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RIEMANNIAN GEOMETRY OF GROUPS OF DIFFEOMORPHISMS PRESERVING A STABLE HAMILTONIAN STRUCTURE

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Kathrin Helmsauer: *Riemannian geometry of groups of diffeomorphisms preserving a stable Hamiltonian structure*

ABSTRACT

We study the Riemannian geometry of the group of diffeomorphisms of principal S^1-bundles M^{2n+1} preserving a stable Hamiltonian structure (ω,λ) or a Hamiltonian structure ω such that the kernel foliation $\ker\omega$ is periodic with some generator R. Herein, we extend results mainly by Ebin and Marsden [EM70], and more recent work by Ebin [Ebi12], and Ebin and Preston [EP13]. We first determine conditions under which the structure-preserving Sobolev diffeomorphisms $\mathrm{Diff}^s_{\omega,\lambda}(M)$ and $\mathrm{Diff}^s_{R,\omega}(M)$ are smooth submanifolds of $\mathrm{Diff}^s(M)$. Following the strategy used in [EM70], we show that for the S^1-bundle over the cylinder $B = S^1 \times [-1,1]$, the orthogonal projection of the tangent bundles projecting $T\mathrm{Diff}^s(M)|_{\mathrm{Diff}^s_{\omega,\lambda}(M)}$ to $T\mathrm{Diff}^s_{\omega,\lambda}(M)$ is a smooth bundle map. As a consequence, local geodesics and therefore, local solutions to the Euler equation exist. Furthermore, we show long-time existence for solutions to the Euler equation on M preserving R and ω for trivial S^1-bundles $M^{2n+1} = B^{2n} \times S^1$ and compute the Euler equation for the general case.

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CONTENTS

1

INTRODUCTION

The Euler equations in hydrodynamics are a set of quasilinear hyperbolic differential equations to describe the motion of an ideal fluid. On a Riemannian manifold M – possibly with boundary ∂M –, Levi-Civita connection ∇ and (not necessarily Riemannian) volume form vol, the Euler equations are:

$$\begin{aligned} \partial_t v + \nabla_v v &= -\nabla p, \\ \operatorname{div}_{\mathrm{vol}} v &= 0, \end{aligned}$$

for the time-dependent velocity vector field v tangent to the boundary ∂M of some ideal fluid and for the pressure function p. As a special case of the more general Navier-Stokes equations, which deal with viscous fluids, they are of great interest to both mathematicians and physicists. One of the Millenium Prize problems by the Clay Mathematics Institute offers \$1 million to the first person to prove or give a counterexample for the following statement:

> In three space dimensions and time, given an initial velocity field, there exists a vector velocity and a scalar pressure field, which are both smooth and globally defined, that solve the Navier–Stokes equations.

This Millenium Prize problem is still open. To get closer to an answer, mathematicians have been trying to prove or find counterexamples to similar statements for the Euler equation.

Vladimir Arnold [Arn66] showed in 1966 that many equations, in particular the Euler equations of an inviscid incompressible fluid, can be viewed as geodesic flows on the infinite dimensional manifold of volume-preserving diffeomorphisms of M. In his blog, Terence Tao [Tao10] provides a summary of this paper. We will also describe in Section 2.3 how to get from the geodesic equation on the manifold of volume-preserving diffeomorphisms to the Euler equations. Arnold's idea has been used extensively in the past, most notably by Ebin and Marsden [EM70], who study the Hilbert manifold of volume-preserving Sobolev diffeomorphisms and prove existence and uniqueness theorems for solutions to the Euler equations on a compact oriented manifold, possibly with boundary. We summarize the important results in Section 2.4. To apply this to other diffeomorphism groups $D(M)$ of some manifold

M, one has to show that $D(M) \subset \mathrm{Diff}^s(M)$ is a smooth submanifold and that for $\eta \in D(M)$, the orthogonal projections

$$P_\eta : T_\eta \mathrm{Diff}^s(M) \to T_\eta D(M)$$

induced by the given metric on M form a smooth bundle map

$$P : T\mathrm{Diff}^s(M)|_{D(M)} \to TD(M).$$

Further work by Ebin and his coauthors includes long-time existence of solutions to the Euler equation for volume-preserving diffeomorphisms of two-dimensional manifolds [Ebi84], long-time existence for symplectomorphisms [Ebi12], and local existence for contactomorphisms of certain contact manifolds [EP15], with some results concerning the long-time existence for strict contactomorphisms (quantomorphisms) of S^1-principal bundles already published in [EP13]. For more details, see Section 2.5.

This thesis proves some results in a similiar spirit for principal bundles $S^1 \to M^{2n+1} \xrightarrow{\pi} B^{2n}$ with a stable or stabilizable Hamiltonian structure and their structure-preserving diffeomorphisms. A stable Hamiltonian structure is a pair (ω, λ) such that the closed two-form $\omega \in \Omega^2(M)$ has maximal rank, $\lambda \in \Omega^1(M)$ satisfies $\ker \omega \subset \ker \mathrm{d}\lambda$ and $\lambda \wedge \omega^n$ is a volume form. In Sections 3.1 and 3.2, we start by defining manifolds with a (stable or stabilizable) Hamiltonian structure and their structure-preserving diffeomorphisms. In Sections 3.3 and 3.4, we restrict our manifolds to S^1-principal bundles such that the Reeb vector field defined by the stable Hamiltonian structure generates the S^1-action. In this case, the stabilizing one-form λ is also a connection form for our circle bundle and $\tau \in \Omega^2(B)$ defined by $\mathrm{d}\lambda = \pi^*\tau$ is the curvature form. For trivial S^1-bundles, which we discuss in Section 3.5, the curvature form τ is exact, i. e. $\tau = \mathrm{d}\mu$ for some $\mu \in \Omega^1(B)$. The form μ is uniquely defined by the identity $\lambda = \mathrm{d}\theta + \pi^*\mu$, where we denote the S^1-coordinate of $M = B \times S^1$ by θ. While it is well known that the classical Sobolev diffeomorphism groups discussed in Section 2.5 are smooth submanifolds of the full diffeomorphism groups, we have to formulate and prove conditions such that the diffeomorphisms preserving the stable Hamiltonian structure (ω, λ) are indeed a smooth submanifold of the full diffeomorphism group. To that end, we identify the diffeomorphisms $\mathcal{D}^s$ of the base manifold B that lift to diffeomorphisms preserving (ω, λ) on $M = B \times S^1$ as

$$\mathcal{D}^s = \left\{ \nu \in \mathrm{Diff}^s_{\sigma,\tau}(B) \;\middle|\; \int_\gamma (\mu - \nu^*\mu) \in \mathbb{Z} \text{ for any } \gamma \in H_1(B;\mathbb{Z}) \right\}.$$

In particular, we show in Theorem 3.29 that $\mathcal{D}^S \subset \mathrm{Diff}^s(B)$ is a smooth Hilbert submanifold iff

$$\mathcal{D}^s \times S^1 \cong \mathrm{Diff}^s_{\omega,\lambda}(B \times S^1) \subset \mathrm{Diff}^s(B \times S^1)$$

is also a smooth Hilbert submanifold. In Section 3.6, we describe the metrics we consider on $M = B \times S^1$ and how results for smooth bundle maps transfer under diffeomorphisms of manifolds with a different stable Hamiltonian structure. Similarly to the trivial bundle case, we show in Section 3.7 (specifically Theorem 3.43) for general S^1-principal bundles $S^1 \to M \xrightarrow{\pi} B$ that there also is a subset $\mathcal{D}^s \subset \mathrm{Diff}^s(B)$ of diffeomorphisms of B that lift to $\mathrm{Diff}^s_{\omega,\lambda}(M)$. Then, $\mathrm{Diff}^s_{\omega,\lambda}(M)$ is also an S^1-bundle

$$S^1 \to \mathrm{Diff}^s_{\omega,\lambda}(M) \to \mathcal{D}^s.$$

In particular, we again get that $\mathrm{Diff}^s_{\omega,\lambda}(M) \subset \mathrm{Diff}^s(M)$ is a smooth submanifold iff $\mathcal{D}^s \subset \mathrm{Diff}^s(B)$ is a smooth submanifold.

Chapter 4 fully proves all the statements for the cylinder $B = S^1 \times [-1,1]$ and the trivial circle bundle over the cylinder

$$M = B \times S^1 = (S^1 \times [-1,1]) \times S^1.$$

Any stable Hamiltonian structure (ω,λ) on M induces two two-forms (σ,τ) on $B = S^1 \times [-1,1]$ by $\omega = \pi^*\sigma$ and $\mathrm{d}\lambda = \pi^*\tau$. Since B is two-dimensional, σ is an area form and τ is a multiple of σ, i. e. $\tau = h\sigma$ for some $h \in C^\infty(B,\mathbb{R})$. Section 4.1 deals with the standard metric on B with its induced area form σ and $\tau = z\sigma$, where $z \in [-1,1]$ denotes the height coordinate on B. We prove both that $\mathcal{D}^S \subset \mathrm{Diff}^s(B)$ is a smooth submanifold and that the projection $P : T\mathrm{Diff}^s(B)|_{\mathcal{D}^s} \to T\mathcal{D}^s$ is a smooth bundle map. In Section 4.2, we compute the Euler equation on B with respect to the standard metric and its area form for vector fields in $T_{\mathrm{id}}\mathrm{Diff}^s_{\sigma,\tau}(B)$, which turns out to be trivial. Similarly, in Section 4.3, we lift the two-forms (σ,τ) from Section 4.1 to a stable Hamiltonian structure (ω,λ) on M and prove that $\mathrm{Diff}^s_{\omega,\lambda}(M) \subset \mathrm{Diff}^s(M)$ is a smooth submanifold and the projection $P : T\mathrm{Diff}^s(M)|_{\mathrm{Diff}^s_{\omega,\lambda}(M)} \to T\mathrm{Diff}^s_{\omega,\lambda}(M)$ is a smooth submanifold. As before, in Section 4.4 we show that the corresponding Euler equation is trivial. In Sections 4.5 and 4.6, we then extend those results to any metric on B, its Riemannian area form $\sigma_b := b\sigma$ for some $b \in C^\infty(B,\mathbb{R})$ and $\tau_b = z\sigma_b$ on B. For the S^1-bundle $M = B \times S^1$ in Sections 4.7 and 4.8, we consider the bundle metric induced by the given metric on B. We let $\omega_b := \pi^*\sigma_b$ and for $\lambda = \mathrm{d}\theta + \pi^*\mu$ such that $\mathrm{d}\lambda = \pi^*\tau_b$, we choose one representative for each possible cohomology class of μ. In Section 4.9, we now also include any possible primitive μ for τ_b, i. e. we explain how to transform the metric on M such that we can change μ by exact one-forms to end up in one of the cases of the previous section. Finally, in Section 4.10, we also allow more general submersions $h \in C^\infty(B,\mathbb{R})$ and consider $\tau = h\sigma$. The last two sections in this chapter, Sections 4.11 and 4.12, provide a brief outlook on how to possibly construct an example where $\mathrm{Diff}^s_{\omega,\lambda}(M)$ is not a smooth submanifold of $\mathrm{Diff}^s(M)$ and what happens with two-dimensional base manifolds other than the cylinder $B = S^1 \times [-1,1]$.

In Chapter 5, we also discuss S^1-principal bundles M with a Hamiltonian structure ω such that the kernel foliation $\ker\omega$ is periodic with some generating vector field R. Such a Hamiltonian structure is always stabilizable but, in contrast to the earlier chapters, we now consider the diffeomorphisms preserving only ω and R, and not neccessarily the stabilizing one-form. In Section 5.1, we recall our results on the diffeomorphism group $\mathrm{Diff}^s_{R,\omega}(M)$, which are already shown in Chapter 3. For trivial bundles $M = B \times S^1$ with the standard S^1-invariant bundle metric (Section 5.2) and a general S^1-invariant bundle metric (Section 5.3), we compute the Euler equation given by variation of the energy of paths in the diffeomorphism group $\mathrm{Diff}^s_{R,\omega}(B \times S^1)$. In the standard case, we can also prove long-time existence of solutions to the Euler equation.

2

THE EULER EQUATION

2.1 The Hilbert manifold $\mathrm{Diff}^s(M)$

Let M be a compact Riemannian manifold. For now, we will assume that M has no boundary even though we will later extend the results to manifolds with boundary. We will denote the Riemannian metric on M by $g(\cdot,\cdot)$ or $\langle\cdot,\cdot\rangle$. Let also $s \in \mathbb{N}$, $s > \frac{\dim M}{2} + 1$, so that by the Sobolev Lemma, $H^s(M,M) \hookrightarrow C^1(M,M)$. In particular, any element of $H^s(M,M)$ is differentiable.

Definition. Let $C^1\mathrm{Diff}(M)$ denote the group of C^1-diffeomorphisms of M, i. e.

$$C^1\mathrm{Diff}(M) := \left\{\eta \in C^1(M,M) \,\middle|\, \eta \text{ is bijective and } \eta^{-1} \in C^1(M,M)\right\}$$

and define the *H^s-diffeomorphisms* $\mathrm{Diff}^s(M)$ as the connected component containing the identity in $H^s(M,M) \cap C^1\mathrm{Diff}(M)$.

Equivalently, using the Sobolev Lemma, we can identify $\mathrm{Diff}^s(M)$ as the connected component containing the identity in

$$\left\{\eta \in H^s(M,M) \,\middle|\, \eta \text{ is bijective and } \eta^{-1} \in H^s(M,M)\right\}. \tag{2.1}$$

We first want to prove that $\mathrm{Diff}^s(M)$ is a Hilbert manifold. To that end, we will construct charts for the continuous maps $C(N,M)$ for compact manifolds N (possibly with boundary) and then restrict those to $H^s(N,M)$ and then finally to $\mathrm{Diff}^s(M)$. This section follows the computations in [Cie92], which in turn is based on the results in [Elí67]. There is also a short summary in Section 2 of [EM70].

Note that the Riemannian metric on M induces an exponential map on a neighbourhood $U_p \subset T_pM$ of the origin for every $p \in M$, i. e. we have $\exp_p : U_p \to M$, which sends $x \in T_pM$ onto $\gamma(1)$ for the unique geodesic γ satisfying $\gamma(0) = p$ and $\gamma'(0) = x$. Those exponential maps fit together to a smooth bundle map $\exp : U \to M \times M$, $(p,x) \mapsto (p, \exp_p x)$ defined on an open neighbourhood $U \subset TM$ of the zero section. We can choose U sufficiently small such that $\exp : U \to M \times M$ is a diffeomorphism onto an open neighbourhood of the diagonal and such that the image $\exp(U) \subset M \times M$ is invariant under the diffeomorphism $(p,q) \mapsto (q,p)$ of $M \times M$. We can further choose $U_p = U \cap T_pM$.

Let $\eta \in C(N,M)$. The space $E_\eta := C(N,\eta^*TM)$ of continuous sections in the pullback bundle $\eta^*TM \to N$ is a Banach space with norm $|\xi| := \max_{p\in M} |\xi(p)|$. The pullback

$$\eta^*U := \Big\{(p,x) \,\Big|\, \big(\eta(p),x\big) \in U\Big\} \subset \eta^*TM$$

is an open neighbourhood of the zero section and

$$\mathcal{V}_\eta := C(N,\eta^*U) = \Big\{\xi \in E_\eta \,\Big|\, \big(\eta(p),\xi(p)\big) \in U \text{ for all } q \in N\Big\}$$

is an open neighbourhood of the origin in E_η. The exponential map induces a continuous map

$$\exp_\eta : \mathcal{V}_\eta \to C(N,M), \quad (\exp_\eta \xi)(p) := \exp_{\eta(p)} \xi(p)$$

which is a homeomorphism onto its image

$$\mathcal{U}_\eta := \Big\{\rho \in C(N,M) \,\Big|\, \big(\eta(p),\rho(p)\big) \in \exp(U) \text{ for all } q \in N\Big\}.$$

Proposition 2.1. *The charts* $\exp_\eta^{-1} : \mathcal{U}_\eta \to \mathcal{V}_\eta$ *for* $\eta \in C(N,M)$ *define a smooth Banach atlas on* $C(N,M)$*. A different Riemannian metric induces an equivalent atlas. The Banach manifold* $C(N,M)$ *is covered by the chart domains* U_η *centered at smooth maps* $\eta \in C^\infty(N,M)$*.*

Let $VB(N)$ denote the category of smooth vector bundles over N and $\mathcal{B}$ the category of Banachable spaces.

Definition. A covariant functor $\mathfrak{T} : VB(N) \to \mathcal{B}$ is a *section functor over* N if for all vector bundles $E,F \in VB(N)$,

(a) elements of $\mathfrak{T}(E)$ are equivalence classes of sections in E, and

(b) the map $\mathfrak{T} : C^\infty(\mathrm{Hom}(E,F)) \to \mathcal{L}(\mathfrak{T}(E),\mathfrak{T}(F))$, $\phi \mapsto \mathfrak{T}(\phi)$ is continuous linear, where $\mathfrak{T}(\phi)(\xi) = \phi \circ \xi$.

Definition. A section functor $\mathfrak{S} : VB(N) \to B$ is a *manifold model*, if for all $E,F \in VB(N)$

(a) $\mathfrak{S}(E) \hookrightarrow C(N,E)$ is continuous linear.

(b) $\mathfrak{S}(\mathrm{Hom}(E,F)) \hookrightarrow \mathcal{L}(\mathfrak{S}(E),\mathfrak{S}(F))$ is continuous linear.

(c) Let $\mathcal{O} \subset E$ be an open subset projecting onto N and $\psi : \mathcal{O} \to F$ be a smooth fibre preserving map. Then for each $\xi \in \mathfrak{S}(\mathcal{O}) := \{\xi \in \mathfrak{S}(E) | \xi(N) \subset \mathcal{O}\}$, we have $\psi \circ \xi \in \mathfrak{S}(F)$ and the corresponding map

$$\mathfrak{S}(\psi) : \mathfrak{S}(\mathcal{O}) \to \mathfrak{S}(F), \quad \xi \mapsto \phi \circ \xi$$

is continuous.

Definition. A section functor $\mathfrak{T} : VB(N) \to \mathcal{B}$ is *compact* with respect to a manifold model $\mathfrak{S}$ if for any $E, F \in VB(N)$,

(a) $\mathfrak{S}(\mathrm{Hom}(E,F)) \hookrightarrow \mathcal{L}(\mathfrak{T}(E), \mathfrak{T}(F))$ is continuous linear.

(b) $\mathfrak{T}(\mathrm{Hom}(E,F)) \hookrightarrow \mathcal{L}(\mathfrak{S}(E), \mathfrak{S}(F))$ is continuous linear.

Theorem 2.2. *Let N be a compact n-dimensional manifold (possibly with boundary) and M be an m-dimensional manifold without boundary. Let further $\mathfrak{S}$ be a manifold model over N. Then the charts $\mathfrak{S}(\exp_\eta^{-1}) : \mathfrak{S}(\mathcal{U}_\eta) \to \mathfrak{S}(\mathcal{V}_\eta)$ for $\eta \in C^\infty(N,M)$ define the structure of a smooth Banach manifold on $\mathfrak{S}(N,M)$.*

Let $\tau : TM \to M$ denote the canonical bundle projection.

Corollary 2.3. *Let M, N be as in the previous theorem. The space $H^s(N,M)$ of Sobolev maps for $s \in \mathbb{N}$ and $s > \frac{n}{2}$ is a separable smooth Hilbert manifold with tangent bundle*

$$TH^s(N,M) = H^s(N,TM) = \bigcup_{\eta \in H^s(N,M)} T_\eta H^s(N,M)$$

for

$$T_\eta H^s(N,M) = \{V \in H^s(N,TM) | \tau \circ V = \eta\}.$$

The C^1-diffeomorphisms $C^1\mathrm{Diff}(M)$ are open in $C^1(M,M)$. For $s > \frac{\dim M}{2} + 1$, the Sobolev lemma implies that $H^s(M,M) \subset C^1(M,M)$ is a continuous linear inclusion, hence $\mathrm{Diff}^s(M) \subset H^s(M,M)$ is open and $\mathrm{Diff}^s(M)$ is a Hilbert (sub-)manifold, see §3 in [Ebi70].

Now let M have boundary. We consider the double $\tilde{M} = M \cup_{\partial M} M$ and choose a metric such that ∂M is totally geodesic. Then the image of the exponential charts on $H^s(M,\tilde{M})$ is always already contained in M and, similarly to Eq. (2.1), we can define $\mathrm{Diff}^s(M)$ as the identity component in

$$\{\eta \in H^s(M,\tilde{M}) \mid \mathrm{im}(\eta) \subset M,\ \eta \text{ is bijective and } \eta^{-1} \in H^s(M,M)\}.$$

Using this, one can show

Corollary 2.4 (§3 in [Ebi70], §6 in [EM70]). *Let M be a compact manifold with or without boundary and $s > \frac{\dim M}{2} + 1$, then $\mathrm{Diff}^s(M)$ is a smooth Hilbert manifold.*

Theorem 2.5 ([EM70], Proofs of Theorems 6.1 and 6.2). *(a) Let M be a compact manifold without boundary and $N \subset M$ a closed submanifold without boundary. Then,*

$$\mathrm{Diff}^s_N(M) := \{\eta \in \mathrm{Diff}^s(M) \mid \eta(N) \subset N\}$$

and

$$\mathrm{Diff}^s_{N,p}(M) := \{\eta \in \mathrm{Diff}^s(M) \mid \eta(x) = x \text{ for any } x \in N\}$$

are smooth submanifolds of $\operatorname{Diff}^s(M)$.

(b) *Let M be a compact manifold with boundary ∂M, then* $\operatorname{Diff}^s(M)$ *is a smooth manifold and*

$$\operatorname{Diff}^s_p(M) := \{\eta \in \operatorname{Diff}^s(M) \mid \eta(x) = x \text{ for all } x \in \partial M\}$$

is a smooth submanifold of $\operatorname{Diff}^s(M)$.

Now we will describe an atlas of the tangent bundle $T\operatorname{Diff}^s(M) \to \operatorname{Diff}^s(M)$ over the given atlas on $\operatorname{Diff}^s(M)$ using the exponential maps. The metric on M induces a Levi-Civita connection ∇. For any $(p,x) \in TM$, let V be a neighbourhood of p in M such that $\exp_p : T_pM \to M$ maps some neighbourhood V' of 0 in T_pM diffeomorphically onto V. Recall the canonical projection $\tau : TM \to M$. Let further denote $\gamma_p : \tau^{-1}(V) \to T_pM$ the smooth fibrewise isometry such that for $(q,y) \in \tau^{-1}(V) \subset TM$, we parallelly transport y from q to p along the unique geodesic in V. For $u \in T_pM$, define the translation $R_{-u} : T_pM \to T_pM$, $R_{-u}(x) = x - u$. Then we define the *connection map*

$$\begin{aligned} K_{(p,x)} : T_{(p,x)}TM &\to T_pM, \\ A &\mapsto T_{(p,x)}(\exp_p \circ R_{-x} \circ \gamma_p)(A). \end{aligned}$$

If we write $A = T_pX(Y_p)$ for some $X \in \mathfrak{X}(M)$, which we view as a map $X : M \to TM$ such that $X_p = X(p) = x$, and $Y_p \in T_pM$, then

$$K_{(p,x)}(A) = K_{(p,x)}(T_pX(Y_p)) = (\nabla_{Y_p}X)_p,$$

see also [Dom62, §§2–4]. The map $\tau : TM \to M$ also induces the bundle $T\tau : TTM \to TM$ with vertical bundle $T^vTM := \ker T\tau \subset TTM$. The map $T(\exp_p \circ R_{-x})$ is an isomorphism $T_{(p,x)}T_pM \to T_pM$. Let $\iota_p : T_pM \to TM$ denote the inclusion map, then

$$T^v_{(p,x)}TM = T\iota_p(T_{(p,x)}T_pM)$$

and

$$T(\iota \circ \gamma)(A) = A$$

for any $A \in T^v_{(p,x)}TM$. Hence, $K_{(p,x)}|_{T^v_{(p,x)}TM} : T^v_{(p,x)}TM \to T_pM$ is an isomorphism. Further, we define

$$(T_{(p,x)}\exp)|_{T^v_{(p,x)}TM} = T_{(p,x)}(\exp|_{T_pM}) : T^v_{(p,x)}TM \to T_{\exp_p(x)}TM.$$

Finally, we let

$$\nabla_2 \exp_{(p,x)} := (T_x \exp)|_{T^v_{(p,x)}TM} \circ (K|_{T^v_{(p,x)}TM})^{-1} : T_pM \to T_{\exp_p(x)}M. \tag{2.2}$$

Proposition 2.6 ([Elí67], Theorem 5.2). *Let $s \geq 4$. The bundle $\tau : TM \to M$ induces a vector bundle*

$$\begin{aligned} \mathfrak{S}(\tau) : \mathfrak{S}(N,TM) &\to \mathfrak{S}(N,M) \\ \alpha &\mapsto \tau \circ \alpha \end{aligned}$$

of class C^{s-3}, which is naturally equivalent to the tangent bundle of $\mathfrak{S}(N,M)$. Moreover, given any connection on M, let $\mathfrak{S}(\exp) : \mathfrak{S}(\mathcal{D}_\eta) \to \mathfrak{S}(N,M)$ be the natural chart centered at $\eta \in C^r(N,M)$. Then,

$$\begin{aligned} \mathfrak{S}(\nabla_2 \exp) : \mathfrak{S}(\mathcal{D}_\eta) \times \mathfrak{S}(E_\eta) &\to \mathfrak{S}(N,TM) \\ (\alpha,\beta) &\mapsto \nabla_2 \exp \circ (\alpha,\beta) \end{aligned}$$

is a trivialization of $\mathfrak{S}(\tau)$ over $\mathfrak{S}(\exp)$ corresponding to the tangent trivialization $T\mathfrak{S}(\exp)$ under the bundle equivalence.

Since $\mathrm{Diff}^s(M) \subset H^s(M,M)$ is an open subset, we have local charts for any $\eta \in \mathrm{Diff}^s(M)$ given by

$$\begin{aligned} T_\eta \mathrm{Diff}^s(M) = \{X \in H^s(M,TM) \mid \tau \circ X = \eta\} &\to \mathrm{Diff}^s(M) \\ X &\mapsto \Big(\exp_\eta X : M \to M, \\ &\qquad\qquad p \mapsto \exp_{\eta(p)} X(p)\Big). \end{aligned}$$

Finally, we want to adapt the last proposition to the tangent bundle $T\mathrm{Diff}^s(M)$. To that end, note that for any $p \in M$, the map $\nabla_2 \exp_{(\eta(p),X(p))}$ maps $T_{\eta(p)}M$ to the space $T_{\exp_{\eta(p)} X(p)}M$. For any $Y \in T_\eta \mathrm{Diff}^s(M)$, we define the map

$$\begin{aligned} (\nabla_2 \exp_{(\eta,X)})(Y) : M &\to TM \\ p &\mapsto \big(\nabla_2 \exp_{(\eta(p),X(p))}\big)(Y(p)), \end{aligned}$$

hence $(\nabla_2 \exp_{(\eta,X)})(Y)(p) \in T_{\exp_\eta X} \mathrm{Diff}^s(M)$.

Corollary 2.7. *Local charts for the Hilbert bundle $T\mathrm{Diff}^s(M) \to \mathrm{Diff}^s(M)$ in a neighbourhood of any $\eta \in \mathrm{Diff}^s(M)$ are given by*

$$\begin{aligned} T_\eta \mathrm{Diff}^s(M) \times T_\eta \mathrm{Diff}^s(M) &\to T\mathrm{Diff}^s(M) \\ (X,Y) &\mapsto \Big(\exp_\eta X,\ \big(\nabla_2 \exp_{(\eta,X)}\big)(Y)\Big). \end{aligned}$$

2.2 Riemannian metrics on $\mathrm{Diff}^s(M)$ and $\mathrm{Diff}^s_{\mathrm{vol}}(M)$

We first recite the standard proof using the implicit function theorem to show that $\mathrm{Diff}^s_{\mathrm{vol}}(M) \subset \mathrm{Diff}^s(M)$ is a smooth submanifold, which can also be found in [EM70].

Theorem 2.8 ([EM70], Theorems 4.2 and 8.1)**.** *Let*

$$\mathrm{Diff}^s_{\mathrm{vol}}(M) := \{\eta \in \mathrm{Diff}^s(M) \mid \eta^*\mathrm{vol} = \mathrm{vol}\}.$$

Then $\mathrm{Diff}^s_{\mathrm{vol}}(M) \subset \mathrm{Diff}^s(M)$ *is a smooth Hilbert submanifold.*

Proof. Define

$$[\mathrm{vol}]^{s-1} := \mathrm{vol} + \mathrm{d}H^s(\Lambda^{n-1}M) \subset H^{s-1}(\Lambda^n M).$$

This is a closed affine subspace of $H^{s-1}(\Lambda^n M)$ because of the Hodge decomposition of n-forms. Now let $\eta \in \mathrm{Diff}^s(M)$. Then $\eta^*\mathrm{vol} = \mathrm{vol} + \alpha$ for some n-form α and we can compute

$$0 = \int_M (\eta^*\mathrm{vol} - \mathrm{vol}) = \int_M \alpha,$$

hence α is exact. This implies $[\eta^*\mathrm{vol}]^{s-1} = [\mathrm{vol}]^{s-1}$, or equivalently $\eta^*\mathrm{vol} \in [\mathrm{vol}]^{s-1}$. We want to use the implicit function theorem for Hilbert manifolds, so we define the smooth map

$$\psi : \mathrm{Diff}^s(M) \to H^{s-1}(\Lambda^n M), \quad \eta \mapsto \eta^*\mathrm{vol}$$

with tangent map

$$T_\eta\psi : T_\eta\mathrm{Diff}^s(M) \to H^{s-1}(\Lambda^n M), \quad V \mapsto \eta^*(\mathcal{L}_{V\circ\eta^{-1}}\mathrm{vol}).$$

At the identity, we get for any vector field $X \in T_{\mathrm{id}}\mathrm{Diff}^s(M)$

$$\begin{aligned} T_{\mathrm{id}}\psi(X) &= \mathrm{id}^*(\mathcal{L}_{X\circ\mathrm{id}^{-1}}\mathrm{vol}) \\ &= \mathcal{L}_X\mathrm{vol} = \mathrm{d}\iota_X\mathrm{vol}. \end{aligned}$$

We first want to show that $T_{\mathrm{id}}\psi$ is surjective. To that end, let $\mathrm{d}\alpha \in \mathrm{d}H^s(\Lambda^{n-1}M) = T_{\mathrm{vol}}[\mathrm{vol}]^{s-1}$. Since vol is non-degenerate, there is an isomorphism

$$H^s(TM) \to H^s(\Lambda^{n-1}M), \quad X \mapsto \iota_X\mathrm{vol}.$$

Hence, there is $X \in H^s(\Lambda^{n-1}M)$ such that $\iota_X\mathrm{vol} = \alpha$ and

$$T_{\mathrm{id}}\psi(X) = \mathrm{d}\iota_X\mathrm{vol} = \mathrm{d}\alpha.$$

For any other diffeomorphism $\eta \in \mathrm{Diff}^s(M)$, both η^* and the right translation by η are isomorphisms and therefore, $T_\eta\psi$ is also surjective. Finally, $\mathrm{Diff}^s_{\mathrm{vol}}(M) = \psi^{-1}(\mathrm{vol}) \subset \mathrm{Diff}^s(M)$ is a closed submanifold. □

Theorem 2.9 ([EM70], Theorem 3.1)**.** *Let M be a compact n-dimensional manifold without boundary, $s > \frac{n}{2} + 2$ and* $\mathrm{Diff}^s(M)$ *the group of H^s diffeomorphisms.*

(a) If V is an H^s vector field on M, its flow η_t is a one parameter subgroup of $\mathrm{Diff}^s(M)$.

(b) The curve $t \mapsto \eta_t$ *is of class* C^1.

(c) The map $\mathrm{E} : T_e\mathrm{Diff}^s(M) \to \mathrm{Diff}^s(M)$, $V \mapsto \eta_1$ *is continuous (but not* C^1*).*

Theorem 2.10 ([EM70], Theorem 6.3). *For* $s > \frac{n}{2} + 2$, *the two groups* $\mathrm{Diff}^s_N(M)$ *and* $\mathrm{Diff}^s_{N,p}(M)$ *as well as* $\mathrm{Diff}^s(M)$ *and* $\mathrm{Diff}^s_p(M)$ *of the previous theorem admit exponential maps. That is in (a), if* V *is an* H^s *vector field on* M *which is tangent to* N *(resp. 0 on* N*) the flow of* V *is a one parameter subgroup of* $\mathrm{Diff}^s_N(M)$ *(resp.* $\mathrm{Diff}^s_{N,p}(M)$*). In (b), if* V *is an* H^s *vector field on* M *parallel to* ∂M *(resp. 0 on* ∂M*), the flow of* V *is a one parameter subgroup of* $\mathrm{Diff}^s(M)$ *(resp.* $\mathrm{Diff}^s_p(M)$*). A similar result holds for time dependent vector fields.*

Definition. A *weak pseudo-Riemannian metric* on some manifold M is a symmetric (0,2)-tensor field g such that at any point $x \in M$, $g_x(v_x, w_x) = 0$ for all $w_x \in T_xM$ implies that $v_x = 0$. A *weak Riemannian structure* or *weak Riemannian metric* is a weak pseudo-Riemannian metric that is also positive definite.

Note that the non-degeneracy condition given in the definition of a weak Riemannian structure only implies that the linear map $T_xM \to T_x^*M$, $v_x \mapsto g_x(v_x, \cdot)$ is injective but not necessarily an isomorphism.

Now let $\tau : TM \to M$ denote the canoncial projection of the tangent bundle of M onto M. Note that for $\eta \in \mathrm{Diff}^s(M)$ and $s > \frac{n}{2} + 1$, we have

$$T_\eta \mathrm{Diff}^s(M) = \{V \in H^s(M, TM) \mid \tau \circ V = \eta\}.$$

At the identity, we will also use the notation

$$\mathfrak{X}^s(M) := T_{\mathrm{id}}\mathrm{Diff}^s(M),$$

and we can define a metric for $V, W \in T_{\mathrm{id}}\mathrm{Diff}^s(M)$ by

$$\langle V, W \rangle := \int_M \langle V(x), W(x) \rangle_x \,\mathrm{vol}. \tag{2.3}$$

There are two natural extensions to weak Riemannian structures on $\mathrm{Diff}^s(M)$, which coincide for $\eta \in \mathrm{Diff}^s_{\mathrm{vol}}(M)$: First, we can extend Eq. (2.3) to a right-invariant weak Riemannian structure on the full tangent space, i. e. for $V, W \in T_\eta\mathrm{Diff}^s(M)$, we let

$$\langle V, W \rangle := \int_M \langle V(x), W(x) \rangle_{\eta(x)} \,\eta^*\mathrm{vol}. \tag{2.4}$$

We will use the second choice, namely for $V, W \in T_\eta\mathrm{Diff}^s(M)$, we let

$$\langle V, W \rangle := \int_M \langle V(x), W(x) \rangle_{\eta(x)} \,\mathrm{vol}. \tag{2.5}$$

The first part of Theorem 2.11 shows that this also defines a weak Riemannian structure on $\mathrm{Diff}^s(M)$, although it is only right-invariant under the action of $\mathrm{Diff}^s_{\mathrm{vol}}(M)$ and not the full diffeomorphism group.

Note that for $\eta \in \operatorname{Diff}^s_{\mathrm{vol}}(M)$ and $V, W \in T_\eta \operatorname{Diff}^s(M)$, η satisfies $\eta^*\mathrm{vol} = \mathrm{vol}$ and hence, the two options (2.4) and (2.5) coincide on $\operatorname{Diff}^s_{\mathrm{vol}}(M)$.

Theorem 2.11 ([EM70], Theorem 9.1)**.** *Let M be compact without boundary with a Riemannian metric $\langle\cdot,\cdot\rangle$ given. We define a bilinear form on $T_\eta \operatorname{Diff}^s(M)$ by*

$$(V,W) = \int_M \langle V(x), W(x)\rangle_{\eta(x)} \,\mathrm{vol}(x). \tag{2.5 rev.}$$

Then:

(a) $(\cdot,\cdot)$ defines a weak Riemannian structure on $\operatorname{Diff}^s(M)$,

(b) $(\cdot,\cdot)$ has associated a unique torsion free affine connection $\bar\nabla$; that is, for smooth vector fields X, Y, Z on $\operatorname{Diff}^s(M)$, we have

  i) $X(Y,Z) = (\bar\nabla_X Y, Z) + (Y, \bar\nabla_X Z)$ and

  ii) $\bar\nabla_X Y - \bar\nabla_Y X = [X,Y]$.

(c) Let $\exp : TM \to M$ be the exponential map corresponding to the connection ∇ on M. Then $\mathrm{E} : T\operatorname{Diff}^s(M) \to \operatorname{Diff}^s(M)$ defined by $\mathrm{E}(V) = \exp \circ V$ is the exponential map of $\bar\nabla$; E is defined only on a neighbourhood of the zero section of $T\operatorname{Diff}^s(M)$, and is a C^∞ mapping onto a neighbourhood of $\mathrm{id} \in \operatorname{Diff}^s(M)$.

2.3 Derivation of the Euler equation

Let $\eta(t) : [0,T] \to \operatorname{Diff}^s_{\mathrm{vol}}(M)$ be a path in $\operatorname{Diff}^s_{\mathrm{vol}}(M)$ with tangent vector $\dot\eta(t) \in T_{\eta(t)} \operatorname{Diff}^s_{\mathrm{vol}}(M)$. We define a time-dependent, divergence-free vector field

$$v(t) \in \mathfrak{X}^s_{\mathrm{div}}(M) := T_{\mathrm{id}} \operatorname{Diff}^s_{\mathrm{vol}}(M) = \left\{ u \in \mathfrak{X}^s(M) \,\middle|\, \operatorname{div}_{\mathrm{vol}} u = 0 \right\}$$

via

$$\dot\eta(t) = v(t) \circ \eta(t)$$

and the energy

$$\begin{aligned} E(\eta(t)) &= \frac{1}{2} \int_{[0,T]} \Big(\dot\eta(t), \dot\eta(t)\Big) \mathrm{d}t \\ &\overset{(2.5)}{=} \frac{1}{2} \int_{[0,T]} \int_M \langle \dot\eta(t)(x), \dot\eta(t)(x)\rangle_{\eta(t)(x)} \,\mathrm{vol}\,\mathrm{d}t. \end{aligned}$$

The path $\eta(t)$ is a geodesic in $\operatorname{Diff}^s(M)$ iff it is an extremal point of the variation of the energy. We consider a variation $\eta(t,\tau)$ of $\eta(t,0) = \eta(t)$ with fixed end points $\eta(0,\tau) = \eta(0)$ and $\eta(T,\tau) = \eta(T)$, i. e. a variation in the direction

$$\sigma(t) = \partial_\tau \eta(t,\tau)|_{\tau=0} \in T_{\eta(t)} \operatorname{Diff}^s_{\mathrm{vol}}(M).$$

Again, we define a corresponding time-dependent, divergence-free vector field $w(t) \in \mathfrak{X}^s_{\text{div}}(M)$ via

$$\sigma(t) = w(t) \circ \eta(t).$$

Because of the fixed end points of the variation $\eta(t,\tau)$, σ satisfies $\sigma(0) = 0 = \sigma(T)$. This yields

$$\begin{aligned}
0 &= \partial_\tau E(\eta(t,\tau))|_{\tau=0} \\
&= \frac{1}{2}\int_0^T \int_M \partial_\tau \langle \dot\eta(t,\tau)(x),\ \dot\eta(t,\tau)(x)\rangle_{\eta(t,\tau)(x)} \text{ vol d}t\ |_{\tau=0} \\
&= \frac{1}{2}\int_0^T \int_M \partial_\tau \langle \dot\eta(t,\tau)(\eta^{-1}(t,\tau)(x)),\ \dot\eta(t,\tau)(\eta^{-1}(t,\tau)(x))\rangle_x \text{ vol d}t\ |_{\tau=0}
\end{aligned}$$

since the metric on $\text{Diff}^s_{\text{vol}}(M)$ is right-invariant

$$= \int_0^T \int_M \langle \partial_\tau\dot\eta(t,\tau)(\eta^{-1}(t,\tau)(x)),\ \dot\eta(t,\tau)(\eta^{-1}(t,\tau)(x))\rangle_x \text{ vol d}t\ |_{\tau=0} \tag{2.6}$$

To improve readability, we will supress the dependence on t, τ and x for the next few steps. Note that for the first vector field in Eq. (2.6), we can compute

$$\begin{aligned}
&\partial_t\big(\partial_\tau\eta\circ\eta^{-1}\big) = \partial_t\partial_\tau\eta\circ\eta^{-1} - \big(\partial_\tau\eta\circ\eta^{-1}\big)\big(\partial_t\eta\circ\eta^{-1}\big) \\
\Rightarrow\quad &\partial_\tau\big(\dot\eta\circ\eta^{-1}\big) = \partial_\tau\partial_t\eta\circ\eta^{-1} - \big(\partial_t\eta\circ\eta^{-1}\big)\big(\partial_\tau\eta\circ\eta^{-1}\big) \\
&\qquad = \partial_t\big(\partial_\tau\eta\circ\eta^{-1}\big) + \big(\partial_\tau\eta\circ\eta^{-1}\big)\big(\partial_t\eta\circ\eta^{-1}\big) \\
&\qquad\qquad - \big(\partial_t\eta\circ\eta^{-1}\big)\big(\partial_\tau\eta\circ\eta^{-1}\big) \\
&\qquad = \partial_t\big(\partial_\tau\eta\circ\eta^{-1}\big) + \big[\partial_\tau\eta\circ\eta^{-1}, \partial_t\eta\circ\eta^{-1}\big] \\
&\qquad \overset{\tau=0}{=} \partial_t w + [w,v].
\end{aligned}$$

The second entry in the metric of Eq. (2.6) is just equal to $v(t)$ and therefore,

$$\begin{aligned}
0 &\overset{(2.6)}{=} \int_0^T \int_M \langle \dot w(t) + [w(t),v(t)],\ v(t)\rangle \text{ vol d}t \\
&= \int_0^T \int_M \langle \dot w(t),\ v(t)\rangle \text{ vol d}t + \int_0^T \int_M \langle [w(t),v(t)],\ v(t)\rangle \text{ vol d}t
\end{aligned}$$

Using integration by parts for the first summand, we get

$$\begin{aligned}
\int_0^T \int_M \langle \dot w(t),\ v(t)\rangle \text{ vol d}t &= \int_M \langle w(t), v(t)\rangle\Big|_{t=0}^T \text{vol} \\
&\quad - \int_0^T \int_M \langle w(t),\ \dot v(t)\rangle \text{ vol d}t \\
&= \int_0^T \int_M \langle w(t),\ -\dot v(t)\rangle \text{ vol d}t
\end{aligned}$$

since $w(0) = 0 = w(T)$. For the second integral, the compatibility of the metric and the covariant derivative implies $[w,v] = \nabla_w v - \nabla_v w$. Hence, we have

$$\begin{aligned}\langle [w,v],v\rangle &= \langle \nabla_w v, v\rangle - \langle \nabla_v w, v\rangle \\ &= \frac{1}{2}\Big(\langle \nabla_w v, v\rangle + \langle v, \nabla_w v\rangle\Big) - \Big(v\langle w,v\rangle - \langle w, \nabla_v v\rangle\Big) \\ &= \frac{1}{2} w\langle v,v\rangle - v\langle w,v\rangle + \langle w, \nabla_v v\rangle\end{aligned}$$

The first summand is equal to

$$\frac{1}{2} w\langle v,v\rangle = \frac{1}{2}\langle w, \operatorname{grad}\langle v,v\rangle\rangle = \langle w, \frac{1}{2}\operatorname{grad}\langle v,v\rangle\rangle,$$

whereas integrating the second term yields

$$\begin{aligned}\int_M v\langle w,v\rangle \operatorname{vol} &= \int_M (\mathcal{L}_v\langle v,w\rangle) \operatorname{vol} \\ &= \int_M \mathcal{L}_v(\langle v,w\rangle \operatorname{vol}) - \int_M \langle v,w\rangle \mathcal{L}_v \operatorname{vol} \\ &= \int_M \mathrm{d}\iota_v(\langle v,w\rangle \operatorname{vol}) - \int_M \langle v,w\rangle \underbrace{\operatorname{div} v}_{=0} \operatorname{vol} \\ &= \int_{\partial M} \underbrace{\iota_v(\langle v,w\rangle \operatorname{vol})}_{=\langle v,w\rangle \iota_v \operatorname{vol} = 0 \text{ on } \partial M} \\ &= 0. \end{aligned} \tag{2.7}$$

Combining all these computations, we get

$$\begin{aligned}\int_M \langle [w,v],v\rangle \operatorname{vol} &= \int_M \langle w, \frac{1}{2}\operatorname{grad}\langle v,v\rangle\rangle \operatorname{vol} + \int_M \langle w, \nabla_v v\rangle \operatorname{vol} \\ &= \int_M \langle w, \nabla_v v + \frac{1}{2}\operatorname{grad}\langle v,v\rangle\rangle \operatorname{vol}.\end{aligned}$$

The full equation is

$$0 = \int_0^T \int_M \langle w, -\dot{v} + \nabla_v v + \frac{1}{2}\operatorname{grad}\langle v,v\rangle\rangle \operatorname{vol} \mathrm{d}t \tag{2.8}$$

for any $w \in \mathfrak{X}^s_{\mathrm{div}}(M)$.

Remark. If we used the right-invariant metric as in Eq. (2.4) instead of Eq. (2.5) to define the energy E on the full diffeomorphism group, there would also be a contribution from the summand computed in Eq. (2.7). In this case, the full equation is

$$0 = \int_0^T \int_M \langle w, -\dot{v} + \nabla_v v + \frac{1}{2}\operatorname{grad}\langle v,v\rangle + v \operatorname{div} v\rangle \operatorname{vol} \mathrm{d}t.$$

Restricting this to divergence-free vector fields, i. e. to the volume-preserving diffeomorphisms $\mathrm{Diff}^s_{\mathrm{vol}}(M)$, also yields Eq. (2.8).

To further simplify Eq. (2.8), we recall the Hodge decomposition for (smooth) forms

$$\Omega^1(M) = \mathrm{d}\Omega^0(M) \oplus \Big(\delta\Omega^2(M) \oplus \mathcal{H}^1(M)\Big).$$

It has a Sobolev equivalent given by

$$H^s(\Lambda^1 M) = \mathrm{d}H^{s+1}(\Lambda^0 M) \oplus \Big(\delta H^{s-1}(\Lambda^2 M) \oplus \ker \Delta|_{H^s(\Lambda^1 M)}\Big),$$

which carries over to vector fields via the given metric on M and we get

$$\mathfrak{X}^s(M) = \nabla H^{s+1}(M,\mathbb{R}) \oplus \underbrace{\{w \in \mathfrak{X}^s(M) \mid \operatorname{div} w = 0\}}_{=\mathfrak{X}^s_{\mathrm{div}}(M)}.$$

This implies that $\operatorname{grad}\langle v,v\rangle = \nabla\langle v,v\rangle$ is always perpendicular to the space of divergence-free vector fields and hence,

$$0 = \langle w, \frac{1}{2}\operatorname{grad}\langle v,v\rangle\rangle$$

for any w satisfying $\operatorname{div} w = 0$. Therefore, the full equation Eq. (2.8) reduces to

$$0 = \int_0^T \int_M \langle w, -\dot{v} + \nabla_v v\rangle\rangle \operatorname{vol} \mathrm{d}t.$$

Replacing v with $-v$ yields

$$0 = \int_0^T \int_M \langle w, \dot{v} + \nabla_v v\rangle \operatorname{vol} \mathrm{d}t. \tag{2.9}$$

Finally, for $\dot{v} + \nabla_v v$ to be perpendicular to the space of divergence-free vector fields, it has to be an element of $\nabla H^{s+1}(M,\mathbb{R})$, i. e. there is a so-called *pressure function* p (unique up to constants) such that

$$\dot{v} + \nabla_v v = -\nabla p,$$

which is the well-known *Euler equation for incompressible fluids.*

2.4 Strategy to prove local existence of solutions

Ebin and Marsden [EM70] have a series of arguments showing that geodesics exist at least locally on certain Hilbert manifolds of Sobolev diffeomorphisms.

Let $P_\eta : T_\eta \mathrm{Diff}^s(M) \to T_\eta \mathrm{Diff}^s_{\mathrm{vol}}(M)$ denote the orthogonal projection induced by $(\cdot,\cdot)$, which form an (a priori not neccessarily smooth) bundle map

$$P : T\mathrm{Diff}^s(M)|_{\mathrm{Diff}^s_{\mathrm{vol}}(M)} \to T\mathrm{Diff}^s_{\mathrm{vol}}(M).$$

Since the metric is right-invariant on the tangent spaces of $\mathrm{Diff}^s_{\mathrm{vol}}(M)$, this projection is given by

$$P_\eta = TR_\eta \circ P_{\mathrm{id}} \circ TR_{\eta^{-1}}, \tag{2.10}$$

where R_η denotes the right-translation by η, so it is completely determined by the projection at the identity P_{id}. Unfortunately, the right-translation is *not* smooth in the base point. Hence, in general, not any bundle map of the form (2.10) will be smooth in the base point. Whether P is a smooth bundle map depends on the specific form of P_{id}.

Theorem 2.12 ([EM70], Theorem 9.6). *Let M be compact without boundary. Then $(\cdot,\cdot)$ defined on $\mathrm{Diff}^s_{\mathrm{vol}}(M)$ is a $\mathrm{Diff}^s_{\mathrm{vol}}(M)$ right invariant weak Riemannian metric. It induces a smooth affine connection $P \circ \nabla$ and an exponential map $\tilde{\mathrm{E}}$ on $\mathrm{Diff}^s_{\mathrm{vol}}(M)$ defined on a neighbourhood of the zero section of $T\mathrm{Diff}^s_{\mathrm{vol}}(M)$. Both $\tilde{\nabla}$ and $\tilde{\mathrm{E}}$ are invariant under right multiplication by $\mathrm{Diff}^s_{\mathrm{vol}}(M)$, and $\tilde{\mathrm{E}}|_{T_{\mathrm{id}}\mathrm{Diff}^s_{\mathrm{vol}}(M)}$ covers a neighbourhood of the identity $\mathrm{id} \in \mathrm{Diff}^s_{\mathrm{vol}}(M)$.*

Since we want to use similar theorems to extend the diffeomorphism groups of manifolds on which solutions to the Euler equation exist, we will recall the main ideas needed for the proof.

Proposition 2.13. *Let X be a Riemannian manifold with connection ∇, $Y \subset X$ a smooth submanifold and $P : TX|_Y \to TY$ the orthogonal projection on each fibre over Y. Then $\tilde{\nabla} = P \circ \nabla$ is the Riemannian connection on Y, i. e. $\tilde{\nabla}$ satisfies the conditions (i) and (ii) in Theorem 2.11(b). If P is a smooth bundle map, then $\tilde{\nabla} = P \circ \nabla$ will be a smooth connection on Y which is compatible with the Riemannian structure.*

Proof of Thm. 2.12. We apply the previous proposition to the manifolds $X = \mathrm{Diff}^s(M)$, $Y = \mathrm{Diff}^s_{\mathrm{vol}}(M)$ and the orthogonal projection

$$P_\eta : T_\eta \mathrm{Diff}^s(M)|_{\mathrm{Diff}^s_{\mathrm{vol}}(M)} \to T_\eta \mathrm{Diff}^s_{\mathrm{vol}}(M)$$

as above. In particular, we can show that P is smooth as in [EM70, §14], so $\tilde{\nabla} = P \circ \nabla$ is the (smooth) Riemannian connection on $\mathrm{Diff}^s_{\mathrm{vol}}(M)$. Hence, the exponential map on $\mathrm{Diff}^s(M)$ induces an exponential map on $\mathrm{Diff}^s_{\mathrm{vol}}(M)$. ☐

Note that this really only relies on the fact that the orthogonal projection

$$P_\eta : T_\eta \mathrm{Diff}^s(M)|_{\mathrm{Diff}^s_{\mathrm{vol}}(M)} \to T_\eta \mathrm{Diff}^s_{\mathrm{vol}}(M)$$

is smooth in η.

A similar result holds for manifolds with boundary. If M is a compact manifold with boundary ∂M such that ∂M is totally geodesic in M, the exponential map exp will also be defined on TM and we can extend the previous theorems to also cover those manifolds.

If ∂M is not totally geodesic in M, i. e. we do not necessarily have an exponential map on TM, we have to adapt the projection. We will fix this by considering the smooth manifold $H^s(M,\tilde{M})$ instead, where

$$\tilde{M} := M \times \{0,1\} \Big/ (x,0) \sim (x,1) \text{ for } x \in \partial M$$

denotes the double of M. Then

$$T_\eta H^s(M,\tilde{M}) = \Big\{ X \in H^s(M,T\tilde{M}) \,\Big|\, \tau \circ X = \eta \Big\}$$

for $\eta \in H^s(M,\tilde{M})$ and bundle projection $\tau : T\tilde{M} \to M$. As before,

$$(X,Y) = \int_M \langle X(m), Y(m) \rangle_{\eta(m)} \mathrm{vol}(m)$$

for $X,Y \in T_\eta \mathrm{Diff}^s_{\mathrm{vol}}(M)$ defines a weak Riemannian metric, where $\langle , \rangle$ denotes the metric on $\tilde{M}$ induced by the metric on M, and $H^s(M,\tilde{M})$ inherits an affine connection $\bar{\nabla}$ and exponential map $\mathrm{E}(X) = \exp \circ X$, where $\exp : T\tilde{M} \to \tilde{M}$ is the exponential map of $\tilde{M}$.

Using this notation, we can extend Theorem 2.12 to manifolds with boundaries.

Theorem 2.14 ([EM70], Theorem 10.2)**.** *Let M be a compact manifold with smooth boundary ∂M. Then $(\cdot,\cdot)$ is a right invariant Riemannian metric on $\mathrm{Diff}^s_{\mathrm{vol}}(M)$ and induces a smooth affine connection $\tilde{\nabla} = P \circ \bar{\nabla}$ and smooth exponential map $\tilde{\mathrm{E}}$ defined on a neighbourhood of the zero section of $T\mathrm{Diff}^s_{\mathrm{vol}}(M)$. Both $\tilde{\nabla}$ and $\tilde{\mathrm{E}}$ are invariant under right multiplication by $\mathrm{Diff}^s_{\mathrm{vol}}(M)$ and $\tilde{\mathrm{E}}|_{T_{\mathrm{id}}\mathrm{Diff}^s_{\mathrm{vol}}(M)}$ covers a neighbourhood of the identity $\mathrm{id} \in \mathrm{Diff}^s_{\mathrm{vol}}(M)$.*

Following [EM70, §§11, 14 and 15], we will now describe how Theorems 2.12 and 2.14 are sufficient to get solutions to the Euler equation. To that end, we first introduce (geodesic) sprays following [Lan02, Chapter VII, §7].

Definition. (a) A *second-order vector field* over M is a a vector field F on the tangent space TM, i. e. $F : TM \to T^2M$, such that $\tau_* \circ F = \mathrm{id}_{TM}$ for the canonical projection $\tau : TM \to M$ and its differential $\tau_* : T^2M \to TM$.

(b) Let $I \subset \mathbb{R}$ be an interval. A curve $\gamma : I \to M$ is a *geodesic with respect to F* if its derivative $\gamma' : I \to TM$ is an integral curve of F. This is equivalent to the condition $\gamma'' = F(\gamma')$, which is called *second-order differential equation for γ determined by F.*

Conversely, if β is an integral curve of F in TM, then $\tau(\beta)$ is a geodesic with respect to F.

Now let s be a real number. Let $s_{TM} : TM \to TM$ and $s_{T^2M} : T^2M \to T^2M$ denote the multiplication by s on TM and T^2M, resp., and we also get the differential $(s_{TM})_* : T^2M \to T^2M$.

Definition. The second-order vector field F is a *spray* if it satisfies the homogeneous quadratic condition

$$F(s_{TM}v) = (s_{TM})_* s_{T^2M} F(v).$$

The geodesic (or canonical) spray is a special kind of spray associated to geodesics on the Riemannian manifold M.

Definition. Let $v \in TM$ with $x := \tau(v) \in M$. By $\gamma_v(t)$, we denote the geodesic on M with initial data $\gamma(0) = x$ and $\dot{\gamma}_v(0) = v$. Then $\dot{\gamma}_v(t)$ defines a curve in TM which projects onto γ_v. We define $Z(v)$ to be the tangent vector to this curve at $t = 0$. This defines the *geodesic spray* $Z : TM \to T^2M$.

In particular, geodesics on M are geodesics with respect to the geodesic spray Z, as defined above. We can now use the geodesic spray associated to the metric on M to compute the geodesic spray associated to the metric $(.,.)$ on $\mathrm{Diff}^s_{\mathrm{vol}}(M)$.

Theorem 2.15 ([EM70], Theorem 11.1). *Let M be compact (possibly with boundary) and let $Z : TM \to T^2M$ be the geodesic spray associated to the metric on M. Let*

$$P : H^s(M, TM)|_{\mathrm{Diff}^s_{\mathrm{vol}}(M)} \to T\mathrm{Diff}^s_{\mathrm{vol}}(M)$$

be the orthogonal projection as before. Then the spray associated to the metric $(.,.)$ on $\mathrm{Diff}^s_{\mathrm{vol}}(M)$ is given by

$$\begin{aligned} S : T\mathrm{Diff}^s_{\mathrm{vol}}(M) &\to T^2\mathrm{Diff}^s_{\mathrm{vol}}(M) \\ X &\mapsto TP(Z \circ X) \end{aligned}$$

and S is a smooth map.

In particular, S is a smooth vector field on $T\mathrm{Diff}^s_{\mathrm{vol}}(M)$ and defines a second order equation, so it has a unique smooth local flow.

The geodesic spray S is explicitly computed in §14 of [EM70]:

Theorem 2.16 ([EM70], Theorem 14.2). *Let $X \in T_\eta \mathrm{Diff}^s_{\mathrm{vol}}(M)$. Then*

$$S(X) = T(X \circ \eta^{-1}) \circ X - \Big(P_{\mathrm{id}}[\nabla_{X \circ \eta^{-1}} X \circ \eta^{-1}]\Big)^l_0 \circ \eta,$$

where $(w)^l_0$ denotes the canonical vertical lift of $w \in T_xM$ to T^2_0M, i. e. $(w)^l_0$ satisfies $T\tau\big((w)^l_0\big) = 0$ for the canonical projection $\tau : TM \to M$ and $T\tau : T^2M \to TM$.

Theorem 2.17 ([EM70], Theorem 14.4). *Let $\tilde{\tau} : T\mathrm{Diff}^s_{\mathrm{vol}}(M) \to \mathrm{Diff}^s_{\mathrm{vol}}(M)$ denote the canonical projection. If v_t is an integral curve of S in $T\mathrm{Diff}^s_{\mathrm{vol}}(M)$, define $\eta_t := \tilde{\tau}(v_t)$ and*

$$\hat{v}_t = v_t \circ \eta_t^{-1}$$

then $\hat{v}_t$ *is an integral curve of the vector field on* $T_{\mathrm{id}}\mathrm{Diff}^s_{\mathrm{vol}}(M)$ *given by*

$$Y(u) = -P_{\mathrm{id}}(\nabla_u u).$$

Conversely, if u_t *is an integral curve of* $Y(u)$ *in* H^s *with flow* η_t*, then* $u_t \circ \eta_t$ *is an integral curve of* S *in* $T\mathrm{Diff}^s_{\mathrm{vol}}(M)$*.*

Since integral curves of S are geodesics in $\mathrm{Diff}^s_{\mathrm{vol}}(M)$, this is sufficient to get solutions to the Euler equation.

Theorem 2.18 ([EM70], parts of Theorem 15.2)**.** *Let* $s > \dfrac{\dim M}{2} + 1$.

(i) *(Existence and uniqueness) If* u_0 *is an* H^s *vector field,* $\operatorname{div} u_0 = 0$ *and* u_0 *parallel to* ∂M*, there is a unique solution* u_t *defined for* $-\delta < t < \delta$ *for some* $\delta > 0$*. The solution* u_t *is an* H^s*-vector field and is* C^1 *as a function of* (t,x) *for* $-\delta < t < \delta$ *and* $x \in M$*. It's flow* η_t *is a volume-preserving* H^s*-diffeomorphism.*

(ii) *(Continuous dependence on initial conditions) For each* u_0*, the* $\delta > 0$ *in (i) is uniform in a whole* H^s *neighbourhood of* u_0 *and the map* $u_0 \mapsto u_t$ *is continuous for each* t*,* $-\delta < t < \delta$*. Each* u_t *is a continuous curve in* H^s *and, in particular,* $\lim_{t\to 0} u_t = u_0$ *in the* H^s *topology.*

(iii) *(Regularity of solutions) If* u_0 *is smooth, so is* u_t *on* $\mathrm{int}(M)$ *and* u_t *is smooth as a function of* (t,x) *as long as* u_t *is defined in* H^s*. The map* $u_0 \mapsto u_t$ *is smooth in the* C^∞ *topology.*

(v) *(Extendability for all t) Let* (a,b) *be the maximal open interval on which a solution* u_t *is defined. Then* $a = -\infty$ *and* $b = \infty$ *if and only if for any finite subinterval* $(a_1, b_1) \subset (a,b)$*,* $\sup_{a_1<t<b_1} \|u_t\|_{H^s} < \infty$*. If solutions are extendible for all t for some s, they are for all s as well, if* $\partial M = \emptyset$*.*

If we now want to show the existence of solutions to the Euler equation for submanifolds D of $\mathrm{Diff}^s_{\mathrm{vol}}(M)$ (for M with or without smooth boundary), we only need to check that the bundle projection $P : T\mathrm{Diff}^s_{\mathrm{vol}}(M)|_D \to TD$ induced by orthogonal projection in each tangent space is a smooth bundle map, i. e. is smooth in the base point.

We will use this method to show the local existence of solutions to the Euler equation for other diffeormorphism groups: First, we show that the diffeomorphism group is a smooth subgroup of some group where we already have an exponential map (e. g. $\mathrm{Diff}^s_{\mathrm{vol}}(M)$ for M with or without smooth boundary). This will be done by either the implicit function theorem (see Proposition 2.19 below) or the image of a known smooth Hilbert submanifold under some embedding (see Proposition 2.20 below).

Proposition 2.19 (Implicit Function Theorem for Hilbert manifolds)**.** *Let* A*,* B *be Hilbert manifolds and* $f : A \to B$ *smooth. Let further* $b \in B$ *be a regular value, i. e. for any* $a \in f^{-1}(b)$*, the differential* $T_a f : T_a A \to T_b B$ *is surjective. Then* $f^{-1}(b) \subset A$ *is a smooth Hilbert submanifold.*

Remark. The implicit function theorem for Banach spaces also requires the kernel $\ker T_a f$ to be complemented. Since any closed subspace of a Hilbert space has an orthogonal complement, this condition is not necessary for Hilbert (sub-)manifolds.

Proposition 2.20 ([Upm85], Prop. 8.7). *Let A, B be Hilbert manifolds and $f : A \to B$ a smooth embedding, i. e. f is a homeomorphism onto its image* $\operatorname{im}(f)$ *such that $T_a f$ is injective for any $a \in A$. Then* $\operatorname{im}(f) \subset B$ *is a smooth submanifold and* $f : A \to \operatorname{im}(f)$ *is a diffeomorphism.*

In the second step, we show that the orthogonal projection of the tangent bundles is smooth in the base point and finally apply an adapted version of Theorem 2.12 resp. 2.14.

To extend those local solutions to global ones, it remains to show that the local solutions and its derivatives are bounded in time. To that end, one can follow and extend the computation on page 15 to find an explicit equation and use that to estimate the vector field and its derivatives.

2.5 Previous results

As mentionend in the introduction, there already exist results regarding local and sometimes even global existence of solutions to the Euler equation, i. e. of geodesics in the (structure-preserving) diffeomorphism group.

A few years after the results on the volume-preserving diffeomorphisms of general compact manifolds, Ebin [Ebi84] also explictly showed the long-time existence of solutions to the Euler equation for two-dimensional manifolds, which we review in Section 2.5.1. More recently, in 2012, Ebin has used similar methods to also show long-time existence of geodesics on the symplectomorphism group in [Ebi12], see also Section 2.5.2. A year later, Ebin and Preston published a preprint [EP13] for quantomorphisms/strict contactomorphisms for contact manifolds that are also principal S^1-bundles such that the Reeb vector field generates the S^1-action. Their preprint uses very similar methods to this thesis, which are described in Section 2.5.3 They also proved the local existence of geodesics on the contactomorphism group of contact manifolds in [EP15]. Since the contactomorphisms are not a smooth submanifold of the H^s-diffeomorphisms, they used the so-called padded contactomorphisms instead. Unfortunately, it has not yet been proven whether the geodesic equation is a smooth ODE for the padded contactomorphism group, so they cannot rely on the results in [EM70]. Therefore, this paper is mathematically very different from the rest and we will only present a very brief summary in Section 2.5.4.

2.5.1 Volume-preserving diffeomorphisms of two-dimensional manifolds

Let M be a two-dimensional manifold, possibly with smooth boundary and unit outward normal vector ν. We further have a Riemannian metric with Riemannian volume form vol. As before, the Euler equation is

$$\begin{aligned} \dot{v} + \nabla_v v &= -\nabla p \\ \operatorname{div}_{\mathrm{vol}} v &= 0 \end{aligned}$$

for a pressure function p (unique up to constants), and boundary condition $\langle \nu, v\rangle = 0$. Let $v^\flat$ denote the one-form associated to the vector field v via the metric, i. e. $v^\flat = \langle v, \cdot\rangle$. Then the Euler equation is equivalent to

$$\dot{v}^\flat + \nabla_v v^\flat = -\mathrm{d}p,$$

which can be rewritten as

$$\dot{v}^\flat + \mathcal{L}_v v^\flat = \mathrm{d}\Big(\frac{1}{2}|v|^2 - p\Big)$$

and, hence, any vector field v satisfying the Euler equation with flow η also satisfies

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\eta(t)^* v^\flat(t) &= \eta(t)^*\Big(\mathcal{L}_{v(t)} v^\flat(t) + \partial_t v^\flat(t)\Big) \\ &= \mathrm{d}\Big(\eta(t)^*\Big(\frac{1}{2}|v(t)|^2 - p\Big)\Big). \end{aligned} \tag{2.11}$$

There is a projection

$$P : C^{k+\alpha}(TM) \to \Big\{ v \in C^{k+\alpha}(TM) \;\Big|\; \operatorname{div}_{\mathrm{vol}} v = 0 \text{ and } \langle \nu, v\rangle = 0 \Big\}$$

given by

$$v \mapsto v - \nabla f,$$

where f is a solution to the *Neumann problem*

$$\Delta f = \operatorname{div} v, \quad \langle \nabla f, \nu\rangle = \langle v, \nu\rangle.$$

Using the metric, we can define the corresponding projection

$$\begin{aligned} \tilde{P} : C^{k+\alpha}(\Lambda^1) \to \Big\{ \alpha \in C^{k+\alpha}(\Lambda^1) \;\Big|\; \delta\alpha = 0 \text{ and } \alpha(\nu) = 0 \Big\} &\cong \mathcal{H} \oplus \delta\mathrm{d}\Delta^{-1}(\mathcal{H}^\perp), \\ \alpha = v^\flat \mapsto (Pv)^\flat = \langle Pv, \cdot\rangle. & \end{aligned}$$

which maps into the first two summands of the Hodge decomposition

$$C^{k+\alpha}(\Lambda^1) = \mathcal{H} \oplus \delta\mathrm{d}\Delta^{-1}(\mathcal{H}^\perp) \oplus \mathrm{d}\delta\Delta^{-1}(\mathcal{H}^\perp),$$

where $\mathcal{H}$ denotes the Hodge forms. Applying this projection to the form $\eta(t)^* v^\flat(t)$ and using Eq. (2.11) yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{P}\Big(\eta(t)^* v^\flat(t)\Big) &= \tilde{P}\Big(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t)^* v^\flat(t)\Big)\\ &\overset{(2.11)}{=} \tilde{P}\Big(\mathrm{d}\Big(\eta(t)^*\Big(\frac{1}{2}|v(t)|^2 - p\Big)\Big)\Big)\\ &= 0.\end{aligned}$$

In particular, $\tilde{P}\big(\eta(t)^* v^\flat(t)\big)$ is independent of t and

$$\begin{aligned}\tilde{P}\Big(\eta(t)^* v^\flat(t)\Big) &= \tilde{P}\Big(\eta(0)^* v^\flat(0)\Big)\\ &= \tilde{P}\Big(\mathrm{id}^* v^\flat(0)\Big)\\ &= v^\flat(0),\end{aligned}$$

since $\operatorname{div} v(0) = 0$ implies that $\delta v^\flat(0) = 0$. Hence, we can define $f_t \in C^{2+\alpha}(M)$ such that

$$\begin{aligned}&\eta(t)^* v^\flat(t) = v_0^\flat + \mathrm{d}f_t\\ \Rightarrow\quad & v^\flat(t) = \Big(\eta(t)^{-1})\Big)^* v_0^\flat + \mathrm{d}\Big(\eta(t)^{-1})\Big)^* f_t\\ \Rightarrow\quad & v^\flat(t) = \tilde{P}(v^\flat(t))\\ &\qquad = \tilde{P}\Big(\Big(\eta(t)^{-1})\Big)^* v_0^\flat\Big) + \underbrace{\tilde{P}\Big(\mathrm{d}\Big(\eta(t)^{-1})\Big)^* f_t\Big)}_{=0}.\end{aligned}$$

Since η is the flow of v, this implies

$$\langle\dot{\eta},\cdot\rangle = v^\flat(t)\circ\eta(t) = \tilde{P}\Big((\eta(t)^{-1})^* v_0^\flat\Big)\circ\eta(t)$$

Splitting $\tilde{P} : C^{k+\alpha}(\Lambda^1) \to \mathcal{H} \oplus \delta\mathrm{d}\Delta^{-1}(\mathcal{H}^\perp)$ into the two projections

$$\begin{aligned}&\tilde{P}_1 : C^{k+\alpha}(\Lambda^1) \to \mathcal{H}, \quad \text{and}\\ &\tilde{P}_2 : C^{k+\alpha}(\Lambda^1) \to \delta\mathrm{d}\Delta^{-1}(\mathcal{H}^\perp)\end{aligned}$$

yields

$$\begin{aligned}&\langle\dot{\eta},\cdot\rangle = \underbrace{\tilde{P}_1\Big(\eta(t)^{-1})\Big)^* v_0^\flat\Big)\circ\eta(t)}_{=:\tilde{F}_1(\eta)} + \underbrace{\tilde{P}^2\Big(\eta(t)^{-1})\Big)^* v_0^\flat\Big)\circ\eta(t)}_{=:\tilde{F}_2(\eta)}\\ \Rightarrow\quad & \dot{\eta} = F_1(\eta) + F_2(\eta). \qquad (2.12)\end{aligned}$$

In particular, solving the Euler equation with initial condition $v(0) = v_0$ is equivalent to solving Eq. (2.12) with initial condition $\eta(0) = \mathrm{id}$ and parameter v_0.

Theorem 2.21 ([Ebi84], Prop. 4.1 and Local Theorem 4.9). *The projections F_1 and F_2 are smooth in η. Hence, the Euler equation has at least local solutions.*

The proof uses the explicitly-known integral kernel of Δ^{-1} and, for F_2, also relies on the fact that

$$\begin{aligned} \tilde{F}_2(\eta) &= \left(\delta \mathrm{d}\Delta^{-1}(\eta^{-1})^* v_0^\flat\right) \circ \eta \\ &= \left(\delta \Delta^{-1}(\eta^{-1})^* \mathrm{d} v_0^\flat\right) \circ \eta, \end{aligned} \tag{2.13}$$

i. e. since the exterior derivative commutes with the pull back, we can shift one derivative from η^{-1} to the initial condition $v_0^\flat$.

To show global existence of solutions, one has to estimate both $F_1(\eta)$ and $F_2(\eta)$. As with the proof of the previous theorem, computing the norm of $F_1(\eta)$ is fairly straightforward whereas the norm of $F_2(\eta)$ is more work but not necessarily more difficult when using Eq. (2.13).

2.5.2 Symplectomorphisms

Let (M^{2n}, ω) be a compact, oriented symplectic manifold with Riemannian metric such that the Riemannian volume form is $\mathrm{vol} = \omega^n$. Ebin [Ebi12] also needs the metric g and the symplectic form ω to be *compatible*, i. e. that there exists an almost complex structure J that satisfies $\omega(v,w) = g(Jv,w)$ and $J^2 = -\mathrm{id}$. Let further

$$\mathrm{Diff}^s_\omega(M) = \{\eta \in \mathrm{Diff}^s(M) \mid \eta^*\omega = \omega\}.$$

Recall the Hodge decomposition

$$H^s(T^*M) = \mathcal{H} \oplus \mathrm{d}\delta H^{s+2}(T^*M) \oplus \delta \mathrm{d} H^{s+2}(T^*M).$$

Then,

$$\begin{aligned} T_{\mathrm{id}}\mathrm{Diff}^s_\omega(M) &= \left\{v \in \mathfrak{X}^s(M) \,\middle|\, \mathcal{L}_v \omega = 0\right\} \\ &= \left\{v \in \mathfrak{X}^s(M) \,\middle|\, \mathrm{d}\omega^\flat(v) = 0\right\} \\ &= \omega^\sharp\left(\mathcal{H} \oplus \mathrm{d}\delta H^{s+2}(T^*M)\right). \end{aligned}$$

Hence, the variation of energy yields

$$\begin{aligned} &\partial_t v + \nabla_v v \perp T_{\mathrm{id}}\mathrm{Diff}^s_\omega(M), \\ \text{i. e.}\quad &\partial_t v + \nabla_v v = \omega^\sharp(\delta \mathrm{d}\alpha) \in \omega^\sharp(\delta \mathrm{d} H^{s+2}(T^*M)) \end{aligned}$$

for some $\alpha \in H^{s+2}(T^*M)$. Let $\Delta = \mathrm{d}\delta + \delta\mathrm{d}$ denote the isomorphism of the orthogonal complement of $\mathcal{H}$ in $H^{s+1}(T^*M)$ to the orthogonal complement of $\mathcal{H}$ in $H^{s-1}(T^*M)$, then we can rewrite this equation as

$$\partial_t v + \nabla_v v = \omega^\sharp \delta \Delta^{-1}[\mathrm{d}\omega^\flat, \nabla_v] v,$$

where, notably, the right-hand side

$$F(v) := \omega^\sharp \delta \Delta^{-1}[\mathrm{d}\omega^\flat, \nabla_v]v$$

is a smooth operator of order 0 for v.

Let us view geodesics on $H^s(M,M)$ and, in turn, $\mathrm{Diff}^s(M)$ as integral curves of a vector field on $TH^s(M,M) \cong H^s(M,TM)$. As before, a vector field on $TH^s(M,M)$ is a smooth map

$$\mathcal{Z} : TH^s(M,M) \to TTH^s(M,M) \cong H^s(M,TTM)$$

such that $T \circ \mathcal{Z} = \mathrm{id}_{TH^s(M,M)}$ for the canonical bundle projection

$$T : TTH^s(M,M) \to TH^s(M,M).$$

If we let $\tau_1 : TTM \to TM$ be the canonical bundle projection and view T as a map $T : H^s(M,TTM) \to H^s(M,TM)$, then $T(v) = \tau_1 \circ v$. We further let $Z : TM \to TTM$ be the spray of the metric on M, i. e. Z is the vector field on TM whose integral curves are $\dot\gamma(t)$ for γ a geodesic on M. In local coordinates $x = (x^1,\dots,x^{2n})$ on M, we get Christoffel symbols Γ_{ij} and for $v = \sum_i v^i \partial_i$, we define $\Gamma(v,v) = \Gamma_{ij}v^iv^j$. Then we can write $Z(v,x) = \big(v, -\Gamma(x)(v,v)\big)$ and $\mathcal{Z}(v) = Z \circ v$ has integral curves $\dot\eta(t)$, where for each $x \in M$, the curve $\dot\gamma(t) := \dot\eta(t)(x)$ is the lift of a geodesic. As a consequence, $\mathcal{Z}$ is the spray for the L^2-metric on $H^s(M,M)$.

Theorem 2.22 (Theorem 5.2 in [Ebi12]). *Since the geodesic spray*

$$\begin{aligned}\tilde{\mathcal{Z}}(\eta, v\circ\eta) &= (v\circ\eta, -\Gamma_{ij}v^iv^j \circ \eta + (\partial_t v + \nabla_v v)\circ\eta) \\ &= \big(v\circ\eta, -\Gamma_{ij}v^iv^j \circ\eta + \omega^\sharp\delta\Delta^{-1}[\mathrm{d}\omega^\flat, \nabla_v]v\circ\eta\big)\end{aligned}$$

is a smooth vector field on $T\mathrm{Diff}^s_\omega(M)$, *local geodesics exist on* $\mathrm{Diff}^s_\omega(M)$.

Estimating $\|\dot\eta\|_{H^s}$ yields that it remains bounded for all times, hence geodesics exist for all times.

Khesin [Khe12] extends those result to symplectic manifolds with Riemannian metrics that are not necessarily compatible.

2.5.3 Quantomorphisms/strict contactomorphisms

Let (M^{2n+1}, λ) be a contact manifold with Riemmanian metric such that the Riemannian volume form vol is a constant multiple of $\lambda \wedge (\mathrm{d}\lambda)^n$. We further assume that the Reeb vector field R is also Killing and regular with all orbits of the same length 1, hence M is a principal S^1-bundle with S^1-action induced by R. We define the *strict contactomorphisms* or *quantomorphisms* as

$$\mathrm{Diff}^s_\lambda(M) := \{\eta \in \mathrm{Diff}^s(M) \mid \eta^*\lambda = \lambda\}.$$

Theorem 2.23 ([EP13], Section 2). *The following inclusions are actually smooth submanifolds:*

$$\mathrm{Diff}^s_R(M) \subset \mathrm{Diff}^s(M), \qquad \mathrm{Diff}^s_{R,\mathrm{vol}} \subset \mathrm{Diff}^s_{\mathrm{vol}}(M), \qquad \mathrm{Diff}^s_{R,\mathrm{vol}} \subset \mathrm{Diff}^s_R(M),$$
$$\mathrm{Diff}^s_\lambda(M) \subset \mathrm{Diff}^s_R(M), \qquad \mathrm{Diff}^s_\lambda(M) \subset \mathrm{Diff}^s_{R,\mathrm{vol}}(M).$$

Theorem 2.24 ([EP13], Theorem 3.1). $\mathrm{Diff}^s_{R,\mathrm{vol}}(M)$ *is a totally geodesic submanifold of* $\mathrm{Diff}^s_{\mathrm{vol}}(M)$.

Theorem 2.25 ([EP13], Theorem 3.4). *The orthogonal projection*

$$P : T\mathrm{Diff}^s_{R,\mathrm{vol}}(M)|_{\mathrm{Diff}^s_\lambda(M)} \to T\mathrm{Diff}^s_\lambda(M)$$

is a smooth bundle map.

Corollary 2.26 ([EP13], Theorem 4.1). *The geodesic equation is a smooth ODE on the diffeomorphism group* $\mathrm{Diff}^s_\lambda(M)$ *and hence, there is a smooth exponential map* $\exp_{\mathrm{id}} : \Omega \to \mathrm{Diff}^s_\lambda(M)$ *for some neighbourhood* $0 \in \Omega \subset T_{\mathrm{id}}\mathrm{Diff}^s_\lambda(M)$ *such that* $\exp_{\mathrm{id}}(v)$ *is the geodesic* $\eta(1)$, *where* $\eta(0) = \mathrm{id}$ *and* $\eta'(0) = v$.

Proof. The geodesic equation on $\mathrm{Diff}^s_{R,\mathrm{vol}}(M)$ is given by $\frac{D}{\mathrm{d}t}\frac{\mathrm{d}\eta}{\mathrm{d}t} = 0$, where $\frac{D}{\mathrm{d}t}$ denotes the covariant derivative. Using that $\mathrm{Diff}^s_\lambda(M) \subset \mathrm{Diff}^s_{R,\mathrm{vol}}(M)$ is a smooth submanifold, the geodesic equation on $\mathrm{Diff}^s_\lambda(M)$ is then given by $P_\eta\big(\frac{D}{\mathrm{d}t}\frac{\mathrm{d}\eta}{\mathrm{d}t}\big) = 0$. Since P is smooth, this ODE is smooth on $\mathrm{Diff}^s_\lambda(M)$ and, hence, we have local solutions, i. e. an exponential map. □

By finding an explicit representation of the tangent spaces $T_\eta\mathrm{Diff}^s_\lambda(M)$, they use the fact that $\dot{v}_t + \nabla_{v_t} v_t$ has to be perpendicular to $T_{\mathrm{id}}\mathrm{Diff}^s_\lambda(M)$ to explictly compute this ODE. Using this description, they can show that solutions stay bounded for all times and, hence, solutions exist for all times, see Section 4 in [EP13].

Those theorems can also be found in Section 4.1 of [EP15] with proofs relying on the corresponding results for contactomorphisms.

2.5.4 Contactomorphisms

Let M^{2n+1} be an oriented manifold with contact structure ξ and some contact form λ. The proofs in [EP15] use an *associated* Riemannian metric (i. e. for any $u, v \in TM$, we have $\lambda(u) = \langle u, R\rangle$ and there is a (1,1)-tensor ϕ such that $\phi^2(u) = -u + \lambda(u)R$ and $\mathrm{d}\lambda(u,v) = \langle u, \phi v\rangle$) but the authors claim that the results are also true for any Riemannian metric on M. The group of contactomorphisms is

$$\mathrm{Diff}^s_\xi(M) = \Big\{\eta \in \mathrm{Diff}^s(M) \;\Big|\; \eta^*\lambda = \mathrm{e}^\Lambda \lambda \text{ for some function } \Lambda \in H^s(M, \mathbb{R})\Big\},$$

and the group of *padded contactormophisms*

$$\widetilde{\mathrm{Diff}}^s_\xi(M) = \Big\{(\eta, \Lambda) \;\Big|\; \eta^*\lambda = \mathrm{e}^\Lambda \lambda\Big\},$$

which is not just a subgroup but also a smooth submanifold of $\widetilde{\mathrm{Diff}}^s(M) := \mathrm{Diff}^s(M) \ltimes H^s(M)$. Unfortunately, since not much is known about geodesics on the padded diffeomorphisms, they cannot rely on the results in [EM70] to deduce the existence of local geodesics but have to work with explicit descriptions of the tangent space $T_{\mathrm{id}}\widetilde{\mathrm{Diff}}^s_\xi(M)$ and compute the Euler-Arnold equation for geodesics. They then show that one can rewrite the geodesic equation as a first-order ODE on $\widetilde{\mathrm{Diff}}^s_\xi(M)$ and show in Theorem 3.1 that the expression one gets for the derivative $\frac{\mathrm{d}}{\mathrm{d}t}(\eta, \Lambda)$ is smooth in (η, Λ).

Theorem 2.27 (Corollary 3.2 in [EP15]). *There is a smooth, locally invertible Riemannian exponential map which takes sufficiently small tangent vectors in* $T_{\mathrm{id}}\widetilde{\mathrm{Diff}}^s_\xi(M)$ *to the time-one solution* $\big(\eta(1), \Lambda(1)\big) \in \widetilde{\mathrm{Diff}}^s_\xi(M)$.

This gives local solutions to the Euler equation.

3

DIFFEOMORPHISMS OF MANIFOLDS WITH A STABLE HAMILTONIAN STRUCTURE

3.1 Manifolds with a stable Hamiltonian structure

Definition. A *Hamiltonian structure* on an oriented $(2n+1)$-dimensional manifold M is a closed two-form ω of maximal rank, i. e. such that ω^n vanishes nowhere. Associated to ω is its one-dimensional *kernel distribution (foliation)* $\ker\omega$. A *stabilizing one-form* for ω is a one-form λ such that $\lambda \wedge \omega^n$ is a volume form and $\ker\omega \subset \ker \mathrm{d}\lambda$.

A Hamiltonian structure ω is called *stabilizable* if it admits a stabilizing one-form λ, and the pair (ω, λ) is called a *stable Hamiltonian structure (SHS)* on M.

Examples. (a) For a contact manifold (M, λ), the pair $(\omega := \mathrm{d}\lambda, \lambda)$ is an SHS on M and finding geodesics on $\mathrm{Diff}^s_{\omega,\lambda}(M) = \mathrm{Diff}^s_{\lambda}(M)$ is equivalent to the quantomorphism case.

(b) Let (B, σ) be a symplectic manifold with a Riemannian metric. Define a trivial bundle $\pi : S^1 \times B \to B$ with S^1-coordinate θ. Then $(\omega := \pi^*\sigma, \lambda := \mathrm{d}\theta)$ is an SHS on $S^1 \times B$ with Reeb vector field $R = \partial_\theta$. Define a Riemannian metric on $S^1 \times B$ by $|R| = 1$, $R \perp TB$ and the given metric on TB. Finding geodesics on $\mathrm{Diff}^s_\sigma(B)$ is equivalent to the existence of solutions on $\mathrm{Diff}^s_{\pi^*\sigma,\mathrm{d}\theta}(S^1 \times B)$.

Additionally, we need a *compatible* Riemannian metric g on M, i. e. we assume that the volume form induced by g is a constant multiple of the volume form $\lambda \wedge \omega^n$.

Definition. Similarly to contact manifolds, we can define a *Reeb vector field* R by

$$\iota_R\omega = 0 \quad \text{and} \quad \lambda(R) = 1.$$

Because $\lambda \wedge \omega$ is nowhere 0, the kernel of ω is one-dimensional and $\ker\omega \cap \ker\lambda = \{0\}$. The condition $\lambda(R) = 1$ then normalizes R. Hence, the Reeb vector field is well defined.

Lemma 3.1. *There is an isomorphism of $C^\infty(M)$-modules*

$$\begin{aligned} \omega^\flat : \ker\lambda &\to \mathrm{ann}(R) = \left\{ \alpha \in \Omega^1(M) \;\middle|\; \alpha(R) = 0 \right\} \\ u &\mapsto \iota_u\omega. \end{aligned}$$

Its inverse is denoted by $\omega^\sharp : \mathrm{ann}(R) \to \ker\lambda$.

Proof. This homomorphism is injective: Let $u \in \ker\lambda$ be a vector field in the kernel of this map, i. e. $\iota_u\lambda = 0$ and $\iota_u\omega = 0$. The second condition implies that we can write $u = fR$ for some function $f \in C^\infty(M)$. Since, furthermore, $u \in \ker\lambda$, we know

$$0 = \lambda(u) = \lambda(fR) = f\lambda(R) = f,$$

hence $u = fR = 0 \cdot R = 0$.

The map is surjective: Let $\alpha \in \operatorname{ann}(R)$, i. e. $\alpha(R) = 0$. Since $\lambda\wedge\omega^n$ is a volume form, ω is non-degenerate on any complement of $\ker\omega$ in $\Gamma(TM)$. Therefore, we can find a vector field $v \in \mathfrak{X}(M)$ such that $\iota_v\omega = \alpha$. Define $u := v - \lambda(v)R$. Then, $\iota_u\omega = \iota_v\omega = \alpha$ and

$$\lambda(u) = \lambda(v - \lambda(v)R) = \lambda(v) - \lambda(v)\lambda(R) = 0,$$

hence $u \in \ker\lambda$ and u is a preimage of $\alpha \in \operatorname{ann}(R)$. □

Remark. If $\dim M = 3$, then $\ker\omega \subset \ker \mathrm{d}\lambda$ implies that we can find a unique function $h \in C^\infty(M)$ such that $\mathrm{d}\lambda = h\omega$.

This thesis deals with manifolds M with stable Hamiltonian structure (ω, λ) that are also equipped with a Riemannian metric g such that

- the Reeb vector field is regular, i. e. all orbits are periodic and of constant period (w. l. o. g. of period 1),
- the Reeb vector field R for (ω, λ) is also a Killing field for g, i. e. $\mathcal{L}_R g = 0$, and
- the Riemannian volume form vol induced by g is a constant multiple of the volume form $\lambda \wedge \omega^n$ (w. l. o. g. $\mathrm{vol} = \lambda \wedge \omega^n$).

Since all orbits of the vector field R are periodic of period 1, we get an S^1-action that induces a principal bundle $S^1 \longrightarrow M \xrightarrow{\pi} B$ for some $2n$-dimensional base manifold B.

3.2 Diffeomorphisms preserving the stable Hamiltonian structure

As defined on page 5, $C^1\mathrm{Diff}(M)$ denotes the group of C^1-diffeomorphisms of M, and $\mathrm{Diff}^s(M)$ denotes the identity component of $H^s(M,M) \cap C^1\mathrm{Diff}(M)$ for $s > \frac{\dim M}{2} + 1$. The H^s-diffeomorphism group of M preserving the stable Hamiltonian structure is given by

$$\mathrm{Diff}^s_{\omega,\lambda}(M) = \{\eta \in \mathrm{Diff}^s(M) \mid \eta^*\lambda = \lambda,\ \eta^*\omega = \omega\} \subset \mathrm{Diff}^s(M).$$

In the previous examples, the groups of volume-preserving diffeomorphisms, symplectomorphisms and quantomorphisms all are smooth submanifolds of $\mathrm{Diff}^s(M)$. Unfortunately, this might generally not be true for the diffeomorphism groups preserving the stable Hamiltonian structure as will be discussed in Section 4.11. We will devote a significant portion of this thesis to examples where we can explicitly show that $\mathrm{Diff}^s_{\omega,\lambda}(M)$ is not just a subgroup of some known Hilbert manifold like $\mathrm{Diff}^s(M)$, but also a smooth submanifold.

Instead of the very restrictive group $\mathrm{Diff}^s_{\omega,\lambda}(M)$, one might also consider only preserving the Hamiltonian structure ω. Any such diffeomorphism will automatically preserve the kernel of ω, i. e. the subspace generated by the Reeb vector field R. Since we might not be able to control R with those diffeomorphisms and in turn cannot be sure about the long-time existence of solutions to the Euler equation, one might want to also preserve R itself. We will discuss those diffeomorphism groups in Chapter 5.

3.3 Principal circle bundles

Let $S^1 \to M \xrightarrow{\pi} B$ be a circle bundle with SHS (ω, λ) on M and Reeb vector field R. We also assume that the flow of the Reeb vector field generates the S^1-action on M. Following Geiges [Gei08, Def. 7.2.3ff], the stabilizing one-form λ is also a *connection-1-form* for our S^1-bundle, since it is *invariant*, i. e. $\mathcal{L}_R\lambda = \mathrm{d}\iota_R\lambda + \iota_R\mathrm{d}\lambda = 0$, and normalized by $\lambda(R) = 1$.

Remark. The usual definition of a connection form is a one-form with values in the Lie algebra $\mathrm{i}\mathbb{R}$ of $S^1 = U(1)$. This corresponds to our definition by identifying $\mathrm{i}\mathbb{R}$ with $\mathbb{R}$ and, hence, viewing connection forms as regular, real-valued differential forms on M.

Definition. Let $S^1 \longrightarrow M \xrightarrow{\pi} B$ be a fibre bundle. The kernel of $\pi_* : TM \to TB$ is called the *vertical bundle* $T^vM := \ker \pi_*$. At each point $x \in M$, we can choose a (not necessarily unique) *horizontal space*, i. e. a complement T^h_xM of T^v_xM in T_xM and we get

$$T_xM = T^h_xM \oplus T^v_xM.$$

A form α on M is called *horizontal* if $v \in T^vM$ implies that $\iota_v\alpha = 0$.

Note that the definition of a horizontal form is independent of the choice of the horizontal bundle, and that the projection $\pi : M \to B$ induces isomorphisms $\pi_* : (T^h_xM) \xrightarrow{\cong} T_{\pi(x)}B$. With our assumptions (see page 28), the kernel of π_* is generated by the Reeb vector field R. Hence, R generates the vertical tangent space T^vM.

Definition. A *connection* in M is a smooth distribution $T^hM = \coprod_{x\in M} T^h_xM$ of S^1-equivariant horizontal spaces, i. e. the horizontal spaces satisfy

$$(\phi_\theta)_*(T^h_xM) = T_{\phi_\theta(x)}M$$

for the flow ϕ_θ of R for $\theta \in S^1$.

The choice of a connection is equivalent to choosing a connection form, for details see [KN96, Prop. II.1.1]. In particular, we have the following lemma:

Lemma 3.2. *Any connection form λ induces an S^1-equivariant connection by $T^hM := \ker\lambda$.*

Proof. To prove that $\ker\lambda$ is a connection, we need to show that any $u \in TM$ can be uniquely written as the sum of two elements in T^hM and T^vM. To that end, let $u \in TM$ and we need to find its components in T^hM and T^vM. Define $f := \lambda(u)$. Then $fR \in T^vM$ and $v := u - fR$ satisfies $u = fR + v$ and

$$\lambda(u - fR) = \lambda(u) - f\lambda(R) = f - f = 0,$$

i. e. $v \in \ker\lambda$. Now assume that also $u = f'R + v'$ for some smooth function f' and $v' \in \ker\lambda$. Then

$$f = \lambda(u) = \lambda(f'R + v') = f'\lambda(R) = f'$$

and hence also

$$v' = u - f'R = u - fR = v. \qquad \square$$

Remark. For manifolds with SHS (ω, λ) we can also use Lemma 3.1 to show that $\ker\lambda$ is a connection: Let $u \in TM$ and define $\alpha := \iota_u\omega$. Since $\iota_R\alpha = \iota_R\iota_u\omega = 0$, α is an element of $\operatorname{ann}(R)$ and we can apply Lemma 3.1 to get a vector field $v \in \ker\lambda$ such that $\alpha = \iota_v\omega$. Also, $\iota_{u-v}\omega = \iota_u\omega - \iota_v\omega = \alpha - \alpha = 0$. Hence, $u - v \in \ker\omega$ and there is some function $f \in C^\infty(M)$ such that $u - v = fR \in T^vM$.

Lemma 3.3 ([KN96], Prop. II.1.2)**.** *Given a connection in M and a vector field v on B, there is a unique horizontal lift v^* of v on M. The lift v^* is invariant by the induced S^1-action on TM.*

Corollary 3.4. *If a differential form α on M is invariant ($\mathcal{L}_R\alpha = 0$) and horizontal ($\iota_R\alpha = 0$), then α descends to a form on B, i. e. there is a form $\bar{\alpha}$ on B such that $\alpha = \pi^*\bar{\alpha}$.* $\square$

Corollary 3.5. *Let $\lambda, \tilde{\lambda} \in \Omega^1(M)$ be two connection forms for the same circle bundle $S^1 \to M \xrightarrow{\pi} B$. Then there is $\rho \in \Omega^1(B)$ such that*

$$\tilde{\lambda} = \lambda + \pi^*\rho.$$

Proof. $\tilde{\lambda} - \lambda$ is both invariant ($\mathcal{L}_R(\tilde{\lambda} - \lambda) = 0$) and horizontal ($\iota_R(\tilde{\lambda} - \lambda) = 0$). $\square$

Since λ is a connection form, we also have that $d\lambda$ is both invariant ($\mathcal{L}_R d\lambda = d\mathcal{L}_R\lambda = 0$) and horizontal ($R \in \ker\omega \subset \ker d\lambda$). Hence, $d\lambda$ also descends to a two-form τ on B. We call τ the *curvature form* of the connection form λ. Since $\mathbb{R}$ as the Lie algebra of S^1 is abelian, this again corresponds to the usual definiton of the curvature form of a principal bundle, which otherwise would also include a commutator term.

Further, since π is a bundle projection, it is also a submersion. Therefore π_* is surjective and π^* is injective. Then the computation

$$\pi^* d\tau = d\pi^*\tau = d^2\lambda = 0$$

implies that τ is closed.

Note that Corollary 3.5 also implies that the cohomology class $[\tau] \in H_{\mathrm{dR}}(B)$ does not depend on the choice of the connection form for the bundle $M \to B$. This is a special case of the Theorem of Chern-Weil: If we identify $S^1 \cong \mathbb{R}/\mathbb{Z}$, then the *first characteristic class* or *Euler class* of M,

$$c_1(M) := -[\tau] \in H^2(B;\mathbb{Z}),$$

is an invariant of the bundle $M \to B$ up to (continuous) isomorphisms. For S^1-principal bundles, $H^2(B;\mathbb{Z})$ actually classifies the principal bundles over B up to (continuous) isomorphisms, see also [Hat17, Prop. 3.10]. Furthermore, ω is also both invariant and horizontal, and can therefore be written as the pullback $\omega = \pi^*\sigma$ of some $\sigma \in \Omega^2(B)$. Again, $\mathrm{d}\omega = 0$ implies that $\mathrm{d}\sigma = 0$. We also have

$$\pi^*\sigma^n = \omega^n \neq 0,$$

since $\lambda \wedge \omega^n$ is a volume form on M, hence $\sigma^n \neq 0$ and σ is a symplectic form on B.

Remark. The list of conditions on page 28 do *not* imply that (M, λ) is a contact manifold. In the contact case, i. e. if ω is such that $\omega = \mathrm{d}\lambda$, we need to have σ, τ on the base B such that

$$\pi^*\sigma = \omega = \mathrm{d}\lambda = \pi^*\tau$$

on M. Since π^* is injective, this implies $\sigma = \tau$ on B. Conversely, $\sigma = \tau$ implies $\mathrm{d}\lambda = \pi^*\tau = \pi^*\sigma = \omega$, hence (M, λ) is contact.

3.4 Structure-preserving diffeomorphisms a submanifold?

We already showed in Theorem 2.8 that $\mathrm{Diff}^s(M)$ is a smooth Hilbert manifold with smooth submanifold $\mathrm{Diff}^s_{\mathrm{vol}}(M) = \{\eta \in \mathrm{Diff}^s(M) \mid \eta^*\mathrm{vol} = \mathrm{vol}\} \subset \mathrm{Diff}^s(M)$.

We first expand the results already cited in Theorem 2.23 with all the necessary conditions so that we can apply them to our situation.

Lemma 3.6 ([EP13], Lemma 2.1). *Let N be a C^∞ Hilbert manifold with C^∞ Hilbert submanifolds L, M. If $L \subset M$, then L is also a C^∞ Hilbert submanifold of M.*

Theorem 3.7 ([EP13], Theorem 2.2). *Let R be a vector field on M with closed orbits all of the same period. Then*

$$\mathrm{Diff}^s_R(M) := \{\eta \in \mathrm{Diff}^s(M) \mid \eta_* R = R\} \subset \mathrm{Diff}^s(M)$$

is a smooth Hilbert submanifold.

Theorem 3.8 ([EP13], Theorem 2.3). *Let M be compact, R a smooth vector field with closed orbits all of the same period,* vol *a volume form which is invariant under the flow of R (i. e.* $\operatorname{div}_{\mathrm{vol}} R = 0$*). Then*

$$\mathrm{Diff}^s_{R,\mathrm{vol}}(M) := \{\eta \in \mathrm{Diff}^s(M) \mid \eta^*\mathrm{vol} = \mathrm{vol},\ \eta_* R = R\} \subset \mathrm{Diff}^s_R(M)$$

is a smooth Hilbert submanifold.

Corollary 3.9 ([EP13], Corollary 2.4). *Let M be compact, R a smooth vector field with closed orbits all of the same period,* vol *a volume form which is invariant under the flow of R (i. e.* $\operatorname{div}_{\mathrm{vol}} R = 0$*). Then* $\mathrm{Diff}^s_{R,\mathrm{vol}}(M) \subset \mathrm{Diff}^s_{\mathrm{vol}}(M)$ *is a C^∞ submanifold.*

Theorem 3.10 ([EP13], Theorem 3.1). *Suppose M is a compact Riemannian manifold with Killing field R with all orbits closed and of the same period. Then in the metric induced by Eq.* (2.5)*, the submanifold* $\mathrm{Diff}^s_{R,\mathrm{vol}}(M)$ *is a totally geodesic Riemannian submanifold of* $\mathrm{Diff}^s_{\mathrm{vol}}(M)$.

We want to figure out when the diffeomorphisms $\mathrm{Diff}^s_{\omega,\lambda}(M)$ for M satisfying the conditions on page 28 is a smooth submanifold of some known Hilbert manifold, e. g. of $\mathrm{Diff}^s_R(M)$.

Lemma 3.11. $\mathrm{Diff}^s_{\omega,\lambda}(M)$ *is a subgroup (but not necessarily a submanifold) of* $\mathrm{Diff}^s_{\mathrm{vol}}(M)$, $\mathrm{Diff}^s_R(M)$ *and* $\mathrm{Diff}^s_{R,\mathrm{vol}}(M)$.

Proof. Let $\eta \in \mathrm{Diff}^s_{\omega,\lambda}(M)$. Since we assume that the volume form vol is a constant multiple of $\lambda \wedge \omega^n$, any diffeomorphism preserving λ and ω also preserves vol.

Also, since R is uniquely determined by $\iota_R\omega = 0$ and the normalization $\lambda(R) = 1$, we only need to compute

$$\begin{aligned}\iota_{\eta_*R}\omega &= \iota_R(\eta^*\omega)\circ\eta^{-1} = \iota_R\omega\circ\eta^{-1} = 0,\\ \lambda(\eta_*R) &= (\eta^*\lambda)(R)\circ\eta^{-1} = \lambda(R)\circ\eta^{-1} = 1.\end{aligned}$$

This yields $\eta_* R = R$. □

Lemma 3.12. *Let $S^1 \to M \xrightarrow{\pi} B$ be a principal circle bundle with vector field $R \in \mathfrak{X}(M)$ generating the S^1-action. Any $\eta \in \mathrm{Diff}^s_R(M)$ is a lift of some $\nu \in \mathrm{Diff}^s(B)$ and we can define a smooth projection*

$$q : \mathrm{Diff}^s_R(M) \to \mathrm{Diff}^s(B)$$

Proof. Let $\eta \in \mathrm{Diff}^s_R(M)$, i e. $\eta_* R = R$. This is equivalent to $\eta \circ \phi_\theta = \phi_\theta \circ \eta$ for the flow ϕ_θ of R. As a consequence, for any x, x' in $\pi^{-1}(\{b\})$, there is ϕ_θ such that $x' = \phi_\theta(x)$ and we have

$$\pi(\eta(x')) = \pi(\eta(\phi_\theta x)) = \pi(\phi_\theta(\eta(x))) = \pi(\eta(x)).$$

Hence, we can define a diffeomorphism $\nu := q(\eta) \in \mathrm{Diff}^s(B)$ by

$$\nu(b) := \pi(\eta(x)) \text{ for any } x \in \pi^{-1}(\{b\})$$

and ν satisfies $\pi \circ \eta = \nu \circ \pi$, i. e. η is a lift of ν. $\square$

Let $\sigma, \tau \in \Omega^2(B)$ such that $\pi^*\sigma = \omega$ and $\pi^*\tau = \mathrm{d}\lambda$ as explained in the previous section.

Lemma 3.13. *(a)* $\eta^*\omega = \omega \Leftrightarrow \nu^*\sigma = \sigma$.

(b) $\eta^*\lambda = \lambda \Rightarrow \eta^*\mathrm{d}\lambda = \mathrm{d}\lambda \Leftrightarrow \nu^*\tau = \tau$.
In particular, if $\eta \in \mathrm{Diff}^s_{\omega,\lambda}(M) \subset \mathrm{Diff}^s_R(M)$, *then* $\nu := q(\eta) \in \mathrm{Diff}^s_{\sigma,\tau}(B)$. *Conversely, if* $\nu \in \mathrm{Diff}^s_{\sigma,\tau}(B)$ *and* $\eta \in q^{-1}(\nu) \subset \mathrm{Diff}^s_R(M)$, *then* $\eta \in \mathrm{Diff}^s_{\omega,\mathrm{d}\lambda}(M)$.

Proof. (a) $\nu^*\sigma = \sigma$ implies that

$$\eta^*\omega = \eta^*\pi^*\sigma = (\pi \circ \eta)^*\sigma = (\nu \circ \pi)^*\sigma = \pi^*\nu^*\sigma = \pi^*\sigma = \omega.$$

Conversely, if $\eta^*\omega = \omega$, then

$$\pi^*\nu^*\sigma = (\nu \circ \pi)^*\sigma = (\pi \circ \eta)^*\sigma = \eta^*\pi^*\sigma = \eta^*\omega = \omega = \pi^*\sigma$$

and since π^* is injective, this yields $\nu^*\sigma = \sigma$.

(b) $\nu^*\tau = \tau$ implies that

$$\eta^*\mathrm{d}\lambda = \eta^*\pi^*\tau = (\pi \circ \eta)^*\tau = (\nu \circ \pi)^*\tau = \pi^*\nu^*\tau = \pi^*\tau = \mathrm{d}\lambda.$$

Conversely, if $\eta^*\lambda = \lambda$, then $\eta^*\mathrm{d}\lambda = \mathrm{d}\eta^*\lambda = \mathrm{d}\lambda$,

$$\pi^*\nu^*\tau = (\nu \circ \pi)^*\tau = (\pi \circ \eta)^*\tau = \eta^*\pi^*\tau = \eta^*\mathrm{d}\lambda = \mathrm{d}\lambda = \pi^*\tau$$

and since π^* is injective, this yields $\nu^*\tau = \tau$. $\square$

3.5 Special case: Trivial circle bundles

Let $S^1 \longrightarrow B \times S^1 \xrightarrow{\pi} B$ be the trivial principal S^1-bundle over some even-dimensional manifold B with S^1-coordinate θ, i. e. the S^1-action is generated by the flow of $R = \partial_\theta$. Let (ω, λ) be a stable Hamiltonian structure on $B \times S^1$. According to the discussion in Section 3.3, we know that ω and $\mathrm{d}\lambda$ descend to two-forms σ and τ on B, respectively. We know (Lemma 3.13) that if $\eta \in \mathrm{Diff}_{\omega,\lambda}(B \times S^1)$ is a lift of some $\nu \in \mathrm{Diff}(B)$, i. e. $\pi \circ \eta = \nu \circ \pi$, then ν also preserves σ and τ, i. e. ν is actually an element of $\mathrm{Diff}_{\sigma,\tau}(B)$. Conversely, we know that any lift $\eta \in \mathrm{Diff}(B \times S^1)$ of $\nu \in \mathrm{Diff}_{\sigma,\tau}(B)$ preserves ω and $\mathrm{d}\lambda$, i. e. satisfies $\eta^*\omega = \omega$ and $\eta^*\mathrm{d}\lambda = \mathrm{d}\lambda$.

Since $[\tau] \in H^2(B)$ is the Euler class of the (trivial) bundle, $[\tau] = 0$ and hence, τ is exact. Further, if θ denotes the S^1-coordinate, then $\mathrm{d}\theta$ is a connection form of the trivial bundle: It satisfies $\iota_R \mathrm{d}\theta = \iota_{\partial_\theta}\mathrm{d}\theta = 1$ and is also invariant ($\mathcal{L}_R \mathrm{d}\theta = \mathrm{d}\iota_R \mathrm{d}\theta = \mathrm{d}1 = 0$). Since λ is also a connection form of the trivial bundle, Corollary 3.5 yields a 1-form $\mu \in \Omega^1(B)$ such that $\lambda = \mathrm{d}\theta + \pi^*\mu$. Then, $\pi^*\tau = \mathrm{d}\lambda = \mathrm{d}^2\theta + \mathrm{d}\pi^*\mu = \pi^*\mathrm{d}\mu$ and since π^* is injective, this actually yields $\tau = \mathrm{d}\mu$, i. e. μ is a primitive of τ.

Lemma 3.14. *The map*

$$\begin{aligned}\Phi : \mathrm{Diff}^s(B) \times H^s(B,S^1) &\to \mathrm{Diff}^s_R(B \times S^1)\\ (\nu,k) &\mapsto \big((b,\theta) \mapsto (\nu(b),\ \theta + k(b))\big)\end{aligned}$$

is a smooth diffeomorphism with inverse

$$\begin{aligned}\mathrm{Diff}^s_R(B \times S^1) &\to \mathrm{Diff}^s(B) \times H^s(B,S^1)\\ \eta = (\eta^1,\eta^2) &\mapsto (q(\eta) = \eta^1, \eta^2 - \theta)\end{aligned}$$

Hence, $\mathrm{Diff}^s(B) \times H^s(B,S^1)$ *and* $\mathrm{Diff}^s_R(B \times S^1)$ *are diffeomorphic.*

Proof. If well defined, the two maps are obviously smooth inverses to each other. We only need to check that the map $\mathrm{Diff}^S_R(B \times S^1) \to \mathrm{Diff}^s(B) \times H^s(B,S^1)$ is well defined. To that end, let $\eta \in \mathrm{Diff}^s_R(B)$, i. e. $\eta = (\eta^1,\eta^2)$ for some $\eta^1(b,\theta) \in H^s(B \times S^1, B)$ and $\eta^2(b,\theta) \in H^s(B \times S^1, S^1)$. Let $b^1,\dots,b^{2n}$ be local coordinates on B and write $\eta^1 = (\eta^{1,1},\dots,\eta^{1,2n})$. Since η preserves $R = \partial_\theta$, we have

$$\begin{aligned}\partial_\theta = R &\overset{!}{=} \eta_* R = \eta_* \partial_\theta\\ &= \sum_i \frac{\partial \eta^{1,i}}{\partial\theta}\partial_{b^i} + \frac{\partial\eta^2}{\partial\theta}\partial_\theta.\end{aligned}$$

Hence, $\dfrac{\partial\eta^{1,i}}{\partial\theta} = 0$ for any $i \in \{1,\dots,2i\}$ and $\dfrac{\partial\eta^2}{\partial\theta} = 1$. Equivalently, $\eta^1(b,\theta) = \eta^1(b)$ defines an element in $\mathrm{Diff}^s(B)$ and $\eta^2(b,\theta) - \theta$ defines an element in $H^s(B,S^1)$. □

Corollary 3.15. *Any element of* $\mathrm{Diff}^s_R(B \times S^1)$ *is the lift of some element in* $\mathrm{Diff}^s(B)$ *(see Lemma 3.12), and if we have a lift* $\eta \in \mathrm{Diff}^s_R(B \times S^1)$ *of some* $\nu \in \mathrm{Diff}^s(B)$*, then* η *is of the form*

$$\eta(b,\theta) = (\nu(b),\ \theta + k(b)).$$ □

Now let σ be a symplectic form on B and let $\omega = \pi^*\sigma$ on $M = B \times S^1$. Note that the symplectomorphisms $\mathrm{Diff}^s_\sigma(B) \subset \mathrm{Diff}^s(B)$ are a smooth submanifold.

Corollary 3.16. *If we consider the restrictions*

$$\Phi|_{\mathrm{Diff}^s_\sigma(B)\times H^s(B,S^1)} : \mathrm{Diff}^s_\sigma(B) \times H^s(B,S^1) \to \mathrm{Diff}^s_{R,\omega}(B \times S^1)$$

and

$$\Phi^{-1}|_{\mathrm{Diff}^s_{R,\omega}(M)} : \mathrm{Diff}^s_{R,\omega}(B \times S^1) \to \mathrm{Diff}^s_\sigma(B) \times H^s(B,S^1),$$

then those define diffeomorphisms and, in particular, $\mathrm{Diff}^s_{R,\omega}(B \times S^1)$ *is a smooth submanifold of* $\mathrm{Diff}^s_R(B \times S^1)$. □

Lemma 3.17. *Let $\eta \in \mathrm{Diff}^s_\lambda(B \times S^1)$ be the lift of some $\nu \in \mathrm{Diff}^s(B)$, i. e. $\eta = (\nu, \eta^2)$. Then η^2 is also of the form $\eta^2(b,\theta) = \theta + k(b)$ for some $k \in H^s(B, S^1)$ and the map k satisfies $\mu - \nu^*\mu = \mathrm{d}k$.*

Proof. Let $b^1, \dots, b^{2n}$ denote local coordinates on B. We compute

$$\begin{aligned} \mathrm{d}\theta + \pi^*\mu = \lambda \overset{!}{=} \eta^*\lambda &= \eta^*(\mathrm{d}\theta + \pi^*\mu) \\ &= \mathrm{d}\eta^2 + \underbrace{\eta^*\pi^*\mu}_{= (\pi\circ\eta)^*\mu = (\nu\circ\pi)^*\mu = \pi^*\nu^*\mu} \\ &= \frac{\partial\eta^2}{\partial b^i}\mathrm{d}b^i + \frac{\partial \eta^2}{\partial\theta}\mathrm{d}\theta + \pi^*\nu^*\mu. \end{aligned}$$

Comparing the coefficients of $\mathrm{d}\theta$ on both sides of the equation yields $\dfrac{\partial\eta^2}{\partial\theta} = 1$ and we can write $\eta^2(b,\theta) = \theta + k(b)$ for some map $k : B \to S^1$. The equation $\lambda = \eta^*\lambda$ then becomes

$$\mathrm{d}\theta + \pi^*\mu = \mathrm{d}(\theta + k(b)) + \eta^*\pi^*\mu = \mathrm{d}\theta + \mathrm{d}k + \pi^*\nu^*\mu,$$

i. e. $\mu - \nu^*\mu = \mathrm{d}k$. □

Lemma 3.18. *Let $\eta \in \mathrm{Diff}^s_{\omega,\lambda}(B \times S^1) \subset \mathrm{Diff}^s_R(B \times S^1)$. By Lemma 3.14 (or Lemma 3.17), η is of the form $\eta(b,\theta) = (\nu(b), \theta + k(b))$ for $\nu := \eta^1 \in \mathrm{Diff}^s(B)$ and some $k \in H^s(B, S^1)$. Since η preserves ω und λ, ν preserves σ and τ, i. e. $\nu \in \mathrm{Diff}^s_{\sigma,\tau}(B)$. Then there is exactly an S^1-collection of lifts of ν in $\mathrm{Diff}^s_{\omega,\lambda}(B \times S^1)$. More precisely, we have:*

(a) Any $\theta_0 \in S^1$ defines an element $\tilde\nu \in \mathrm{Diff}^s_{\omega,\lambda}(B \times S^1)$ by

$$\tilde\eta(b,\theta) := (\nu(b), \theta + \theta_0 + k(b)) \in \mathrm{Diff}^s_{\omega,\lambda}(B \times S^1).$$

(b) Let $\tilde\eta \in \mathrm{Diff}^s_{\omega,\lambda}(B \times S^1)$ be some other lift of ν, i. e. using Lemma 3.14 we can write

$$\begin{aligned} \eta(b,\theta) &= (\nu(b), \theta + k(b)), \\ \tilde\eta(b,\theta) &= (\nu(b), \theta + \tilde k(b)). \end{aligned}$$

Then $\tilde k(b) = k(b) + \theta_0$ for some constant $\theta_0 \in S^1$.

Proof. (a) The map $\tilde\eta$ is clearly a lift of ν. Since ν preserves both σ and τ, $\tilde\eta$ automatically preserves ω and $\mathrm{d}\lambda$ by Lemma 3.13. We only need to check that $\tilde\eta$ also preserves λ. To that end, we compute

$$\begin{aligned} \tilde\eta^*\lambda &= \tilde\eta^*(\mathrm{d}\theta + \pi^*\mu) \\ &= \mathrm{d}(\theta + \theta_0 + k) + \tilde\eta^*\pi^*\mu \\ &= \mathrm{d}\theta + \mathrm{d}k + \pi^*\nu^*\mu \\ &= \eta^*\lambda \\ &= \lambda. \end{aligned}$$

(b) Using Lemma 3.17, we know that

$$\mathrm{d}k = \pi^*(\mu - \nu^*\mu) = \mathrm{d}\tilde{k},$$

hence k is equal to $\tilde{k}$ up to some additive constant $\theta_0 \in S^1$. □

In Lemma 3.23, we will see that $\mathrm{Diff}^s_{\omega,\lambda}(B \times S^1)$ really is homeomorphic to $\mathcal{D}^s \times S^1$ for some subspace $\mathcal{D}^s \subset \mathrm{Diff}^s_{\sigma,\tau}(B)$. We will first discuss the definition of $\mathcal{D}^s$.

Let now $\nu \in \mathrm{Diff}^s_\tau(B)$. If ν is at least a C^2-diffeomorphism so that $\mu - \nu^*\mu$ is still C^1, we can compute

$$\mathrm{d}\nu^*\mu = \nu^*\mathrm{d}\mu = \nu^*\tau = \tau$$

and hence,

$$\mathrm{d}(\mu - \nu^*\mu) = \tau - \tau = 0,$$

i. e. $\mu - \nu^*\mu$ is a closed form. Using Stokes' Theorem for a null-homologous loop γ in B bounding some disk $u : D^2 \to B$, we get

$$\int_{\gamma = \partial u} (\mu - \nu^*\mu) = \int_u \mathrm{d}(\mu - \nu^*\mu) = \int_u (\tau - \tau) = 0. \tag{3.1}$$

Hence, if ν is C^2, then $\mu - \nu^*\mu$ immediately defines a cohomology class in $H^1_{\mathrm{dR}}(B)$.

In general, we might not be able to take the differential of $\mu - \nu^*\mu$, but using the next lemma, we will be able to show that it still defines a cohomology class.

Lemma 3.19. *Let $\gamma : S^1 \to B$ be a null-homologous loop, i. e. $\gamma = \partial u$ is the boundary of some disk $u : D^2 \to B$. Let $\mu \in \Omega^1(B)$ and $\nu \in \mathrm{Diff}^s(B)$ be at least C^1 (but not neccesarily C^2). Then*

$$\int_\gamma \nu^*\mu = \int_u \nu^*\mathrm{d}\mu.$$

Proof. Define $f := \nu \circ u \in C^1(D^2, B)$. Then there exists a sequence of smooth functions $f_n \in C^\infty(D^2, B)$ such that $f_n \xrightarrow[n\to\infty]{C^1} f$ and

$$\begin{tikzcd}
\int_\gamma \nu^*\mu = \int_{\partial D^2} f^*\mu & & \int_D f^*\mathrm{d}\mu = \int_u \nu^*\mathrm{d}\mu \\
& \int_{\partial D^2} f_n^*\mu = \int_{D^2} \mathrm{d}(f_n^*\mu) = \int_{D^2} f_n^*\mathrm{d}\mu \arrow[ul, "n\to\infty", "C^0"'] \arrow[ur, "n\to\infty"', "C^0"] &
\end{tikzcd}$$

□

Corollary 3.20. *$\mu - \nu^*\mu$ defines a cohomology class in $H^1_{\mathrm{dR}}(B)$.*

Proof. The Theorem of de Rham and the Universal Coefficient Theorem yield isomorphisms

$$H^1_{\mathrm{dR}}(B) \cong H^1(B;\mathbb{R}) \cong \mathrm{Hom}_{\mathbb{Z}}(H_1(B),\mathbb{R}).$$

Hence, it suffices to prove that $\mu - \nu^*\mu$ defines a homomorphism

$$\langle [\mu - \nu^*\mu], \cdot \rangle \in \mathrm{Hom}_{\mathbb{Z}}(H_1(B),\mathbb{R}).$$

For any representative $\gamma : S^1 \to B$ of a homology class $[\gamma] \in H_1(B)$, we let

$$\langle [\mu - \nu^*\mu], [\gamma] \rangle := \int_\gamma (\mu - \nu^*\mu).$$

To show that this is well defined, we need to check that

$$\int_\gamma (\mu - \nu^*\mu) = 0$$

for any null-homologous loop γ in B. To that end, let γ be such a loop, i. e. $\gamma = \partial u$ is the boundary of some disk $u : D^2 \to B$. Lemma 3.19 shows that

$$\int_\gamma \nu^*\mu = \int_u \nu^* \mathrm{d}\mu.$$

Hence, the same computation as in Eq. (3.1) shows that $\int_\gamma (\mu - \nu^*\mu) = 0$ if γ is null-homologous. □

Lemma 3.21.

$$H^1_{\mathrm{dR}}(B)\Big/\Big\{[\mathrm{d}k] \;\Big|\; k : B \to S^1\Big\} \cong \mathrm{Hom}\Big(H_1(B;\mathbb{Z}),\mathbb{R}\Big)\Big/\mathrm{Hom}\Big(H_1(B;\mathbb{Z}),\mathbb{Z}\Big). \tag{3.2}$$

Proof. De Rham's Theorem says that integration is an isomorphism

$$H^1_{\mathrm{dR}}(B) \underset{\cong}{\xrightarrow{\int}} \mathrm{Hom}\Big(H_1(B),\mathbb{R}\Big). \tag{3.3}$$

Restricting this map to $\Big\{[\mathrm{d}k] \;\Big|\; k : B \to S^1\Big\} \subset H^1_{\mathrm{dR}}(B)$ yields a map

$$\Big\{[\mathrm{d}k] \;\Big|\; k : B \to S^1\Big\} \xrightarrow{\int} \mathrm{Hom}\Big(H_1(B),\mathbb{Z}\Big).$$

We claim that this is also an isomorphism: For injectivity, let $k, \tilde{k} : B \to S^1$ such that $\int_\gamma \mathrm{d}k = \int_\gamma \mathrm{d}\tilde{k}$ for any $\gamma \in H_1(B)$. Hence,

$$\int_\gamma (\mathrm{d}k - \mathrm{d}\tilde{k}) = 0 \quad \text{for any } \gamma \in H_1(B),$$

i. e. $\mathrm{d}k - \mathrm{d}\tilde{k}$ is an exact 1-form and there is a function $l : B \to \mathbb{R}$ such that $\mathrm{d}k - \mathrm{d}\tilde{k} = \mathrm{d}l$. This implies $[\mathrm{d}\tilde{k}] = [\mathrm{d}\tilde{k} + \mathrm{d}l] = [\mathrm{d}k]$ and integration is injective.

To show surjectivity, we let $f \in \mathrm{Hom}\big(H_1(B), \mathbb{Z}\big) \subset \mathrm{Hom}\big(H_1(B), \mathbb{R}\big)$. By Eq. (3.3), there is a cohomology class $[\alpha] \in H^1_{\mathrm{dR}}(B)$ for some closed $\alpha \in \Omega^1(B)$ such that for any $\gamma \in H_1(B)$, we have

$$f(\gamma) = \int_\gamma \alpha.$$

Fix some base point $b_0 \in B$ and define

$$k : B \to S^1 = \mathbb{R}/\mathbb{Z}, \quad b \mapsto \int_{b_0}^{b} \alpha \mod 1$$

for $b \in B_i$. This definition is independent of the path from b_0 to b: Let β_1, β_2 be two such paths, then $\beta_1 \#(-\beta_2)$ is a closed path and defines an element $\gamma := [\beta_1 \#(-\beta_2)] \in H_1(B)$. Hence,

$$\int_{\beta_1} \alpha - \int_{\beta_2} \alpha \mod 1 = \int_\gamma \alpha \mod 1 = f(\gamma) \mod 1 = 0,$$

since $f(\gamma) \in \mathbb{Z}$. Finally, $[\alpha] = [\mathrm{d}k]$ is a preimage of f in $\Big\{[\mathrm{d}k] \,\Big|\, k : B \to S^1\Big\}$.

We have shown that we have a commuting diagram

$$\begin{tikzcd}
H^1_{\mathrm{dR}}(B) \arrow[r, "\int", "\cong"'] & \mathrm{Hom}\big(H_1(B), \mathbb{R}\big) \\
\Big\{[\mathrm{d}k] \,\Big|\, k : B \to S^1\Big\} \arrow[u, hook] \arrow[r, "\int", "\cong"'] & \mathrm{Hom}\big(H_1(B), \mathbb{Z}\big) \arrow[u, hook]
\end{tikzcd}$$

and this implies the lemma. ☐

Proposition 3.22. *A diffeomorphism* $\nu \in \mathrm{Diff}^s_{\sigma,\tau}(B)$ *has a lift* $\eta \in \mathrm{Diff}^s_{\omega,\lambda}(B \times S^1)$. $\iff$ $\displaystyle\int_\gamma (\mu - \nu^*\mu) \in \mathbb{Z}$ *for any loop* $\gamma \in H_1(B; \mathbb{Z})$.

Remark. In particular, if B is a surface of genus g, those are just $2g$ conditions for the $2g$ generators of $H_1(B; \mathbb{Z})$.

Example. The condition in the previous proposition is not always satisfied, i. e. not any element of $\mathrm{Diff}^s_{\sigma,\tau}(B)$ has a lift in $\mathrm{Diff}^s_{\omega,\lambda}(B \times S^1)$. As an example, let $B = \Sigma = T^2$ be the two-torus and choose coordinates (b_1, b_2) such that $\sigma = \mathrm{d}b_1 \wedge \mathrm{d}b_2$ is an area form. Let further $a_1, a_2 \in \mathbb{R}$ and define $\mu := a_1\,\mathrm{d}b_1 + a_2\,\mathrm{d}b_2$. Then $\nu : T^2 \to T^2$, $(b_1, b_2) \mapsto (b_2, -b_1)$ is an element of $\mathrm{Diff}^s_{\sigma,\tau}(T^2)$: It is a (smooth) diffeomorphism of T^2 and preserves $\tau = \mathrm{d}\mu = 0$ and σ since $\nu^*\sigma = \mathrm{d}b_2 \wedge \mathrm{d}(-b_1) = \mathrm{d}b_1 \wedge \mathrm{d}b_2 = \sigma$. The cohomology class of

$$\begin{aligned}
\mu - \nu^*\mu &= a_1\,\mathrm{d}b_1 + a_2\,\mathrm{d}b_2 - (a_1\,\mathrm{d}b_2 + a_2\,\mathrm{d}(-b_1)) \\
&= (a_1 + a_2)\,\mathrm{d}b_1 + (a_2 - a_1)\,\mathrm{d}b_2
\end{aligned}$$

has no integer period if $a_1 + a_2$, $a_2 - a_1 \notin \mathbb{Z}$, hence we can apply the previous Lemma in those cases to get that ν does not have a lift in $\mathrm{Diff}_{\omega,\lambda}(T^2 \times S^1)$ for $\omega = \pi^*\sigma$ and $\lambda = \mathrm{d}\theta + \pi^*\mu$.

Proof of Proposition 3.22. "$\Rightarrow$": Let η be a lift of ν. Using Lemma 3.17, we know that

$$\mu - \nu^*\mu = \mathrm{d}k$$

for some $k : B \to S^1$. By Corollary 3.20, we can consider the cohomology class $[\mathrm{d}k] = [\mu - \nu^*\mu] \in H^1(B;\mathbb{R})$. Using the isomomorphism in Eq. (3.2), this implies that $\int_\gamma \mu - \nu^*\mu = \int_\gamma \mathrm{d}k \in \mathbb{Z}$ for any loop $\gamma \in H_1(B;\mathbb{Z})$.

"$\Leftarrow$": Let $\mu - \nu^*\mu$ be such that $\int_\gamma \mu - \nu^*\mu \in \mathbb{Z}$ for any loop $\gamma \in H_1(B;\mathbb{Z})$. Again, using the isomorphism in Eq. (3.2), we can find $l_1 : B \to S^1$ such that $[\mu - \nu^*\mu] = [\mathrm{d}l_1]$. This implies that there is a function $l_2 : B \to \mathbb{R}$ such that $\mu - \nu^*\mu = \mathrm{d}l_1 + \mathrm{d}l_2$. Let us project $l_2 : B \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \cong S^1$ and define $k := l_1 + l_2 : B \to S^1$. Then $\mu - \nu^*\mu = \mathrm{d}k$ and we claim that

$$\begin{aligned} \eta : B \times S^1 &\to B \\ (b,\theta) &\mapsto (\nu(b),\ \theta + k(b)) \end{aligned}$$

is a lift of $\nu \in \mathrm{Diff}_{\sigma,\tau}(B)$ in $\mathrm{Diff}_{\omega,\lambda}(B \times S^1)$. The map η clearly satisfies $\pi \circ \eta = \nu \circ \pi$, i. e. it is a lift of ν in $\mathrm{Diff}^s(M)$. Lemma 3.13 implies that $\eta \in \mathrm{Diff}^s_{\omega,\mathrm{d}\lambda}(M)$. It only remains to check that $\eta^*\lambda = \lambda$. To that end, we compute

$$\begin{aligned} \eta^*\lambda &= \eta^*(\mathrm{d}\theta + \pi^*\mu) \\ &= \mathrm{d}\theta + \mathrm{d}k + \underbrace{\eta^*\pi^*\mu}_{= \pi^*\nu^*\mu = \pi^*(\mu - \mathrm{d}k)} \\ &= \mathrm{d}\theta + \pi^*\mu = \lambda. \end{aligned}$$ ☐

Remark. As a special case of the previous theorem, we can show that if B satisfies $H^1(B) = 0$ (e. g. if $B = S^{2n}$), then any diffeomorphism $\nu \in \mathrm{Diff}^s_{\tau,\sigma}(B)$ has a lift $\eta \in \mathrm{Diff}^s_{\omega,\lambda}(B \times S^1)$: To that end, let $\nu \in \mathrm{Diff}^s_{\tau,\sigma}(B)$. Since $H^1_{\mathrm{dR}}(B) = 0$, any form representing a first cohomology class is exact. In particular, $\mu - \nu^*\mu$ is exact and hence, $\int_\gamma (\mu - \nu^*\mu) = 0$ for any loop $\gamma \in H_1(B;\mathbb{Z})$. Using Proposition 3.22, we get that ν has a lift $\eta \in \mathrm{Diff}^s_{\omega,\lambda}(B \times S^1)$.

This proposition motivates the definition

$$\mathcal{D}^s := \left\{ \nu \in \mathrm{Diff}^s_{\sigma,\tau}(B) \;\middle|\; \int_\gamma (\mu - \nu^*\mu) \in \mathbb{Z} \text{ for all } \gamma \in H_1(B;\mathbb{Z}) \right\}$$

for the diffeomorphisms in $\mathrm{Diff}^s_{\sigma,\tau}(B)$ that admit a lift to $\mathrm{Diff}^s_{\omega,\lambda}(B \times S^1)$. According to Lemma 3.18, there is a S^1-collection of lifts for any $\nu \in \mathrm{Diff}^s_{\sigma,\tau}(B)$, i. e. we expect

$\mathrm{Diff}^s_{\omega,\lambda}(B\times S^1)$ to be diffeomorphic to $\mathcal{D}^s\times S^1$ if $\mathcal{D}^s\subset\mathrm{Diff}^s_\sigma(B)$ is a smooth submanifold. We will make this statement precise in the rest of this section by trying to further restrict the diffeomorphisms given in Lemma 3.14.

The set $\mathrm{Diff}^s_{\omega,\lambda}(B\times S^1)$ is contained in $\mathrm{Diff}^s_{R,\omega}(B\times S^1)$. We will now discuss a continuous map $\iota:\mathcal{D}^s\times S^1\hookrightarrow\mathrm{Diff}^s_\sigma(B)\times H^s(B,S^1)$ such that we can restrict Φ to $\mathcal{D}^s\times S^1$ via ι.

Lemma 3.23. *There is a continuous embedding* $\iota:\mathcal{D}^s\times S^1\hookrightarrow\mathrm{Diff}^s_\sigma(B)\times H^s(B,S^1)$ *such that the image of the composition* $\Psi:=\Phi\circ\iota:\mathrm{Diff}^s_{\sigma,\tau}(B)\times S^1\to\mathrm{Diff}^s_{R,\omega}(B\times S^1)$ *actually lies in* $\mathrm{Diff}^s_{\omega,\lambda}(B\times S^1)$*, i. e. the following diagram commutes:*

$$\begin{tikzcd}
\mathrm{Diff}^s_\sigma\times H^s(B,S^1) \arrow[r, "\Phi"] & \mathrm{Diff}^s_{R,\omega}(B\times S^1) \\
\mathcal{D}^s\times S^1 \arrow[u, hook, "\iota"] \arrow[r, "\Psi"'] & \mathrm{Diff}^s_{\omega,\lambda}(B\times S^1) \arrow[u, phantom, "\bigcup"]
\end{tikzcd}$$

The map Ψ *is a homeomorphism.*

Proof. **Step 1.** Let $\nu\in\mathcal{D}^s$ and $\theta_0\in S^1$. We will define a continuous map

$$\begin{aligned}k:\mathcal{D}^s&\to H^s(B,S^1),\\ \nu&\mapsto k_\nu\end{aligned}$$

and then let

$$\iota(\nu,\theta_0):=(\nu,\theta_0+k_\nu(b)).$$

To that end, we start with $\nu\in\mathcal{D}^s$, i. e. $\nu\in\mathrm{Diff}^s_{\sigma,\tau}(B)$ such that $\int_\gamma(\mu-\nu^*\mu)\in\mathbb{Z}$ for all $\gamma\in H_1(B;\mathbb{Z})$. Corollary 3.20 implies that $\mu-\nu^*\mu$ represents a cohomology class $[\mu-\nu^*\mu]\in H^1(B;\mathbb{Z})$. In particular, the map $\mathcal{D}^s\to H^{s-1}(\Lambda^1B)$, $\nu\mapsto\mu-\nu^*\mu$ has image $\coprod_{h\in H^1(B;\mathbb{Z})}H^{s-1}_h(\Lambda^1B)$, where

$$H^{s-1}_h(\Lambda^1B):=\left\{\alpha\in H^{s-1}(\Lambda^1B)\;\middle|\;\alpha\text{ is a representative of }h\right\}.$$

If ν is at least C^2, then this definition is equivalent to

$$H^{s-1}_h(\Lambda^1B)=\left\{\alpha\in H^{s-1}(\Lambda^1B)\;\middle|\;d\alpha=0,\ [\alpha]=h\right\}.$$

For every cohomology class $h\in H^1(B;\mathbb{Z})$, fix some map $k_h\in C^\infty(B,S^1)$ such that $h=[\mathrm{d}k_h]$ (see Lemma 3.21) and define $\alpha_h:=\mathrm{d}k_h\in\Omega^1(B)$. Any other element $\alpha\in H^{s-1}_h(\Lambda^1B)$ can then be written as

$$\alpha=\alpha_h+\beta$$

for some exact $\beta \in H^{s-1}(\Lambda^1 B)$. In particular, the one-form

$$\mu_\nu := \mu - \nu^*\mu - \alpha_{[\mu-\nu^*\mu]}$$

is exact. Fix some base point $b_0 \in B$ and define a map $H^{s-1}_{\mathrm{exact}}(\Lambda^1 B) \to H^s(B,S^1)$ by mapping an exact one-form β to the function k_β defined by

$$k_\beta(b) := \int_{b_0}^{b} \beta \quad \text{for any path from } b_0 \text{ to } b.$$

This is well defined since β is exact. Since $\mathrm{d}k_\beta = \beta \in H^{s-1}_{\mathrm{exact}}(\Lambda^1 B)$, Lemma 3.24 (after this proof) implies that $k_\beta \in H^s(B,S^1)$. In particular, we let

$$k_{\mu_\nu}(b) := \int_{b_0}^{b} \mu_\nu \quad \text{for any path from } b_0 \text{ to } b.$$

Then we define $k_\nu := k_{\mu_\nu} + k_{[\mu-\nu^*\mu]} \in H^s(B,S^1)$. Note that the map

$$B \times S^1 \to B \times S^1, \quad (b,\theta) \mapsto \big(\nu(b), \theta + k_\nu(b)\big)$$

is a lift of ν in $\mathrm{Diff}^s_{\omega,\lambda}(B \times S^1)$.

In summary, for every cohomology class $h \in H^1(B;\mathbb{Z})$, we fixed some map $k_h \in C^\infty(B,S^1)$ such that $h = [\mathrm{d}k_h]$ and defined $\alpha_h \in \Omega^1(B)$ by $\alpha_h := \mathrm{d}k_h$. Then we let

$$\begin{tikzcd}
\mathcal{D}^s \arrow[d] & \nu \arrow[d, mapsto] \\
\coprod_{h\in H^1(B;\mathbb{Z})} H^{s-1}_h(\Lambda^1 B) \arrow[d] & \mu - \nu^*\mu \arrow[d, mapsto] \\
\coprod_{h\in H^1(B;\mathbb{Z})} \{h\} \times H^{s-1}_{\mathrm{exact}}(\Lambda^1 B) \arrow[d] & \big([\mu-\nu^*\mu], \underbrace{\mu - \nu^*\mu - \alpha_{[\mu-\nu^*\mu]}}_{=:\mu_\nu}\big) \arrow[d, mapsto] \\
H^s(B,S^1) & k_\nu := k_{[\mu-\nu^*\mu]} + k_{\mu_\nu}
\end{tikzcd}$$

The map $\nu \mapsto k_\nu$ is continuous since $H^1(B;\mathbb{Z})$ is discrete.

Step 2. The image of the composition $\Psi := \Phi \circ \iota : \mathcal{D}^s \times S^1 \to \mathrm{Diff}^s_{R,\omega}(B \times S^1)$ lies in $\mathrm{Diff}^s_{\omega,\lambda}(B \times S^1)$.

Let $\nu \in \mathrm{Diff}^s_{\sigma,\tau}(B)$ and $\kappa \in S^1$. Then $\eta := \Psi(\nu,\theta_0) \in \mathrm{Diff}^s_{R,\omega}(B \times S^1)$ is of the form

$$\eta(b,\theta) = \big(\nu(b), \theta + \kappa + k_\nu(b)\big) \in B \times S^1$$

and it remains to check that η also preserves λ. Write $\lambda = \mathrm{d}\theta + \pi^*\mu$ for some $\mu \in \Omega^1(B)$ and first compute

$$\begin{aligned}\mathrm{d}k_\nu &= \mathrm{d}k_{\mu_\nu} + \mathrm{d}k_{[\mu-\nu^*\mu]}\\ &= \mu_\nu + \alpha_{[\mu-\nu^*\mu]}\\ &= \left(\mu - \nu^*\mu - \alpha_{[\mu-\nu^*\mu]}\right) + \alpha_{[\mu-\nu^*\mu]}\\ &= \mu - \nu^*\mu.\end{aligned}$$

Now we can check that

$$\begin{aligned}\eta^*\lambda = \eta^*(\mathrm{d}\theta + \pi^*\mu) &= \mathrm{d}\theta + \mathrm{d}k_\nu + \pi^* \underbrace{\nu^*\mu}_{=\mu-\mathrm{d}k_\nu}\\ &= \mathrm{d}\theta + \mathrm{d}k_\nu + \pi^*\mu - \pi^*\mathrm{d}k_\nu\\ &= \mathrm{d}\theta + \pi^*\mu\\ &= \lambda.\end{aligned}$$

Step 3. The (continuous) inverse of Ψ is given by

$$\begin{aligned}&\mathrm{Diff}^s_{\omega,\lambda}(B\times S^1) \to \mathcal{D}^s \times S^1\\ &\quad \eta = (\eta^1,\eta^2) \mapsto (\eta^1, \eta^2 - \theta - k_{\eta^1}).\end{aligned}$$ □

Remark. There is a similar theorem describing the quantomorphisms of a contact S^1-principal bundle $S^1 \to M \xrightarrow{\pi} B$ with contact form λ as an S^1-principal bundle over the Hamiltonian diffeomorphisms of B with symplectic form ω defined by $\pi^*\omega = \mathrm{d}\lambda$, see also Theorem 3.1 in [RS81].

To really complete this proof, we need to provide the next lemma.

Lemma 3.24. *Let $k \in H^{s-1}(B,\mathbb{R})$ such that $\mathrm{d}k$ is of the same Sobolev class $s-1$, i.e. $\mathrm{d}k \in H^{s-1}(\Lambda^1 B)$. Then $k \in H^s(B,\mathbb{R})$.*

The same result holds for maps to S^1.

Proof. Let B have coordinates b^i. Since $\mathrm{d}k \in H^{s-1}(\Lambda^1 B)$, all the coefficent functions of $\mathrm{d}k = \sum_i \frac{\partial k}{\partial b^i}\mathrm{d}b^i$ satisfy $\frac{\partial k}{\partial b^i} \in H^{s-1}(B,\mathbb{R})$ for all i. Hence, $k \in H^s(B,\mathbb{R})$. □

We will now define a group structure on $\mathcal{D}^s \times S^1$, which induces the regular group structure given by the composition of maps in $\mathrm{Diff}^s_{\omega,\lambda}(M)$ via $\Psi : \mathcal{D}^s \times S^1 \to \mathrm{Diff}^s_{\omega,\lambda}(M)$ as defined in Lemma 3.23.

Lemma 3.25. *The composition*

$$(\nu_2,\kappa_2)\circ(\nu_1,\kappa_1) := \Big(\nu_2\circ\nu_1,\ \kappa_1 + \kappa_2 - k_{\nu_2}(\nu_1(b_0))\Big)$$

defines a group structure on $\mathcal{D}^s \times S^1$.

Proof. The identity element is given by $(\mathrm{id}_B, 0)$: For any $(\nu, \kappa) \in \mathcal{D}^s \times S^1$, we have

$$\begin{aligned}
&(\nu,\kappa) \circ (\mathrm{id}_B, 0) = (\nu \circ \mathrm{id}_B,\ 0 + \kappa - k_\nu(b_0)) = (\nu, \kappa),\\
&(\mathrm{id}_B, 0) \circ (\nu,\kappa) = (\mathrm{id}_B \circ \nu,\ \kappa + 0 - k_{\mathrm{id}_B}(\nu(b_0)) = (\nu,\kappa).
\end{aligned}$$

The inverse of (ν, κ) is given by $(\nu^{-1}, k_\nu(\nu^{-1}(b_0)) - \kappa)$:

$$\begin{aligned}
(\nu,\kappa) \circ (\nu^{-1}, k_\nu(\nu^{-1}(b_0)) - \kappa) &= (\nu \circ \nu^{-1},\ k_\nu(\nu^{-1}(b_0)) - \kappa + \kappa - k_\nu(\nu^{-1}(b_0)))\\
&= (\mathrm{id}_B, 0),\\
(\nu^{-1}, k_\nu(\nu^{-1}(b_0)) - \kappa) \circ (\nu,\kappa) &= (\nu^{-1} \circ \nu,\ \kappa + k_\nu(\nu^{-1}(b_0)) - \kappa - k_{\nu^{-1}}(\nu(b_0)))\\
&= (\mathrm{id}_B, 0)
\end{aligned}$$

since

$$\begin{aligned}
k_{\nu^{-1}}(\nu(b_0))) &= \int_{b_0}^{\nu(b_0)} \Big(\mu - (\nu^{-1})^*\mu\Big)\\
&= \int_{\nu^{-1}(b_0)}^{\nu^{-1}(\nu^{(}b_0))} \nu^*\Big(\mu - (\nu^{-1})^*\mu\Big)\\
&= \int_{\nu^{-1}(b_0)}^{b_0} (\nu^*\mu - \mu)\\
&= \int_{b_0}^{\nu^{-1}(b_0)} (\mu - \nu^*\mu)\\
&= k_\nu(\nu^{-1}(b_0)).
\end{aligned}$$

Finally, the composition is associative:

$$\begin{aligned}
\Big((\nu_3,\kappa_3) \circ (\nu_2,\kappa_2)\Big) \circ (\nu_1,\kappa_1) &= \Big(\nu_3 \circ \nu_2,\ \kappa_2 + \kappa_3 - k_{\nu_3}(\nu_2(b_0))\Big) \circ (\nu_1,\kappa_1)\\
&= \Big((\nu_3 \circ \nu_2) \circ \nu_1,\ \kappa_1 + \kappa_2 + \kappa_3 - k_{\nu_3}(\nu_2(b_0)) - k_{\nu_3\circ\nu_2}(\nu_1(b_0))\Big). \qquad (3.4)
\end{aligned}$$

We compute

$$\begin{aligned}
k_{\nu_3\circ\nu_2}(\nu_1(b_0)) &= \int_{b_0}^{\nu_1(b_0)} \Big(\mu - (\nu_3 \circ \nu_2)^*\mu\Big)\\
&= \int_{b_0}^{\nu_1(b_0)} (\mu - \nu_2^*\nu_3^*\mu)\\
&= \int_{b_0}^{\nu_1(b_0)} \Big(\mu - \nu_2^*\mu + \int_{b_0}^{\nu_1(b_0)} \nu_2^*(\mu - \nu_3^*\mu)\Big)\\
&= k_{\nu_2}(\nu_1(b_0)) + \int_{\nu_2(b_0)}^{\nu_2(\nu_1(b_0))} (\mu - \nu_3^*\mu)\\
&= k_{\nu_2}(\nu_1(b_0)) + \int_{b_0}^{\nu_2(\nu_1(b_0))} (\mu - \nu_3^*\mu) - \int_{b_0}^{\nu_2(b_0)} (\mu - \nu_3^*\mu)\\
&= k_{\nu_2}(\nu_1(b_0)) + k_{\nu_3}(\nu_2(\nu_1(b_0))) - k_{\nu_3}(\nu_2(b_0)),
\end{aligned}$$

hence

$$-k_{\nu_3}(\nu_2(b_0)) - k_{\nu_3\circ\nu_2}(\nu_1(b_0)) = -k_{\nu_2}(\nu_1(b_0)) - k_{\nu_3}(\nu_2(\nu_1(b_0)))$$

and continuing Eq. (3.4) yields

$$\begin{aligned}
&\big((\nu_3,\kappa_3)\circ(\nu_2,\kappa_2)\big)\circ(\nu_1,\kappa_1)\\
&\qquad= \big(\nu_3\circ(\nu_2\circ\nu_1),\ \kappa_1+\kappa_2-k_{\nu_2}(\nu_1(b_0))+\kappa_3-k_{\nu_3}(\nu_2(\nu_1)(b_0))\big)\\
&\qquad= (\nu_3,\kappa_3)\circ\big(\nu_2\circ\nu_1,\ \kappa_1+\kappa_2-k_{\nu_2}(\nu_1(b_0))\big)\\
&\qquad= (\nu_3,\kappa_3)\circ\big((\nu_2,\kappa_2)\circ(\nu_1,\kappa_1)\big).
\end{aligned}$$

□

Proposition 3.26. *The map* $\Psi : \mathcal{D}^s\times S^1 \to \mathrm{Diff}^s_{\omega\lambda}(B\times S^1)$ *as defined in Lemma 3.23 is a group homomorphism.*

Proof. For $(\nu_1,\kappa_1),(\nu_2,\kappa_2)\in\mathcal{D}^s\times S^1$, we have

$$\begin{aligned}
\big(\Psi(\nu_2,\kappa_2)\circ\Psi(\nu_1,\kappa_1)\big)(b,\theta) &= \Psi(\nu_2,\kappa_2)\big(\nu_1(b),\ \theta+k_{\nu_1}(b)+\kappa_1\big)\\
&= \big(\nu_2(\nu_1(b)),\ \theta+k_{\nu_1}(b)+\kappa_1+k_{\nu_2}(\nu_1(b))+\kappa_2\big).
\end{aligned} \tag{3.5}$$

We compute

$$\begin{aligned}
k_{\nu_2\circ\nu_1}(b) &= \int_{b_0}^{b}\mu-(\nu_2\circ\nu_1)^*\mu\\
&= \int_{b_0}^{b}\mu-\nu_1^*\nu_2^*\mu\\
&= \int_{b_0}^{b}\mu-\nu_1^*\mu+\int_{b_0}^{b}\nu_1^*(\mu-\nu_2^*\mu)\\
&= k_{\nu_1}(b)+\int_{\nu_1(b_0)}^{\nu_1(b)}\mu-\nu_2^*\mu\\
&= k_{\nu_1}(b)+\int_{b_0}^{\nu_1(b)}\mu-\nu_2^*\mu-\int_{b_0}^{\nu_1(b_0)}\mu-\nu_2^*\mu\\
&= k_{\nu_1}(b)+k_{\nu_2}(\nu_1(b))-k_{\nu_2}(\nu_1(b_0)),
\end{aligned}$$

hence

$$k_{\nu_1}(b)+k_{\nu_2}(\nu_1(b)) = k_{\nu_2\circ\nu_1}(b)+k_{\nu_2}(\nu_1(b_0))$$

and continuing Eq. (3.5) yields

$$\begin{aligned}
&\big(\Psi(\nu_2,\kappa_2)\circ\Psi(\nu_1,\kappa_1)\big)(b,\theta)\\
&\qquad= \big((\nu_2\circ\nu_1)(b),\ \theta+(\kappa_1+\kappa_2+k_{\nu_2}(\nu_1(b_0)))+k_{\nu_2\circ\nu_1}(b)\big)\\
&\qquad= \Psi\big(\nu_2\circ\nu_1,\ \kappa_1+\kappa_2+k_{\nu_2}(\nu_1(b_0))\big)
\end{aligned}$$

$$= \Psi\Big((\nu_2, \kappa_2) \circ (\nu_1, \kappa_1)\Big). \qquad \square$$

Up to now, we have only discussed the continuous structure of the bundle $\Psi : \mathcal{D}^s \times S^1 \xrightarrow{\cong} \mathrm{Diff}^s_{\omega,\lambda}(M)$, so we will spend the rest of this section prove that if $\mathcal{D}^s \subset \mathrm{Diff}^s_{\sigma,\tau}(B)$ is a smooth submanifold, then the map $k : \mathcal{D}^s \to H^s(B, S^1)$ is smooth and Ψ is actually a diffeomorphism.

A candidate for the differential of k is the directional derivative. Let $\nu_0 \in \mathcal{D}^s$ and for any path $\nu(t) \in \mathcal{D}^s$ for $t \in (-\epsilon, \epsilon)$ such that $\nu(0) = \nu_0$, we have

$$\begin{aligned}
T_{\nu_0} k(\dot{\nu}_0)(b) &= \lim_{t \to 0} \frac{k_{\nu(t)}(b) - k_{\nu_0}(b)}{t} \\
&= \lim_{t \to 0} \frac{1}{t} \int_{b_0}^{b} (\mu - \nu(t)^*\mu) - (\mu - \nu_0^*\mu) \\
&= \lim_{t \to 0} \frac{1}{t} \int_{b_0}^{b} \nu_0^*\mu - \nu(t)^*\mu \\
&= \int_{b_0}^{b} \lim_{t \to 0} \frac{1}{t} \Big(\nu_0^*\mu - \nu(t)^*\mu\Big) \\
&= -\int_{b_0}^{b} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \nu(t)^*\mu \\
&= -\int_{b_0}^{b} \nu(t)^* \mathcal{L}_{\dot{\nu}(t) \circ \nu(t)^{-1}} \mu \Big|_{t=0} \\
&= -\int_{b_0}^{b} \nu_0^* \mathcal{L}_{\underbrace{\dot{\nu}(0) \circ \nu_0^{-1}}_{=:X}} \mu \\
&= -\int_{\nu_0(b_0)}^{\nu_0(b)} \mathcal{L}_X \mu \\
&= -\int_{\nu_0(b_0)}^{\nu_0(b)} \mathrm{d}\iota_X \mu + \iota_X \underbrace{\mathrm{d}\mu}_{=\tau}.
\end{aligned}$$

Since both the full integral and

$$\begin{aligned}
\int_{\nu_0(b_0)}^{\nu_0(b)} \mathrm{d}\iota_X \mu &= \iota_X \mu \Big|_{\nu_0(b_0)}^{\nu_0(b)} \\
&= \mu(X)(\nu_0(b)) - \mu(X)(\nu_0(b_0)) \\
&= \mu_{\nu_0(b)}(X(\nu_0(b))) - \mu_{\nu_0(b_0)}(X(\nu_0(b_0))) \\
&= \mu_{\nu_0(b)}(\dot{\nu}_0(b)) - \mu_{\nu_0(b_0)}(\dot{\nu}_0(b_0))
\end{aligned}$$

are independent of the path from b_0 to b, also $\displaystyle\int_{\nu_0(b_0)}^{\nu_0(b)} \iota_X \tau$ is and we get

$$T_{\nu_0} k(\dot{\nu}_0)(b) = -\mu_{\nu_0(b)}(\dot{\nu}_0(b)) + \mu_{\nu_0(b_0)}(\dot{\nu}_0(b_0)) - \int_{\nu_0(b_0)}^{\nu_0(b)} \iota_{\dot{\nu}_0 \circ \nu_0^{-1}} \tau.$$

In particular, at the identity we have

$$T_{\mathrm{id}}k(X) = -\mu(X) + \mu(X)(b_0) - \int_{b_0}^{b} \iota_X \tau. \tag{3.6}$$

Lemma 3.27. *If $\mathcal{D}^s \subset \mathrm{Diff}^s(B)$ is a smooth submanifold, then the map*

$$\begin{aligned} k : \mathcal{D}^s &\to H^s(B, S^1) \\ \nu &\mapsto k_\nu \end{aligned}$$

is differentiable with tangent map

$$\begin{aligned} T_\nu k : T_\nu \mathcal{D}^s &\to H^s(B, \mathbb{R}) \\ X &\mapsto \left(b \mapsto -\mu_{\nu(b)}(X(b)) + \mu_{\nu(b_0)}(X(b_0)) - \int_{\nu(b_0)}^{\nu(b)} \iota_{X \circ \nu^{-1}} \tau \right). \end{aligned}$$

Proof. We have to verify that

$$\lim_{X \to 0} \frac{\|k(\exp_\nu X) - k(\nu) - T_\nu k(X)\|_{H^s}}{\|X\|_{H^s}} \to 0.$$

We will omit the computation as this lemma also follows from the corresponding statement for general S^1-bundles, see the proof of Theorem 3.43 and the remark on page 63. □

Inductively, one can show

Corollary 3.28. *If $\mathcal{D}^s \subset \mathrm{Diff}^s(B)$ is a smooth submanifold, then the map k is smooth.*

This also follows directly from Theorem 3.43.

We are now in a position to find out when the diffeomorphisms preserving the stable Hamiltonian structure of a trivial S^1-bundle are a smooth submanifold of the full diffeomorphism group.

Theorem 3.29. *Assume that $\mathcal{D}^s \subset \mathrm{Diff}^s(B)$ is a smooth submanifold. Then also $\mathrm{Diff}^s_{\omega,\lambda}(B \times S^1) \subset \mathrm{Diff}^s(B \times S^1)$ is a smooth submanifold and*

$$\begin{aligned} \Psi : \mathcal{D}^s \times S^1 &\to \mathrm{Diff}^s_{\omega,\lambda}(B \times S^1) \\ (\nu, \theta_0) &\mapsto \big((b, \theta) \mapsto (\nu(b), \theta + k_\nu(b) + \theta_0) \big) \end{aligned}$$

is a diffeomorphism with inverse

$$\eta = (\eta^1, \eta^2) \mapsto \big(p(\eta) = \eta^1, \eta^2(b, \theta) - k_{\eta^1}(b) - \theta \big).$$

Proof. If we view Ψ as a map

$$\begin{aligned} \Psi : \mathcal{D}^s \times S^1 &\to \mathrm{Diff}^s_R(B \times S^1) \\ (\nu, \kappa) &\to \big((b, \theta) \mapsto (\nu(b),\ \theta + k_\nu(b) + \kappa) \big) \end{aligned}$$

then Ψ is a homeomorphism onto its image $\operatorname{im}(\Psi) = \operatorname{Diff}^s_{\omega,\lambda}(B \times S^1)$ by Lemma 3.23. Let $v \in T_\nu\mathcal{D}$ and $x \in \mathbb{R} \cong T_\kappa S^1$. The tangent map of Ψ is given by

$$T_{(\nu,\kappa)}\Psi(v,x) = v + \big(T_\nu k(v) + x\big)\partial_\theta.$$

This is an injective map: Let $v_1, v_2 \in T_\nu\mathcal{D}$ and $x_1, x_2 \in \mathbb{R}$ such that $T_{(\nu,\kappa)}\Psi(v_1,x_1) = T_{(\nu,\kappa)}\Psi(v_2,x_2)$, i. e.

$$v_1 + \big(T_\nu k(v_1) + x_1\big)\partial_\theta = v_2 + \big(T_\nu k(v_2) + x_2\big)\partial_\theta.$$

Since v_1 and v_2 only depend on the coordinates of B, this yields $v_1 = v_2$. Then also $T_\nu k(v_1) = T_\nu k(v_2)$, which in turn implies $x_1 = x_2$.

Therefore, we can apply Proposition 2.20 to find that $\operatorname{im}(\Psi) = \operatorname{Diff}^s_{\omega,\lambda}(B \times S^1)$ is a smooth submanifold of $\operatorname{Diff}^s(B \times S^1)$. ☐

3.6 Metrics on trivial circle bundles

As in the previous section, let M^{2n+1} be a trivial circle bundle

$$S^1 \longrightarrow M = B \times S^1 \xrightarrow{\pi} B$$

with S^1-coordinate θ, and we let (ω, λ) be a stable Hamiltonian structure on $B \times S^1$ such that the Reeb vector field is $R = \partial_\theta$. The discussion in the previous section implies that $\omega = \pi^*\sigma$ for some symplectic 2-form σ on B and $\lambda = \mathrm{d}\theta + \pi^*\mu$ for some one-form μ on B. Furthermore, there is $\tau \in \Omega^2(B)$ such that $\mathrm{d}\lambda = \pi^*\tau$, namely $\tau := \mathrm{d}\mu$.

Now let $(\tilde\omega, \tilde\lambda = \mathrm{d}\theta + \pi^*\tilde\mu)$ be another such stable Hamiltonian structure on $M = B \times S^1$, which also induces $\tilde\sigma, \tilde\tau \in \Omega^2(B)$ by $\tilde\omega = \pi^*\tilde\sigma$ and $\tilde\tau = \mathrm{d}\tilde\mu$. We further choose a metric $\langle\cdot,\cdot\rangle^B$ on B.

Lemma 3.30. *Let $\rho : B \to B$ be a smooth diffeomorphism such that $\rho^*\sigma = \tilde\sigma$ and $\rho^*\tau = \tilde\tau$.*

(a) The map

$$\begin{aligned} C_\rho := R_{\rho^{-1}} \circ L_\rho : \operatorname{Diff}^s_{\tilde\sigma,\tilde\tau}(B) &\to \operatorname{Diff}^s_{\sigma,\tau}(B) \\ \tilde\nu &\mapsto \rho \circ \tilde\nu \circ \rho^{-1} \end{aligned}$$

is a group isomorphism with inverse

$$\begin{aligned} C_\rho^{-1} = C_{\rho^{-1}} = R_\rho \circ L_{\rho^{-1}} : \operatorname{Diff}^s_{\sigma,\tau}(B) &\to \operatorname{Diff}^s_{\tilde\sigma,\tilde\tau}(B) \\ \nu &\mapsto \rho^{-1} \circ \nu \circ \rho. \end{aligned}$$

In particular, $\operatorname{Diff}^s_{\sigma,\tau}(B) \subset \operatorname{Diff}^s(B)$ is a smooth submanifold iff the corresponding diffeomorphism group $\operatorname{Diff}^s_{\tilde\sigma,\tilde\tau}(B) \subset \operatorname{Diff}^s(B)$ is a smooth submanifold. In this case, C_ρ is a smooth diffeomorphism.

(b) *Let* $P_\nu : T_\nu \mathrm{Diff}^s(B) \to T_\nu \mathrm{Diff}^s_{\sigma,\tau}(B)$ *for* $\nu \in \mathrm{Diff}^s_{\sigma,\tau}(B)$ *be the orthogonal projection with respect to the metric induced by* $\langle \cdot , \cdot \rangle^B$*. Then*

$$\begin{aligned} \tilde{P}_{\tilde{\nu}} : T_{\tilde{\nu}} \mathrm{Diff}^s(B) &\to T_{\tilde{\nu}} \mathrm{Diff}^s_{\tilde{\sigma},\tilde{\tau}}(B) \\ v &\mapsto (TC_{\rho^{-1}} \circ P_{C_\rho(\tilde{\nu})} \circ TC_\rho)(v) \end{aligned}$$

is the orthogonal projection with respect to the metric induced by the pullback metric of $\langle \cdot , \cdot \rangle^B$ *under* ρ*. In particular,* P *is a smooth bundle map iff* $\tilde{P}$ *is a smooth bundle map.*

Proof. (a) It only remains to show that this map is well defined. Let $\nu \in \mathrm{Diff}^s_{\tilde{\sigma},\tilde{\tau}}(B)$, i. e. $\tilde{\nu}^* \tilde{\sigma} = \tilde{\sigma}$ and $\tilde{\nu}^* \tilde{\tau} = \tilde{\tau}$. Then

$$\begin{aligned} (\rho \circ \tilde{\nu} \circ \rho^{-1})^* \sigma &= (\rho^{-1})^* \tilde{\nu}^* \rho^* \sigma \\ &= (\rho^{-1})^* \tilde{\nu}^* \tilde{\sigma} \\ &= (\rho^{-1})^* \tilde{\sigma} \\ &= \sigma, \end{aligned}$$

and similarly for τ. The same computation shows that if ν preserves σ and τ, then the preimage $\rho^{-1} \circ \nu \circ \rho$ preserves $\tilde{\sigma}$ and $\tilde{\tau}$.

(b) We first show that the L^2-metric on $T\mathrm{Diff}^s(B)$ induced by the pullback metric on B with respect to ρ is equal to the pullback metric with respect to C_ρ of the L^2-metric on $T\mathrm{Diff}^s(B)$ induced by the chosen metric on B: The pullback of the L^2-metric with respect to C_ρ is given by

$$\begin{aligned} \big(u, v\big)^*_\nu &= \big((C_\rho)_* u, (C_\rho)_* v\big)_{C_\rho(\nu)} \\ &= \int_B \langle (C_\rho)_* u, (C_\rho)_* v \rangle_{C_\rho(\nu)(b)} \, \sigma^n(b) \\ &= \int_B \langle TR_{\rho^{-1}} TL_\rho u, TR_{\rho^{-1}} TL_\rho v \rangle_{\rho(\nu(\rho^{-1}(b)))} \, \sigma^n(b) \\ &= \int_B \langle (TL_\rho u) \circ \rho^{-1}, (TL_\rho v) \circ \rho^{-1} \rangle_{\rho(\nu(\rho^{-1}(b)))} \, \sigma^n(b) \\ &\overset{b=\rho(b')}{=} \int_B \langle TL_\rho u, TL_\rho v \rangle_{\rho(\nu(b'))} \, \underbrace{(\rho^* \sigma^n)}_{\tilde{\sigma}^n}(b') \\ &= \int_B \langle \rho_* u, \rho_* v \rangle_{\rho(\nu(b'))} \, \tilde{\sigma}^n(b') \\ &= \int_B \langle u, v \rangle^*_{\nu(b')} \, \tilde{\sigma}^n(b'), \end{aligned}$$

which is the L^2-metric induced by the pullback metric with respect to ρ. $\tilde{P}_\nu$ is a projection if P_ν is a projection since

$$\begin{aligned}
\tilde{P}_\nu^2 &= (TC_{\rho^{-1}} \circ P_{C_\rho(\nu)} \circ TC_\rho)^2 \\
&= TC_{\rho^{-1}} \circ P_{C_\rho(\nu)} \circ TC_\rho \circ TC_{\rho^{-1}} \circ P_{C_\rho(\nu)} \circ TC_\rho \\
&= TC_{\rho^{-1}} \circ P^2_{C_\rho(\nu)} \circ TC_\rho \\
&= TC_{\rho^{-1}} \circ P_{C_\rho(\nu)} \circ TC_\rho \\
&= \tilde{P}_\nu .
\end{aligned}$$

It remains to check that $\tilde{P}_\nu$ is the orthogonal projection. By definition, P_ν satisfies

$$\Big(u - P_\nu(u), v\Big)_\nu = 0 \tag{3.7}$$

for any $u \in T_\nu \mathrm{Diff}^s(B)$ and $v \in T_\nu \mathrm{Diff}^s_{\sigma,\tau}(B)$, where

$$\Big(u - P_\nu(u), v\Big)_\nu = \int_B \langle u - P_\nu(u), v \rangle_{\nu(b)}\, \sigma^n(b).$$

We have to show that $\tilde{P}_\nu$ satisfies the same equation for the pull back metric. To that end, let $\tilde{u} \in T_{\tilde{\nu}} \mathrm{Diff}^s(B)$ and $\tilde{v} \in T_{\tilde{\nu}} \mathrm{Diff}^s_{\tilde{\sigma},\tilde{\tau}}(B)$, then

$$\begin{aligned}
\Big(\tilde{u} - \tilde{P}(\tilde{u}), \tilde{v}\Big)^*_{\tilde{\nu}} &= \Big(TC_\rho \tilde{u} - TC_\rho \tilde{P}(\tilde{u}), TC_\rho \tilde{v}\Big)_{C_\rho(\tilde{\nu})} \\
&= \Big(\underbrace{TC_\rho \tilde{u}}_{=: u \in T_{C_\rho(\nu)} \mathrm{Diff}^s(B)} - \underbrace{TC_\rho TC_{\rho^{-1}}}_{= \mathrm{id}} P_{C_\rho(\tilde{\nu})} \underbrace{TC_\rho(\tilde{u})}_{= u},\ \underbrace{TC_\rho \tilde{v}}_{=: v \in T_{C_\rho(\nu)} \mathrm{Diff}^s_{\sigma,\tau}(B)}\Big)_{C_\rho(\tilde{\nu})} \\
&= \Big(u - P_{C_\rho(\tilde{\nu})}(u), v\Big)_{C_\rho(\tilde{\nu})} \\
&\overset{\nu := C_\rho(\tilde{\nu})}{=} \Big(u - P_\nu(u), v\Big)_\nu \\
&\overset{(3.7)}{=} 0.
\end{aligned}$$

□

Recall the diffeomorphisms of B that have a lift to $\mathrm{Diff}^s_{\omega,\lambda}(M)$ and $\mathrm{Diff}^s_{\tilde{\omega},\tilde{\lambda}}(M)$, resp., given by

$$\mathcal{D}^s_{\sigma,\mu} = \Big\{\nu \in \mathrm{Diff}^s_{\sigma,\tau = \mathrm{d}\mu}(B) \;\Big|\; \int_\gamma (\mu - \nu^*\mu) \in \mathbb{Z} \text{ for any } \gamma \in H_1(B;\mathbb{Z})\Big\}$$

and

$$\mathcal{D}^s_{\tilde{\sigma},\tilde{\mu}} = \Big\{\nu \in \mathrm{Diff}^s_{\tilde{\sigma},\tilde{\tau} = \mathrm{d}\tilde{\mu}}(B) \;\Big|\; \int_\gamma (\tilde{\mu} - \nu^*\tilde{\mu}) \in \mathbb{Z} \text{ for any } \gamma \in H_1(B;\mathbb{Z})\Big\}.$$

Corollary 3.31. *If we further assume that* $\mathrm{im}(C_\rho|_{\mathcal{D}^s_{\tilde{\sigma},\tilde{\mu}}}) = \mathcal{D}^s_{\sigma,\mu}$, *i. e.* C_ρ *induces a group isomorphism*

$$C_\rho|_{\mathcal{D}^s_{\tilde{\sigma},\tilde{\mu}}} : \mathcal{D}^s_{\tilde{\sigma},\tilde{\mu}} \xrightarrow{\cong} \mathcal{D}^s_{\sigma,\mu},$$

then the previous lemma is still true if we replace $\mathrm{Diff}^s_{\tilde\sigma,\tilde\tau}(B)$ *by* $\mathcal{D}^s_{\tilde\sigma,\tilde\mu}$ *and* $\mathrm{Diff}^s_{\sigma,\tau}(B)$ *by* $\mathcal{D}^s_{\sigma,\mu}$*, respectively.* □

We further have to choose a Riemannian metric on M such that the induced Riemannian volume form is given by $\mathrm{vol} = \lambda \wedge \omega^n = \mathrm{d}\theta \wedge \omega^n$. To that end, we denote by $\langle\cdot,\cdot\rangle^B$ some given metric on B with area form σ^n. On the horizontal bundle, i. e. for $v, w \in \ker \lambda_x \subset T_xM$, we use the isomorphism $\pi_* : \ker\lambda \to TB$ and pull the metric back to

$$\langle v, w\rangle_x := \langle \pi_* v, \pi_* w\rangle^B_{\pi(x)}.$$

Its complement, the horizontal bundle, is generated by $R = \partial_\theta$. We let R have length 1 and be perpendicular to the vertical bundle.

Proposition 3.32. *Let* $(\tilde\omega, \tilde\lambda)$ *be another such stable Hamiltonian structure on* $M = B\times S^1$ *and assume that we have a bundle diffeomorphism* $\rho : B\times S^1 \to B \times S^1$*, i. e.* ρ *satisfies* $\rho_* R = R$*. We further assume that* $\rho^*\omega = \tilde\omega$ *and* $\rho^*\lambda = \tilde\lambda$*. Then:*

(a) The map

$$\begin{aligned} C_\rho := R_{\rho^{-1}} \circ L_\rho : \mathrm{Diff}^s_{\tilde\omega,\tilde\lambda}(M) &\to \mathrm{Diff}^s_{\omega,\lambda}(M) \\ \eta &\mapsto \rho\circ\eta\circ\rho^{-1} \end{aligned}$$

is a group isomorphism. In particular, $\mathrm{Diff}^s_{\tilde\omega,\tilde\lambda}(M) \subset \mathrm{Diff}^s(M)$ *is a smooth submanifold iff* $\mathrm{Diff}^s_{\omega,\lambda}(M) \subset \mathrm{Diff}^s(M)$ *is a smooth submanifold. In this case,* C_ρ *is a smooth diffeomorphism.*

(b) The pullback metric $\langle\cdot,\cdot\rangle^*$ *of* $\langle\cdot,\cdot\rangle$ *under* ρ *is of the same form as* $\langle\cdot,\cdot\rangle$*, i. e.* ∂_θ *has length 1,* ∂_θ *is perpendicular to* $\ker\tilde\lambda$ *and on* $\ker\tilde\lambda$*, the metric is the pull back of some metric on B via the projection* π_**.*

(c) Let $P_\eta : T_\eta\mathrm{Diff}^s(M) \to T_\eta\mathrm{Diff}^s_{\omega,\lambda}(M)$ *for* $\eta \in \mathrm{Diff}^s_{\omega,\lambda}(M)$ *be the orthogonal projection with respect to the metric induced by* $\langle\cdot,\cdot\rangle$ *on M. Then*

$$\begin{aligned} \tilde P_{\tilde\eta} : T_\eta \mathrm{Diff}^s(M)|_{\mathrm{Diff}^s_{\tilde\omega,\tilde\lambda}(M)} &\to T_\eta\mathrm{Diff}^s_{\tilde\omega,\tilde\lambda}(M) \\ \tilde v &\mapsto (TC_{\rho^{-1}} \circ P_{C_\rho(\eta)} \circ TC_\rho)(\tilde v) \end{aligned}$$

is the orthogonal projection with respect to the metric induced by the pullback metric of $\langle\cdot,\cdot\rangle$ *under* ρ*. In particular, P is a smooth bundle map iff* $\tilde P$ *is a smooth bundle map.*

Proof. (a) It only remains to show that this map is well defined. Let $\eta \in \mathrm{Diff}^s_{\tilde\omega,\tilde\lambda}(M)$, i. e. $\eta^*\tilde\omega = \tilde\omega$ and $\eta^*\tilde\lambda = \tilde\lambda$. Then

$$\begin{aligned} (\rho\circ\eta\circ\rho^{-1})^*\omega &= (\rho^{-1})^*\eta^*\rho^*\omega \\ &= (\rho^{-1})^*\eta^*\tilde\omega \\ &= (\rho^{-1})^*\tilde\omega \\ &= \omega \end{aligned}$$

and similarly for λ. The same computation shows that if η preserves ω and λ, then the preimage $\rho^{-1} \circ \eta \circ \rho$ preserves $\tilde{\omega}$ and $\tilde{\lambda}$.

(b) We compute

$$\begin{aligned} \langle \partial_\theta, \partial_\theta \rangle^* &= \langle \rho_* \partial_\theta, \rho_* \partial_\theta \rangle \\ &= \langle \partial_\theta, \partial_\theta \rangle \\ &= 1. \end{aligned}$$

Now let $v \in \ker \tilde{\lambda}$. Then $\rho_* v \in \ker \lambda$ since

$$\lambda(\rho_* v) = (\rho^* \lambda)(v) = \tilde{\lambda}(v) = 0,$$

and we have

$$\begin{aligned} \langle \partial_\theta, v \rangle^* &= \langle \rho_* \partial_\theta, \rho_* v \rangle \\ &= \langle \partial_\theta, \underbrace{\rho_* v}_{\in \ker \lambda} \rangle \\ &= 0. \end{aligned}$$

Finally, for $v, w \in \ker \tilde{\lambda}$ and $x = (b, \theta) \in M$,

$$\begin{aligned} \langle v, w \rangle_x^* &= \langle \rho_* v, \rho_* w \rangle_{\rho(x)} \\ &= \langle \pi_* \rho_* v, \pi_* \rho_* w \rangle^B_{\pi(\rho(x))} \\ &= \langle \rho^B_* \pi_* v, \rho^B_* \pi_* w \rangle^B_{\rho^B(b)}. \end{aligned}$$

In particular, the metric on $\ker \tilde{\lambda}$ is the pullback (via π_*) of the pullback of the chosen metric $\langle \cdot, \cdot \rangle^B$ on B via ρ^B.

(c) As in the proof of Lemma 3.30, we first show that he L^2-metric on $T\mathrm{Diff}^s(M)$ induced by the pullback metric on M with respect to ρ is equal to the pullback metric with respect to C_ρ of the L^2-metric on $T\mathrm{Diff}^s(M)$ induced by the chosen metric on M: The pullback of the L^2-metric is given by

$$\begin{aligned} \big(u, v\big)^*_\eta &= \big((C_\rho)_* u, (C_\rho)_* v\big)_{C_\rho(\eta)} \\ &= \int_M \langle (C_\rho)_* u, (C_\rho)_* v \rangle_{C_\rho(\eta)(x)} \, \lambda \wedge \omega^n(x) \\ &= \int_M \langle TR_{\rho^{-1}} TL_\rho u, TR_{\rho^{-1}} TL_\rho v \rangle_{\rho(\eta(\rho^{-1}(x)))} \, \lambda \wedge \omega^n(x) \\ &= \int_M \langle (TL_\rho u) \circ \rho^{-1}, (TL_\rho v) \circ \rho^{-1} \rangle_{\rho(\eta(\rho^{-1}(x)))} \, \lambda \wedge \omega^n(x) \\ &\overset{x=\rho(x')}{=} \int_M \langle TL_\rho u, TL_\rho v \rangle_{\rho(\eta(x'))} \, (\rho^*(\lambda \wedge \omega^n))(x') \\ &= \int_M \langle TL_\rho u, TL_\rho v \rangle_{\rho(\eta(x'))} \, \tilde{\lambda} \wedge \tilde{\omega}^n(x') \end{aligned}$$

$$
\begin{aligned}
&= \int_M \langle \rho_* u, \rho_* v\rangle_{\rho(\eta(x'))}\, \tilde{\lambda} \wedge \tilde{\omega}^n(x') \\
&= \int_M \langle u, v\rangle^*_{\eta(x')}\, \tilde{\lambda} \wedge \tilde{\omega}^n(x'),
\end{aligned}
$$

which is the L^2-metric induced by the pullback metric. $\tilde{P}_\eta$ is a projection if P_η is a projection since

$$
\begin{aligned}
\tilde{P}_\eta^2 &= (TC_{\rho^{-1}} \circ P_{C_\rho(\eta)} \circ TC_\rho)^2 \\
&= TC_{\rho^{-1}} \circ P_{C_\rho(\eta)} \circ TC_\rho \circ TC_{\rho^{-1}} \circ P_{C_\rho(\eta)} \circ TC_\rho \\
&= TC_{\rho^{-1}} \circ P^2_{C_\rho(\eta)} \circ TC_\rho \\
&= TC_{\rho^{-1}} \circ P_{C_\rho(\eta)} \circ TC_\rho \\
&= \tilde{P}_\eta.
\end{aligned}
$$

It remains to check that $\tilde{P}_\eta$ is the orthogonal projection. By definition, P_η satisfies

$$
\big(u - P_\eta(u), v\big)_\eta = 0 \tag{3.8}
$$

for any $u \in T_\eta \mathrm{Diff}^s(M)$ and $v \in T_\eta \mathrm{Diff}^s_{\omega,\lambda}(M)$, where

$$
\big(u - P_\eta(u), v\big)_\eta = \int_M \langle u - P_\eta(u), v\rangle_{\eta(x)}\, \lambda \wedge \omega^n(x).
$$

We have to show that $\tilde{P}_\eta$ satisfies the same equation for the pullback metric. To that end, let $\tilde{u} \in T_{\tilde{\eta}} \mathrm{Diff}^s(M)$ and $\tilde{v} \in T_{\tilde{\eta}} \mathrm{Diff}^s_{\tilde{\omega},\tilde{\lambda}}(M)$, then

$$
\begin{aligned}
\big(\tilde{u} - \tilde{P}(\tilde{u}), \tilde{v}\big)^*_{\tilde{\eta}} &= \big(TC_\rho \tilde{u} - TC_\rho \tilde{P}(\tilde{u}), TC_\rho \tilde{v}\big)_{C_\rho(\tilde{\eta})} \\
&= \big(\underbrace{TC_\rho \tilde{u}}_{=:u \in T_{C_\rho(\eta)} \mathrm{Diff}^s(M)} - \underbrace{TC_\rho TC_{\rho^{-1}}}_{=\mathrm{id}} P_{C_\rho(\tilde{\eta})} \underbrace{TC_\rho(\tilde{u})}_{=u},\ \underbrace{TC_\rho \tilde{v}}_{=:v \in T_{C_\rho(\eta)} \mathrm{Diff}^s_{\omega,\lambda}(M)}\big)_{C_\rho(\tilde{\eta})} \\
&= \big(u - P_{C_\rho(\tilde{\eta})}(u), v\big)_{C_\rho(\tilde{\eta})} \\
&\overset{\eta := C_\rho(\tilde{\eta})}{=} \big(u - P_\eta(u), v\big)_\eta \\
&\overset{(3.8)}{=} 0.
\end{aligned}
$$

□

Corollary 3.33. *Let $(\omega, \lambda = \mathrm{d}\theta + \pi^*\mu)$ and $(\tilde{\omega}, \tilde{\lambda} = \mathrm{d}\theta + \pi^*\tilde{\mu})$ be two SHS on $M = B \times S^1$. They define two-forms $(\sigma, \tau = \mathrm{d}\mu)$ and $(\tilde{\sigma}, \tilde{\tau} = \mathrm{d}\tilde{\mu})$ on B, resp. Let further $\rho \in \mathrm{Diff}(B)$ as in Lemma 3.30, i. e. $\rho^*\sigma = \tilde{\sigma}$ and $\rho^*\tau = \tilde{\tau}$, and assume that*

$$
\int_\gamma (\tilde{\mu} - \rho^*\mu) \in \mathbb{Z} \text{ for any } \gamma \in H_1(B;\mathbb{Z}).
$$

Then there is a lift $\rho^M \in \mathrm{Diff}^s(M)$ satisfying the conditions of Proposition 3.32.

Proof. Since $\int_\gamma (\tilde{\mu} - \rho^*\mu) \in \mathbb{Z}$ for any $\gamma \in H_1(B;\mathbb{Z})$, the map

$$\begin{aligned} k_\rho : B &\to \mathbb{R} \\ b &\mapsto \int_{b_0}^{b} (\tilde{\mu} - \rho^*\mu) \end{aligned}$$

is well defined. Then the lift

$$\begin{aligned} \rho^M : M &\to M \\ (b,\theta) &\mapsto \big(\rho(b), \theta + k_\rho(b) \mod 1\big) \end{aligned}$$

satisfies the conditions of Proposition 3.32: We have

$$\begin{aligned} (\rho^M)_*\partial_\theta &= \frac{\partial(\theta + k(b))}{\partial\theta}\partial_\theta = \partial_\theta, \\ (\rho^M)^*\omega &= (\rho^M)^*\pi^*\sigma \\ &= \pi^*\rho^*\sigma \\ &= \pi^*\tilde{\sigma} \\ &= \tilde{\omega}, \end{aligned}$$

and

$$\begin{aligned} (\rho^M)^*\lambda &= (\rho^M)^*(\mathrm{d}\theta + \pi^*\mu) \\ &= \mathrm{d}(\rho^M)^*\theta + \pi^*\rho^*\mu \\ &= \mathrm{d}(\theta + \pi^*k) + \pi^*\rho^*\mu \\ &= \mathrm{d}\theta + \pi^*\underbrace{(\mathrm{d}k + \rho^*\mu)}_{=\tilde{\mu}} \\ &= \mathrm{d}\theta + \pi^*\tilde{\mu} \\ &= \tilde{\lambda}. \end{aligned}$$

□

3.7 General circle bundles

In this section, let M be a manifold with SHS (ω, λ) such that the flow ϕ_θ, $\theta \in S^1$, of the Reeb vector field R induces a free S^1-action and $M \xrightarrow{\pi} B$ is the corresponding principal S^1-bundle. Let further $\nu \in \mathrm{Diff}^s_{\sigma,\tau}(B)$ be an H^s-diffeomorphism of the base manifold B which, in particular, also preserves the curvature form τ. We first assume that ν has at least one S^1-equivariant lift $\tilde{\eta}_\nu : M \to M$. As before, Lemma 3.13 shows that $\tilde{\eta}_\nu$ as a lift of $\nu \in \mathrm{Diff}^s_{\sigma,\tau}(B)$ already preserves ω and $\mathrm{d}\lambda$, i.e. it is actually an element of $\mathrm{Diff}^s_{\omega,\mathrm{d}\lambda}(M)$.

Lemma 3.34. *Since λ is a connection form on $\pi : M \to B$ and $\tilde{\eta}_\nu$ preserves R, the pullback $\tilde{\eta}_\nu^*\lambda$ is again a connection form on $\pi : M \to B$.*

Proof. We compute

$$\begin{aligned}
&\mathcal{L}_R(\tilde{\eta}_\nu^*\lambda) = \mathcal{L}_{\tilde{\eta}_\nu^* R}(\lambda) = \mathcal{L}_R\lambda = 0,\\
&(\tilde{\eta}_\nu^*\lambda)(R) = \lambda\big((\tilde{\eta}_\nu)_* R\big) = \lambda(R) = 1.
\end{aligned}$$

Hence, $\tilde{\eta}_\nu^*\lambda$ satisfies the conditions given in the beginning of Section 3.3. □

By Corollary 3.5, there is a unique $\tilde{\rho}_\nu \in H^{s-1}(\Lambda^1 B)$ such that

$$\tilde{\eta}_\nu^*\lambda = \lambda + \pi^*\tilde{\rho}_\nu. \tag{3.9}$$

Remark. If $\tilde{\eta}_\nu$ is at least C^2, then the form $\tilde{\rho}$ is closed since

$$\begin{aligned}
\pi^*\mathrm{d}\tilde{\rho} = \mathrm{d}\pi^*\tilde{\rho} &= \mathrm{d}(\tilde{\eta}_\nu^*\lambda - \lambda) = \tilde{\eta}_\nu^*\mathrm{d}\lambda - \mathrm{d}\lambda\\
&= \tilde{\eta}_\nu^*(\pi^*\tau) - \pi^*\tau = \pi^*(\nu^*\tau - \tau) = \pi^*0 = 0
\end{aligned}$$

and π^* is injective. A computation similarly to the one for the trivial bundle in Corollary 3.20 shows that $\tilde{\rho}$ always defines a cohomology class in $H^1_{\mathrm{dR}}(B)$.

Now consider an H^s-map $k : B \to S^1$. Any such map induces a lift $\tilde{\eta}_{\nu,k} \in \mathrm{Diff}^s(M)$ of ν by setting

$$\tilde{\eta}_{\nu,k}(x) = \underbrace{k\big(\pi(x)\big)}_{\in S^1}.\tilde{\eta}_\nu(x) = \phi_{k(\pi(x))}\big(\tilde{\eta}_\nu(x)\big), \tag{3.10}$$

where ϕ denotes the flow of the Reeb vector field R. This defines an action of $H^s(B,S^1)$ on $\mathrm{Diff}^s(M)$. To show that $\tilde{\eta}_{\nu,k}$ still preserves R, which implies that $\tilde{\eta}_{\nu,k}$ is also a bundle diffeomorphism, it suffices to show that $\tilde{\eta}_{\nu,k} \circ \phi_\theta = \phi_\theta \circ \tilde{\eta}_{\nu,k}$. To that end, we compute

$$\begin{aligned}
(\phi_\theta \circ \tilde{\eta}_{\nu,k})(x) &= \phi_\theta\big(\tilde{\eta}_{\nu,k}(x)\big) = \phi_\theta\big(\phi_{k(\pi(x))}(\tilde{\eta}_\nu(x))\big)\\
&= \phi_{\theta+k(\pi(x))}(\tilde{\eta}_\nu(x))\\
&= \phi_{k(\pi(x))}(\phi_\theta(\tilde{\eta}_\nu(x)))\\
&= \phi_{k(\pi(x))}\tilde{\eta}_\nu(\phi_\theta(x)) \quad \text{since } \tilde{\eta}_\nu \text{ preserves } R\\
&= \tilde{\eta}_{\nu,k}(\phi_\theta(x)).
\end{aligned}$$

Hence, $\tilde{\eta}_{\nu,k}$ is an H^s-diffeomorphism of principal S^1-bundles and $H^s(B,S^1)$ acts on $\mathrm{Diff}^s_R(M)$. Furthermore, since $\tilde{\eta}_{\nu,k}$ also satisfies

$$\begin{aligned}
\pi\big(\tilde{\eta}_{\nu,k}(x)\big) &= \pi\big(\phi_{k(\pi(x))}(\tilde{\eta}_\nu(x))\big)\\
&= \pi\big(\tilde{\eta}_\nu(x)\big)\\
&= \nu(\pi(x))
\end{aligned}$$

for every $x \in M$, it is still a lift of $\nu \in \mathrm{Diff}^s_{\sigma,\tau}(B)$. Hence, also $\tilde{\eta}_{\nu,k} \in \mathrm{Diff}^s_{\omega,\mathrm{d}\lambda}(M)$. We now identify a condition such that $\tilde{\eta}_{\nu,k}$ preserves λ instead of just $\mathrm{d}\lambda$.

Lemma 3.35. *Let* $\eta \in \mathrm{Diff}^s_{\omega,\mathrm{d}\lambda}(M)$ *be a lift of some* $\nu \in \mathrm{Diff}^s_{\sigma,\tau}(B)$ *and* $k \in H^s(B,S^1)$. *The lift* $\eta_k \in \mathrm{Diff}^s_{\omega,\mathrm{d}\lambda}(M)$ *preserves* λ, *i. e.* η_k *is an element of* $\mathrm{Diff}^s_{\omega,\lambda}(M)$, *iff* $\lambda - \tilde{\eta}^*_\nu \lambda = \pi^*\mathrm{d}k$.

Remark. This condition is very similar to the trivial bundle case as in Proposition 3.22: If α is a form on M that descends to a form on B, we use $\bar{\alpha}$ for the form on B that satisfies $\alpha = \pi^*\bar{\alpha}$. For $\lambda - \eta^*\lambda$, we get

$$\begin{aligned}
&\lambda - \eta^*\lambda = \pi^*\mathrm{d}k \quad \text{for some } k \in H^s(B,S^1)\\
\Leftrightarrow\quad &\overline{\lambda - \eta^*\lambda} = \mathrm{d}k \quad \text{as forms on } B \text{ for some } k \in H^s(B,S^1)\\
\Leftrightarrow &\int_\gamma \overline{\lambda - \eta^*\lambda} \in \mathbb{Z} \quad \forall \gamma \in H_1(B;\mathbb{Z}).
\end{aligned}$$

Proof. We let $v \in T_xM$ and compute

$$\begin{aligned}
(\mathrm{d}_x\eta_k)\cdot v &= \mathrm{d}_x(\phi_{k(\pi(x))}(\eta(x)))\cdot v\\
&= (\mathrm{d}_{\eta(x)}\phi_{k(\pi(x))})\cdot(\mathrm{d}_x\eta)\cdot v + R_{\eta_k(x)}\cdot d_x(k\circ\pi)\cdot v\\
&= (\mathrm{d}_{\eta(x)}\phi_{k(\pi(x))})\cdot(\mathrm{d}_x\eta)\cdot v + R_{\eta_k(x)}\cdot(\pi^*\mathrm{d}_{\pi(x)}k)\cdot v.
\end{aligned}$$

Applying λ to this expression yields

$$\begin{aligned}
\lambda_{\eta_k(x)}((\mathrm{d}_x\eta_k)\cdot v) &= \lambda_{\eta_k(x)}((\mathrm{d}_{\eta(x)}\phi_{k(\pi(x))})\cdot(\mathrm{d}_x\eta)\cdot v)\\
&\qquad + \underbrace{\lambda_{\eta_k(x)}(R_{\eta_k(x)})}_{=1}\cdot(\pi^*\mathrm{d}_{\pi(x)}k)\cdot v\\
&= (\eta^*\phi^*_{k(\pi(x))}\lambda)_x(v) + (\pi^*\mathrm{d}k)_x\cdot v.
\end{aligned}$$

Since $\mathcal{L}_R\lambda = 0$ implies $\phi^*_\theta\lambda = \lambda$ for any $\theta \in S^1$, we know that

$$\phi^*_{k(\pi(x))}\lambda = \lambda$$

for any $x \in M$ and, in particular,

$$\phi^*_{k(\pi(x))}\lambda_x = \lambda_{\phi_{k(\pi(x))}(x)},$$

hence

$$(\eta_k^*\lambda)(v) = (\eta^*\lambda)(v) + (\pi^*\mathrm{d}k)\cdot v \tag{3.11}$$

or, equivalently,

$$\eta_k^*\lambda = \eta^*\lambda + \pi^*\mathrm{d}k.$$

Therefore, $\eta_k^*\lambda = \lambda$ iff $\lambda - \eta^*\lambda = \pi^*\mathrm{d}k$. ☐

Now let $\nu \in \mathrm{Diff}^s_{\sigma,\tau}(B)$ and suppose that $\tilde{\eta}_\nu \in \mathrm{Diff}^s_{\omega,\mathrm{d}\lambda}(B)$ is an S^1-equivariant lift of ν. By Lemma 3.34 and Corollary 3.5, there exists a unique one-form $\tilde{\rho}_\nu \in H^{s-1}(\Lambda^1 B)$ such that $\lambda - \tilde{\eta}_\nu^* \lambda = \pi^* \tilde{\rho}_\nu$ as in (3.9). Similarly to the trivial bundle case, we define

$$\mathcal{D}^s := \Big\{ \nu \in \mathrm{Diff}^s_{\sigma,\tau}(B) \;\Big|\; \nu \text{ has at least one } S^1\text{-equivariant lift } \tilde{\eta}_\nu \in \mathrm{Diff}^s(M) \text{ and } \int_\gamma \tilde{\rho}_\nu \in \mathbb{Z} \text{ for any } \gamma \in H_1(B;\mathbb{Z}) \Big\}. \tag{3.12}$$

can use Lemma 3.35 to identify the diffeomorphisms of B that have a lift to $\mathrm{Diff}^s_{\omega,\lambda}(M)$ as

Conversely, we can also show that any other lift of ν is of the form given by Eq. (3.10):

Lemma 3.36. *For any lift*

$$\begin{tikzcd} M \arrow[r, "\eta'"] \arrow[d, "\pi"'] & M \arrow[d, "\pi"] \\ B \arrow[r, "\nu"] & B \end{tikzcd}$$

of $\nu \in \mathrm{Diff}^s_{\sigma,\tau}(B)$ as maps of principal S^1-bundles, there is an H^s-map $k : B \to S^1$ such that $\eta' = \tilde{\eta}_{\nu,k}$.

Proof. For any $x \in M$, $\tilde{\eta}_\nu(x)$ and $\eta'(x)$ lie in the same fibre of M over B. Hence, we can define a (possibly not H^s) map $k : B \to S^1$ such that $\eta' = \tilde{\eta}_{\nu,k}$ and it remains to check that k is H^s. To that end, for any point $b \in B$, choose an open set $b \in U \subset B$ such that for $V := \pi^{-1}(U)$, we have a trivial bundle $\pi|_V : V \to U$. We also have a local section $s : U \to V$ of $\pi|_V : V \to U$ and can define an H^s-map $\theta : U \to S^1$ for any $c \in U$ by the equation

$$V \ni s(c) = (c, \theta(c)) \in U \times S^1.$$

Further, for any $c \in U$, we have

$$(\tilde{\eta}_\nu^{-1} \circ \eta')(s(c)) = \phi_{k(c)}(s(c)) = (c, \theta(c) + k(c)) \in U \times S^1.$$

Since the left hand side is in $H^s(U, U \times S^1)$, the right hand side is aswell, and in particular, $k|_U$ is an element of $H^s(U, S^1)$. Hence, $k \in H^s(B, S^1)$. ☐

There are conditions under which we can guarantee the existence of an S^1-equivariant lift $\tilde{\eta}_\nu$ of $\nu \in \mathrm{Diff}^s_{\sigma,\tau}(B)$: Consider the pullback bundle

$$\nu^* M = \Big\{ (b,x) \in B \times M \;\Big|\; \nu(b) = \pi(x) \Big\} \subset B \times M,$$

with projections p_1 and p_2 onto the first and second component, respectively, which is defined such that

$$\begin{tikzcd} \nu^* M \arrow[r, "p_2"] \arrow[d, "\pi' := p_1"'] & M \arrow[d, "\pi"] \\ B \arrow[r, "\nu"'] & B \end{tikzcd}$$

commutes. This construction yields a principal S^1-bundle $\nu^*M \xrightarrow{\pi'} B$ such that p_2 is S^1-equivariant. Note that by [Hat17, Prop. 3.10], the first Chern class in $H^2_{\mathrm{sing}}(B;\mathbb{Z})$ determines circle bundles over a given base manifold up to continuous isomorphisms. While $\nu^*M \to B$ and $M \to B$ have the same curvature form $\nu^*\tau = \tau \in H^2_{\mathrm{dR}}(B)$, their first Chern classes might differ.[1] To determine the connection between the first Chern class and the curvature form, let $T_i \subset H_i(B;\mathbb{Z})$ denote the corresponding torsion subgroups of the singular homology groups and β_2 the second Betti number of B, so that

$$H_2(B;\mathbb{Z}) \cong \mathbb{Z}^{\beta_2} \oplus T_2 \qquad \text{and} \qquad H^2_{\mathrm{sing}}(B;\mathbb{Z}) \cong \mathbb{Z}^{\beta_2} \oplus T_1, \tag{3.13}$$

then

$$\begin{aligned}
H^2_{\mathrm{dR}}(B) &\cong H^2_{\mathrm{sing}}(B;\mathbb{R}) && \text{by de Rham's Theorem} \\
&\cong \mathrm{Hom}_{\mathbb{R}}\big(H_2(B;\mathbb{R}),\mathbb{R}\big) && \text{by the Universal Coefficient Theorem} \\
& && \text{as given on page 198 of [Hat02]} \\
&\cong H_2(B;\mathbb{R}) && \text{since } H_2(B;\mathbb{R}) \text{ is finite dimensional} \\
&\cong \big(H_2(B;\mathbb{Z})\otimes\mathbb{R}\big)\oplus \mathrm{Tor}\big(H_1(B;\mathbb{Z}),\mathbb{R}\big) && \\
& && \text{by the Universal Coefficient Theorem} \\
& && \text{for homology [Hat02, Theorem 3A.3]} \\
&\cong H_2(B;\mathbb{Z})\otimes\mathbb{R} && \text{since } \mathbb{R} \text{ is flat and hence, Tor vanishes} \\
&\cong \big(H_2(B;\mathbb{Z})/T_2\big)\otimes\mathbb{R} && \text{by (3.13)} \\
&\cong \big(H^2_{\mathrm{sing}}(B;\mathbb{Z})/T_1\big)\otimes\mathbb{R} && \text{also by (3.13).}
\end{aligned}$$

Hence, the curvature form determines the non-torsion component of the Chern class. In particular, we also get the following lemma:

Lemma 3.37. *The curvature form of a principal S^1-bundle uniquely determines the Chern class iff $T_1 = 0$, i. e. iff $H^2_{sing}(B;\mathbb{Z})$ has no torsion elements.* ☐

Recall that we have $\nu \in \mathrm{Diff}^s_{\sigma,\tau}(B)$, so that $\nu^*\tau = \tau$ implies that $\nu^*M \to B$ and $M \to B$ have the same curvature form. If we assume that $H^2_{\mathrm{sing}}(B;\mathbb{Z})$ has no torsion elements, this uniquely determines the bundle and by [Hat17, Prop. 3.10], there is a continuous isomorphism $\tilde{F}_\nu : M \to \nu^*M$

$$\begin{tikzcd}
M \arrow[rr, "\tilde{F}_\nu"] \arrow[dr, "\pi"'] & & \nu^*M \arrow[dl, "\pi'"] \\
& B &
\end{tikzcd}$$

---

[1] Many thanks go to Thorsten Hertl for questioning my use of Hatcher's Prop. 3.10 for just the curvature form with the following counterexample: The two bundles $S^1 \to S^1 \times \mathbb{R}P^2 \to \mathbb{R}P^2$ and $S^1 \to g \oplus g \to \mathbb{R}P^2$ for the Whitney sum of the tautological bundle g both admit a flat connection, but are not isomorphic, because they have different Chern classes.

of principal S^1-bundles. We can smoothen $\tilde{F}_\nu$ to get a smooth bundle diffeomorphism $F_\nu : M \to \nu^*M$. In particular, F_ν is also S^1-equivariant.

Lemma 3.38. *If $H^2_{sing}(B;\mathbb{Z})$ has no torsion elements, then $\tilde{\eta}_\nu := p_2 \circ F_\nu : M \to M$ is well defined. The map $\tilde{\eta}_\nu$ is an S^1-equivariant diffeomorphism which is a lift of ν and satisfies $(\tilde{\eta}_\nu)_* R = R$.*

Proof. Since

$$\begin{tikzcd} M \arrow[r, "F_\nu"] \arrow[d, "\pi"'] & \nu^*M \arrow[r, "p_2"] \arrow[dl, "\pi'"] & M \arrow[d, "\pi"] \\ B \arrow[rr, "\nu"'] & & B \end{tikzcd}$$

commutes and both p_2 and F_ν are S^1-equivariant, i. e. they commute with the flow ϕ_θ of R, we compute

$$\begin{aligned}(\tilde{\eta}_\nu)_*(R_x) &= (\tilde{\eta}_\nu)_*\Big(\frac{\mathrm{d}}{\mathrm{d}\theta}\Big|_{\theta=0}\phi_\theta(x)\Big) = \frac{\mathrm{d}}{\mathrm{d}\theta}\Big|_{\theta=0}\tilde{\eta}_\nu(\phi_\theta(x)) \\ &= \frac{\mathrm{d}}{\mathrm{d}\theta}\Big|_{\theta=0}\phi_\theta(\tilde{\eta}_\nu(x)) = R_{\eta(x)}.\end{aligned}$$

□

Corollary 3.39. *If $H^2_{sing}(B;\mathbb{Z})$ has no torsion elements, then any $\nu \in \mathrm{Diff}^s_{\sigma,\tau}(B)$ has some lift $\tilde{\eta}_\nu \in \mathrm{Diff}^s_{\lambda,\omega}(M)$ as constructed in Lemma 3.38. In this case, Eq.* (3.12) *simplifies to*

$$\mathcal{D}^s = \Big\{\nu \in \mathrm{Diff}^s_{\sigma,\tau}(B) \;\Big|\; \int_\gamma \tilde{\rho}_\nu \in \mathbb{Z} \text{ for any } \gamma \in H_1(B;\mathbb{Z})\Big\}.$$

□

Now we get back to discussing the structure of $\mathrm{Diff}^s_{\omega,\lambda}(M)$. Our goal is to show that $\mathrm{Diff}^s_{\omega,\lambda}(M)$ is an S^1-bundle over $\mathcal{D}^s$. First, recall the projection $q : \mathrm{Diff}^s_R(M) \to \mathrm{Diff}^s(B)$ defined in Lemma 3.12.

Lemma 3.40. *The action of $H^s(B,S^1)$ on $\mathrm{Diff}^s_R(M)$ given by Eq.* (3.10) *is free and transitive on each fibre $q^{-1}(\{\nu\})$ for any $\nu \in \mathcal{D}^s(B)$.*

Proof. The action is free: Let $\nu \in \mathcal{D}^s$ and $\eta \in q^{-1}(\{\nu\})$. Let further $k \in H^s(B,S^1)$ and we assume $\eta_k = \eta$, i. e. $\phi_{k(\pi(x))}\big(\eta(x)\big) = \eta(x)$ for any $x \in M$. Locally, for any $b \in B$, choose an open set $U \subset B$ such that $b \in U$ and for $V := \pi^{-1}(U) \subset M$, the restriction $\pi|_V : V \to U$ is a local trivialization. Any $x \in V$ can be written as $(b,\theta) \in U \times S^1$ and η is of the form $\eta(x) = \eta(b,\theta) = \big(\eta^1(b,\theta), \eta^2(b,\theta)\big) = \big(\nu(b), \eta^2(b,\theta)\big)$. Then we have

$$\begin{aligned}\big(\nu(b), \eta^2(b,\theta)\big) &= \eta(b,\theta) = \eta(x) \\ &\overset{!}{=} \phi_{k(\pi(x))}\big(\eta(x)\big) \\ &= \phi_{k(b)}\big(\nu(b), \eta^2(b,\theta)\big) \\ &= \big(\nu(b), \eta^2(b,\theta) + k(b)\big),\end{aligned}$$

i. e. $k(b) = 0$.

The action is transitive: Let $\nu \in \mathcal{D}^s$ and $\eta, \eta' \in q^{-1}(\{\nu\})$. Recall that by the definition of $\mathcal{D}^s$, we can also fix a lift $\tilde{\eta}_\nu \in \mathrm{Diff}^s_{\omega,\mathrm{d}\lambda}(M)$. By Lemma 3.36, there exist $k, k' \in H^s(B, S^1)$ such that $\eta = (\tilde{\eta}_\nu)_k$ and $\eta' = (\tilde{\eta}_\nu)_{k'}$. Hence, $\tilde{\eta}_\nu = \eta_{-k}$ and

$$\eta' = (\tilde{\eta}_\nu)_{k'} = (\eta_{-k})_{k'} = \eta_{-k+k'}$$

with $-k + k' \in H^s(B, S^1)$. □

As in the trivial bundle case, we define the restriction

$$p := q|_{\mathrm{Diff}^s_{\omega,\lambda}(M)} : \mathrm{Diff}^s_{\omega,\lambda}(M) \to \mathcal{D}^s \tag{3.14}$$

and we show that the fibre over each $\nu \in \mathcal{D}^s$ is isomorphic to S^1: Every $\theta_0 \in S^1$ induces the constant map $k \in H^s(B, S^1)$, $k(b) \equiv \theta_0$ and for every $\eta \in \mathrm{Diff}^s_{\omega,\lambda}(M)$, we also have $\eta_k \in \mathrm{Diff}^s_{\omega,\lambda}(M)$. Conversely, the following lemma shows that any two lifts in $\mathrm{Diff}^s_{\sigma,\lambda}(M)$ of some fixed $\nu \in \mathcal{D}^s$ only differ by a constant map.

Lemma 3.41. *Let $\nu \in \mathcal{D}^s$ and $\eta, \eta' \in p^{-1}(\{\nu\}) \subset \mathrm{Diff}^s_{\omega,\lambda}(M)$. Then there is a constant $\theta_0 \in S^1$ such that $\eta' = \eta_k$ for $k \in H^s(B, S^1)$ with $k(b) \equiv \theta_0$.*

Proof. By Lemma 3.36, there is a H^s-map $k : B \to S^1$ such that $\eta' = \eta_k$. By Eq. (3.11), we have

$$\begin{aligned}
\lambda &= (\eta')^*\lambda = \eta_k^*\lambda \\
&\overset{(3.11)}{=} \eta^*\lambda + \pi^*\mathrm{d}k \\
&= \lambda + \pi^*\mathrm{d}k,
\end{aligned}$$

and get $\mathrm{d}k = 0$. Hence, k is constant. □

As a special case of Lemma 3.40, we get

Corollary 3.42. *The action of S^1 on $\mathrm{Diff}^s_{\omega,\lambda}(M)$, defined by the constant action in* (3.10), *is free and transitive on each fibre $p^{-1}(\{\nu\})$ for any $\nu \in \mathcal{D}^s$.* □

Now, we can finally describe $\mathrm{Diff}^s_{\omega,\lambda}(M)$ as an S^1-bundle over $\mathcal{D}^s$.

Theorem 3.43. *Assume that $\mathcal{D}^s$ is a smooth submanifold of $\mathrm{Diff}^s(B)$. Then there is a smooth principal bundle*

$$S^1 \to \mathrm{Diff}^s_{\omega,\lambda}(M) \xrightarrow{p} \mathcal{D}^s,$$

where the first map is the action of the constant map $k \in H^s(B, S^1)$, $k(b) \equiv \theta_0$ for $\theta_0 \in S^1$ on $\mathrm{Diff}^s_{\omega,\lambda}(M)$ as described in Eq. (3.10), *and the second map is the projection p defined in Eq.* (3.14).

In particular, $\mathrm{Diff}^s_{\omega,\lambda}(M) \subset \mathrm{Diff}^s(M)$ is a smooth submanifold.

Proof. By Corollary 3.42, it only remains to show that for every $\nu_0 \in \mathcal{D}^s$, there is a neighbourhood $\nu_0 \in \mathcal{U} \subset \mathcal{D}^s$ such that there is a smooth section $s : \mathcal{U} \to \mathrm{Diff}^s_{\omega,\lambda}$. Hence, for every $\nu_0 \in \mathcal{D}^s$, let $\mathcal{U} \subset \mathcal{D}^s$ be a sufficiently small, contractible neighbourhood of ν_0 in $\mathcal{D}^s$ and for every $\nu \in \mathcal{U}$, we now want to construct $s(\nu) \in \mathrm{Diff}^s_{\omega,\lambda}(M)$ such that $s(\nu) \in p^{-1}(\{\nu\})$. By Lemma 3.38, there is a smooth bundle diffeomorphism

$$\begin{tikzcd} M \arrow[rr, "F_0"] \arrow[dr, "\pi"'] & & \nu_0^* M. \arrow[dl, "\pi_0 := \pi'"] \\ & B & \end{tikzcd}$$

Define a new bundle $S^1 \to E \overset{\mathrm{pr}}{\to} \mathcal{U} \times B$ by $E_{(\nu,b)} = \nu^* M|_b = M_{\nu(b)}$ for $(\nu, b) \in \mathcal{U} \times B$. Since $\mathcal{U} \subset \mathcal{D}^s$, this bundle also has an infinite-dimensional base space.

Step 1. The bundle $\mathrm{pr} : E \to \mathcal{U} \times B$ is diffeomorphic to the pullback bundle $(\mathrm{id}_{\mathcal{U}}, \pi_0) : \mathcal{U} \times \nu_0^* M \to \mathcal{U} \times B$.

Proof of Step 1. Since $\mathcal{U}$ is contractible, $\mathcal{U} \times B$ is homotopy equivalent to $\{\nu_0\} \times B$. Let $f_t : \mathcal{U} \times B \to \mathcal{U} \times B$ be a homotopy from $f_0 : \mathcal{U} \times B \to \mathcal{U} \times B$, $(\nu, b) \mapsto (\nu_0, b)$ to $f_1 = \mathrm{id}_{\mathcal{U}\times B}$. Using Theorem 3.44 below for the principal S^1-bundle $E \to \mathcal{U} \times B$ yields a (continuous) isomorphism $\tilde{\Sigma} : f_0^* E \to f_1^* E$ over $\mathcal{U} \times B$, which we can then smoothen to a diffeomorphism Σ such that

$$\begin{tikzcd} f_0^* E \arrow[rr, "\Sigma", "\cong"'] \arrow[dr] & & f_1^* E \arrow[dl] \\ & \mathcal{U} \times B & \end{tikzcd} \tag{3.15}$$

commutes. Since $f_1 = \mathrm{id}_{\mathcal{U}\times B}$, the bundle $f_1^* E \to \mathcal{U} \times B$ is just the original bundle $\mathrm{pr} : E \to \mathcal{U} \times B$. For $f_0^* E$, we recall the definition of the pullback bundle

$$f_0^* E = \Big\{ (\nu, b, e) \in \mathcal{U} \times B \times E \;\Big|\; f_0(\nu, b) = \mathrm{pr}(e) \Big\}$$

Since $\mathrm{pr}(e) \overset{!}{=} f_0(\nu, b) = (\nu_0, b)$ is equivalent to $e \in E_{(\nu_0,b)} = \nu_0^* M|_b$, the bundle $f_0^* E \to \mathcal{U} \times B$ is given by $(\mathrm{id}_{\mathcal{U}}, \pi_0) : \mathcal{U} \times \nu_0^* M \to \mathcal{U} \times B$. Hence, the diffeomorphism Σ in Eq. (3.15) is between

$$\begin{tikzcd} f_0^* E = \mathcal{U} \times \nu_0^* M \arrow[rr, "\Sigma"] \arrow[dr, "(\mathrm{id}_{\mathcal{U}}, \pi_0)"'] & & E = f_1^* E. \arrow[dl, "\mathrm{pr}"] \\ & \mathcal{U} \times B & \end{tikzcd}$$

Step 2. There is a smooth map $\tilde{s} : \mathcal{U} \to \mathrm{Diff}^s_{\omega,\mathrm{d}\lambda}(M)$ such that $\tilde{s}(\nu)$ is a lift of $\nu \in \mathcal{U}$.

Proof of Step 2. First define the bundle diffeomorphism $S := \Sigma\circ(\mathrm{id}_{\mathcal{U}}, F_0) : \mathcal{U}\times M \to E$, so that

$$\begin{tikzcd}
\mathcal{U}\times M \arrow[r, "(\mathrm{id}_{\mathcal{U}},F_0)", "\cong"'] \arrow[rr, bend left, "S"] \arrow[dr, "(\mathrm{id}_{\mathcal{U}},\pi)"'] & \mathcal{U}\times \nu_0^*M \arrow[r, "\Sigma", "\cong"'] \arrow[d, "(\mathrm{id}_{\mathcal{U}},\pi_0)"'] & E \arrow[dl, "\mathrm{pr}"] \\
& \mathcal{U}\times B &
\end{tikzcd}$$

commutes. For every $(\nu, x) \in \mathcal{U}\times M$, we have

$$S(\nu,x) \in E|_{(\mathrm{id}_{\mathcal{U}},\pi)(\nu,x)} = E|_{(\nu,\pi(x))} = \nu^*M|_{\pi(x)} = M|_{\nu(\pi(x))}.$$

Therefore, the diffeomorphism $S(\nu,\cdot)M \to M$ fits into the commuting diagram

$$\begin{tikzcd}
M \arrow[r, "S(\nu{,}\cdot)"] \arrow[d, "\pi"'] & M, \arrow[d, "\pi"] \\
B \arrow[r, "\nu"'] & B
\end{tikzcd}$$

i. e. $S(\nu,\cdot)$ is a lift of $\nu \in \mathcal{D}^s$. In particular, $S(\nu,\cdot)$ automatically preserves ω and $\mathrm{d}\lambda$ and we can define

$$\begin{aligned}
\tilde{s} : \mathcal{D}^s &\to \mathrm{Diff}^s_{\omega,\mathrm{d}\lambda}(M)\\
\nu &\mapsto S(\nu,\cdot).
\end{aligned}$$

Step 3. There is a smooth map $k : \mathcal{U} \to H^s(B,S^1)$, $\nu \mapsto k_\nu$ such that the shifted diffeomorphism $\tilde{s}(\nu)_{k_\nu}$ preserves λ, i. e. $\tilde{s}(\nu)_{k_\nu} \in \mathrm{Diff}^s_{\omega,\lambda}(M)$.

Proof of Step 3. Since $\tilde{s}(\nu)$ is a lift of $\nu \in \mathcal{D}^s$, there is $\tilde{k}_\nu \in H^s(B,S^1)$ such that $\tilde{s}(\nu)_{\tilde{k}_\nu} \in \mathrm{Diff}^s_{\omega,\lambda}(M)$. The map $\tilde{k}_\nu$ is unique up to constants in S^1, so we want to normalize this choice: Fix $b_0 \in B$ and $0 \in S^1$ (independent of ν_0). Then define $k_\nu \in H^s(B,S^1)$ by

$$k_\nu(b) := \tilde{k}_\nu(b) - \tilde{k}_\nu(b_0), \tag{3.16}$$

so that $k_\nu(b_0) = 0$.

For this step, it remains to show that $k : \mathcal{U} \to H^s(B,S^1)$, $\nu \mapsto k_\nu$ is smooth. To that end, define $\rho_\nu \in H^{s-1}(\Lambda^1 B)$ by $\pi^*\rho_\nu = \lambda - \tilde{s}(\nu)^*\lambda$. Since $\nu \mapsto \tilde{s}(\nu)$ is smooth, also $\nu \mapsto \rho_\nu$ is smooth. The map $k_\nu \in H^s(B,S^1)$ as defined in Eq. (3.16) is the unique primitive of ρ_ν (i. e. we have $\rho_\nu = \mathrm{d}k_\nu$) satisfying $k_\nu(b_0) = 0$ and we want to prove that $\rho_\nu \mapsto k_\nu$ is smooth. To that end, fix k_{ν_0} such that $\mathrm{d}k_{\nu_0} = \rho_{\nu_0}$ and $k_{\nu_0}(b_0) = 0$, define the Hilbert spaces

$$\mathcal{K} := \left\{ l \in H^s(B,\mathbb{R}) \,\middle|\, l(b_0) = 0 \right\},$$
$$\mathcal{A} := \left\{ \alpha \in H^{s-1}(\Lambda^1 B) \,\middle|\, \int_\gamma \alpha = 0 \text{ for any } \gamma \in H_1(B,\mathbb{Z}) \right\},$$

and let

$$\begin{tikzcd}
\mathcal{K} \arrow[r] \arrow[rr, bend left, "f"] & H^s(B,S^1) \arrow[r] & \mathcal{A} \\
l \arrow[r, mapsto] & k_{\nu_0} + l \arrow[r, mapsto] & f(l) := \mathrm{d}(k_{\nu_0} + l)
\end{tikzcd}$$

Then f is a continuous linear operator that is also bijective:

For surjectivity, let $\alpha \in \mathcal{A}$, i. e. $\alpha \in H^{s-1}(\Lambda^1 B)$ such that $\displaystyle\int_\gamma \alpha = 0$ for any $\gamma \in H_1(B;\mathbb{Z})$. Then there is a unique map $a \in H^s(B,S^1)$ such that $\alpha = \mathrm{d}a$ and $a(b_0) = 0$. Since $T_{k_{\nu_0}} H^s(B,S^1) = H^s(B,\mathbb{R})$, we can find a function $l \in H^s(B,\mathbb{R})$ such that $a = k_{\nu_0} + l$, and can compute

$$l(b_0) = a(b_0) - k_{\nu_0}(b_0) = 0 - 0 = 0,$$

i. e. $l \in \mathcal{K}$.

For injectivity, let $l_1, l_2 \in \mathcal{K}$ such that $f(l_1) = f(l_2)$, i. e. $\mathrm{d}(k_{\nu_0} + l_1) = \mathrm{d}(k_{\nu_0} + l_2)$. This implies $\mathrm{d}l_1 = \mathrm{d}l_2$ and therefore, l_1 is equal to l_2 up to some constant in $\mathbb{R}$. Since $l_1(b_0) = 0 = l_2(b_0)$, this constant has to be 0 and we get $l_1 = l_2$.

Now we can apply the Open Mapping Theorem (see Theorem 3.45 below), which yields that the inverse operator

$$f^{-1} : \mathcal{A} \to \mathcal{K}$$

is continuous linear, and therefore smooth. Since $\rho_\nu \in \mathcal{A}$ and $f^{-1}(\rho_\nu) = k_\nu$, this implies that $k : \mathcal{U} \to H^s(B,S^1)$, $\nu \mapsto k_\nu$ is smooth.

Step 4. For every $\nu_0 \in \mathcal{D}^s$ with (contractible) neighbourhood $\nu_0 \in \mathcal{U} \subset \mathcal{D}^s$, there is a smooth section $s : \mathcal{U} \to \mathrm{Diff}^s_{\omega,\lambda}(M)$ of the bundle $p : \mathrm{Diff}^s_{\omega,\lambda}(M) \to \mathcal{D}^s$.

Proof of Step 4. Define

$$\begin{aligned} s : \mathcal{U} &\to \mathrm{Diff}^s_{\omega,\lambda}(M) \\ \nu &\mapsto \tilde{s}(\nu)_{k_\nu}. \end{aligned}$$

Since both $\tilde{s} : \mathcal{U} \mapsto \mathrm{Diff}^s_{\omega,\mathrm{d}\lambda}(M)$ and $k : \mathcal{U} \to H^s(B,S^1)$ are smooth, s is also smooth.

This completes the proof. □

In the previous proof, we have used the two following theorems:

Definition ([Hus94], Definitions 9.1 and 9.2). (a) An open covering $\{U_i\}_{i\in I}$ of a topological space $\mathcal{B}$ is *numerable* provided there exists a (locally finite) partition of unity $\{u_i\}_{i\in I}$ such that $\overline{u_i^{-1}((0,1])} \subset U_i$ for each $i \in I$.

(b) A principal G-bundle $\xi : \mathcal{X} \to \mathcal{B}$ is numerable provided there is a numerable cover $\{U_i\}_{i\in I}$ of $\mathcal{B}$ such that $\xi|_{U_i}$ is trivial for each $i \in I$.

In particular, a locally trivial principal G-bundle over a paracompact space is numerable.

Theorem 3.44 ([Hus94], Theorem 9.9). *Let G be a group and $\xi : \mathcal{X} \to \mathcal{B}$ a numerable principal G-bundle over $\mathcal{B}$. Let $f_t : \mathcal{B}' \to \mathcal{B}$ be a homotopy. Then the principal G-bundles $f_0^*\mathcal{X} \to \mathcal{B}'$ and $f_1^*\mathcal{X} \to \mathcal{B}'$ are isomorphic over $\mathcal{B}'$.*

Theorem 3.45 (Open Mapping Theorem, see e. g. [Wer11], Theorem IV.3.3 and Korollar IV.3.4). *Let X and Y be Banach spaces and assume that $f : X \to Y$ a bijective continuous linear operator. Then the inverse $f^{-1} : Y \to X$ is also continuous.*

Note that if $M = B \times S^1$ is a trivial bundle with stable Hamiltonian structure $(\omega, \lambda = \mathrm{d}\theta + \pi^*\mu)$, we can add a constant $\theta_0 \in S^1$ to k_{ν_0} (depending on the choice of F_0 and the base point b_0) in the proof of Theorem 3.43 such that the lift of ν_0 coincides with the lift of ν_0 in the proof of Lemma 3.23. By adding this constant θ_0 to any other k_ν (i. e. we normalize to $k_\nu(b_0) = \theta_0$), the two sections $\mathcal{U} \to \mathrm{Diff}^s_{\omega,\lambda}(B \times S^1)$ in the proofs of Lemma 3.23 and Theorem 3.43 coincide. In particular, Lemma 3.27 also follows from Theorem 3.43.

4

S^1-BUNDLES OVER THE CYLINDER $B = S^1 \times [-1,1]$

In this chapter, we discuss the principal circle bundle $S^1 \to M \xrightarrow{\pi} S^1 \times [-1,1]$ over the cylinder $B := S^1 \times [-1,1]$ in detail. Since $H^2(B) = \{0\}$, M is a trivial circle bundle, i. e. $M \cong B \times S^1$. Denote by $\theta \in \mathbb{R}/\mathbb{Z} \cong S^1$ the S^1-bundle coordinate in the trivial bundle $M = B \times S^1 = (S^1 \times [-1,1]) \times S^1$ and let $(\varphi, z) \in \mathbb{R}/\mathbb{Z} \times [-1,1]$ denote the coordinates on the cylinder $B = S^1 \times [-1,1]$. In Sections 4.1 and 4.2, which only deal with the cylinder itself, we let $\langle .,. \rangle$ be the standard metric in which $(\partial_\varphi, \partial_z)$ is an orthonormal basis. The corresponding Riemannian area form is $\sigma := \mathrm{d}\varphi \wedge \mathrm{d}z$. We further let $h : B \to \mathbb{R}$, $(\varphi, z) \mapsto z$ and define smooth forms on B by

$$\mu := -\frac{z^2}{2}\mathrm{d}\varphi, \quad \tau := \mathrm{d}\mu = z\,\mathrm{d}\varphi \wedge \mathrm{d}z = h(\varphi, z)\sigma.$$

In Sections 4.3 and 4.4, we consider the stable Hamiltonian structure on M given by

$$\omega := \pi^*\sigma, \quad \lambda := \mathrm{d}\theta + \pi^*\mu.$$

This notation matches the one in the previous chapters. In particular, we have

$$\mathrm{d}\lambda = \mathrm{d}(\mathrm{d}\theta + \pi^*\mu) = \pi^*\mathrm{d}\mu = \pi^*\tau,$$

as before.

We will show for both (B, σ, τ) and (M, ω, λ) that the structure-preserving diffeomorphisms are smooth submanifolds of the full diffeomorphism groups and that the projections of the tangent bundles induced by the Riemannian metrics on B and M, resp., are smooth bundle maps. We will also explicitly compute all solutions to the Euler equation using variational principles as in Section 2.3, which only yields trivial solutions in those cases.

In Sections 4.5 and 4.6, we generalize this to an arbitrary metric on the cylinder B. We will show that we can reduce this case to a Riemannian area form given by $\sigma_a := a(z)\sigma$ for some smooth function $a \in C^\infty([-1,1], \mathbb{R})$. We use $\tau_a := h\sigma_a$ with primitive

$$\mu_a := -m_a(z)\,\mathrm{d}\varphi \quad \text{for} \quad m_a(z) = \int_{-1}^{z} \zeta a(\zeta)\,\mathrm{d}\zeta.$$

That is, we have

$$\begin{aligned} \mathrm{d}\mu_a &= \mathrm{d}(-m_a \, \mathrm{d}\varphi) \\ &= -\frac{\partial m_a}{\partial z} \, \mathrm{d}z \wedge \mathrm{d}\varphi \\ &= za(z) \, \mathrm{d}\varphi \wedge \mathrm{d}z \\ &= \tau_a. \end{aligned}$$

Note that this choice for μ_a differs from the standard metric, where we start integrating at 0 instead of -1. The stable Hamiltonian structure on the bundle $B \times S^1$ is then

$$\omega_a := \pi^* \sigma_a \quad \text{and} \quad \lambda_a = \mathrm{d}\theta + \pi^* \mu_a.$$

In Section 4.10, we also generalize the standard situation to

$$\omega := \pi^* \sigma, \quad \tilde{\lambda} := \mathrm{d}\theta + \pi^* \tilde{\mu}$$

for some $\tilde{\mu} \in \Omega^1(B)$ such that $\tilde{\tau} := \mathrm{d}\tilde{\mu} = \tilde{h}\sigma$ for any smooth submersion $\tilde{h} : B \to [-1,1]$ which maps $S^1 \times \{\pm 1\}$ to ± 1, respectively.

4.1 $B = S^1 \times [-1,1]$, standard metric

Our goal in this section is to show:

Theorem 4.1. *(a)* $\mathrm{Diff}^s_{\sigma,\tau}(S^1 \times [-1,1]) = \mathrm{Diff}^s_{\sigma,h}(S^1 \times [-1,1])$ *is a smooth Hilbert submanifold of* $\mathrm{Diff}^s(S^1 \times [-1,1])$.

(b) The orthogonal projection

$$P : T\mathrm{Diff}^s(B)|_{\mathrm{Diff}^s_{\sigma,\tau}} \to T\mathrm{Diff}^s_{\sigma,\tau}(B)$$

induced by the standard metric on B is a smooth bundle map.

In the first subsection, we will prove Theorem 4.1(a). In Section 4.1.2, we compute local bundle trivializations for $T\mathrm{Diff}^s(S^1 \times [-1,1])$ following the steps in Section 2.1. To verify Theorem 4.1(b), we will compute the orthogonal projection of the tangent bundle

$$P : T\mathrm{Diff}^s(S^1 \times [-1,1])|_{\mathrm{Diff}^s_{\sigma,h}(S^1 \times [-1,1])} \to T\mathrm{Diff}^s_{\sigma,h}(S^1 \times [-1,1])$$

using the local bundle trivializations of Section 4.1.2.

4.1.1 Smooth submanifold $\mathrm{Diff}^s_{\sigma,\tau}(B) \subset \mathrm{Diff}^s(B)$

First note that by definition, $\mathrm{Diff}^s(S^1 \times [-1,1])$ only consists of the connected component containing the identity map. Since any diffeomorphism of $S^1 \times [-1,1]$ preserves its boundary $S^1 \times \{\pm 1\}$, any element of $\mathrm{Diff}^s(S^1 \times [-1,1])$ preserves both $S^1 \times \{1\}$ and $S^1 \times \{-1\}$.

Furthermore, the boundary ∂B is totally geodesic in $B = S^1 \times [-1,1]$. This implies that $\mathrm{Diff}^s(S^1 \times [-1,1])$ is a smooth manifold with an exponential function as described in Section 2.4.

We now start with $\mathrm{Diff}^s_{\sigma,h}(S^1 \times [-1,1])$ and want to show that it is a smooth submanifold of $\mathrm{Diff}^s(S^1 \times [-1,1])$. A first idea might be to use that the volume-preserving diffeomorphisms $\mathrm{Diff}^s_\sigma(S^1 \times [-1,1]) \subset \mathrm{Diff}^s(S^1 \times [-1,1])$ are a smooth submanifold (see Theorem 2.8) so that we only have to show that $\mathrm{Diff}^s_{\sigma,h}(S^1 \times [-1,1]) \subset \mathrm{Diff}^s_\sigma(S^1 \times [-1,1])$ is also a smooth submanifold, e. g. by using the implicit function theorem for Hilbert manifolds.

Unfortunately, this approach does not work. If we define

$$\begin{aligned} F : \mathrm{Diff}^s_\sigma(S^1 \times [-1,1]) &\to H^s(S^1 \times [-1,1], \mathbb{R}) \\ v = (v^1, v^2) &\mapsto v^* h = v^2 \end{aligned}$$

to get $\mathrm{Diff}^s_{\sigma,h}(S^1 \times [-1,1]) = F^{-1}(h) = F^{-1}(z)$, then the tangent space at $\mathrm{id} \in F^{-1}(z)$ is given by

$$\begin{aligned} T_{\mathrm{id}}\mathrm{Diff}^s_\sigma(S^1 \times [-1,1]) &= \left\{ v \in \mathfrak{X}^s(S^1 \times [-1,1]) \,\middle|\, \mathrm{div}_\sigma v = 0 \right\} \\ &= \left\{ v = v^1 \partial_\varphi + v^2 \partial_z \in \mathfrak{X}^s(S^1 \times [-1,1]) \,\middle|\, \frac{\partial v^1}{\partial \varphi} + \frac{\partial v^2}{\partial z} = 0 \right\} \end{aligned}$$

and the tangent map by

$$\begin{aligned} T_{\mathrm{id}}F : T_{\mathrm{id}}\mathrm{Diff}^s_\sigma(S^1 \times [-1,1]) &\to T_z H^s(S^1 \times [-1,1], \mathbb{R}) = H^s(S^1 \times [-1,1], \mathbb{R}) \\ v = v^1 \partial_\varphi + v^2 \partial_z &\mapsto v^2. \end{aligned}$$

We would now have to show that $T_{\mathrm{id}}F$ is surjective. To that end, let $g \in H^s(S^1 \times [-1,1], \mathbb{R})$ and we need to find $f \in H^s(S^1 \times [-1,1], \mathbb{R})$ such that $v := f\partial_\varphi + g\partial_z$ satisfies $\mathrm{div}_\sigma v = 0$, i. e. that $\frac{\partial f}{\partial \varphi} + \frac{\partial g}{\partial z} = 0$. This implies that f has to be of the form

$$f_c(\varphi, z) = -\int_0^\varphi \frac{\partial g}{\partial z}(\psi, z)\,\mathrm{d}\psi + c(z).$$

Since we cannot control $\frac{\partial^{s+1} g}{\partial z^{s+1}}$, we cannot guarantee the existence of a function $c(z) : [-1,1] \to \mathbb{R}$ such hat $f_c(\varphi, z)$ is of Sobolev class s. This implies that for any such map f, the vector field $f\partial_\varphi + g\partial_z$ is generally *not* an element of $T_{\mathrm{id}}\mathrm{Diff}^s_\sigma(S^1 \times [-1,1])$ and hence, $T_{\mathrm{id}}F$ is not necessarily surjective.

Changing the function F for the implicit function theorem runs into the same problem: If we copy the proof for Theorem 2.8 and define

$$\begin{aligned} F : \mathrm{Diff}^s_\sigma(S^1 \times [-1,1]) &\to z\sigma + H^s(S^1 \times [-1,1], \mathbb{R})\sigma \\ \nu = (\nu^1, \nu^2) &\mapsto \nu^*(\tau) = \nu^*(z\sigma) = \nu^2\sigma, \end{aligned}$$

then $\mathrm{Diff}^s_{\sigma,\tau}(S^1 \times [-1,1]) = F^{-1}(\tau)$. The map F is well defined, i. e. the image of F is really contained in $z\sigma + H^s(B,\mathbb{R})\sigma$ because any map $\nu^2(\varphi, z)$ can be written as $z + (\nu^2 - z)$ with $\nu^2 - z \in H^s(B,\mathbb{R})$. At the identity, the tangent map is given by

$$\begin{aligned} T_{\mathrm{id}}F : T_{\mathrm{id}}\mathrm{Diff}^s_\sigma(S^1 \times [-1,1]) &\to H^s(S^1 \times [-1,1], \mathbb{R})\sigma \\ v = v^1\partial_\varphi + v^2\partial_z &\mapsto \mathcal{L}_v(\tau), \end{aligned}$$

for any v satifying $\mathrm{div}_\sigma(v) = 0$. Computing this map yields

$$\begin{aligned} \mathcal{L}_v(\tau) &= \mathcal{L}_v(z\sigma) \\ &= (\mathcal{L}_v z)\sigma + z\mathcal{L}_v\sigma \\ &= (\iota_v \mathrm{d}z)\sigma + z\,\mathrm{div}(v)\sigma \\ &= v^2\sigma. \end{aligned}$$

To show that $T_{\mathrm{id}}F$ is surjective, we let $g\sigma \in H^s(S^1 \times [-1,1], \mathbb{R})\sigma$. Again, finding $f \in H^s(S^1 \times [-1,1], \mathbb{R})$ such that $v := f\partial_\varphi + g\partial_z$ satisfies $\mathrm{div}_\sigma(v) = 0$ has the exact same problem as in the previous approach.

Instead, we will show that $\mathrm{Diff}^s_{\sigma,h}(S^1 \times [-1,1]) \subset \mathrm{Diff}^s(S^1 \times [-1,1])$ is a smooth submanifold by using the implicit function theorem for the inclusion

$$\mathrm{Diff}^s_h(S^1 \times [-1,1]) \subset \mathrm{Diff}^s(S^1 \times [-1,1])$$

and then explicitly compute a local description of $\mathrm{Diff}^s_{\sigma,h}(S^1 \times [-1,1])$ in $\mathrm{Diff}^s_h(S^1 \times [-1,1])$.

Proposition 4.2.

$$\mathrm{Diff}^s_h(S^1 \times [-1,1]) \subset \mathrm{Diff}^s(S^1 \times [-1,1])$$

is a smooth submanifold.

Proof. We let $\mathrm{Subm}(S^1 \times [-1,1], \mathbb{R})$ denote the C^1-submersions, define the H^s-submersions as

$$\mathrm{Subm}^s(S^1 \times [-1,1], \mathbb{R}) := \mathrm{Subm}(S^1 \times [-1,1], \mathbb{R}) \cap H^s(S^1 \times [-1,1], \mathbb{R}),$$

and let

$$\mathcal{F} := \{f \in \mathrm{Subm}^s(S^1 \times [-1,1], \mathbb{R}) \mid f|_{S^1 \times \{\pm 1\}} = \pm 1\}$$

to be the set of H^s-submersions $f : S^1 \times [-1,1] \to \mathbb{R}$ such that $f|_{S^1\times\{\pm 1\}} = \pm 1$. Because we only consider submersions, any such f satisfies $\operatorname{im}(f) = [-1,1]$ and so,

$$\mathcal{F} = \{f \in \operatorname{Subm}^s(S^1 \times [-1,1],[-1,1]) \mid f|_{S^1\times\{\pm 1\}} = \pm 1\}.$$

We want to use the implicit function theorem for

$$\begin{aligned} F : \operatorname{Diff}^s(S^1 \times [-1,1]) &\to \mathcal{F} \subset H^s(S^1 \times [-1,1],\mathbb{R}) \\ \nu = (\nu^1(\varphi,z),\nu^2(\varphi,z)) &\mapsto \nu^* h = h \circ \nu = \nu^2(\varphi,z). \end{aligned} \tag{4.1}$$

Hence, we first have to show that $\mathcal{F}$ is a smooth submanifold of $H^s(S^1 \times [-1,1],\mathbb{R})$: Since $s > \frac{1}{2}\dim(M) + 1 > \frac{1}{2}\dim(B) + 1$, i. e. any element of $H^s(S^1 \times [-1,1],\mathbb{R})$ is also differentiable, this is an open subset of

$$\mathcal{A} := \left\{f \in H^s(S^1 \times [-1,1],\mathbb{R}) \,\middle|\, f|_{S^1\times\{\pm 1\}} = \pm 1\right\}.$$

We further define

$$\mathcal{B} := \left\{g \in H^s(S^1 \times [-1,1],\mathbb{R}) \,\middle|\, g|_{S^1\times\{\pm 1\}} = 0\right\},$$

which is a closed subspace of the Hilbert space $H^s(S^1 \times [-1,1],\mathbb{R})$. In particular, $\mathcal{B}$ is also a smooth Hilbert submanifold of the Hilbert manifold $H^s(S^1 \times [-1,1],\mathbb{R})$. For any $f \in \mathcal{A}$, we have $\mathcal{A} = f + \mathcal{B}$. We now fix $f \in \mathcal{A}$. Since

$$\begin{aligned} H^s(S^1 \times [-1,1],\mathbb{R}) = \mathcal{B} \oplus \mathcal{B}^\perp &\to H^s(S^1 \times [-1,1],\mathbb{R}) \\ g + g^\perp &\mapsto f + g + g^\perp \end{aligned}$$

is a diffeomorphism which maps $\mathcal{B} \oplus 0$ onto $\mathcal{A}$, $\mathcal{A}$ is also a smooth submanifold of $H^s(S^1 \times [-1,1],\mathbb{R})$. Hence, $\mathcal{F}$ is a smooth submanifold of $H^s(S^1 \times [-1,1],\mathbb{R})$.

Since $\operatorname{Diff}^s_h(S^1 \times [-1,1]) = F^{-1}(h)$ for Eq. (4.1), it only remains to show that h is a regular value of F, i. e. that all preimages ν of h under F are regular points. To that end, we need to show that for any preimage ν of h under F, $T_\nu F$ is surjective. We first compute

$$\begin{aligned} T_{\mathrm{id}}\operatorname{Diff}^s(S^1 \times [-1,1]) &= \Big\{X = X^1\partial_\varphi + X^2\partial_z \in \mathfrak{X}^s(S^1 \times [-1,1]) \;\Big|\; \\ &\qquad\qquad X \text{ is tangent to } S^1 \times \{\pm 1\}\Big\} \\ &= \Big\{X = X^1\partial_\varphi + X^2\partial_z \in \mathfrak{X}^s(S^1 \times [-1,1]) \;\Big|\; \\ &\qquad\qquad X^2|_{S^1\times\{\pm 1\}} = 0\Big\} \\ &= \Big\{X = (X^1,X^2) \;\Big|\; X^2|_{S^1\times\{\pm 1\}} = 0\Big\}. \end{aligned}$$

Recall that we can describe the tangent spaces of $\mathrm{Diff}^s(S^1 \times [-1,1])$ by the isomorphisms

$$\begin{aligned} T_{\mathrm{id}}\mathrm{Diff}^s(S^1\times[-1,1]) &\to T_\nu \mathrm{Diff}^s(S^1\times[-1,1]) = T_{\mathrm{id}}\mathrm{Diff}^s(S^1\times[-1,1])\circ\nu \\ X &\mapsto X\circ\nu. \end{aligned}$$

Also,

$$T_h\mathcal{F} = \Big\{ g \in H^s(S^1\times[-1,1],\mathbb{R}) \;\Big|\; g|_{S^1\times\{\pm1\}} = 0 \Big\} = \mathcal{B}$$

and

$$\begin{aligned} T_\nu F : T_\nu\mathrm{Diff}^s(S^1\times[-1,1]) &\to T_h\mathcal{F} \\ X\circ\nu = (X^1\partial_\varphi + X^2\partial_z)\circ\nu &\mapsto X^2\circ\nu. \end{aligned}$$

Now let $\nu \in \mathrm{Diff}^s(S^1\times[-1,1])$ be some preimage of h under F. For any $g \in T_h\mathcal{F}$, we can define $X := g(\partial_z\circ\nu) \in T_\nu\mathrm{Diff}^s(S^1\times[-1,1])$. Then $T_\nu F(X) = g$ and $T_\nu F$ is surjective. ☐

Proposition 4.3. $\mathrm{Diff}^s_{\sigma,h}(S^1\times[-1,1]) \subset \mathrm{Diff}^s_h(S^1\times[-1,1])$ *is a smooth submanifold.*

Remark. Again, using the implicit function theorem as in the proof of Theorem 2.8 does not work. Recall the closed affine subspace of $H^{s-1}(\Lambda^n)$,

$$[\sigma]^{s-1} = \sigma + \mathrm{d}H^s(\Lambda^{n-1})$$

from the proof of Theorem 2.8. Let $[\sigma]^{s-1}_h \subset [\sigma]^{s-1}$ denote the subset we can use for the image of

$$\begin{aligned} \psi_h : \mathrm{Diff}^s_h(S^1\times[-1,1]) &\to [\sigma]^{s-1}_h \\ \nu &\mapsto \nu^*\sigma. \end{aligned}$$

We want to show that $\mathrm{Diff}^s_{h,\sigma}(S^1\times[-1,1]) = \psi_h^{-1}(\sigma)$ is a smooth submanifold, i. e. that the tangent map

$$\begin{aligned} T_\nu\psi_h : T_\nu\mathrm{Diff}^s_h(S^1\times[-1,1]) &\to T_{\nu^*\sigma}[\sigma]^{s-1}_h \\ V &\mapsto \nu^*(\mathcal{L}_{V\circ\nu^{-1}}\sigma) \end{aligned}$$

is surjective for any $\nu \in \psi_h^{-1}(\sigma)$. At the identity, any $X \in T_{\mathrm{id}}\mathrm{Diff}^s_h(S^1\times[-1,1])$ can be written as the vector field $X = X^1\partial_\varphi$ and we can compute

$$\begin{aligned} T_{\mathrm{id}}\psi_h(X) &= \mathcal{L}_X\sigma = \mathrm{d}\iota_X\sigma \\ &= \mathrm{d}(\iota_{X^1\partial_\varphi}\mathrm{d}\varphi\wedge\mathrm{d}z) \\ &= \mathrm{d}(X^1\,\mathrm{d}z). \end{aligned}$$

Now let $\mathrm{d}\alpha \in T_\sigma[\sigma]_h^{s-1}$, i. e. $\alpha = f\,\mathrm{d}z + g\,\mathrm{d}\varphi$ for some $f,g \in H^s(S^1 \times [-1,1],\mathbb{R})$. If we chose $[\sigma]_h^{s-1} = [\sigma]^{s-1}$, then we would have to let

$$X^1 = f - \int_{S^1} \frac{\partial g}{\partial z}\mathrm{d}\varphi,$$

which generally is not an H^s-map. If we want to ensure that $T_{\mathrm{id}}\psi_h$ is surjective, we would have to restrict to

$$[\sigma]_h^{s-1} := \sigma + \mathrm{d}\{\alpha \in H^s(\Lambda^1) \mid \alpha = f(\varphi,z)\,\mathrm{d}z\},$$

since then we can let $X^1 = f \in H^s(S^1 \times [-1,1],\mathbb{R})$. Unfortunately, this space is equal to

$$\begin{aligned}
[\sigma]_h^{s-1} &:= \sigma + \mathrm{d}\Big\{\alpha \;\Big|\; \alpha = f(\varphi,z)\,\mathrm{d}z \in H^s(\Lambda^1)\Big\} \\
&= \sigma + \Big\{\mathrm{d}(f(\varphi,z)\,\mathrm{d}z) \;\Big|\; f \in H^s(B,\mathbb{R})\Big\} \\
&= \sigma + \Big\{\frac{\partial f}{\partial \varphi}\mathrm{d}\varphi \wedge \mathrm{d}z \;\Big|\; f \in H^s(B,\mathbb{R})\Big\} \\
&= \sigma + \Big\{\frac{\partial f}{\partial \varphi}\sigma \;\Big|\; f \in H^s(B,\mathbb{R})\Big\} \\
&= \sigma + \Big\{\frac{\partial f}{\partial \varphi} \;\Big|\; f \in H^s(B,\mathbb{R})\Big\}\sigma,
\end{aligned}$$

but $\Big\{\frac{\partial f}{\partial \varphi} \;\Big|\; f \in H^s(B,\mathbb{R})\Big\} \subset H^{s-1}(B,\mathbb{R})$ is not a closed Hilbert space.

Proof of Proposition 4.3. Let $\nu \in \mathrm{Diff}_h^s(S^1 \times [-1,1])$, i. e. ν is of the form $\nu = (\nu^1, z)$. For ν to be an element of $\mathrm{Diff}_{\sigma,h}^s(S^1 \times [-1,1])$, it has to also satisfy $\nu^*\sigma = \sigma$, which is equivalent to

$$\mathrm{d}\varphi \wedge \mathrm{d}z = \sigma \overset{!}{=} \nu^*\sigma = \mathrm{d}\nu^1 \wedge \mathrm{d}z = \frac{\partial \nu^1}{\partial \varphi}\mathrm{d}\varphi \wedge \mathrm{d}z, \tag{4.2}$$

i. e. $\frac{\partial \nu^1}{\partial \varphi} \equiv 1$. Since being a smooth submanifold is a local condition, we first consider a small neighourhood U around the identity $\mathrm{id} \in \mathrm{Diff}_h^s(S^1 \times [-1,1])$. We can uniquely write any $\nu \in U$ as $\nu(\varphi,z) = (\varphi + f(\varphi,z), z)$ for some small $f \in H^s(S^1 \times [-1,1],\mathbb{R})$ and U is isomorphic to some neighbourhood V of 0 in $H^s(S^1 \times [-1,1],\mathbb{R})$. Then,

$$\nu \in U \cap \mathrm{Diff}_{\sigma,h}^s(S^1 \times [-1,1]) \quad \Leftrightarrow \quad \frac{\partial \nu^1}{\partial \varphi} \equiv 1 \quad \Leftrightarrow \quad \frac{\partial f}{\partial \varphi} = 0,$$

i. e. $f \in H^s([-1,1],\mathbb{R})$ only depends on z. Hence, $U \cong \Big\{f \in V \;\Big|\; \frac{\partial f}{\partial \varphi} = 0\Big\}$. Since the space $\Big\{f \in V \;\Big|\; \frac{\partial f}{\partial \varphi} = 0\Big\} = \ker\Big(\frac{\partial}{\partial \varphi}\Big) < V$ is a closed Hilbert subspace, $\mathrm{Diff}_{\sigma,h}^s(S^1 \times [-1,1])$ is a smooth Hilbert submanifold of $\mathrm{Diff}^s_h(S^1 \times [-1,1])$ *close to the identiy* with tangent space

$$\begin{aligned} T_{\mathrm{id}}\mathrm{Diff}^s_{h,\sigma}(B \times S^1) &\cong T_{\mathrm{id}}U \\ &\cong T_0\Big\{f \in H^s(B,\mathbb{R}) \;\Big|\; \frac{\partial f}{\partial \varphi} = 0\Big\} \\ &\cong \Big\{f \in H^s(B,\mathbb{R}) \;\Big|\; \frac{\partial f}{\partial \varphi} = 0\Big\}. \end{aligned}$$

By right translation, the same local situation occurs at any other $\nu \in \mathrm{Diff}^s_{\sigma,h}(S^1 \times [-1,1])$. Therefore

$$\mathrm{Diff}^s_{\sigma,h}(S^1 \times [-1,1]) \subset \mathrm{Diff}^s_h(S^1 \times [-1,1])$$

is a smooth submanifold. ☐

Eq. (4.2) also implies that any $\nu =\in \mathrm{Diff}^s_{\sigma,h}(S^1 \times [-1,1])$ can be written as

$$\nu(\varphi, z) = \big(\varphi + f(z), z\big)$$

for some $f \in H^s(B,\mathbb{R})$.

Corollary 4.4 (=Theorem 4.1(a)). *Propositions 4.2 and 4.3 show that*

$$\mathrm{Diff}^s_{\sigma,h}(S^1 \times [-1,1]) \subset \mathrm{Diff}^s(S^1 \times [-1,1])$$

is a smooth submanifold. ☐

Even though we have not proved that $\mathrm{Diff}^s_{\sigma,h}(S^1 \times [-1,1]) \subset \mathrm{Diff}^s_\sigma(S^1 \times [-1,1])$ is a smooth submanifold, it now follows from Corollary 4.4 and the next lemma.

Lemma 4.5 ([EP13], Lemma 2.1). *Let A and B be smooth Hilbert submanifolds of some smooth Hilbert manifold C. If $A \subset B$ is a subset, then A is a smooth Hilbert submanifold of B.*

Since

$$\begin{tikzcd}
\mathrm{Diff}^s_{\sigma,h}(S^1 \times [-1,1]) \arrow[r, hook, "\substack{\text{smth submfd}\\ \text{(Proposition 4.3)}}"] \arrow[dr, hook, "\text{subset}"'] & \mathrm{Diff}^s_h(S^1 \times [-1,1]) \arrow[r, hook, "\substack{\text{smth submfd}\\ \text{(Proposition 4.2)}}"] & \mathrm{Diff}^s(S^1 \times [-1,1]) \\
& \mathrm{Diff}^s_\sigma(S^1 \times [-1,1]) \arrow[ur, hook, "\substack{\text{smth submfd}\\ \text{(Theorem 2.8)}}"'] &
\end{tikzcd}$$

it follows that also

$$\mathrm{Diff}^s_{\sigma,h}(S^1 \times [-1,1]) \xhookrightarrow[\text{submfd}]{\text{smooth}} \mathrm{Diff}^s_\sigma(S^1 \times [-1,1]).$$

Since we have shown that $\mathrm{Diff}^s_{\sigma,h}(S^1 \times [-1,1]) \subset \mathrm{Diff}^s(S^1 \times [-1,1])$ is a smooth submanifold, we can now continue with the tangent bundle maps. Recall that we have to show that the bundle projection

$$P : T\mathrm{Diff}^s(S^1 \times [-1,1])|_{\mathrm{Diff}^s_{\sigma,h}(S^1 \times [-1,1])} \to T\mathrm{Diff}^s_{\sigma,h}(S^1 \times [-1,1]),$$

which is the orthogonal projection in each tangent space

$$P_\nu : T_\nu \mathrm{Diff}^s(S^1 \times [-1,1]) \to T_\nu \mathrm{Diff}^s_{\sigma,h}(S^1 \times [-1,1]),$$

is smooth in the base point $\nu \in \mathrm{Diff}^s_{\sigma,h}(S^1 \times [-1,1])$. To check smoothness, we will need to compute P in local charts of $T\mathrm{Diff}^s(S^1 \times [-1,1])$.

4.1.2 Charts for $T\mathrm{Diff}^s(B)$ and its submanifolds

Adapting Corollary 2.7 to our situation yields the local bundle trivializations:

$$\begin{aligned}\Phi : T_\nu \mathrm{Diff}^s(S^1 \times [-1,1]) \times T_\nu \mathrm{Diff}^s(S^1 \times [-1,1]) &\to T\mathrm{Diff}^s(S^1 \times [-1,1]) \\ (X,Y) &\mapsto \Big(\exp_\nu X,\ \big(\nabla_2 \exp_{(\nu,X)}\big)(Y)\Big).\end{aligned}$$

Recall that

$$T_\nu \mathrm{Diff}^s(S^1 \times [-1,1]) = T_{\mathrm{id}} \mathrm{Diff}^s(S^1 \times [-1,1]) \circ \nu = \mathfrak{X}^s(S^1 \times [-1,1]) \circ \nu,$$

hence $\partial_\varphi \circ \nu$ and $\partial_z \circ \nu$ generate $T_\nu \mathrm{Diff}^s(S^1 \times [-1,1])$. Write $X = X^1(\partial_\varphi \circ \nu) + X^2(\partial_z \circ \nu)$. Since $(\partial_\varphi, \partial_z)$ is an orthonormal basis, the map $\exp_\nu X$ maps (φ, z) to

$$\begin{aligned}\big(\exp_\nu X\big)(\varphi,z) &= \exp_{\nu(\varphi,z)} X(\varphi,z) \\ &= \nu(\varphi,z) + \big(X^1(\varphi,z),\ X^2(\varphi,z)\big) \\ &= \big(\nu^1(\varphi,z) + X^1(\varphi,z),\ \nu^2(\varphi,z) + X^2(\varphi,z)\big) \\ &=: (\nu + X)(\varphi,z).\end{aligned}$$

We now compute $\nabla_2 \exp_{(\nu(\varphi,z), X(\varphi,z))}$. Let $p := (\varphi,z) \in S^1 \times [-1,1]$ and $x \in T_p(S^1 \times [-1,1])$, i. e. $(p,x) \in T(S^1 \times [-1,1])$. Recall the definition in Eq. (2.2),

$$\begin{aligned}\nabla_2 \exp_{(p,x)} : T_p(S^1 \times [-1,1]) &\to T_{\exp_p(x)}(S^1 \times [-1,1]) \\ \nabla_2 \exp_{(p,x)} &:= (T_x \exp)|_{T^v_{(p,x)} T(S^1 \times [-1,1])} \circ \big(K|_{T^v_{(p,x)} T(S^1 \times [-1,1])}\big)^{-1}.\end{aligned} \tag{4.3}$$

Following [Dom62], let φ, z be the coordinates on $S^1 \times [-1,1]$ and let $\tau : T(S^1 \times [-1,1]) \to S^1 \times [-1,1]$ denote the canonical projection. Then

$$v^1 := \varphi \circ \tau, \quad v^2 := z \circ \tau, \quad v^3 := \mathrm{d}\varphi, \quad v^4 := \mathrm{d}z$$

are coordinates on $T(S^1 \times [-1,1])$ and $\frac{\partial}{\partial v^i}$ for $i = 1,\dots,4$ is a basis of $TT(S^1 \times [-1,1])$. Since

$$T^v_{(p,x)} T(S^1 \times [-1,1]) = \ker(T\tau|_{T_{(p,x)}T(S^1\times[-1,1])}),$$

we let $A = \sum_{i=1}^{4} a^i \frac{\partial}{\partial v^i} \in T_{(p,x)} T(S^1 \times [-1,1])$, $f(\varphi,z) \in C^\infty(S^1 \times [-1,1],\mathbb{R})$ and we compute

$$\begin{aligned}
(T\tau)(A)(f) &= A(f \circ \tau) \\
&= \Big(\sum_{i=1}^{4} a^i \frac{\partial}{\partial v^i}\Big)(f \circ \tau) \\
&= \Big(a^1 \frac{\partial}{\partial v^1} + a^2 \frac{\partial}{\partial v^2}\Big)(f \circ \tau) \\
&\quad \text{since } (f \circ \tau)(v^1,v^2,v^3,v^4) = f(v^1,v^2) \\
&= a^1 \frac{\partial f}{\partial \varphi} \circ \tau \cdot \frac{\partial v^1}{\partial v^1} + a^2 \frac{\partial f}{\partial z} \circ \tau \cdot \frac{\partial v^2}{\partial v^2} \\
&= a^1 \frac{\partial f}{\partial \varphi} \circ \tau + a^2 \frac{\partial f}{\partial z} \circ \tau.
\end{aligned}$$

This yields

$$T^v_{(p,x)} T(S^1 \times [-1,1]) = \ker(T\tau|_{T_{(p,x)}T(S^1\times[-1,1])}) = \operatorname{span}\Big\{\frac{\partial}{\partial v^3}, \frac{\partial}{\partial v^4}\Big\}.$$

To compute the connection map K, first note that since our metric is constant on $S^1 \times [-1,1]$, all Christoffel symbols vanish. Eq. (11) in [Dom62] states for $A = \sum_{i=1}^{4} a^i \frac{\partial}{\partial v^i} \in T_{(p,x)} T(S^1 \times [-1,1])$

$$K_{(p,x)}(A) = a^3 \frac{\partial}{\partial \varphi} + a^4 \frac{\partial}{\partial z}.$$

Restricting $K_{(p,x)}$ to $T^v_{(p,x)} T(S^1 \times [-1,1]) = \operatorname{span}\Big\{\frac{\partial}{\partial v^3}, \frac{\partial}{\partial v^4}\Big\}$ yields an isomorphism

$$\begin{aligned}
K_{(p,x)} : T^v_{(p,x)} T(S^1 \times [-1,1]) &\to T_pM \\
a^3 \frac{\partial}{\partial v^3} + a^4 \frac{\partial}{\partial v^4} &\mapsto a^3 \frac{\partial}{\partial \varphi} + a^4 \frac{\partial}{\partial z}
\end{aligned}$$

with inverse

$$\begin{aligned}
K^{-1}_{(p,x)} : T_pM &\to T^v_{(p,x)} T(S^1 \times [-1,1] \\
X^1 \frac{\partial}{\partial \varphi} + X^2 \frac{\partial}{\partial z} &\mapsto X^1 \frac{\partial}{\partial v^3} + X^2 \frac{\partial}{\partial v^4}.
\end{aligned}$$

Finally, we compute

$$(T_x \exp_p)|_{T^v_{(p,x)}T(S^1\times[-1,1])} : T^v_{(p,x)} T(S^1 \times [-1,1]) \to T_{\exp_p(x)} S^1 \times [-1,1].$$

To that end, let $a^3 \frac{\partial}{\partial v^3} + a^4 \frac{\partial}{\partial v^4} \in T^v_{(p,x)} T(S^1 \times [-1,1])$, $f(\varphi,z) \in C^\infty(S^1 \times [-1,1],\mathbb{R})$. Then

$$\begin{aligned}
(T_x \exp_p)\Big(a^3 \frac{\partial}{\partial v^3} + a^4 \frac{\partial}{\partial v^4}\Big)(f) &= \Big(a^3 \frac{\partial}{\partial v^3} + a^4 \frac{\partial}{\partial v^4}\Big)(f \circ \exp_p) \\
&= \Big(a^3 \frac{\partial}{\partial v^3} + a^4 \frac{\partial}{\partial v^4}\Big) f \circ (v^1 + v^3, v^2 + v^4) \\
&= a^3 \frac{\partial f}{\partial \varphi} \circ (v^1 + v^3, v^2 + v^4) \cdot \frac{\partial (v^1 + v^3)}{\partial v^3} \\
&\quad + a^4 \frac{\partial f}{\partial z} \circ (v^1 + v^3, v^2 + v^4) \cdot \frac{\partial (v^2 + v^4)}{\partial v^4} \\
&= a^3 \frac{\partial f}{\partial \varphi} \circ \exp_p + a^4 \frac{\partial f}{\partial z} \circ \exp_p
\end{aligned}$$

and hence

$$(T_x \exp_p)\Big(a^3 \frac{\partial}{\partial v^3} + a^4 \frac{\partial}{\partial v^4}\Big) = a^3 \frac{\partial}{\partial \varphi} \circ \exp_p + a^4 \frac{\partial}{\partial z} \circ \exp_p .$$

Combining our results for $K^{-1}_{(p,x)}$ and $T_x \exp_p$ yields for Eq. (4.3)

$$\begin{aligned}
\nabla_2 \exp_p : T_{\nu(\varphi,z)} S^1 \times [-1,1] &\to T_{\exp_{\nu(\varphi,z)} X(\nu(\varphi,z))} S^1 \times [-1,1] \\
v^1 \partial_\varphi + v^2 \partial_z &\mapsto v^1 \partial_\varphi + v^2 \partial_z,
\end{aligned}$$

where the tangent vectors ∂_φ and ∂_z are evaluated at the respective base points $\nu(\varphi,z)$ and $\exp_{\nu(\varphi,z)} X(\nu(\varphi,z)) = (\nu + X)(\varphi,z)$. Finally, the local bundle trivializations are given by

$$\Phi(X,Y) = \big(\nu + X,\ Y^1 \partial_\varphi \circ (\nu + X) + Y^2 \partial_z \circ (\nu + X)\big). \tag{4.4}$$

Theorem 4.6. *(a) For any $\nu \in \mathrm{Diff}^s_h(B)$, the restriction of Φ to a map*

$$\Phi : T_\nu \mathrm{Diff}^s_h(B) \times T_\nu \mathrm{Diff}^s(B) \to T\mathrm{Diff}^s(B)$$

is a local bundle trivialization for a neighbourhood of ν in $T\mathrm{Diff}^s(B)|_{\mathrm{Diff}^s_h(B)}$.

(b) Similarly, for any $\nu \in \mathrm{Diff}^s_{\sigma,\tau}(B)$, the restriction of Φ to a map

$$\Phi : T_\nu \mathrm{Diff}^s_{\sigma,\tau}(B) \times T_\nu \mathrm{Diff}^s_h(B) \to T\mathrm{Diff}^s(B)$$

is a local bundle trivialization for a neighbourhood of ν in $T\mathrm{Diff}^s_h(B)|_{\mathrm{Diff}^s_{\sigma,\tau}(B)}$.

Proof. For part (a), we have to show

- that
$$\operatorname{im}(\Phi|_{T_\nu \mathrm{Diff}_h^s(B) \times T_\nu \mathrm{Diff}^s(B)}) \subset T\mathrm{Diff}^s(B)|_{\mathrm{Diff}_h^s(B)},$$
i. e. that for $(X,Y) \in T_\nu \mathrm{Diff}_h^s \times T_\nu \mathrm{Diff}^s(B)$, we get $\Phi(X,Y) \in T\mathrm{Diff}^s(B)|_{\mathrm{Diff}_h^s(B)}$, where
$$\Phi(X,Y) = \big(\nu + X,\ Y^1 \partial_\varphi \circ (\nu + X) + Y^2 \partial_z \circ (\nu + X)\big),$$
- and that for any $\tilde{\nu} \in \mathrm{Diff}_h^s(B)$ and $Z \in T_{\tilde{\nu}} \mathrm{Diff}^s(B)$, there is $(X,Y) \in T_\nu \mathrm{Diff}_h^s(B) \times T_\nu \mathrm{Diff}^s(B)$ such that $Z = \Phi(X,Y)$.

For the first step, since $Y^1 \partial_\varphi \circ (\nu + X) + Y^2 \partial_z \circ (\nu + X) \in T_{\nu + X} \mathrm{Diff}^s(B)$, we only need to check that $\nu + X \in \mathrm{Diff}_h^s(B)$. To that end, we compute

$$(\nu + X)^* z = \nu^2 + X^2 = z + 0 = z$$

since $X \in T_\nu \mathrm{Diff}_h^s(B)$.

For the second step, let $\tilde{\nu} \in \mathrm{Diff}_h^s(B)$ and $Z = Z^1(\partial_\varphi \circ \tilde{\nu}) + Z^2(\partial_z \circ \tilde{\nu}) \in T_{\tilde{\nu}} \mathrm{Diff}^s(B)$. The map

$$(\varphi, z) \mapsto \tilde{\nu}^1(\varphi, z) - \nu^1(\varphi, z)$$

then defines an element of $H^s(S^1 \times [-1,1], S^1)$ and we choose a lift $X^1 \in H^s(S^1 \times [-1,1], \mathbb{R})$. We let $X := X^1(\partial_\varphi \circ \nu) \in T_\nu \mathrm{Diff}_h^s(B)$, such that

$$\begin{aligned}(\nu + X)(\varphi, z) &= \big(\nu^1(\varphi, z) + X^1(\varphi, z),\ z\big)\\ &= \big(\tilde{\nu}^1(\varphi, z),\ z\big)\\ &= \tilde{\nu}(\varphi, z),\end{aligned}$$

and we further let $Y := Z^1(\partial_\varphi \circ \nu) + Z^2(\partial_z \circ \nu) \in T_\nu \mathrm{Diff}^s(B)$. Then we get

$$\begin{aligned}\Phi(X,Y) &= \big(\nu + X,\ Z^1 \partial_\varphi \circ (\nu + X) + Z^2 \partial_z \circ (\nu + X)\big)\\ &= \big(\tilde{\nu},\ Z^1 \partial_\varphi \circ \tilde{\nu} + Z^2 \partial_z \circ \tilde{\nu}\big)\\ &= \big(\tilde{\nu},\ Z\big).\end{aligned}$$

A similar computation proves part (b). □

Remark. The previous theorem is true because of the specific form of Φ on $\mathrm{Diff}^s(B)$. In general, for a submanifold $D \subset \mathrm{Diff}^s(B)$ there is no reason for $\exp_\nu X$ with $\nu \in D$, $X \in T_\nu \mathcal{D}$ to define an element of D.

4.1.3 Smooth orthogonal bundle projection

Similarly to our Section 4.1.1 on the submanifolds, we split the map

$$P : T\mathrm{Diff}^s(B)|_{\mathrm{Diff}^s_{\sigma,\tau}(B)} \to T\mathrm{Diff}^s_{\sigma,\tau}(B)$$

into the two projections $P = P^2 \circ P^1$ with

$$P^1_\nu : T_\nu\mathrm{Diff}^s(S^1 \times [-1,1]) \to T_\nu\mathrm{Diff}^s_h(S^1 \times [-1,1])$$

at $\nu \in \mathrm{Diff}^s_h(S^1 \times [-1,1])$ and

$$P^2_\nu : T_\nu\mathrm{Diff}^s_h(S^1 \times [-1,1]) \to T_\nu\mathrm{Diff}^s_{\sigma,h}(S^1 \times [-1,1])$$

at $\nu \in \mathrm{Diff}^s_{\sigma,h}(S^1 \times [-1,1]) = \mathrm{Diff}^s_{\sigma,\tau}(S^1 \times [-1,1])$. We first compute P^1 at the identity id: Let

$$X = (X^1, X^2) = X^1\partial_\varphi + X^2\partial_z \in T_{\mathrm{id}}\mathrm{Diff}^s(S^1 \times [-1,1]).$$

Then we must have $P^1_{\mathrm{id}}(X) \in T_{\mathrm{id}}\mathrm{Diff}^s_h(S^1 \times [-1,1])$, i. e. we can write

$$P^1_{\mathrm{id}}(X) = p^1_{\mathrm{id}}(X)\partial_\varphi$$

for some operator p^1_{id} such that $p^1_{\mathrm{id}}(X) : S^1 \times [-1,1] \to \mathbb{R}$ and for any vector field $Y^1\partial_\varphi \in T_{\mathrm{id}}\mathrm{Diff}^s_h(S^1 \times [-1,1])$, we need to have

$$\begin{aligned}
0 &\overset{!}{=} \left(P^1_{\mathrm{id}}(X) - X,\ Y^1\partial_\varphi\right) \\
&= \int_{S^1\times[-1,1]} \langle P^1_{\mathrm{id}}(X) - X,\ Y^1\partial_\varphi\rangle_{(\varphi,z)}\, \mathrm{d}\varphi \wedge \mathrm{d}z \\
&= \int_{S^1\times[-1,1]} \langle p^1_{\mathrm{id}}(X)\partial_\varphi - X^1\partial_\varphi - X^2\partial_z,\ Y^1\partial_\varphi\rangle_{(\varphi,z)}\, \mathrm{d}\varphi \wedge \mathrm{d}z \\
&= \int_{S^1\times[-1,1]} \left(\left(p^1_{\mathrm{id}}(X) - X^1\right)Y^1 \underbrace{\langle\partial_\varphi,\partial_\varphi\rangle}_{\equiv 1} - X^2Y^1 \underbrace{\langle\partial_z,\partial_\varphi\rangle}_{\equiv 0}\right)\mathrm{d}\varphi \wedge \mathrm{d}z \\
&= \int_{S^1\times[-1,1]} \left(p^1_{\mathrm{id}}(X) - X^1\right)Y^1\, \mathrm{d}\varphi \wedge \mathrm{d}z.
\end{aligned}$$

This is solved by

$$p^1_{\mathrm{id}}(X) = X^1$$

and hence,

$$\begin{aligned}
P^1_{\mathrm{id}} : T_{\mathrm{id}}\mathrm{Diff}^s(S^1 \times [-1,1])|_{\mathrm{Diff}^s_h(S^1\times[-1,1])} &\to T_{\mathrm{id}}\mathrm{Diff}^s_h(S^1 \times [-1,1]) \\
(X^1, X^2) = X^1\partial_\varphi + X^2\partial_z &\mapsto X^1\partial_\varphi.
\end{aligned}$$

Since

$$T_\nu \mathrm{Diff}^s(S^1 \times [-1,1]) = T_{\mathrm{id}} \mathrm{Diff}^s(S^1 \times [-1,1]) \circ \nu,$$

we can similarly compute the projection P^1_ν. Let

$$X = (X^1, X^2) = X^1(\partial_\varphi \circ \nu) + X^2(\partial_z \circ \nu) \in T_\nu \mathrm{Diff}^s(S^1 \times [-1,1]),$$

then $P^1_\nu(X) \in T_\nu \mathrm{Diff}^s_h(S^1 \times [-1,1])$, i. e. we can write $P^1_\nu(X) = p^1_\nu(X)\partial_\varphi \circ \nu$ and for any $Y^1 \partial_\varphi \circ \nu$, we need to have

$$\begin{aligned}
0 &\overset{!}{=} \big(P^1_\nu(X) - X,\ Y^1 \partial_\varphi \circ \nu\big) \\
&= \int_{S^1 \times [-1,1]} \langle P^1_\nu(X) - X,\ Y^1 \partial_\varphi \circ \nu \rangle_{\nu(\varphi,z)} \, \mathrm{d}\varphi \wedge \mathrm{d}z \\
&= \int_{S^1 \times [-1,1]} \langle p^1_\nu(X)\partial_\varphi \circ \nu - X^1 \partial_\varphi \circ \nu - X^2 \partial_z \circ \nu,\ Y^1 \partial_\varphi \circ \nu \rangle_{\nu(\varphi,z)} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{d}\varphi \wedge \mathrm{d}z \\
&= \int_{S^1 \times [-1,1]} Y^1 \Big((p^1_\nu(X) - X^1) \underbrace{\langle \partial_\varphi \circ \nu, \partial_\varphi \circ \nu \rangle_{\nu(\varphi,z)}}_{= \langle \partial_\varphi, \partial_\varphi \rangle \circ \nu = 1} \\
&\qquad\qquad - X^2 \underbrace{\langle \partial_z \circ \nu, \partial_\varphi \circ \nu \rangle_{\nu(\varphi,z)}}_{= \langle \partial_z, \partial_\varphi \rangle \circ \nu = 0} \Big) \, \mathrm{d}\varphi \wedge \mathrm{d}z \\
&= \int_{S^1 \times [-1,1]} Y^1 (p^1_\nu(X) - X^1) \, \mathrm{d}\varphi \wedge \mathrm{d}z.
\end{aligned}$$

This is solved by

$$p^1_\nu(X^1) = X^1,$$

which implies

$$P^1_\nu\Big(X^1(\partial_\varphi \circ \nu) + X^2(\partial_z \circ \nu)\Big) = X^1(\partial_\varphi \circ \nu). \tag{4.5}$$

To show that P^1 is smooth in the base point, we will use the local trivializations $T\mathrm{Diff}^s(S^1 \times [-1,1])$ as computed in Section 4.1.2, more specifically Eq. (4.4).

Proposition 4.7. *$P^1 : T\mathrm{Diff}^s(S^1 \times [-1,1])|_{\mathrm{Diff}^s_h(S^1 \times [-1,1])} \to T\mathrm{Diff}^s_h(S^1 \times [-1,1])$ induced by Eq. (4.5) is a smooth bundle map, i. e. P^1 is smooth in the base point.*

Proof. Our trivializations for $T\mathrm{Diff}^s(S^1 \times [-1,1])|_{\mathrm{Diff}^s_h(S^1 \times [-1,1])}$ are given by

$$\begin{aligned}
\Phi : T_\nu \mathrm{Diff}^s_h(S^1 \times [-1,1]) \times T_\nu \mathrm{Diff}^s(S^1 \times [-1,1]) \\
\to T\mathrm{Diff}^s(S^1 \times [-1,1])|_{\mathrm{Diff}^s_h(S^1 \times [-1,1])} \\
(X, Y) = Y^1 \partial_\varphi \circ \nu + Y^2 \partial_z \circ \nu) \mapsto \Big(\nu + X,\ Y^1 \partial_\varphi \circ (\nu + X) + Y^2 \partial_z \circ (\nu + X)\Big).
\end{aligned} \tag{4.4 revisited}$$

In a neighbourhood around any $\nu \in \mathrm{Diff}^s_h(S^1 \times [-1,1])$, P^1 therefore takes the form

$$\begin{aligned} T_\nu \mathrm{Diff}^s_h(S^1 \times [-1,1]) \times T_\nu \mathrm{Diff}^s(S^1 \times [-1,1]) \\ \to T_\nu \mathrm{Diff}^s_h(S^1 \times [-1,1]) \times T_\nu \mathrm{Diff}^s(S^1 \times [-1,1]) \\ (X,Y) \mapsto (\Phi^{-1} \circ P^1 \circ \Phi)(X,Y). \end{aligned}$$

We get for $Y = Y^1 \partial_\varphi \circ \nu + Y^2 \partial_z \circ \nu$

$$\begin{aligned} (\Phi^{-1} \circ P^1 \circ \Phi)(X,Y) &= \Phi^{-1}(P^1(\Phi(X,Y))) \\ &= \Phi^{-1}\Big(P^1\big(\nu + X,\ Y^1 \partial_\varphi \circ (\nu + X) + Y^2 \partial_z \circ (\nu + X)\big)\Big) \\ &= \Phi^{-1}\big(\nu + X,\ Y^1 \partial_\varphi \circ (\nu + X)\big) \\ &= (X,\ Y^1 \partial_\varphi \circ \nu). \end{aligned}$$

This map is smooth in the base point X and hence, P^1 is a smooth bundle map. □

Our next goal is to show that P^2 is also a smooth bundle map. At the identity, P^2_{id} is a map of the form

$$\begin{aligned} P^2_{\mathrm{id}} : T_{\mathrm{id}} \mathrm{Diff}^s_h(S^1 \times [-1,1]) &\to T_{\mathrm{id}} \mathrm{Diff}^s_{\sigma,h}(S^1 \times [-1,1]) \\ X = X^1 \partial_\varphi &\mapsto p^2_{\mathrm{id}}(X) \partial_\varphi \end{aligned}$$

for some smooth map p^2_{id} such that $p^2_{\mathrm{id}}(X)$ only depends on z. For any $Y = Y^1(z)\partial_\varphi \in T_{\mathrm{id}} \mathrm{Diff}^s_{\sigma,h}(S^1 \times [-1,1])$, we must have

$$\begin{aligned} 0 &\overset{!}{=} \big(P^2_{\mathrm{id}}(X) - X,\ Y\big) \\ &= \int_{S^1 \times [-1,1]} \langle P^2_{\mathrm{id}}(X) - X,\ Y \rangle \, \mathrm{d}\varphi \wedge \mathrm{d}z \\ &= \int_{S^1 \times [-1,1]} \langle p^2_{\mathrm{id}}(X) \partial_\varphi - X^1 \partial_\varphi,\ Y^1 \partial_\varphi \rangle \, \mathrm{d}\varphi \wedge \mathrm{d}z \\ &= \int_{S^1 \times [-1,1]} (p^2_{\mathrm{id}}(X) - X^1) Y^1 \underbrace{\langle \partial_\varphi, \partial_\varphi \rangle}_{\equiv 1} \, \mathrm{d}\varphi \wedge \mathrm{d}z \\ &= \int_{-1}^{1} Y^1 \Big(\int_0^1 (p^2_{\mathrm{id}}(X) - X^1) \, \mathrm{d}\varphi \Big) \mathrm{d}z \\ &= \int_{-1}^{1} Y^1 \Big(p^2_{\mathrm{id}}(X) - \int_0^1 X^1 \, \mathrm{d}\varphi \Big) \mathrm{d}z \\ \Rightarrow \quad & p^2_{\mathrm{id}}(X) = \int_0^1 X^1 \, \mathrm{d}\varphi \end{aligned}$$

Let now $\nu \in \mathrm{Diff}^s_{\sigma,h}(S^1 \times [-1,1])$. Since ν preserves the area form σ, both the metric and orthogonal projection are right invariant and we can extend P^2_{id} to

$$P^2_\nu : T_\nu \mathrm{Diff}^s_h(S^1 \times [-1,1]) \to T_\nu \mathrm{Diff}^s_{\sigma,h}(S^1 \times [-1,1])$$

by

$$\begin{aligned}
P^2_\nu(X) &= (TR_\nu \circ P^2_{\mathrm{id}} \circ TR_{\nu^{-1}})(X)\\
&= TR_\nu(P^2_{\mathrm{id}}(TR_{\nu^{-1}}(X)))\\
&= TR_\nu(P^2_{\mathrm{id}}(X \circ \nu^{-1}))\\
&= TR_\nu(p^2_{\mathrm{id}}(X \circ \nu^{-1})\partial_\varphi)\\
&= p^2_{\mathrm{id}}(X \circ \nu^{-1}) \circ \nu \; (\partial_\varphi \circ \nu).
\end{aligned}$$

Since the map $p^2_{\mathrm{id}}(X \circ \nu^{-1})$ only depends on z and ν preserves z, we can compute

$$\begin{aligned}
p^2_{\mathrm{id}}(X \circ \nu^{-1}) \circ \nu &= p^2_{\mathrm{id}}(X \circ \nu^{-1})\\
&= p^2_{\mathrm{id}}(X^1 \circ \nu^{-1}\, \partial_\varphi)\\
&= \int_0^1 X^1 \circ \nu^{-1} \, \mathrm{d}\varphi.
\end{aligned}$$

We know that $\nu(\varphi, z) = \big(\nu^1(\varphi, z), z\big)$, hence for fixed z, we can write $\nu_z(\varphi) = \big(\nu^1_z(\varphi), z\big)$ and we also have $\nu_z^{-1}(\varphi) = \big((\nu^1_z)^{-1}(\varphi), z\big)$. Hence, we can change coordinates to

$$\begin{aligned}
p^2_{\mathrm{id}}(X \circ \nu^{-1}) \circ \nu &= \int_0^1 X^1 \circ (\nu^1_z)^{-1} \, \mathrm{d}\varphi\\
&= \int_0^1 X^1 \underbrace{(\nu^1_z)^*(\mathrm{d}\varphi)}_{= \mathrm{d}\nu^1_z = \frac{\partial \nu^1}{\partial \varphi} \mathrm{d}\varphi = \mathrm{d}\varphi}\\
&= \int_0^1 X^1 \, \mathrm{d}\varphi\\
&= p^2_{\mathrm{id}}(X^1 \partial_\varphi).
\end{aligned}$$

We define an operator

$$\begin{aligned}
p^2 : H^s(S^1 \times [-1,1], \mathbb{R}) &\to H^s([-1,1], \mathbb{R})\\
X^1 &\mapsto p^2_{\mathrm{id}}(X^1 \partial_\varphi) = \int_0^1 X^1 \, \mathrm{d}\varphi.
\end{aligned}$$

Then we can rewrite the previous computation as

$$p^2(X^1 \circ \nu^{-1}) = p^2(X^1)$$

and we finally get for $X = X^1(\partial_\varphi \circ \nu)$ that

$$P^2_\nu(X) = p^2(X^1)(\partial_\varphi \circ \nu)$$

for any $\nu \in \mathrm{Diff}^s_{\sigma,h}(S^1 \times [-1,1])$.

Proposition 4.8. *P^2 is a smooth bundle map, i. e. it is smooth in the base point.*

Proof. Again using the trivializations

$$\begin{aligned}\Phi : T_\nu \mathrm{Diff}^s_{\sigma,h}(S^1\times[-1,1])\times T_\nu \mathrm{Diff}^s_h(S^1\times[-1,1]) \\ \to T\mathrm{Diff}^s_h(S^1\times[-1,1])|_{\mathrm{Diff}^s_{\sigma,h}(S^1\times[-1,1])} \\ (X, Y = Y^1\partial_\varphi\circ\nu)\mapsto\Big(\nu+X,\ Y^1\partial_\varphi\circ(\nu+X)\Big),\end{aligned}$$

we can write

$$\begin{aligned}T_\nu \mathrm{Diff}^s_{\sigma,h}(S^1\times[-1,1])\times T_\nu \mathrm{Diff}^s_h(S^1\times[-1,1]) \\ \to T_\nu \mathrm{Diff}^s_{\sigma,h}(S^1\times[-1,1])\times T_\nu \mathrm{Diff}^s_h(S^1\times[-1,1]) \\ (X,Y)\mapsto(\Phi^{-1}\circ P^2\circ\Phi)(X,Y).\end{aligned}$$

We compute for $Y = Y^1(\partial_\varphi\circ\nu)$

$$\begin{aligned}(\Phi^{-1}\circ P^2\circ\Phi)(X,Y) &= \Phi^{-1}(P^2(\Phi(X,Y))) \\ &= \Phi^{-1}\Big(P^2\Big(\nu+X,\ Y^1\partial_\varphi\circ(\nu+X)\Big)\Big) \\ &= \Phi^{-1}\Big(\nu+X,\ P^2_{\nu+X}(Y^1\partial_\varphi\circ(\nu+X))\Big) \\ &= \Phi^{-1}\Big(\nu+X, p^2(Y^1)\partial_\varphi\circ(\nu+X)\Big) \\ &= (X, p^2(Y^1)\partial_\varphi\circ\nu).\end{aligned}$$

Since the map

$$(X,Y)\mapsto p^2(Y^1)\partial_\varphi\circ\nu$$

is constant in X, it is in particular also smooth in X. ☐

Corollary 4.9. *The previous two propositions show that*

$$P = P^2\circ P^1 : T\mathrm{Diff}^s(S^1\times[-1,1])|_{\mathrm{Diff}^s_{\sigma,h}(S^1\times[-1,1])}\to T\mathrm{Diff}^s_{\sigma,h}(S^1\times[-1,1])$$

is a smooth bundle projection. ☐

4.2 Euler equation on $\mathrm{Diff}^s_{\sigma,\tau}(B)$

Recall the result of the variation of energy in Section 2.3: Let $v_t\in T_{\mathrm{id}}\mathrm{Diff}^s_{\sigma,\tau}(S^1\times[-1,1])$ be a time-dependent vector field, i. e. v_t is of the form $v_t = v_t(z)\partial_\varphi$. If

$$0 = \int_0^T\int_B\langle w_t, \dot{v}_t + \nabla_{v_t}v_t\rangle\,\sigma\,\mathrm{d}t \qquad (2.9\text{ rev.})$$

for any time-dependent $w_t = w_t(z)\partial_\varphi \in T_{\mathrm{id}}\mathrm{Diff}^s_{\sigma,\tau}(S^1 \times [-1,1])$, then v_t is a solution to the Euler equation. We compute

$$\begin{aligned}\nabla_{v_t} v_t &= \nabla_{v_t(z)\partial_\varphi} v_t(z)\partial_\varphi \\ &= v_t(z)\nabla_{\partial_\varphi} v_t(z)\partial_\varphi \\ &= v_t(z)\Big(\underbrace{(\partial_\varphi v_t(z))}_{=0}\partial_\varphi + v_t(z)\underbrace{\nabla_{\partial_\varphi}\partial_\varphi}_{=0 \text{ since all the metric coefficients are constant}}\Big) \\ &= 0\end{aligned}$$

Then

$$\begin{aligned}\langle w_t, \dot v_t + \nabla_{v_t} v_t\rangle &= \langle w_t, \dot v_t + 0\rangle \\ &= \langle w_t(z)\partial_\varphi, \dot v_t(z)\partial_\varphi\rangle \\ &= w_t(z)\dot v_t(z)\underbrace{\langle \partial_\varphi, \partial_\varphi\rangle}_{=1} \\ &= w_t(z)\dot v_t(z). \end{aligned} \tag{4.6}$$

Equation (2.9) becomes

$$\begin{aligned}0 &\overset{(2.9)}{=} \int_0^T \int_M \langle w_t, \dot v_t + \nabla_{v_t} v_t\rangle \,\mathrm{vol}\,\mathrm{d}t \\ &\overset{(4.6)}{=} \int_0^T \int_M w_t(z)\dot v_t(z)\,\mathrm{vol}\,\mathrm{d}t\end{aligned}$$

for any $w_t(z) \in H^s([-1,1],\mathbb{R})$. This is equivalent to

$$\dot v_t(z) = 0.$$

Proposition 4.10. *The previous computation shows that the only solutions to the Euler equation on $S^1 \times [-1,1]$ preserving σ and τ are all stationary vector fields of the form $v_t = v = v(z)\partial_\varphi$.* □

The corresponding path ν_t in $\mathrm{Diff}^s_{\sigma,\tau}(B)$ then satisfies

$$\dot\nu_t = v_t \circ \nu_t = (v_t(z)\partial_\varphi) \circ \nu_t = v_t(z)(\partial_\varphi \circ \nu_t)$$

since ν_t preserves z. Hence,

$$\nu_t(\varphi, z) = \big(\varphi + t v_t(z),\, z\big).$$

and geodesics on $\mathrm{Diff}^s_{\sigma,\tau}(B)$ are given by straight lines.

4.3 $M = B \times S^1$, standard metric

Let $M = (S^1 \times [-1,1]) \times S^1 \xrightarrow{\pi} B = S^1 \times [-1,1]$ be the trivial S^1-bundle with stable Hamiltonian structure $\omega = \pi^*\sigma$ and $\lambda = \mathrm{d}\theta + \pi^*\mu$ for $\mu = -\frac{z^2}{2}\mathrm{d}\varphi$. The Reeb vector field is given by $R = \partial_\theta$. We will first consider the standard metric $\langle \cdot, \cdot \rangle^B$ on B, in which $(\partial_\varphi, \partial z)$ is an orthonormal basis as in Section 4.1. Then we get two-forms $\sigma = \mathrm{d}\varphi \wedge \mathrm{d}z$ and $\tau = z\sigma = z\,\mathrm{d}\varphi \wedge \mathrm{d}z = \mathrm{d}\mu$ on B. We further consider the metric on $M = B \times S^1$ defined by

- $\ker\lambda \perp R$, i. e. $\ker\lambda \perp \partial_\theta$,
- $|R| = 1$,
- and for any $v, w \in \ker\lambda_x$, we have

  $$\langle v, w \rangle_x = \langle \pi_* v, \pi_* w \rangle^B_{\pi(x)}.$$

Using this metric, the Riemannian volume form on M is given by

$$\mathrm{vol} = \omega \wedge \lambda = \mathrm{d}\varphi \wedge \mathrm{d}z \wedge \mathrm{d}\theta.$$

We will also follow the same steps as in Section 4.1: In Section 4.3.1, we first show that $\mathrm{Diff}^s_{\omega,\lambda}(M) \subset \mathrm{Diff}^s(M)$ is a smooth submanifold (which is independent of the chosen metric). In Section 4.3.2, we compute local charts for the tangent bundle. In Section 4.3.3, we finally prove for this specific metric, that the induced projection on each tangent space of $\mathrm{Diff}^s_{\omega,\lambda}(M)$ defines a smooth bundle map.

4.3.1 Smooth submanifold $\mathrm{Diff}^s_{\omega,\lambda}(M) \subset \mathrm{Diff}^s(M)$

Our first goal is to use Theorem 3.29 to prove

Theorem 4.11. $\mathrm{Diff}^s_{\omega,\lambda}(M) \subset \mathrm{Diff}^s(M)$ *is a smooth submanifold.*

Recall that

$$\mathrm{Diff}^s_{\omega,\lambda}(M) \cong \mathcal{D}^s \times S^1$$

for

$$\mathcal{D}^s = \left\{ \nu \in \mathrm{Diff}^s_{\sigma,\tau}(B) \,\middle|\, \int_\gamma \mu - \nu^*\mu \in \mathbb{Z} \quad \text{for any } \gamma \in H_1(B;\mathbb{Z}) \right\}.$$

We will start with results on $\mu - \nu^*\mu$.

Lemma 4.12. *Let* $\nu \in \mathrm{Diff}^s_{\sigma,\tau}(B)$. *Then* $\mu - \nu^*\mu$ *is exact.*

Proof. Recall that $\nu = (\nu^1, \nu^2) \in \mathrm{Diff}^s_{\sigma,\tau}(B)$ is equivalent to $\dfrac{\partial \nu^1}{\partial \varphi} \equiv 1$ and $\nu^2(\varphi, z) \equiv z$, hence

$$\begin{aligned}
\mu - \nu^*\mu &= -\frac{z^2}{2}\,\mathrm{d}\varphi + \frac{(\nu^2)^2}{2}\,\mathrm{d}\nu^1 \\
&= -\frac{z^2}{2}\,\mathrm{d}\varphi + \frac{z^2}{2}\Big(\underbrace{\frac{\partial \nu^1}{\partial \varphi}}_{=1}\,\mathrm{d}\varphi + \frac{\partial \nu^1}{\partial z}\,\mathrm{d}z\Big) \\
&= \frac{z^2}{2}\frac{\partial \nu^1}{\partial z}\,\mathrm{d}z. \qquad (4.7)
\end{aligned}$$

Define

$$M(\varphi, z) := \int_0^z \frac{\zeta^2}{2}\frac{\partial \nu^1}{\partial z}(\varphi, \zeta)\,\mathrm{d}\zeta$$

so that

$$\begin{aligned}
\mathrm{d}M &= \frac{\partial M}{\partial \varphi}\,\mathrm{d}\varphi + \frac{\partial M}{\partial z}\,\mathrm{d}z \\
&= \Big(\int_0^z \frac{\zeta^2}{2}\frac{\partial}{\partial \varphi}\frac{\partial \nu^1}{\partial z}(\varphi, \zeta)\,\mathrm{d}\zeta\Big)\mathrm{d}\varphi + \underbrace{\frac{z^2}{2}\frac{\partial \nu^1}{\partial z}\,\mathrm{d}z}_{\overset{(4.7)}{=}\mu - \nu^*\mu} \\
&= \Big(\int_0^z \underbrace{\frac{\zeta^2}{2}\frac{\partial}{\partial z}\underbrace{\frac{\partial \nu^1}{\partial \varphi}}_{\equiv 1}(\varphi, \zeta)}_{\equiv 0}\,\mathrm{d}\zeta\Big)\mathrm{d}\varphi + \mu - \nu^*\mu \\
&= \mu - \nu^*\mu.
\end{aligned}$$

☐

Proof of Theorem 4.11. The previous lemma implies that $\int_\gamma \mu - \nu^*\mu = 0$ for any $\gamma \in H_1(B;\mathbb{Z})$, hence

$$\begin{aligned}
\mathcal{D}^s &= \Big\{\nu \in \mathrm{Diff}^s_{\sigma,\tau}(B) \,\Big|\, \int_\gamma \mu - \nu^*\mu \in \mathbb{Z} \text{ for all } \gamma \in H_1(B;\mathbb{Z})\Big\} \\
&= \mathrm{Diff}^s_{\sigma,\tau}(B).
\end{aligned}$$

In particular, $\mathcal{D}^s = \mathrm{Diff}^s_{\sigma,\tau}(B)$ is a smooth submanifold of $\mathrm{Diff}^s(B)$, so by Theorem 3.29, also $\mathrm{Diff}^s_{\omega,\lambda}(B \times S^1) \subset \mathrm{Diff}^s_R(B \times S^1) \subset \mathrm{Diff}^s(B \times S^1)$ are smooth submanifolds. ☐

Recall the map $k : \mathcal{D}^s \to H^s(B, S^1)$ used in Theorem 3.29 defined by

$$k_\nu(b) = \int_{b_0}^b \mu_\nu = \int_{b_0}^b \mu - \nu^*\mu$$

for $b_0 = (0,-1) \in S^1 \times [-1,1]$.

Corollary 4.13 (see Theorem 3.29). *We have smooth diffeomorphisms*

$$\begin{aligned} \mathrm{Diff}^s_{\omega,\lambda}(B \times S^1) &\cong \mathrm{Diff}^s_{\sigma,\tau}(B) \times S^1 \\ \eta = (\eta^1,\, \eta^2) &\mapsto \big(\eta^1,\, \eta^2(b,\theta) - k_{\eta^1}(b) - \theta\big) \\ \big(\nu(b),\, k_\nu(b) + \theta + \theta_0\big) &\leftarrow\!\shortmid (\nu,\, \theta_0) \end{aligned}$$

☐

We will use the rest of this section to explicitly compute the map $k : \mathcal{D}^s \to H^s(B,S^1)$ used in Theorem 3.29 and verify Corollary 3.28, i. e. that k is smooth. Following the construction of k in Lemma 3.23, we start with the cohomology class defined by $\mu - \nu^*\mu$ for $\nu \in \mathcal{D}^s$. Since $[\mu - \nu^*\mu] = [0]$, we only need to choose $\alpha_{[0]} := 0 \in \Omega_{[0]}(B)$ and the constant map $k_{[0]} := 0$. As required, $\alpha_{[0]} = dk_{[0]}$. Then,

$$\mu_\nu := \mu - \nu^*\mu - \alpha_{[0]} = \mu - \nu^*\mu.$$

With the base point $b_0 = (0,-1) \in S^1 \times [-1,1] = B$, we get

$$k_\nu(b) = \int_{b_0}^{b} \mu_\nu = \int_{b_0}^{b} \mu - \nu^*\mu.$$

Then

$$\begin{aligned} &\eta_\nu : B \times S^1 \to B \times S^1, \\ &\quad \eta_\nu(b,\theta) := (\nu(b), \theta + k_\nu(b)) \end{aligned}$$

is a lift of ν in $\mathrm{Diff}^s_{\omega,\lambda}(B \times S^1)$.

To compute η_ν, recall that any $\nu = (\nu^1,\nu^2) \in \mathrm{Diff}^s_{\sigma,\tau}(B) = \mathrm{Diff}^s_{\sigma,h}(B)$ for $h(\varphi,z) = z$ satisfies

$$\nu^2(\varphi,z) = z \quad \text{and} \quad \frac{\partial \nu^1}{\partial \varphi} = 1.$$

In particular, ν^1 is of the form $\nu^1(\varphi,z) = \varphi + g(z) \mod 1$ for some $g \in H^s([-1,1],\mathbb{R})$. This yields

$$\begin{aligned} k_\nu(\varphi,z) &= \int_{(0,-1)}^{(\varphi,z)} \mu - \nu^*\mu \\ &\overset{(4.7)}{=} \int_{-1}^{z} \frac{\zeta^2}{2} \frac{\partial \nu^1}{\partial z}(\varphi,\zeta)\,\mathrm{d}\zeta \\ &= \int_{-1}^{z} \frac{\zeta^2}{2} g'(\zeta)\,\mathrm{d}\zeta. \end{aligned}$$

Then

$$\begin{aligned} \eta_\nu : (S^1 \times [-1,1]) \times S^1 = M &\to M \\ (b,\theta) &\mapsto (\nu(b), \theta + k_\nu(b)) \end{aligned}$$

or explicitly for $\nu(\varphi,z) = (\varphi + g(z),z)$,

$$((\varphi,z),\theta) \mapsto \Big(\underbrace{(\varphi + g(z),z)}_{=\nu(\varphi,z)},\ \theta + \int_{-1}^{z} \frac{\zeta^2}{2} g'(\zeta)\,\mathrm{d}\zeta\Big)$$

is an element of $\mathrm{Diff}^s_{\omega,\lambda}(M)$. Note that this also proves that for $\nu \in \mathrm{Diff}^s_{\sigma,\tau}(B)$, i. e. $g \in H^s([-1,1],\mathbb{R})$, we get η_ν of the same Sobolev class.

Lemma 4.14. *The operator*

$$\begin{aligned} &H^s([-1,1],\mathbb{R}) \to H^s([-1,1],\mathbb{R}) \\ &\qquad g \mapsto \Big(z \mapsto \int_{-1}^{z} \frac{\zeta^2}{2} g'(\zeta)\,\mathrm{d}\zeta\Big) \end{aligned}$$

is smooth.

Proof. First note that this is a linear map. To show smoothness, we only need to check continuity. Integration by parts yields

$$\begin{aligned} \int_{-1}^{z} \frac{\zeta^2}{2} g'(\zeta)\,\mathrm{d}\zeta &= \frac{\zeta^2}{2} g(\zeta)|_{-1}^{z} - \int_{-1}^{z} \zeta g(\zeta)\,\mathrm{d}\zeta \\ &= \frac{z^2}{2} g(z) - \frac{1}{2} g(-1) - \int_{-1}^{z} \zeta g(\zeta)\,\mathrm{d}\zeta. \end{aligned}$$

Both $g \mapsto \frac{z^2}{2} g(z)$ and the evaluation $g \mapsto \frac{1}{2} g(-1)$ are continuous. It remains to compute the H^s-norm of $g \mapsto \int_{-1}^{z} \zeta g(\zeta)\,\mathrm{d}\zeta$.

$$\begin{aligned} \Big\| \int_{-1}^{z} \zeta g(\zeta)\,\mathrm{d}\zeta \Big\|_{H^s}^2 &= \Big\| \int_{-1}^{z} \zeta g(\zeta)\,\mathrm{d}\zeta \Big\|_{H^0}^2 + \Big\| \frac{\partial}{\partial z} \int_{-1}^{z} \zeta g(\zeta)\,\mathrm{d}\zeta \Big\|_{H^{s-1}}^2 \\ &= \Big\| \int_{-1}^{z} \zeta g(\zeta)\,\mathrm{d}\zeta \Big\|_{L^2}^2 + \| z g(z) \|_{H^{s-1}}^2 . \end{aligned}$$

The first term can be estimated using the Cauchy-Schwarz inequality (CSI)

$$\begin{aligned}
\left\| \int_{-1}^{z} \zeta g(\zeta)\, \mathrm{d}\zeta \right\|_{H^0}^2 &= \left\| \int_{-1}^{z} \zeta g(\zeta)\, \mathrm{d}\zeta \right\|_{L^2}^2 \\
&= \int_{-1}^{1} \left(\int_{-1}^{z} \zeta g(\zeta)\, \mathrm{d}\zeta \right)^2 \mathrm{d}z \\
&\overset{\text{CSI}}{\leq} \int_{-1}^{1} \left(\int_{-1}^{z} \zeta^2\, \mathrm{d}\zeta \right) \left(\int_{-1}^{z} g^2(\zeta)\, \mathrm{d}\zeta \right) \mathrm{d}z \\
&\leq \int_{-1}^{1} \underbrace{\left(\int_{-1}^{1} \zeta^2\, \mathrm{d}\zeta \right)}_{= \frac{\zeta^3}{3}|_{-1}^{1} = \frac{2}{3}} \underbrace{\left(\int_{-1}^{1} g^2(\zeta)\, \mathrm{d}\zeta \right)}_{= \|g\|_{L^2}^2 \leq \|g\|_{H^s}^2} \mathrm{d}z \\
&\leq \frac{2}{3} \|g\|_{H^s}^2 \int_{-1}^{1} \mathrm{d}z \\
&= \frac{4}{3} \|g\|_{H^s}^2
\end{aligned}$$

Since s is sufficiently large, $H^s([-1,1],\mathbb{R})$ is a Hilbert algebra and hence

$$\begin{aligned}
\|zg(z)\|_{H^{s-1}}^2 &\leq \underbrace{\|z\|_{H^{s-1}}^2}_{\leq \int_{-1}^{1} \left((z^2 + \left(\frac{\partial z}{\partial z} \right)^2 + + \left(\frac{\partial^2 z}{\partial z^2} \right)^2 + \ldots \right) \mathrm{d}z = \int_{-1}^{1} (z^2+1)\, \mathrm{d}z = \left(\frac{z^3}{3} + z \right)|_{-1}^{1} = \frac{8}{3}} \overbrace{\|g\|_{H^{s-1}}^2}^{\leq \|g\|_{H^s}^2} \\
&\leq \frac{8}{3} \|g\|_{H^s}^2 .
\end{aligned}$$

Using the two previous results yields

$$\begin{aligned}
\left\| \int_{-1}^{z} \zeta g(\zeta)\, \mathrm{d}\zeta \right\|_{H^s}^2 &\leq \left\| \int_{-1}^{z} \zeta g(\zeta)\, \mathrm{d}\zeta \right\|_{L^2}^2 + \left\| zg(z) \right\|_{H^{s-1}}^2 \\
&\leq \frac{4}{3} \|g\|_{H^s}^2 + \frac{8}{3} \|g\|_{H^s}^2 \\
&= 4 \|g\|_{H^s}^2 .
\end{aligned}$$

□

Corollary 4.15. *The map*

$$k : \mathrm{Diff}_{\sigma,\tau}^{s}(B) \to H^s(B, \mathbb{R})$$

$$\Big(\nu : (\varphi, z) \mapsto (\varphi + g(z), z) \Big) \mapsto \left(k_\nu : (\varphi, z) \mapsto \int_{(0,-1)}^{(\varphi,z)} (\mu - \nu^* \mu) = \int_{-1}^{z} \frac{\zeta^2}{2} g'(\zeta) \mathrm{d}\zeta \right)$$

is smooth. □

4.3.2 Charts for $T\mathrm{Diff}^s(M)$ and its submanifolds

In this subsection (and only in this subsection), we consider the standard (orthonormal) metric on $(S^1 \times [-1,1]) \times S^1$, i. e. ∂_φ, ∂_z and ∂_θ form an orthonormal basis.

Adapting Corollary 2.7 to our situation yields the local bundle trivializations

$$\begin{aligned}\Phi : T_\eta\mathrm{Diff}^s((S^1 \times [-1,1]) \times S^1) \times T_\eta\mathrm{Diff}^s((S^1 \times [-1,1]) \times S^1)&\\ \to T\mathrm{Diff}^s((S^1 \times [-1,1]) \times S^1)&\\ (X,Y) \mapsto \Big(\exp_\eta X,\ \big(\nabla_2 \exp_{(\eta,X)}\big)(Y)\Big)&\end{aligned}$$

around $\eta \in \mathrm{Diff}^s((S^1 \times [-1,1]) \times S^1)$. Recall that

$$\begin{aligned}T_\eta\mathrm{Diff}^s((S^1 \times [-1,1]) \times S^1) &= T_{\mathrm{id}}\mathrm{Diff}^s((S^1 \times [-1,1]) \times S^1) \circ \eta\\ &= \mathfrak{X}^s((S^1 \times [-1,1]) \times S^1) \circ \eta,\end{aligned}$$

hence $\partial_\varphi \circ \eta$, $\partial_z \circ \eta$ and $\partial_\theta \circ \eta$ generate $T_\eta\mathrm{Diff}^s((S^1 \times [-1,1]) \times S^1)$. Write $X = X^\varphi(\partial_\varphi \circ \eta) + X^z(\partial_z \circ \eta) + X^\theta(\partial_\theta \circ \eta)$. Since $(\partial_\varphi, \partial_z, \partial_\theta)$ is an orthonormal basis, the map $\exp_\eta X$ maps (φ,z,θ) to

$$\begin{aligned}\big(\exp_\eta X\big)(\varphi,z,\theta) &= \exp_{\eta(\varphi,z,\theta)} X(\varphi,z,\theta)\\ &=: (\eta + X)(\varphi,z,\theta),\end{aligned}$$

where we define the addition component wise.

We now compute $\nabla_2 \exp_{(\eta(\varphi,z,\theta),X(\varphi,z,\theta))}$. Let $p := (\varphi,z,\theta) \in S^1 \times [-1,1]$ and $x \in T_p((S^1 \times [-1,1]) \times S^1)$, i. e. $(p,x) \in T((S^1 \times [-1,1]) \times S^1)$. Recall the definition in Eq. (2.2),

$$\begin{aligned}\nabla_2 \exp_{(p,x)} : T_p((S^1 \times [-1,1]) \times S^1) &\to T_{\exp_p(x)}((S^1 \times [-1,1]) \times S^1)\\ \nabla_2 \exp_{(p,x)} &:= (T_x \exp)|_{T^v_{(p,x)}T((S^1\times[-1,1])\times S^1)}\\ &\quad\circ \big(K|_{T^v_{(p,x)}T((S^1\times[-1,1])\times S^1)}\big)^{-1}.\end{aligned}$$

Following [Dom62], let φ,z,θ be the coordinates on $(S^1 \times [-1,1]) \times S^1$ and let $\tau : T((S^1 \times [-1,1]) \times S^1) \to (S^1 \times [-1,1]) \times S^1$ denote the canonical projection. Then

$$\begin{aligned}&v^1 := \varphi \circ \tau, && v^2 := z \circ \tau, && v^3 := \theta \circ \tau,\\ &v^4 := \mathrm{d}\varphi, && v^5 := \mathrm{d}z, && v^6 := \mathrm{d}\theta\end{aligned}$$

are coordinates on $T((S^1 \times [-1,1]) \times S^1)$ and $\dfrac{\partial}{\partial v^i}$ for $i = 1,\dots,6$ is a basis of $TT((S^1 \times [-1,1]) \times S^1)$. Since

$$T^v_{(p,x)}T((S^1 \times [-1,1]) \times S^1) = \ker(T\tau|_{T_{(p,x)}T((S^1\times[-1,1])\times S^1)}),$$

we let $A = \sum_{i=1}^{6} a^i \frac{\partial}{\partial v^i} \in T_{(p,x)}T((S^1 \times [-1,1]) \times S^1)$, $f(\varphi, z, \theta) \in C^\infty((S^1 \times [-1,1]) \times S^1, \mathbb{R})$ and compute for

$$\begin{aligned}(T\tau)(A)(f) &= A(f \circ \tau)\\ &= \Big(\sum_{i=1}^{6} a^i \frac{\partial}{\partial v^i}\Big)(f \circ \tau)\\ &= \Big(a^1 \frac{\partial}{\partial v^1} + a^2 \frac{\partial}{\partial v^2} + a^3 \frac{\partial}{\partial v^2}\Big)(f \circ \tau)\end{aligned}$$

since $(f \circ \tau)(v^1, \dots, v^6) = f(v^1, v^2, v^3)$,

$$\begin{aligned}&= a^1 \frac{\partial f}{\partial \varphi} \circ \tau \cdot \frac{\partial v^1}{\partial v^1} + a^2 \frac{\partial f}{\partial z} \circ \tau \cdot \frac{\partial v^2}{\partial v^2} + a^3 \frac{\partial f}{\partial \theta} \circ \tau \cdot \frac{\partial v^3}{\partial v^3}\\ &= a^1 \frac{\partial f}{\partial \varphi} \circ \tau + a^2 \frac{\partial f}{\partial z} \circ \tau + a^3 \frac{\partial f}{\partial \theta} \circ \tau.\end{aligned}$$

This yields

$$\begin{aligned}T^v_{(p,x)}T((S^1 \times [-1,1]) \times S^1) &= \ker\big(T\tau|_{T_{(p,x)}T((S^1 \times [-1,1]) \times S^1)}\big)\\ &= \operatorname{span}\Big\{\frac{\partial}{\partial v^4}, \frac{\partial}{\partial v^5}, \frac{\partial}{\partial v^6}\Big\}.\end{aligned}$$

To compute the connection map K, first note that since our metric is constant on $(S^1 \times [-1,1]) \times S^1$, all Christoffel symbols vanish. Eq. (11) in [Dom62] states for $A = \sum_{i=1}^{6} a^i \frac{\partial}{\partial v^i} \in T_{(p,x)}T((S^1 \times [-1,1]) \times S^1)$,

$$K_{(p,x)}(A) = a^4 \frac{\partial}{\partial \varphi} + a^5 \frac{\partial}{\partial z} + a^6 \frac{\partial}{\partial \theta}.$$

Restricting $K_{(p,x)}$ to $T^v_{(p,x)}T((S^1 \times [-1,1]) \times S^1) = \operatorname{span}\Big\{\frac{\partial}{\partial v^4}, \frac{\partial}{\partial v^5}, \frac{\partial}{\partial v^6}\Big\}$ yields an isomorphism

$$\begin{aligned}K_{(p,x)} : T^v_{(p,x)}T((S^1 \times [-1,1]) \times S^1) &\to T_pM\\ a^4 \frac{\partial}{\partial v^4} + a^5 \frac{\partial}{\partial v^5} + a^6 \frac{\partial}{\partial v^6} &\mapsto a^4 \frac{\partial}{\partial \varphi} + a^5 \frac{\partial}{\partial z} + a^6 \frac{\partial}{\partial \theta}\end{aligned}$$

with inverse

$$\begin{aligned}K^{-1}_{(p,x)} : T_pM &\to T^v_{(p,x)}T(S^1 \times [-1,1]\\ X^1 \frac{\partial}{\partial \varphi} + X^2 \frac{\partial}{\partial z} + X^3 \frac{\partial}{\partial \theta} &\mapsto X^1 \frac{\partial}{\partial v^4} + X^2 \frac{\partial}{\partial v^5} + X^3 \frac{\partial}{\partial v^6}.\end{aligned}$$

Finally, we compute

$$(T_x \exp_p)|_{T^v_{(p,x)}T((S^1\times[-1,1])\times S^1)} : \\ T^v_{(p,x)}T((S^1\times[-1,1])\times S^1) \to T_{\exp_p(x)}(S^1\times[-1,1])\times S^1.$$

To that end, let $a^4\frac{\partial}{\partial v^4} + a^5\frac{\partial}{\partial v^5} + a^6\frac{\partial}{\partial v^6} \in T^v_{(p,x)}T((S^1\times[-1,1])\times S^1)$ and a function $f(\varphi,z,\theta) \in C^\infty((S^1\times[-1,1])\times S^1,\mathbb{R})$. Then

$$\begin{aligned}
(T_x\exp_p)\Big(a^4\frac{\partial}{\partial v^4} + a^5\frac{\partial}{\partial v^5} + a^6\frac{\partial}{\partial v^6}\Big)(f) = \\
&= \Big(a^4\frac{\partial}{\partial v^4} + a^5\frac{\partial}{\partial v^5} + a^6\frac{\partial}{\partial v^6}\Big)(f\circ\exp_p) \\
&= \Big(a^4\frac{\partial}{\partial v^4} + a^5\frac{\partial}{\partial v^5} + a^6\frac{\partial}{\partial v^6}\Big)f\circ(v^1+v^4,v^2+v^5,v^3+v^6) \\
&= a^4\frac{\partial f}{\partial\varphi}\circ(v^1+v^4,v^2+v^5,v^3+v^6)\cdot\frac{\partial(v^1+v^4)}{\partial v^4} \\
&\quad + a^5\frac{\partial f}{\partial z}\circ(v^1+v^4,v^2+v^5,v^3+v^6)\cdot\frac{\partial(v^2+v^5)}{\partial v^5} \\
&\quad + a^6\frac{\partial f}{\partial\theta}\circ(v^1+v^4,v^2+v^5,v^3+v^6)\cdot\frac{\partial(v^3+v^6)}{\partial v^6} \\
&= a^4\frac{\partial f}{\partial\varphi}\circ\exp_p + a^5\frac{\partial f}{\partial z}\circ\exp_p + a^6\frac{\partial f}{\partial\theta}\circ\exp_p
\end{aligned}$$

and hence

$$(T_x\exp_p)\Big(a^4\frac{\partial}{\partial v^4} + a^5\frac{\partial}{\partial v^5} + a^6\frac{\partial}{\partial v^6}\Big) = a^4\frac{\partial f}{\partial\varphi}\circ\exp_p + a^5\frac{\partial f}{\partial z}\circ\exp_p + a^6\frac{\partial f}{\partial\theta}\circ\exp_p.$$

Combining our results for $K^{-1}_{(p,x)}$ and $T_x\exp_p$ yields for Eq. (4.3)

$$\begin{aligned}
\nabla_2\exp_p : T_{\eta(\varphi,z,\theta)}(S^1\times[-1,1])\times S^1 &\to T_{\exp_{\eta(\varphi,z,\theta)}X(\eta(\varphi,z,\theta))}(S^1\times[-1,1])\times S^1 \\
v^1\partial_\varphi + v^2\partial_z + v^3\partial_\theta &\mapsto v^1\partial_\varphi + v^2\partial_z + v^3\partial_\theta,
\end{aligned}$$

where the tangent vectors ∂_φ, ∂_z and ∂_θ are evaluated at the respective base points $\eta(\varphi,z,\theta)$ and $\exp_{\eta(\varphi,z,\theta)}X(\eta(\varphi,z,\theta)) = (\eta+X)(\varphi,z,\theta)$. Finally, the local bundle trivializations are given by

$$\Phi(X,Y) = \big(\eta+X,\ Y^1\partial_\varphi\circ(\eta+X) + Y^2\partial_z\circ(\eta+X) + Y^3\partial_\theta\circ(\eta+X)\big).$$

Theorem 4.16. *(a) For any $\eta\in\mathrm{Diff}^s_R((S^1\times[-1,1])\times S^1)$, the restriction of Φ to a map*

$$\Phi : T_\eta\mathrm{Diff}^s_R((S^1\times[-1,1])\times S^1)\times T_\eta\mathrm{Diff}^s_R((S^1\times[-1,1])\times S^1) \\ \to T\mathrm{Diff}^s_R((S^1\times[-1,1])\times S^1)$$

is a local bundle trivialization for a neighbourhood of η in $T\mathrm{Diff}^s_R((S^1\times[-1,1])\times S^1)$.

(b) For any $\eta \in \mathrm{Diff}^s_{\omega,\lambda}((S^1 \times [-1,1]) \times S^1)$*, the restriction of* Φ *to a map*

$$\Phi : T_\eta \mathrm{Diff}^s_{\omega,\lambda}((S^1 \times [-1,1]) \times S^1) \times T_\eta \mathrm{Diff}^s_R((S^1 \times [-1,1]) \times S^1) \\ \rightarrow T\mathrm{Diff}^s_R((S^1 \times [-1,1]) \times S^1)$$

is a local bundle trivialization for a neighbourhood of η *in* $T\mathrm{Diff}^s_R((S^1 \times [-1,1]) \times S^1)|_{\mathrm{Diff}^s_{\omega,\lambda}((S^1 \times [-1,1]) \times S^1)}$*.*

Proof. For part (a), we have to show

- that for $(X,Y) \in T_\eta \mathrm{Diff}^s_R((S^1 \times [-1,1]) \times S^1) \times T_\eta \mathrm{Diff}^s_R((S^1 \times [-1,1]) \times S^1)$, we get $\Phi(X,Y) \in \mathrm{Diff}^s_R((S^1 \times [-1,1]) \times S^1)$ with
$$\Phi(X,Y) = \big(\eta + X,\ Y^\varphi \partial_\varphi \circ (\eta + X) + Y^z \partial_z \circ (\eta + X) + Y^\theta \partial_\theta \circ (\eta + X)\big)$$
for $Y = Y^\varphi \partial_\varphi \circ \eta + Y^z \partial_z \circ \eta + Y^\theta \partial_\theta \circ \eta$.
- and that for any $\tilde{\eta} \in \mathrm{Diff}^s_R(B \times S^1)$ and $Z \in T_{\tilde{\eta}} \mathrm{Diff}^s_R(B \times S^1)$, there is $(X,Y) \in T_\eta \mathrm{Diff}^s_R(B \times S^1) \times T_\eta \mathrm{Diff}^s_R(B \times S^1)$ such that $Z = \Phi(X,Y)$.

For the first step, since the tangent vector of $\Phi(X,Y)$ satisfies $Y^\varphi \partial_\varphi \circ (\eta + X) + Y^z \partial_z \circ (\eta + X) + Y^\theta \partial_\theta \circ (\eta + X) \in T_{\eta+X} \mathrm{Diff}^s((S^1 \times [-1,1]) \times S^1)$, we need to check that

$$\eta + X \in \mathrm{Diff}^s_R((S^1 \times [-1,1]) \times S^1)$$

and

$$\mathcal{L}_R\big(Y^\varphi \partial_\varphi \circ (\eta + X) + Y^z \partial_z \circ (\eta + X) + Y^\theta \partial_\theta \circ (\eta + X)\big) = 0.$$

To that end, we compute

$$\begin{aligned}
(\eta + X)_* R &= (\eta + X)_* \frac{\partial}{\partial \theta} \\
&= \underbrace{\frac{\partial(\eta^1 + X^\varphi)}{\partial \theta}}_{=0} \partial_\varphi + \underbrace{\frac{\partial(\eta^2 + X^z)}{\partial \theta}}_{=0} \partial_z + \underbrace{\frac{\partial(\eta^3 + X^\theta)}{\partial \theta}}_{=\frac{\partial \eta^3}{\partial \theta} = 1} \partial_\theta \\
&= \partial_\theta \\
&= R.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\mathcal{L}_R\big(Y^\varphi \partial_\varphi \circ (\eta + X) + Y^z \partial_z \circ (\eta + X) + Y^\theta \partial_\theta \circ (\eta + X)\big) = \\
&\quad = \big[R, Y^\varphi \partial_\varphi \circ (\eta + X) + Y^z \partial_z \circ (\eta + X) + Y^\theta \partial_\theta \circ (\eta + X)\big] \\
&\quad = \big[\partial_\theta \circ (\eta + X), Y^\varphi \partial_\varphi \circ (\eta + X) + Y^z \partial_z \circ (\eta + X) + Y^\theta \partial_\theta \circ (\eta + X)\big] \\
&\quad = \underbrace{\frac{\partial Y^\varphi}{\partial \theta}}_{=0} \partial_\varphi \circ (\eta + X) + \underbrace{\frac{\partial Y^z}{\partial \theta}}_{=0} \partial_z \circ (\eta + X) + \underbrace{\frac{\partial Y^\theta}{\partial \theta}}_{=0} \partial_\theta \circ (\eta + X) \\
&\quad = 0.
\end{aligned}$$

For the second part, let $\tilde{\eta} \in \mathrm{Diff}^s_R(B \times S^1)$ and $Z \in T_{\tilde{\eta}}\mathrm{Diff}^s_R(B \times S^1)$. Note that since both $\eta, \tilde{\eta} \in \mathrm{Diff}^s_R(B \times S^1)$, the first two components η^φ, η^z and $\tilde{\eta}^\varphi, \tilde{\eta}^z$, resp., only depend on φ and z, whereas the last components η^θ and $\tilde{\eta}^\theta$ are of the form $\theta + k_{(\eta^\varphi, \eta^z)}(\varphi, z)$ and $\theta + k_{(\tilde{\eta}^\varphi, \tilde{\eta}^z)}(\varphi, z)$, resp. This implies that all three of the maps

$$\begin{aligned}
&(\varphi, z, \theta) \mapsto \tilde{\eta}^\varphi(\varphi, z, \theta) - \eta^\varphi(\varphi, z, \theta) \\
&(\varphi, z, \theta) \mapsto \tilde{\eta}^z(\varphi, z, \theta) - \eta^z(\varphi, z, \theta) \\
&(\varphi, z, \theta) \mapsto \tilde{\eta}^\theta(\varphi, z, \theta) - \eta^\theta(\varphi, z, \theta)
\end{aligned}$$

only depend on φ and z, and *not* on θ. The first and last define elements of $H^s(B, S^1) = H^s(S^1 \times [-1,1], S^1)$, which we can lift to $X^\varphi, X^z \in H^s(S^1 \times [-1,1], \mathbb{R})$. The second already maps into $\mathbb{R}$, i.e. defines an element $X^z \in H^s(S^1 \times [-1,1], \mathbb{R})$. We let $X := X^\varphi(\partial_\varphi \circ \eta) + X^z(\partial_z \circ \eta) + X^\theta(\partial_\theta \circ \eta) \in T_\eta \mathrm{Diff}^s_R(B \times S^1)$, such that

$$\begin{aligned}
(\eta + X)(\varphi, z, \theta) &= \big(\eta^\varphi(\varphi, z, \theta) + X^\varphi(\varphi, z, \theta),\ \eta^z(\varphi, z, \theta) + X^z(\varphi, z, \theta), \\
&\qquad\qquad \eta^\theta(\varphi, z, \theta) + X^\theta(\varphi, z, \theta)\big) \\
&= \tilde{\eta}(\varphi, z, \theta).
\end{aligned}$$

We further let $Y := Z^\varphi(\partial_\varphi \circ \eta) + Z^z(\partial_z \circ \eta) + Z^\theta(\partial_\theta \circ \eta) \in T_\eta \mathrm{Diff}^s_R(B \times S^1)$. Then we get

$$\begin{aligned}
\Phi(X, Y) &= \big(\eta + X,\ Z^\varphi \partial_\varphi \circ (\eta + X) + Z^z \partial_z \circ (\eta + X) + Z^\theta \partial_\theta \circ (\eta + X)\big) \\
&= \big(\tilde{\eta},\ Z^\varphi \partial_\varphi \circ \tilde{\eta} + Z^z \partial_z \circ \tilde{\eta} + Z^\theta \partial_\theta \circ \tilde{\eta}\big) \\
&= \big(\tilde{\eta},\ Z\big).
\end{aligned}$$

A similar computation proves part (b). ☐

4.3.3 Smooth orthogonal bundle projection

Since $\mathrm{Diff}^s_R(B \times S^1) \subset \mathrm{Diff}^s(B \times S^1)$ is totally geodesic (see Theorem 2.2 in [EP13]), it only remains to show that the orthogonal projection

$$P : T\mathrm{Diff}^s_R(B \times S^1)|_{\mathrm{Diff}^s_{\omega,\lambda}(B\times S^1)} \to T\mathrm{Diff}^s_{\omega,\lambda}(B \times S^1)$$

is a smooth bundle map. Recall Corollary 4.13

$$\begin{aligned} \mathrm{Diff}^s_{\sigma,\tau}(B) \times S^1 &\xrightarrow{\cong} \mathrm{Diff}^s_{\omega,\lambda}(B \times S^1) \\ (\nu,\kappa) &\mapsto \big((\varphi,z,\theta) \mapsto (\nu(\varphi,z), \theta + k_\nu(z) + \kappa)\big), \end{aligned}$$

which implies

$$\begin{aligned} T_{\mathrm{id}}\mathrm{Diff}^s_{\sigma,\tau}(B) \times T_0 S^1 &\xrightarrow{\cong} T_{\mathrm{id}}\mathrm{Diff}^s_{\omega,\lambda}(B \times S^1) \\ \big(v,\, c\partial_\kappa\big) &\mapsto v + \big(T_{\mathrm{id}}k(v) + c\big)\partial_\theta. \end{aligned} \tag{4.8}$$

We let $V \in T_{\mathrm{id}}\mathrm{Diff}^s_R(B \times S^1)$, i. e. $V = V^\varphi(\varphi,z)\partial_\varphi + V^z(\varphi,z)\partial_z + V^\theta(\varphi,z)\partial_\theta$. Since any element $v \in T_{\mathrm{id}}\mathrm{Diff}^s_{\sigma,\tau}(B)$ is of the form $v = v(z)\partial_\varphi$, we further define $p^B_{\mathrm{id}}(V) \in H^s([-1,1],\mathbb{R})$ and $p^R_{\mathrm{id}}(V) \in \mathbb{R}$ by

$$P_{\mathrm{id}}(V) = p^B_{\mathrm{id}}(V)\partial_\varphi + \big(T_{\mathrm{id}}k(p^B_{\mathrm{id}}(V)\partial_\varphi) + p^R_{\mathrm{id}}(V)\big)\partial_\theta.$$

For any $V \in T_{\mathrm{id}}\mathrm{Diff}^s_R(B \times S^1)$, we have $P_{\mathrm{id}}(V) \in T_{\mathrm{id}}\mathrm{Diff}^s_{\omega,\lambda}(B \times S^1)$, i. e. the coefficient $p^B_{\mathrm{id}}(V) \in H^s([-1,1],\mathbb{R})$ only depends on z and $p^R_{\mathrm{id}}(V) \in \mathbb{R}$ is constant. Then for any $W \in T_{\mathrm{id}}\mathrm{Diff}^s_{\omega,\lambda}(B \times S^1)$, i. e.

$$W = \underbrace{w(z)\partial_\varphi}_{=:w} + \big(T_{\mathrm{id}}k(w) + x\big)\partial_\theta,$$

we need to have

$$\begin{aligned} 0 &\overset{!}{=} \big(V - P_{\mathrm{id}}(V), W\big) \\ &= \int_{B\times S^1} \big\langle V - P_{\mathrm{id}}(V), W\big\rangle \mathrm{d}\theta \wedge \mathrm{d}\varphi \wedge \mathrm{d}z \\ &= \int_{B\times S^1} \Big\langle V^\varphi\partial_\varphi + V^z\partial_z + V^\theta\partial_\theta - p^B_{\mathrm{id}}(V)\partial_\varphi \\ &\qquad - \big(T_{\mathrm{id}}k(p^B_{\mathrm{id}}(V)\partial_\varphi) + p^R_{\mathrm{id}}(V)\big)\partial_\theta, \\ &\qquad\qquad W\Big\rangle \mathrm{d}\theta \wedge \mathrm{d}\varphi \wedge \mathrm{d}z \\ &= \int_{B\times S^1} \Big[\big(V^\varphi - p^B_{\mathrm{id}}(V)\big)\langle\partial_\varphi,\, W\rangle + V^z\langle\partial_z,\, W\rangle \\ &\qquad + \big(V^\theta - T_{\mathrm{id}}k(p^B_{\mathrm{id}}(V)\partial_\varphi) - p^R_{\mathrm{id}}(V)\big)\langle\partial_\theta,\, W\rangle\Big] \\ &\qquad\qquad \mathrm{d}\theta \wedge \mathrm{d}\varphi \wedge \mathrm{d}z \end{aligned}$$

$$
\begin{aligned}
&= \int_{B\times S^1} \Big\{ \big(V^\varphi - p^B_{\mathrm{id}}(V)\big)\Big[w(z)\langle\partial_\varphi,\partial_\varphi\rangle + \big(T_{\mathrm{id}}k(w)+x\big)\langle\partial_\varphi,\partial_\theta\rangle\Big] \\
&\qquad\qquad + V^z\Big[w(z)\langle\partial_z,\partial_\varphi\rangle + \big(T_{\mathrm{id}}k(w)+x\big)\langle\partial_z,\partial_\theta\rangle\Big] \\
&\qquad\qquad\quad + \big(V^\theta - T_{\mathrm{id}}k(p^B_{\mathrm{id}}(V)\partial_\varphi) - p^R_{\mathrm{id}}(V)\big)\cdot \\
&\qquad\qquad \cdot\Big[w(z)\langle\partial_\theta,\partial_\varphi\rangle + \big(T_{\mathrm{id}}k(w)+x\big)\langle\partial_\theta,\partial_\theta\rangle\Big]\Big\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{d}\theta\wedge\mathrm{d}\varphi\wedge\mathrm{d}z \\
&= \int_{B\times S^1} \Big\{ \big(V^\varphi - p^B_{\mathrm{id}}(V)\big)\Big[w(z)\big(1+\mu(\partial_\varphi)^2\big) + \big(T_{\mathrm{id}}k(w)+x\big)\mu(\partial_\varphi)\Big] \\
&\qquad\qquad\qquad + V^z\Big[w(z)\cdot 0 + \big(T_{\mathrm{id}}k(w)+x\big)\cdot 0\Big] \\
&\qquad\qquad\qquad + \big(V^\theta - T_{\mathrm{id}}k(p^B_{\mathrm{id}}(V)\partial_\varphi) - p^R_{\mathrm{id}}(V)\big)\cdot \\
&\qquad\qquad\qquad\quad \cdot\Big[w(z)\mu(\partial_\varphi) + T_{\mathrm{id}}k(w) + x\Big]\Big\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \mathrm{d}\theta\wedge\mathrm{d}\varphi\wedge\mathrm{d}z \\
&= \int_B w(z)\Big\{\Big[\big(V^\varphi - p^B_{\mathrm{id}}(V)\big)\big(1+\mu(\partial_\varphi)^2\big) \\
&\quad + \big(V^\theta - T_{\mathrm{id}}k(p^B_{\mathrm{id}}(V)\partial_\varphi) - p^R_{\mathrm{id}}(V)\big)\mu(\partial_\varphi)\Big] \\
&\qquad\quad + T_{\mathrm{id}}k(w)\Big[\big(V^\varphi - p^B_{\mathrm{id}}(V)\big)\mu(\partial_\varphi) \\
&\qquad + \big(V^\theta - T_{\mathrm{id}}k(p^B_{\mathrm{id}}(V)\partial_\varphi) - p^R_{\mathrm{id}}(V)\big)\Big] \\
&\qquad\qquad + x\Big[\big(V^\varphi - p^B_{\mathrm{id}}(V)\big)\mu(\partial_\varphi) \\
&\qquad + \big(V^\theta - T_{\mathrm{id}}k(p^B_{\mathrm{id}}(V)\partial_\varphi) - p^R_{\mathrm{id}}(V)\big)\Big]\Big\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{d}\varphi\wedge\mathrm{d}z \\
&= \int_{-1}^{1} w(z)\Big[\Big(\int_{S^1} V^\varphi\,\mathrm{d}\varphi - p^B_{\mathrm{id}}(V)\Big)\big(1+\mu(\partial_\varphi)^2\big) \\
&\quad + \Big(\int_{S^1} V^\theta\,\mathrm{d}\varphi - T_{\mathrm{id}}k(p^B_{\mathrm{id}}(V)\partial_\varphi) - p^R_{\mathrm{id}}(V)\Big)\mu(\partial_\varphi)\Big]\mathrm{d}z \\
&\qquad\quad + \int_{-1}^{1} T_{\mathrm{id}}k(w)\Big[\Big(\int_{S^1} V^\varphi\,\mathrm{d}\varphi - p^B_{\mathrm{id}}(V)\Big)\mu(\partial_\varphi) \\
&\qquad\qquad + \int_{S^1} V^\theta\,\mathrm{d}\varphi - T_{\mathrm{id}}k(p^B_{\mathrm{id}}(V)\partial_\varphi) - p^R_{\mathrm{id}}(V)\Big]\mathrm{d}z \\
&\qquad\quad + x\int_{-1}^{1}\Big[\Big(\int_{S^1} V^\varphi\,\mathrm{d}\varphi - p^B_{\mathrm{id}}(V)\Big)\mu(\partial_\varphi) \\
&\qquad\qquad + \int_{S^1} V^\theta\,\mathrm{d}\varphi - T_{\mathrm{id}}k(p^B_{\mathrm{id}}(V)\partial_\varphi) - p^R_{\mathrm{id}}(V)\Big]\mathrm{d}z.
\end{aligned}
\tag{4.9}
$$

For the coefficient of x to vanish, we get

$$\begin{aligned}
0 &= \int_{-1}^{1} \Big[\Big(\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi - p^{B}_{\mathrm{id}}(V)\Big)\mu(\partial_\varphi) \\
&\qquad + \int_{S^1} V^{\theta}\,\mathrm{d}\varphi - T_{\mathrm{id}}k(p^{B}_{\mathrm{id}}(V)\partial_\varphi) - p^{R}_{\mathrm{id}}(V)\Big]\mathrm{d}z \\
&= \int_{-1}^{1} \Big[\mu(\partial_\varphi)\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi - \mu(\partial_\varphi)p^{B}_{\mathrm{id}}(V) \\
&\qquad + \int_{S^1} V^{\theta}\,\mathrm{d}\varphi - T_{\mathrm{id}}k(p^{B}_{\mathrm{id}}(V)\partial_\varphi)\Big]\mathrm{d}z - 2p^{R}_{\mathrm{id}}(V),
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
p^{R}_{\mathrm{id}}(V) = \frac{1}{2}\int_{-1}^{1} \Big[\mu(\partial_\varphi)\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi - \mu(\partial_\varphi)p^{B}_{\mathrm{id}}(V)& \\
+ \int_{S^1} V^{\theta}\,\mathrm{d}\varphi - T_{\mathrm{id}}k(p^{B}_{\mathrm{id}}(V)\partial_\varphi)\Big]\mathrm{d}z.&
\end{aligned} \tag{4.10}$$

Lemma 4.17. *For any $z \in [-1,1]$ and functions $b(\zeta)$ and $u(\zeta)$, we have*

$$\begin{aligned}
\int_{z}^{1} b(\zeta)\cdot T_{\mathrm{id}}k(u\partial_\varphi)\,\mathrm{d}\zeta &= \\
&= -\frac{1}{2}u(-1)\int_{z}^{1} b(\zeta)\,\mathrm{d}\zeta - \int_{z}^{1} b(\alpha)\,\mathrm{d}\alpha \int_{-1}^{1} \zeta u(\zeta)\,\mathrm{d}\zeta \\
&\qquad + \int_{z}^{1}\Big[b(\zeta)\frac{\zeta^2}{2} + \int_{z}^{\zeta} b(\alpha)\,\mathrm{d}\alpha\cdot\zeta\Big]u(\zeta)\,\mathrm{d}\zeta
\end{aligned} \tag{4.11}$$

Proof. First note that by Eq. (3.6),

$$\begin{aligned}
T_{\mathrm{id}}k(u\partial_\varphi) &\overset{(3.6)}{=} -\mu(u\partial_\varphi) + \mu(u\partial_\varphi)(0,-1) - \int_{(0,-1)}^{(\varphi,z)} \iota_{u\partial_\varphi}\tau \\
&= \frac{z^2}{2}\,\mathrm{d}\varphi(u\partial_\varphi) - \Big(\frac{z^2}{2}\,\mathrm{d}\varphi(u\partial_\varphi)\Big)(0,-1) - \int_{(0,-1)}^{(\varphi,z)} \iota_{u\partial_\varphi}\zeta\,\mathrm{d}\varphi\wedge\mathrm{d}\zeta \\
&= \frac{z^2}{2}u(z) - \frac{1}{2}u(-1) - \int_{-1}^{z}\zeta u(\zeta)\,\mathrm{d}\zeta \qquad (4.12) \\
&= \int_{-1}^{z}\frac{\zeta^2}{2}u'(\zeta)\,\mathrm{d}\zeta \qquad (4.13)
\end{aligned}$$

by integration by parts. Then

$$\begin{aligned}
\int_{z}^{1} b(\zeta)\cdot T_{\mathrm{id}}k(u\partial_\varphi)\,\mathrm{d}\zeta &\overset{(4.12)}{=} \int_{z}^{1} b(\zeta)\Big[\frac{\zeta^2}{2}u(\zeta) - \frac{1}{2}u(-1) - \int_{-1}^{\zeta}\alpha u(\alpha)\,\mathrm{d}\alpha\Big]\mathrm{d}\zeta \\
&= \int_{z}^{1} b(\zeta)\frac{\zeta^2}{2}u(\zeta)\,\mathrm{d}\zeta - \frac{1}{2}u(-1)\int_{z}^{1} b(\zeta)\,\mathrm{d}\zeta - \int_{z}^{1} b(\zeta)\int_{-1}^{\zeta}\alpha u(\alpha)\,\mathrm{d}\alpha\,\mathrm{d}\zeta.
\end{aligned} \tag{4.14}$$

Integrating the last term by parts yields

$$\begin{aligned}
&-\int_z^1 b(\zeta)\int_{-1}^{\zeta}\alpha u(\alpha)\,\mathrm{d}\alpha\,\mathrm{d}\zeta = \\
&\quad = -\int_z^{\zeta} b(\alpha)\,\mathrm{d}\alpha\int_{-1}^{\zeta}\alpha u(\alpha)\,\mathrm{d}\alpha\,\Big|_{\zeta=z}^{1} + \int_z^1\int_z^{\zeta} b(\alpha)\,\mathrm{d}\alpha\cdot\zeta u(\zeta)\,\mathrm{d}\zeta \\
&\quad = -\int_z^1 b(\alpha)\,\mathrm{d}\alpha\int_{-1}^{1}\alpha u(\alpha)\,\mathrm{d}\alpha + \int_z^1\int_z^{\zeta} b(\alpha)\,\mathrm{d}\alpha\cdot\zeta u(\zeta)\,\mathrm{d}\zeta \\
&\quad = -\int_z^1 b(\alpha)\,\mathrm{d}\alpha\int_{-1}^{1}\zeta u(\zeta)\,\mathrm{d}\zeta + \int_z^1\int_z^{\zeta} b(\alpha)\,\mathrm{d}\alpha\cdot\zeta u(\zeta)\,\mathrm{d}\zeta. \qquad (4.15)
\end{aligned}$$

Plugging Eq. (4.15) back into Eq. (4.14) yield

$$\begin{aligned}
&\int_z^1 b(\zeta)\cdot T_{\mathrm{id}}k(u\partial_\varphi)\,\mathrm{d}\zeta = \\
&\quad = \int_z^1 b(\zeta)\frac{\zeta^2}{2}u(\zeta)\,\mathrm{d}\zeta - \frac{1}{2}u(-1)\int_z^1 b(\zeta)\,\mathrm{d}\zeta \\
&\qquad - \int_z^1 b(\alpha)\,\mathrm{d}\alpha\int_{-1}^{1}\zeta u(\zeta)\,\mathrm{d}\zeta + \int_z^1\int_z^{\zeta} b(\alpha)\,\mathrm{d}\alpha\cdot\zeta u(\zeta)\,\mathrm{d}\zeta \\
&\quad = -\frac{1}{2}u(-1)\int_z^1 b(\zeta)\,\mathrm{d}\zeta - \int_z^1 b(\alpha)\,\mathrm{d}\alpha\int_{-1}^{1}\zeta u(\zeta)\,\mathrm{d}\zeta \\
&\qquad + \int_z^1\Big[b(\zeta)\frac{\zeta^2}{2} + \int_z^{\zeta} b(\alpha)\,\mathrm{d}\alpha\cdot\zeta\Big]u(\zeta)\,\mathrm{d}\zeta. \qquad \square
\end{aligned}$$

In particular, for $b \equiv 1$, Eq. (4.11) yields

$$\begin{aligned}
\int_z^1 T_{\mathrm{id}}k(u\partial_\varphi)\,\mathrm{d}\zeta &= -\frac{1}{2}u(-1)\int_z^1 1\,\mathrm{d}\zeta - \int_z^1 1\,\mathrm{d}\alpha\int_{-1}^{1}\zeta u(\zeta)\,\mathrm{d}\zeta \\
&\qquad + \int_z^1\Big[1\cdot\frac{\zeta^2}{2} + \int_z^{\zeta} 1\,\mathrm{d}\alpha\cdot\zeta\Big]u(\zeta)\,\mathrm{d}\zeta \\
&= -\frac{1-z}{2}u(-1) - (1-z)\int_{-1}^{1}\zeta u(\zeta)\,\mathrm{d}\zeta \\
&\qquad + \int_z^1\Big[\frac{\zeta^2}{2} + (\zeta - z)\zeta\Big]u(\zeta)\,\mathrm{d}\zeta \\
&= -\frac{1-z}{2}u(-1) - \int_{-1}^{1}\zeta u(\zeta)\,\mathrm{d}\zeta + z\int_{-1}^{1}\zeta u(\zeta)\,\mathrm{d}\zeta \\
&\qquad + \int_z^1 \frac{3}{2}\zeta^2 u(\zeta)\,\mathrm{d}\zeta - z\int_z^1 \zeta u(\zeta)\,\mathrm{d}\zeta \\
&= -\frac{1-z}{2}u(-1) - \int_{-1}^{1}\zeta u(\zeta)\,\mathrm{d}\zeta + z\int_{-1}^{z}\zeta u(\zeta)\,\mathrm{d}\zeta \\
&\qquad + 3\int_z^1 \frac{\zeta^2}{2}u(\zeta)\,\mathrm{d}\zeta. \qquad (4.16)
\end{aligned}$$

We will also need Eq. (4.11) for $z = -1$:

$$
\begin{aligned}
&\int_{-1}^{1} b(\zeta)\cdot \mathrm{T}_{\mathrm{id}}k(u\partial_\varphi)\,\mathrm{d}\zeta = \\
&\quad = -\frac{1}{2}u(-1)\int_{-1}^{1} b(\zeta)\,\mathrm{d}\zeta - \int_{-1}^{1} b(\alpha)\,\mathrm{d}\alpha \int_{-1}^{1} \zeta u(\zeta)\,\mathrm{d}\zeta \\
&\qquad\qquad + \int_{-1}^{1}\Big[b(\zeta)\frac{\zeta^2}{2} + \int_{-1}^{\zeta} b(\alpha)\,\mathrm{d}\alpha\cdot\zeta\Big]u(\zeta)\,\mathrm{d}\zeta \\
&\quad = -\frac{1}{2}u(-1)\int_{-1}^{1} b(\zeta)\,\mathrm{d}\zeta \\
&\qquad + \int_{-1}^{1}\Big[b(\zeta)\frac{\zeta^2}{2} + \int_{-1}^{\zeta} b(\alpha)\,\mathrm{d}\alpha\cdot\zeta - \int_{-1}^{1} b(\alpha)\,\mathrm{d}\alpha\cdot\zeta\Big]u(\zeta)\,\mathrm{d}\zeta \\
&\quad = -\frac{1}{2}u(-1)\int_{-1}^{1} b(\zeta)\,\mathrm{d}\zeta + \int_{-1}^{1}\Big[b(\zeta)\frac{\zeta^2}{2} - \int_{\zeta}^{1} b(\alpha)\,\mathrm{d}\alpha\cdot\zeta\Big]u(\zeta)\,\mathrm{d}\zeta. \qquad (4.17)
\end{aligned}
$$

Again for $b \equiv 1$, this simplifies to

$$
\begin{aligned}
\int_{-1}^{1} \mathrm{T}_{\mathrm{id}}k(u\partial_\varphi)\,\mathrm{d}\zeta &= -\frac{1}{2}u(-1)\int_{-1}^{1} 1\,\mathrm{d}\zeta + \int_{-1}^{1}\Big[1\cdot\frac{\zeta^2}{2} - \int_{\zeta}^{1} 1\,\mathrm{d}\alpha\cdot\zeta\Big]u(\zeta)\,\mathrm{d}\zeta \\
&= -u(-1) + \int_{-1}^{1}\Big[\frac{\zeta^2}{2} - (1-\zeta)\zeta\Big]u(\zeta)\,\mathrm{d}\zeta \\
&= -u(-1) - \int_{-1}^{1}\Big(\zeta - \frac{3}{2}\zeta^2\Big)u(\zeta)\,\mathrm{d}\zeta. \qquad (4.18)
\end{aligned}
$$

Plugging Eq. (4.18) for $u = p^B_{\mathrm{id}}(V)$ into Eq. (4.10) yields

$$
\begin{aligned}
p^R_{\mathrm{id}}(V) &\overset{(4.10)}{=} \frac{1}{2}\int_{-1}^{1}\Big[\mu(\partial_\varphi)\int_{S^1} V^\varphi\,\mathrm{d}\varphi - \mu(\partial_\varphi)p^B_{\mathrm{id}}(V) \\
&\qquad\qquad + \int_{S^1} V^\theta\,\mathrm{d}\varphi - \mathrm{T}_{\mathrm{id}}k(p^B_{\mathrm{id}}(V)\partial_\varphi)\Big]\mathrm{d}z \\
&= \frac{1}{2}\int_{-1}^{1}\Big[\mu(\partial_\varphi)\int_{S^1} V^\varphi\,\mathrm{d}\varphi - \mu(\partial_\varphi)p^B_{\mathrm{id}}(V) \\
&\qquad + \int_{S^1} V^\theta\,\mathrm{d}\varphi\Big]\mathrm{d}z - \frac{1}{2}\int_{-1}^{1} \mathrm{T}_{\mathrm{id}}k(p^B_{\mathrm{id}}(V)\partial_\varphi)\,\mathrm{d}z \\
&\overset{(4.18)}{=} \frac{1}{2}\int_{-1}^{1}\Big[-\frac{z^2}{2}\int_{S^1} V^\varphi\,\mathrm{d}\varphi + \int_{S^1} V^\theta\,\mathrm{d}\varphi\Big]\mathrm{d}z \\
&\qquad\qquad + \frac{1}{2}\int_{-1}^{1} \frac{z^2}{2}p^B_{\mathrm{id}}(V)\,\mathrm{d}z \\
&\qquad\qquad\quad + \frac{1}{2}p^B_{\mathrm{id}}(V)(-1) \\
&\qquad\qquad\quad + \frac{1}{2}\int_{-1}^{1}\Big(z - \frac{3}{2}z^2\Big)p^B_{\mathrm{id}}(V)\,\mathrm{d}z
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{-1}^{1}\Big[-\frac{z^2}{2}\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi + \int_{S^1} V^{\theta}\,\mathrm{d}\varphi\Big]\mathrm{d}z \\
&\quad + \frac{1}{2}\int_{-1}^{1}(z-z^2)p^B_{\mathrm{id}}(V)\,\mathrm{d}z + \frac{1}{2}p^B_{\mathrm{id}}(V)(-1)
\end{aligned}
\tag{4.19}
$$

Similarly, all terms containing w in Eq. (4.9) are

$$
\begin{aligned}
0 = \int_{-1}^{1} w(z)\Big[&\Big(\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi - p^B_{\mathrm{id}}(V)\Big)\Big(1+\mu(\partial_\varphi)^2\Big) \\
&+ \Big(\int_{S^1} V^{\theta}\,\mathrm{d}\varphi - T_{\mathrm{id}}k(p^B_{\mathrm{id}}(V)\partial_\varphi) - p^R_{\mathrm{id}}(V)\Big)\mu(\partial_\varphi)\Big]\mathrm{d}z \\
&\quad + \int_{-1}^{1} T_{\mathrm{id}}k(w\partial_\varphi)\Big[\Big(\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi - p^B_{\mathrm{id}}(V)\Big)\mu(\partial_\varphi) \\
&\qquad + \int_{S^1} V^{\theta}\,\mathrm{d}\varphi - T_{\mathrm{id}}k(p^B_{\mathrm{id}}(V)\partial_\varphi) - p^R_{\mathrm{id}}(V)\Big]\mathrm{d}z.
\end{aligned}
\tag{4.20}
$$

For the second integral, we use Eq. (4.17) with $u = w$ and

$$
\begin{aligned}
b &= \Big(\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi - p^B_{\mathrm{id}}(V)\Big)\mu(\partial_\varphi) \\
&\qquad + \int_{S^1} V^{\theta}\,\mathrm{d}\varphi - T_{\mathrm{id}}k(p^B_{\mathrm{id}}(V)\partial_\varphi) - p^R_{\mathrm{id}}(V) \\
&= -\frac{z^2}{2}\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi + \int_{S^1} V^{\theta}\,\mathrm{d}\varphi \\
&\qquad + \frac{z^2}{2}p^B_{\mathrm{id}}(V) - T_{\mathrm{id}}k(p^B_{\mathrm{id}}(V)\partial_\varphi) - p^R_{\mathrm{id}}(V)
\end{aligned}
$$

to get

$$
\begin{aligned}
&\int_{-1}^{1} T_{\mathrm{id}}k(w\partial_\varphi)\Big[\Big(\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi - p^B_{\mathrm{id}}(V)\Big)\mu(\partial_\varphi) \\
&\quad + \int_{S^1} V^{\theta}\,\mathrm{d}\varphi - T_{\mathrm{id}}k(p^B_{\mathrm{id}}(V)\partial_\varphi) - p^R_{\mathrm{id}}(V)\Big]\mathrm{d}z = \\
&\overset{(4.17)}{=} -\frac{w(-1)}{2}\int_{-1}^{1}\Big[-\frac{z^2}{2}\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi + \int_{S^1} V^{\theta}\,\mathrm{d}\varphi \\
&\qquad + \frac{z^2}{2}p^B_{\mathrm{id}}(V) - T_{\mathrm{id}}k(p^B_{\mathrm{id}}(V)\partial_\varphi) - p^R_{\mathrm{id}}(V)\Big]\mathrm{d}z \\
&\qquad + \int_{-1}^{1} w(z)\Big[\Big(-\frac{z^2}{2}\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi + \int_{S^1} V^{\theta}\,\mathrm{d}\varphi \\
&\qquad + \frac{z^2}{2}p^B_{\mathrm{id}}(V) - T_{\mathrm{id}}k(p^B_{\mathrm{id}}(V)\partial_\varphi) - p^R_{\mathrm{id}}(V)\Big)\frac{z^2}{2} \\
&\qquad\quad - z\int_{z}^{1}\Big(-\frac{\zeta^2}{2}\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi + \int_{S^1} V^{\theta}\,\mathrm{d}\varphi \\
&\qquad\qquad + \frac{\zeta^2}{2}p^B_{\mathrm{id}}(V) - T_{\mathrm{id}}k(p^B_{\mathrm{id}}(V)\partial_\varphi) \\
&\qquad\qquad\qquad - p^R_{\mathrm{id}}(V)\Big)\mathrm{d}\zeta\Big]\mathrm{d}z.
\end{aligned}
$$

Note that the coefficient of $w(-1)$ vanishes because of the definition of $p^R_{\mathrm{id}}(V)$ in Eq. (4.10), hence we are left with

$$\begin{aligned}
&\int_{-1}^{1} T_{\mathrm{id}}k(w\partial_\varphi)\Big[\Big(\int_{S^1} V^\varphi \,\mathrm{d}\varphi - p^B_{\mathrm{id}}(V)\Big)\mu(\partial_\varphi) \\
&\qquad\qquad + \int_{S^1} V^\theta \,\mathrm{d}\varphi - T_{\mathrm{id}}k(p^B_{\mathrm{id}}(V)\partial_\varphi) - p^R_{\mathrm{id}}(V)\Big]\mathrm{d}z = \\
&\qquad\qquad\qquad = \int_{-1}^{1} w(z)\Big[\Big(-\frac{z^2}{2}\int_{S^1} V^\varphi \,\mathrm{d}\varphi + \int_{S^1} V^\theta \,\mathrm{d}\varphi \\
&\qquad\qquad\qquad\quad + \frac{z^2}{2}p^B_{\mathrm{id}}(V) - T_{\mathrm{id}}k(p^B_{\mathrm{id}}(V)\partial_\varphi) - p^R_{\mathrm{id}}(V)\Big)\frac{z^2}{2} \\
&\qquad\qquad\qquad\quad - z\int_{z}^{1}\Big(-\frac{\zeta^2}{2}\int_{S^1} V^\varphi \,\mathrm{d}\varphi + \int_{S^1} V^\theta \,\mathrm{d}\varphi \\
&\qquad\qquad\qquad\quad + \frac{\zeta^2}{2}p^B_{\mathrm{id}}(V) - T_{\mathrm{id}}k(p^B_{\mathrm{id}}(V)\partial_\varphi) \\
&\qquad\qquad\qquad\qquad - p^R_{\mathrm{id}}(V)\Big)\mathrm{d}\zeta\Big]\mathrm{d}z.
\end{aligned}$$

Going back to Eq. (4.20), we get

$$\begin{aligned}
0 = &\int_{-1}^{1} w(z)\Big[\Big(\int_{S^1} V^\varphi \,\mathrm{d}\varphi - p^B_{\mathrm{id}}(V)\Big)\Big(1 + \mu(\partial_\varphi)^2\Big) \\
&+ \Big(\int_{S^1} V^\theta \,\mathrm{d}\varphi - T_{\mathrm{id}}k(p^B_{\mathrm{id}}(V)\partial_\varphi) - p^R_{\mathrm{id}}(V)\Big)\mu(\partial_\varphi)\Big]\mathrm{d}z \\
&\quad + \int_{-1}^{1} w(z)\Big[\Big(-\frac{z^2}{2}\int_{S^1} V^\varphi \,\mathrm{d}\varphi + \int_{S^1} V^\theta \,\mathrm{d}\varphi \\
&\quad + \frac{z^2}{2}p^B_{\mathrm{id}}(V) - T_{\mathrm{id}}k(p^B_{\mathrm{id}}(V)\partial_\varphi) - p^R_{\mathrm{id}}(V)\Big)\frac{z^2}{2} \\
&\quad - z\int_{z}^{1}\Big(-\frac{\zeta^2}{2}\int_{S^1} V^\varphi \,\mathrm{d}\varphi + \int_{S^1} V^\theta \,\mathrm{d}\varphi \\
&\quad + \frac{\zeta^2}{2}p^B_{\mathrm{id}}(V) - T_{\mathrm{id}}k(p^B_{\mathrm{id}}(V)\partial_\varphi) - p^R_{\mathrm{id}}(V)\Big)\mathrm{d}\zeta\Big]\mathrm{d}z \\
= &\int_{-1}^{1} w(z)\Big[\Big(\int_{S^1} V^\varphi \,\mathrm{d}\varphi - p^B_{\mathrm{id}}(V)\Big)\Big(1 + \frac{z^4}{4}\Big) \\
&- \frac{z^2}{2}\int_{S^1} V^\theta \,\mathrm{d}\varphi + \frac{z^2}{2}T_{\mathrm{id}}k(p^B_{\mathrm{id}}(V)\partial_\varphi) + \frac{z^2}{2}p^R_{\mathrm{id}}(V)\Big]\mathrm{d}z \\
&\quad + \int_{-1}^{1} w(z)\Big[-\frac{z^4}{4}\int_{S^1} V^\varphi \,\mathrm{d}\varphi + \frac{z^2}{2}\int_{S^1} V^\theta \,\mathrm{d}\varphi \\
&\quad + \frac{z^4}{4}p^B_{\mathrm{id}}(V) - \frac{z^2}{2}T_{\mathrm{id}}k(p^B_{\mathrm{id}}(V)\partial_\varphi) - \frac{z^2}{2}p^R_{\mathrm{id}}(V) \\
&\quad - z\int_{z}^{1}\Big(-\frac{\zeta^2}{2}\int_{S^1} V^\varphi \,\mathrm{d}\varphi + \int_{S^1} V^\theta \,\mathrm{d}\varphi \\
&\quad + \frac{\zeta^2}{2}p^B_{\mathrm{id}}(V) - T_{\mathrm{id}}k(p^B_{\mathrm{id}}(V)\partial_\varphi) - p^R_{\mathrm{id}}(V)\Big)\mathrm{d}\zeta\Big]\mathrm{d}z
\end{aligned}$$

$$
\begin{aligned}
&= \int_{-1}^{1} w(z)\Big[\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi - p^B_{\mathrm{id}}(V) \\
&\qquad - z\int_z^1 \Big(-\frac{\zeta^2}{2}\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi + \int_{S^1} V^{\theta}\,\mathrm{d}\varphi \\
&\qquad + \frac{\zeta^2}{2}p^B_{\mathrm{id}}(V) - T_{\mathrm{id}}k(p^B_{\mathrm{id}}(V)\partial_{\varphi}) - p^R_{\mathrm{id}}(V)\Big)\,\mathrm{d}\zeta\Big]\,\mathrm{d}z.
\end{aligned}
$$

This expression has to vanish for every choice of w, hence the coefficient of w has to vanish. This yields

$$
\begin{aligned}
0 &= \int_{S^1} V^{\varphi}\,\mathrm{d}\varphi - p^B_{\mathrm{id}}(V) \\
&\qquad - z\int_z^1 \Big(-\frac{\zeta^2}{2}\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi + \int_{S^1} V^{\theta}\,\mathrm{d}\varphi \\
&\qquad + \frac{\zeta^2}{2}p^B_{\mathrm{id}}(V) - T_{\mathrm{id}}k(p^B_{\mathrm{id}}(V)\partial_{\varphi}) - p^R_{\mathrm{id}}(V)\Big)\,\mathrm{d}\zeta \\
&= \int_{S^1} V^{\varphi}\,\mathrm{d}\varphi - p^B_{\mathrm{id}}(V) - z\int_z^1 \Big[-\frac{\zeta^2}{2}\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi + \int_{S^1} V^{\theta}\,\mathrm{d}\varphi\Big]\,\mathrm{d}\zeta \\
&\qquad - z\int_z^1 \frac{\zeta^2}{2}p^B_{\mathrm{id}}(V)\,\mathrm{d}\zeta + z\int_z^1 T_{\mathrm{id}}k(p^B_{\mathrm{id}}(V)\partial_{\varphi})\,\mathrm{d}\zeta \\
&\qquad\qquad + z\int_z^1 p^R_{\mathrm{id}}(V)\,\mathrm{d}\zeta \\
&\overset{(4.16)}{=} \int_{S^1} V^{\varphi}\,\mathrm{d}\varphi - p^B_{\mathrm{id}}(V) - z\int_z^1 \Big[-\frac{\zeta^2}{2}\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi + \int_{S^1} V^{\theta}\,\mathrm{d}\varphi\Big]\,\mathrm{d}\zeta \\
&\qquad - z\int_z^1 \frac{\zeta^2}{2}p^B_{\mathrm{id}}(V)\Big)\,\mathrm{d}\zeta \\
&\qquad + z\Big[-\frac{1-z}{2}p^B_{\mathrm{id}}(V)(-1) - \int_{-1}^1 \zeta p^B_{\mathrm{id}}(V)\,\mathrm{d}\zeta \\
&\qquad + z\int_{-1}^z \zeta p^B_{\mathrm{id}}(V)\,\mathrm{d}\zeta + 3\int_z^1 \frac{\zeta^2}{2}p^B_{\mathrm{id}}(V)\,\mathrm{d}\zeta\Big] \\
&\qquad\qquad + z(1-z)p^R_{\mathrm{id}}(V) \\
&\overset{(4.19)}{=} \int_{S^1} V^{\varphi}\,\mathrm{d}\varphi - p^B_{\mathrm{id}}(V) - z\int_z^1 \Big[-\frac{\zeta^2}{2}\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi + \int_{S^1} V^{\theta}\,\mathrm{d}\varphi\Big]\,\mathrm{d}\zeta \\
&\qquad - z\int_z^1 \frac{\zeta^2}{2}p^B_{\mathrm{id}}(V)\Big)\,\mathrm{d}\zeta \\
&\qquad + z\Big[-\frac{1-z}{2}p^B_{\mathrm{id}}(V)(-1) - \int_{-1}^1 \zeta p^B_{\mathrm{id}}(V)\,\mathrm{d}\zeta \\
&\qquad + z\int_{-1}^z \zeta p^B_{\mathrm{id}}(V)\,\mathrm{d}\zeta + 3\int_z^1 \frac{\zeta^2}{2}p^B_{\mathrm{id}}(V)\,\mathrm{d}\zeta\Big] \\
&\qquad + z(1-z)\Big[\frac{1}{2}\int_{-1}^1 \Big[-\frac{\zeta^2}{2}\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi + \int_{S^1} V^{\theta}\,\mathrm{d}\varphi\Big]\,\mathrm{d}\zeta \\
&\qquad\qquad + \frac{1}{2}\int_{-1}^1 (\zeta-\zeta^2)p^B_{\mathrm{id}}(V)\,\mathrm{d}\zeta + \frac{1}{2}p^B_{\mathrm{id}}(V)(-1)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{S^1} V^{\varphi}\,\mathrm{d}\varphi - p^B_{\mathrm{id}}(V) \\
&\qquad -z\int_{-1}^{1}\Big[-\frac{\zeta^2}{2}\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi + \int_{S^1} V^{\theta}\,\mathrm{d}\varphi\Big]\mathrm{d}\zeta \\
&\qquad + z\int_{-1}^{z}\Big[-\frac{\zeta^2}{2}\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi + \int_{S^1} V^{\theta}\,\mathrm{d}\varphi\Big]\mathrm{d}\zeta \\
&\qquad\qquad\qquad -z\int_{z}^{1}\frac{\zeta^2}{2}p^B_{\mathrm{id}}(V)\Big)\,\mathrm{d}\zeta \\
&\qquad -\frac{z(1-z)}{2}p^B_{\mathrm{id}}(V)(-1) - z\int_{-1}^{1}\zeta p^B_{\mathrm{id}}(V)\,\mathrm{d}\zeta \\
&\qquad + z^2\int_{-1}^{z}\zeta p^B_{\mathrm{id}}(V)\,\mathrm{d}\zeta + 3z\int_{z}^{1}\frac{\zeta^2}{2}p^B_{\mathrm{id}}(V)\,\mathrm{d}\zeta \\
&\quad + z(1-z)\frac{1}{2}\int_{-1}^{1}\Big[-\frac{\zeta^2}{2}\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi + \int_{S^1} V^{\theta}\,\mathrm{d}\varphi\Big]\mathrm{d}\zeta \\
&\qquad\qquad + z(1-z)\frac{1}{2}\int_{-1}^{1}(\zeta-\zeta^2)p^B_{\mathrm{id}}(V)\,\mathrm{d}\zeta \\
&\qquad\qquad\qquad\qquad + z(1-z)\frac{1}{2}p^B_{\mathrm{id}}(V)(-1) \\
&= \int_{S^1} V^{\varphi}\,\mathrm{d}\varphi - p^B_{\mathrm{id}}(V) \\
&\quad + z(-1-z)\frac{1}{2}\int_{-1}^{1}\Big[-\frac{\zeta^2}{2}\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi + \int_{S^1} V^{\theta}\,\mathrm{d}\varphi\Big]\mathrm{d}\zeta \\
&\qquad + z\int_{-1}^{z}\Big[-\frac{\zeta^2}{2}\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi + \int_{S^1} V^{\theta}\,\mathrm{d}\varphi\Big]\mathrm{d}\zeta \\
&\qquad + z\Big[\int_{-1}^{1}\zeta^2 p^B_{\mathrm{id}}(V)\,\mathrm{d}\zeta - \int_{-1}^{z}\zeta^2 p^B_{\mathrm{id}}(V)\,\mathrm{d}\zeta\Big] \\
&\qquad - z\int_{-1}^{1}\zeta p^B_{\mathrm{id}}(V)\,\mathrm{d}\zeta + z^2\int_{-1}^{z}\zeta p^B_{\mathrm{id}}(V)\,\mathrm{d}\zeta \\
&\qquad\qquad + \frac{1}{2}(z-z^2)\int_{-1}^{1}\zeta p^B_{\mathrm{id}}(V)\,\mathrm{d}\zeta \\
&\qquad\qquad\qquad\qquad - \frac{1}{2}(z-z^2)\int_{-1}^{1}\zeta^2 p^B_{\mathrm{id}}(V)\,\mathrm{d}\zeta
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{S^1} V^\varphi \,\mathrm{d}\varphi - p^B_{\mathrm{id}}(V) \\
&\quad + (-z - z^2)\frac{1}{2}\int_{-1}^{1}\Big[-\frac{\zeta^2}{2}\int_{S^1} V^\varphi \,\mathrm{d}\varphi + \int_{S^1} V^\theta \,\mathrm{d}\varphi\Big]\mathrm{d}\zeta \\
&\qquad + z\int_{-1}^{z}\Big[-\frac{\zeta^2}{2}\int_{S^1} V^\varphi \,\mathrm{d}\varphi + \int_{S^1} V^\theta \,\mathrm{d}\varphi\Big]\mathrm{d}\zeta \\
&\qquad\qquad - z\int_{-1}^{z} \zeta^2 p^B_{\mathrm{id}}(V)\,\mathrm{d}\zeta \\
&\qquad\qquad + z^2\int_{-1}^{z} \zeta p^B_{\mathrm{id}}(V)\,\mathrm{d}\zeta \\
&\qquad\qquad + \frac{1}{2}(-z - z^2)\int_{-1}^{1} \zeta p^B_{\mathrm{id}}(V)\,\mathrm{d}\zeta \\
&\qquad\qquad\qquad - \frac{1}{2}(-z - z^2)\int_{-1}^{1} \zeta^2 p^B_{\mathrm{id}}(V)\,\mathrm{d}\zeta \\
&= \int_{S^1} V^\varphi \,\mathrm{d}\varphi - p^B_{\mathrm{id}}(V) \\
&\quad - (z + z^2)\frac{1}{2}\Big[\int_{-1}^{1}\Big[-\frac{\zeta^2}{2}\int_{S^1} V^\varphi \,\mathrm{d}\varphi + \int_{S^1} V^\theta \,\mathrm{d}\varphi\Big]\mathrm{d}\zeta \\
&\qquad + \int_{-1}^{1} \zeta p^B_{\mathrm{id}}(V)\,\mathrm{d}\zeta - \int_{-1}^{1} \zeta^2 p^B_{\mathrm{id}}(V)\,\mathrm{d}\zeta \\
&\qquad + z\int_{-1}^{z}\Big[-\frac{\zeta^2}{2}\int_{S^1} V^\varphi \,\mathrm{d}\varphi + \int_{S^1} V^\theta \,\mathrm{d}\varphi\Big]\mathrm{d}\zeta \\
&\qquad\quad - z\int_{-1}^{z} \zeta^2 p^B_{\mathrm{id}}(V)\,\mathrm{d}\zeta + z^2\int_{-1}^{z} \zeta p^B_{\mathrm{id}}(V)\,\mathrm{d}\zeta.
\end{aligned}
$$

This is equivalent to

$$
\begin{aligned}
p^B_{\mathrm{id}}(V) &+ \frac{1}{2}(z + z^2)\Big[\int_{-1}^{1} \zeta p^B_{\mathrm{id}}(V)\,\mathrm{d}\zeta - \int_{-1}^{1} \zeta^2 p^B_{\mathrm{id}}(V)\,\mathrm{d}\zeta\Big] \\
&+ z\int_{-1}^{z} \zeta^2 p^B_{\mathrm{id}}(V)\,\mathrm{d}\zeta - z^2\int_{-1}^{z} \zeta p^B_{\mathrm{id}}(V)\,\mathrm{d}\zeta \\
&= -\frac{1}{2}(z + z^2)\Big[\int_{-1}^{1}\Big[-\frac{\zeta^2}{2}\int_{S^1} V^\varphi \,\mathrm{d}\varphi + \int_{S^1} V^\theta \,\mathrm{d}\varphi\Big]\mathrm{d}\zeta\Big] \\
&+ z\int_{-1}^{z}\Big[-\frac{\zeta^2}{2}\int_{S^1} V^\varphi \,\mathrm{d}\varphi + \int_{S^1} V^\theta \,\mathrm{d}\varphi\Big]\mathrm{d}\zeta. \qquad (4.21)
\end{aligned}
$$

Define a linear operator $K : H^s([-1,1],\mathbb{R}) \to H^s([-1,1],\mathbb{R})$ by the green part of the previous equation, i. e.

$$
\begin{aligned}
K(u)(z) = \frac{1}{2}(z + z^2)&\Big[\int_{-1}^{1} \zeta u(\zeta)\,\mathrm{d}\zeta - \int_{-1}^{1} \zeta^2 u(\zeta)\,\mathrm{d}\zeta\Big] \\
&+ z\int_{-1}^{z} \zeta^2 u(\zeta)\,\mathrm{d}\zeta - z^2\int_{-1}^{z} \zeta u(\zeta)\,\mathrm{d}\zeta,
\end{aligned}
$$

so that Eq. (4.21) becomes

$$\begin{aligned}
&(\mathrm{id}+K)(p^B_{\mathrm{id}}(V))(z) = \\
&\quad = -\frac{1}{2}(z+z^2)\Big[\int_{-1}^{1}\big[-\frac{\zeta^2}{2}\int_{S^1} V^\varphi \,\mathrm{d}\varphi + \int_{S^1} V^\theta \,\mathrm{d}\varphi\big]\mathrm{d}\zeta\Big] \\
&\qquad\qquad + z\int_{-1}^{z}\big[-\frac{\zeta^2}{2}\int_{S^1} V^\varphi \,\mathrm{d}\varphi + \int_{S^1} V^\theta \,\mathrm{d}\varphi\big]\mathrm{d}\zeta. \qquad (4.22)
\end{aligned}$$

Lemma 4.18. *The operator* $\mathrm{id}+K : H^s([-1,1],\mathbb{R}) \to H^s([-1,1],\mathbb{R})$ *is injective.*

Proof. Since $\mathrm{id}+K$ is linear, this is equivalent to showing that $(\mathrm{id}+K)(u)(z) \equiv 0$ implies $u(z) \equiv 0$. To that end, we try to solve

$$\begin{aligned}
0 &= (\mathrm{id}+K)(u)(z) \\
&= u(z) + \frac{1}{2}(z+z^2)\Big[\int_{-1}^{1} \zeta u(\zeta)\,\mathrm{d}\zeta - \int_{-1}^{1} \zeta^2 u(\zeta)\,\mathrm{d}\zeta\Big] \\
&\qquad + z\int_{-1}^{z} \zeta^2 u(\zeta)\,\mathrm{d}\zeta - z^2\int_{-1}^{z} \zeta u(\zeta)\,\mathrm{d}\zeta. \qquad (4.23)
\end{aligned}$$

Note that this equation immediately implies

$$\begin{aligned}
0 &= u(-1), \\
0 &= u(0), \\
0 &= u(1) + \frac{1}{2}\cdot 2\Big[\int_{-1}^{1} \zeta u(\zeta)\,\mathrm{d}\zeta - \int_{-1}^{1} \zeta^2 u(\zeta)\,\mathrm{d}\zeta\Big] \\
&\qquad + \int_{-1}^{1} \zeta^2 u(\zeta)\,\mathrm{d}\zeta - \int_{-1}^{1} \zeta u(\zeta)\,\mathrm{d}\zeta \\
&= u(1).
\end{aligned}$$

Furthermore, for $z \neq 0$, Eq. (4.23) is equivalent to

$$\begin{aligned}
-\frac{u(z)}{z} &= \frac{1}{2}(1+z)\Big[\int_{-1}^{1} \zeta u(\zeta)\,\mathrm{d}\zeta - \int_{-1}^{1} \zeta^2 u(\zeta)\,\mathrm{d}\zeta\Big] \\
&\qquad + \int_{-1}^{z} \zeta^2 u(\zeta)\,\mathrm{d}\zeta - z\int_{-1}^{z} \zeta u(\zeta)\,\mathrm{d}\zeta.
\end{aligned}$$

Hence, let $w(z) := \dfrac{u(z)}{z}$ with initial conditions $w(1) = 0 = w(-1)$ and the previous equation can be written as

$$\begin{aligned}
-w(z) &= \frac{1}{2}(1+z)\Big[\int_{-1}^{1} \zeta^2 w(\zeta)\,\mathrm{d}\zeta - \int_{-1}^{1} \zeta^3 w(\zeta)\,\mathrm{d}\zeta\Big] \\
&\qquad + \int_{-1}^{z} \zeta^3 w(\zeta)\,\mathrm{d}\zeta - z\int_{-1}^{z} \zeta^2 w(\zeta)\,\mathrm{d}\zeta
\end{aligned}$$

with first derivative

$$\begin{aligned} -w'(z) = \frac{1}{2}\Big[\int_{-1}^{1} \zeta^2 w(\zeta)\,\mathrm{d}\zeta - \int_{-1}^{1} \zeta^3 w(\zeta)\,\mathrm{d}\zeta\Big] \\ +z^3 w(z) - \int_{-1}^{z} \zeta^2 w(\zeta)\,\mathrm{d}\zeta - z \cdot z^2 w(z) \\ = \frac{1}{2}\Big[\int_{-1}^{1} \zeta^2 w(\zeta)\,\mathrm{d}\zeta - \int_{-1}^{1} \zeta^3 w(\zeta)\,\mathrm{d}\zeta\Big] \\ - \int_{-1}^{z} \zeta^2 w(\zeta)\,\mathrm{d}\zeta, \end{aligned}$$

and second derivative

$$-w''(z) = -z^2 w(z).$$

The last equation is equivalent to

$$0 = w''(z) - z^2 w(z). \tag{4.24}$$

This is a special case of *Weber's equation* with general solution

$$w(z) = c_1 D_{-1/2}(\sqrt{2}z) + c_2 D_{-1/2}(\sqrt{2}iz)$$

for the *parabolic cylinder function* $D_{-1/2}(z)$. Using the intial conditions $w(1) = 0 = w(-1)$, we have

$$\begin{aligned} 0 &\overset{!}{=} w(1) = c_1 D_{-1/2}(\sqrt{2}) + c_2 D_{-1/2}(\sqrt{2}i) \\ 0 &\overset{!}{=} w(-1) = c_1 D_{-1/2}(-\sqrt{2}) + c_2 D_{-1/2}(-\sqrt{2}i). \end{aligned}$$

Since $D_{-1/2}(\sqrt{2}) + D_{-1/2}(-\sqrt{2}) = D_{-1/2}(\sqrt{2}i) + D_{-1/2}(-\sqrt{2}i) \in \mathbb{R}\backslash\{0\}$, adding those two equations yields

$$0 = c_1 + c_2$$

and the first equation can be written as

$$\begin{aligned} 0 = w(1) &= c_1 D_{-1/2}(\sqrt{2}) - c_1 D_{-1/2}(\sqrt{2}i) \\ &= c_1 \underbrace{\Big(D_{-1/2}(\sqrt{2}) - D_{-1/2}(\sqrt{2}i)\Big)}_{\neq 0}. \end{aligned}$$

This implies $c_2 = -c_1 = 0$. Therefore, the only possible solution is $w(z) \equiv 0$ and hence $u(z) \equiv 0$.

Instead of solving Eq. (4.24) explicitly, we can also multiply it with $w(z)$ and take the full integral

$$\begin{aligned} 0 &= \int_{-1}^{1} w''(z)w(z) - z^2 w^2(z)\,\mathrm{d}z \\ &= w'(z)w(z)\Big|_{z=-1}^{1} - \int_{-1}^{1} (w')^2(z)\,\mathrm{d}z - \int_{-1}^{1} z^2 w^2(z)\,\mathrm{d}z \\ &= -\int_{-1}^{1} (w')^2(z) + z^2 w^2(z)\,\mathrm{d}z, \end{aligned}$$

hence $(w')^2(z) + z^2 w^2(z) \equiv 0$, which implies $w(z) \equiv 0$ and $u(z) \equiv 0$. $\square$

At $\eta \in \mathrm{Diff}^s_{\omega,\lambda}(B \times S^1)$, we can consider the projection

$$\begin{aligned} P_\eta : T_\eta \mathrm{Diff}^s_R(B \times S^1) &\to T_\eta \mathrm{Diff}^s_{\omega,\lambda}(B \times S^1) \\ V &\mapsto P_\eta(V) = (TR_\eta \circ P_{\mathrm{id}} \circ TR_{\eta^{-1}})(V). \end{aligned}$$

If we write

$$V = V^\varphi(\partial_\varphi \circ \eta) + V^z(\partial_z \circ \eta) + V^\theta(\partial_\theta \circ \eta),$$

then

$$TR_{\eta^{-1}}(V) = (V^\varphi \circ \eta^{-1})\partial_\varphi + (V^z \circ \eta^{-1})\partial_z + (V^\theta \circ \eta^{-1})\partial_\theta$$

and

$$\begin{aligned} P_\eta(V) &= (TR_\eta \circ P_{\mathrm{id}} \circ TR_{\eta^{-1}})(V) \\ &= TR_\eta\Big(P_{\mathrm{id}}(TR_{\eta^{-1}}(V))\Big) \\ &= TR_\eta\Big(p^B_{\mathrm{id}}(TR_{\eta^{-1}}(V))\partial_\varphi + \\ &\qquad + \big(T_{\mathrm{id}}k(p^B_{\mathrm{id}}(TR_{\eta^{-1}}(V))\partial_\varphi) + p^R_{\mathrm{id}}(TR_{\eta^{-1}}(V))\big)\partial_\theta\Big) \\ &= p^B_{\mathrm{id}}(TR_{\eta^{-1}}(V)) \circ \eta \cdot (\partial_\varphi \circ \eta) + \\ &\qquad + \big(T_{\mathrm{id}}k(p^B_{\mathrm{id}}(TR_{\eta^{-1}}(V))\partial_\varphi) \circ \eta + p^R_{\mathrm{id}}(TR_{\eta^{-1}}(V)) \circ \eta\big)(\partial_\theta \circ \eta) \\ &= p^B_{\mathrm{id}}(TR_{\eta^{-1}}(V))(\partial_\varphi \circ \eta) + \\ &\qquad + \big(T_{\mathrm{id}}k(p^B_{\mathrm{id}}(TR_{\eta^{-1}}(V))\partial_\varphi) + p^R_{\mathrm{id}}(TR_{\eta^{-1}}(V))\big)(\partial_\theta \circ \eta). \end{aligned}$$

Lemma 4.19.

$$p^B_{\mathrm{id}}(V^\varphi\partial_\varphi + V^z\partial_z + V^\theta\partial_\theta) = p^B_{\mathrm{id}}(TR_{\eta^{-1}}(V)). \tag{4.25}$$

Proof. Note that

$$
\begin{aligned}
&(\mathrm{id}+K)(p^B_{\mathrm{id}}(TR_{\eta^{-1}}(V)))(z) = \\
&\overset{(4.22)}{=} -\frac{1}{2}(z+z^2)\Big[\int_{-1}^{1}\big[-\frac{\zeta^2}{2}\int_{S^1} V^\varphi\circ\eta^{-1}\,\mathrm{d}\varphi + \int_{S^1} V^\theta\circ\eta^{-1}\,\mathrm{d}\varphi\big]\mathrm{d}\zeta\Big] \\
&\qquad + z\int_{-1}^{z}\big[-\frac{\zeta^2}{2}\int_{S^1} V^\varphi\circ\eta^{-1}\,\mathrm{d}\varphi + \int_{S^1} V^\theta\circ\eta^{-1}\,\mathrm{d}\varphi\big]\mathrm{d}\zeta \\
&= -\frac{1}{2}(z+z^2)\Big[\int_{-1}^{1}\big[-\frac{\zeta^2}{2}\int_{S^1} V^\varphi\,\mathrm{d}\varphi + \int_{S^1} V^\theta\,\mathrm{d}\varphi\big]\mathrm{d}\zeta\Big] \\
&\qquad + z\int_{-1}^{z}\big[-\frac{\zeta^2}{2}\int_{S^1} V^\varphi\,\mathrm{d}\varphi + \int_{S^1} V^\theta\,\mathrm{d}\varphi\big]\mathrm{d}\zeta \\
&\overset{(4.22)}{=} (\mathrm{id}+K)(p^B_{\mathrm{id}}(V^\varphi\partial_\varphi + V^z\partial_z + V^\theta\partial_\theta))(z).
\end{aligned}
$$

Since $\mathrm{id}+K$ is injective (Lemma 4.18), this implies the statement of the lemma. □

Lemma 4.20.

$$
p^R_{\mathrm{id}}(V^\varphi\partial_\varphi + V^z\partial_z + V^\theta\partial_\theta) = p^R_{\mathrm{id}}(TR_{\eta^{-1}}(V)). \tag{4.26}
$$

Proof.

$$
\begin{aligned}
p^R_{\mathrm{id}}(TR_{\eta^{-1}}(V)) &\overset{(4.19)}{=} \frac{1}{2}\int_{-1}^{1}\big[-\frac{z^2}{2}\int_{S^1} V^\varphi\circ\eta^{-1}\,\mathrm{d}\varphi + \int_{S^1} V^\theta\circ\eta^{-1}\,\mathrm{d}\varphi\big]\mathrm{d}z \\
&\quad + \frac{1}{2}\int_{-1}^{1}(z-z^2)p^B_{\mathrm{id}}(V^\varphi\circ\eta^{-1}, V^z\circ\eta^{-1}, V^\theta\circ\eta^{-1})\,\mathrm{d}z \\
&\quad + \frac{1}{2}p^B_{\mathrm{id}}(V^\varphi\circ\eta^{-1}, V^z\circ\eta^{-1}, V^\theta\circ\eta^{-1})(-1) \\
&= \frac{1}{2}\int_{-1}^{1}\big[-\frac{z^2}{2}\int_{S^1} V^\varphi\,\mathrm{d}\varphi + \int_{S^1} V^\theta\,\mathrm{d}\varphi\big]\mathrm{d}z \\
&\quad + \frac{1}{2}\int_{-1}^{1}(z-z^2)p^B_{\mathrm{id}}(V^\varphi\partial_\varphi + V^z\partial_z + V^\theta\partial_\theta)\,\mathrm{d}z \\
&\quad + \frac{1}{2}p^B_{\mathrm{id}}(V^\varphi\partial_\varphi + V^z\partial_z + V^\theta\partial_\theta)(-1) \\
&= p^R_{\mathrm{id}}(V^\varphi\partial_\varphi + V^z\partial_z + V^\theta\partial_\theta)
\end{aligned}
$$

□

Theorem 4.21. *The projection* $P : T\mathrm{Diff}^s_R(B\times S^1)|_{\mathrm{Diff}^s_{\omega,\lambda}(B\times S^1)} \to T\mathrm{Diff}^s_{\omega,\lambda}(B\times S^1)$ *is a smooth bundle map.*

Proof. Around any $\eta\in\mathrm{Diff}^s_R(S^1\times[-1,1])$, P takes the form

$$
\begin{aligned}
&T_\eta\mathrm{Diff}^s_{\omega,\lambda}(S^1\times[-1,1])\times T_\eta\mathrm{Diff}^s_R(S^1\times[-1,1]) \\
&\qquad\to T_\eta\mathrm{Diff}^s_{\omega,\lambda}(S^1\times[-1,1])\times T_\eta\mathrm{Diff}^s_R(S^1\times[-1,1]) \\
&\qquad (X,Y)\mapsto(\Phi^{-1}\circ P\circ\Phi)(X,Y).
\end{aligned}
$$

and we get for $Y = Y^\varphi \partial_\varphi \circ \eta + Y^z \partial_z \circ \eta + Y^\theta \partial_\theta \circ \eta$ that

$$\begin{aligned}
(\Phi^{-1} \circ P \circ \Phi)(X,Y) &= \Phi^{-1}(P(\Phi(X,Y))) \\
&= \Phi^{-1}\Big(P\big(\eta + X,\ Y^\varphi \partial_\varphi \circ (\eta + X) + Y^z \partial_z \circ (\eta + X) + Y^\theta \partial_\theta \circ (\eta + X)\big)\Big) \\
&= \Phi^{-1}\big(P_{\eta+X}(Y^\varphi \partial_\varphi \circ (\eta + X) + Y^z \partial_z \circ (\eta + X) + Y^\theta \partial_\theta \circ (\eta + X))\big) \\
&= \Phi^{-1}\Bigg(p^B_{\mathrm{id}}\Big(TR_{(\eta+X)^{-1}}\big(Y^\varphi \partial_\varphi \circ (\eta + X) + Y^z \partial_z \circ (\eta + X) \\
&\qquad\qquad + Y^\theta \partial_\theta \circ (\eta + X)\big)\Big)\partial_\varphi \circ (\eta + X) \\
&\quad + \Big(T_{\mathrm{id}}k\Big(p^B_{\mathrm{id}}\big(TR_{(\eta+X)^{-1}}\big(Y^\varphi \partial_\varphi \circ (\eta + X) + Y^z \partial_z \circ (\eta + X) \\
&\qquad\qquad + Y^\theta \partial_\theta \circ (\eta + X)\big)\big)\partial_\varphi\Big) \\
&\quad + p^R_{\mathrm{id}}\Big(TR_{(\eta+X)^{-1}}\big(Y^\varphi \partial_\varphi \circ (\eta + X) + Y^z \partial_z \circ (\eta + X) \\
&\qquad\qquad + Y^\theta \partial_\theta \circ (\eta + X)\big)\Big)\Big)\partial_\theta \circ (\eta + X)\Bigg)
\end{aligned}$$

Since

$$\begin{aligned}
&p^B_{\mathrm{id}}\big(TR_{(\eta+X)^{-1}}(Y^\varphi \partial_\varphi \circ (\eta + X) + Y^z \partial_z \circ (\eta + X) + Y^\theta \partial_\theta \circ (\eta + X))\big) = \\
&\overset{(4.25)}{=} p^B_{\mathrm{id}}\big(Y^\varphi \partial_\varphi + Y^z \partial_z + Y^\theta \partial_\theta\big) \\
&\overset{(4.25)}{=} p^B_{\mathrm{id}}(TR_{\eta^{-1}} Y)
\end{aligned}$$

and similarly

$$\begin{aligned}
&p^R_{\mathrm{id}}\big(TR_{(\eta+X)^{-1}}(Y^\varphi \partial_\varphi \circ (\eta + X) + Y^z \partial_z \circ (\eta + X) + Y^\theta \partial_\theta \circ (\eta + X))\big) = \\
&\overset{(4.26)}{=} p^R_{\mathrm{id}}\big(Y^\varphi \partial_\varphi + Y^z \partial_z + Y^\theta \partial_\theta\big) \\
&\overset{(4.26)}{=} p^R_{\mathrm{id}}(TR_{\eta^{-1}} Y),
\end{aligned}$$

we get

$$\begin{aligned}
&(\Phi^{-1} \circ P \circ \Phi)(X,Y) = \\
&\qquad = \Phi^{-1}\Bigg(p^B_{\mathrm{id}}(TR_{\eta^{-1}} Y)\partial_\varphi \circ (\eta + X) \\
&\qquad\quad + \Big(T_{\mathrm{id}}k\big(p^B_{\mathrm{id}}(TR_{\eta^{-1}} Y)\partial_\varphi\big) + p^R_{\mathrm{id}}(TR_{\eta^{-1}} Y)\Big)\partial_\theta \circ (\eta + X)\Bigg) \\
&\qquad = \Big(X,\ p^B_{\mathrm{id}}(TR_{\eta^{-1}} Y)\partial_\varphi \circ \eta \\
&\qquad\quad + \big(T_{\mathrm{id}}k(p^B_{\mathrm{id}}(TR_{\eta^{-1}} Y)) + p^R_{\mathrm{id}}(TR_{\eta^{-1}} Y)\big)\partial_\theta \circ \eta\Big).
\end{aligned}$$

Since the first component of this map is the identity and the second component is independent of X, this map is smooth in the base point X. Hence, P is a smooth bundle map. $\square$

4.4 Euler equation on $\mathrm{Diff}^s_{\omega,\lambda}(M)$

Recall Eq. (4.8):

$$\begin{aligned} T_{\mathrm{id}}\mathrm{Diff}^s_{\sigma,\tau}(B) \times T_0 S^1 &\xrightarrow{\cong} T_{\mathrm{id}}\mathrm{Diff}^s_{\omega,\lambda}(B \times S^1) \\ \big(v,\, c\partial_\kappa\big) &\mapsto V_t + \big(T_{\mathrm{id}}k(v) + c\big)\partial_\theta. \end{aligned} \tag{4.8 rev.}$$

We already know that $v = v(z)\partial_\varphi$ and $T_{\mathrm{id}}k$ is of the form

$$T_{\mathrm{id}}k(v(z)\partial_\varphi) = \int_{-1}^{z} \frac{\zeta^2}{2} v'(\zeta)\,\mathrm{d}\zeta. \tag{4.13 rev.}$$

Hence, we can write any $V \in T_{\mathrm{id}}\mathrm{Diff}^s_{\omega,\lambda}(B \times S^1)$ as

$$V = v(z)\partial_\varphi + \big(T_{\mathrm{id}}k(v) + c\big)\partial_\theta.$$

Recall the result of the variation of energy in Section 2.3: Let $V_t \in T_{\mathrm{id}}\mathrm{Diff}^s_{\omega,\lambda}(B \times S^1)$ be a time-dependent vector field, i. e. V_t is of the form $V_t = v_t(z)\partial_\varphi + (T_{\mathrm{id}}k(v_t) + c_t)\partial_\theta$. If

$$0 = \int_0^T \int_M \langle W_t, \dot{V}_t + \nabla_{V_t} V_t \rangle \operatorname{vol} \mathrm{d}t \tag{2.9 rev.}$$

for any time-dependent $W_t = w_t(z)\partial_\varphi + (T_{\mathrm{id}}k(w_t) + d_t)\partial_\theta \in T_{\mathrm{id}}\mathrm{Diff}^s_{\omega,\lambda}(B \times S^1)$, then V_t is a solution to the Euler equation. We first compute

$$\begin{aligned} \langle \partial_\varphi, \partial_\theta \rangle &= \underbrace{\underbrace{\langle \partial_\varphi - \mu(\partial_\varphi)\partial_\theta, \partial_\theta \rangle}_{\in \ker \lambda}}_{=0} + \underbrace{\langle \mu(\partial_\varphi)\partial_\theta, \partial_\theta \rangle}_{=\mu(\partial_\varphi)} \\ &= \mu(\partial_\varphi), \\ \langle \partial_\varphi, \partial_\varphi \rangle &= \langle \partial_\varphi - \mu(\partial_\varphi)\partial_\theta, \partial_\varphi - \mu(\partial_\varphi)\partial_\theta \rangle + 2\mu(\partial_\varphi)\langle \partial_\varphi, \partial_\theta \rangle - \mu(\partial_\varphi)^2 \langle \partial_\theta, \partial_\theta \rangle \\ &= \langle \partial_\varphi, \partial_\varphi \rangle^B + 2\mu(\partial_\varphi)\mu(\partial_\varphi) - \mu(\partial_\varphi)^2 \\ &= \langle \partial_\varphi, \partial_\varphi \rangle^B + \mu(\partial_\varphi)^2. \end{aligned}$$

In particular, $\langle\partial_\varphi,\partial_\varphi\rangle$ is independent of θ. We use these computations for

$$\begin{aligned}
\nabla_{V_t}V_t &= \nabla_{v_t(z)\partial_\varphi+(T_{\mathrm{id}}k(v_t)+c_t)\partial_\theta}\big(v_t(z)\partial_\varphi+(T_{\mathrm{id}}k(v_t)+c_t)\partial_\theta\big)\\
&= v_t(z)\nabla_{\partial_\varphi}\big(v_t(z)\partial_\varphi+(T_{\mathrm{id}}k(v_t)+c_t)\partial_\theta\big)\\
&\qquad+(T_{\mathrm{id}}k(v_t)+c_t)\nabla_{\partial_\theta}\big(v_t(z)\partial_\varphi+(T_{\mathrm{id}}k(v_t)+c_t)\partial_\theta\big)\\
&= v_t(z)\big(v_t(z)\nabla_{\partial_\varphi}\partial_\varphi+(T_{\mathrm{id}}k(v_t)+c_t)\nabla_{\partial_\varphi}\partial_\theta\big)\\
&\qquad+(T_{\mathrm{id}}k(v_t)+c_t)\big(v_t(z)\nabla_{\partial_\theta}\partial_\varphi+(T_{\mathrm{id}}k(v_t)+c_t)\nabla_{\partial_\theta}\partial_\theta\big)\\
&= v_t^2(z)\nabla_{\partial_\varphi}\partial_\varphi+v_t(z)(T_{\mathrm{id}}k(v_t)+c_t)\nabla_{\partial_\varphi}\partial_\theta\\
&\qquad(T_{\mathrm{id}}k(v_t)+c_t)v_t(z)\nabla_{\partial_\theta}\partial_\varphi+(T_{\mathrm{id}}k(v_t)+c_t)^2\nabla_{\partial_\theta}\partial_\theta.
\end{aligned}$$

Pairing the covariant derivatives with ∂_φ and ∂_θ yields

$$\begin{aligned}
2\langle\nabla_{\partial_\varphi}\partial_\varphi,\partial_\varphi\rangle &= \partial_\varphi\langle\partial_\varphi,\partial_\varphi\rangle,\\
&= \partial_\varphi\big(\langle\partial_\varphi,\partial_\varphi\rangle^B+\mu(\partial_\varphi)^2\big)\\
&= \partial_\varphi\big(1+\frac{z^4}{4}\big)\\
&= 0, && (4.27)\\
2\langle\nabla_{\partial_\varphi}\partial_\theta,\partial_\varphi\rangle &= \partial_\theta\langle\partial_\varphi,\partial_\varphi\rangle = 0,\\
2\langle\nabla_{\partial_\theta}\partial_\varphi,\partial_\varphi\rangle &= \partial_\theta\langle\partial_\varphi,\partial_\varphi\rangle = 0,\\
2\langle\nabla_{\partial_\theta}\partial_\theta,\partial_\varphi\rangle &= \underbrace{2\,\partial_\theta\underbrace{\langle\partial_\theta,\partial_\varphi\rangle}_{=\mu(\partial_\varphi)}}_{=0}-\underbrace{\partial_\varphi\underbrace{\langle\partial_\theta,\partial_\theta\rangle}_{\equiv 1}}_{=0}=0,\\
2\langle\nabla_{\partial_\varphi}\partial_\varphi,\partial_\theta\rangle &= 2\partial_\varphi\langle\partial_\theta,\partial_\varphi\rangle-\partial_\theta\langle\partial_\varphi,\partial_\varphi\rangle\\
&= 2\partial_\varphi\underbrace{\mu(\partial_\varphi)}_{=-\frac{z^2}{2}}=0, && (4.28)\\
2\langle\nabla_{\partial_\varphi}\partial_\theta,\partial_\theta\rangle &= \partial_\varphi\langle\partial_\theta,\partial_\theta\rangle = 0,\\
2\langle\nabla_{\partial_\theta}\partial_\varphi,\partial_\theta\rangle &= \partial_\varphi\langle\partial_\theta,\partial_\theta\rangle = 0,\\
2\langle\nabla_{\partial_\theta}\partial_\theta,\partial_\theta\rangle &= \partial_\theta\langle\partial_\theta,\partial_\theta\rangle = 0.
\end{aligned}$$

Note that all these computations – except for Eqs. (4.27) and (4.28) – do not rely on the specific form of μ or the chosen metric on B, but just on the fact that $\langle\partial_\varphi,\partial_\varphi\rangle^B$ and $\mu(\partial_\varphi)$ are functions on B and do not depend on θ. Then

$$\langle W_t,\nabla_{V_t}V_t\rangle = 0$$

and the full equation is

$$0=\int_0^T\int_{B\times S^1}\langle W_t,\dot V_t\rangle\lambda\wedge\omega\,\mathrm{d}t.$$

In particular, for $W_t = \dot{V}_t$, we get

$$0 = \int_0^T \int_{B\times S^1} \langle \dot{V}_t, \dot{V}_t \rangle \lambda \wedge \omega \, \mathrm{d}t,$$

hence

$$\begin{aligned} 0 &= \dot{V}_t \\ &= \dot{v}_t(z)\partial_\varphi + \Big(T_{\mathrm{id}} k(\dot{v}_t(z)) + \dot{c}_t\Big)\partial_\theta. \end{aligned}$$

This implies $\dot{v}_t = 0$ (as the coefficient of ∂_φ) and then also $\dot{c}_t = 0$.

Proposition 4.22. *The previous computation shows that the only solutions to the Euler equation on $M = B \times S^1$ preserving ω and λ are all stationary vector fields of the form $V_t = V = v(z)\partial_\varphi + \Big(T_{\mathrm{id}} k(v(z)\partial_\varphi) + c\Big)\partial_\theta$.* □

4.5 $B = S^1 \times [-1,1]$, general metric

In the following sections, we will generalize the situation to an arbitrary Riemannian metric $\langle\cdot,\cdot\rangle$ on $B = S^1 \times [-1,1]$. The Riemannian area form is then given by $\sigma_b := b(\varphi,z)\sigma$ for some smooth map $b \in C^\infty(B,\mathbb{R})$, which is nowhere 0. We will still let $\tau_b := h\sigma_b = zb(\varphi,z)\sigma$.

Proposition 4.23. *Let $\langle\cdot,\cdot\rangle$ be a Riemannian metric on B with Riemannian area form $\sigma_b = b(\varphi,z)\,\mathrm{d}\varphi \wedge \mathrm{d}z$. There is a diffeomorphism ρ of B such that ρ preserves z and the Riemannian area form $\rho^*\sigma_b$ of the pullback metric satisfies $\rho^*\sigma_b = a(z)\sigma =: \sigma_a$ for some smooth function $a \in C^\infty([-1,1],\mathbb{R})$ that only depends on z.*

Proof. We first lift $b : S^1 \times [-1,1] \to \mathbb{R}$ to a smooth function $b_\mathbb{R} : \mathbb{R} \times [-1,1] \to \mathbb{R}$ satisfying $b_\mathbb{R}(x+1,z) = b_\mathbb{R}(x,z)$ and define a smooth map $B : \mathbb{R} \times [-1,1] \to \mathbb{R}$,

$$B(x,z) := \int_0^x b_\mathbb{R}(y,z)\,\mathrm{d}y.$$

This map satisfies

$$\begin{aligned} B(x+1,z) &= \int_0^{x+1} b_\mathbb{R}(y,z)\,\mathrm{d}y \\ &= \underbrace{\int_0^1 b_\mathbb{R}(y,z)\,\mathrm{d}y}_{=:a(z)} + \int_1^{x+1} b_\mathbb{R}(y,z)\,\mathrm{d}y \\ &= a(z) + \int_0^x b_\mathbb{R}(y+1,z)\,\mathrm{d}y \\ &= a(z) + \int_0^x b_\mathbb{R}(y,z)\,\mathrm{d}y \\ &= a(z) + B(x,z). \end{aligned}$$

For fixed $z \in [-1,1]$, let $B_z : \mathbb{R} \to \mathbb{R}, x \mapsto B(x,z)$. Since

$$\frac{\mathrm{d}B_z}{\mathrm{d}x}(x) = \frac{\partial B}{\partial x}(x,z) = b_{\mathbb{R}}(x,z) \neq 0,$$

B_z is an isomorphism with inverse B_z^{-1}. We define a diffeomorphism $\rho_{\mathbb{R}}$ by

$$\begin{aligned}\rho_{\mathbb{R}} : \mathbb{R} \times [-1,1] &\to \mathbb{R} \times [-1,1] \\ (x,z) &\mapsto \big(B_z^{-1}(a(z)\cdot x), z\big).\end{aligned}$$

Then the first component of $\rho_{\mathbb{R}}$ satisfies

$$\begin{aligned}\rho_{\mathbb{R}}^1(x+1,z) &= B_z^{-1}(a(z)\cdot(x+1)) \\ &= B_z^{-1}(a(z)\cdot x + a(z)) \\ &= B_z^{-1}(a(z)\cdot x) + 1 \\ &= \rho_{\mathbb{R}}^1(x,z) + 1,\end{aligned}$$

hence $\rho_{\mathbb{R}}$ descends to a diffeomorphism ρ of the cylinder $S^1 \times [-1,1]$, defined by

$$\begin{aligned}\rho : S^1 \times [-1,1] &\to S^1 \times [-1,1] \\ (\varphi,z) &\mapsto \big(B_z^{-1}(a(z)\cdot x) \mod 1,\ z\big)\end{aligned}$$

for any representative $x \in \mathbb{R}$ of $\varphi \in S^1 \cong \mathbb{R}/\mathbb{Z}$. Then ρ preserves z and for any representative $x \in \mathbb{R}$ of $\varphi \in S^1 \cong \mathbb{R}/\mathbb{Z}$, we have that

$$\begin{aligned}\rho^*\sigma_b &= (\rho^* b)\rho^*(\mathrm{d}\varphi \wedge \mathrm{d}z) \\ &= (b\circ\rho)(\varphi,z)\,\mathrm{d}\rho^1 \wedge \mathrm{d}\rho^2 \\ &= (b\circ\rho)(\varphi,z)\frac{\partial\rho^1}{\partial\varphi}(\varphi,z)\,\underbrace{\mathrm{d}\varphi\wedge\mathrm{d}z}_{=\sigma} \\ &= b_{\mathbb{R}}(\rho_{\mathbb{R}}(x,z))\frac{\partial\rho_{\mathbb{R}}^1}{\partial x}(x,z)\cdot\sigma \\ &= \frac{\partial B}{\partial x}(\rho_{\mathbb{R}}(x,z))\frac{\partial\rho_{\mathbb{R}}^1}{\partial x}(x,z)\cdot\sigma \\ &= \frac{\mathrm{d}}{\mathrm{d}x}B(\rho_{\mathbb{R}}^1(x,z),z)\cdot\sigma \\ &= \frac{\mathrm{d}}{\mathrm{d}x}B_z(B_z^{-1}(a(z)\cdot x))\cdot\sigma \\ &= \frac{\mathrm{d}}{\mathrm{d}x}a(z)\cdot x\cdot\sigma \\ &= a(z)\cdot\sigma\end{aligned}$$

is independent of φ. □

Using this proposition and Lemma 3.30, we can w. l. o. g. assume that we have a Riemannian metric on $S^1 \times [-1,1]$ such that the Riemannian area form is of the form $\sigma_a = a(z)\sigma$, and $\tau_a = za(z)\sigma$.

Proposition 4.24.

$$\mathrm{Diff}^s_{\sigma_a,\tau_a}(B) = \mathrm{Diff}^s_{\sigma,\tau}(B)$$

Proof. First note that

$$\begin{aligned}\mathrm{Diff}^s_{\sigma_a,\tau_a}(B) &= \mathrm{Diff}^s_{\sigma_a,h}(B)\\ &= \left\{\nu \in \mathrm{Diff}^s_h(B) \,\middle|\, \nu^*\sigma_a = \sigma_a\right\}.\end{aligned}$$

We let $\nu \in \mathrm{Diff}^s_h(S^1 \times [-1,1])$, i. e. $\nu(\varphi,z) = (\nu^1(\varphi,z),z)$ and analyze the condition $\nu^*\sigma_a = \sigma_a$:

$$\begin{aligned}a\,\mathrm{d}\varphi \wedge \mathrm{dz} = \sigma_a &\overset{!}{=} \nu^*\sigma_a\\ &= \nu^*(a\,\mathrm{d}\varphi \wedge \mathrm{dz})\\ &= \underbrace{a \circ \nu}_{=a \text{ since } a \text{ only depends on } z}\, \mathrm{d}\nu^1 \wedge \mathrm{dz}\\ &= a\frac{\partial \nu^1}{\partial \varphi}\mathrm{d}\varphi \wedge \mathrm{dz}.\end{aligned}$$

This is equivalent to $\dfrac{\partial \nu^1}{\partial \varphi} \equiv 1$. Hence,

$$\begin{aligned}\mathrm{Diff}^s_{\sigma_a,\tau_a}(B) &= \left\{\nu \in \mathrm{Diff}^s_h(B) \,\middle|\, \frac{\partial \nu^1}{\partial \varphi} = 1\right\}\\ &= \mathrm{Diff}^s_{\sigma,\tau}(B)\end{aligned}$$

with the last identity being shown in the proof of Proposition 4.3. □

Corollary 4.4 then shows that $\mathrm{Diff}^s_{\sigma_a,\tau_a}(B) = \mathrm{Diff}^s_{\sigma,h}(B)$ is a smooth submanifold of $\mathrm{Diff}^s(B)$.

In the second part of this section, we have to show that the orthogonal projections in each fibre form a smooth bundle map

$$P : T\mathrm{Diff}^s(S^1 \times [-1,1])|_{\mathrm{Diff}^s_{\sigma_a,h}(S^1\times[-1,1])} \to T\mathrm{Diff}^s_{\sigma_a,h}(S^1 \times [-1,1]).$$

Again, we split the map in two projections $P = P^2 \circ P^1$ for

$$P^1 : T\mathrm{Diff}^s(S^1 \times [-1,1])|_{\mathrm{Diff}^s_h(S^1\times[-1,1])} \to T\mathrm{Diff}^s_h(S^1 \times [-1,1])$$

and

$$P^2 : T\mathrm{Diff}^s_h(S^1 \times [-1,1])|_{\mathrm{Diff}^s_{\sigma_a,h}(S^1\times[-1,1])} \to T\mathrm{Diff}^s_{\sigma_a,h}(S^1 \times [-1,1]).$$

We first compute P^1_{id}. Let $X = X^1\partial_\varphi + X^2\partial_z \in T_{\mathrm{id}}\mathrm{Diff}^s(S^1\times[-1,1])$. Then we must have

$$P^1_{\mathrm{id}}(X) = p^1_{\mathrm{id}}(X)\partial_\varphi$$

for some operator p^1_{id} such that $p^1_{\mathrm{id}}(X) \in H^s(S^1\times[-1,1],\mathbb{R})$. For any $Y = Y^1\partial_\varphi \in T_{\mathrm{id}}\mathrm{Diff}^s_h(S^1\times[-1,1])$, P^1_{id} has to satisfy

$$\begin{aligned}
0 &\overset{!}{=} \left(P^1_{\mathrm{id}}(X) - X,\ Y\right) \\
&= \int_{S^1\times[-1,1]} \langle P^1_{\mathrm{id}}(X) - X,\ Y\rangle_{(\varphi,z)}\,\sigma_a \\
&= \int_{S^1\times[-1,1]} \langle p^1_{\mathrm{id}}(X)\partial_\varphi - X^1\partial_\varphi - X^2\partial_z,\ Y^1\partial_\varphi\rangle_{(\varphi,z)}\,a(z)\,\mathrm{d}\varphi\wedge\mathrm{d}z \\
&= \int_{S^1\times[-1,1]} Y^1\Big(\big(p^1_{\mathrm{id}}(X) - X^1\big)\langle\partial_\varphi,\partial_\varphi\rangle - X^2\langle\partial_z,\partial_\varphi\rangle\Big)a(z)\,\mathrm{d}\varphi\wedge\mathrm{d}z \\
\Rightarrow\quad & p^1_{\mathrm{id}}(X) = X^1 + X^2\frac{\langle\partial_z,\partial_\varphi\rangle}{\langle\partial_\varphi,\partial_\varphi\rangle}.
\end{aligned}$$

Hence,

$$\begin{aligned}
P^1_{\mathrm{id}} : T_{\mathrm{id}}\mathrm{Diff}^s(S^1\times[-1,1])|_{\mathrm{Diff}^s_h(S^1\times[-1,1])} &\to T_{\mathrm{id}}\mathrm{Diff}^s_h(S^1\times[-1,1]) \\
X = X^1\partial_\varphi + X^2\partial_z &\mapsto \left(X^1 + X^2\frac{\langle\partial_z,\partial_\varphi\rangle}{\langle\partial_\varphi,\partial_\varphi\rangle}\right)\partial_\varphi.
\end{aligned}$$

Recall that

$$T_\nu\mathrm{Diff}^s(S^1\times[-1,1]) = T_{\mathrm{id}}\mathrm{Diff}^s(S^1\times[-1,1])\circ\nu.$$

For any $X = X^1\partial_\varphi\circ\nu + X^2\partial_z\circ\nu \in T_\nu\mathrm{Diff}^s(S^1\times[-1,1])$, the projection $P^1_\nu(X)$ has to be of the form

$$P^1_\nu(X) = p^1_\nu(X)\partial_\varphi\circ\nu \in T_\nu\mathrm{Diff}^s_h(S^1\times[-1,1])$$

for $p^1_\nu(X) \in H^s(S^1\times[-1,1],\mathbb{R})$. For any $Y^1\partial_\varphi\circ\nu$, we need to have

$$\begin{aligned}
0 &\overset{!}{=} \left(P^1_\nu(X) - X,\ Y^1\partial_\varphi\circ\nu\right) \\
&= \int_{S^1\times[-1,1]} \langle P^1_\nu(X) - X,\ Y^1\partial_\varphi\circ\nu\rangle_{\nu(\varphi,z)}\,\sigma_a \\
&= \int_{S^1\times[-1,1]} \langle p^1_\nu(X)\partial_\varphi\circ\nu - X^1\partial_\varphi\circ\nu - X^2\partial_z\circ\nu,\ Y^1\partial_\varphi\circ\nu\rangle_{\nu(\varphi,z)} \\
&\qquad\qquad a(z)\,\mathrm{d}\varphi\wedge\mathrm{d}z \\
&= \int_{S^1\times[-1,1]} Y^1\Big(\big(p^1_\nu(X) - X^1\big)\langle\partial_\varphi\circ\nu,\partial_\varphi\circ\nu\rangle_{\nu(\varphi,z)} \\
&\qquad - X^2\langle\partial_z\circ\nu,\partial_\varphi\circ\nu\rangle_{\nu(\varphi,z)}\Big)\ a(z)\,\mathrm{d}\varphi\wedge\mathrm{d}z
\end{aligned}$$

$$\begin{aligned}
&= \int_{S^1\times[-1,1]} Y^1\Big(\big(p^1_\nu(X) - X^1\big)\langle\partial_\varphi,\partial_\varphi\rangle \circ \nu \\
&\qquad\qquad - X^2\langle\partial_z,\partial_\varphi\rangle \circ \nu\Big)\; a(z)\,\mathrm{d}\varphi\wedge\mathrm{d}z \\
\Rightarrow \quad & p^1_\nu(X) = X^1 + X^2\cdot\left(\frac{\langle\partial_z,\partial_\varphi\rangle}{\langle\partial_\varphi,\partial_\varphi\rangle}\right)\circ\nu
\end{aligned}$$

and we get for $X = X^1\partial_\varphi \circ \nu + X^2\partial_z \circ \nu$,

$$P^1_\nu(X) = \left(X^1 + X^2\cdot\left(\frac{\langle\partial_z,\partial_\varphi\rangle}{\langle\partial_\varphi,\partial_\varphi\rangle}\right)\circ\nu\right)\partial_\varphi\circ\nu.$$

We now want to show that combining all P^1_ν yields a smooth bundle projection. Note that even though we used the standard metric to compute the trivializations (4.4), they are still trivializations even if we work with a different Riemannian metric in this section.

Proposition 4.25. $P^1 : T\mathrm{Diff}^s(S^1\times[-1,1])|_{\mathrm{Diff}^s_h(S^1\times[-1,1])} \to T\mathrm{Diff}^s_h(S^1\times[-1,1])$ *is a smooth bundle projection, i. e.* P^1 *is smooth in the base point.*

Proof. In those coordinates, P^1 takes the form

$$\begin{aligned}
T_\nu\mathrm{Diff}^s_h(S^1\times[-1,1]) \times T_\nu\mathrm{Diff}^s(S^1\times[-1,1]) \\
\to T_\nu\mathrm{Diff}^s_h(S^1\times[-1,1]) \times T_\nu\mathrm{Diff}^s(S^1\times[-1,1]) \\
(X,Y) \mapsto (\Phi^{-1}\circ P^1\circ\Phi)(X,Y)
\end{aligned}$$

and

$$\begin{aligned}
(\Phi^{-1}\circ P^1\circ\Phi)(X,Y) &= \Phi^{-1}(P^1(\Phi(X,Y))) \\
&= \Phi^{-1}\Big(P^1\big(\nu+X,\ Y^1\partial_\varphi\circ(\nu+X) + Y^2\partial_z\circ(\nu+X)\big)\Big) \\
&= \Phi^{-1}\Big(\nu+X,\ P^1_{\nu+X}\big(Y^1\partial_\varphi\circ(\nu+X) + Y^2\partial_z\circ(\nu+X)\big)\Big) \\
&= \Phi^{-1}\left(\nu+X,\ \left(Y^1 + Y^2\left(\frac{\langle\partial_z,\partial_\varphi\rangle}{\langle\partial_\varphi,\partial_\varphi\rangle}\right)\circ(\nu+X)\right)\partial_\varphi\circ(\nu+X)\right) \\
&= \left(X,\ \left(Y^1 + Y^2\left(\frac{\langle\partial_z,\partial_\varphi\rangle}{\langle\partial_\varphi,\partial_\varphi\rangle}\right)\circ(\nu+X)\right)\partial_\varphi\circ\nu\right).
\end{aligned}$$

Theorem 1.2 in [IKT13] shows that for any smooth $f \in C^\infty(S^1\times[-1,1],\mathbb{R})$, the left translation

$$\begin{aligned}
\mathrm{Diff}^s(S^1\times[-1,1]) &\to H^s(S^1\times[-1,1],\mathbb{R}) \\
\nu &\mapsto f\circ\nu
\end{aligned}$$

is smooth. Since both $\dfrac{\langle \partial_z, \partial_\varphi \rangle}{\langle \partial_\varphi, \partial_\varphi \rangle} : S^1 \times [-1,1] \to \mathbb{R}$ and the exponential function are smooth, also the composition

$$\begin{array}{rcccl} T_\nu \mathrm{Diff}^s_{\sigma,h}(S^1 \times [-1,1]) & \to & \mathrm{Diff}^s_{\sigma,h}(S^1 \times [-1,1]) & \to & H^s(S^1 \times [-1,1], \mathbb{R}) \\ X & \mapsto & \exp_\nu X = \nu + X & \mapsto & \left(\dfrac{\langle \partial_z, \partial_\varphi \rangle}{\langle \partial_\varphi, \partial_\varphi \rangle} \right) \circ (\nu + X) \end{array}$$

is smooth. Since $s > \frac{1}{2} \dim(S^1 \times [-1,1]) + 1$, the product of two H^s-functions is again an H^s-map. This implies that

$$X \mapsto Y^1 + Y^2 \left(\frac{\langle \partial_z, \partial_\varphi \rangle}{\langle \partial_\varphi, \partial_\varphi \rangle} \right) \circ (\nu + X)$$

is smooth and hence, P^1 is a smooth bundle map. □

We now let $P^2 : T\mathrm{Diff}^s_h(B)|_{\mathrm{Diff}^s_{\sigma_a,h}(B)} \to T\mathrm{Diff}^s_{\sigma_a h}(B)$ denote the orthogonal projection of the tangent bundle with restriction $P^2_{\mathrm{id}} := P^2|_{T_{\mathrm{id}}\mathrm{Diff}^s_h(S^1 \times [-1,1])}$. Recall that $\mathrm{Diff}^s_{\sigma_a,h}(B)$ is locally diffeomorphic to $H^s([-1,1], \mathbb{R})$ as in the proof of Proposition 4.3. Therefore, we have

$$T_{\mathrm{id}} \mathrm{Diff}^s_{\sigma_a,h}(S^1 \times [-1,1]) = H^s([-1,1], \mathbb{R}) \partial_\varphi$$

and for $\nu \in \mathrm{Diff}^s_{\sigma_a,h}(S^1 \times [-1,1])$, i. e. $\nu(\varphi, z) = (\nu^1(\varphi, z), \nu^2(\varphi, z)) = (\nu^1(\varphi, z), z)$,

$$\begin{aligned} T_\nu \mathrm{Diff}^s_{\sigma_a,h}(S^1 \times [-1,1]) &= T_{\mathrm{id}} \mathrm{Diff}^s_{\sigma_a,h}(S^1 \times [-1,1]) \circ \nu \\ &= H^s([-1,1], \mathbb{R}) \circ \nu^2 \cdot (\partial_\varphi \circ \nu) \\ &= H^s([-1,1], \mathbb{R}) \circ z \cdot (\partial_\varphi \circ \nu) \\ &= H^s([-1,1], \mathbb{R}) \cdot (\partial_\varphi \circ \nu). \end{aligned}$$

Lemma 4.26. *The orthogonal projection P^2_{id} is given by*

$$\begin{aligned} P^2_{\mathrm{id}} : T_{\mathrm{id}} \mathrm{Diff}^s_h(S^1 \times [-1,1]) &\to T_{\mathrm{id}} \mathrm{Diff}^s_{\sigma,h}(S^1 \times [-1,1]) \\ X = X^1 \partial_\varphi &\mapsto p^2(X^1) \partial_\varphi \end{aligned}$$

for

$$\begin{aligned} p^2 : H^s(S^1 \times [-1,1], \mathbb{R}) &\to H^s([-1,1], \mathbb{R}) \\ f &\mapsto \left(z \mapsto \frac{\int_0^1 f(\varphi, z) \langle \partial_\varphi, \partial_\varphi \rangle \, \mathrm{d}\varphi}{\int_0^1 \langle \partial_\varphi, \partial_\varphi \rangle \, \mathrm{d}\varphi} \right). \end{aligned}$$

Proof. We first note that for any $X \in T_{\mathrm{id}} \mathrm{Diff}^s_h(S^1 \times [-1,1])$, the image under P^2_{id} is an element of $T_{\mathrm{id}} \mathrm{Diff}^s_{\sigma,h}(S^1 \times [-1,1])$, hence it can be written in the form $p^2(X^1) \partial_\varphi$ for some map $p^2(X^1) \in H^s([-1,1],\mathbb{R})$. Furthermore, for any $X^1\partial_\varphi \in T_{\mathrm{id}}\mathrm{Diff}^s_h(S^1 \times [-1,1])$ and any $Y = Y^1\partial_\varphi \in T_{\mathrm{id}}\mathrm{Diff}^s_{\sigma,h}(S^1 \times [-1,1])$, we have

$$
\begin{aligned}
(P^2_{\mathrm{id}}(X) - X,\ Y) &= \int_{S^1\times[-1,1]} \langle P^2_{\mathrm{id}}(X) - X,\ Y\rangle a \mathrm{d}\varphi \wedge \mathrm{dz} \\
&= \int_{S^1\times[-1,1]} \langle p^2(X^1)\partial_\varphi - X^1\partial\varphi, Y^1\partial\varphi\rangle a \mathrm{d}\varphi \wedge \mathrm{dz} \\
&= \int_{S^1\times[-1,1]} \langle \Big(\frac{\int_0^1 X^1(\psi,z)\langle\partial_\varphi,\partial_\varphi\rangle_{(\psi,z)}\,\mathrm{d}\psi}{\int_0^1 \langle\partial_\varphi,\partial_\varphi\rangle_{(\psi,z)}\,\mathrm{d}\psi} \Big)\partial_\varphi \\
&\qquad\qquad - X^1(\varphi,z)\partial_\varphi,\ Y^1(z)\partial_\varphi\rangle \cdot a(z)\mathrm{d}\varphi \wedge \mathrm{dz} \\
&= \int_{S^1\times[-1,1]} \Big(\frac{\int_0^1 X^1(\psi,z)\langle\partial_\varphi,\partial_\varphi\rangle_{(\psi,z)}\,\mathrm{d}\psi}{\int_0^1 \langle\partial_\varphi,\partial_\varphi\rangle_{(\psi,z)}\,\mathrm{d}\psi} - X^1(\varphi,z) \Big) \\
&\qquad\qquad\qquad Y^1(z)\langle\partial_\varphi,\partial_\varphi\rangle \cdot a(z)\,\mathrm{d}\varphi \wedge \mathrm{dz} \\
&= \int_{-1}^1 Y^1(z)a(z)\Bigg[\int_0^1 \Big(\frac{\int_0^1 X^1(\psi,z)\langle\partial_\varphi,\partial_\varphi\rangle_{(\psi,z)}\,\mathrm{d}\psi}{\int_0^1 \langle\partial_\varphi,\partial_\varphi\rangle_{(\psi,z)}\,\mathrm{d}\psi} \Big)\langle\partial_\varphi,\partial_\varphi\rangle \\
&\qquad\qquad\qquad - X^1(\varphi,z)\langle\partial_\varphi,\partial_\varphi\rangle\mathrm{d}\varphi\Bigg]\mathrm{dz} \\
&= \int_{-1}^1 Y^1(z)a(z)\Bigg[\int_0^1 \Big(\frac{\int_0^1 X^1(\psi,z)\langle\partial_\varphi,\partial_\varphi\rangle_{(\psi,z)}\,\mathrm{d}\psi}{\int_0^1 \langle\partial_\varphi,\partial_\varphi\rangle_{(\psi,z)}\,\mathrm{d}\psi} \Big)\langle\partial_\varphi,\partial_\varphi\rangle\mathrm{d}\varphi \\
&\qquad\qquad\qquad - \int_0^1 X^1(\varphi,z)\Big)\langle\partial_\varphi,\partial_\varphi\rangle\mathrm{d}\varphi\Bigg]\mathrm{dz} \\
&= \int_{-1}^1 Y^1(z)a(z)\Bigg[\frac{\int_0^1 X^1(\psi,z)\langle\partial_\varphi,\partial_\varphi\rangle_{(\psi,z)}\,\mathrm{d}\psi}{\int_0^1 \langle\partial_\varphi,\partial_\varphi\rangle_{(\psi,z)}\,\mathrm{d}\psi} \int_0^1 \langle\partial_\varphi,\partial_\varphi\rangle\mathrm{d}\varphi \\
&\qquad\qquad\qquad - \int_0^1 X^1(\varphi,z)\langle\partial_\varphi,\partial_\varphi\rangle\mathrm{d}\varphi\Bigg]\mathrm{dz} \\
&= \int_{-1}^1 Y^1(z)a(z)\Bigg[\int_0^1 X^1(\psi,z)\langle\partial_\varphi,\partial_\varphi\rangle_{(\psi,z)}\,\mathrm{d}\psi \\
&\qquad\qquad - \int_0^1 X^1(\varphi,z)\langle\partial_\varphi,\partial_\varphi\rangle\mathrm{d}\varphi\Bigg]\mathrm{dz} \\
&= 0.
\end{aligned}
$$

□

Let $\nu \in \mathrm{Diff}^s_{\sigma,h}(S^1 \times [-1,1])$. Since ν preserves the area form σ, both the metric and orthogonal projection are right invariant and we can compute

$$
\begin{aligned}
P^2_\nu : T_\nu\mathrm{Diff}^s_h(S^1 \times [-1,1]) &\to T_\nu\mathrm{Diff}^s_{\sigma,h}(S^1 \times [-1,1]) \\
X = X^1(\partial_\varphi \circ \nu) &\mapsto (TR_\nu \circ P^2_{\mathrm{id}} \circ TR_{\nu^{-1}})(X)
\end{aligned}
$$

which equals for $X = X^1(\partial_\varphi \circ \nu)$

$$\begin{aligned}
P^2_\nu(X) &= (TR_\nu \circ P^2_{\mathrm{id}} \circ TR_{\nu^{-1}})(X)\\
&= (TR_\nu \circ P^2_{\mathrm{id}})(X^1 \circ \nu^{-1}\partial_\varphi)\\
&= TR_\nu(P^2_{\mathrm{id}}(X \circ \nu^{-1}\partial_\varphi))\\
&= TR_\nu(p^2(X^1 \circ \nu^{-1})\partial_\varphi)\\
&= p^2(X^1 \circ \nu^{-1}) \circ \nu(\partial_\varphi \circ \nu)\\
&= p^2(X^1 \circ \nu^{-1})(\partial_\varphi \circ \nu)
\end{aligned}$$

since $p^2(X^1 \circ \nu^{-1})$ only depends on z and ν preserves z. Furthermore,

$$\begin{aligned}
p^2(X^1 \circ \nu^{-1}) &= \frac{\int_0^1 (X^1 \circ \nu^{-1})(\psi,z)\langle \partial_\varphi, \partial_\varphi\rangle_{(\psi,z)}\,\mathrm{d}\psi}{\int_0^1 \langle \partial_\varphi, \partial_\varphi\rangle_{(\psi,z)}\,\mathrm{d}\psi}\\
&= \frac{\int_0^1 X^1(\psi,z)(\langle \partial_\varphi, \partial_\varphi\rangle \circ \nu)(\psi,z)\,\nu^*\mathrm{d}\psi}{\int_0^1 \langle \partial_\varphi, \partial_\varphi\rangle_{(\psi,z)}\,\mathrm{d}\psi}\\
&= \frac{\int_0^1 X^1(\psi,z)(\langle \partial_\varphi, \partial_\varphi\rangle \circ \nu)(\psi,z)\,\mathrm{d}\psi}{\int_0^1 \langle \partial_\varphi, \partial_\varphi\rangle_{(\psi,z)}\,\mathrm{d}\psi}.
\end{aligned}$$

Hence,

$$\begin{aligned}
P^2_\nu(X) &= p^2(X^1 \circ \nu^{-1})(\partial_\varphi \circ \nu)\\
&= \left(\frac{\int_0^1 X^1(\psi,z)(\langle \partial_\varphi, \partial_\varphi\rangle \circ \nu)(\psi,z)\,\mathrm{d}\psi}{\int_0^1 \langle \partial_\varphi, \partial_\varphi\rangle_{(\psi,z)}\,\mathrm{d}\psi}\right)(\partial_\varphi \circ \nu).
\end{aligned}$$

Proposition 4.27. *P^2 is a smooth bundle map, i. e. it is smooth in the base point.*

Proof. Using the trivializations

$$\begin{aligned}
\Phi : T_\nu \mathrm{Diff}^s_{\sigma,h}(S^1 \times [-1,1]) \times T_\nu \mathrm{Diff}^s_h(S^1 \times [-1,1])&\\
\to T\mathrm{Diff}^s_h(S^1 \times [-1,1])|_{\mathrm{Diff}^s_{\sigma,h}(S^1\times[-1,1])}&\\
(X, Y = Y^1\partial_\varphi \circ \nu) \mapsto \big(\nu + X,\ Y^1\partial_\varphi \circ (\nu + X)\big),&
\end{aligned}$$

we can write

$$\begin{aligned}
T_\nu \mathrm{Diff}^s_h(S^1 \times [-1,1]) \times T_\nu \mathrm{Diff}^s_h(S^1 \times [-1,1])&\\
\to T_\nu \mathrm{Diff}^s_h(S^1 \times [-1,1]) \times T_\nu \mathrm{Diff}^s_h(S^1 \times [-1,1])&\\
(X, Y) \mapsto (\Phi^{-1} \circ P^2 \circ \Phi)(X, Y).&
\end{aligned}$$

We compute for $X, Y = Y^1\partial_\varphi \circ \nu \in T_\nu \mathrm{Diff}_h^s(S^1 \times [-1,,1])$

$$\begin{aligned}(\Phi^{-1} \circ P^2 \circ \Phi)(X,Y) &= \Phi^{-1}(P^2(\Phi(X,Y)))\\ &= \Phi^{-1}\Big(P^2\big(\nu + X,\ Y^1\partial_\varphi \circ (\nu + X)\big)\Big)\\ &= \Phi^{-1}\big(\nu + X, P^2_{\nu+X}(Y^1\partial_\varphi \circ (\nu + X))\big)\\ &= \Phi^{-1}\big(\nu + X,\ p^2(Y^1 \circ (\nu + X)^{-1})\partial_\varphi \circ (\nu + X)\big)\\ &= \big(X,\ p^2(Y^1 \circ (\nu + X)^{-1})\partial_\varphi \circ \nu\big).\end{aligned}$$

Hence, we need to check whether the map

$$X \mapsto p^2(Y^1 \circ (\nu + X)^{-1}) = \frac{\int_0^1 Y^1(\psi,z)\big(\langle\partial_\varphi,\partial_\varphi\rangle \circ (\nu + X)(\psi,z)\big)\,\mathrm{d}\psi}{\int_0^1 \langle\partial_\varphi,\partial_\varphi\rangle_{(\psi,z)}\,\mathrm{d}\psi}$$

is smooth in X. Since $\langle\partial_\varphi,\partial_\varphi\rangle : S^1 \times [-1,1] \to \mathbb{R}$ is smooth, Theorem 1.2 in [IKT13] implies that

$$X \mapsto \langle\partial_\varphi,\partial_\varphi\rangle \circ (\nu + X)$$

is smooth as in the proof of Proposition 4.25. Hence, P^2 is a smooth bundle map. □

Corollary 4.28. *The previous two lemmas show that*

$$P = P^2 \circ P^1 : T\mathrm{Diff}^s(S^1 \times [-1,1])|_{\mathrm{Diff}^s_{\sigma_a,\tau_a}(S^1\times[-1,1])} \to T\mathrm{Diff}^s_{\sigma_a,\tau_a}(S^1 \times [-1,1])$$

is a smooth bundle map. □

4.6 Euler equation on $\mathrm{Diff}^s_{\sigma_a,\tau_a}(B)$

Recall the result of the variation of energy in Section 2.3: Let $v_t \in T_{\mathrm{id}}\mathrm{Diff}^s_{\sigma_a,\tau_a}(S^1 \times [-1,1])$ be a time-dependent vector field, i. e. v_t is of the form $v_t = v_t(z)\partial_\varphi$. If

$$0 = \int_0^T \int_B \langle w, \dot{v} + \nabla_v v\rangle\, \sigma_a\, \mathrm{d}t \tag{2.9 rev.}$$

for any time-dependent $w_t = w_t(z)\partial_\varphi \in T_{\mathrm{id}}\mathrm{Diff}^s_{\sigma_a,\tau_a}(S^1 \times [-1,1])$, then v_t is a solution to the Euler equation. Let us use this information to compute

$$\begin{aligned}
\langle w, \dot{v} + \nabla_v v\rangle &= \langle w_t\partial_\varphi,\ \dot{v}_t\partial_\varphi + \underbrace{\nabla_{v_t\partial_\varphi} v_t\partial_\varphi}_{=v_t\nabla_{\partial_\varphi} v_t\partial_\varphi = v_t^2\nabla_{\partial_\varphi}\partial_\varphi}\rangle \\
&= w_t(z)\Big[\dot{v}_t\langle\partial_\varphi,\partial_\varphi\rangle + v_t^2\underbrace{\langle\partial_\varphi,\nabla_{\partial_\varphi}\partial_\varphi\rangle}_{=\frac{1}{2}\frac{\partial}{\partial\varphi}\langle\partial_\varphi,\partial_\varphi\rangle}\Big] \\
&= w_t\Big[\dot{v}_t\langle\partial_\varphi,\partial_\varphi\rangle + \frac{v_t^2}{2}\frac{\partial}{\partial\varphi}\langle\partial_\varphi,\partial_\varphi\rangle\Big].
\end{aligned}$$

Since the coefficients of w_t and v_t only depend on z, we compute the integral

$$\begin{aligned}
&\int_{S^1\times[-1,1]} \langle w_t, \dot{v}_t + \nabla_{v_t} v_t\rangle\,(a\,\mathrm{d}\varphi\wedge\mathrm{d}z) = \\
&\qquad = \int_{S^1\times[-1,1]} w_t\Big[\dot{v}_t\langle\partial_\varphi,\partial_\varphi\rangle + \frac{v_t^2}{2}\frac{\partial}{\partial\varphi}\langle\partial_\varphi,\partial_\varphi\rangle\Big](a\,\mathrm{d}\varphi\wedge\mathrm{d}z) \\
&\qquad = \int_{-1}^{1} w_t a\Big[\int_0^1 \dot{v}_t\langle\partial_\varphi,\partial_\varphi\rangle + \frac{v_t^2}{2}\frac{\partial\langle\partial_\varphi,\partial_\varphi\rangle}{\partial\varphi}\,\mathrm{d}\varphi\Big]\mathrm{d}z \\
&\qquad = \int_{-1}^{1} w_t a\Big[\dot{v}_t\int_0^1 \langle\partial_\varphi,\partial_\varphi\rangle\mathrm{d}\varphi + \frac{v_t^2}{2}\underbrace{\int_0^1 \frac{\partial\langle\partial_\varphi,\partial_\varphi\rangle}{\partial\varphi}\Big)\mathrm{d}\varphi}_{=0}\Big]\mathrm{d}z \\
&\qquad = \int_{-1}^{1} w_t a\dot{v}_t\underbrace{\Big[\int_0^1 \langle\partial_\varphi,\partial_\varphi\rangle\mathrm{d}\varphi\Big]}_{>0}\mathrm{d}z.
\end{aligned}$$

The Euler equation

$$0 = \int_{S^1\times[-1,1]} \langle w_t, \dot{v}_t + \nabla_{v_t} v_t\rangle\,(a\,\mathrm{d}\varphi\wedge\mathrm{d}z) \qquad \text{for any } w_t(z)$$

is then equivalent to

$$0 = \underbrace{a(z)}_{\neq 0}\ \dot{v}_t\underbrace{\int_0^1 \langle\partial_\varphi,\partial_\varphi\rangle\mathrm{d}\varphi}_{\neq 0},$$

or

$$0 = \dot{v}_t.$$

Proposition 4.29. *The previous computation shows that the only solutions to the Euler equation on $S^1 \times [-1,1]$ preserving σ_a and τ_a are all stationary vector fields of the form $v_t = v = v(z)\partial_\varphi$.* □

4.7 $M = B \times S^1$, general metric

Let $M = (S^1 \times [-1,1]) \times S^1 \xrightarrow{\pi} B = S^1 \times [-1,1]$ be the trivial S^1-bundle with stable Hamiltonian structure $\omega_b = \pi^*\sigma_b = \pi^*(b\sigma)$ and $\lambda_{b,r} = \mathrm{d}\theta + \pi^*\mu_{b,r}$ for

$$\mu_{b,\tilde{r}} := -m_{b,\tilde{r}}(\varphi,z)\mathrm{d}\varphi \quad \text{for} \quad m_{b,\tilde{r}}(\varphi,z) := \int_{-1}^{z} \zeta b(\varphi,\zeta)\,\mathrm{d}\zeta + \tilde{r}$$

with $\tilde{r} \in \mathbb{R}$. In particular, $m_{b,\tilde{r}}(\varphi,-1) \equiv \tilde{r}$. Then we get two-forms $\sigma_b = b(\varphi,z)\,\mathrm{d}\varphi \wedge \mathrm{d}z$ and

$$\tau_b = \mathrm{d}\mu_{b,\tilde{r}} = -\frac{\partial m_{b,\tilde{r}}}{\partial z}\mathrm{d}z \wedge \mathrm{d}\varphi = zb(\varphi,z)\,\mathrm{d}\varphi \wedge \mathrm{d}z = h(z)b(\varphi,z)\,\mathrm{d}\varphi \wedge \mathrm{d}z$$

on B, as in the last section.

This seemingly random choice for $\mu_{b,r}$ corresponds to dealing with one representative of each cohomology class in the following sense: Let μ, $\tilde{\mu}$ be two one-forms corresponding to the same two-form τ on B, then $\mathrm{d}\mu = \tau = \mathrm{d}\tilde{\mu}$, hence those two one-forms differ by a closed form. Any closed form can be written as the sum of an exact form and an element of $H^1(B)$. We will deal with adding exact one-forms in Section 4.9 and $H^1(B) \cong \mathbb{R}$ is generated by $\tilde{r}\mathrm{d}\varphi$ for $\tilde{r} \in \mathbb{R}$.

Remark. Please note that this choice for $\mu_{b,1/2}$ is equal to the one in Section 4.3. Here, for $b(\varphi,z) \equiv 1$, we get

$$m_{1,1/2}(\varphi,z) = \int_{-1}^{z} \zeta\,\mathrm{d}\zeta + \frac{1}{2} = \frac{z^2}{2} - \frac{1}{2} + \frac{1}{2} = \frac{z^2}{2},$$
$$\mu_{1,1/2} = -m_{1,1/2}(\varphi,z)\,\mathrm{d}\varphi = -\frac{z^2}{2}\,\mathrm{d}\varphi,$$

which is equal to $\mu = -\frac{z^2}{2}\,\mathrm{d}\varphi$.

As in Section 4.3, we consider the metric on $M = B \times S^1$ defined by

- $\ker\lambda_{b,\tilde{r}} \perp R = \partial_\theta$,
- $|R| = 1$,
- and for any $v, w \in \ker(\lambda_{b,\tilde{r}})_x \subset T_xM$, we have

$$\langle v,w\rangle_x = \langle \pi_* v, \pi_* w\rangle^B_{\pi(x)}.$$

Using this metric, the Riemannian volume form on M is given by

$$\mathrm{vol} = \lambda_{b,\tilde{r}} \wedge \omega_b = b(\varphi,z)\,\mathrm{d}\theta \wedge \mathrm{d}z \wedge \mathrm{d}\varphi.$$

We want to lift the diffeomorphism $\rho : S^1 \times [-1,1] \to S^1 \times [-1,1]$ in Section 4.5 (constructed in Proposition 4.23) as in Corollary 3.33. Recall the construction in Proposition 4.23

$$\begin{aligned}\rho : S^1 \times [-1,1] &\to S^1 \times [-1,1]\\ (\varphi, z) &\mapsto \left(B_z^{-1}(a(z)\cdot x) \mod 1,\ z\right)\end{aligned}$$

for any representative $x \in \mathbb{R}$ of $\varphi \in S^1 \cong \mathbb{R}/\mathbb{Z}$ and where we define $B_z(x) = B(x,z)$, $B(x,z) = \int_0^x b_{\mathbb{R}}(y,z)\,\mathrm{d}y$ and $a(z) = B(1,z) = \int_0^1 b(\varphi,z)\,\mathrm{d}\varphi$. For $\gamma = [S^1 \times \{-1\}]$, let

$$r := \tilde{r} \int_\gamma \rho^*(\mathrm{d}\varphi) \in \mathbb{R}.$$

Lemma 4.30. *The diffeomorphism ρ in Proposition 4.23 lifts to a diffeomorphism $\rho^M : M \to M$ such that $(\rho^M)^*\omega_b = \omega_a$ and $(\rho^M)^*\lambda_{b,\tilde{r}} = \lambda_{a,r}$.*

Proof. Since $H_1(B;\mathbb{Z})$ is generated by $\gamma = [S^1 \times \{-1\}]$, it suffices to compute

$$\begin{aligned}\int_\gamma (\mu_{a,r} - \rho^*\mu_{b,\tilde{r}}) &= \int_0^1 -\underbrace{m_{a,r}(-1)}_{=-r}\,\mathrm{d}\varphi + \int_0^1 (m_{b,\tilde{r}} \circ \rho)(\varphi,-1)\,\rho^*(\mathrm{d}\varphi)\\ &= -r + \int_0^1 \underbrace{m_{b,\tilde{r}}(\rho^1(\varphi,-1),-1)}_{=\tilde{r}}\,\rho^*(\mathrm{d}\varphi)\\ &= -r + \tilde{r}\int_0^1 \rho^*(\mathrm{d}\varphi)\\ &= 0 \in Z,\end{aligned}$$

as required by Corollary 3.33. ☐

Hence, we can wlog assume that our stable Hamiltonian structure is given by $\omega_a = \pi^*\sigma_a$ and $\lambda_{a,r} = \mathrm{d}\theta + \pi^*\mu_{a,r}$ for $a(z) \in C^\infty([-1,1],\mathbb{R})$ and $r \in \mathbb{R}$.

Our first goal is to use Theorem 3.29 to prove

Theorem 4.31. $\mathrm{Diff}^s_{\omega_a,\lambda_{a,r}}(M) \subset \mathrm{Diff}^s(M)$ *is a smooth submanifold.*

Recall that

$$\mathrm{Diff}^s_{\omega_a,\lambda_{a,r}}(M) \cong \mathcal{D}^s_{a,r} \times S^1$$

for

$$\mathcal{D}^s_{a,r} = \left\{\nu \in \mathrm{Diff}^s_{\sigma_a,\tau_a}(B) \;\middle|\; \int_\gamma (\mu_{a,r} - \nu^*\mu_{a,r}) \in \mathbb{Z} \quad \text{for any } \gamma \in H_1(B;\mathbb{Z})\right\}.$$

We will start with results on $\mu_{a,r} - \nu^*\mu_{a,r}$.

Lemma 4.32. *Let $\nu \in \mathrm{Diff}^s_{\sigma_a,\tau_a}(B)$. Then $\mu_{a,r} - \nu^*\mu_{a,r}$ is exact.*

Proof. Recall that $\nu = (\nu^1, \nu^2) \in \mathrm{Diff}^s_{\sigma_a,\tau_a}(B) = \mathrm{Diff}^s_{\sigma,\tau}(B)$ is equivalent to $\dfrac{\partial \nu^1}{\partial \varphi} \equiv 1$ and $\nu^2(\varphi, z) = z$, hence we can write $\nu^1(\varphi, z) = \varphi + g(z)$ and get

$$\begin{aligned}\mu_{a,r} - \nu^* \mu_{a,r} &= -m_{a,r}(z)\,\mathrm{d}\varphi + \big((\nu^2)^* m_{a,r}\big)(z)\,\mathrm{d}\nu^1 \\ &= -m_{a,r}(z)\,\mathrm{d}\varphi + m_{a,r}(z)\Big(\underbrace{\frac{\partial \nu^1}{\partial \varphi}}_{\equiv 1}\,\mathrm{d}\varphi + \frac{\partial \nu^1}{\partial z}\,\mathrm{d}z\Big) \\ &= m_{a,r}(z) g'(z)\,\mathrm{d}z. \end{aligned} \tag{4.29}$$

Define

$$M_{a,r}(z) := \int_{-1}^{z} m_{a,r}(\zeta) g'(\zeta)\,\mathrm{d}\zeta$$

so that

$$\mathrm{d}M_{a,r} = m_{a,r}(z) g'(z)\,\mathrm{d}z \overset{(4.29)}{=} \mu_{a,r} - \nu^* \mu_{a,r}. \qquad \square$$

Proof of Theorem 4.31. The previous lemma implies that $\displaystyle\int_\gamma (\mu_{a,r} - \nu^* \mu_{a,r}) = 0$ for any $\gamma \in H_1(B; \mathbb{Z})$, hence

$$\begin{aligned}\mathcal{D}^s_{a,r} &= \left\{ \nu \in \mathrm{Diff}^s_{\sigma_a,\tau_a}(B) \;\middle|\; \int_\gamma (\mu_{a,r} - \nu^* \mu_{a,r}) \in \mathbb{Z} \text{ for all } \gamma \in H_1(B;\mathbb{Z}) \right\} \\ &= \mathrm{Diff}^s_{\sigma_a,\tau_a}(B) \\ &= \mathrm{Diff}^s_{\sigma,\tau}(B)\end{aligned}$$

by Proposition 4.24. In particular, $\mathcal{D}^s_{a,r} = \mathrm{Diff}^s_{\sigma,\tau}(B)$ is a smooth submanifold of the full diffeomorphism group $\mathrm{Diff}^s(B)$, so by Theorem 3.29 also $\mathrm{Diff}^s_{\omega_a,\lambda_{a,r}}(B \times S^1) \subset \mathrm{Diff}^s_R(B \times S^1) \subset \mathrm{Diff}^s(B \times S^1)$ are smooth submanifolds. $\square$

Recall the map $k_{a,r} : \mathcal{D}^s_{a,r} \to H^s(B, S^1)$ used in Theorem 3.29. Following the construction of $k_{a,r}$ in Lemma 3.23, we start with the cohomology class defined by $\mu_{a,r} - \nu^* \mu_{a,r}$ for $\nu \in \mathcal{D}^s$. Since $[\mu_{a,r} - \nu^* \mu_{a,r}] = [0]$, we only need to choose $\alpha_{[0]} := 0 \in \Omega_{[0]}(B)$ and the constant function $(k_{a,r})_{[0]} := 0$. As required, $\alpha_{[0]} = d(k_{a,r})_{[0]}$. Then,

$$(\mu_{a,r})_\nu := \mu_{a,r} - \nu^* \mu_{a,r} - \alpha_{[0]} = \mu_{a,r} - \nu^* \mu_{a,r}.$$

With the base point $b_0 = (0, -1) \in S^1 \times [-1, 1] = B$, we get

$$(k_{a,r})_\nu(b) := \int_{b_0}^{b} (\mu_{a,r})_\nu = \int_{b_0}^{b} (\mu_{a,r} - \nu^* \mu_{a,r}).$$

Corollary 4.33 (see Theorem 3.29). *We have smooth diffeomorphisms*

$$\begin{aligned} \operatorname{Diff}^s_{\omega_a,\lambda_{a,r}}(B\times S^1) &\cong \operatorname{Diff}^s_{\sigma_a,\tau_a}(B)\times S^1\\ \eta=(\eta^1,\,\eta^2) &\mapsto \big(\eta^1,\,\eta^2(b,\theta)-(k_{a,r})_{\eta^1}(b)-\theta\big)\\ \big(\nu(b),\,(k_{a,r})_\nu(b)+\theta+\theta_0\big) &\leftarrow\!\shortmid (\nu,\,\theta_0) \end{aligned}$$ ☐

We will now explicitly verify Corollary 3.28, i. e. that $k_{a,r}$ is smooth. To compute the lift η_ν

$$\begin{aligned}&\eta_\nu : B\times S^1 \to B\times S^1,\\ &\quad\eta_\nu(x,\theta) := (\nu(x),\theta+(k_{a,r})_\nu(x))\end{aligned}$$

of ν in $\operatorname{Diff}^s_{\omega_a,\lambda_{a,r}}(B\times S^1)$, recall that any

$$\nu(\varphi,z)=(\nu^1(\varphi,z),\nu^2(\varphi,z))\in \operatorname{Diff}^s_{\sigma_a,\tau_a}(B)=\operatorname{Diff}^s_{\sigma_a,h}(B)$$

satisfies

$$\nu^2(\varphi,z)=z \quad\text{and}\quad \frac{\partial\nu^1}{\partial\varphi}=1.$$

In particular, ν^1 is of the form $\nu^1(\varphi,z)=\varphi+g(z)\mod 1$ for some $g\in H^s([-1,1],\mathbb{R})$. This yields

$$\begin{aligned}(k_{a,r})_\nu(\varphi,z) &= \int_{(0,-1)}^{(\varphi,z)}(\mu_{a,r}-\nu^*\mu_{a,r})\\ &\overset{(4.29)}{=}\int_{-1}^{z} m_{a,r}(\zeta)\frac{\partial\nu^1}{\partial\zeta}\,\mathrm{d}\zeta\\ &=\int_{-1}^{z} m_{a,r}(\zeta)g'(\zeta)\,\mathrm{d}\zeta.\end{aligned}$$

Then

$$\begin{aligned}\eta_\nu : (S^1\times[-1,1])\times S^1 = M &\to M\\ (b,\theta)&\mapsto(\nu(b),\theta+(k_{a,r})_\nu(x))\end{aligned}$$

or explicitly for $\nu(\varphi,z)=(\nu^1(\varphi,z),z)$,

$$((\varphi,z),\theta)\mapsto\Big(\underbrace{(\nu^1(\varphi,z),z)}_{=\nu(\varphi,z)},\,\theta+\int_{-1}^{z}m_{a,r}(\zeta)\frac{\partial\nu^1}{\partial\zeta}\mathrm{d}\zeta\Big)$$

is an element of $\operatorname{Diff}^s_{\omega_a,\lambda_{a,r}}(M)$.

Lemma 4.34. *The operator*

$$\begin{aligned} H^s([-1,1],\mathbb{R}) &\to H^s([-1,1],\mathbb{R}) \\ g &\mapsto \left(z \mapsto \int_{-1}^{z} m_{a,r}(\zeta)g'(\zeta)\mathrm{d}\zeta\right) \end{aligned}$$

is smooth.

Proof. We will use the same argument as in the proof of Lemma 4.14. Since this map is linear, we only have to check continuity to prove smoothness. Integration by parts yields

$$\begin{aligned} \int_{-1}^{z} m_{a,r}(\zeta)g'(\zeta)\mathrm{d}\zeta &= m_{a,r}(\zeta)g(\zeta)\,\Big|_{\zeta=-1}^{z} - \int_{-1}^{z} \underbrace{m'_{a,r}(\zeta)}_{=a(\zeta)\zeta} g(\zeta)\,\mathrm{d}\zeta \\ &= m_{a,r}(z)g(z) - \underbrace{m_{a,r}(-1)}_{=r} g(-1) + \int_{-1}^{z} a(\zeta)\zeta g(\zeta)\,\mathrm{d}\zeta. \end{aligned}$$

The maps $g \mapsto m_{a,r} \cdot g$ and $g \mapsto rg(-1)$ are continuous, so it only remains to compute the H^s-norms of $g \mapsto \int_{-1}^{z} a(\zeta)\zeta g(\zeta)\,\mathrm{d}\zeta$.

$$\begin{aligned} \left\| \int_{-1}^{z} a(\zeta)\zeta g(\zeta)\,\mathrm{d}\zeta \right\|_{H^s}^2 &= \left\| \int_{-1}^{z} a(\zeta)\zeta g(\zeta)\,\mathrm{d}\zeta \right\|_{H^0}^2 \\ &\quad + \left\| \frac{\partial}{\partial z} \int_{-1}^{z} a(\zeta)\zeta g(\zeta)\,\mathrm{d}\zeta \right\|_{H^{s-1}}^2 \\ &= \left\| \int_{-1}^{z} a(\zeta)\zeta g(\zeta)\,\mathrm{d}\zeta \right\|_{L^2}^2 + \left\| a(z)zg(z) \right\|_{H^{s-1}}^2. \end{aligned}$$

The first term can be estimated using the Cauchy-Schwarz inequality

$$\begin{aligned} \left\| \int_{-1}^{z} a(\zeta)\zeta g(\zeta)\,\mathrm{d}\zeta \right\|_{L^2}^2 &= \int_{-1}^{1} \left(\int_{-1}^{z} a(\zeta)\zeta g(\zeta)\,\mathrm{d}\zeta \right)^2 \mathrm{d}z \\ &\overset{\text{CSI}}{\leq} \int_{-1}^{1} \left(\int_{-1}^{z} a^2(\zeta)\zeta^2\,\mathrm{d}\zeta \right) \left(\int_{-1}^{z} \big(g(\zeta)\big)^2\,\mathrm{d}\zeta \right) \mathrm{d}z \\ &\leq \int_{-1}^{1} \underbrace{\left(\int_{-1}^{1} a^2(\zeta)\zeta^2\,\mathrm{d}\zeta \right)}_{=\|a(z)z\|_{L^2}^2} \left(\int_{-1}^{1} g^2(\zeta)\,\mathrm{d}\zeta \right) \mathrm{d}z \\ &= \|a(z)z\|_{L^2}^2 \int_{-1}^{1} \underbrace{\left(\int_{-1}^{1} g^2(\zeta)\,\mathrm{d}\zeta \right)}_{=\|g\|_{L^2}^2} \mathrm{d}z \\ &= 2\|a(z)z\|_{L^2}^2 \|g\|_{L^2}^2 \\ &\leq 2\|a(z)z\|_{H^{s-1}}^2 \|g\|_{H^s}^2. \end{aligned}$$

Since s is sufficiently large, $H^s([-1,1],\mathbb{R})$ is a Hilbert algebra and hence

$$\begin{aligned}\|a(z)zg(z)\|^2_{H^{s-1}} &\le \|a(z)z\|^2_{H^{s-1}}\|g\|^2_{H^{s-1}}\\ &\le \|a(z)z\|^2_{H^{s-1}}\|g\|^2_{H^s}.\end{aligned}$$

Using the two previous results yields

$$\begin{aligned}\left\|\int_{-1}^{z} a(\zeta)\zeta g(\zeta)\,\mathrm{d}\zeta\right\|^2_{H^s} &\le \left\|\int_{-1}^{z} a(\zeta)\zeta g(\zeta)\,\mathrm{d}\zeta\right\|^2_{L^2} + \left\|a(z)zg(z)\right\|^2_{H^{s-1}}\\ &\le 2\|a(z)z\|^2_{H^{s-1}}\|g\|^2_{H^s} + \|a(z)z\|^2_{H^{s-1}}\|g\|^2_{H^s}\\ &= 3\|a(z)z\|^2_{H^{s-1}}\|g\|^2_{H^s}.\end{aligned}$$

□

Corollary 4.35. *The map*

$$\begin{aligned}&k_{a,r} : \mathrm{Diff}^s_{\sigma_a,\tau_a}(B) \to H^s(B,\mathbb{R})\\ &\Big(\nu : (\varphi,z) \mapsto (\varphi + g(z),z)\Big) \mapsto \Big((k_{a,r})_\nu : (\varphi,z) \mapsto \int_{(0,-1)}^{(\varphi,z)} (\mu_{a,r} - \nu^*\mu_{a,r}) =\\ &\qquad\qquad\qquad\qquad\qquad\qquad \int_{-1}^{z} m_{a,r}(\zeta)g'(\zeta)\mathrm{d}\zeta\Big)\end{aligned}$$

is smooth. □

In the second part of this section, we want to show that the orthogonal projection

$$P : T\mathrm{Diff}^s_R(B\times S^1)|_{\mathrm{Diff}^s_{\omega_a,\lambda_{a,r}}(B\times S^1)} \to T\mathrm{Diff}^s_{\omega_a,\lambda_{a,r}}(B\times S^1)$$

is a smooth bundle map.

To that end, we first compute all the metric coefficients. Recall from Section 3.6 that $R = \partial_\theta$ has length 1 and is perpendicular to $\ker\lambda_{a,r}$ for $\lambda_{a,r} = \mathrm{d}\theta + \pi^*\mu_{a,r}$ with

$$\mu_{a,r} = -m_{a,r}(z)\,\mathrm{d}\varphi \quad \text{and } m_{a,r} = \int_{-1}^{z} \zeta a(\zeta)\,\mathrm{d}\zeta + r.$$

Hence, any element of $\ker\lambda_{a,r}$ is of the form $v - \mu_{a,r}(v)\partial_\theta$ for $v \in \mathfrak{X}(B)$. Then we can compute

$$\begin{aligned}\langle\partial_\theta,\partial_\theta\rangle &= 1,\\ \langle\partial_\varphi,\partial_\theta\rangle &= \underbrace{\langle\partial_\varphi - \mu_{a,r}(\partial_\varphi)\partial_\theta,\ \partial_\theta\rangle}_{=0} + \mu_{a,r}(\partial_\varphi)\underbrace{\langle\partial_\theta,\partial_\theta\rangle}_{=1}\\ &= 0 + \mu_{a,r}(\partial_\varphi),\\ \langle\partial_\varphi,\partial_\varphi\rangle &= \langle\partial_\varphi - \mu_{a,r}(\partial_\varphi)\partial_\theta,\ \partial_\varphi - \mu_{a,r}(\partial_\varphi)\partial_\theta\rangle\\ &\qquad + 2\mu_{a,r}(\partial_\varphi)\langle\partial_\varphi,\partial_\theta\rangle - \mu_{a,r}(\partial_\varphi)^2\langle\partial_\theta,\partial_\theta\rangle\\ &= \langle\partial_\varphi,\partial_\varphi\rangle^B + 2\mu_{a,r}(\partial_\varphi)^2 - \mu_{a,r}(\partial_\varphi)^2\\ &= \langle\partial_\varphi,\partial_\varphi\rangle^B + \mu_{a,r}(\partial_\varphi)^2,\end{aligned}$$

$$\begin{aligned}
\langle \partial_z, \partial_\theta \rangle &= \langle \partial_z - \mu_{a,r}(\partial_z)\partial_\theta,\ \partial_\theta \rangle + \mu_{a,r}(\partial_z)\langle \partial_\theta, \partial_\theta \rangle \\
&= 0 + \mu_{a,r}(\partial_z) \\
&= 0, \\
\langle \partial_z, \partial_\varphi \rangle &= \langle \partial_z - \mu_{a,r}(\partial_z)\partial_\theta, \partial_\varphi - \mu_{a,r}(\partial_\varphi)\partial_\theta \rangle \\
&\quad + \mu_{a,r}(\partial_z)\langle \partial_\theta, \partial_\varphi \rangle + \mu_{a,r}(\partial_\varphi)\langle \partial_z, \partial_\theta \rangle \\
&\qquad - \mu_{a,r}(\partial_z)\mu_{a,r}(\partial_\varphi)\langle \partial_\theta, \partial_\theta \rangle \\
&= \langle \partial_z, \partial_\varphi \rangle^B \\
&\quad + \mu_{a,r}(\partial_z)\mu_{a,r}(\partial_\varphi) + \mu_{a,r}(\partial_\varphi)\mu_{a,r}(\partial_z) \\
&\qquad - \mu_{a,r}(\partial_\varphi)\mu_{a,r}(\partial_z) \\
&= \langle \partial_z, \partial_\varphi \rangle^B + \mu_{a,r}(\partial_z)\mu_{a,r}(\partial_\varphi) \\
&= \langle \partial_z, \partial_\varphi \rangle^B
\end{aligned}$$

and also for $b = (\varphi, z)$ and $b_0 = (0, -1)$

$$\begin{aligned}
T_{\mathrm{id}} k_{a,r}(v(z)\partial_\varphi) &= -\mu_{a,r}(v(z)\partial_\varphi) + \mu_{a,r}(v(z)\partial_\varphi)(b_0) - \int_{b_0}^{b} \iota_{v(\zeta)\partial_\psi} \underbrace{\tau_a}_{=\zeta a(\zeta)\,\mathrm{d}\psi \wedge \mathrm{d}\zeta} \\
&= -v(z)\mu_{a,r}(\partial_\varphi) + v(-1)\mu_{a,r}(\partial_\varphi)(0,-1) - \int_{(0,-1)}^{(\varphi,z)} v(\zeta)\zeta a(\zeta)\,\mathrm{d}\zeta \\
&= v(z)m_{a,r}(z) - v(-1)\underbrace{m_{a,r}(-1)}_{=r} - \int_{-1}^{z} v(\zeta)\zeta a(\zeta)\,\mathrm{d}\zeta \qquad (4.30) \\
&= \int_{-1}^{z} m_{a,r}(\zeta)v'(\zeta)\,\mathrm{d}\zeta.
\end{aligned}$$

Let now $V \in T_{\mathrm{id}}\mathrm{Diff}_R^s(B \times S^1)$, i. e. $V = V^\varphi(\varphi,z)\partial_\varphi + V^z(\varphi,z)\partial_z + V^\theta(\varphi,z)\partial_\theta$. We further define $p_{\mathrm{id}}^B : T_{\mathrm{id}}\mathrm{Diff}_R^s(B \times S^1) \to H^s([-1,1], \mathbb{R})$ and $p_{\mathrm{id}}^R : T_{\mathrm{id}}\mathrm{Diff}_R^s(B \times S^1) \to \mathbb{R}$ by

$$P_{\mathrm{id}}(V) = p_{\mathrm{id}}^B(V)(z)\partial_\varphi + \Big(T_{\mathrm{id}} k_{a,r}(p_{\mathrm{id}}^B(V)(z)\partial_\varphi) + p_{\mathrm{id}}^R(V)\Big)\partial_\theta. \qquad (4.31)$$

For any $V \in T_{\mathrm{id}}\mathrm{Diff}_{R,\mathrm{vol}}^s(B \times S^1)$, we have $P_{\mathrm{id}}(V) \in T_{\mathrm{id}}\mathrm{Diff}_{\omega_a,\lambda_{a,r}}^s(B \times S^1)$, i. e. $p_{\mathrm{id}}^B(V)(z)$ only depends on z and $p_{\mathrm{id}}^R(V) \in \mathbb{R}$. Then for any $W \in T_{\mathrm{id}}\mathrm{Diff}_{\omega_a,\lambda_{a,r}}^s(B \times S^1)$, i. e.

$$W = w(z)\partial_\varphi + \Big(T_{\mathrm{id}} k_{a,r}(w(z)\partial_\varphi) + x\Big)\partial_\theta$$

with $w \in H^s([-1,1],\mathbb{R})$ and $x \in \mathbb{R}$ arbitrary, we need to have

$$
\begin{aligned}
0 &\overset{!}{=} \big(V - P_{\mathrm{id}}(V), W\big) \\
&= \int_{B\times S^1} \big\langle V - P_{\mathrm{id}}(V), W \big\rangle \lambda_{a,r} \wedge \omega_a \\
&= \int_{B\times S^1} \Big\langle V^\varphi \partial_\varphi + V^z \partial_z + V^\theta \partial_\theta - p^B_{\mathrm{id}}(V)(z)\partial_\varphi \\
&\qquad\qquad - \Big(T_{\mathrm{id}} k_{a,r}(p^B_{\mathrm{id}}(V)(z)\partial_\varphi) + p^R_{\mathrm{id}}(V)\Big)\partial_\theta, \\
&\qquad\qquad\qquad\qquad\qquad W\Big\rangle \big(a(z)\,\mathrm{d}\theta \wedge \mathrm{d}\varphi \wedge \mathrm{d}z\big) \\
&= \int_{B\times S^1} \Big(V^\varphi - p^B_{\mathrm{id}}(V)(z)\Big)\langle \partial_\varphi,\ W\rangle + V^z \langle \partial_z,\ W\rangle \\
&\quad + \Big(V^\theta - T_{\mathrm{id}} k_{a,r}(p^B_{\mathrm{id}}(V)(z)\partial_\varphi) - p^R_{\mathrm{id}}(V)\Big)\langle \partial_\theta,\ W\rangle \\
&\qquad\qquad\qquad\qquad\qquad\qquad a(z)\,\mathrm{d}\theta \wedge \mathrm{d}\varphi \wedge \mathrm{d}z \\
&= \int_{B\times S^1} \Big(V^\varphi - p^B_{\mathrm{id}}(V)(z)\Big)\Big[w(z)\langle \partial_\varphi, \partial_\varphi\rangle + \Big(T_{\mathrm{id}} k_{a,r}(w(z)\partial_\varphi) + x\Big)\langle \partial_\varphi, \partial_\theta\rangle\Big] \\
&\qquad\qquad + V^z \Big[w(z)\langle \partial_z, \partial_\varphi\rangle + \Big(T_{\mathrm{id}} k_{a,r}(w(z)\partial_\varphi) + x\Big)\langle \partial_z, \partial_\theta\rangle\Big] \\
&\qquad\qquad\quad + \Big(V^\theta - T_{\mathrm{id}} k_{a,r}(p^B_{\mathrm{id}}(V)(z)\partial_\varphi) - p^R_{\mathrm{id}}(V)\Big) \cdot \\
&\qquad\qquad \cdot \Big[w(z)\langle \partial_\theta, \partial_\varphi\rangle + \Big(T_{\mathrm{id}} k_{a,r}(w(z)\partial_\varphi) + x\Big)\langle \partial_\theta, \partial_\theta\rangle\Big] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad a(z)\,\mathrm{d}\theta \wedge \mathrm{d}\varphi \wedge \mathrm{d}z \\
&= \int_{B\times S^1} \Big(V^\varphi - p^B_{\mathrm{id}}(V)(z)\Big)\Big[w(z)\Big(\langle \partial_\varphi, \partial_\varphi\rangle^B + \mu_{a,r}(\partial_\varphi)^2\Big) \\
&\qquad\qquad + \Big(T_{\mathrm{id}} k_{a,r}(w(z)\partial_\varphi) + x\Big)\mu_{a,r}(\partial_\varphi)\Big] \\
&\quad + V^z \Big[w(z) \cdot \langle \partial_z, \partial_\varphi\rangle^B + \Big(T_{\mathrm{id}} k_{a,r}(w(z)\partial_\varphi) + x\Big) \cdot 0\Big] \\
&\quad + \Big(V^\theta - T_{\mathrm{id}} k_{a,r}(p^B_{\mathrm{id}}(V)(z)\partial_\varphi) - p^R_{\mathrm{id}}(V)\Big) \cdot \\
&\quad \cdot \Big[w(z)\mu_{a,r}(\partial_\varphi) + T_{\mathrm{id}} k_{a,r}(w(z)\partial_\varphi) + x\Big] \\
&\qquad\qquad\qquad\qquad\qquad\qquad a(z)\,\mathrm{d}\theta \wedge \mathrm{d}\varphi \wedge \mathrm{d}z \\
&= \int_B w(z)\Big[\Big(V^\varphi - p^B_{\mathrm{id}}(V)(z)\Big)\Big(\langle \partial_\varphi, \partial_\varphi\rangle^B + \mu_{a,r}(\partial_\varphi)^2\Big) \\
&\qquad\qquad\qquad + V^z \cdot \langle \partial_z, \partial_\varphi\rangle^B \\
&\quad + \Big(V^\theta - T_{\mathrm{id}} k_{a,r}(p^B_{\mathrm{id}}(V)(z)\partial_\varphi) - p^R_{\mathrm{id}}(V)\Big)\mu_{a,r}(\partial_\varphi)\Big] \\
&\quad + T_{\mathrm{id}} k_{a,r}(w(z)\partial_\varphi)\Big[\Big(V^\varphi - p^B_{\mathrm{id}}(V)(z)\Big)\mu_{a,r}(\partial_\varphi) \\
&\quad + \Big(V^\theta - T_{\mathrm{id}} k_{a,r}(p^B_{\mathrm{id}}(V)(z)\partial_\varphi) - p^R_{\mathrm{id}}(V)\Big)\Big] \\
&\qquad + x\Big[\Big(V^\varphi - p^B_{\mathrm{id}}(V)(z)\Big)\mu_{a,r}(\partial_\varphi) \\
&\quad + \Big(V^\theta - T_{\mathrm{id}} k_{a,r}(p^B_{\mathrm{id}}(V)(z)\partial_\varphi) - p^R_{\mathrm{id}}(V)\Big)\Big] \\
&\qquad\qquad\qquad\qquad\qquad a(z)\,\mathrm{d}\varphi \wedge \mathrm{d}z
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{-1}^{1} w(z)\Big[\int_{S^1} V^{\varphi}\langle\partial_\varphi,\partial_\varphi\rangle^B \,\mathrm{d}\varphi - \int_{S^1}\langle\partial_\varphi,\partial_\varphi\rangle^B\,\mathrm{d}\varphi\cdot p^B_{\mathrm{id}}(V)(z) \\
&\qquad + \Big(\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi - p^B_{\mathrm{id}}(V)(z)\Big)\mu_{a,r}(\partial_\varphi)^2 \\
&\qquad\quad + \int_{S^1} V^{z}\langle\partial_z,\partial_\varphi\rangle^B\,\mathrm{d}\varphi \\
&\quad + \Big(\int_{S^1} V^{\theta}\,\mathrm{d}\varphi - T_{\mathrm{id}}k_{a,r}(p^B_{\mathrm{id}}(V)(z)\partial_\varphi) - p^R_{\mathrm{id}}(V)\Big)\mu_{a,r}(\partial_\varphi)\Big]a(z)\,\mathrm{d}z \\
&\quad + \int_{-1}^{1} T_{\mathrm{id}}k_{a,r}(w(z)\partial_\varphi)\Big[\Big(\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi - p^B_{\mathrm{id}}(V)(z)\Big)\mu_{a,r}(\partial_\varphi) \\
&\qquad + \int_{S^1} V^{\theta}\,\mathrm{d}\varphi - T_{\mathrm{id}}k_{a,r}(p^B_{\mathrm{id}}(V)(z)\partial_\varphi) - p^R_{\mathrm{id}}(V)\Big]a(z)\,\mathrm{d}z \\
&\quad + x\int_{-1}^{1}\Big[\Big(\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi - p^B_{\mathrm{id}}(V)(z)\Big)\mu_{a,r}(\partial_\varphi) \\
&\qquad + \int_{S^1} V^{\theta}\,\mathrm{d}\varphi - T_{\mathrm{id}}k_{a,r}(p^B_{\mathrm{id}}(V)(z)\partial_\varphi) - p^R_{\mathrm{id}}(V)\Big]a(z)\,\mathrm{d}z.
\end{aligned}
\tag{4.32}
$$

For the coefficient of x to vanish, we need to have

$$
\begin{aligned}
0 &= \int_{-1}^{1}\Big[\Big(\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi - p^B_{\mathrm{id}}(V)(z)\Big)\mu_{a,r}(\partial_\varphi) \\
&\qquad + \int_{S^1} V^{\theta}\,\mathrm{d}\varphi - T_{\mathrm{id}}k_{a,r}(p^B_{\mathrm{id}}(V)(z)\partial_\varphi) - p^R_{\mathrm{id}}(V)\Big]a(z)\,\mathrm{d}z \\
&= \int_{-1}^{1}\Big[\mu_{a,r}(\partial_\varphi)\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi - \mu_{a,r}(\partial_\varphi)p^B_{\mathrm{id}}(V)(z) \\
&\qquad + \int_{S^1} V^{\theta}\,\mathrm{d}\varphi - T_{\mathrm{id}}k_{a,r}(p^B_{\mathrm{id}}(V)(z)\partial_\varphi)\Big]a(z)\,\mathrm{d}z - \underbrace{\int_{-1}^{1} a(z)\,\mathrm{d}z}_{=:\mathrm{vol}_a(B\times S^1)}\cdot p^R_{\mathrm{id}}(V),
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
&\mathrm{vol}_a(B\times S^1)\cdot p^R_{\mathrm{id}}(V) = \\
&\qquad = \int_{-1}^{1}\Big[\mu_{a,r}(\partial_\varphi)\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi - \mu_{a,r}(\partial_\varphi)p^B_{\mathrm{id}}(V)(z) \\
&\qquad\quad + \int_{S^1} V^{\theta}\,\mathrm{d}\varphi - T_{\mathrm{id}}k_{a,r}(p^B_{\mathrm{id}}(V)(z)\partial_\varphi)\Big]a(z)\,\mathrm{d}z.
\end{aligned}
\tag{4.33}
$$

Note that for any $z \in [-1,1]$ and functions $b(\zeta)$ and $u(\zeta)$, we have

$$\begin{aligned}
&\int_z^1 b(\zeta) \cdot T_{\mathrm{id}} k_{a,r}(u)\,\mathrm{d}\zeta = \\
&\overset{(4.30)}{=} \int_z^1 b(\zeta)\Big[u(\zeta) m_{a,r}(\zeta) - r u(-1) - \int_{-1}^{\zeta} u(\beta)\beta a(\beta)\,\mathrm{d}\beta\Big]\mathrm{d}\zeta \\
&= \int_z^1 b(\zeta) u(\zeta) m_{a,r}(\zeta)\,\mathrm{d}\zeta - r u(-1) \int_z^1 b(\zeta)\,\mathrm{d}\zeta \\
&\qquad\qquad - \int_z^1 b(\zeta) \int_{-1}^{\zeta} \beta u(\beta) a(\beta)\,\mathrm{d}\beta\,\mathrm{d}\zeta. \qquad (4.34)
\end{aligned}$$

Integrating the last term by parts yields

$$\begin{aligned}
&- \int_z^1 b(\zeta) \int_{-1}^{\zeta} \beta u(\beta) a(\beta)\,\mathrm{d}\beta\,\mathrm{d}\zeta = \\
&\quad = - \int_z^{\zeta} b(\beta)\,\mathrm{d}\beta \int_{-1}^{\zeta} \beta u(\beta) a(\beta)\,\mathrm{d}\beta\,\Big|_{\zeta=z}^{1} + \int_z^1 \int_z^{\zeta} b(\beta)\,\mathrm{d}\beta \cdot \zeta u(\zeta) a(\zeta)\,\mathrm{d}\zeta \\
&\quad = - \int_z^1 b(\beta)\,\mathrm{d}\beta \int_{-1}^1 \beta u(\beta) a(\beta)\,\mathrm{d}\beta + \int_z^1 \int_z^{\zeta} b(\beta)\,\mathrm{d}\beta \cdot \zeta u(\zeta) a(\zeta)\,\mathrm{d}\zeta \\
&\quad = - \int_z^1 b(\beta)\,\mathrm{d}\beta \int_{-1}^1 \zeta u(\zeta) a(\zeta)\,\mathrm{d}\zeta + \int_z^1 \int_z^{\zeta} b(\beta)\,\mathrm{d}\beta \cdot \zeta u(\zeta) a(\zeta)\,\mathrm{d}\zeta.
\end{aligned} \qquad (4.35)$$

Plugging Eq. (4.35) back into Eq. (4.34) yields

$$\begin{aligned}
&\int_z^1 b(\zeta) \cdot T_{\mathrm{id}} k_{a,r}(u(\zeta)\partial_\varphi)\,\mathrm{d}\zeta = \\
&\quad = \int_z^1 b(\zeta) u(\zeta) m_{a,r}(\zeta)\,\mathrm{d}\zeta - r u(-1) \int_z^1 b(\zeta)\,\mathrm{d}\zeta \\
&\qquad - \int_z^1 b(\beta)\,\mathrm{d}\beta \int_{-1}^1 \zeta u(\zeta) a(\zeta)\,\mathrm{d}\zeta + \int_z^1 \int_z^{\zeta} b(\beta)\,\mathrm{d}\beta \cdot \zeta u(\zeta) a(\zeta)\,\mathrm{d}\zeta \\
&\quad = -r u(-1) \int_z^1 b(\zeta)\,\mathrm{d}\zeta - \int_z^1 b(\beta)\,\mathrm{d}\beta \int_{-1}^1 \zeta u(\zeta) a(\zeta)\,\mathrm{d}\zeta \\
&\qquad + \int_z^1 \Big[b(\zeta) m_{a,r}(\zeta) + \int_z^{\zeta} b(\beta)\,\mathrm{d}\beta \cdot \zeta a(\zeta)\Big] u(\zeta)\,\mathrm{d}\zeta. \qquad (4.36)
\end{aligned}$$

For $z = -1$, this is

$$\begin{aligned}
&\int_{-1}^{1} b(\zeta)\cdot T_{\mathrm{id}} k_{a,r}(u(\zeta)\partial_\varphi)\,\mathrm{d}\zeta = \\
&\quad = -ru(-1)\int_{-1}^{1} b(\zeta)\,\mathrm{d}\zeta - \int_{-1}^{1} b(\beta)\,\mathrm{d}\beta \int_{-1}^{1} \zeta u(\zeta) a(\zeta)\,\mathrm{d}\zeta \\
&\qquad\qquad + \int_{-1}^{1}\Big[b(\zeta) m_{a,r}(\zeta) + \int_{-1}^{\zeta} b(\beta)\,\mathrm{d}\beta\cdot\zeta a(\zeta)\Big] u(\zeta)\,\mathrm{d}\zeta \\
&\quad = -ru(-1)\int_{-1}^{1} b(\zeta)\,\mathrm{d}\zeta \\
&\qquad + \int_{-1}^{1}\Big[b(\zeta) m_{a,r}(\zeta) + \int_{-1}^{\zeta} b(\beta)\,\mathrm{d}\beta\cdot\zeta a(\zeta) - \int_{-1}^{1} b(\beta)\,\mathrm{d}\beta\cdot\zeta a(\zeta)\Big] u(\zeta)\,\mathrm{d}\zeta \\
&\quad = -ru(-1)\int_{-1}^{1} b(\zeta)\,\mathrm{d}\zeta + \int_{-1}^{1}\Big[b(\zeta) m_{a,r}(\zeta) - \int_{\zeta}^{1} b(\beta)\,\mathrm{d}\beta\cdot\zeta a(\zeta)\Big] u(\zeta)\,\mathrm{d}\zeta.
\end{aligned} \tag{4.37}$$

Plugging Eq. (4.37) for $b = a$ and $u = p^B_{\mathrm{id}}(V)(z)$ into Eq. (4.33) yields

$$\begin{aligned}
&\mathrm{vol}_a(B\times S^1)\cdot p^R_{\mathrm{id}}(V) = \\
&\quad \overset{(4.33)}{=} \int_{-1}^{1}\Big[\mu_{a,r}(\partial_\varphi)\int_{S^1} V^\varphi\,\mathrm{d}\varphi - \mu_{a,r}(\partial_\varphi) p^B_{\mathrm{id}}(V)(z) \\
&\qquad\qquad + \int_{S^1} V^\theta\,\mathrm{d}\varphi - T_{\mathrm{id}} k_{a,r}(p^B_{\mathrm{id}}(V)(z)\partial_\varphi)\Big] a(z)\,\mathrm{d}z \\
&\quad = \int_{-1}^{1}\Big[\mu_{a,r}(\partial_\varphi)\int_{S^1} V^\varphi\,\mathrm{d}\varphi - \mu_{a,r}(\partial_\varphi) p^B_{\mathrm{id}}(V)(z) \\
&\qquad\qquad + \int_{S^1} V^\theta\,\mathrm{d}\varphi\Big] a(z)\,\mathrm{d}z - \int_{-1}^{1} T_{\mathrm{id}} k_{a,r}(p^B_{\mathrm{id}}(V)(z)\partial_\varphi) a(z)\,\mathrm{d}z \\
&\quad \overset{(4.37)}{=} \int_{-1}^{1}\Big[-m_{a,r}(z)\int_{S^1} V^\varphi\,\mathrm{d}\varphi + \int_{S^1} V^\theta\,\mathrm{d}\varphi\Big] a(z)\,\mathrm{d}z \\
&\qquad\qquad\qquad + \int_{-1}^{1} m_{a,r}(z) p^B_{\mathrm{id}}(V)(z) a(z)\,\mathrm{d}z \\
&\qquad\qquad\qquad + r p^B_{\mathrm{id}}(V)(-1)\underbrace{\int_{-1}^{z} a(\zeta)\,\mathrm{d}\zeta}_{=\mathrm{vol}_a(B\times S^1)} \\
&\qquad\quad - \int_{-1}^{1}\Big[a(z) m_{a,r}(z) - \int_{z}^{1} a(\zeta)\,\mathrm{d}\zeta\cdot z a(z)\Big] p^B_{\mathrm{id}}(V)(z)\,\mathrm{d}z \\
&\quad = \int_{-1}^{1}\Big[-m_{a,r}(z)\int_{S^1} V^\varphi\,\mathrm{d}\varphi + \int_{S^1} V^\theta\,\mathrm{d}\varphi\Big] a(z)\,\mathrm{d}z \\
&\qquad\quad + \int_{-1}^{1}\int_{z}^{1} a(\zeta)\,\mathrm{d}\zeta\cdot z a(z) p^B_{\mathrm{id}}(V)(z)\,\mathrm{d}z \\
&\qquad\qquad\qquad\qquad + r\mathrm{vol}_a(B\times S^1)\cdot p^B_{\mathrm{id}}(V)(-1).
\end{aligned} \tag{4.38}$$

Similarly, all terms containing w in Eq. (4.32) are

$$
\begin{aligned}
0 = \int_{-1}^{1} w(z) \Big[\int_{S^1} V^{\varphi} \langle \partial_\varphi, \partial_\varphi \rangle^B \, \mathrm{d}\varphi - \int_{S^1} \langle \partial_\varphi, \partial_\varphi \rangle^B \, \mathrm{d}\varphi \cdot p^B_{\mathrm{id}}(V)(z) \\
+ \Big(\int_{S^1} V^{\varphi} \, \mathrm{d}\varphi - p^B_{\mathrm{id}}(V)(z) \Big) \mu_{a,r}(\partial_\varphi)^2 \\
+ \int_{S^1} V^{z} \langle \partial_z, \partial_\varphi \rangle^B \, \mathrm{d}\varphi \\
+ \Big(\int_{S^1} V^{\theta} \, \mathrm{d}\varphi - T_{\mathrm{id}} k_{a,r}(p^B_{\mathrm{id}}(V)(z)\partial_\varphi) - p^R_{\mathrm{id}}(V) \Big) \mu_{a,r}(\partial_\varphi) \Big] a(z) \, \mathrm{d}z \\
+ \int_{-1}^{1} T_{\mathrm{id}} k_{a,r}(w(z)\partial_\varphi) \Big[\Big(\int_{S^1} V^{\varphi} \, \mathrm{d}\varphi - p^B_{\mathrm{id}}(V)(z) \Big) \mu_{a,r}(\partial_\varphi) \\
+ \int_{S^1} V^{\theta} \, \mathrm{d}\varphi - T_{\mathrm{id}} k_{a,r}(p^B_{\mathrm{id}}(V)(z)\partial_\varphi) - p^R_{\mathrm{id}}(V) \Big] a(z) \, \mathrm{d}z.
\end{aligned}
\tag{4.39}
$$

For the second integral (i. e. the last two lines in the previous equation), we use Eq. (4.37) with $u = w$ and

$$
\begin{aligned}
b = \Big[\Big(\int_{S^1} V^{\varphi} \, \mathrm{d}\varphi - p^B_{\mathrm{id}}(V)(z) \Big) \mu_{a,r}(\partial_\varphi) \\
+ \int_{S^1} V^{\theta} \, \mathrm{d}\varphi - T_{\mathrm{id}} k_{a,r}(p^B_{\mathrm{id}}(V)(z)\partial_\varphi) - p^R_{\mathrm{id}}(V) \Big] a(z)
\end{aligned}
$$

to get

$$
\begin{aligned}
\int_{-1}^{1} T_{\mathrm{id}} k_{a,r}(w(z)\partial_\varphi) \Big[\Big(\int_{S^1} V^{\varphi} \, \mathrm{d}\varphi - p^B_{\mathrm{id}}(V)(z) \Big) \mu_{a,r}(\partial_\varphi) \\
+ \int_{S^1} V^{\theta} \, \mathrm{d}\varphi - T_{\mathrm{id}} k_{a,r}(p^B_{\mathrm{id}}(V)(z)\partial_\varphi) - p^R_{\mathrm{id}}(V) \Big] a(z) \, \mathrm{d}z = \\
\overset{(4.37)}{=} \int_{-1}^{1} \Big[\Big[\Big(\int_{S^1} V^{\varphi} \, \mathrm{d}\varphi - p^B_{\mathrm{id}}(V)(z) \Big) \mu_{a,r}(\partial_\varphi) \\
+ \int_{S^1} V^{\theta} \, \mathrm{d}\varphi - T_{\mathrm{id}} k_{a,r}(p^B_{\mathrm{id}}(V)(z)\partial_\varphi) - p^R_{\mathrm{id}}(V) \Big] a(z) m_{a,r}(z) \\
- \int_{z}^{1} \Big[\Big(\int_{S^1} V^{\varphi} \, \mathrm{d}\varphi - p^B_{\mathrm{id}}(V)(\zeta) \Big) \mu_{a,r}(\partial_\varphi) \\
+ \int_{S^1} V^{\theta} \, \mathrm{d}\varphi - T_{\mathrm{id}} k_{a,r}(p^B_{\mathrm{id}}(V)(\zeta)\partial_\varphi) - p^R_{\mathrm{id}}(V) \Big] a(\zeta) \, \mathrm{d}\zeta \cdot z a(z) \Big] \cdot \\
\cdot w(z) \, \mathrm{d}z \\
- r w(-1) \int_{-1}^{1} \Big[\Big(\int_{S^1} V^{\varphi} \, \mathrm{d}\varphi - p^B_{\mathrm{id}}(V)(z) \Big) \mu_{a,r}(\partial_\varphi) \\
+ \int_{S^1} V^{\theta} \, \mathrm{d}\varphi - T_{\mathrm{id}} k_{a,r}(p^B_{\mathrm{id}}(V)(z)\partial_\varphi) - p^R_{\mathrm{id}}(V) \Big] a(z) \, \mathrm{d}z.
\end{aligned}
\tag{4.40}
$$

Trying to simplify the coefficient of $-rw(-1)$ in this equation yields

$$
\begin{aligned}
&\int_{-1}^{1}\Big[\Big(\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi - p^B_{\mathrm{id}}(V)(z)\Big)\mu_{a,r}(\partial_\varphi)\\
&\quad + \int_{S^1} V^{\theta}\,\mathrm{d}\varphi - T_{\mathrm{id}}k_{a,r}(p^B_{\mathrm{id}}(V)(z)\partial_\varphi) - p^R_{\mathrm{id}}(V)\Big]a(z)\,\mathrm{d}z =\\
&\quad = \int_{-1}^{1}\Big[-m_{a,r}(z)\Big(\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi - p^B_{\mathrm{id}}(V)(z)\Big) + \int_{S^1} V^{\theta}\,\mathrm{d}\varphi\Big]a(z)\,\mathrm{d}z\\
&\qquad\qquad - \int_{-1}^{1} T_{\mathrm{id}}k_{a,r}(p^B_{\mathrm{id}}(V)(z)\partial_\varphi)a(z)\,\mathrm{d}z - p^R_{\mathrm{id}}(V)\underbrace{\int_{-1}^{1} a(z)\,\mathrm{d}z}_{=\mathrm{vol}_a(B\times S^1)}\\
&\quad \overset{(4.38)}{=} \int_{-1}^{1}\Big[-m_{a,r}(z)\Big(\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi - p^B_{\mathrm{id}}(V)(z)\Big) + \int_{S^1} V^{\theta}\,\mathrm{d}\varphi\Big]a(z)\,\mathrm{d}z\\
&\qquad\qquad - \int_{-1}^{1}\Big[-m_{a,r}(z)\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi + \int_{S^1} V^{\theta}\,\mathrm{d}\varphi\Big]a(z)\,\mathrm{d}z\\
&\qquad\qquad\quad - \int_{-1}^{1}\int_{z}^{1} a(\zeta)\,\mathrm{d}\zeta\cdot za(z)p^B_{\mathrm{id}}(V)(z)\,\mathrm{d}z\\
&\qquad\qquad\qquad - r\mathrm{vol}_a(B\times S^1)\cdot p^B_{\mathrm{id}}(V)(-1)\\
&\qquad\qquad\qquad\qquad - \int_{-1}^{1} T_{\mathrm{id}}k_{a,r}(p^B_{\mathrm{id}}(V)(z)\partial_\varphi)a(z)\,\mathrm{d}z\\
&\quad \overset{(4.37)}{=} \int_{-1}^{1} m_{a,r}(z)p^B_{\mathrm{id}}(V)(z)a(z)\,\mathrm{d}z\\
&\qquad\qquad - \int_{-1}^{1}\int_{z}^{1} a(\zeta)\,\mathrm{d}\zeta\cdot za(z)p^B_{\mathrm{id}}(V)(z)\,\mathrm{d}z\\
&\qquad\qquad\quad - r\mathrm{vol}_a(B\times S^1)\cdot p^B_{\mathrm{id}}(V)(-1)\\
&\qquad\qquad\quad + rp^B_{\mathrm{id}}(V)(-1)\underbrace{\int_{-1}^{1} a(\zeta)\,\mathrm{d}\zeta}_{=\mathrm{vol}_a(B\times S^1)}\\
&\qquad - \int_{-1}^{1}\Big[a(\zeta)m_{a,r}(\zeta) - \int_{\zeta}^{1} a(\beta)\,\mathrm{d}\beta\cdot\zeta a(\zeta)\Big]p^B_{\mathrm{id}}(V)(\zeta)\,\mathrm{d}\zeta\\
&\quad = 0,
\end{aligned}
$$

hence the previous equation (4.40) becomes

$$\begin{aligned}
&\int_{-1}^{1} T_{\mathrm{id}} k_{a,r}(w(z)\partial_\varphi)\Big[\Big(\int_{S^1} V^\varphi \,\mathrm{d}\varphi - p_{\mathrm{id}}^B(V)(z)\Big)\mu_{a,r}(\partial_\varphi) \\
&\quad + \int_{S^1} V^\theta \,\mathrm{d}\varphi - T_{\mathrm{id}} k_{a,r}(p_{\mathrm{id}}^B(V)(z)\partial_\varphi) - p_{\mathrm{id}}^R(V)\Big]a(z)\,\mathrm{d}z = \\
&\quad \overset{(4.40)}{=} \int_{-1}^{1}\Big[\Big[\Big(\int_{S^1} V^\varphi \,\mathrm{d}\varphi - p_{\mathrm{id}}^B(V)(z)\Big)\mu_{a,r}(\partial_\varphi) \\
&\qquad + \int_{S^1} V^\theta \,\mathrm{d}\varphi - T_{\mathrm{id}} k_{a,r}(p_{\mathrm{id}}^B(V)(z)\partial_\varphi) - p_{\mathrm{id}}^R(V)\Big]a(z)m_{a,r}(z) \\
&\qquad\qquad - \int_{z}^{1}\Big[\Big(\int_{S^1} V^\varphi \,\mathrm{d}\varphi - p_{\mathrm{id}}^B(V)(\zeta)\Big)\mu_{a,r}(\partial_\varphi) \\
&\qquad + \int_{S^1} V^\theta \,\mathrm{d}\varphi - T_{\mathrm{id}} k_{a,r}(p_{\mathrm{id}}^B(V)(\zeta)\partial_\varphi) - p_{\mathrm{id}}^R(V)\Big]a(\zeta)\,\mathrm{d}\zeta \cdot za(z)\Big]\cdot \\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot w(z)\,\mathrm{d}z.
\end{aligned} \tag{4.41}$$

Going back to Eq. (4.39), we get

$$\begin{aligned}
0 \overset{(4.39)}{=} &\int_{-1}^{1} w(z)\Big[\int_{S^1} V^\varphi \langle\partial_\varphi, \partial_\varphi\rangle^B \,\mathrm{d}\varphi - \int_{S^1}\langle\partial_\varphi, \partial_\varphi\rangle^B \,\mathrm{d}\varphi \cdot p_{\mathrm{id}}^B(V)(z) \\
&\qquad + \Big(\int_{S^1} V^\varphi \,\mathrm{d}\varphi - p_{\mathrm{id}}^B(V)(z)\Big)\mu_{a,r}(\partial_\varphi)^2 \\
&\qquad\qquad + \int_{S^1} V^z \langle\partial_z, \partial_\varphi\rangle^B \,\mathrm{d}\varphi \\
&\quad + \Big(\int_{S^1} V^\theta \,\mathrm{d}\varphi - T_{\mathrm{id}} k_{a,r}(p_{\mathrm{id}}^B(V)(z)\partial_\varphi) - p_{\mathrm{id}}^R(V)\Big)\mu_{a,r}(\partial_\varphi)\Big]a(z)\,\mathrm{d}z \\
&+ \int_{-1}^{1} T_{\mathrm{id}} k_{a,r}(w(z)\partial_\varphi)\Big[\Big(\int_{S^1} V^\varphi \,\mathrm{d}\varphi - p_{\mathrm{id}}^B(V)(z)\Big)\mu_{a,r}(\partial_\varphi) \\
&\quad + \int_{S^1} V^\theta \,\mathrm{d}\varphi - T_{\mathrm{id}} k_{a,r}(p_{\mathrm{id}}^B(V)(z)\partial_\varphi) - p_{\mathrm{id}}^R(V)\Big]a(z)\,\mathrm{d}z
\end{aligned}$$

$$
\begin{aligned}
&\overset{(4.41)}{=} \int_{-1}^{1} w(z)\Big[\int_{S^1} V^{\varphi}\langle\partial_\varphi,\partial_\varphi\rangle^B \,\mathrm{d}\varphi - \int_{S^1}\langle\partial_\varphi,\partial_\varphi\rangle^B \,\mathrm{d}\varphi\cdot p^B_{\mathrm{id}}(V)(z)\\
&\qquad\qquad + \Big(\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi - p^B_{\mathrm{id}}(V)(z)\Big)\mu_{a,r}(\partial_\varphi)^2\\
&\qquad\qquad\qquad + \int_{S^1} V^{z}\langle\partial_z,\partial_\varphi\rangle^B\,\mathrm{d}\varphi\\
&\quad + \Big(\int_{S^1} V^{\theta}\,\mathrm{d}\varphi - T_{\mathrm{id}}k_{a,r}(p^B_{\mathrm{id}}(V)(z)\partial_\varphi) - p^R_{\mathrm{id}}(V)\Big)\mu_{a,r}(\partial_\varphi)\Big]a(z)\,\mathrm{d}z\\
&\quad + \int_{-1}^{1}\Big[\Big[\Big(\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi - p^B_{\mathrm{id}}(V)(z)\Big)\mu_{a,r}(\partial_\varphi)\\
&\qquad + \int_{S^1} V^{\theta}\,\mathrm{d}\varphi - T_{\mathrm{id}}k_{a,r}(p^B_{\mathrm{id}}(V)(z)\partial_\varphi) - p^R_{\mathrm{id}}(V)\Big]a(z)m_{a,r}(z)\\
&\qquad\qquad - \int_{z}^{1}\Big[\Big(\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi - p^B_{\mathrm{id}}(V)(\zeta)\Big)\mu_{a,r}(\partial_\varphi)\\
&\quad + \int_{S^1} V^{\theta}\,\mathrm{d}\varphi - T_{\mathrm{id}}k_{a,r}(p^B_{\mathrm{id}}(V)(\zeta)\partial_\varphi) - p^R_{\mathrm{id}}(V)\Big]a(\zeta)\,\mathrm{d}\zeta\cdot za(z)\Big]w(z)\,\mathrm{d}z\\
&= \int_{-1}^{1} w(z)\Big[\int_{S^1} V^{\varphi}\langle\partial_\varphi,\partial_\varphi\rangle^B \,\mathrm{d}\varphi - \int_{S^1}\langle\partial_\varphi,\partial_\varphi\rangle^B \,\mathrm{d}\varphi\cdot p^B_{\mathrm{id}}(V)(z)\\
&\qquad\qquad + \Big(\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi - p^B_{\mathrm{id}}(V)(z)\Big)m^2_{a,r}(z)\\
&\qquad\qquad\qquad + \int_{S^1} V^{z}\langle\partial_z,\partial_\varphi\rangle^B\,\mathrm{d}\varphi\\
&\quad + \Big(\int_{S^1} V^{\theta}\,\mathrm{d}\varphi - T_{\mathrm{id}}k_{a,r}(p^B_{\mathrm{id}}(V)(z)\partial_\varphi) - p^R_{\mathrm{id}}(V)\Big)\Big(-m_{a,r}(z)\Big)\Big]a(z)\,\mathrm{d}z\\
&\quad + \int_{-1}^{1}\Big[\Big[-m_{a,r}(z)\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi + \int_{S^1} V^{\theta}\,\mathrm{d}\varphi\\
&\qquad + m_{a,r}(z)p^B_{\mathrm{id}}(V)(z) - T_{\mathrm{id}}k_{a,r}(p^B_{\mathrm{id}}(V)(z)\partial_\varphi) - p^R_{\mathrm{id}}(V)\Big]a(z)m_{a,r}(z)\\
&\qquad\qquad - \int_{z}^{1}\Big[-m_{a,r}(\zeta)\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi + \int_{S^1} V^{\theta}\,\mathrm{d}\varphi\\
&\quad + m_{a,r}(\zeta)p^B_{\mathrm{id}}(V)(\zeta) - T_{\mathrm{id}}k_{a,r}(p^B_{\mathrm{id}}(V)(\zeta)\partial_\varphi) - p^R_{\mathrm{id}}(V)\Big]a(\zeta)\,\mathrm{d}\zeta\cdot za(z)\Big]w(z)\,\mathrm{d}z
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{-1}^{1} w(z)a(z)\Big[\Big[\int_{S^1} V^{\varphi}\langle\partial_\varphi,\partial_\varphi\rangle^B \,\mathrm{d}\varphi - \int_{S^1}\langle\partial_\varphi,\partial_\varphi\rangle^B \,\mathrm{d}\varphi \cdot p^B_{\mathrm{id}}(V)(z)\\
&\qquad + m^2_{a,r}(z)\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi - m^2_{a,r}(z)p^B_{\mathrm{id}}(V)(z)\\
&\qquad + \int_{S^1} V^{z}\langle\partial_z,\partial_\varphi\rangle^B \,\mathrm{d}\varphi\\
&\qquad - m_{a,r}(z)\int_{S^1} V^{\theta}\,\mathrm{d}\varphi + m_{a,r}(z)T_{\mathrm{id}}k_{a,r}(p^B_{\mathrm{id}}(V)(z)\partial_\varphi) + m_{a,r}(z)p^R_{\mathrm{id}}(V)\\
&\qquad - m^2_{a,r}(z)\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi + m_{a,r}(z)\int_{S^1} V^{\theta}\,\mathrm{d}\varphi\\
&\qquad + m^2_{a,r}(z)p^B_{\mathrm{id}}(V)(z) - m_{a,r}(z)T_{\mathrm{id}}k_{a,r}(p^B_{\mathrm{id}}(V)(z)\partial_\varphi) - m_{a,r}(z)p^R_{\mathrm{id}}(V)\\
&\qquad - z\int_{z}^{1}\Big[-m_{a,r}(\zeta)\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi + \int_{S^1} V^{\theta}\,\mathrm{d}\varphi\\
&\qquad + m_{a,r}(\zeta)p^B_{\mathrm{id}}(V)(\zeta) - T_{\mathrm{id}}k_{a,r}(p^B_{\mathrm{id}}(V)(\zeta)\partial_\varphi) - p^R_{\mathrm{id}}(V)\Big]a(\zeta)\,\mathrm{d}\zeta\Big]\,\mathrm{d}z\\
&= \int_{-1}^{1} w(z)a(z)\Big[\Big[\int_{S^1} V^{\varphi}\langle\partial_\varphi,\partial_\varphi\rangle^B \,\mathrm{d}\varphi - \int_{S^1}\langle\partial_\varphi,\partial_\varphi\rangle^B \,\mathrm{d}\varphi \cdot p^B_{\mathrm{id}}(V)(z)\\
&\qquad + \int_{S^1} V^{z}\langle\partial_z,\partial_\varphi\rangle^B \,\mathrm{d}\varphi\\
&\qquad - z\int_{z}^{1}\Big[-m_{a,r}(\zeta)\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi + \int_{S^1} V^{\theta}\,\mathrm{d}\varphi\\
&\qquad + m_{a,r}(\zeta)p^B_{\mathrm{id}}(V)(\zeta) - T_{\mathrm{id}}k_{a,r}(p^B_{\mathrm{id}}(V)(\zeta)\partial_\varphi) - p^R_{\mathrm{id}}(V)\Big]a(\zeta)\,\mathrm{d}\zeta\Big]\,\mathrm{d}z.
\end{aligned}
$$

This expression has to vanish for every choice of w, hence the coefficient of w has to vanish. This yields

$$
\begin{aligned}
0 &= \int_{S^1} V^{\varphi}\langle\partial_\varphi,\partial_\varphi\rangle^B \,\mathrm{d}\varphi - \int_{S^1}\langle\partial_\varphi,\partial_\varphi\rangle^B \,\mathrm{d}\varphi \cdot p^B_{\mathrm{id}}(V)(z)\\
&\qquad + \int_{S^1} V^{z}\langle\partial_z,\partial_\varphi\rangle^B \,\mathrm{d}\varphi\\
&\qquad - z\int_{z}^{1}\Big[-m_{a,r}(\zeta)\int_{S^1} V^{\varphi}\,\mathrm{d}\varphi + \int_{S^1} V^{\theta}\,\mathrm{d}\varphi\\
&\qquad + m_{a,r}(\zeta)p^B_{\mathrm{id}}(V)(\zeta) - T_{\mathrm{id}}k_{a,r}(p^B_{\mathrm{id}}(V)(\zeta)\partial_\varphi) - p^R_{\mathrm{id}}(V)\Big]a(\zeta)\,\mathrm{d}\zeta
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{S^1} V^{\varphi} \langle \partial_\varphi, \partial_\varphi \rangle^B \,\mathrm{d}\varphi - \int_{S^1} \langle \partial_\varphi, \partial_\varphi \rangle^B \,\mathrm{d}\varphi \cdot p^B_{\mathrm{id}}(V)(z) \\
&\qquad + \int_{S^1} V^z \langle \partial_z, \partial_\varphi \rangle^B \,\mathrm{d}\varphi \\
&\qquad - z \int_z^1 \Big[-m_{a,r}(\zeta) \int_{S^1} V^{\varphi} \,\mathrm{d}\varphi + \int_{S^1} V^{\theta} \,\mathrm{d}\varphi \Big] a(\zeta) \,\mathrm{d}\zeta \\
&\qquad - z \int_z^1 m_{a,r}(\zeta) p^B_{\mathrm{id}}(V)(\zeta) a(\zeta) \,\mathrm{d}\zeta \\
&\qquad + z \int_z^1 T_{\mathrm{id}} k_{a,r}(p^B_{\mathrm{id}}(V)(\zeta) \partial_\varphi) a(\zeta) \,\mathrm{d}\zeta \\
&\qquad + z \int_z^1 p^R_{\mathrm{id}}(V) a(\zeta) \,\mathrm{d}\zeta \\
&\overset{(4.36)}{=} \int_{S^1} V^{\varphi} \langle \partial_\varphi, \partial_\varphi \rangle^B \,\mathrm{d}\varphi - \int_{S^1} \langle \partial_\varphi, \partial_\varphi \rangle^B \,\mathrm{d}\varphi \cdot p^B_{\mathrm{id}}(V)(z) \\
&\qquad + \int_{S^1} V^z \langle \partial_z, \partial_\varphi \rangle^B \,\mathrm{d}\varphi \\
&\qquad - z \int_z^1 \Big[-m_{a,r}(\zeta) \int_{S^1} V^{\varphi} \,\mathrm{d}\varphi + \int_{S^1} V^{\theta} \,\mathrm{d}\varphi \Big] a(\zeta) \,\mathrm{d}\zeta \\
&\qquad - z \int_z^1 m_{a,r}(\zeta) p^B_{\mathrm{id}}(V)(\zeta) a(\zeta) \,\mathrm{d}\zeta \\
&\qquad + z \cdot \Big[-r p^B_{\mathrm{id}}(V)(-1) \int_z^1 a(\zeta) \,\mathrm{d}\zeta - \int_z^1 a(\beta) \,\mathrm{d}\beta \int_{-1}^1 \zeta p^B_{\mathrm{id}}(V)(\zeta) a(\zeta) \,\mathrm{d}\zeta \\
&\qquad + \int_z^1 \Big[a(\zeta) m_{a,r}(\zeta) + \int_z^{\zeta} a(\beta) \,\mathrm{d}\beta \cdot \zeta a(\zeta) \Big] p^B_{\mathrm{id}}(V)(\zeta) \,\mathrm{d}\zeta \Big] \\
&\qquad + z \int_z^1 a(\zeta) \,\mathrm{d}\zeta \cdot p^R_{\mathrm{id}}(V)
\end{aligned}
$$

$$
\begin{aligned}
\overset{(4.38)}{=} & \int_{S^1} V^{\varphi} \langle \partial_\varphi, \partial_\varphi \rangle^B \,\mathrm{d}\varphi - \int_{S^1} \langle \partial_\varphi, \partial_\varphi \rangle^B \,\mathrm{d}\varphi \cdot p^B_{\mathrm{id}}(V)(z) \\
& + \int_{S^1} V^{z} \langle \partial_z, \partial_\varphi \rangle^B \,\mathrm{d}\varphi \\
& - z \int_z^1 \Big[-m_{a,r}(\zeta) \int_{S^1} V^{\varphi} \,\mathrm{d}\varphi + \int_{S^1} V^{\theta} \,\mathrm{d}\varphi \Big] a(\zeta) \,\mathrm{d}\zeta \\
& - z \int_z^1 m_{a,r}(\zeta) p^B_{\mathrm{id}}(V)(\zeta) a(\zeta) \,\mathrm{d}\zeta \\
& \underbrace{- z r p^B_{\mathrm{id}}(V)(-1) \int_z^1 a(\zeta) \,\mathrm{d}\zeta}_{} - z \int_z^1 a(\beta) \,\mathrm{d}\beta \int_{-1}^1 \zeta p^B_{\mathrm{id}}(V)(\zeta) a(\zeta) \,\mathrm{d}\zeta \\
& + z \int_z^1 a(\zeta) m_{a,r}(\zeta) p^B_{\mathrm{id}}(V)(\zeta) \,\mathrm{d}\zeta + z \int_z^1 \int_z^{\zeta} a(\beta) \,\mathrm{d}\beta \cdot \zeta a(\zeta) p^B_{\mathrm{id}}(V)(\zeta) \,\mathrm{d}\zeta \\
& + z \int_z^1 a(\zeta) \,\mathrm{d}\zeta \cdot \frac{1}{\mathrm{vol}_a(B \times S^1)} \Big[\int_{-1}^1 \Big[-m_{a,r}(\zeta) \int_{S^1} V^{\varphi} \,\mathrm{d}\varphi \\
& + \int_{S^1} V^{\theta} \,\mathrm{d}\varphi \Big] a(\zeta) \,\mathrm{d}\zeta + \int_{-1}^1 \int_{\zeta}^1 a(\beta) \,\mathrm{d}\beta \cdot \zeta a(\zeta) p^B_{\mathrm{id}}(V)(\zeta) \,\mathrm{d}\zeta \Big] \\
& \underbrace{+ z \int_z^1 a(\zeta) \,\mathrm{d}\zeta \cdot r p^B_{\mathrm{id}}(V)(-1)}_{} \\
= & \int_{S^1} V^{\varphi} \langle \partial_\varphi, \partial_\varphi \rangle^B \,\mathrm{d}\varphi - \int_{S^1} \langle \partial_\varphi, \partial_\varphi \rangle^B \,\mathrm{d}\varphi \cdot p^B_{\mathrm{id}}(V)(z) \\
& + \int_{S^1} V^{z} \langle \partial_z, \partial_\varphi \rangle^B \,\mathrm{d}\varphi \\
& - z \int_z^1 \Big[-m_{a,r}(\zeta) \int_{S^1} V^{\varphi} \,\mathrm{d}\varphi + \int_{S^1} V^{\theta} \,\mathrm{d}\varphi \Big] a(\zeta) \,\mathrm{d}\zeta \\
& - z \int_z^1 a(\beta) \,\mathrm{d}\beta \int_{-1}^1 \zeta p^B_{\mathrm{id}}(V)(\zeta) a(\zeta) \,\mathrm{d}\zeta \\
& + z \int_z^1 \int_z^{\zeta} a(\beta) \,\mathrm{d}\beta \cdot \zeta a(\zeta) p^B_{\mathrm{id}}(V)(\zeta) \,\mathrm{d}\zeta \\
& + z \int_z^1 a(\zeta) \,\mathrm{d}\zeta \cdot \frac{1}{\mathrm{vol}_a(B \times S^1)} \Big[\int_{-1}^1 \Big[-m_{a,r}(\zeta) \int_{S^1} V^{\varphi} \,\mathrm{d}\varphi \\
& + \int_{S^1} V^{\theta} \,\mathrm{d}\varphi \Big] a(\zeta) \,\mathrm{d}\zeta + \int_{-1}^1 \int_{\zeta}^1 a(\beta) \,\mathrm{d}\beta \cdot \zeta a(\zeta) p^B_{\mathrm{id}}(V)(\zeta) \,\mathrm{d}\zeta \Big]
\end{aligned}
$$

Let $A(z) := \int_{-1}^{z} a(\zeta)\,\mathrm{d}\zeta$, i. e. $A(z)$ is the antiderivative of $a(z)$ satisfying $A(-1) = 0$. Then also $A(1) = \operatorname{vol}_a(B \times S^1)$ and we have

$$
\begin{aligned}
&= \int_{S^1} V^{\varphi}\langle\partial_\varphi,\partial_\varphi\rangle^B\,\mathrm{d}\varphi - \int_{S^1}\langle\partial_\varphi,\partial_\varphi\rangle^B\,\mathrm{d}\varphi\cdot p^B_{\mathrm{id}}(V)(z)\\
&\qquad + \int_{S^1} V^{z}\langle\partial_z,\partial_\varphi\rangle^B\,\mathrm{d}\varphi\\
&\qquad - z\int_z^1\Big[-m_{a,r}(\zeta)\int_{S^1}V^{\varphi}\,\mathrm{d}\varphi + \int_{S^1}V^{\theta}\,\mathrm{d}\varphi\Big]a(\zeta)\,\mathrm{d}\zeta\\
&\qquad - z\big(A(1)-A(z)\big)\int_{-1}^{1}\zeta p^B_{\mathrm{id}}(V)(\zeta)a(\zeta)\,\mathrm{d}\zeta\\
&\qquad + z\int_z^1\big(A(\zeta)-A(z)\big)\zeta a(\zeta)p^B_{\mathrm{id}}(V)(\zeta)\,\mathrm{d}\zeta\\
&\qquad + z\big(A(1)-A(z)\big)\frac{1}{A(1)}\Big[\int_{-1}^{1}\Big[-m_{a,r}(\zeta)\int_{S^1}V^{\varphi}\,\mathrm{d}\varphi+\int_{S^1}V^{\theta}\,\mathrm{d}\varphi\Big]a(\zeta)\,\mathrm{d}\zeta\\
&\qquad\qquad + \int_{-1}^{1}\big(A(1)-A(\zeta)\big)\zeta a(\zeta)p^B_{\mathrm{id}}(V)(\zeta)\,\mathrm{d}\zeta\Big]\\
&= \int_{S^1} V^{\varphi}\langle\partial_\varphi,\partial_\varphi\rangle^B\,\mathrm{d}\varphi - \int_{S^1}\langle\partial_\varphi,\partial_\varphi\rangle^B\,\mathrm{d}\varphi\cdot p^B_{\mathrm{id}}(V)(z)\\
&\qquad + \int_{S^1} V^{z}\langle\partial_z,\partial_\varphi\rangle^B\,\mathrm{d}\varphi\\
&\qquad - z\int_z^1\Big[-m_{a,r}(\zeta)\int_{S^1}V^{\varphi}\,\mathrm{d}\varphi + \int_{S^1}V^{\theta}\,\mathrm{d}\varphi\Big]a(\zeta)\,\mathrm{d}\zeta\\
&\qquad - z\big(A(1)-A(z)\big)\int_{-1}^{1}\zeta p^B_{\mathrm{id}}(V)(\zeta)a(\zeta)\,\mathrm{d}\zeta\\
&\qquad + z\int_z^1 A(\zeta)\zeta a(\zeta)p^B_{\mathrm{id}}(V)(\zeta)\,\mathrm{d}\zeta - zA(z)\int_z^1\zeta a(\zeta)p^B_{\mathrm{id}}(V)(\zeta)\,\mathrm{d}\zeta\\
&\qquad + z\frac{A(1)}{A(1)}\int_{-1}^{1}\Big[-m_{a,r}(\zeta)\int_{S^1}V^{\varphi}\,\mathrm{d}\varphi+\int_{S^1}V^{\theta}\,\mathrm{d}\varphi\Big]a(\zeta)\,\mathrm{d}\zeta\\
&\qquad - z\frac{A(z)}{A(1)}\int_{-1}^{1}\Big[-m_{a,r}(\zeta)\int_{S^1}V^{\varphi}\,\mathrm{d}\varphi+\int_{S^1}V^{\theta}\,\mathrm{d}\varphi\Big]a(\zeta)\,\mathrm{d}\zeta\\
&\qquad + z\big(A(1)-A(z)\big)\frac{A(1)}{A(1)}\int_{-1}^{1}\zeta a(\zeta)p^B_{\mathrm{id}}(V)(\zeta)\,\mathrm{d}\zeta\\
&\qquad - zA(1)\frac{1}{A(1)}\int_{-1}^{1}A(\zeta)\zeta a(\zeta)p^B_{\mathrm{id}}(V)(\zeta)\,\mathrm{d}\zeta\\
&\qquad\qquad + zA(z)\frac{1}{A(1)}\int_{-1}^{1}A(\zeta)\zeta a(\zeta)p^B_{\mathrm{id}}(V)(\zeta)\,\mathrm{d}\zeta
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{S^1} V^{\varphi} \langle \partial_\varphi, \partial_\varphi \rangle^B \,\mathrm{d}\varphi - \int_{S^1} \langle \partial_\varphi, \partial_\varphi \rangle^B \,\mathrm{d}\varphi \cdot p^B_{\mathrm{id}}(V)(z) \\
&\qquad + \int_{S^1} V^{z} \langle \partial_z, \partial_\varphi \rangle^B \,\mathrm{d}\varphi \\
&\qquad + z \int_{-1}^{z} \Big[-m_{a,r}(\zeta) \int_{S^1} V^{\varphi} \,\mathrm{d}\varphi + \int_{S^1} V^{\theta} \,\mathrm{d}\varphi \Big] a(\zeta) \,\mathrm{d}\zeta \\
&\qquad - z \frac{A(z)}{A(1)} \int_{-1}^{1} \Big[-m_{a,r}(\zeta) \int_{S^1} V^{\varphi} \,\mathrm{d}\varphi + \int_{S^1} V^{\theta} \,\mathrm{d}\varphi \Big] a(\zeta) \,\mathrm{d}\zeta \\
&\qquad - z \int_{-1}^{z} A(\zeta) \zeta a(\zeta) p^B_{\mathrm{id}}(V)(\zeta) \,\mathrm{d}\zeta - z A(z) \int_{z}^{1} \zeta a(\zeta) p^B_{\mathrm{id}}(V)(\zeta) \,\mathrm{d}\zeta \\
&\qquad + z A(z) \frac{1}{A(1)} \int_{-1}^{1} A(\zeta) \zeta a(\zeta) p^B_{\mathrm{id}}(V)(\zeta) \,\mathrm{d}\zeta,
\end{aligned}
$$

i. e. $p^B_{\mathrm{id}}(V)(z)$ is defined by

$$
\begin{aligned}
0 &= \int_{S^1} V^{\varphi} \langle \partial_\varphi, \partial_\varphi \rangle^B \,\mathrm{d}\varphi - \int_{S^1} \langle \partial_\varphi, \partial_\varphi \rangle^B \,\mathrm{d}\varphi \cdot p^B_{\mathrm{id}}(V)(z) \\
&\quad + \int_{S^1} V^{z} \langle \partial_z, \partial_\varphi \rangle^B \,\mathrm{d}\varphi \\
&\quad + z \int_{-1}^{z} \Big[-m_{a,r}(\zeta) \int_{S^1} V^{\varphi} \,\mathrm{d}\varphi + \int_{S^1} V^{\theta} \,\mathrm{d}\varphi \Big] a(\zeta) \,\mathrm{d}\zeta \\
&\quad - z \frac{A(z)}{A(1)} \int_{-1}^{1} \Big[-m_{a,r}(\zeta) \int_{S^1} V^{\varphi} \,\mathrm{d}\varphi + \int_{S^1} V^{\theta} \,\mathrm{d}\varphi \Big] a(\zeta) \,\mathrm{d}\zeta \\
&\quad - z \int_{-1}^{z} A(\zeta) \zeta a(\zeta) p^B_{\mathrm{id}}(V)(\zeta) \,\mathrm{d}\zeta - z A(z) \int_{z}^{1} \zeta a(\zeta) p^B_{\mathrm{id}}(V)(\zeta) \,\mathrm{d}\zeta \\
&\quad + z A(z) \frac{1}{A(1)} \int_{-1}^{1} A(\zeta) \zeta a(\zeta) p^B_{\mathrm{id}}(V)(\zeta) \,\mathrm{d}\zeta.
\end{aligned}
\tag{4.42}
$$

Let $f(z) := \int_{S^1} \langle \partial_\varphi, \partial_\varphi \rangle^B \,\mathrm{d}\varphi$ and define a linear operator $K : H^s \to H^s$ by

$$
\begin{aligned}
f(z) \cdot K(u)(z) &:= z \int_{-1}^{z} A(\zeta) \zeta a(\zeta) u(\zeta) \,\mathrm{d}\zeta + z A(z) \int_{z}^{1} \zeta a(\zeta) u(\zeta) \,\mathrm{d}\zeta \\
&\qquad - z A(z) \frac{1}{A(1)} \int_{-1}^{1} A(\zeta) \zeta a(\zeta) u(\zeta) \,\mathrm{d}\zeta
\end{aligned}
\tag{4.43}
$$

and a linear operator $R : H^s(B, \mathbb{R}) \times H^s(B, \mathbb{R}) \times H^s(B, \mathbb{R}) \to H^s([-1,1], \mathbb{R})$ by

$$
\begin{aligned}
R(V^{\varphi}, V^{z}, V^{\theta})(z) &:= \int_{S^1} V^{\varphi} \langle \partial_\varphi, \partial_\varphi \rangle^B \,\mathrm{d}\varphi + \int_{S^1} V^{z} \langle \partial_z, \partial_\varphi \rangle^B \,\mathrm{d}\varphi \\
&\qquad + z \int_{-1}^{z} \Big[-m_{a,r}(\zeta) \int_{S^1} V^{\varphi} \,\mathrm{d}\varphi + \int_{S^1} V^{\theta} \,\mathrm{d}\varphi \Big] a(\zeta) \,\mathrm{d}\zeta \\
&\qquad - z \frac{A(z)}{A(1)} \int_{-1}^{1} \Big[-m_{a,r}(\zeta) \int_{S^1} V^{\varphi} \,\mathrm{d}\varphi + \int_{S^1} V^{\theta} \,\mathrm{d}\varphi \Big] a(\zeta) \,\mathrm{d}\zeta,
\end{aligned}
$$

so that Eq. (4.42) is equivalent to

$$p^B_{\mathrm{id}}(V)(z) + K(p^B_{\mathrm{id}}(V)(z))(z) = \frac{1}{f(z)} R(V^\varphi, V^z, V^\theta)(z).$$

Note that

$$\begin{aligned}
&R\big(V^\varphi \circ \eta^{-1}, V^z \circ \eta^{-1}, V^\theta \circ \eta^{-1}\big)(z) = \\
&= \int_{S^1} V^\varphi \circ \eta^{-1} \langle \partial_\varphi, \partial_\varphi \rangle^B \,\mathrm{d}\varphi + \int_{S^1} V^z \circ \eta^{-1} \langle \partial_z, \partial_\varphi \rangle^B \,\mathrm{d}\varphi \\
&\qquad + z \int_{-1}^{z} \Big[-m_{a,r}(\zeta) \int_{S^1} V^\varphi \circ \eta^{-1} \,\mathrm{d}\varphi + \int_{S^1} V^\theta \circ \eta^{-1} \,\mathrm{d}\varphi \Big] a(\zeta)\,\mathrm{d}\zeta \\
&\qquad - z \frac{A(z)}{A(1)} \int_{-1}^{1} \Big[-m_{a,r}(\zeta) \int_{S^1} V^\varphi \circ \eta^{-1} \,\mathrm{d}\varphi + \int_{S^1} V^\theta \circ \eta^{-1} \,\mathrm{d}\varphi \Big] a(\zeta)\,\mathrm{d}\zeta \\
&= \int_{S^1} V^\varphi \cdot \langle \partial_\varphi, \partial_\varphi \rangle^B \circ \eta \,\mathrm{d}\varphi + \int_{S^1} V^z \cdot \langle \partial_z, \partial_\varphi \rangle^B \circ \eta \,\mathrm{d}\varphi \\
&\qquad + z \int_{-1}^{z} \Big[-m_{a,r}(\zeta) \int_{S^1} V^\varphi \,\mathrm{d}\varphi + \int_{S^1} V^\theta \,\mathrm{d}\varphi \Big] a(\zeta)\,\mathrm{d}\zeta \\
&\qquad - z \frac{A(z)}{A(1)} \int_{-1}^{1} \Big[-m_{a,r}(\zeta) \int_{S^1} V^\varphi \,\mathrm{d}\varphi + \int_{S^1} V^\theta \,\mathrm{d}\varphi \Big] a(\zeta)\,\mathrm{d}\zeta
\end{aligned}$$

is smooth in η.

Lemma 4.36. *Let $k_1, k_2 \in C^\infty([-1,1], \mathbb{R})$ be smooth. Any operator $F : H^s([-1,1], \mathbb{R}) \to H^s([-1,1], \mathbb{R})$ of the form*

(a) $F(u)(z) = k_2(z) \int_{-1}^{1} k_1(\zeta) u(\zeta)\,\mathrm{d}\zeta$

(b) $F(u)(z) = k_2(z) \int_{-1}^{z} k_1(\zeta) u(\zeta)\,\mathrm{d}\zeta$

is compact.

Proof. (a) Since F has its image generated by k_2, it is an operator of rank 1 and therefore compact.

(b) Since multiplication with a smooth function is continuous, we only have to check that $\bar{F}(u)(z) = \int_{-1}^{z} u(\zeta)\,\mathrm{d}\zeta$ is compact. Note that $\bar{F}$ is actually a bounded linear operator $H^s([-1,1], \mathbb{R}) \to H^{s+1}([-1,1], \mathbb{R})$ because we can estimate

$$\begin{aligned}
\|\bar{F}(u)\|^2_{H^{s+1}} &= \int_{-1}^{1} F(u)(z)^2\,\mathrm{d}z + \Big\| \frac{\partial F(u)}{\partial z} \Big\|^2_{H^s} \\
&= \int_{-1}^{1} \Big(\int_{-1}^{z} u(\zeta)\,\mathrm{d}\zeta \Big)^2 \mathrm{d}z + \|u\|^2_{H^s} \\
&\le \int_{-1}^{1} \Big(\int_{-1}^{z} 1^2\,\mathrm{d}\zeta \Big) \Big(\int_{-1}^{z} u^2(\zeta)\,\mathrm{d}\zeta \Big) \mathrm{d}z + \|u\|^2_{H^s}
\end{aligned}$$

$$\leq \int_{-1}^{1} \underbrace{\left(\int_{-1}^{1} 1^2 \,\mathrm{d}\zeta\right)}_{=2} \underbrace{\left(\int_{-1}^{1} u^2(\zeta)\,\mathrm{d}\zeta\right)}_{=\|u\|_{L^2}^2 \leq \|u\|_{H^s}^2} \mathrm{d}z + \|u\|_{H^s}^2$$
$$\leq 5\|u\|_{H^s}^2.$$

Hence, $\bar{F} : H^s \to H^{s+1}$ is continuous. Furthermore, the inclusion $H^{s+1} \hookrightarrow H^s$ is compact by the Sobolev lemma. Therefore, we can write F as the composition of a compact operator with continuous operators, which implies that F is compact. ☐

Corollary 4.37. *The operator K defined in Eq.* (4.43) *is compact.*

Proof. If we rewrite

$$\begin{aligned} K(u)(z) &= \frac{z}{f(z)} \int_{-1}^{z} A(\zeta)\zeta a(\zeta) u(\zeta)\,\mathrm{d}\zeta + \frac{zA(z)}{f(z)} \int_{z}^{1} \zeta a(\zeta) u(\zeta)\,\mathrm{d}\zeta \\ &\qquad - \frac{zA(z)}{f(z)A(1)} \int_{-1}^{1} A(\zeta)\zeta a(\zeta) u(\zeta)\,\mathrm{d}\zeta \\ &= \frac{z}{f(z)} \int_{-1}^{z} A(\zeta)\zeta a(\zeta) u(\zeta)\,\mathrm{d}\zeta - \frac{zA(z)}{f(z)A(1)} \int_{-1}^{1} A(\zeta)\zeta a(\zeta) u(\zeta)\,\mathrm{d}\zeta \\ &\qquad + \frac{zA(z)}{f(z)} \int_{-1}^{1} \zeta a(\zeta) u(\zeta)\,\mathrm{d}\zeta - \frac{zA(z)}{f(z)} \int_{-1}^{z} \zeta a(\zeta) u(\zeta)\,\mathrm{d}\zeta, \end{aligned}$$

then each of the summands is compact by the previous lemma. ☐

Hence, $\mathrm{id} + K$ is a Fredholm operator of degree 0 and our goal is to invert it. To that end, we first compute its kernel.

Lemma 4.38. *The operator* $\mathrm{id} + K$ *is injective.*

Proof. Since K is linear, we have to check that the only solution to $\big(\mathrm{id} + K\big)(u) \equiv 0$ is $u \equiv 0$. To that end, let $u \in H^s([-1,1],\mathbb{R})$ such that $\big(\mathrm{id} + K\big)(u) = 0$. Multiplying this equation with $f(z) \neq 0$ yields

$$\begin{aligned} 0 &= f(z)\big(\mathrm{id} + K\big)(u)(z) \\ &= \underbrace{f(z)}_{\neq 0} \cdot u(z) + z \int_{-1}^{z} A(\zeta)\zeta a(\zeta) u(\zeta)\,\mathrm{d}\zeta \\ &\quad + zA(z) \int_{z}^{1} \zeta a(\zeta) u(\zeta)\,\mathrm{d}\zeta - zA(z) \frac{1}{A(1)} \int_{-1}^{1} A(\zeta)\zeta a(\zeta) u(\zeta)\,\mathrm{d}\zeta. \end{aligned} \tag{4.44}$$

In particular, we immediately get

$$0 = \underbrace{f(-1)}_{\neq 0} \cdot u(-1) \quad \Rightarrow \quad 0 = u(-1),$$
$$0 = \underbrace{f(0)}_{\neq 0} \cdot u(0) \quad \Rightarrow \quad 0 = u(0),$$

$$0 = \underbrace{f(1)}_{\neq 0} \cdot u(1) \quad \Rightarrow \quad 0 = u(1).$$

Since $u(0) = 0$, we can rewrite Eq. (4.44) to

$$\begin{aligned}\frac{f(z)u(z)}{z} = -\int_{-1}^{z} A(\zeta)\zeta a(\zeta)u(\zeta)\,\mathrm{d}\zeta - A(z)\int_{z}^{1} \zeta a(\zeta)u(\zeta)\,\mathrm{d}\zeta \\ + A(z)\frac{1}{A(1)}\int_{-1}^{1} A(\zeta)\zeta a(\zeta)u(\zeta)\,\mathrm{d}\zeta.\end{aligned}$$

Taking the derivative yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}\Big(\frac{f(z)}{z}u(z)\Big) &= -A(z)za(z)u(z) \\ &\quad - a(z)\int_{z}^{1} \zeta a(\zeta)u(\zeta)\,\mathrm{d}\zeta + A(z)za(z)u(z) \\ &\quad + a(z)\frac{1}{A(1)}\int_{-1}^{1} A(\zeta)\zeta a(\zeta)u(\zeta)\,\mathrm{d}\zeta \\ &= -a(z)\int_{z}^{1} \zeta a(\zeta)u(\zeta)\,\mathrm{d}\zeta \\ &\quad + a(z)\frac{1}{A(1)}\int_{-1}^{1} A(\zeta)\zeta a(\zeta)u(\zeta)\,\mathrm{d}\zeta,\end{aligned}$$

or, equivalently,

$$\begin{aligned}\frac{1}{a(z)}\frac{\mathrm{d}}{\mathrm{d}z}\Big(\frac{f(z)}{z}u(z)\Big) = -\int_{z}^{1} \zeta a(\zeta)u(\zeta)\,\mathrm{d}\zeta \\ + \frac{1}{A(1)}\int_{-1}^{1} A(\zeta)\zeta a(\zeta)u(\zeta)\,\mathrm{d}\zeta.\end{aligned}$$

Again taking a derivative yields

$$\frac{\mathrm{d}}{\mathrm{d}z}\Big(\frac{1}{a(z)}\frac{\mathrm{d}}{\mathrm{d}z}\Big(\frac{f(z)}{z}u(z)\Big)\Big) = za(z)u(z).$$

Let $\tilde{w}(z) := \frac{f(z)}{z}u(z)$, then this is equivalent to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}\Big(\frac{1}{a(z)}\frac{\mathrm{d}}{\mathrm{d}z}\tilde{w}(z)\Big) &= za(z)\frac{z}{f(z)}\tilde{w}(z) \\ &= \frac{z^2 a(z)}{f(z)}\tilde{w}(z)\end{aligned}$$

and our initial conditions become $\tilde{w}(-1) = 0 = \tilde{w}(1)$. We change coordinates from z to $y := A(z)$ and define $w(y) := \tilde{w}(A(y))$. Then $\mathrm{d}y = A'(z)\,\mathrm{d}z = a(z)\,\mathrm{d}z$ and we get

$$w''(y) = \underbrace{\frac{\big(A^{-1}(y)\big)^2}{f(A^{-1}(y))}}_{=:F(y)^2} w(y),$$

or, equivalently,

$$0 = w''(y) - F(y)^2 w(y) \tag{4.46}$$

with initial conditions $w(0) = 0 = w(A(1))$. We multiply this equation by $w(y)$ to get

$$0 = w''(y)w(y) - F(y)^2 w(y)^2.$$

Integrating from 0 to $A(1)$ yields

$$\begin{aligned}
0 &= \int_0^{A(1)} \Big(w''(y)w(y) - F(y)^2 w(y)^2\Big)\mathrm{d}y \\
&= w'(y)w(y)\,\Big|_{y=0}^{A(1)} - \int_0^{A(1)} \Big(w'(y)^2 + F(y)^2 w(y)^2\Big)\mathrm{d}y \\
&= -\int_0^{A(1)} \Big(w'(y)^2 + F(y)^2 w(y)^2\Big)\mathrm{d}y.
\end{aligned} \tag{4.47}$$

Hence, any solution to Eq. (4.46) also satisfies Eq. (4.47). Since the integrand $w'(y)^2 + F(y)^2 w(y)^2 \geq 0$, we in particular get

$$0 = w'(y)^2 + F(y)^2 w(y)^2,$$

which is equivalent to $w(y) \equiv 0$. Then also $\tilde{w} \equiv 0$ and $u(z) = \dfrac{\tilde{w}(z)z}{f(z)} \equiv 0$. □

By the Fredholm alternative, $\mathrm{id} + K$ is invertible and $(\mathrm{id} + K)^{-1} : H^s \to H^s$ is a bounded linear operator and hence smooth. Equation (4.42) is now equivalent to

$$p_{\mathrm{id}}^B(V)(z) = (\mathrm{id} + K)^{-1}\Big(\frac{1}{f(z)} R(V^\varphi, V^z, V^\theta)(z)\Big),$$

which can be used to define $p_{\mathrm{id}}^B(V)$. Then, Eq. (4.38) defines $p_{\mathrm{id}}^R(V)$:

$$\begin{aligned}
\mathrm{vol}_a(B \times S^1) \cdot p_{\mathrm{id}}^R(V) = \int_{-1}^1 \Big[-m_{a,r}(z) \int_{S^1} V^\varphi\,\mathrm{d}\varphi + \int_{S^1} V^\theta\,\mathrm{d}\varphi\Big] a(z)\,\mathrm{d}z \\
+ \int_{-1}^1 \int_z^1 a(\zeta)\,\mathrm{d}\zeta \cdot z a(z) p_{\mathrm{id}}^B(V)(z)\,\mathrm{d}z
\end{aligned} \tag{4.38 rev.}$$

and, finally, we can define

$$P_{\mathrm{id}}(V) = p^B_{\mathrm{id}}(V)(z)\partial_\varphi + \Big(T_{\mathrm{id}} k_{a,r}(p^B_{\mathrm{id}}(V)(z)\partial_\varphi) + p^R_{\mathrm{id}}(V)\Big)\partial_\theta. \tag{4.31 rev.}$$

Theorem 4.39. *The fibrewise orthogonal projection*

$$\begin{aligned} P_\eta : T_\eta \mathrm{Diff}^s_R(M) &\to T_\eta \mathrm{Diff}^s_{\omega_a,\lambda_{a,r}}(M) \\ V &\mapsto (TR_\eta \circ P_{\mathrm{id}} \circ TR_{\eta^{-1}})(V) \end{aligned}$$

defines a smooth bundle map

$$P : T\mathrm{Diff}^s_R(M)|_{\mathrm{Diff}^s_{\omega_a,\lambda_{a,r}}(M)} \to T\mathrm{Diff}^s_{\omega_a,\lambda_{a,r}}(M).$$

Proof. We first compute

$$\begin{aligned} P_\eta(V) &= (TR_\eta \circ P_{\mathrm{id}} \circ TR_{\eta^{-1}})(V) \\ &= \Big(P_{\mathrm{id}}(V \circ \eta^{-1})\Big) \circ \eta \\ &= \Big(p^B_{\mathrm{id}}(V \circ \eta^{-1})\partial_\varphi + \Big(T_{\mathrm{id}} k_{a,r}(p^B_{\mathrm{id}}(V \circ \eta^{-1})\partial_\varphi) + p^R_{\mathrm{id}}(V \circ \eta^{-1})\Big)\partial_\theta\Big) \circ \eta \\ &= p^B_{\mathrm{id}}(V \circ \eta^{-1})\partial_\varphi \circ \eta + \Big(T_{\mathrm{id}} k_{a,r}(p^B_{\mathrm{id}}(V \circ \eta^{-1})\partial_\varphi) + p^R_{\mathrm{id}}(V \circ \eta^{-1})\Big)\partial_\theta \circ \eta, \end{aligned}$$

since all coefficients either only depend on z, which is preserved by η, or are constant. If we write $V = V^\varphi \partial_\varphi \circ \eta + V^z \partial_z \circ \eta + V^\theta \partial_\theta \circ \eta$, then $V \circ \eta^{-1} = V^\varphi \circ \eta^{-1} \partial_\varphi + V^z \circ \eta^{-1} \partial_z + V^\theta \eta^{-1} \partial_\theta$. The right hand side of

$$p^B_{\mathrm{id}}(V \circ \eta^{-1}) = (\mathrm{id} + K)^{-1}\Big(\frac{1}{f(z)} R(V^\varphi \circ \eta^{-1}, V^z \circ \eta^{-1}, V^\theta \circ \eta^{-1})(z)\Big)$$

is smooth in η (see page 140). Also,

$$\begin{aligned} &\mathrm{vol}_a(B \times S^1) \cdot p^R_{\mathrm{id}}(V \circ \eta^{-1}) = \\ &\qquad = \int_{-1}^1 \Big[-m_{a,r}(z) \int_{S^1} V^\varphi \circ \eta^{-1} \,\mathrm{d}\varphi + \int_{S^1} V^\theta \circ \eta^{-1} \,\mathrm{d}\varphi\Big] a(z) \,\mathrm{d}z \\ &\qquad\qquad + \int_{-1}^1 \int_z^1 a(\zeta) \,\mathrm{d}\zeta \cdot z a(z) p^B_{\mathrm{id}}(V \circ \eta^{-1})(z) \,\mathrm{d}z \\ &\qquad = \int_{-1}^1 \Big[-m_{a,r}(z) \int_{S^1} V^\varphi \,\mathrm{d}\varphi + \int_{S^1} V^\theta \,\mathrm{d}\varphi\Big] a(z) \,\mathrm{d}z \\ &\qquad\qquad + \int_{-1}^1 \int_z^1 a(\zeta) \,\mathrm{d}\zeta \cdot z a(z) p^B_{\mathrm{id}}(V \circ \eta^{-1})(z) \,\mathrm{d}z \end{aligned}$$

is smooth in η, hence P_η is smooth in η. □

4.8 Euler equation on $\mathrm{Diff}^s_{\omega_a,\lambda_{a,r}}(M)$

Recall the result of the variation of energy in Section 2.3: Let $V_t \in T_{\mathrm{id}}\mathrm{Diff}^s_{\omega_a,\lambda_{a,r}}(B\times S^1)$ be a time-dependent vector field, i. e. V_t is of the form

$$V_t = v_t(z)\partial_\varphi + \Big(T_{\mathrm{id}}k_{a,r}(v_t(z)\partial_\varphi) + c_t\Big)\partial_\theta.$$

If

$$0 = \int_0^T \int_M \langle W_t, \dot V_t + \nabla_{V_t} V_t\rangle \,\mathrm{vol}\, \mathrm{d}t \tag{2.9 revisited}$$

for any time-dependent $W_t = w_t(z)\partial_\varphi + \Big(T_{\mathrm{id}}k_{a,r}(w_t(z)\partial_\varphi) + d_t\Big)\partial_\theta \in T_{\mathrm{id}}\mathrm{Diff}^s_{\omega,\lambda}(B\times S^1)$, then V_t is a solution to the Euler equation. We now compute

$$\begin{aligned}
\nabla_{V_t} V_t &= \nabla_{v_t(z)\partial_\varphi + (T_{\mathrm{id}}k_{a,r}(v_t(z)\partial_\varphi)+c_t)\partial_\theta}\Big(v_t(z)\partial_\varphi + (T_{\mathrm{id}}k_{a,r}(v_t(z)\partial_\varphi)+c_t)\partial_\theta\Big)\\
&= v_t(z)\nabla_{\partial_\varphi}\Big(v_t(z)\partial_\varphi + (T_{\mathrm{id}}k_{a,r}(v_t(z)\partial_\varphi)+c_t)\partial_\theta\Big)\\
&\qquad + (T_{\mathrm{id}}k_{a,r}(v_t(z)\partial_\varphi)+c_t)\nabla_{\partial_\theta}\Big(v_t(z)\partial_\varphi + (T_{\mathrm{id}}k_{a,r}(v_t(z)\partial_\varphi)+c_t)\partial_\theta\Big)\\
&= v_t(z)\Big(v_t(z)\nabla_{\partial_\varphi}\partial_\varphi + (T_{\mathrm{id}}k_{a,r}(v_t(z)\partial_\varphi)+c_t)\nabla_{\partial_\varphi}\partial_\theta\Big)\\
&\qquad + (T_{\mathrm{id}}k_{a,r}(v_t(z)\partial_\varphi)+c_t)\Big(v_t(z)\nabla_{\partial_\theta}\partial_\varphi + (T_{\mathrm{id}}k_{a,r}(v_t(z)\partial_\varphi)+c_t)\nabla_{\partial_\theta}\partial_\theta\Big)\\
&= v_t(z)\Big(v_t(z)\nabla_{\partial_\varphi}\partial_\varphi + (T_{\mathrm{id}}k_{a,r}(v_t(z)\partial_\varphi)+c_t)\nabla_{\partial_\varphi}\partial_\theta\Big)\\
&\qquad + (T_{\mathrm{id}}k_{a,r}(v_t(z)\partial_\varphi)+c_t)\Big(v_t(z)\nabla_{\partial_\theta}\partial_\varphi + (T_{\mathrm{id}}k_{a,r}(v_t(z)\partial_\varphi)+c_t)\nabla_{\partial_\theta}\partial_\theta\Big)\\
&= v_t^2(z)\nabla_{\partial_\varphi}\partial_\varphi + v_t(z)(T_{\mathrm{id}}k_{a,r}(v_t(z)\partial_\varphi)+c_t)\nabla_{\partial_\varphi}\partial_\theta\\
&\qquad + (T_{\mathrm{id}}k_{a,r}(v_t(z)\partial_\varphi)+c_t)v_t(z)\nabla_{\partial_\theta}\partial_\varphi + (T_{\mathrm{id}}k_{a,r}(v_t(z)\partial_\varphi)+c_t)^2\nabla_{\partial_\theta}\partial_\theta.
\end{aligned} \tag{4.48}$$

Recall from page 109 that for pairing the covariant derivatives with ∂_φ and ∂_θ, the only possibly nonzero terms are

$$\begin{aligned}
2\langle\nabla_{\partial_\varphi}\partial_\varphi, \partial_\varphi\rangle &= \partial_\varphi\langle\partial_\varphi,\partial_\varphi\rangle,\\
2\langle\nabla_{\partial_\varphi}\partial_\varphi, \partial_\theta\rangle &= 2\partial_\varphi\langle\partial_\theta,\partial_\varphi\rangle - \partial_\theta\langle\partial_\varphi,\partial_\varphi\rangle\\
&= 2\partial_\varphi\mu_{a,r}(\partial_\varphi)\\
&= -2\partial_\varphi m_{a,r}(z)\\
&= 0.
\end{aligned}$$

Hence, ∂_φ only yields nonzero metric terms when paired with $\nabla_{\partial_\varphi}\partial_\varphi$, i. e. the first summand of Eq. (4.48), the remaining terms pair to 0. Furthermore, all of the summands of Eq. (4.48) pair to 0 with ∂_θ. Hence,

$$\begin{aligned}\langle W_t, \nabla_{V_t} V_t\rangle &= w_t(z)\langle \partial_\varphi,\ \nabla_{V_t} V_t\rangle + \big(T_{\mathrm{id}} k_{a,r}(w_t(z)\partial_\varphi) + d_t\big)\underbrace{\langle \partial_\theta,\ \nabla_{V_t} V_t\rangle}_{=0} \\ &= w_t(z)\cdot v_t(z)^2 \langle \partial_\varphi, \nabla_{\partial_\varphi}\partial_\varphi\rangle \\ &= w_t(z) v_t(z)^2 \frac{1}{2}\partial_\varphi \langle \partial_\varphi, \partial_\varphi\rangle\end{aligned}$$

and in turn

$$\begin{aligned}\int_{B\times S^1} \langle W_t, \nabla_{V_t} V_t\rangle \lambda_a \wedge \sigma_a &= \int_{B\times S^1} w_t(z) v_t(z)^2 \frac{1}{2}\partial_\varphi\langle\partial_\varphi,\partial_\varphi\rangle\,(a(z)\,\mathrm{d}\theta\wedge\mathrm{d}\varphi\wedge\mathrm{d}z) \\ &= \int_{-1}^{1} w_t(z) v_t(z)^2 a(z) \frac{1}{2} \underbrace{\int_{S^1} \partial_\varphi\langle\partial_\varphi,\partial_\varphi\rangle \mathrm{d}\varphi}_{=0} \quad \mathrm{d}z \\ &= 0.\end{aligned}$$

Then the full equation is

$$0 = \int_0^T \int_{B\times S^1} \langle W_t, \dot{V}_t\rangle \lambda_a \wedge \omega_a \quad \mathrm{d}t.$$

Again for $W_t = \dot{V}_t$, this is

$$0 = \int_0^T \int_{B\times S^1} \langle \dot{V}_t, \dot{V}_t\rangle \lambda_a \wedge \omega_a \quad \mathrm{d}t,$$

which implies $\dot{V}_t = 0$ and in turn $\dot{v}_t = 0$ and $\dot{c}_t = 0$.

Proposition 4.40. *The previous computation shows that the only solutions to the Euler equation on $M = B \times S^1$ preserving ω_a and λ_a are all stationary vector fields of the form $V_t = V = v(z)\partial_\varphi + \big(T_{\mathrm{id}} k_{a,r}(v(z)\partial_\varphi) + c\big)\partial_\theta$.* □

4.9 Generalization: any SHS on M descending to $(\sigma, \tau = h\sigma)$ on B

Let $(\omega_a, \lambda_{a,r} = \mathrm{d}\theta + \pi^*\mu_{a,r})$ be a stable Hamiltonian structure on $M = B \times S^1$, as in Section 4.7. This determines unique two-forms (σ_a, τ_a) on B by $\omega_a = \pi^*\sigma_a$ and $\tau_a = \mathrm{d}\mu_a$. Note that when given (σ_a, τ_a) on B, then *not* every possible associated SHS on M is of the form $(\omega_a, \lambda_{a,r})$: Let $(\tilde{\omega}, \tilde{\lambda} = \mathrm{d}\theta + \pi^*\tilde{\mu})$ be some other choice that also descends to (σ_a, τ_a) on B, i. e. $\tilde{\omega} = \pi^*\sigma_a = \omega_a$ and $\tau_a = \mathrm{d}\tilde{\mu}$. Since

$$\mathrm{d}\tilde{\mu} = \tau_a = \mathrm{d}\mu_{a,r},$$

there is a closed $\beta \in \Omega^1(B)$ such that $\tilde{\mu} = \mu_{a,r} + \beta$.

Note that since $H^1_{\mathrm{dR}}(B) \cong \mathbb{R}$ with representatives $\tilde{r}\,\mathrm{d}\varphi$ for any $\tilde{r} \in \mathbb{R}$, we can write

$$\beta = \mathrm{d}f + \tilde{r}\,\mathrm{d}\varphi$$

for some $f \in C^\infty(B, \mathbb{R})$ and $\tilde{r} \in \mathbb{R}$. Then

$$\tilde{\lambda} = \lambda_{a,r} + \pi^*\beta = \lambda_{a,r+\tilde{r}} + \mathrm{d}f.$$

Lemma 4.41. *The diffeomorphism*

$$\begin{aligned} \rho : M &\to M \\ (\varphi, z, \theta) &\mapsto \big(\varphi, z, \theta + f(\varphi, z) \mod 1\big) \end{aligned}$$

satisfies $\rho_* R = R$, $\rho^*\omega_a = \tilde{\omega} = \omega_a$ and $\rho^*\lambda_{a,r+\tilde{r}} = \tilde{\lambda}$, i. e. the conditions of Proposition 3.32.

Proof. We compute

$$\begin{aligned} \rho_* R &= \rho_* \partial_\theta = \frac{\partial(\theta + f(b))}{\partial\theta} \partial_\theta = \partial_\theta, \\ \rho^*\omega_a &= \rho^*\big(a(z)\,\mathrm{d}\varphi \wedge \mathrm{d}z\big) \\ &= a(z)\,\mathrm{d}\varphi \wedge \mathrm{d}z \\ &= \omega_a \\ &= \tilde{\omega} \end{aligned}$$

and

$$\begin{aligned} \rho^*\lambda_{a,r+\tilde{r}} &= \rho^*\big(\mathrm{d}\theta + \pi^*\mu_{a,r+\tilde{r}}\big) \\ &= \mathrm{d}(\theta + f) + \pi^*\mathrm{id}^*(\mu_{a,r} + \tilde{r}\,\mathrm{d}\varphi) \\ &= \mathrm{d}\theta + \mathrm{d}f + \pi^*\mu_{a,r} + \pi^*(\tilde{r}\,\mathrm{d}\varphi) \\ &= \mathrm{d}\theta + \pi^*\tilde{\mu} \\ &= \tilde{\lambda}. \end{aligned}$$

☐

Corollary 4.42. *Let $(\omega, \lambda = \mathrm{d}\theta + \pi^*\mu)$ be a stable Hamiltonian structure on $M = B \times S^1$ such that $\omega = \pi^*\sigma$ for some area form $\sigma \in \Omega^2(B)$ and $\tau := \mathrm{d}\mu = h(\varphi, z)\sigma$ with $h(\varphi, z) = z$. Then $\mathrm{Diff}^s_{\omega,\lambda}(M) \subset \mathrm{Diff}^s(M)$ is a smooth submanifold and the orthogonal projection in each tangent space $P_\eta : T_\eta\mathrm{Diff}^s_R(M) \to T_\eta\mathrm{Diff}^s_{\omega,\lambda}(M)$ for $\eta \in \mathrm{Diff}^s_{\omega,\lambda}(M)$ yields a smooth bundle map $P : T\mathrm{Diff}^s_R(M)|_{\mathrm{Diff}^s_{\omega,\lambda}(M)} \to T\mathrm{Diff}^s_{\omega,\lambda}(M)$.*

Proof. Combine the diffeomorphisms in Lemma 4.41 and Lemma 4.30 with the result in Proposition 3.32. ☐

4.10 Generalization: h any submersion

The most general stable Hamiltonian structure on a cylinder bundle we will consider in this thesis is some two-form

$$\tilde{\omega} = \pi^* \tilde{\sigma}$$

for some area form $\tilde{\sigma}$ on $B = S^1 \times [-1,1]$ and $\tilde{\lambda} = \mathrm{d}\theta + \pi^* \tilde{\mu}$ for some one-form $\tilde{\mu} \in \Omega^1(B)$. Since $\tilde{\tau} = \mathrm{d}\tilde{\mu}$ is another two-form on B, there is a smooth function $\tilde{h} : B \to \mathbb{R}$ such that $\tilde{\tau} = \tilde{h}\tilde{\sigma}$. In this section, we assume that $\tilde{h}$ is a submersion satisfying $\tilde{h}(S^1 \times \{-1\}) = \{-1\}$ and $\tilde{h}(S^1 \times \{1\}) = \{1\}$.

Proposition 4.43. *Let $\tilde{h}$ be a submersion satisfying $\tilde{h}(S^1 \times \{-1\}) = \{-1\}$ and $\tilde{h}(S^1 \times \{1\}) = \{1\}$. Then there is a diffeomorphism $\rho : B \to B$ such that $(\rho^* \tilde{h})(\varphi, z) = z = h(\varphi, z)$.*

Proof. Since $\tilde{h}$ is a submersion, the gradient vector field $\nabla \tilde{h}$ is transversal to the level sets $\tilde{h}^{-1}(c)$ for any $c \in [-1,1]$ with respect to some metric on B. Let $(\varphi, z) \in S^1 \times [-1,1]$. The point $(\varphi, -1)$ corresponds to the endpoint of the flow line of $\nabla(-h)$. Now consider the flow line of $\nabla \tilde{h}$ starting at $(\varphi, -1)$. There is a unique point in the intersection of this flow line and $\tilde{h}^{-1}(z)$. Define this point to be the image of (φ, z) under ρ, see Fig. 4.1

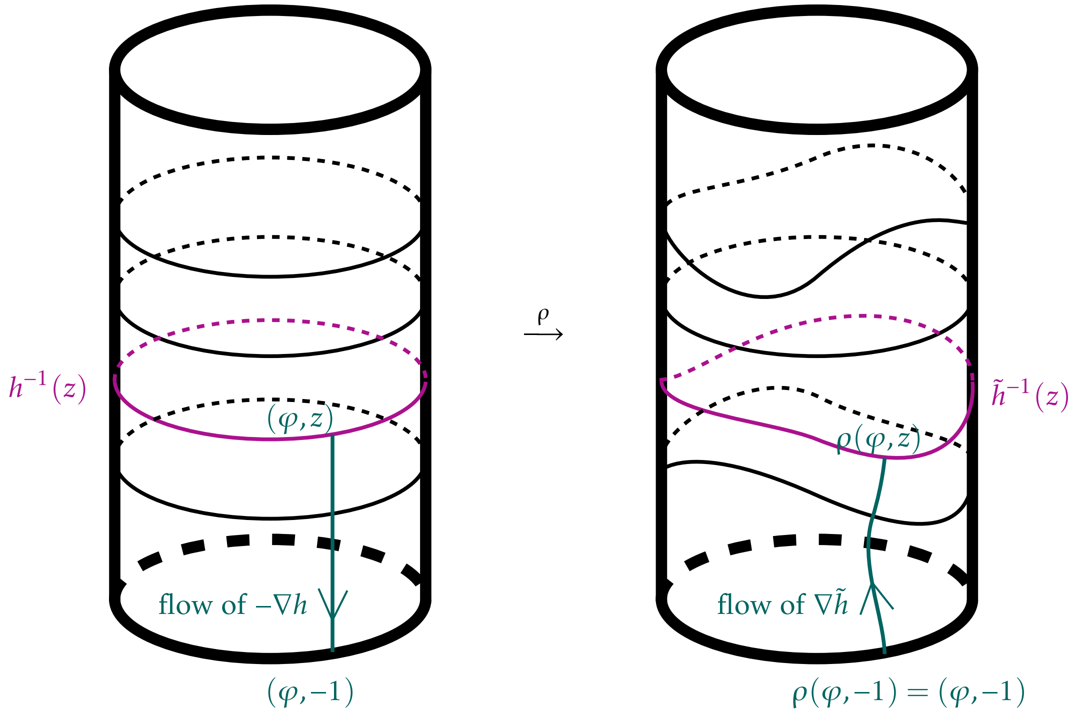

Figure 4.1: Definition of $\rho : B \to B$

By construction,

$$(\rho^* \tilde{h})(\varphi, z) = \tilde{h}(\rho(\varphi, z)) = z = h(\varphi, z). \qquad \square$$

Proposition 4.44. *Let* $(\tilde{\omega} = \pi^*\tilde{\sigma}, \tilde{\lambda} = \mathrm{d}\theta + \pi^*\tilde{\mu})$ *be a SHS on* $M = B \times S^1$ *such that* $\tilde{\tau} := \mathrm{d}\tilde{\mu} = \tilde{h}\tilde{\sigma}$ *for some submersion* $\tilde{h} : B \to \mathbb{R}$ *such that* $\tilde{h}(S^1\times\{-1\}) = -1$ *and* $\tilde{h}(S^1\times\{1\}) = 1$*. Then* $\mathrm{Diff}^s_{\tilde{\omega},\tilde{\lambda}}(M) \subset \mathrm{Diff}^s_R(M)$ *is a smooth submanifold and the orthogonal projection* $P_\eta : T_\eta\mathrm{Diff}^s_R(M) \to T_\eta\mathrm{Diff}^s_{\tilde{\omega},\tilde{\lambda}}(M)$ *for* $\eta \in \mathrm{Diff}^s_{\tilde{\omega},\tilde{\lambda}}(M)$ *is a smooth bundle map.*

Proof. We extend ρ defined in Proposition 4.43 to a diffeomorphism ρ^M on $M = B\times S^1$ by the identity on $\theta \in S^1$, i. e.

$$\rho^M(\varphi, z, \theta) = \big(\rho(\varphi, z), \theta\big).$$

We define $\sigma := \rho^*\tilde{\sigma}$ and

$$\omega := (\rho^M)^*\tilde{\omega} = (\rho^M)^*\pi^*\tilde{\sigma} = \pi^*\rho^*\tilde{\sigma} = \pi^*\sigma.$$

We further let $\mu := \rho^*\tilde{\mu}$ and get

$$\begin{aligned}\lambda &:= (\rho^M)^*\tilde{\lambda}\\ &= (\rho^M)^*(\mathrm{d}\theta + \pi^*\tilde{\mu})\\ &= \mathrm{d}\theta + \pi^*\rho^*\tilde{\mu}\\ &= \mathrm{d}\theta + \pi^*\mu.\end{aligned}$$

Now, $(\omega_b, \lambda = \mathrm{d}\theta + \pi^*\mu)$ is a stable Hamiltonian structure on $M = B\times S^1$ that descends to σ and

$$\begin{aligned}\tau &:= \mathrm{d}\mu = \mathrm{d}\rho^*\tilde{\mu}\\ &= \rho^*\mathrm{d}\tilde{\mu}\\ &= \rho^*\tilde{\tau}\\ &= \rho^*(\tilde{h}\tilde{\sigma})\\ &= h\sigma\end{aligned}$$

on B and we can apply Corollary 4.42. ☐

4.11 Outlook: counterexample

We tried finding an example for a manifold M with a stable Hamiltonian structure (ω, λ) such that $\mathrm{Diff}^s_{\omega,\lambda}(M) \subset \mathrm{Diff}^s(M)$ is not a smooth submanifold. We suspect that, varying examples of this section, for the cylinder bundle $M = B \times S^1$ with $B = S^1 \times [-1, 1]$, choosing a stable Hamiltonian structure (ω, λ) on M that descends to the two-forms $(\sigma, \tau = h\sigma)$ on B such that h has at least one critical point, may provide such an example. The results in the previous section already show that if h has no cricital points, i. e. it is a submersion, then for all such choices, the diffeomorphism groups are smooth submanifolds and h being a submersion was critical for our proof. As a candidate, we tried $h : S^1 \times [-1, 1] \to \mathbb{R}$, $(\varphi, z) \mapsto \sin(2\pi\varphi) \cdot z$, which has level sets roughly shown in Fig. 4.2. In particular, the green level set $h^{-1}(\{0\})$ looks suspiciously

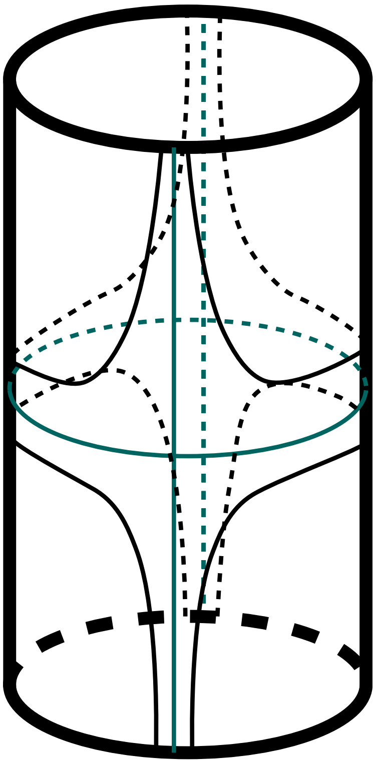

Figure 4.2: Level sets of $h(\varphi,z) = \sin(2\pi\varphi) \cdot z$

non-smooth and strongly restricts the structure-preserving diffeomorphisms of $S^1 \times [-1,1]$. Unfortunately, there is no nice criterion to show that something is *not* a smooth submanifold and we could not come up with a rigorous proof.

4.12 Outlook: other 2-dimensional base manifolds

The cylinder as discussed in this chapter is a very specific choice of base manifold. We expect the results to also hold for the standard two-torus as the computations are very similar. It is an open question as to what happens with other 2-dimensional base manifolds. A natural choice might also be the sphere S^2 with the standard metric. In cylindrical coordinates (φ,z) for $\varphi \in S^1 \cong \mathbb{R}/\mathbb{Z}$ and $z \in [-1,1]$, we have the Riemannian volume form $\sigma = \mathrm{d}\varphi \wedge \mathrm{d}z$ and for $\tau = h\sigma$, we can also consider the height function $h(\varphi,z) = z$ similar to the cylinder. Unfortunately, this height function has critical points at the two poles of the sphere, which might already cause problems with the submanifold structure of $\mathrm{Diff}^s_{\sigma,\tau}(S^2)$ as discussed in the previous section.

5

DIFFEOMORPHISMS OF MANIFOLDS WITH A (STABILIZABLE) HAMILTONIAN STRUCTURE

Recall from the definition at the beginning of in Section 3.1 that a Hamiltonian structure on a compact, oriented $(2n+1)$-dimensional manifold is a closed two-form ω of maximal rank, i. e. such that ω^n vanishes nowhere. We further assume that the kernel foliation $\ker\omega$ is periodic, so that we can choose a vector field R generating $\ker\omega$ all of whose orbits have period 1. As before, this implies that M is an S^1-principal bundle over some compact $2n$-dimensional base manifold B, i. e.

$$S^1 \to M \xrightarrow{\pi} B,$$

where the S^1-action is generated by R. Associated to this bundle, we can choose a connection form $\lambda \in \Omega^1(M)$.

Lemma 5.1. *The connection form λ is a stabilizing one-form for the Hamiltonian structure defined by ω on M. In particular, any Hamiltonian structure ω with periodic kernel foliation* $\ker\omega$ *is stabilizable.*

Proof. The S^1-action on M is generated by R, hence the connection form λ satisfies $\mathcal{L}_R\lambda = 0$ and $\lambda(R) = 1$. This implies $R \in \ker \mathrm{d}\lambda$, i. e. $\ker\omega \subset \ker \mathrm{d}\lambda$:

$$\iota_R \mathrm{d}\lambda = \mathrm{d}\underbrace{\iota_R\lambda}_{\equiv 1} + \iota_R \mathrm{d}\lambda = \mathcal{L}_R\lambda = 0.$$

Furthermore, since R generates $\ker\omega$ and $\lambda(R) = 1$, we also know that $\lambda \wedge \omega^n$ is a volume form. □

5.1 Structure-preserving diffeomorphisms of principal circle bundles

For such a stabilizable Hamiltonian structure on a prinicpal circle bundle $S^1 \to M \to B$, we consider the diffeomorphisms preserving the Hamiltonian structure ω and the chosen generator R of the kernel foliation $\ker\omega$, i. e.

$$\mathrm{Diff}^s_{R,\omega}(M) := \left\{\eta \in \mathrm{Diff}^s(M) \;\middle|\; \eta_* R = R,\ \eta^*\omega = \omega\right\}.$$

In contrast to the previous sections, we do *not* require that the diffeomorphisms also preserve the stabilizing one-form λ.

By Theorem 2.23 and Corollary 3.16, we already know that

$$\mathrm{Diff}^s_{R,\omega}(M) \subset \mathrm{Diff}^s_R(M) \subset \mathrm{Diff}^s(M)$$

are smooth submanifolds.

Now choose an S^1-invariant metric $\langle .,. \rangle$ on M such that R has length 1. Then this metric descends to a metric $\langle .,. \rangle^B$ on B and we assume that its Riemmanian volume form is given by σ^n, where $\sigma \in \Omega^2(B)$ is defined by $\omega = \pi^*\sigma$. Furthermore, this defines an orthogonal complement of $\ker \omega$ in TM, i. e. choosing a metric automatically defines a stabilizing one-form λ. Locally, λ can be written as $\lambda = \mathrm{d}\theta + \pi^*\mu$ for the S^1-bundle coordinate θ and a one-form μ on some subset of B. For any such choice of metric, the Riemannian volume form is locally given by $\mathrm{vol} = \lambda \wedge \omega^n = \mathrm{d}\theta \wedge \omega^n$.

Lemma 5.2. $\mathrm{Diff}^s_{R,\omega}(M) \subset \mathrm{Diff}^s_{\mathrm{vol}}(M)$ *is a smooth submanifold.*

Proof. We first check that any $\eta \in \mathrm{Diff}^s_{R,\omega}(M)$ also preserves $\mathrm{vol} = \mathrm{d}\theta \wedge \omega^n$: In local coordinates around $(b,\theta) \in U \times S^1$ for $b \in U \subset B$, we can use Corollary 3.16 to write $\eta(b,\theta) = \big(\nu(b), \theta + k(b)\big)$ for some $\nu \in \mathrm{Diff}^s_\sigma(B)$ and $k \in H^s(U,S^1)$. Then, we compute

$$\begin{aligned}
\eta^*\mathrm{vol} &= \eta^*(\mathrm{d}\theta \wedge \omega^n) \\
&= \mathrm{d}(\eta^*\theta) \wedge (\eta^*\omega)^n \\
&= \mathrm{d}(\theta + k) \wedge \omega^n \\
&= \mathrm{d}\theta \wedge \omega^n + \underbrace{\mathrm{d}k \wedge \omega^n}_{=0} \\
&= \mathrm{vol}.
\end{aligned}$$

This implies that $\mathrm{Diff}^s_{R,\omega}(M)$ is a subset of $\mathrm{Diff}^s_{\mathrm{vol}}(M)$. Since both are also smooth Hilbert submanifolds of $\mathrm{Diff}^s(M)$, the statement follows from Lemma 4.5. □

In particular, the induced metric defined by Eq. (2.5) on $T\mathrm{Diff}^s_{R,\omega}(M)$ is right-invariant and we can use the computation in Section 2.3 for the derivation of the Euler equation.

For trivial circle bundles $M = B \times S^1$, we denote the S^1-coordinate by θ and get $R = \partial_\theta$. As in Section 3.5, we can write $\lambda = \mathrm{d}\theta + \pi^*\mu$ for some (fixed) $\mu \in \Omega^1(B)$ and the Riemannian volume forms are $\mathrm{vol} = \lambda \wedge \omega^n = \mathrm{d}\theta \wedge \omega^n$ on M and σ^n on B.

Also recall Corollary 3.16, which yields the diffeomorphism

$$\begin{aligned}
\Phi|_{\mathrm{Diff}^s_\sigma(B)\times H^s(B,S^1)} : \mathrm{Diff}^s_\sigma(B) \times H^s(B,S^1) &\to \mathrm{Diff}^s_{R,\omega}(M) \\
(\nu,k) &\mapsto \big((b,\theta) \mapsto (\nu(b),\, \theta + k(b))\big)
\end{aligned}$$

with tangent map

$$\begin{aligned}
T_{(\nu,k)}\Phi|_{\mathrm{Diff}^s_\sigma(B)\times H^s(B,S^1)} : T_\nu\mathrm{Diff}^s_\sigma(B) \times H^s(B,\mathbb{R}) &\to T_{\Phi(\nu,k)}\mathrm{Diff}^s_{R,\omega}(M) \\
(v,l) &\mapsto v + l\partial_\theta.
\end{aligned}$$

Hence, every tangent vector $V \in T_{\mathrm{id}}\mathrm{Diff}^s_{R,\omega}(M)$ can be written as

$$V = v + l\partial_\theta$$

for some $v \in T_{\mathrm{id}}\mathrm{Diff}^s_\sigma(B)$ and $l \in H^s(B,\mathbb{R})$.

5.2 Euler equation on $\mathrm{Diff}^s_{R,\omega}(B \times S^1)$, standard bundle metric

As in the previous sections on the Euler equation, we start by recalling the result of the variation of energy in Section 2.3: Let $V_t \in T_{\mathrm{id}}\mathrm{Diff}^s_{R,\omega}(B \times S^1)$ be a time-dependent vector field, i. e. V_t is of the form $V_t = v_t + l_t\partial_\theta$ with $v_t \in T_{\mathrm{id}}\mathrm{Diff}^s_\sigma(B)$ and $l_t \in H^s(B,\mathbb{R})$. If

$$0 = \int_0^T \int_M \langle W_t, \dot{V}_t + \nabla_{V_t} V_t \rangle \,\mathrm{vol}\, \mathrm{d}t \tag{2.9 revisited}$$

for any time-dependent $W_t = w_t + m_t\partial_\theta \in T_{\mathrm{id}}\mathrm{Diff}^s_{R,\omega}(B \times S^1)$, then V_t is a solution to the Euler equation.

Still considering a general bundle metric, which induces a stabilizing one-form $\lambda = \mathrm{d}\theta + \pi^*\mu$, we compute

$$\begin{aligned}
\langle W_t, \dot{V}_t \rangle &= \langle w_t + m_t\partial_\theta, \dot{V}_t \rangle \\
&= \langle w_t - \mu(w_t)\partial_\theta, \dot{v}_t + \dot{l}_t\partial_\theta \rangle + \big(\mu(w_t) + m_t\big)\langle \partial_\theta, \dot{v}_t + \dot{l}_t\partial_\theta \rangle \\
&= \langle w_t - \mu(w_t)\partial_\theta, \dot{v}_t - \mu(\dot{v}_t)\partial_\theta \rangle + \big(\mu(\dot{v}_t) + \dot{l}_t\big)\underbrace{\langle \underbrace{w_t - \mu(w_t)\partial_\theta}_{\in \ker\lambda}, \partial_\theta \rangle}_{=0} \\
&\qquad + \big(\mu(w_t) + m_t\big)\Big(\underbrace{\langle \partial_\theta, \underbrace{\dot{v}_t - \mu(\dot{v}_t)\partial_\theta}_{\in \ker\lambda} \rangle}_{=0} + \big(\mu(\dot{v}_t) + \dot{l}_t\big)\underbrace{\langle \partial_\theta, \partial_\theta \rangle}_{=1}\Big) \\
&= \langle w_t, \dot{v}_t \rangle^B + \big(\mu(w_t) + m_t\big)\big(\mu(\dot{v}_t) + \dot{l}_t\big).
\end{aligned} \tag{5.1}$$

The covariant derivative is

$$\begin{aligned}
\nabla_{V_t} V_t &= \nabla_{V_t}\big(v_t + l_t\partial_\theta\big) \\
&= \nabla_{V_t} v_t + l_t \nabla_{V_t}\partial_\theta + V_t(l_t)\partial_\theta \\
&= \nabla_{v_t + l_t\partial_\theta} v_t + l_t \nabla_{v_t + l_t\partial_\theta}\partial_\theta + \big(v_t + l_t\partial_\theta\big)(l_t)\partial_\theta \\
&= \nabla_{v_t} v_t + l_t \nabla_{\partial_\theta} v_t + l_t \nabla_{v_t}\partial_\theta + l_t^2 \nabla_{\partial_\theta}\partial_\theta + v_t(l_t)\partial_\theta.
\end{aligned} \tag{5.2}$$

Using the Koszul formula for the vector fields X, Y and Z,

$$2\langle X, \nabla_Y Z \rangle = Y\langle Z, X \rangle + Z\langle X, Y \rangle - X\langle Y, Z \rangle + \langle X, [Y, Z] \rangle - \langle Y, [Z, X] \rangle - \langle Z, [Y, X] \rangle,$$

then pairing these covariant derivatives with w_t yields

$$\begin{aligned}
2\langle w_t, \nabla_{v_t} v_t\rangle &= 2v_t\langle w_t, v_t\rangle - w_t\langle v_t, v_t\rangle + \langle w_t, \underbrace{[v_t, v_t]}_{=0}\rangle - 2\langle v_t, [w_t, v_t]\rangle \\
&= 2v_t\Big(\langle w_t, v_t\rangle^B + \mu(w_t)\mu(v_t)\Big) - w_t\Big(\langle v_t, v_t\rangle^B + \mu(v_t)^2\Big) \\
&\qquad\qquad - 2\Big(\langle v_t, [w_t, v_t]\rangle^B + \mu(v_t)\mu([w_t, v_t])\Big) \\
&= 2v_t\langle w_t, v_t\rangle^B + 2\mu(v_t)v_t\Big(\mu(w_t)\Big) + 2\mu(w_t)v_t\Big(\mu(v_t)\Big) \\
&\qquad - w_t\langle v_t, v_t\rangle^B - 2\mu(v_t)w_t\Big(\mu(v_t)\Big) \\
&\qquad\qquad - 2\langle v_t, [w_t, v_t]\rangle^B - 2\mu(v_t)\mu([w_t, v_t]), \\
2\langle w_t, \nabla_{\partial_\theta} v_t\rangle &= \partial_\theta\langle w_t, v_t\rangle + v_t\langle w_t, \partial_\theta\rangle - w_t\langle \partial_\theta, v_t\rangle \\
&\qquad + \langle w_t, \underbrace{[\partial_\theta, v_t]}_{=0}\rangle - \langle \partial_\theta, [v_t, w_t]\rangle - \langle v_t, \underbrace{[\partial_\theta, w_t]}_{=0}\rangle \\
&= v_t\Big(\mu(w_t)\Big) - w_t\Big(\mu(v_t)\Big) - \mu([v_t, w_t]), \\
2\langle w_t, \nabla_{v_t} \partial_\theta\rangle &= v_t\langle w_t, \partial_\theta\rangle + \partial_\theta\langle w_t, v_t\rangle - w_t\langle v_t, \partial_\theta\rangle \\
&\qquad + \langle w_t, \underbrace{[v_t, \partial_\theta]}_{=0}\rangle - \langle v_t, \underbrace{[\partial_\theta, w_t]}_{=0}\rangle - \langle \partial_\theta, [v_t, w_t]\rangle \\
&= v_t\Big(\mu(w_t)\Big) - w_t\Big(\mu(v_t)\Big) - \mu([v_t, w_t]), \\
2\langle w_t, \nabla_{\partial_\theta} \partial_\theta\rangle &= 2\partial_\theta\langle w_t, \partial_\theta\rangle - w_t\underbrace{\langle \partial_\theta, \partial_\theta\rangle}_{\equiv 1} - 2\langle \partial_\theta, \underbrace{[\partial_\theta, w_t]}_{=0}\rangle \\
&= 2\partial_\theta \mu(w_t) \\
&= 0,
\end{aligned}$$

$$\langle w_t, \partial_\theta\rangle = \mu(w_t),$$

whereas pairing them with ∂_θ yields

$$\begin{aligned}
2\langle \partial_\theta, \nabla_{v_t} v_t\rangle &= 2v_t\langle \partial_\theta, v_t\rangle - \partial_\theta\langle v_t, v_t\rangle + \langle \partial_\theta, [v_t, v_t]\rangle - 2\langle v_t, [\partial_\theta, v_t]\rangle \\
&= 2v_t\Big(\mu(v_t)\Big), \\
2\langle \partial_\theta, \nabla_{\partial_\theta} v_t\rangle &= v_t\langle \partial_\theta, \partial_\theta\rangle - \langle v_t, \underbrace{[\partial_\theta, \partial_\theta]}_{=0}\rangle \\
&= 0, \\
2\langle \partial_\theta, \nabla_{v_t} \partial_\theta\rangle &= v_t\langle \partial_\theta, \partial_\theta\rangle - \langle v_t, \underbrace{[\partial_\theta, \partial_\theta]}_{=0}\rangle \\
&= 0, \\
2\langle \partial_\theta, \nabla_{\partial_\theta} \partial_\theta\rangle &= \partial_\theta\langle \partial_\theta, \partial_\theta\rangle - \langle \partial_\theta, [\partial_\theta, \partial_\theta]\rangle \\
&= 0,
\end{aligned}$$

$$\langle \partial_\theta, \partial_\theta \rangle = 1.$$

We now restrict to the standard product metric on $M = B{\times}S^1$, i. e. we assume that ∂_θ is perpendicular to any tangent vector to B. This corresponds to $\mu = 0 \in \Omega^1(M)$ and $\lambda = \mathrm{d}\theta$. The previous computation simplifies to

$$\begin{aligned}
2\langle w_t, \nabla_{v_t} v_t \rangle &= 2v_t\langle w_t, v_t\rangle - w_t\langle v_t, v_t\rangle - 2\langle v_t, [w_t, v_t]\rangle \\
&= 2v_t\langle w_t, v_t\rangle^B - w_t\langle v_t, v_t\rangle^B - 2\langle v_t, [w_t, v_t]\rangle^B \\
&= 2\langle w_t, \nabla_{v_t} v_t\rangle^B, \\
2\langle w_t, \nabla_{\partial_\theta} v_t \rangle &= v_t\big(\mu(w_t)\big) - w_t\big(\mu(v_t)\big) - \mu([v_t, w_t]) \\
&= 0, \\
2\langle w_t, \nabla_{v_t} \partial_\theta \rangle &= v_t\big(\mu(w_t)\big) - w_t\big(\mu(v_t)\big) - \mu([v_t, w_t]) \\
&= 0, \\
2\langle w_t, \nabla_{\partial_\theta} \partial_\theta \rangle &= 0, \\
\langle w_t, \partial_\theta \rangle &= \mu(w_t) = 0,
\end{aligned}$$

and

$$\begin{aligned}
2\langle \partial_\theta, \nabla_{v_t} v_t \rangle &= 2v_t\big(\mu(v_t)\big) = 0, \\
2\langle \partial_\theta, \nabla_{\partial_\theta} v_t \rangle &= 0, \\
2\langle \partial_\theta, \nabla_{v_t} \partial_\theta \rangle &= 0, \\
2\langle \partial_\theta, \nabla_{\partial_\theta} \partial_\theta \rangle &= 0, \\
\langle \partial_\theta, \partial_\theta \rangle &= 1.
\end{aligned}$$

Then we get for the full covariant derivative

$$\begin{aligned}
\langle W_t, \nabla_{V_t} V_t \rangle &\overset{(5.2)}{=} \langle W_t, \nabla_{v_t} v_t \rangle + l_t\langle W_t, \nabla_{\partial_\theta} v_t\rangle + l_t\langle W_t, \nabla_{v_t}\partial_\theta\rangle \\
&\qquad + l_t^2\langle W_t, \nabla_{\partial_\theta}\partial_\theta\rangle + v_t(l_t)\langle W_t, \partial_\theta\rangle \\
&= \langle w_t, \nabla_{v_t} v_t\rangle + l_t\underbrace{\langle w_t, \nabla_{\partial_\theta} v_t\rangle}_{=0} + l_t\underbrace{\langle w_t, \nabla_{v_t}\partial_\theta\rangle}_{=0} \\
&\qquad + l_t^2\underbrace{\langle w_t, \nabla_{\partial_\theta}\partial_\theta\rangle}_{=0} + v_t(l_t)\underbrace{\langle w_t, \partial_\theta\rangle}_{=0} \\
&\quad + m_t\underbrace{\langle \partial_\theta, \nabla_{v_t} v_t\rangle}_{=0} + l_t m_t\underbrace{\langle \partial_\theta, \nabla_{\partial_\theta} v_t\rangle}_{=0} + l_t m_t\underbrace{\langle \partial_\theta, \nabla_{v_t}\partial_\theta\rangle}_{=0} \\
&\qquad + l_t^2 m_t\underbrace{\langle \partial_\theta, \nabla_{\partial_\theta}\partial_\theta\rangle}_{=0} + v_t(l_t) m_t\underbrace{\langle \partial_\theta, \partial_\theta\rangle}_{=1} \\
&= \langle w_t, \nabla_{v_t} v_t\rangle^B + v_t(l_t) m_t.
\end{aligned}$$

Combining this result with Eq. (5.1) yields

$$\langle W_t, \dot{V}_t + \nabla_{V_t} V_t \rangle = \langle w_t, \dot{v}_t \rangle^B + m_t \dot{l}_t + \langle w_t, \nabla_{v_t} v_t \rangle^B + v_t(l_t) m_t$$

and for the Euler equation

$$\begin{aligned} 0 &= \int_0^T \int_M \langle W_t, \dot{V}_t + \nabla_{V_t} V_t \rangle \quad \mathrm{vol} \quad \mathrm{d}t \\ &= \int_0^T \int_{B \times S^1} \Big(\langle w_t, \dot{v}_t \rangle^B + m_t \dot{l}_t + \langle w_t, \nabla_{v_t} v_t \rangle^B + v_t(l_t) m_t \Big) \quad \mathrm{d}\theta \wedge \omega^n \quad \mathrm{d}t \\ &= \int_0^T \int_B \Big(\langle w_t, \dot{v}_t \rangle^B + m_t \dot{l}_t + \langle w_t, \nabla_{v_t} v_t \rangle^B + v_t(l_t) m_t \Big) \quad \sigma^n \quad \mathrm{d}t \\ &= \int_0^T \int_B \langle w_t, \dot{v}_t + \nabla_{v_t} v_t \rangle \sigma^n \, \mathrm{d}t + \int_0^T \int_B m_t \Big(\dot{l}_t + v_t(l_t) \Big) \sigma^n \, \mathrm{d}t \end{aligned}$$

for any $w_t \in T_{\mathrm{id}}\mathrm{Diff}^s_\sigma(B)$ and $m_t \in H^s(B, \mathbb{R})$. Hence,

$$0 = \int_0^T \int_B \langle w_t, \dot{v}_t + \nabla_{v_t} v_t \rangle \sigma^n \, \mathrm{d}t, \tag{5.3}$$

i. e. v_t is a solution to the Euler equation on the symplectomorphisms of (B, σ), and l_t then solves

$$0 = \dot{l}_t + v_t(l_t). \tag{5.4}$$

Theorem 5.3. *For the standard product metric on $M = B \times S^1$ Hamiltonian structure ω, generator $R = \partial_\theta$ of* $\ker \omega$ *and Riemannian volume form given by* $\mathrm{vol} = \mathrm{d}\theta \wedge \omega^n$*, the Euler equations preserving R and ω is given by Eqs.* (5.3) *and* (5.4)*. For any initial condition $(v_0, l_0) \in T_{\mathrm{id}}\mathrm{Diff}^s_\sigma(B), H^s(B, \mathbb{R})$, solutions exist for any time $t \in \mathbb{R}$ and depend smoothly on (v_0, l_0).*

Proof. Using the results by Ebin [Ebi12] (see Section 2.5.2), we have long-time existence of solutions v_t to the Euler equation on the symplectomorphism group of (B, σ). Their paper also includes smooth dependence of the solution v_t on the initial condition v_0. Let ν_t denote the flow of v_t, i. e. ν_t satisfies $\frac{\mathrm{d}}{\mathrm{d}t} \nu_t = v_t \circ \nu_t$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\nu_t^* l_t \Big) = \nu_t^* \Big(\dot{l}_t + \mathcal{L}_{v_t} l_t \Big) = \nu^* \Big(\underbrace{\dot{l}_t + v_t(l_t)}_{\overset{(5.4)}{=} 0} \Big) = 0,$$

hence $l_t \circ \nu_t = \nu_t^* l_t \equiv \nu_0^* l_0 = l_0$ and given the initial condition l_0, we can solve $l_t = l_0 \circ \nu_t^{-1}$. Then also l_t depends smoothly on v_0 and l_0. ☐

5.3 Outlook: Euler equation on $\mathrm{Diff}^s_{R,\omega}(B\times S^1)$, general bundle metric

Going back to a general metric, we have

$$
\begin{aligned}
\langle W_t, \nabla_{V_t} V_t\rangle &\overset{(5.2)}{=} \langle W_t, \nabla_{v_t} v_t\rangle + l_t\langle W_t, \nabla_{\partial_\theta} v_t\rangle + l_t\langle W_t, \nabla_{v_t}\partial_\theta\rangle \\
&\qquad + l_t^2\langle W_t, \nabla_{\partial_\theta}\partial_\theta\rangle + v_t(l_t)\langle W_t, \partial_\theta\rangle \\
&= \langle w_t, \nabla_{v_t} v_t\rangle + l_t\langle w_t, \nabla_{\partial_\theta} v_t\rangle + l_t\underbrace{\langle w_t, \nabla_{v_t}\partial_\theta\rangle}_{=\langle w_t, \nabla_{\partial_\theta} v_t\rangle} \\
&\qquad + l_t^2\underbrace{\langle w_t, \nabla_{\partial_\theta}\partial_\theta\rangle}_{=0} + v_t(l_t)\underbrace{\langle w_t, \partial_\theta\rangle}_{=\mu(w_t)} \\
&\quad + m_t\underbrace{\langle \partial_\theta, \nabla_{v_t} v_t\rangle}_{=v_t\left(\mu(v_t)\right)} + l_t m_t\underbrace{\langle \partial_\theta, \nabla_{\partial_\theta} v_t\rangle}_{=0} + l_t m_t\underbrace{\langle \partial_\theta, \nabla_{v_t}\partial_\theta\rangle}_{=0} \\
&\qquad + l_t^2 m_t\underbrace{\langle \partial_\theta, \nabla_{\partial_\theta}\partial_\theta\rangle}_{=0} + v_t(l_t)m_t\underbrace{\langle \partial_\theta, \partial_\theta\rangle}_{=1} \\
&= v_t\langle w_t, v_t\rangle^B + \mu(v_t)v_t\big(\mu(w_t)\big) + \mu(w_t)v_t\big(\mu(v_t)\big) \\
&\qquad - \frac{1}{2}w_t\langle v_t, v_t\rangle^B - \mu(v_t)w_t\big(\mu(v_t)\big) \\
&\qquad - \langle v_t, [w_t, v_t]\rangle^B - \mu(v_t)\mu([w_t, v_t]) \\
&\qquad + l_t\Big(v_t\big(\mu(w_t)\big) - w_t\big(\mu(v_t)\big) - \mu([v_t, w_t])\Big) \\
&\qquad\qquad + \mu(w_t) + m_t v_t\big(\mu(v_t)\big) + v_t(l_t)m_t.
\end{aligned} \tag{5.5}
$$

Plugging Eqs. (5.1) and (5.5) into the variation of the energy, we get

$$
\begin{aligned}
0 &= \int_0^T \int_{B\times S^1} \langle W_t, \dot V_t + \nabla_{V_t} V_t\rangle \quad \mathrm{vol} \quad \mathrm{d}t \\
&= \int_0^T \int_{B\times S^1} \Big(\langle w_t, \dot v_t\rangle^B + \big(\mu(w_t) + m_t\big)\big(\mu(\dot v_t) + \dot l_t\big) \\
&\quad + v_t\langle w_t, v_t\rangle^B + \mu(v_t)v_t\big(\mu(w_t)\big) + \mu(w_t)v_t\big(\mu(v_t)\big) \\
&\qquad - \frac{1}{2}w_t\langle v_t, v_t\rangle^B - \mu(v_t)w_t\big(\mu(v_t)\big) \\
&\qquad - \langle v_t, [w_t, v_t]\rangle^B - \mu(v_t)\mu([w_t, v_t]) \\
&\quad + l_t\Big(v_t\big(\mu(w_t)\big) - w_t\big(\mu(v_t)\big) - \mu([v_t, w_t])\Big) \\
&\quad + \mu(w_t) + m_t v_t\big(\mu(v_t)\big) + v_t(l_t)m_t\Big)\mathrm{d}\theta \wedge \omega^n \quad \mathrm{d}t
\end{aligned}
$$

$$\begin{aligned}
&= \int_0^T \int_{B\times S^1} \Big(\langle w_t, \dot{v}_t \rangle^B + \mu(w_t)\big(\mu(\dot{v}_t) + \dot{l}_t\big) \\
&\qquad\qquad + v_t\langle w_t, v_t\rangle^B + \mu(v_t)v_t\big(\mu(w_t)\big) + \mu(w_t)v_t\big(\mu(v_t)\big) \\
&\qquad\qquad\qquad - \frac{1}{2} w_t\langle v_t, v_t\rangle^B - \mu(v_t)w_t\big(\mu(v_t)\big) \\
&\qquad\qquad\qquad - \langle v_t, [w_t, v_t]\rangle^B - \mu(v_t)\mu([w_t, v_t]) \\
&\quad + l_t\Big(v_t\big(\mu(w_t)\big) - w_t\big(\mu(v_t)\big) - \mu([v_t, w_t])\Big) + \mu(w_t)\Big) \mathrm{d}\theta \wedge \omega^n \quad \mathrm{d}t \\
&\quad + \int_0^T \int_{B\times S^1} m_t\Big(\mu(\dot{v}_t) + \dot{l}_t + v_t\big(\mu(v_t)\big) + v_t(l_t)\Big) \mathrm{d}\theta \wedge \omega^n \quad \mathrm{d}t \\
&= \int_0^T \int_B \Big(\langle w_t, \dot{v}_t \rangle^B + \mu(w_t)\big(\mu(\dot{v}_t) + \dot{l}_t\big) \\
&\qquad\quad + v_t\langle w_t, v_t\rangle^B + \mu(v_t)v_t\big(\mu(w_t)\big) + \mu(w_t)v_t\big(\mu(v_t)\big) \\
&\qquad\qquad\qquad - \frac{1}{2} w_t\langle v_t, v_t\rangle^B - \mu(v_t)w_t\big(\mu(v_t)\big) \\
&\qquad\qquad\qquad - \langle v_t, [w_t, v_t]\rangle^B - \mu(v_t)\mu([w_t, v_t]) \\
&\quad + l_t\Big(v_t\big(\mu(w_t)\big) - w_t\big(\mu(v_t)\big) - \mu([v_t, w_t])\Big) + \mu(w_t)\Big) \sigma^n \quad \mathrm{d}t \\
&\qquad + \int_0^T \int_B m_t\Big(\mu(\dot{v}_t) + \dot{l}_t + v_t\big(\mu(v_t)\big) + v_t(l_t)\Big) \sigma^n \quad \mathrm{d}t
\end{aligned}$$

for any w_t and m_t. Hence, $v_t \in \mathrm{Diff}^s_\sigma(B)$ satisfies

$$\begin{aligned}
0 = \int_0^T \int_B \Big(&\langle w_t, \dot{v}_t \rangle^B + \mu(w_t)\big(\mu(\dot{v}_t) + \dot{l}_t\big) \\
&+ v_t\langle w_t, v_t\rangle^B + \mu(v_t)v_t\big(\mu(w_t)\big) + \mu(w_t)v_t\big(\mu(v_t)\big) \\
&- \frac{1}{2} w_t\langle v_t, v_t\rangle^B - \mu(v_t)w_t\big(\mu(v_t)\big) - \langle v_t, [w_t, v_t]\rangle^B - \mu(v_t)\mu([w_t, v_t]) \\
&\qquad + l_t\Big(v_t\big(\mu(w_t)\big) - w_t\big(\mu(v_t)\big) - \mu([v_t, w_t])\Big) + \mu(w_t)\Big) \sigma^n \quad \mathrm{d}t
\end{aligned} \tag{5.6}$$

for any $w_t \in T_{\mathrm{id}}\mathrm{Diff}^s_\sigma(B)$, which is an Euler-type equation on the symplectomorphisms of (B, σ), and then, l_t satisfies

$$\dot{l}_t + v_t(l_t) = -\mu(\dot{v}_t) - v_t\big(\mu(v_t)\big), \tag{5.7}$$

which is an inhomogeneous linear PDE corresponding to the homogeneous equation (5.4).

Proposition 5.4. *On a Hamiltonian manifold* $(M = B \times S^1, \omega)$ *with generator* $R = \partial_\theta$ *of* $\ker \omega$ *and Riemannian metric with volume form* $\mathrm{vol} = \mathrm{d}\theta \wedge \omega^n$*, the Euler equations preserving* R *and* ω *are given by Eqs.* (5.6) *and* (5.7). □

To prove that solutions exist for short times, one can try to follow the strategies in [EM70] and this thesis, i. e. one can compute the orthogonal projections on each of the tangent spaces $T_\eta\mathrm{Diff}^s(B \times S^1) \to T_\eta\mathrm{Diff}^s_{R,\omega}(B \times S^1)$ for any $\eta \in \mathrm{Diff}^s_{R,\omega}(B \times S^1)$ and determine whether these maps are smooth in the base point η.

By the results in Section 2.4, we have (local) geodesics on $\mathrm{Diff}^s_{\mathrm{vol}}(M)$ for any (not necessarily trivial) circle bundle $S^1 \to M \to B$. By Theorem 2.24, $\mathrm{Diff}^s_{R,\mathrm{vol}}(M) \subset \mathrm{Diff}^s_{\mathrm{vol}}(M)$ is a totally geodesic submanifold. In our case, it therefore remains to compute the projections $T_\eta \mathrm{Diff}^s_{R,\mathrm{vol}}(B\times S^1) \to T_\eta \mathrm{Diff}^s_{R,\omega}(B\times S^1)$.

For the long-time existence of solutions, one can then use Eqs. (5.6) and (5.7) to estimate the H^s-norms of $V_t = v_t + l_t\partial_\theta$.

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