically distributed time steps, such that $\mathcal{P}(\{\tau_k\}) = \prod_k P(\tau_k)$. This can be understood as the effect of an inaccurate clock. This enables us to consider the average $\langle\cdots\rangle_\tau \equiv \int\cdots\prod_k\mathcal{P}(\tau_k)\mathrm{d}\tau_k$ with respect to the ensemble of random time steps. Here we do not specify the distribution. For instance, $\mathcal{P}(\tau_k)$ can be a Dirac delta, which would recover the stroboscopic ME [3, 9, 13, 18], it could be an exponential distribution or any other distribution, which allows us to perform the average $\langle\cdots\rangle_\tau$.

Besides the averaging with respect to times steps $\{\tau_k\}$ we also need to define the averages

$$\overline{k^m} := \sum_{k\geqslant 1} k^m\langle|\phi_k|^2\rangle_\tau, \qquad \overline{t^m} := \sum_{k\geqslant 1}\langle t_k^m|\phi_k|^2\rangle_\tau \quad (m = 1, 2, \ldots). \tag{3}$$

Here it is assumed that $\sum_k |\phi_k|^2 = 1$, which is justified for a finite-dimensional Hilbert space (cf appendix A). This equation means that the state is eventually detected [9]. Thus, the overline represents a double average, namely an average with respect to the number of time steps $k = 1, 2, \ldots$ with weight $|\phi_k|^2$, followed by the average $\langle\cdots\rangle_\tau$. For $m = 1$ both expressions in equation (3) will be used to characterize the ME: $\overline{k}$ is the MNM for the FDR and $\overline{t}$ is the mean FDR time.

Since both evolutions are defined on an N-dimensional Hilbert space by the Hamiltonian H, we consider its eigenstates $\{|E_j\rangle\}_{j=1,\ldots,N}$ and its corresponding eigenvalues $\{E_j\}_{j=1,\ldots,N}$. Then the return amplitude ϕ_k of the ME can be expressed as a matrix product, which is efficiently written in the energy representation as a sum over all energy levels $\{E_j\}_{j=1,2,\ldots,N}$ as

$$\phi_k = \sum_{j_1,j_2,\ldots,j_k=1}^{N} \langle\Psi|E_{j_k}\rangle\mathrm{e}^{-\mathrm{i}E_{j_k}\tau_k}\langle E_{j_k}|P|E_{j_{k-1}}\rangle\mathrm{e}^{-\mathrm{i}E_{j_{k-1}}\tau_{k-1}}\ldots\langle E_{j_2}|P|E_{j_1}\rangle\mathrm{e}^{-\mathrm{i}E_{j_1}\tau_1}\langle E_{j_1}|\Psi\rangle. \tag{4}$$

In principle, there is the possibility of degenerate eigenvalues or of a vanishing overlap $p_j = |\langle E_j|\Psi\rangle|^2$, which must be treated with care [9]. To understand the effect of degenerate energy levels on ϕ_k we consider the levels E_j, $E_{j'}$ and assume at first that they are not degenerate. Then we can write

$$\sum_{j_k=1}^{N} \langle E_{j_{k-1}}|P|E_{j_k}\rangle e^{-iE_{j_k}\tau_k}\langle E_{j_k}|P|E_{j_{k+1}}\rangle = \sum_{j_k=1, j_k=j,j'}^{N} e^{-iE_{j_k}\tau_k}\langle E_{j_{k-1}}|P|E_{j_k}\rangle\langle E_{j_k}|P|E_{j_{k+1}}\rangle$$

$$+ \langle E_{j_{k-1}}|P[e^{-iE_j\tau_k}|E_j\rangle\langle E_j| + e^{-iE_{j'}\tau_k}|E_{j'}\rangle\langle E_{j'}|]P|E_{j_{k+1}}\rangle,$$

where the second term on the right-hand side describes the evolution of the state $P|E_{j_{k+1}}\rangle$ with the evolution operator

$$P[e^{-iE_j\tau_k}|E_j\rangle\langle E_j| + e^{-iE_{j'}\tau_k}|E_{j'}\rangle\langle E_{j'}|]$$

over the time period τ_k. The evolution creates a superposition of the states $|E_j\rangle$ and $|E_{j'}\rangle$, which changes in time due to the time dependent coefficients, provided that $E_j = E_{j'}$. On the other hand, for the degenerate case $E_j = E_{j'}$, the superposition of $|E_j\rangle$ and $|E_{j'}\rangle$ is fixed during the time period τ_k:

$$e^{-iE_j\tau_k}P[|E_j\rangle\langle E_j| + |E_{j'}\rangle\langle E_{j'}|]$$

and only the global phase changes. This reflects a dimensional reduction of the accessible Hilbert space by 1, implying that we should use the replacement

$$|E_j\rangle, |E_{j'}\rangle \to |E_{jj'}\rangle := |E_j\rangle\langle E_j|\Psi\rangle + |E_{j'}\rangle\langle E_{j'}|\Psi\rangle$$

and the simultaneous elimination of j' from the summation of j_k in equation (4).

Another special case is a vanishing overlap $p_j = |\langle E_j|\Psi\rangle|^2 = 0$. Beginning with the initial state on the right-hand side of equation (4) we get

$$\sum_{j_1=1}^{N} \langle E_{j_2}|P|E_{j_1}\rangle e^{-iE_{j_1}\tau_1}\langle E_{j_1}|\Psi\rangle = \sum_{j_1=1; j_1=j}^{N} \langle E_{j_2}|P|E_{j_1}\rangle e^{-iE_{j_1}\tau_1}\langle E_{j_1}|\Psi\rangle.$$

Next, in the summation with respect to j_2 the special value j does not contribute again, since $j_2 = j$ gives

$$\langle E_j|P|E_{j_1}\rangle = \langle E_j|E_{j_1}\rangle - \langle E_j|\Psi\rangle\langle\Psi|E_{j_1}\rangle = 0$$

due to $j_1 = j$ and due to $\langle E_j|\Psi\rangle = 0$. Repeating this argument we find that the value j drops out of all summations in equation (4), reducing the accessible Hilbert space by 1 again.

With these arguments we have removed the degeneracy of the energy levels and the vanishing overlaps p_j. Therefore, the remaining return amplitude ϕ_k depends only non-degenerate energy levels and on overlaps with $p_j > 0$. For the subsequent analysis we use the convention that N is the dimensionality of the accessible Hilbert space.

For the subsequent calculations it is useful to introduce two types of discrete Fourier transformations, where one is based on the phase factors $e^{i\omega k}$

$$\tilde{\phi}(\omega) = \sum_{k\geqslant 1} e^{i\omega k}\,\phi_k \tag{5}$$

Table 1. Comparison between stroboscopic ME (SME) and random ME (RME) with the Hilbert space dimensionality N. The FDR agrees for both approaches, provided that we average over the random measurements.

	SME	RME
Probability $\lvert\tilde{\phi}(\omega)\rvert^2$	1	Random
Probability $\langle\lvert\tilde{\phi}(\omega)\rvert^2\rangle_\tau$	—	1
Winding number w of $\tilde{\phi}(\omega)$	N	Random
Winding number $\langle w\rangle_\tau$ of $\tilde{\phi}(\omega)$	—	N
Mean number of measurements for FDR $\bar{k}$	N	N
Mean FDR time $\bar{t}$	$N\tau$	$N\langle\tau\rangle_\tau$

and the other is based on the random phase factors $e^{i\omega(\tau_1+\cdots+\tau_k)} \equiv e^{i\omega t_k}$

$$\tilde{\phi}_\tau(\omega) = \sum_{k\geqslant 1} e^{i\omega t_k}\,\phi_k, \tag{6}$$

provided that these series exist. Both $\tilde{\phi}(\omega)$ and $\tilde{\phi}_\tau(\omega)$ are still functions of the random variables $\{\tau_k\}$.

3. Summary and results

It will be shown that for a quantum system with energy levels $\{E_j\}$ and eigenstates $\{|E_j\rangle\}$ of a given Hamiltonian H the FDR probability $|\phi_k|^2$ in the case of an ME is determined by the random phase factors $\{e^{-iE_j\tau_k}\}$ and the overlaps $\{|\langle E_j|\Psi\rangle|^2\} \equiv \{p_j\}$ alone. After averaging with respect to the random measurements we get $\langle|\phi_k|^2\rangle_\tau$, which will turn out to be a function of the N parameters $\{\langle e^{-iE_j\tau}\rangle_\tau\}$ and of the N^2 parameters $\{\langle e^{-i(E_j-E_{j'})\tau}\rangle_\tau\}$. (Details are given in section 5.) Using the Fourier transformations (5) and (6), we define the generating functions $F(\omega) = \sum_{k\geqslant 1} e^{ik\omega}\langle|\phi_k|^2\rangle_\tau$ and $F_\tau(\omega) = \sum_{k\geqslant 1}\langle e^{i\omega t_k}|\phi_k|^2\rangle_\tau$. By differentiation with respect to ω we get the MNM for the FDR and the mean FDR time as

$$\bar{k} = -i\partial_\omega F(\omega)|_{\omega=0}, \qquad \bar{t} = -i\partial_\omega F_\tau(\omega)|_{\omega=0}.$$

Then we will derive the relation $\bar{t} = \langle\tau\rangle_\tau\bar{k}$, where $\langle\tau\rangle_\tau$ is the mean time interval between successive measurements.

Besides the average FDR probability $\langle|\phi_k|^2\rangle_\tau$ we will also calculate the corresponding expressions of the Fourier transform $\tilde{\phi}(\omega)$, namely $\langle|\tilde{\phi}(\omega)|^2\rangle_\tau$. This will turn out to be 1, as shown in equation (33), which is essential for calculating the average Berry phase and the mean values of the FDR as

$$\bar{k} = N, \qquad \bar{t} = \langle\tau\rangle_\tau N. \tag{7}$$

The main result are listed in table 1.

4. Fixed time step τ

In this section we briefly recapitulate what is known about the FDR problem in the case of non-random τ to set the stage for our investigation of random time steps $\{\tau_k\}$. Stroboscopic measurement with $\tau_k = \tau$ has been studied extensively in the literature [3, 9, 11, 13, 14, 18].

Next we summarize relevant information from previous works, in particular, some results of reference [18].

At fixed τ the Laplace transformation for the return amplitude of the ME reads

$$\hat{\phi}(z) \equiv \sum_{k\geqslant 1} z^k \phi_k = \sum_{k\geqslant 1} z^k \langle \Psi | (\mathrm{e}^{-\mathrm{i}H\tau} P)^{k-1} \mathrm{e}^{-\mathrm{i}H\tau} | \Psi \rangle = \langle \Psi | (\mathrm{e}^{\mathrm{i}\tau H}/z - P)^{-1} | \Psi \rangle. \tag{8}$$

Due to the relation of equation (B3) in appendix B, we obtain for $K = \mathrm{e}^{\mathrm{i}\tau H}/z - \mathbf{1}$ the following identity

$$\langle \Psi | (K + |\Psi\rangle\langle\Psi|)^{-1} | \Psi \rangle = 1 - \frac{1}{1 + \langle \Psi | K^{-1} | \Psi \rangle}. \tag{9}$$

This identity is important because it allows us to represent the projector-dependent left-hand side by the expression $\langle \Psi | K^{-1} | \Psi \rangle$, which is diagonal in terms of the energy eigenstates of H and independent of the projector $|\Psi\rangle\langle\Psi|$. A corresponding identity exists in the case of random time steps, which will be central for our subsequent calculations.

With u_k of equation (1) we can write $\langle \Psi | K^{-1} | \Psi \rangle$ as the Laplace transform of u_k: $\langle \Psi | K^{-1} | \Psi \rangle = \sum_{k\geqslant 1} z^k u_k \equiv \hat{u}(z)$. Then $\hat{\phi}(z)$ in equation (8), together with equation (9), becomes

$$\hat{\phi}(z) = 1 - \frac{1}{1 + \hat{u}(z)}. \tag{10}$$

By analytic continuation to the unit circle $z \to \mathrm{e}^{\mathrm{i}\omega}$ we get $\hat{u}(z) \to \tilde{u}(\omega)$ and $\hat{\phi}(z) \to \tilde{\phi}(\omega)$. Since $\mathrm{Re}[\tilde{u}(\omega)] = -1/2$, the expression $\tilde{\phi}$ is unimodular:

$$\tilde{\phi}(\omega) = \frac{-1/2 + \mathrm{i}\,\mathrm{Im}[\tilde{u}]}{1/2 + \mathrm{i}\,\mathrm{Im}[\tilde{u}]} = -\frac{\tilde{u}}{\tilde{u}^*}, \tag{11}$$

such that we can write

$$\tilde{\phi}(\omega) = -\mathrm{e}^{2\mathrm{i}\arg[\tilde{u}(\omega)]} \equiv \mathrm{e}^{\mathrm{i}\varphi(\omega)}. \tag{12}$$

This result for fixed time steps indicates that the UE and the ME have the same phase change with ω except for a factor 2. The winding number of $\tilde{\phi}(\omega)$ around the unit circle (i.e. for $0 \leqslant \omega < 2\pi$) is identical with $\sum_{k\geqslant 1} k|\phi_k|^2$, and it is known that the winding number is equal to the dimensionality of the Hilbert space [9].

5. Matrix products

Our goal is to calculate the probability $|\phi_k|^2$ of the return amplitude ϕ_k of equation (4) for the general case of random time steps. We would expect that the calculations of the previous section can be extended to this situation. As we will see though it requires some additional steps to calculate quantities, such as the mean FDR time, that are averaged with respect to the random time steps. Our calculation starts with the matrix representation of the projector P of equation (2) in terms of energy eigenstates

$$\langle E_j | P | E_{j'} \rangle = \delta_{j,j'} - q_j q_{j'}^*, \quad q_j = \langle E_j | \Psi \rangle, \tag{13}$$

since the eigenstates are orthonormal: $\langle E_j | E_{j'} \rangle = \delta_{j,j'}$. This is automatically fulfilled for non-degenerate eigenvalues. The above expression is inserted in equation (4) and yields for the return amplitude a trace of a matrix product:

$$\phi_k = \mathrm{Tr}\left[D_k(\mathbf{1} - QEQ^*)D_{k-1}(\mathbf{1} - QEQ^*)\ldots D_2(\mathbf{1} - QEQ^*)D_1 QEQ^*\right]$$

with the $N \times N$ matrix E, whose elements are all 1, and with the diagonal matrices $D_k = \mathrm{diag}(\exp(-iE_1\tau_k), \exp(-iE_2\tau_k), \ldots, \exp(-iE_N\tau_k))$ and $Q = \mathrm{diag}(q_1, q_2, \ldots, q_N)$.

Now $QQ^* = \Pi$ is the diagonal matrix $\Pi = \mathrm{diag}(p_1, p_2, \ldots, p_N)$, which enables us to rewrite ϕ_k as

$$\phi_k = \mathrm{Tr}\,[D_k(\mathbf{1} - E\Pi)D_{k-1}(\mathbf{1} - E\Pi)\ldots D_2(\mathbf{1} - E\Pi)D_1 E\Pi]\,, \tag{14}$$

since Π and D_j as diagonal matrices commute. This means that ϕ_k depends only on the spectral weights $\{p_j\}$ through Π and on the energy levels $\{E_j\}$ through D_k.

For the calculation of the return probability we need the product of two traces

$$|\phi_k|^2 = \mathrm{Tr}\,[D_k(\mathbf{1} - E\Pi)D_{k-1}(\mathbf{1} - E\Pi)\ldots D_2(\mathbf{1} - E\Pi)D_1 E\Pi]^* \\ \times \mathrm{Tr}\,[D_k(\mathbf{1} - E\Pi)D_{k-1}(\mathbf{1} - E\Pi)\ldots D_2(\mathbf{1} - E\Pi)D_1 E\Pi]\,. \tag{15}$$

In order to express this product it is convenient to use the notation of the Kronecker product of matrices

$$\hat{A} = A_1 \times A_2$$

with the properties

$$\hat{A}\hat{B} = (A_1 \times A_2)(B_1 \times B_2) = A_1B_1 \times A_2B_2,\ \mathrm{Tr}(\hat{A}) = \mathrm{Tr}(A_1)\,\mathrm{Tr}(A_2),\ (A_1 \times A_2)^{-1} = A_1^{-1} \times A_2^{-1}. \tag{16}$$

The second identity, or trace 'disentanglement' relation, is relevant for equation (15). With the matrix $\hat{C} = (\mathbf{1} - E\Pi) \times (\mathbf{1} - E\Pi)$ it gives us

$$|\phi_k|^2 = \mathrm{Tr}(\hat{D}_k\hat{C}\ldots\hat{D}_2\hat{C}\hat{D}_1\hat{E}\hat{\Pi}) \tag{17}$$

with $\hat{E} = E \times E$, $\hat{\Pi} = \Pi \times \Pi$ and $\hat{D}_k = D_k^* \times D_k$. For the matrix elements we use the notation

$$[A \times B]_{ij,kl} = A_{ik}B_{jl}. \tag{18}$$

6. Averaging over the distribution of random time steps

In the previous section we obtained a random distribution of return probabilities $\{|\phi_k|^2\}$. Here we are interested in the mean values $\{\langle|\phi_k|^2\rangle_\tau\}$, the subsequent calculation of the time average is based on the fact that the random matrices $\{\hat{D}_k\}$ are statistically independent and identically distributed. Thus, from equation (17) we get

$$\langle|\phi_k|^2\rangle_\tau = \mathrm{Tr}(\langle\hat{D}_k\rangle_\tau\hat{C}\ldots\langle\hat{D}_2\rangle_\tau\hat{C}\langle\hat{D}_1\rangle_\tau\hat{E}\hat{\Pi}) = \mathrm{Tr}([\langle\hat{D}\rangle_\tau\hat{C}]^{k-1}\langle\hat{D}\rangle_\tau\hat{E}\hat{\Pi}). \tag{19}$$

With the matrices

$$\hat{\Gamma} = \langle D^*(\mathbf{1} - E\Pi) \times D(\mathbf{1} - E\Pi)\rangle_\tau = \langle D^* \times D\rangle_\tau(\mathbf{1} - E\Pi) \times (\mathbf{1} - E\Pi) = \langle\hat{D}\rangle_\tau\hat{C} \tag{20}$$

and

$$\hat{G} = \langle D^* E\Pi \times DE\Pi \rangle_\tau = \langle D^* \times D \rangle_\tau E\Pi \times E\Pi = \langle \hat{D} \rangle_\tau \hat{E}\hat{\Pi} \tag{21}$$

we obtain the compact expression

$$\langle |\phi_k|^2 \rangle_\tau = \mathrm{Tr}[\hat{\Gamma}^{k-1}\hat{G}] \ . \tag{22}$$

$\hat{\Gamma}$ depends on the averaged product $\langle D^* \times D \rangle_\tau$. The latter cannot be expressed as a Kronecker product, which prevents us also from applying the trace 'disentanglement' relation of equation (16). This reflects a robust 'entanglement' due to $\langle D^* \times D \rangle_\tau$. We will return to this fact in the next section.

6.1. The generating functions

First, from equation (22) we obtain, after a discrete Fourier transformation, the generating functions

$$F(\omega) = \sum_{k \geqslant 1} \mathrm{e}^{\mathrm{i}k\omega} \langle |\phi_k|^2 \rangle_\tau = \mathrm{e}^{\mathrm{i}\omega} \ \mathrm{Tr}[(\hat{\mathbf{1}} - \mathrm{e}^{\mathrm{i}\omega} \ \hat{\Gamma})^{-1}\hat{G}], \tag{23}$$

and

$$F_\tau(\omega) = \sum_{k \geqslant 1} \langle \mathrm{e}^{\mathrm{i}\omega(\tau_1 + \cdots + \tau_k)} |\phi_k|^2 \rangle_\tau = \mathrm{Tr}[(\hat{\mathbf{1}} - \hat{\Gamma}_\omega)^{-1}\hat{G}_\omega] \tag{24}$$

with

$$\hat{\Gamma}_\omega = \langle \mathrm{e}^{\mathrm{i}\omega\tau}\hat{D} \rangle_\tau \hat{C} \tag{25}$$

and

$$\hat{G}_\omega = \langle \mathrm{e}^{\mathrm{i}\omega\tau}\hat{D} \rangle_\tau E\Pi \times E\Pi. \tag{26}$$

As shown in appendix E, the matrix $(\hat{\mathbf{1}} - z\hat{\Gamma})^{-1}$ is analytic for $|z| < 1$. This means that we should consider the discrete Fourier summation as an analytic continuation $z \to \mathrm{e}^{\mathrm{i}\omega}$ from $|z| < 1$.

From $F(\omega)$ and $F_\tau(\omega)$ we can calculate moments of k and $t_k = \tau_1 + \cdots + \tau_k$ with respect to the weight $\langle |\phi_k|^2 \rangle_\tau$ and $\langle \tau |\phi_k|^2 \rangle_\tau$, respectively, as

$$\begin{aligned} \overline{k^m} &= \sum_{k \geqslant 1} k^m \langle |\phi_k|^2 \rangle_\tau = (-i\partial_\omega)^m F(\omega)|_{\omega=0}, \\ \overline{t^m} &= \sum_{k \geqslant 1} \langle t_k^m |\phi_k|^2 \rangle_\tau = (-i\partial_\omega)^m F_\tau(\omega)|_{\omega=0}. \end{aligned} \tag{27}$$

The property $F(\omega = 0) = F_\tau(\omega = 0) = 1$, discussed in appendix A, indicates a close relation between the two generating functions. Then the ω dependence of the generating functions is through the fact that (i) $F(\omega)$ and $F_\tau(\omega)$ depend on ω only through $\langle \mathrm{e}^{\mathrm{i}\omega}\hat{D} \rangle_\tau$ and $\langle \mathrm{e}^{\mathrm{i}\omega\tau}\hat{D} \rangle_\tau$, respectively, and (ii) the matrix $\langle \mathrm{e}^{\mathrm{i}\omega\tau}\hat{D} \rangle_\tau$ is a function of $\omega + E_j - E_{j'}$. It implies that we can replace a derivative with respect to ω by a derivative with respect to the difference of energy levels if $E_j - E_{j'} = 0$ ($j' = j$). Since the latter is implicitly assumed here for all energy levels, we can write for the first moment in equation (27)

$$-i\partial_\omega F(\omega)\bigg|_{\omega=0} = \sum_{j,j'}[\partial_{\bar{D}_{jj'}} F(\omega)]\partial_\omega \bar{D}_{jj'}(\omega)\bigg|_{\omega=0}$$

$$= \sum_{j=j'}[\partial_{\bar{D}_{jj'}(0)} F(0)]\partial_{E_j - E_{j'}} \bar{D}_{jj'}(0) + \sum_j [\partial_{\bar{D}_{jj}(\omega)} F(\omega)]\partial_\omega \bar{D}_{jj}(\omega)\bigg|_{\omega=0}$$

with $\bar{D}_{jj'}(\omega) = \mathrm{e}^{\mathrm{i}\omega}\langle \hat{D}_{jj'}\rangle_\tau$. The first sum on the right-hand side vanishes due to $F(0) = 1$ and consequently $\partial_{E_j - E_{j'}} F(0) = 0$, such that we obtain

$$\partial_\omega F(\omega)\bigg|_{\omega=0} = \sum_j \partial_{\bar{D}_{jj}(\omega)} F(\omega)\bigg|_{\omega=0} i\langle \hat{D}_{jj}(0)\rangle_\tau = i\sum_j \partial_{\bar{D}_{jj}(\omega)} F(\omega)\bigg|_{\omega=0}. \tag{28}$$

The analog calculation is valid for $F_\tau(\omega)$ and gives

$$\partial_\omega F_\tau(\omega)\bigg|_{\omega=0} = \sum_j \partial_{\bar{D}_{jj}(\omega)} F_\tau(\omega)\bigg|_{\omega=0} i\langle \tau \hat{D}_{jj}(0)\rangle_\tau = i\langle\tau\rangle_\tau \sum_j \partial_{\bar{D}_{jj}(\omega)} F(\omega)\Big|_{\omega=0}. \tag{29}$$

Comparing the expressions in equations (28) and (29) implies for the first moments in equation (27) the relation

$$\bar{t} = \sum_{k\geqslant 1}\langle t_k|\phi_k|^2\rangle_\tau = \langle\tau\rangle_\tau \sum_{k\geqslant 1} k\langle|\phi_k|^2\rangle_\tau = \langle\tau\rangle_\tau \bar{k}. \tag{30}$$

6.2. Evaluation of $\langle|\tilde{\phi}(\omega)|^2\rangle_\tau$

Next we will show that $\langle|\tilde{\phi}(\omega)|^2\rangle_\tau = 1$ holds in general for any integer N due to

$$T_{j_1 j_2} = \sum_{j_3,j_4}[\langle\hat{D}\rangle_\tau^{-1} - \hat{C}]^{-1}_{j_1 j_2, j_3 j_4} = \frac{1}{p_{j_1}}\delta_{j_1 j_2}, \tag{31}$$

where $\hat{C}$ was defined in section 5. To derive this property and to calculate the mean FDR time we use the matrix relations

$$\hat{E}\hat{W}[\langle\hat{D}\rangle_\tau^{-1} - \hat{C}] = \hat{E}\hat{\Pi}, \quad [\langle\hat{D}\rangle_\tau^{-1} - \hat{C}]\hat{T}\hat{E} = \hat{E}, \tag{32}$$

where $\hat{W}$ is the $N^2 \times N^2$ diagonal matrix $\mathrm{diag}(p_1, 0_N, p_2, 0_N, \ldots, 0_N, p_N)$, and 0_N is a sequence of N zeros. $\hat{T}$ is the $N^2 \times N^2$ diagonal matrix with elements $T_{jj'} = \delta_{jj'}/p_j$ of equation (31). The $\langle\hat{D}\rangle_\tau$ contribution on the right-hand side disappears, since $\hat{W}\langle\hat{D}\rangle_\tau = \hat{W}$. The first relation of equation (32) is obtained from $W_{j_1 j_2} = p_{j_1}\delta_{j_1 j_2}$ by a direct inspection of the matrix elements:

$$\sum_{j_1,j_2} W_{j_1 j_2}[-E_{j_1 j_3} p_{j_3} E_{j_2 j_4} p_{j_4} + E_{j_1 j_3} p_{j_3} \delta_{j_2 j_4} + \delta_{j_1 j_3} E_{j_2 j_4} p_{j_4}] = p_{j_3} p_{j_4}$$

and the second relation for $T_{j_3 j_4} = \delta_{j_3 j_4}/p_{j_3}$ from

$$\sum_{j_3,j_4}[-E_{j_1 j_3} p_{j_3} E_{j_2 j_4} p_{j_4} + E_{j_1 j_3} p_{j_3} \delta_{j_2 j_4} + \delta_{j_1 j_3} E_{j_2 j_4} p_{j_4}] T_{j_3 j_4} = 1.$$

Provided that the inverse of $\langle \hat{D} \rangle_\tau^{-1} - \hat{C}$ exists, the second relation of equation (32) implies $[\langle \hat{D} \rangle_\tau^{-1} - \hat{C}]^{-1}\hat{E} = \hat{T}\hat{E}$, which gives directly equation (31) and subsequently the normalization

$$\langle |\tilde{\phi}(\omega)|^2 \rangle_\tau = 1 \tag{33}$$

according to equation (D8) of appendix D. This result can be used to reduce the integral of the average winding number

$$\langle w \rangle_\tau := \frac{1}{2\pi} \int_0^{2\pi} \frac{\langle \tilde{\phi}(\omega)^* [-i\partial_\omega] \tilde{\phi}(\omega) \rangle_\tau}{\langle |\tilde{\phi}(\omega)|^2 \rangle_\tau} \, \mathrm{d}\omega, \tag{34}$$

which is discussed in more detail in section 8, with the help of equation (33) to

$$\langle w \rangle_\tau = \frac{1}{2\pi} \int_0^{2\pi} \langle \tilde{\phi}(\omega)^* [-i\partial_\omega] \tilde{\phi}(\omega) \rangle_\tau \mathrm{d}\omega = \sum_{k \geqslant 1} k \langle |\phi_k|^2 \rangle_\tau = \overline{k}. \tag{35}$$

In the next section we will see that $\langle w \rangle_\tau = \overline{k}$ is an integer, equal to the dimensionality of the accessible Hilbert space.

6.2.1. Mean FDR time. Now we return to the first moment in equation (27), using an extension of the previous calculation. Starting with

$$-i\partial_\omega F(\omega)\big|_{\omega=0} = \mathrm{Tr}[(\hat{\mathbf{1}} - \hat{\Gamma})^{-2} \langle \hat{D} \rangle_\tau \hat{E} \hat{\Pi}] \tag{36}$$

we write for the matrix inside the trace

$$\hat{E}\hat{\Pi}[\langle \hat{D} \rangle_\tau^{-1} - \hat{C}]^{-1} \langle \hat{D} \rangle_\tau^{-1} [\langle \hat{D} \rangle_\tau^{-1} - \hat{C}]^{-1}$$

and apply the first relation of equation (32) to the first inverse matrix to obtain

$$= \hat{E}\hat{W} \langle \hat{D} \rangle_\tau^{-1} [\langle \hat{D} \rangle_\tau^{-1} - \hat{C}]^{-1} = \hat{E}\hat{W} [\langle \hat{D} \rangle_\tau^{-1} - \hat{C}]^{-1}, \tag{37}$$

where the last equation is due to $\hat{W} \langle \hat{D} \rangle_\tau^{-1} = \hat{W}$. This can be inserted into equation (36), and with the second relation of equation (32) we get

$$-i\partial_\omega F(\omega)\big|_{\omega=0} = \mathrm{Tr}\{\hat{E}\hat{W}[\langle \hat{D} \rangle_\tau^{-1} - \hat{C}]^{-1}\} = \mathrm{Tr}\{\hat{W}\hat{T}\hat{E}\} = \sum_{j_1=1}^{N} T_{j_1 j_1} p_{j_1} = N, \tag{38}$$

where the last two equations follow from equation (31) and the definition of $\hat{W}$. This result gives us, together with equations (30) and (38), for the MNM of the FDR and the mean FDR time

$$\overline{k} = \sum_{k \geqslant 1} k \langle |\phi_k|^2 \rangle_\tau = \langle w \rangle_\tau = N \quad \text{and} \quad \overline{t} = \sum_{k \geqslant 1} \langle t_k |\phi_k|^2 \rangle_\tau = \langle \tau \rangle_\tau \langle w \rangle_\tau = \langle \tau \rangle_\tau N, \tag{39}$$

which presents an extension of a central result of the seminal work by Grünbaum *et al* [9] of stroboscopic measurements to random time measurements.

As already mentioned in the introduction, higher order moments are not quantized but can be very sensitive to degeneracies of the spectrum, at least for stroboscopic measurements

[3, 11, 18]. In the case of random measurements this is also true near degeneracies of the energy levels when $\langle D_j^* D_{j'} \rangle_\tau = \langle \mathrm{e}^{\mathrm{i}(E_j - E_{j'})\tau} \rangle_\tau$ ($j' = j$) is close to 1. This originates in the fact that for

$$-\partial_\omega^2 F(\omega)\big|_{\omega=0} = \mathrm{Tr}[(\hat{\mathbf{1}} - \hat{\Gamma})^{-3} \langle \hat{D} \rangle_\tau \hat{E} \hat{\Pi}] = \mathrm{Tr}\{\hat{W}(\hat{\mathbf{1}} - \hat{\Gamma})^{-1} \hat{T} \hat{E}\}$$

small eigenvalues of $\hat{\mathbf{1}} - \hat{\Gamma}$ can appear. This can indeed happen when at least one $\langle D_j^* D_{j'} \rangle_\tau$ is close to 1 or when one p_j is close to 0, as shown in appendix E.

7. Example: symmetric two-level system

In the previous section we derived relations between the MNM of the FDR and the mean FDR time, their relation with the average winding number $\langle w \rangle_\tau$ of equation (34) and with the dimensionality of the Hilbert space in equation (39). These results are general and valid for any quantum system on an N-dimensional Hilbert space. Besides these mean values we obtained in equation (27) also higher moments of the number of measurements and the return time for the FDR. For $m > 1$ we have not found a simple expression but can obtain these moments only by calculating the generating functions $F(\omega)$ and $F_\tau(\omega)$ directly. To determine these generating functions would require the inversion of the $N^2 \times N^2$ matrix $\hat{\mathbf{1}} - z\hat{\Gamma}$. This is a tedious task, which goes beyond the scope of this paper. Therefore, we limit ourselves to $N = 2$ and calculate the corresponding 4×4 matrices (cf appendix F). In particular, we consider a symmetric two-level system (2LS) with energy levels $E_\pm = \pm J$ and spectral weights $p_1 = p_2 = 1/2$ for random times. The Hilbert space is two-dimensional with two basis states, e.g. $|0\rangle$ and $|1\rangle$. If the measured state is $|0\rangle$, the projector P reads $P = |1\rangle\langle 1|$. Then we get $\langle 0|\mathrm{e}^{-\mathrm{i}H\tau}|0\rangle = \cos(J\tau)$ and $\langle 0|\mathrm{e}^{-\mathrm{i}H\tau}|1\rangle = \mathrm{i}\sin(J\tau)$ and the return amplitude ϕ_k reads

$$\phi_k = \begin{cases} \cos(J\tau_1) & \text{for } k = 1 \\ -\sin(J\tau_1)\sin(J\tau_2) & \text{for } k = 2 \\ -\sin(J\tau_1)\cos(J\tau_2)\ldots\cos(J\tau_{k-1})\sin(J\tau_k) & \text{for } k \geqslant 3 \end{cases}. \tag{40}$$

The simplicity of the two-level system is manifested in the fact that ϕ_k is a scalar product in contrast to the matrix product in equation (14) of the general case $N > 2$. This simplifies calculations with respect to random $\{\tau_k\}$ substantially. For instance, we can easily perform the summation with respect to k to get

$$\sum_{k=1}^{n} |\phi_k|^2 = 1 - [1 - \cos^2(J\tau_1)]\cos^2(J\tau_2)\ldots\cos^2(J\tau_n) \tag{41}$$

as a special case of the general equation (A1). After averaging with respect to $\{\tau_k\}$, we get for the mean FDR time

$$\bar{t} = \sum_{k\geqslant 1} \langle (\tau_1 + \cdots + \tau_k)|\phi_k|^2 \rangle_\tau = 2\langle \tau \rangle_\tau, \tag{42}$$

which is in agreement with our general result in equation (39). Moreover, the generating functions $F(\omega)$ in equation (23) and $F_\tau(\omega)$ in equation (24) read

$$\begin{aligned} F(\omega) &= \mathrm{e}^{\mathrm{i}\omega}\frac{(2\,\mathrm{e}^{\mathrm{i}\omega}-1)\langle\cos\,2J\tau\rangle_\tau-1}{\mathrm{e}^{\mathrm{i}\omega}(\langle\cos\,2J\tau\rangle_\tau+1)-2}\ , \\ F_\tau(\omega) &= \frac{(2\langle\mathrm{e}^{\mathrm{i}\omega\tau}\rangle_\tau-1)\langle\mathrm{e}^{\mathrm{i}\omega\tau}\,\cos\,2J\tau\rangle_\tau-\langle\mathrm{e}^{\mathrm{i}\omega\tau}\rangle_\tau}{\langle\mathrm{e}^{\mathrm{i}\omega\tau}\,\cos\,2J\tau\rangle_\tau+\langle\mathrm{e}^{\mathrm{i}\omega\tau}\rangle_\tau-2}, \end{aligned} \tag{43}$$

which gives for $\omega = 0$

$$\begin{aligned} \sum_{k\geqslant 1}\langle|\phi_k|^2\rangle_\tau = F(0) = 1, \quad \sum_{k\geqslant 1}k\langle|\phi_k|^2\rangle_\tau = -iF'(0) = 2, \\ \sum_{k\geqslant 1}k^2\langle|\phi_k|^2\rangle_\tau = -F''(0) = 2\frac{3-\langle\cos\,2J\tau\rangle_\tau}{1-\langle\cos\,2J\tau\rangle_\tau}\ , \end{aligned} \tag{44}$$

where the first equation reflects the normalization, the second equation the quantization of the mean FDR time and the last equation the FDR fluctuations. The latter only diverge for $J = 0$ in the case of random times steps, but in the case of a fixed time step τ it also diverges for $J\tau = k\pi$ ($k = \pm 1, \pm 2, \ldots$). For $F_\tau(\omega)$ we get

$$\sum_{k\geqslant 1}\langle(\tau_1+\cdots+\tau_k)|\phi_k|^2\rangle_\tau = -iF'_\tau(0) = 2\langle\tau\rangle_\tau. \tag{45}$$

The two-level systems gives also a direct insight into the effect of random time steps on the return amplitude $\tilde{\phi}(\omega)$ before averaging, since we can calculate these amplitudes from equation (40) for special realizations of $\{\tau_k\}$. A few examples are visualized in figure 1, indicating that $|\tilde{\phi}(\omega)|$ as well as the winding number vary from realization to realization substantially. For fixed time steps, on the other hand, we have $|\tilde{\phi}(\omega)| = 1$ and the winding number is 2 in figure 2, as predicted by the general theory of stroboscopic measurements [9, 18].

8. Discussion

The aim of our work has been to calculate the properties of the ME with random measurements through the return amplitude $\tilde{\phi}(\omega)$. For given eigenvalues $\{E_j\}$ and eigenstates $\{|E_j\rangle\}$ of the Hamiltonian H the ME of the return to the initial state $|\Psi\rangle$ are characterized by the time averaged phase factors $\langle\mathrm{e}^{\pm\mathrm{i}E_j\tau}\rangle_\tau$, $\langle\mathrm{e}^{\mathrm{i}(E_j-E'_j)\tau}\rangle_\tau$ and by the spectral weights $p_j = |\langle E_j|\Psi\rangle|^2$. This allowed us to calculated the mean FDR times and the MNM of the FDR, using the generating functions defined in equations (23) and (24). The surprising result in equation (39) is that the MNM of the FDR is just the dimensionality N of the accessible Hilbert space and that the mean FDR time is $\langle\tau\rangle_\tau N$, where $\langle\tau\rangle_\tau$ is the mean time between two successive measurements. The same was previously observed for fixed time steps τ [9, 11, 18]. The robustness of the average winding number is remarkable, since the winding number of the return amplitude $\tilde{\phi}(\omega)$ fluctuates strongly from realization to realization of the random $\{\tau_k\}$ (cf figure 1).

Other quantities, such as the correlation function $\langle\phi_k^*\phi_{k'}\rangle_\tau$ of the return amplitudes for different discrete times k and k' in equation (C1), can also be calculated. An example is $\sum_{k,k'\geqslant 1}\mathrm{e}^{\mathrm{i}(k'-k)\omega}\langle\phi_k^*\phi_{k'}\rangle_\tau = \langle|\tilde{\phi}(\omega)|^2\rangle_\tau$, which is 1 according to the calculation in appendix D. This enabled us to determine the mean FDR time in equation (39). Its connection with the

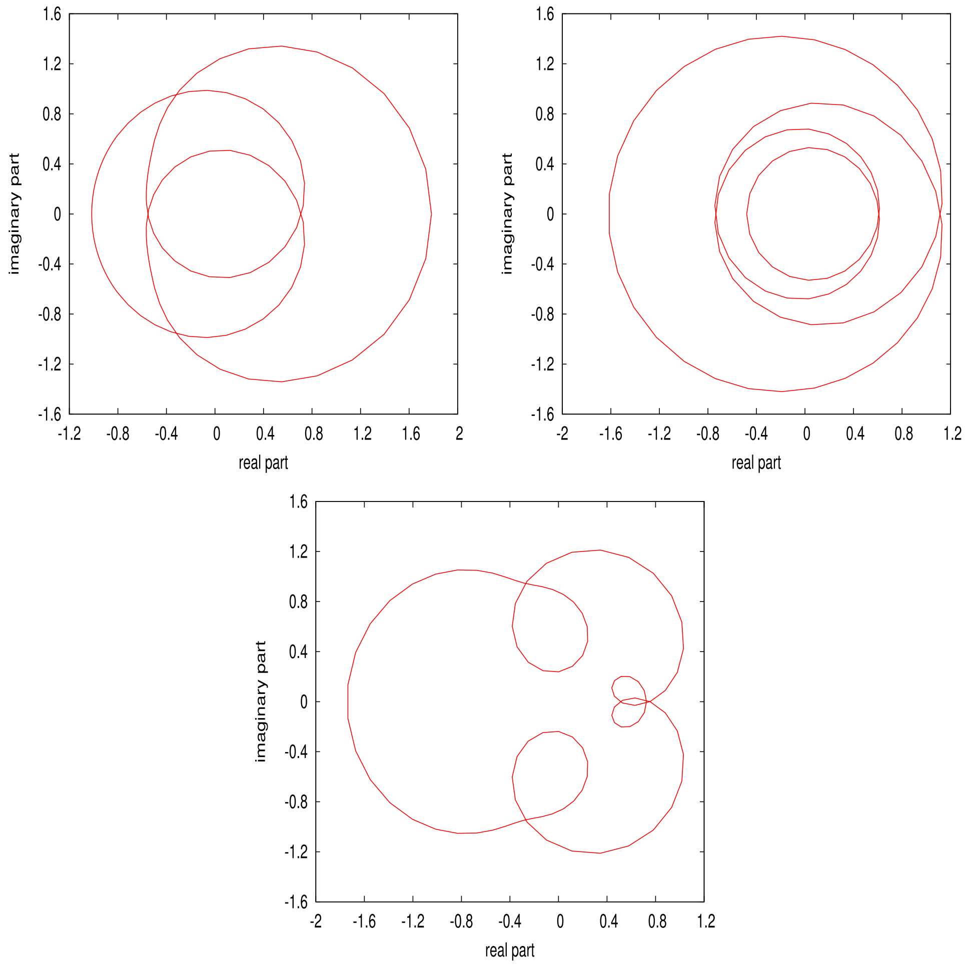

Figure 1. Symmetric two-level system: the return amplitude $\tilde{\phi}_8(\omega) = \sum_{k=1}^{8} \mathrm{e}^{\mathrm{i}k\omega}\, \phi_k$ on the ω interval $[0, 2\pi)$ with three randomly chosen realizations of $\{\tau_1, \ldots, \tau_8\}$ performs a closed trajectory in the complex plane. The winding numbers in these examples are $w_\phi = 3, 4, 1$, respectively.

integral in equation (34) is a generalization of the quantized winding number w_{st} in the case of stroboscopic measurements by Grünbaum *et al* [9, 18]. The latter is based on the fact that $|\tilde{\phi}(\omega)| = 1$ (cf equation (12)). Then the winding number w_{st} simply reads for $\tilde{\phi}(\omega) = \mathrm{e}^{\mathrm{i}\varphi}$ (cf reference [3] and section 4)

$$w_{st} = -\frac{i}{2\pi}\int_0^{2\pi} \tilde{\phi}(\omega)^* \partial_\omega \tilde{\phi}(\omega) \mathrm{d}\omega = \frac{1}{2\pi}\int_0^{2\pi} \partial_\omega \varphi \, \mathrm{d}\omega.$$

In the case of random time measurements $|\tilde{\phi}(\omega; \{\tau_k\})| \neq 1$, such that we must modify the definition of the winding number by normalizing $\tilde{\phi}(\omega, \{\tau_k\})$. A further problem is that the winding number depends on the realization of the time steps $\{\tau_k\}$, as we have demonstrated in figure 1 for the symmetric two-level system. Therefore, we must also perform an average with respect to the time steps. Then the definition of the winding number becomes equation (34),

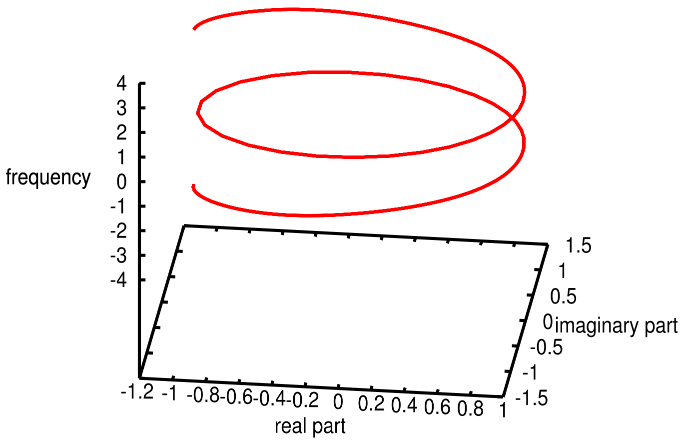

Figure 2. Symmetric two-level system: the return amplitude $\tilde{\phi}_8(\omega) = \sum_{k=1}^{8} \mathrm{e}^{\mathrm{i}k\omega}\, \phi_k$ on the ω interval $[0, 2\pi)$ with fixed time step $\tau \approx \pi/2J$. The winding number in this example is $w_\phi = 2$.

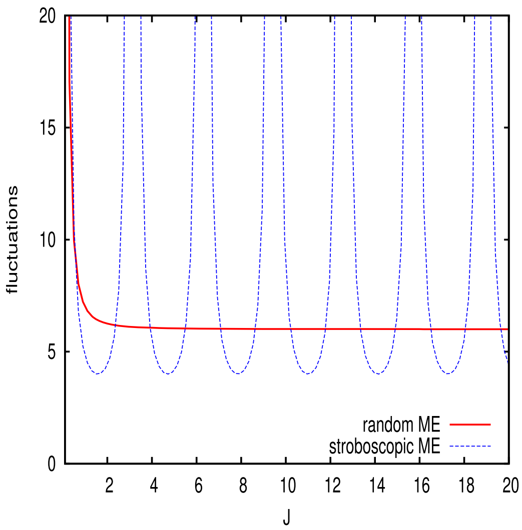

Figure 3. Symmetric two-level system: fluctuations of the FDR for stroboscopic ME and random time steps from equations (48) and (49).

which reads with $\tilde{\phi}(\omega) = |\tilde{\phi}(\omega)|\mathrm{e}^{\mathrm{i}\varphi}$ as an average differential phase change

$$\langle w\rangle_\tau = \frac{1}{2\pi}\int_0^{2\pi} \frac{\langle|\tilde{\phi}(\omega)|^2\partial_\omega\varphi\rangle_\tau}{\langle|\tilde{\phi}(\omega)|^2\rangle_\tau}\, \mathrm{d}\omega. \qquad (46)$$

Here we note that equation (34) is formally equivalent to the definition of the Berry phase [28] when we replace $\tilde{\phi}(\omega, \{\tau_k\})$ by the spatial wave function $\phi(\omega, \mathbf{r})$ and replace the time average $\langle\cdots\rangle_\tau$ by the usual quantum average in space.

Averaging over random time steps is crucial to obtain a generic winding number. That different special realizations of the random time steps lead to different winding numbers can be seen when we assume a finite sequence of $\{\phi_k\}$ ($k \leqslant M$). The latter is either the result of an approximative truncation of the sequence or when the sequence terminates with $S_M = 0$ in equation (A1). Then the Fourier transformed return amplitude in equation (5) becomes a finite sum

$$\tilde{\phi}_M(\omega) = \sum_{k=1}^{M} \mathrm{e}^{\mathrm{i}\omega k}\, \phi_k$$

for which a winding number w_M can be defined for (random) coefficients $\{\phi_k\}$ as

$$w_M = \frac{1}{2\pi}\int_0^{2\pi} \frac{\tilde{\phi}^*_M(\omega)(-i\partial_\omega)\tilde{\phi}_M(\omega)}{\tilde{\phi}^*_M(\omega)\tilde{\phi}_M(\omega)}\,\mathrm{d}\omega = \frac{1}{2\pi}\int_0^{2\pi} (-i\partial_\omega)\log[\tilde{\phi}_M(\omega)]\mathrm{d}\omega. \quad (47)$$

Then we rewrite the polynomial $\tilde{\phi}_M(\omega)$ as the product

$$\tilde{\phi}_M(\omega) = \phi_M(\mathrm{e}^{\mathrm{i}\omega} - z_1)\ldots(\mathrm{e}^{\mathrm{i}\omega} - z_M),$$

such that we get for the winding number

$$w_M = \sum_{k=1}^{M} \frac{1}{2\pi}\int_0^{2\pi} \frac{\mathrm{e}^{\mathrm{i}\omega}}{\mathrm{e}^{\mathrm{i}\omega} - z_k}\,\mathrm{d}\omega.$$

With $z = \mathrm{e}^{\mathrm{i}\omega}$ this gives

$$w_M = \sum_{k=1}^{M} \frac{1}{2\pi i}\int_{S_1} \frac{1}{z - z_k}\,\mathrm{d}z = M',$$

where the Cauchy integral is performed over the unit circle S_1 and M' ($0 \leqslant M' \leqslant M$) is the number of poles inside the unit circle. A simple example is $M = 2$ with

$$\tilde{\phi}_2(\omega) = \phi_1\,\mathrm{e}^{\mathrm{i}\omega} + \phi_2\,\mathrm{e}^{2\mathrm{i}\omega} = z(\phi_1 + \phi_2 z) = \phi_2 z(z + \phi_1/\phi_2),$$

where we get

$$w_2 = \begin{cases} 1 & |\phi_1/\phi_2| > 1 \\ 2 & |\phi_1/\phi_2| < 1 \end{cases}.$$

A detailed calculation of several quantities was presented in section 7 for the case of a symmetric two-level system with $w = 2$ after averaging with respect to $\{\tau_k\}$. The fluctuations of the return time are finite

$$\sum_{k\geqslant 1} k^2 \langle|\phi_k|^2\rangle_\tau = 2\frac{3 - \langle\cos\, 2J\tau\rangle_\tau}{1 - \langle\cos\, 2J\tau\rangle_\tau} = \frac{1 + 6J^2}{J^2}\ , \quad (48)$$

where the last expression is obtained from the Poisson distribution $\mathrm{e}^{-\tau_k}\,\mathrm{d}\tau_k$. In the limit of a fixed measurement time τ the fluctuations

$$\sum_{k\geqslant 1} k^2 |\phi_k|^2 = 2\frac{3 - \cos\, 2J\tau}{1 - \cos\, 2J\tau} \quad (49)$$

would diverge for $J\tau = \pi n$ ($n = 0, 1, \ldots$). Thus, the random measurements wash out the divergences of the fluctuations. For most values of the level splitting J the fluctuations are stronger for the fixed time steps, as visualized in figure 3.

In this paper we have completely focused on the return of the quantum system to its initial state. A natural extension would be a corresponding analysis of the transition from an initial to a different final state, monitored by random projective measurements. We have addressed this topic in a separate article [23].

In conclusion, the mean FDR time of the ME for random time steps is equal to the dimensionality of the accessible Hilbert space. This is very similar to the ME for fixed time steps. On the other hand, the strong fluctuations of the FDR time, which appear for a small distance of eigenvalues in the case of fixed time steps, are washed out by averaging with respect to the random time steps. This was briefly discussed for a two-level system in this article and more general in reference [23].

Acknowledgments

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A. Normalization

The normalization of the vector $\vec{\phi} = (\phi_1, \phi_2, \ldots, \phi_n)$ with

$$\phi_k = \langle\Psi|\mathrm{e}^{-\mathrm{i}H\tau_k}(P\mathrm{e}^{-\mathrm{i}H\tau_{k-1}})\ldots(P\mathrm{e}^{-\mathrm{i}H\tau_1})|\Psi\rangle$$

in the limit $n \to \infty$ is based on the normalization of $|\Psi\rangle$

$$\langle\Psi|\mathrm{e}^{\mathrm{i}H\tau_1}\,\mathrm{e}^{-\mathrm{i}H\tau_1}|\Psi\rangle = \langle\Psi|\Psi\rangle = 1$$

and will be derived by iteration: for the evaluation of $|\vec{\phi}|^2$ we consider the sequence of projection operators $\{\Pi_k\}_{k=1,2,\ldots,n}$ with

$$\Pi_k := P\mathrm{e}^{\mathrm{i}H\tau_k}\,\mathrm{e}^{-\mathrm{i}H\tau_k}\,P, \quad P = \mathbf{1} - |\Psi\rangle\langle\Psi| \equiv \mathbf{1} - P_0.$$

With $\Pi_k = P^2 = P$ and $|\Psi\rangle\langle\Psi| + P = \mathbf{1}$ we can insert $P_0 + \Pi_2 = \mathbf{1}$ at

$$1 = \langle\Psi|\mathrm{e}^{\mathrm{i}H\tau_1}\,\mathrm{e}^{-\mathrm{i}H\tau_1}|\Psi\rangle = \langle\Psi|\mathrm{e}^{\mathrm{i}H\tau_1}(P_0 + \Pi_2)\mathrm{e}^{-\mathrm{i}H\tau_1}|\Psi\rangle$$

$$= \langle\Psi|\mathrm{e}^{\mathrm{i}H\tau_1}|\Psi\rangle\langle\Psi|\mathrm{e}^{-\mathrm{i}H\tau_1}|\Psi\rangle + \langle\Psi|\mathrm{e}^{\mathrm{i}H\tau_1}\,\Pi_2\,\mathrm{e}^{-\mathrm{i}H\tau_1}|\Psi\rangle.$$

Next we replace Π_2 in the second term by

$$\Pi_2 = P\mathrm{e}^{\mathrm{i}H\tau_2}\,\mathrm{e}^{-\mathrm{i}H\tau_2}\,P = P\mathrm{e}^{\mathrm{i}H\tau_2}(P_0 + \Pi_3)\mathrm{e}^{-\mathrm{i}H\tau_2}\,P$$

to get

$$1 = \langle\Psi|\mathrm{e}^{\mathrm{i}H\tau_1}|\Psi\rangle\langle\Psi|\mathrm{e}^{-\mathrm{i}H\tau_1}|\Psi\rangle + \langle\Psi|\mathrm{e}^{\mathrm{i}H\tau_1}\, P\mathrm{e}^{\mathrm{i}H\tau_2}(P_0+\Pi_3)\mathrm{e}^{-\mathrm{i}H\tau_2}P\mathrm{e}^{-\mathrm{i}H\tau_1}|\Psi\rangle$$
$$= \langle\Psi|\mathrm{e}^{\mathrm{i}H\tau_1}|\Psi\rangle\langle\Psi|\mathrm{e}^{-\mathrm{i}H\tau_1}|\Psi\rangle + \langle\Psi|\mathrm{e}^{\mathrm{i}H\tau_1}P\mathrm{e}^{\mathrm{i}H\tau_2}|\Psi\rangle\langle\Psi|\mathrm{e}^{-\mathrm{i}H\tau_2}P\mathrm{e}^{-\mathrm{i}H\tau_1}|\Psi\rangle$$
$$+ \langle\Psi|\mathrm{e}^{\mathrm{i}H\tau_1}P\mathrm{e}^{iH\tau_2}\,\Pi_3\,\mathrm{e}^{-\mathrm{i}H\tau_2}P\mathrm{e}^{-\mathrm{i}H\tau_1}|\Psi\rangle.$$

The replacement of the operator Π_k by $P\mathrm{e}^{\mathrm{i}H\tau_k}(P_0+\Pi_{k+1})\mathrm{e}^{-\mathrm{i}H\tau_k}P$ can be repeated for $k=3,\ldots,n$ to obtain

$$1 = \langle\Psi|\mathrm{e}^{\mathrm{i}H\tau_1}|\Psi\rangle\langle\Psi|\mathrm{e}^{-\mathrm{i}H\tau_1}|\Psi\rangle + \langle\Psi|\mathrm{e}^{\mathrm{i}H\tau_1}P\mathrm{e}^{\mathrm{i}H\tau_2}|\Psi\rangle\langle\Psi|\mathrm{e}^{-\mathrm{i}H\tau_2}P\mathrm{e}^{-\mathrm{i}H\tau_1}|\Psi\rangle$$
$$+\cdots+\langle\Psi|\mathrm{e}^{\mathrm{i}H\tau_1}(P\mathrm{e}^{\mathrm{i}H\tau_2})\ldots(P\mathrm{e}^{\mathrm{i}H\tau_n})|\Psi\rangle\langle\Psi|(\mathrm{e}^{-\mathrm{i}H\tau_n}P)\ldots(\mathrm{e}^{-\mathrm{i}H\tau_2}P)\mathrm{e}^{-\mathrm{i}H\tau_1}|\Psi\rangle$$
$$+\langle\Psi|\mathrm{e}^{\mathrm{i}H\tau_1}(P\mathrm{e}^{\mathrm{i}H\tau_2})\ldots(P\mathrm{e}^{\mathrm{i}H\tau_n})P(\mathrm{e}^{-\mathrm{i}H\tau_n}P)\ldots(\mathrm{e}^{-\mathrm{i}H\tau_2}P)\mathrm{e}^{-\mathrm{i}H\tau_1}|\Psi\rangle$$
$$=\sum_{k=1}^{n}|\phi_k|^2+S_n,$$

where

$$S_n = \langle\Psi|\mathrm{e}^{\mathrm{i}H\tau_1}(P\mathrm{e}^{\mathrm{i}H\tau_2})\ldots(P\mathrm{e}^{\mathrm{i}H\tau_n})P(\mathrm{e}^{-\mathrm{i}H\tau_n}P)\ldots(\mathrm{e}^{-\mathrm{i}H\tau_2}P)\mathrm{e}^{-\mathrm{i}H\tau_1}|\Psi\rangle.$$

Thus, we have

$$\sum_{k=1}^{n}|\phi_k|^2 = 1 - S_n, \tag{A1}$$

where the probability S_n of not recording the state after n attempts is

$$S_n = \langle\Psi|\mathrm{e}^{\mathrm{i}H\tau_1}(P\mathrm{e}^{\mathrm{i}H\tau_2})\ldots(P\mathrm{e}^{\mathrm{i}H\tau_n})P(\mathrm{e}^{-\mathrm{i}H\tau_n}P)\ldots(\mathrm{e}^{-\mathrm{i}H\tau_2}P)\mathrm{e}^{-\mathrm{i}H\tau_1}|\Psi\rangle.$$

Provided the remainder S_n vanishes in the limit $n\to\infty$, the wave function $\vec{\phi}=(\phi_1,\phi_2,\ldots,\phi_n)$ is normalized in this limit: $|\vec{\phi}|^2=1$. Although we do not have proof that S_n always vanishes with $n\to\infty$, the latter is plausible and is supported by the example of the symmetric two-level system in equation (41), by stroboscopic measurements [9] and in the case of the time averaged sum $\sum_{k\geqslant 1}\langle|\phi_k|^2\rangle_\tau = 1$ [23].

Appendix B. Recursion

For an operator K and the projector $|\Psi\rangle\langle\Psi|$ we assume that the inverse $(K+|\Psi\rangle\langle\Psi|)^{-1}$ exists. Then we get the relation

$$\langle\Psi|(K+|\Psi\rangle\langle\Psi|)^{-1}|\Psi_0\rangle = \langle\Psi|K^{-1}(\mathbf{1}+|\Psi\rangle\langle\Psi|K^{-1})^{-1}|\Psi_0\rangle. \tag{B1}$$

Now we can use the identity

$$(\mathbf{1}+|\Psi\rangle\langle\Psi|K^{-1})^{-1} = \mathbf{1} - |\Psi\rangle\langle\Psi|K^{-1}(\mathbf{1}+|\Psi\rangle\langle\Psi|K^{-1})^{-1}$$

to write for the right-hand side of equation (B1)

$$\langle\Psi|K^{-1}|\Psi_0\rangle - \langle\Psi|K^{-1}|\Psi\rangle\langle\Psi|K^{-1}(\mathbf{1}+|\Psi\rangle\langle\Psi|K^{-1})^{-1}|\Psi_0\rangle,$$

such that equation (B1) becomes

$$\langle\Psi|(K+|\Psi\rangle\langle\Psi|)^{-1}|\Psi_0\rangle=\langle\Psi|K^{-1}|\Psi_0\rangle-\langle\Psi|K^{-1}|\Psi\rangle\langle\Psi|(K+|\Psi\rangle\langle\Psi|)^{-1}|\Psi_0\rangle.$$

This relation is equivalent to

$$\langle\Psi|(K+|\Psi\rangle\langle\Psi|)^{-1}|\Psi_0\rangle=\frac{\langle\Psi|K^{-1}|\Psi_0\rangle}{1+\langle\Psi|K^{-1}|\Psi\rangle}. \tag{B2}$$

In particular, for the diagonal case $|\Psi_0\rangle=|\Psi\rangle$ we have

$$\langle\Psi|(K+|\Psi\rangle\langle\Psi|)^{-1}|\Psi\rangle=1-\frac{1}{1+\langle\Psi|K^{-1}|\Psi\rangle}. \tag{B3}$$

These relations read in the energy (spectral) representation, where we assume that K is diagonal,

$$\begin{aligned}\langle\Psi|(K+|\Psi\rangle\langle\Psi|)^{-1}|\Psi_0\rangle&=\sum_{j,j'}\langle\Psi|E_j\rangle\langle E_j|(K+|\Psi\rangle\langle\Psi|)^{-1}|E_{j'}\rangle\langle E_{j'}|\Psi_0\rangle\\&=\sum_{j,j'}(K+\Pi E)^{-1}_{E_j,E_{j'}}\langle\Psi|E_{j'}\rangle\langle E_{j'}|\Psi_0\rangle\end{aligned}$$

and

$$\begin{aligned}\langle\Psi|(K+|\Psi\rangle\langle\Psi|)^{-1}|\Psi\rangle&=\sum_{j,j'}\langle\Psi|E_j\rangle\langle E_j|(K+|\Psi\rangle\langle\Psi|)^{-1}|E_{j'}\rangle\langle E_{j'}|\Psi\rangle\\&=\sum_{j,j'}(K+\Pi E)^{-1}_{E_j,E_{j'}}\Pi_{E_{j'}},\end{aligned}$$

where K on the right-hand side is in the energy representation $\langle E_j|K|E_{j'}\rangle=K_{E_j,E_j}\delta_{E_j,E_{j'}}$. This implies with equation (B3)

$$\sum_{j,j'}(K+\Pi E)^{-1}_{E_j,E_{j'}}\Pi_{E_{j'}}=1-\frac{1}{1+\sum_j K^{-1}_{E_j,E_j}\Pi_{E_j}} \tag{B4}$$

and with equation (B2)

$$\sum_{j,j'}(K+\Pi E)^{-1}_{E_j,E_{j'}}\langle\Psi|E_{j'}\rangle\langle E_{j'}|\Psi_0\rangle=\frac{\sum_j K^{-1}_{E_j,E_j}\langle\Psi|E_j\rangle\langle E_j|\Psi_0\rangle}{1+\sum_j K^{-1}_{E_j,E_j}\Pi_{E_j}}. \tag{B5}$$

Here $|\Psi_0\rangle$ can be any state, implying that the relation holds for any $Q_{E_{j'}}=\langle\Psi|E_{j'}\rangle h_{E_{j'}}\langle E_{j'}|\Psi\rangle=p_{j'}h_{E_{j'}}$:

$$\sum_{j,j'}(K+\Pi E)^{-1}_{E_j,E_{j'}}Q_{E_{j'}}=\frac{\sum_j K^{-1}_{E_j,E_j}Q_{E_j}}{1+\sum_j K^{-1}_{E_j,E_j}\Pi_{E_j}}. \tag{B6}$$

Appendix C. Product matrices

With the properties of equation (16) we get for $k' \geqslant k$

$$\begin{aligned}\phi_k^*\phi_{k'} &= \mathrm{Tr}[D_k^*(\mathbf{1}-E\Pi)\ldots D_2^*(\mathbf{1}-E\Pi)D_1^*E\Pi]\,\mathrm{Tr}\,[D_{k'}(\mathbf{1}-E\Pi)\ldots D_2\\ &\times(\mathbf{1}-E\Pi)D_1E\Pi]\end{aligned}$$

due to $\mathrm{Tr}(A_1)\mathrm{Tr}(A_2) = \mathrm{Tr}(A_1\times A_2)$

$$\begin{aligned}\phi_k^*\phi_{k'} &= \mathrm{Tr}\,\{[D_k^*(\mathbf{1}-E\Pi)\ldots D_2^*(\mathbf{1}-E\Pi)D_1^*E\Pi]\times[D_{k'}(\mathbf{1}-E\Pi)\ldots D_2\\ &\times(\mathbf{1}-E\Pi)D_1E\Pi]\}\end{aligned}$$

and due to $A_1B_1\times A_2B_2 = [A_1\times A_2][B_1\times B_2]$

$$\begin{aligned}\phi_k^*\phi_{k'} &= \mathrm{Tr}\,\big\{\big[\mathbf{1}\times D_{k'}(\mathbf{1}-E\Pi)\ldots\mathbf{1}\times D_{k+1}(\mathbf{1}-E\Pi)\big]\\ &\big[D_k^*(\mathbf{1}-E\Pi)\times D_k(\mathbf{1}-E\Pi)\ldots D_2^*(\mathbf{1}-E\Pi)\\ &\times D_2(\mathbf{1}-E\Pi)\big]\,D_1^*E\Pi\times D_1E\Pi\big\}\,.\end{aligned}$$

Averaging with respect to independent random times $\{\tau_k\}$ then gives

$$\langle\phi_k^*\phi_{k'}\rangle_\tau = \mathrm{Tr}\{\hat{C}_2^{k'-k}\hat{\Gamma}^{k-1}\hat{G}\}\quad\text{with }\hat{C}_2 = \mathbf{1}\times\langle D\rangle_\tau(\mathbf{1}-E\Pi),$$

since

$$\langle D^*(\mathbf{1}-E\Pi)\times D(\mathbf{1}-E\Pi)\rangle_\tau = \langle D^*\times D\rangle_\tau(\mathbf{1}-E\Pi)\times(\mathbf{1}-E\Pi)$$

from the first relation in equation (16) implies

$$\begin{aligned}&\mathrm{Tr}\,\{\mathbf{1}\times\langle D\rangle_\tau(\mathbf{1}-E\Pi)\ldots\mathbf{1}\times\langle D\rangle_\tau(\mathbf{1}-E\Pi)\langle D^*(\mathbf{1}-E\Pi)\\ &\quad\times D(\mathbf{1}-E\Pi)\rangle_\tau\ldots\langle D^*(\mathbf{1}-E\Pi)\times D(\mathbf{1}-E\Pi)\rangle_\tau\\ &\langle D^*E\Pi\times DE\Pi\rangle_\tau\}\\ &= \mathrm{Tr}\,\{\mathbf{1}\times\langle D\rangle_\tau(\mathbf{1}-E\Pi)\ldots\mathbf{1}\times\langle D\rangle_\tau(\mathbf{1}-E\Pi)\langle D^*\times D\rangle_\tau(\mathbf{1}-E\Pi)\\ &\quad\times(\mathbf{1}-E\Pi)\ldots\langle D^*\times D\rangle_\tau(\mathbf{1}-E\Pi)\times(\mathbf{1}-E\Pi)\\ &\langle D^*\times D\rangle_\tau E\Pi\times E\Pi\} = \mathrm{Tr}(\hat{C}_2^{k'-k}\hat{\Gamma}^{k-1}\hat{G}),\end{aligned}$$

where the last equation follows from equations (20) and (21). An analog expression exists for $k\geqslant k'$, such that we get

$$\begin{aligned}\langle\phi_k^*\phi_{k'}\rangle_\tau &= \begin{cases}\mathrm{Tr}[(\hat{C}_1)^{k-k'}\hat{\Gamma}^{k'-1}\hat{G}] & \text{for } k\geqslant k'\geqslant 1\\ \mathrm{Tr}[(\hat{C}_2)^{k'-k}\hat{\Gamma}^{k-1}\hat{G}] & \text{for } k'\geqslant k\geqslant 1\end{cases},\\ \hat{C}_j &= \begin{cases}\langle D^*\rangle_\tau(\mathbf{1}-E\Pi)\times\mathbf{1} & j=1\\ \mathbf{1}\times\langle D\rangle_\tau(\mathbf{1}-E\Pi) & j=2\end{cases}.\end{aligned}\tag{C1}$$

Appendix D. Generating function

Next we consider the Fourier transform of equation (C1)

$$\langle\tilde{\phi}^*(\omega)\tilde{\phi}(\omega+\omega')\rangle_\tau = \sum_{k\geqslant 1}\sum_{k'\geqslant k} \mathrm{e}^{\mathrm{i}(\omega+\omega')(k'-k)+\mathrm{i}\omega' k}\langle\phi_k^*\phi_{k'}\rangle_\tau$$
$$+\sum_{k'\geqslant 1}\sum_{k>k'} \mathrm{e}^{\mathrm{i}\omega(k'-k)+\mathrm{i}\omega' k'}\langle\phi_k^*\phi_{k'}\rangle_\tau \tag{D1}$$

$$= \mathrm{e}^{\mathrm{i}\omega'}\ \mathrm{Tr}\{[(\hat{\mathbf{1}}-\mathrm{e}^{\mathrm{i}(\omega+\omega')}\hat{C}_2)^{-1}+\mathrm{e}^{-\mathrm{i}\omega}\hat{C}_1(\hat{\mathbf{1}}-\mathrm{e}^{-\mathrm{i}\omega}\ \hat{C}_1)^{-1}](\hat{\mathbf{1}}-\mathrm{e}^{\mathrm{i}\omega'}\ \hat{\Gamma})^{-1}\hat{G}\},$$

which becomes, after rewriting the second term in the trace,

$$= \mathrm{e}^{\mathrm{i}\omega'}\ \mathrm{Tr}\{[(\hat{\mathbf{1}}-\mathrm{e}^{\mathrm{i}(\omega+\omega')}\hat{C}_2)^{-1}+(\hat{\mathbf{1}}-\mathrm{e}^{-\mathrm{i}\omega}\ \hat{C}_1)^{-1}-\hat{\mathbf{1}}](\hat{\mathbf{1}}-\mathrm{e}^{\mathrm{i}\omega'}\ \hat{\Gamma})^{-1}\hat{G}\}$$

and with the expression of $F(\omega)$ in equation (23)

$$= \mathrm{e}^{\mathrm{i}\omega'}\ \mathrm{Tr}\{[(\hat{\mathbf{1}}-\mathrm{e}^{\mathrm{i}(\omega+\omega')}\hat{C}_2)^{-1}+(\hat{\mathbf{1}}-\mathrm{e}^{-\mathrm{i}\omega}\hat{C}_1)^{-1}](\hat{\mathbf{1}}-\mathrm{e}^{\mathrm{i}\omega'}\hat{\Gamma})^{-1}\hat{G}\}-F(\omega')\ . \tag{D2}$$

To calculate the first term we can use the identity (B5). With $A_1\times(A_2+B_2)=A_1\times A_2+A_1\times B_2$ we get the relation

$$(\hat{\mathbf{1}}-z\hat{C}_j)^{-1}=\begin{cases}[\mathbf{1}-z\langle D^*\rangle_\tau(\mathbf{1}-E\Pi)]^{-1}\times\mathbf{1} & j=1\\ \mathbf{1}\times[\mathbf{1}-z\langle D\rangle_\tau(\mathbf{1}-E\Pi)]^{-1} & j=2\end{cases}$$

such that

$$[\mathbf{1}-z\langle D^*\rangle_\tau(\mathbf{1}-E\Pi)]^{-1}=[(z\langle D^*\rangle_\tau)^{-1}-\mathbf{1}+E\Pi)]^{-1}(z\langle D^*\rangle_\tau)^{-1}.$$

This yields for the first term in equation (D2)

$$\mathrm{Tr}\{(\mathbf{1}\times[\mathbf{1}-\mathrm{e}^{\mathrm{i}(\omega+\omega')}\langle D\rangle_\tau(\mathbf{1}-E\Pi)]^{-1})(\widehat{\mathbf{1}}-\mathrm{e}^{\mathrm{i}\omega'}\widehat{\Gamma})^{-1}\widehat{G}\}$$

$$=\mathrm{Tr}\{\widehat{\Pi}(\mathbf{1}\times[\mathbf{1}-\mathrm{e}^{\mathrm{i}(\omega+\omega')}\langle D\rangle_\tau(\mathbf{1}-E\Pi)]^{-1})(\widehat{\mathbf{1}}-\mathrm{e}^{\mathrm{i}\omega'}\ \widehat{\Gamma})^{-1}\langle\widehat{D}\rangle_\tau\widehat{E}\}$$

$$=\mathrm{Tr}\{(\mathbf{1}\times[\mathbf{1}-\mathrm{e}^{\mathrm{i}(\omega+\omega')}\langle D\rangle_\tau(\mathbf{1}-\Pi E)]^{-1})\widehat{\Pi}(\widehat{\mathbf{1}}-\mathrm{e}^{\mathrm{i}\omega'}\ \widehat{\Gamma})^{-1}\langle\widehat{D}\rangle_\tau\widehat{E}\}$$

$$=\sum_{j,j'}[\mathbf{1}-\mathrm{e}^{\mathrm{i}(\omega+\omega')}\langle D\rangle_\tau(\mathbf{1}-\Pi E)]^{-1}_{j,j'}p_{j'}A_{j'}(\omega') \tag{D3}$$

with

$$A_{j'}(\omega')=\sum_{j_1,j_2,j_3}[(\hat{\mathbf{1}}-\mathrm{e}^{\mathrm{i}\omega'}\ \hat{\Gamma})^{-1}\langle D^*\times D\rangle_\tau]_{j_1j',j_2j_3}p_{j_1}.$$

These components are linked to the generating function $F(\omega')$ in equation (23) through $F(\omega')=\sum_j p_jA_j(\omega')$. Now we write

$$\sum_{j,j'}[\mathbf{1}-\mathrm{e}^{\mathrm{i}(\omega+\omega')}\langle D\rangle_\tau(\mathbf{1}-\Pi E)]^{-1}_{j,j'}p_{j'}A_{j'}$$

$$=\sum_{j,j'}[(\mathrm{e}^{\mathrm{i}(\omega+\omega')}\langle D\rangle_\tau)^{-1}-\mathbf{1}+\Pi E)]^{-1}_{j,j'}\frac{p_{j'}A_{j'}(\omega')}{\mathrm{e}^{\mathrm{i}(\omega+\omega')}[\langle D\rangle_\tau]_{j'j'}}$$

and with $K=(\mathrm{e}^{\mathrm{i}(\omega+\omega')}\langle D^*\rangle_\tau)^{-1}-\mathbf{1}$ we can apply equation (B6)

$$=\sum_{j,j'}[K+\Pi E)]^{-1}_{j,j'}\frac{p_{j'}A_{j'}(\omega')}{\mathrm{e}^{\mathrm{i}(\omega+\omega')}\left[\langle D\rangle_\tau\right]_{j'j'}}$$

$$=\mathrm{e}^{-\mathrm{i}(\omega+\omega')}\frac{\sum_j K^{-1}_{j,j}p_jA_j(\omega')/\gamma^*_j}{1+\sum_j K^{-1}_{j,j}p_j}=\frac{\sum_j p_jA_j(\omega')\prod_{k\neq j}(1-\mathrm{e}^{\mathrm{i}(\omega+\omega')}\gamma^*_k)}{\sum_{j'}p_{j'}\prod_{k\neq j'}(1-\mathrm{e}^{\mathrm{i}(\omega+\omega')}\gamma^*_k)} \tag{D4}$$

The corresponding calculation yields for the second term in equation (D2):

$$\mathrm{Tr}\left\{([\mathbf{1}-\mathrm{e}^{-\mathrm{i}\omega}\langle D^*\rangle_\tau(\mathbf{1}-E\Pi)]^{-1}\times\mathbf{1})(\hat{\mathbf{1}}-\mathrm{e}^{-\mathrm{i}\omega'}\hat{\Gamma})^{-1}\hat{G})\right\}$$

$$=\sum_{j,j'}[\mathbf{1}-\mathrm{e}^{-\mathrm{i}\omega}\langle D^*\rangle_\tau(\mathbf{1}-\Pi E)]^{-1}_{j,j'}p_{j'}B_{j'}(\omega')$$

and again with equation (B6)

$$=\sum_{j,j'}(K'+\Pi E)^{-1}_{j,j'}\frac{p_{j'}B_{j'}(\omega')}{\mathrm{e}^{-\mathrm{i}\omega}[\langle D^*\rangle_\tau]_{j'j'}}=\mathrm{e}^{\mathrm{i}\omega}\frac{\sum_j K'^{-1}_{j,j}p_jB_j(\omega')/\gamma_j}{1+\sum_j K'^{-1}_{j,j}p_j}$$

$$=\frac{\sum_j p_jB_j(\omega')\prod_{k\neq j}(1-\mathrm{e}^{-\mathrm{i}\omega}\,\gamma_k)}{\sum_{j'}p_{j'}\prod_{k\neq j'}(1-\mathrm{e}^{-\mathrm{i}\omega}\,\gamma_k)} \tag{D5}$$

with

$$B_{j'}(\omega')=\sum_{j_1,j_2,j_3}[(\hat{\mathbf{1}}-\mathrm{e}^{-\mathrm{i}\omega'}\hat{\Gamma})^{-1}\langle D^*\times D\rangle_\tau]_{j'j_1,j_2j_3}p_{j_1},\quad K'=(\mathrm{e}^{-\mathrm{i}\omega}\langle D\rangle_\tau)^{-1}-\mathbf{1}$$

and $\sum_j p_jB_j(\omega')=F(\omega')$. This allows us to write for the Fourier transform of equation (C1)

$$\langle\tilde{\phi}^*(\omega)\tilde{\phi}(\omega+\omega')\rangle_\tau=\mathrm{e}^{\mathrm{i}\omega'}\sum_{j_1,j_2}p_{j_1}p_{j_2}(h_{j_2}+h'_{j_1})\sum_{j'_1,j'_2}\left[(\hat{\mathbf{1}}-\mathrm{e}^{\mathrm{i}\omega'}\hat{\Gamma})^{-1}\langle\hat{D}\rangle_\tau\right]_{j_1j_2,j'_1j'_2}-F(\omega') \tag{D6}$$

with

$$h_j=\frac{\prod_{k\neq j}(1-\mathrm{e}^{\mathrm{i}(\omega+\omega')}\gamma^*_k)}{\sum_{j'}p_{j'}\prod_{k\neq j'}(1-\mathrm{e}^{\mathrm{i}(\omega+\omega')}\gamma^*_k)},\qquad h'_j=\frac{\prod_{k\neq j}(1-\mathrm{e}^{-\mathrm{i}\omega}\gamma_k)}{\sum_{j'}p_{j'}\prod_{k\neq j'}(1-\mathrm{e}^{-\mathrm{i}\omega}\gamma_k)},\quad \gamma_k=\langle\mathrm{e}^{\mathrm{i}E_k\tau}\rangle_\tau. \tag{D7}$$

These are analytic functions in $\mathrm{e}^{\mathrm{i}(\omega+\omega')}$ and $\mathrm{e}^{-\mathrm{i}\omega}$, respectively, and their special form implies $\sum_j p_jh_j=\sum_j p_jh'_j=1$. For $\omega'=0$ the normalization $\langle|\tilde{\phi}(\omega)|^2\rangle_\tau=1$ can be obtained from equation (D6) with the help of equation (31). This can be shown by the following reasoning. From appendix A we have $F(\omega'=0)=1$. Then we can write with equation (D6)

$$\begin{aligned}\langle|\tilde{\phi}(\omega)|^2\rangle_\tau &= \sum_{j_1,j_2} p_{j_1} p_{j_2}(h_{j_2}+h'_{j_1})\sum_{j_3,j_4}\Big[(\hat{\mathbf{1}}-\hat{\Gamma})^{-1}\langle\hat{D}\rangle_\tau\Big]_{j_1 j_2, j_3 j_4} - 1\\ &= \sum_{j_1,j_2} p_{j_1} p_{j_2}(h_{j_2}+h'_{j_1}) T_{j_1 j_2} - 1. \end{aligned} \tag{D8}$$

Inserting now $T_{j_1 j_2}$ from equation (31) and use $\sum_j p_j h_j = \sum_j p_j h'_j = 1$ we obtain 1 for this expression.

Appendix E. Analytic properties of $F(\omega)$

When we consider the trace term in equation (19) as

$$\langle|\phi_{n+1}|^2\rangle_\tau = \mathrm{Tr}[(\hat{D}\hat{C})^n \hat{D}\hat{E}\hat{\Pi}] \tag{E1}$$

with the short-hand notation $\hat{D} = \langle\hat{D}\rangle_\tau$, we get in the Zeno limit $\hat{D} \to \hat{\mathbf{1}}$ a vanishing expression except for $n = 0$, since $\hat{E}\hat{\Pi}$ and $\hat{C}$ are projectors with

$$(\hat{E}\hat{\Pi})^2 = \hat{E}\hat{\Pi}, \qquad \hat{C}^2 = \hat{C}, \qquad \hat{E}\hat{\Pi}\hat{C} = \hat{E}\hat{\Pi}(\mathbf{1}-E\Pi)\times(\mathbf{1}-E\Pi) = 0,$$

and since for $n = 0$

$$\mathrm{Tr}(\hat{E}\hat{\Pi}) = \sum_{j,j'=1}^{N} p_j p_{j'} = 1.$$

Returning to equation (E1), we can write with $\hat{C}^2 = \hat{C}$ and $\hat{R} = \hat{\mathbf{1}} - \hat{D}$

$$(\hat{D}\hat{C})^n = \hat{D}(\hat{C}\hat{D}\hat{C})^n = \hat{D}[\hat{C}(\hat{\mathbf{1}}-\hat{R})\hat{C}]^n = \hat{D}(\hat{C}-\hat{C}\hat{R}\hat{C})^n = \hat{D}\hat{C}(\hat{\mathbf{1}}-\hat{R}\hat{C})^n, \tag{E2}$$

such that for $n \geqslant 1$

$$\langle|\phi_{n+1}|^2\rangle_\tau = \mathrm{Tr}[\hat{C}(\hat{\mathbf{1}}-\hat{R}\hat{C})^n\hat{D}\hat{E}\hat{\Pi}\hat{D}] = \mathrm{Tr}[\hat{C}(\hat{\mathbf{1}}-\hat{R}\hat{C})^n(\hat{\mathbf{1}}-\hat{R})\hat{E}\hat{\Pi}(\hat{\mathbf{1}}-\hat{R})]$$

and with $\hat{E}\hat{\Pi}\hat{C} = \hat{C}\hat{E}\hat{\Pi} = 0$

$$\langle|\phi_{n+1}|^2\rangle_\tau = \mathrm{Tr}[\hat{C}(\hat{\mathbf{1}}-\hat{R}\hat{C})^n\hat{R}\hat{E}\hat{\Pi}\hat{R}] = \mathrm{Tr}[\hat{R}\hat{C}(\hat{\mathbf{1}}-\hat{R}\hat{C})^n\hat{R}\hat{E}\hat{\Pi}]. \tag{E3}$$

Moreover, with $\hat{R}\hat{C} = \hat{\mathbf{1}} - (\hat{\mathbf{1}} - \hat{R}\hat{C})$ we get

$$\langle|\phi_{n+1}|^2\rangle_\tau = \mathrm{Tr}[(\hat{\mathbf{1}}-\hat{R}\hat{C})^n\hat{R}\hat{E}\hat{\Pi}] - \mathrm{Tr}[(\hat{\mathbf{1}}-\hat{R}\hat{C})^{n+1}\hat{R}\hat{E}\hat{\Pi}].$$

Now we introduce the projector $\hat{P}$ with $\hat{R} = \hat{P}\hat{R}$. Then we can write

$$\begin{aligned}\hat{C}(\hat{\mathbf{1}}-\hat{R}\hat{C})^n\hat{R} = \hat{C}(\hat{\mathbf{1}}-\hat{R}\hat{C}\hat{P})^n\hat{R} = \hat{C}(\hat{\mathbf{1}}-\hat{P}+\hat{P}-\hat{R}\hat{C}\hat{P})^n\hat{R} = \hat{C}\,[(\hat{\mathbf{1}}-\hat{P})^n \\ +(\hat{P}-\hat{R}\hat{C}\hat{P})^n\,]\,\hat{R}\end{aligned}$$

and since $(\hat{\mathbf{1}}-\hat{P})^n\hat{R} = 0$

$$= \hat{C}(\hat{P}-\hat{R}\hat{C}\hat{P})^n\hat{R}. \tag{E4}$$

The eigenvalues of $\hat{R}\hat{C}\hat{P}$ might be complex. Therefore, it is better to calculate the eigenvalues of the Hermitian matrix

$$(\hat{R}\hat{C}\hat{P})^\dagger\hat{R}\hat{C}\hat{P} = \hat{P}\hat{C}\hat{R}^\dagger\hat{R}\hat{C}\hat{P},$$

whose determinant reads

$$\det(\hat{P}\hat{C}\hat{R}^{\dagger}\hat{R}\hat{C}\hat{P}) = \det(\hat{P}\hat{C}\hat{P})^2 \prod_{j,j'=1;j'\neq j}^{N} |1 - \langle D_{jj'}\rangle_\tau|^2. \tag{E5}$$

A necessary condition for a quick decay of $(\hat{P} - \hat{R}\hat{C}\hat{P})^n$ with n is that the product of $|1 - \langle D_{jj'}\rangle_\tau|^2$ is not small, while the sufficient condition requires that $\det(\hat{P}\hat{C}\hat{P})$ also is not small. To see the latter, we analyze the elements of the projected matrix

$$(\hat{P}\hat{C}\hat{P})_{jj',kk'} = (1 - \delta_{jj'})(1 - \delta_{kk'})(\delta_{jk} - p_k)(\delta_{j'k'} - p_{k'}). \tag{E6}$$

We only consider the projected matrix, which has the following matrix elements with $j \neq j'$ and $k \neq k'$:

$$k = j, k' = j' : (\hat{P}\hat{C}\hat{P})_{jj',jj'} = (1 - p_j)(1 - p_{j'}), \quad k \neq j, k' \neq j': \\ (\hat{P}\hat{C}\hat{P})_{jj',kk'} = p_k p_{k'}$$

and

$$k = j, k' \neq j' : (\hat{P}\hat{C}\hat{P})_{jj',jk'} = -(1 - p_j)p_{k'}, \quad k \neq j, k' = j': \\ (\hat{P}\hat{C}\hat{P})_{jj',kj'} = -p_k(1 - p_{j'}).$$

For the special case of $N = 2$ this gives a 2×2 matrix:

$$\begin{pmatrix} 0 & -p_2(1 - p_2) \\ -p_1(1 - p_1) & 0 \end{pmatrix},$$

whose determinant $-p_1(1 - p_1)p_2(1 - p_2)$ vanishes only for $p_1 = 0, 1$ and/or $p_2 = 0, 1$.

Appendix F. Symmetric two-level system

The matrix structure of the symmetric 2LS reads

$$([A \times B]_{ij,kl}) = (A_{ik}B_{jl}) = \begin{pmatrix} A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\ A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\ A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\ A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22} \end{pmatrix}$$

and

$$(\Gamma_{ij,kl}) = \begin{pmatrix} \Gamma_{11,11} & \Gamma_{11,12} & \Gamma_{11,21} & \Gamma_{11,22} \\ \Gamma_{12,11} & \Gamma_{12,12} & \Gamma_{12,21} & \Gamma_{12,22} \\ \Gamma_{21,11} & \Gamma_{21,12} & \Gamma_{21,21} & \Gamma_{21,22} \\ \Gamma_{22,11} & \Gamma_{22,12} & \Gamma_{22,21} & \Gamma_{22,22} \end{pmatrix}$$

Then we have

$$\langle D^* \times D \rangle_\tau = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \langle e^{2iJ}\tau \rangle_\tau & 0 & 0 \\ 0 & 0 & \langle e^{-2iJ\tau} \rangle_\tau & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & y^* & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with $y = \langle e^{2iJ}\tau \rangle_\tau$. Moreover,

$$(\mathbf{1} - E\Pi) \times (\mathbf{1} - E\Pi) = \frac{1}{4} \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

such that

$$\hat{\Gamma} = \langle D^* \times D \rangle_\tau (\mathbf{1} - E\Pi) \times (\mathbf{1} - E\Pi) = \frac{1}{4} \begin{pmatrix} -1 & 1 & 1 & -1 \\ y & -y & -y & y \\ y^* & -y^* & -y^* & y^* \\ -1 & 1 & 1 & -1 \end{pmatrix}$$

with three vanishing eigenvalues and one eigenvalue $(y + y^* + 2)/4$. With the help of maxima we obtain with $c = z/4$

$$\det(\hat{\mathbf{1}} - z\hat{\Gamma}) = 1 - c(2 + y + y^*) = 1 - \frac{z}{2}[1 + \langle \cos(2J\tau) \rangle_\tau] = 1 - z\langle \cos^2(J\tau) \rangle_\tau$$

$$(\hat{\mathbf{1}} - z\hat{\Gamma})^{-1} = \frac{-1}{1 - z\langle \cos^2(J\tau) \rangle_\tau} \begin{pmatrix} c(y + y^* + 1) - 1 & c & c & -c \\ cy & c(y^* + 2) - 1 & -cy & cy \\ cy^* & -cy^* & c(y + 2) - 1 & cy^* \\ -c & c & c & c(y + y^* + 1) - 1 \end{pmatrix}$$

$$(\hat{\mathbf{1}} - z\hat{\Gamma})^{-1} \langle D^* \times D \rangle_\tau$$

$$= \frac{-1}{1 - z\langle \cos^2(J\tau) \rangle_\tau} \begin{pmatrix} c(y + y^* + 1) - 1 & cy & cy^* & -c \\ cy & c(yy^* + 2y) - y & -cyy^* & cy \\ cy^* & -cyy^* & c(yy^* + 2y^*) - y^* & cy^* \\ -c & cy & cy^* & c(y + y^* + 1) - 1 \end{pmatrix}$$

Then the generating function $F(\omega)$ in equation (23) reads

$$F(\omega) = \frac{4e^{2i\omega}\langle \cos 2J\tau \rangle_\tau - 2e^{i\omega}(\langle \cos 2J\tau \rangle_\tau + 1)}{2e^{i\omega}(\langle \cos 2J\tau \rangle_\tau + 1) - 4},$$

which gives for $\omega = 0$

$$F(0) = 1, \qquad -iF'(0) = 2, \qquad -F''(0) = 2\frac{3 - \langle \cos 2J\tau \rangle_\tau}{1 - \langle \cos 2J\tau \rangle_\tau}.$$

For the ϕ-correlator we get the winding number

$$w_\phi = \frac{1}{2\pi i}\int_0^{2\pi} \partial_{\omega'} \log(\langle\tilde{\phi}^*(\omega)\tilde{\phi}(\omega+\omega')\rangle_\tau)\Big|_{\omega'=0} \mathrm{d}\omega$$
$$= \frac{1}{2\pi i}\int \frac{a_2 z^2 + a_1 z + a_0}{4(y+y^*-2)(z-C)(Cz-1)}\frac{1}{z}\,\mathrm{d}z \qquad \text{(F1)}$$

with $C = \langle\cos J\tau\rangle_\tau$ and $y = \langle \mathrm{e}^{2\mathrm{i}J}\,\tau\rangle_\tau$, with poles

$$z_0 = 0, \qquad z_1 = C, \qquad z_2 = 1/C$$

and with the coefficients

$$a_0 = a_2 = (6x^* + 2x)y^* + (2x^* + 6x)y - 8x^* - 8x$$

$$a_1 = 4[(-x^{*2} - xx^* - 2)y^* + (-xx^* - x^2 - 2)y + x^{*2} + 2xx^* + x^2 + 4],$$

where $x = \langle \mathrm{e}^{-\mathrm{i}J\tau}\rangle_\tau$. After performing the Cauchy integration in equation (F1) for the two poles $z_{0,1}$ we get

$$w_\phi = 2.$$

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