

Robust quantum transport at particle-hole symmetry

IPSITA MANDAL^{1,2(a)}  and KLAUS ZIEGLER³

¹ *Institute of Nuclear Physics, Polish Academy of Sciences - 31-342 Kraków, Poland*

² *Department of Physics, Stockholm University, AlbaNova University Center - 106 91 Stockholm, Sweden*

³ *Institut für Physik, Universität Augsburg - D-86135 Augsburg, Germany*

Abstract – We study quantum transport in disordered systems with particle-hole symmetric Hamiltonians. The particle-hole symmetry is spontaneously broken after averaging with respect to disorder, and the resulting massless mode is treated in a random-phase representation of the invariant measure of the symmetry group. We compute the resulting fermionic functional integral of the average two-particle Green’s function in a perturbation theory around the diffusive limit. The results up to two-loop order show that the corrections vanish, indicating that the diffusive quantum transport is robust. On the other hand, the diffusion coefficient depends strongly on the particle-hole symmetric Hamiltonian we choose to study. This reveals a connection between the underlying microscopic theory and the classical long-scale metallic behaviour of these systems.

Introduction. – Recent studies have found that transport in multi-band semimetals is substantially different from conventional transport based on the classical Boltzmann theory. This is due to the particle-hole (PH) symmetry, which is realized, at least approximately, in multi-band systems when the Fermi energy is between two neighboring bands. A typical example is the Dirac node of graphene. The PH symmetry of this two-band model leads to characteristic quantum effects, such as spontaneous particle-hole pair creation on arbitrarily small energy scales, which are only limited by the band width of the material. This is accompanied by strong fluctuations (also known as “zitterbewegung”), which causes a finite dc conductivity even in the absence of disorder. Although graphene is a two-dimensional (2d) material, where fluctuation effects are strong, these quantum fluctuations may also play a crucial role in higher dimensions. Based on this idea, there has been a lot of progress to compute transport properties in systems like the 3d Weyl semimetals [1–22].

The fundamental quantity for the study of quantum transport is the transition probability $P_{\mathbf{r}\mathbf{r}'}$, for a particle to move from site \mathbf{r}' to a site \mathbf{r} on a d -dimensional lattice. This classical quantity can be linked to a quantum model through the average two-particle Green’s function (A2PGF) of a particle of energy E , given by

$$K_{\mathbf{r}\mathbf{r}'} = \frac{1}{\pi} \langle G_{\mathbf{r}\mathbf{r}'}(E + i\epsilon) G_{\mathbf{r}'\mathbf{r}}(E - i\epsilon) \rangle_d \quad (\text{where } \epsilon > 0). \quad (1)$$

Here H is a random Hamiltonian, $G_{\mathbf{r}\mathbf{r}'}(E + i\epsilon) = \langle \mathbf{r}' | (H - E - i\epsilon)^{-1} | \mathbf{r} \rangle$ is the one-particle Green’s function, and $\langle \dots \rangle_d$ denotes averaging with respect to disorder-induced randomness. Then the transition probability is given by $P_{\mathbf{r}\mathbf{r}'} = K_{\mathbf{r}\mathbf{r}'} / \sum_{\mathbf{r}'} K_{\mathbf{r}\mathbf{r}'}$. The time-dependent transition probability is obtained through the Fourier transformation $E \rightarrow t$. The transport properties in the metallic regime can be understood by computing the diffusion coefficient D , which can be obtained from $K_{\mathbf{r}\mathbf{r}'}$ as [23]

$$D = \lim_{\epsilon \rightarrow 0} \epsilon^2 \sum_{\mathbf{r} \in \Lambda} r^2 K_{0\mathbf{r}}, \quad (2)$$

on a lattice Λ . The corresponding dc conductivity is related to D via the Einstein relation.

We can assume that the form of G results from a self-energy approximation in interacting many-body systems. Then ϵ , as the imaginary part of the self-energy, depends on the frequency ω and the Fermi energy E_F . With this, we can bridge the microscopic quantum modeling and the more qualitative hydrodynamic description of transport due to long-lived modes in quantum systems [24–28]. In particular, using this formalism, we can analyze the effect of a vanishing $\epsilon \sim \omega^s$, where s is a positive rational number.

In this paper, we will focus on diffusion in systems with PH symmetry, in the presence of disorder. We assume that the disorder also obeys the PH symmetry. Although systems without PH symmetry can also be treated by the subsequently discussed method (cf. refs. [29,30]),

^(a)E-mail: ipsita.mandal@gmail.com (corresponding author)

we shall focus here only on the PH-symmetric case for simplicity. Taking into account the fact that averaging over a random distribution of disorder leads to spontaneous PH symmetry breaking, we employ a field theory representation to study this effect. Although the PH symmetry is discrete, an underlying global symmetry of the field theory is continuous [31]. Thus, there is a massless mode associated with the spontaneous PH symmetry breaking that leads to long-range correlations. These correlations are the origin of diffusion. They can be calculated from the massless mode, using the integration over the symmetry-related saddle point manifold. This is known as the integration with respect to the invariant measure of the symmetry group, and is often approximated in a leading-order gradient expansion, also known as the nonlinear sigma model approach [32]. Here we will consider the full invariant measure, which was shown to provide a simple expression for the A2PGF in terms of a random-phase model [33]. This is briefly summarized in the next section.

The paper is organized as follows. The representation of the invariant measure by the random-phase model is briefly reviewed in the next section. In the third section we introduce a fermionic functional integral for the description of the A2PGF, and employ a perturbation expansion around the diffusive approximation. It is shown that one- and two-loop corrections vanish, indicating the robustness of diffusion. In the fourth section we study some examples of the microscopic Hamiltonians with PH symmetry, and the results reveal that the diffusion coefficients depend strongly on the details of these microscopic Hamiltonians.

Random-phase representation of the average two-particle Green's function. – For simplicity, we are restricting the discussion to two bands, while an extension to n bands is straightforward. Hence, our starting point is a two-band Hamiltonian H with matrix elements $H_{\mathbf{r}j,\mathbf{r}'j'}$, where \mathbf{r}, \mathbf{r}' are the co-ordinates on a d -dimensional lattice, and j, j' are band indices. In this case, we can assign a Pauli matrix representation for the Hamiltonian as

$$H_{\mathbf{r}j,\mathbf{r}'j'} = \sum_{a=1}^3 \mathcal{H}_{a,\mathbf{r}\mathbf{r}'} \sigma_{jj'}^a, \quad (3)$$

where σ^a is a Pauli matrix, and $\mathcal{H}_{a,\mathbf{r}\mathbf{r}'}$ a matrix element on the lattice. An instructive example is discussed in appendix A. The \mathbf{r} -independent PH transformation S transforms the Hamiltonian H as $SHS^{-1} = -H$, which implies that an eigenvector Ψ_E with energy E is related to the eigenvector Ψ_{-E} with energy $-E$ by $\Psi_{-E} = S\Psi_E$. Thus, S is a symmetry transformation for the Green's function $G(z) \rightarrow -SG(z)S^{-1} = G(-z)$ at $z = 0$, which is exactly at the mirror-symmetric point between the two symmetric bands. Due to the poles of $G(z)$, this symmetry point must be treated with care. We can avoid the poles by choosing $z = i\epsilon$ ($\epsilon > 0$). Then the

difference, in the limit $\epsilon \rightarrow 0$, reads

$$\lim_{\epsilon \rightarrow 0} [-SG(i\epsilon)S^{-1} - G(i\epsilon)] = \lim_{\epsilon \rightarrow 0} [G(-i\epsilon) - G(i\epsilon)]. \quad (4)$$

A nonzero result indicates a spontaneously broken PH symmetry. For the diagonal elements of the Green's functions, the right-hand side is proportional to the density of states at $E = 0$. This reflects the fact that a nonzero density of states at $E = 0$ provides a sufficient condition for spontaneous PH symmetry-breaking.

The disorder-averaged one-particle Green's function can be calculated within the self-consistent Born approximation (or the saddle-point approximation of a functional-integral representation) as

$$\langle H \pm i\epsilon \rangle_d^{-1} \approx [H_0 \pm i(\epsilon + \eta)]^{-1}, \quad H_0 = \langle H \rangle_d + \Sigma', \quad (5)$$

where Σ' is the real part of the self-energy, and η is its imaginary part. The parameter η provides a broadening of the average one-particle Green's function, and also plays the role of an order parameter for spontaneous PH symmetry breaking, since we get $2i\eta(H_0^2 + \eta^2)^{-1}$ for eq. (4).

Transport properties are determined by properties on large time and spatial scales. In ref. [33], it was shown that the long-range part of the A2PGF of eq. (1) can be obtained by reducing the disorder average to the integration with respect to the invariant measure of the saddle-point manifold. This means that in practice, we replace H with the effective random-phase Hamiltonian

$$\mathcal{H}_R = UH_0U^\dagger, \quad U = \text{diag}(e^{i\alpha_{\mathbf{r}j}}), \quad (6)$$

and then average with respect to the independently and identically distributed random phases $\{\alpha_{\mathbf{r}j}\}$. This gives us

$$K_{\mathbf{r}\mathbf{r}'} \sim \mathcal{K}_{\mathbf{r}\mathbf{r}'} = \frac{\langle \text{adj}_{\overline{\mathbf{r}\mathbf{r}'}} C \rangle_\alpha}{\langle \det C \rangle_\alpha}, \quad (7)$$

$$\langle \dots \rangle_\alpha = \frac{1}{2\pi} \int_0^{2\pi} \dots \prod_{\mathbf{r},j} d\alpha_{\mathbf{r}j},$$

with the random-phase matrix

$$C_{\mathbf{r}\mathbf{r}'} = 2\delta_{\mathbf{r}\mathbf{r}'} - \sum_{j,j'} e^{i\alpha_{\mathbf{r}j}} h_{\mathbf{r}j,\mathbf{r}'j'} \sum_{j'',\mathbf{r}''} h_{\mathbf{r}'j',\mathbf{r}''j''}^\dagger e^{-i\alpha_{\mathbf{r}''j''}}, \quad (8)$$

with

$$h_{\mathbf{r}\mathbf{r}'} = \mathcal{I}_2 \delta_{\mathbf{r}\mathbf{r}'} + 2i\eta (H_0 - i\bar{\eta})_{\mathbf{r}\mathbf{r}'}^{-1}, \quad \bar{\eta} = \eta + \epsilon, \quad (9)$$

and $\text{adj}_{\overline{\mathbf{r}\mathbf{r}'}} C$ denoting the elements of the adjugate matrix. Under a PH transformation, we obtain the Hermitian conjugation $ShS^{-1} = h^\dagger$, which implies that C is real and symmetric. The quantity UhU^\dagger represents the effective one-particle Green's function of the generic

system, while C is the corresponding effective two-particle propagator.

Although $C_{\mathbf{r}\mathbf{r}'}$ is still a random matrix, the A2PGF in eq. (7) is much simpler to treat than the A2PGF in eq. (1), because the phase integration is not plagued by poles of the integrand. Nevertheless, this does not mean that the theory becomes simple. For instance, the long-range behaviour of the A2PGF is based on the zero modes of C , which exist for any realization of the random-phases due to the relation

$$hh^\dagger = \mathbf{1} - 4\epsilon(1-\epsilon)\bar{\eta}(H_0^2 + \bar{\eta}^2)^{-1}. \quad (10)$$

This implies that the constant mode Ψ_0 with vanishing wave vector $\mathbf{k} = 0$ obeys

$$\sum_{\mathbf{r}'} C_{\mathbf{r}\mathbf{r}'} \Psi_0 \sim c\epsilon\Psi_0, \quad (11)$$

i.e., Ψ_0 is always a zero-energy eigenmode of the effective two-particle propagator in the limit $\epsilon \rightarrow 0$.

In order to evaluate \mathcal{K} , or the diffusion coefficient D , we can employ two different methods. The first is based on a graphical representation, while the second involves a fermionic functional integral representation. While the former was described and discussed in ref. [33], we will focus on the latter in this paper.

General properties of $\tilde{K}_{\mathbf{q}}$. Before we start with the specific calculations, it is useful to mention an important connection between the two-particle and the one-particle Green's function ("Ward Identity"), which takes the form

$$\tilde{K}_{\mathbf{q}=0} = \frac{\pi}{\epsilon} \langle \rho(0) \rangle_d, \quad (12)$$

after averaging. Here, $\langle \rho(0) \rangle_d$ is the disorder average of the density of states at energy $E = 0$, which is typically nonzero and finite. Although we do not have a proof, this relation should also hold for \mathcal{K} due to $K \sim \mathcal{K}$ on large scales. Therefore, $\tilde{K}_{\mathbf{q}=0} \sim \text{const} \epsilon^{-1}$, which will be confirmed in the subsequent calculation. The second derivative with respect to q_μ at $\mathbf{q} = 0$ is

$$-\partial_{q_\mu}^2 \tilde{K}_{\mathbf{q}} \Big|_{\mathbf{q}=0} \equiv -\tilde{K}_0'' = D\epsilon^{-\beta}, \quad (13)$$

with $\beta = 2$ for diffusion. This is in agreement with eq. (2). Higher-order derivatives of $\tilde{K}_{\mathbf{q}}$ are also of interest, since

$$\sum_{\mathbf{r}} (r_\mu)^{2n} K_{0\mathbf{r}} = (-1)^n \tilde{K}_0^{(2n)} \quad (14)$$

describe higher moments of spatial fluctuations.

Functional integral representation. – The averaged determinant in eq. (7) can be expressed as a fermionic (Grassmann) functional integral [34]

$$\langle \det C \rangle_\alpha = \int_{\Psi} \langle \exp(-\Psi \cdot C \bar{\Psi}) \rangle_\alpha, \quad (15)$$

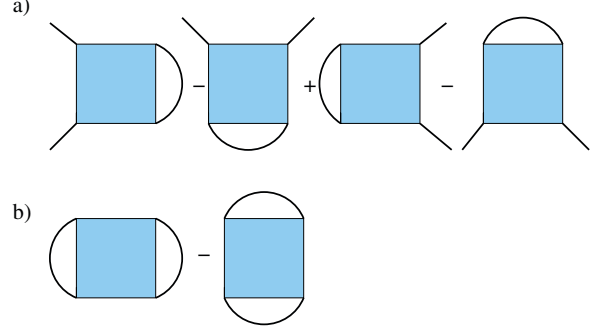


Fig. 1: First-order perturbation theory of \mathcal{K} in eq. (16) around the diffusion propagator G for the (a) numerator, and (b) the denominator. These terms cancel each other due to the symmetry of the blue square vertex.

which implies that

$$\mathcal{K}_{\mathbf{r}\mathbf{r}'} = \frac{\int_{\Psi} \bar{\Psi}_{\mathbf{r}} \Psi_{\mathbf{r}'} \langle \exp(-\Psi \cdot C \bar{\Psi}) \rangle_\alpha}{\int_{\Psi} \langle \exp(-\Psi \cdot C \bar{\Psi}) \rangle_\alpha}. \quad (16)$$

In terms of a perturbation theory around $\bar{C} = \langle C \rangle_\alpha$, with

$$\begin{aligned} \langle C_{\mathbf{r}\mathbf{r}'} \rangle_\alpha &= 2\delta_{\mathbf{r}\mathbf{r}'} - \sum_{j,j'=1,2} h_{\mathbf{r}j,\mathbf{r}'j'} h_{\mathbf{r}'j',\mathbf{r}j}^\dagger = \\ &= 2\delta_{\mathbf{r}\mathbf{r}'} - \text{Tr}_2(h_{\mathbf{r}\mathbf{r}'} h_{\mathbf{r}\mathbf{r}'}^\dagger), \end{aligned} \quad (17)$$

we have

$$\begin{aligned} \langle e^{-\Psi C \bar{\Psi}} \rangle_\alpha &= \langle e^{-\Psi(\bar{C}-C')\bar{\Psi}} \rangle_\alpha = \\ &= [1 + \langle (\Psi C' \bar{\Psi})^2 \rangle_\alpha / 2 + \dots] e^{-\Psi \bar{C} \bar{\Psi}}. \end{aligned} \quad (18)$$

Using this, we obtain

$$\begin{aligned} \int_{\Psi} \bar{\Psi}_{\mathbf{r}} \Psi_{\mathbf{r}'} \langle e^{-\Psi C \bar{\Psi}} \rangle_\alpha &= \\ \int_{\Psi} \bar{\Psi}_{\mathbf{r}} \Psi_{\mathbf{r}'} [1 + \langle (\Psi C' \bar{\Psi})^2 \rangle_\alpha / 2 + \dots] e^{-\Psi \bar{C} \bar{\Psi}}, \end{aligned} \quad (19)$$

and

$$\begin{aligned} \int_{\Psi} \langle e^{-\Psi C \bar{\Psi}} \rangle_\alpha &= \\ \int_{\Psi} [1 + \langle (\Psi C' \bar{\Psi})^2 \rangle_\alpha / 2 + \dots] e^{-\Psi \bar{C} \bar{\Psi}}. \end{aligned} \quad (20)$$

After normalizing both the expressions by $Z_0 = \det \langle C \rangle_\alpha$, we can represent the result graphically by two-loop graphs as depicted in fig. 1 (neglecting higher-order terms indicated by \dots). The square in fig. 1 represents the vertex

$$\begin{aligned} V_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3\mathbf{r}_4} &\equiv \langle C'_{\mathbf{r}_1\mathbf{r}_2} C'_{\mathbf{r}_3\mathbf{r}_4} \rangle_\alpha = \\ &= \text{Tr}_2(h_{\mathbf{r}_1\mathbf{r}_2} h_{\mathbf{r}_2\mathbf{r}_3}^\dagger h_{\mathbf{r}_3\mathbf{r}_4} h_{\mathbf{r}_4\mathbf{r}_1}^\dagger) \\ &\quad - \delta_{\mathbf{r}_1\mathbf{r}_3} \sum_j (h_{\mathbf{r}_1\mathbf{r}_2} h_{\mathbf{r}_2\mathbf{r}_1}^\dagger)_{jj} (h_{\mathbf{r}_1\mathbf{r}_4} h_{\mathbf{r}_4\mathbf{r}_1}^\dagger)_{jj}, \end{aligned} \quad (21)$$

$$\tilde{C}_{\mathbf{q}} \sim \frac{4\eta}{3\lambda^4} \left[-\eta q^2 + 6\eta^2 \epsilon - 3\eta \lambda^2 + 3\lambda^2 \left\{ \frac{\eta (q^2 + 2\eta^2) \ln \left(1 + \frac{q(q + \sqrt{q^2 + 4\eta^2})}{2\eta^2} \right)}{q\sqrt{4\eta^2 + q^2}} - 4\epsilon \ln(\eta/\lambda) \right\} \right], \quad (29)$$

and the thick lines are the unperturbed propagator \bar{C}^{-1} . It turns out that the two-loop corrections cancel each other. This is a consequence of the anti-commuting property of the fermion field. The details of the calculation are given in appendix B. Thus, in the Fourier space only the unperturbed propagator

$$\tilde{\mathcal{K}}_{\mathbf{q}} = \frac{1}{2 - \int_{\mathbf{k}} \text{Tr}_2(\tilde{h}_{\mathbf{k}} \tilde{h}_{\mathbf{k}-\mathbf{q}}^\dagger)} \quad (22)$$

survives in this approximation. Here $\int_{\mathbf{k}}$ denotes the normalized integral with respect to the d -dimensional sphere with radius λ . The denominator of $\tilde{\mathcal{K}}_{\mathbf{q}}$ can be expanded in powers of \mathbf{q} . This leads to $\tilde{\mathcal{K}}_{\mathbf{q}} \sim 1/(A\epsilon + Bq^2)$, provided the expansion exists and the system is isotropic. Due to the mirror-symmetric dispersion ($E_{\mathbf{k}}, -E_{\mathbf{k}}$) of the PH-symmetric averaged Hamiltonian H_0 , we get

$$A = 8\eta \int_{\mathbf{k}} \frac{1}{E_{\mathbf{k}}^2 + \bar{\eta}^2}, \quad B = \frac{1}{2} \int_{\mathbf{k}} \text{Tr}_2(\tilde{h}_{\mathbf{k}} \partial_{k_\mu} \partial_{k_\mu} \tilde{h}_{\mathbf{k}}^\dagger). \quad (23)$$

Thus, $\tilde{\mathcal{K}}_{\mathbf{q}}$ is a diffusion propagator with the diffusion coefficient $D = B/A$. This perturbative result clearly indicates that diffusion is quite robust for a PH-symmetric Hamiltonian. The robustness of diffusion in terms of a perturbation theory was also observed for the special case of 2d Dirac fermions [31,35].

Discussions. – For a better understanding of the result in eqs. (22) and (23), we will consider a simple example of a two-band Hamiltonian. It is defined in the Fourier space with the dispersion $E_{\mathbf{k}}$ as

$$\tilde{h}_{\mathbf{k}} = \sigma_0 + \frac{2i\eta}{E_{\mathbf{k}}^2 + \bar{\eta}^2} \begin{pmatrix} E_{\mathbf{k}} + i\bar{\eta} & 0 \\ 0 & -E_{\mathbf{k}} + i\bar{\eta} \end{pmatrix}, \quad (24)$$

which can also be written as

$$\tilde{h}_{\mathbf{k}} = \begin{pmatrix} \kappa_{\mathbf{k}} & 0 \\ 0 & \kappa_{\mathbf{k}}^* \end{pmatrix}, \quad \kappa_{\mathbf{k}} = \frac{E_{\mathbf{k}}^2 - \bar{\eta}^2 + 2\epsilon\bar{\eta} + 2i\eta E_{\mathbf{k}}}{E_{\mathbf{k}}^2 + \bar{\eta}^2}. \quad (25)$$

In the limit $\epsilon \rightarrow 0$, we get a unimodular function

$$\kappa_{\mathbf{k}} \rightarrow \frac{(E_{\mathbf{k}} + i\eta)^2}{E_{\mathbf{k}}^2 + \eta^2} = e^{i\phi_{\mathbf{k}}}, \quad \phi_{\mathbf{k}} = \arg[(E_{\mathbf{k}}^2 + i\eta)^2]. \quad (26)$$

The Fourier transform of \bar{C} takes the form

$$\tilde{C}_{\mathbf{q}} = \int_{\mathbf{k}} \left[2 - \text{Tr}_2(\tilde{h}_{\mathbf{k}} \tilde{h}_{\mathbf{k}-\mathbf{q}}^\dagger) \right] = \int_{\mathbf{k}} (2 - \kappa_{\mathbf{k}}^* \kappa_{\mathbf{k}-\mathbf{q}} - \kappa_{\mathbf{k}} \kappa_{\mathbf{k}-\mathbf{q}}^*), \quad (27)$$

which gives the expression $\tilde{C}_{\mathbf{q}} \sim A\epsilon + Bq^2$ (as shown in the third section) for small \mathbf{q} . Here A is defined as the integral in eq. (23) and

$$B = 4\eta^2 \int_{\mathbf{k}} \frac{(\partial_{k_\mu} E_{\mathbf{k}})(\partial_{k_\mu} E_{\mathbf{k}})}{(E_{\mathbf{k}}^2 + \eta^2)^2}. \quad (28)$$

Our model has two independent parameters, the effective disorder strength η and the momentum cut-off λ , where $1/\lambda$ defines the shortest wavelength. Moreover, the dimensionality d of the \mathbf{k} -integration plays a crucial role. Although we do not expect that the qualitative behaviour of diffusion is much affected by the short-distance regime, the diffusion coefficient $D = B/A$ might depend on it. In order to study this effect and its relation to different dispersions $E_{\mathbf{k}}$, we calculate it for two characteristic examples, namely $E_{\mathbf{k}} = E_s k^s$ with $s = 1, 2$. The expressions for A and B imply that the energy coefficient E_s can be absorbed in the scaling of η and ϵ . Therefore, we implicitly assume subsequently that these parameters are scaled as $\eta \rightarrow \eta/E_s$ and $\epsilon \rightarrow \epsilon/E_s$.

The integral in eq. (27) can be calculated for $\epsilon \sim 0$, we obtain the following results:

- 1) $E_{\mathbf{k}} = k, d = 2$:

see eq. (29) above

and

$$\tilde{C}_{\mathbf{q}}|_{\mathbf{q} \sim 0} \sim \frac{4\eta}{3\lambda} \left[\left\{ \frac{6\eta^2}{\lambda^2} - 12 \ln(\eta/\lambda) \right\} \frac{\epsilon}{\lambda} + \frac{\lambda^2 - \eta^2}{\eta} q^2 \right]. \quad (30)$$

- 2) $E_{\mathbf{k}} = k, d = 3$:

$$\tilde{C}_{\mathbf{q}} \sim \frac{3\eta}{2\pi\lambda^3 q} \left[(32\lambda\epsilon - 2\pi\eta^2) q + \pi\eta (4\eta^2 + 3q^2) \arctan\left(\frac{q}{2\eta}\right) \right] \quad (31)$$

and

$$\tilde{C}_{\mathbf{q}}|_{\mathbf{q} \sim 0} \sim \frac{3\eta}{2\pi\lambda} \left(\frac{32}{\lambda} \epsilon + \frac{4\pi}{3\lambda^2} q^2 \right). \quad (32)$$

This describes diffusion with a diffusion coefficient $D = \lambda^2(\bar{\eta}^{-1} - \bar{\eta})/(6\bar{\eta}^2 - 12 \ln \bar{\eta})$ with $\bar{\eta} = \eta/\lambda$ in $d = 2$ and $D = \frac{\pi}{24\lambda}$ in $d = 3$.

- 3) $E_{\mathbf{k}} = k^2$:

$$\tilde{C}_{\mathbf{q}} \sim \begin{cases} 4\lambda^{-2} [\epsilon(\pi - 2\eta/\lambda^2) + q^2/\lambda^2], & \text{for } d = 2, \\ \lambda^{-3} \sqrt{2\eta} (6\epsilon + \eta q^2), & \text{for } d = 3, \end{cases} \quad (33)$$

Table 1: The diffusion coefficients for the dispersions $E_{\mathbf{k}}$, and for dimensionalities $d = 2, 3$, with $\tilde{\eta} = \eta/\lambda$. The result for 2d Dirac fermions is from ref. [31].

	$d = 2$	$d = 3$
$E_{\mathbf{k}} = k$	$\frac{\lambda^2 (\tilde{\eta}^{-1} - \tilde{\eta})}{6\tilde{\eta}^2 - 12 \ln \tilde{\eta}}$	$\frac{\pi}{24\lambda}$
$E_{\mathbf{k}} = k^2$	$\frac{1}{\lambda^2 (\pi - 2\eta/\lambda^2)}$	$\eta/6$
Dirac fermions	$-\frac{1}{4\lambda\tilde{\eta} \ln \tilde{\eta}}$	-

for $\epsilon \sim 0$ and $q \sim 0$. This describes diffusion with diffusion coefficients $D = 1/(\pi\lambda^2 - 2\eta)$ ($d = 2$) and $D = \eta/6$ ($d = 3$). It is remarkable that D vanishes with $\eta \rightarrow 0$ only in $d = 3$, but not in $d = 2$.

The results for the diffusion coefficients D are summarized in table 1. They clearly indicate that these diffusion coefficients depend strongly on the dispersion of H_0 . A vanishing order parameter η indicates a transition from the metallic phase to another phase, typically to an insulating phase. The results in table 1 reveal that the properties of such a transition from the metallic side depend strongly on the details of the model and the dimensionality of the system. Although we focus on the metallic phase here, these properties might be interesting and deserve a further analysis to identify the strong influences of the PH-symmetric Hamiltonians.

In conclusion, we have found that diffusion (*i.e.*, metallic behaviour) is very robust if the model Hamiltonian obeys the PH transformation property $H \rightarrow SHS^{-1} = -H$, as the one-loop and two-loop corrections vanish. The reason behind this is the spontaneous PH symmetry breaking, which is associated with a spontaneous breaking of a continuous symmetry, which creates a massless mode. However, the diffusion coefficient D is very sensitive to the spectral properties of H and the dimensionality of the underlying space. This connection between the underlying microscopic details of the model and the classical diffusion coefficient is an advantage of our approach, in comparison to more heuristic approaches (*e.g.*, the Mori-Zwanzig memory matrix formalism [27,36–39]). In particular, it will be interesting to apply this formalism to Luttinger semimetals, where methods like Kubo formula and memory matrix fail unless the PH symmetry is broken by unequal band masses [38–41]. Finally, it will be worthwhile to see if our formalism can be used to compute transport properties in non-Fermi liquids having a critical Fermi surface [42–47].

KZ gratefully acknowledges the support by the Julian Schwinger Foundation.

Data availability statement: No new data were created or analysed in this study.

Appendix A: example of a particle-hole symmetric tight-binding model. – As an example for a system with PH symmetry, we consider the tight-binding Hamiltonian on the honeycomb lattice with nearest-neighbor hopping. The honeycomb lattice is bipartite and consists of two triangular sublattices. The nearest-neighbor hopping connects these two sublattices, such that we can write the hopping Hamiltonian in the sublattice representation as

$$H = \begin{pmatrix} 0 & h \\ h^T & 0 \end{pmatrix},$$

where the hopping term h^T is the transpose of h . H can be expanded in terms of the Pauli matrices as

$$H = \mathcal{H}_1\sigma^1 + \mathcal{H}_2\sigma^2, \quad \mathcal{H}_1 = (h + h^T)/2, \\ \mathcal{H}_2 = i(h - h^T)/2.$$

Thus, we get $H \rightarrow \sigma^3 H \sigma^3 = -H$, *i.e.*, $S = \sigma^3$ in the second section, as the PH transformation. PH-symmetric disorder can be implemented as corrugations or random strain in the hopping terms.

Appendix B: calculation details of the perturbation theory. – The vertex in eq. (21) reads

$$V_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3\mathbf{r}_4} = \sum_{j_1, \dots, j_4=1,2} (1 - \delta_{\mathbf{r}_1j_1, \mathbf{r}_3j_3}) h_{\mathbf{r}_1j_1, \mathbf{r}_2j_2} \\ \times h_{\mathbf{r}_2j_2, \mathbf{r}_3j_3}^\dagger h_{\mathbf{r}_3j_3, \mathbf{r}_4j_4} h_{\mathbf{r}_4j_4, \mathbf{r}_1j_1}^\dagger,$$

which can be decomposed with the help of trace terms as

$$(1 - \delta_{\mathbf{r}_1\mathbf{r}_3}) \text{Tr}_2 (h_{\mathbf{r}_1\mathbf{r}_2} h_{\mathbf{r}_2\mathbf{r}_3}^\dagger h_{\mathbf{r}_3\mathbf{r}_4} h_{\mathbf{r}_4\mathbf{r}_1}^\dagger) \\ + \delta_{\mathbf{r}_1\mathbf{r}_3} \left[\text{Tr}_2 (h_{\mathbf{r}_1\mathbf{r}_2} h_{\mathbf{r}_2\mathbf{r}_3}^\dagger h_{\mathbf{r}_3\mathbf{r}_4} h_{\mathbf{r}_4\mathbf{r}_1}^\dagger) \right. \\ \left. - \sum_j (h_{\mathbf{r}_1\mathbf{r}_2} h_{\mathbf{r}_2\mathbf{r}_1}^\dagger)_{jj} (h_{\mathbf{r}_1\mathbf{r}_4} h_{\mathbf{r}_4\mathbf{r}_1}^\dagger)_{jj} \right] = \\ \text{Tr}_2 (h_{\mathbf{r}_1\mathbf{r}_2} h_{\mathbf{r}_2\mathbf{r}_3}^\dagger h_{\mathbf{r}_3\mathbf{r}_4} h_{\mathbf{r}_4\mathbf{r}_1}^\dagger) \\ - \delta_{\mathbf{r}_1\mathbf{r}_3} \sum_j (h_{\mathbf{r}_1\mathbf{r}_2} h_{\mathbf{r}_2\mathbf{r}_1}^\dagger)_{jj} (h_{\mathbf{r}_1\mathbf{r}_4} h_{\mathbf{r}_4\mathbf{r}_1}^\dagger)_{jj}. \quad (\text{B.1})$$

Since we get $h \rightarrow ShS^{-1} = h^\dagger$ from a PH transformation, the first term obeys the relation

$$\text{Tr}_2 (h_{\mathbf{r}_1\mathbf{r}_2} h_{\mathbf{r}_2\mathbf{r}_3}^\dagger h_{\mathbf{r}_3\mathbf{r}_4} h_{\mathbf{r}_4\mathbf{r}_1}^\dagger) = \text{Tr}_2 (h_{\mathbf{r}_1\mathbf{r}_2}^\dagger h_{\mathbf{r}_2\mathbf{r}_3} h_{\mathbf{r}_3\mathbf{r}_4}^\dagger h_{\mathbf{r}_4\mathbf{r}_1}). \quad (\text{B.2})$$

Moreover, $\langle C \rangle_\alpha^{-1}$ is real symmetric, as mentioned in the second section. With these two properties, the perturbation expansion up to two loops can be rewritten as

$$\frac{1}{Z_0} \int_{\Psi} \bar{\Psi}_{\mathbf{r}} \Psi_{\mathbf{r}'} \langle e^{-\Psi \cdot C \Psi} \rangle_\alpha \approx \\ G_{\mathbf{r}\mathbf{r}'} [1 + V_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3\mathbf{r}_4} (G_{\mathbf{r}_1\mathbf{r}_2} G_{\mathbf{r}_3\mathbf{r}_4} - G_{\mathbf{r}_1\mathbf{r}_4} G_{\mathbf{r}_3\mathbf{r}_2})]. \quad (\text{B.3})$$

Note that

$$V_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3\mathbf{r}_4}[G_{\mathbf{r}\mathbf{r}_1}(G_{\mathbf{r}_2\mathbf{r}_3}G_{\mathbf{r}_4\mathbf{r}'} - G_{\mathbf{r}_2\mathbf{r}'}G_{\mathbf{r}_4\mathbf{r}_3}) + G_{\mathbf{r}\mathbf{r}_3}(G_{\mathbf{r}_2\mathbf{r}'}G_{\mathbf{r}_4\mathbf{r}_1} - G_{\mathbf{r}_2\mathbf{r}_1}G_{\mathbf{r}_4\mathbf{r}'})] = G_{\mathbf{r}\mathbf{r}'}, \quad (\text{B.4})$$

since the vertex is invariant under $V_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3\mathbf{r}_4} \rightarrow V_{\mathbf{r}_4\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3}$ (cyclic permutation of its indices). We also have taken into account the appropriate signs, which reflect the fermionic statistics of the field Ψ . These properties finally lead to the result

$$\frac{1}{Z_0} \int_{\Psi} \langle e^{-\Psi \cdot C \bar{\Psi}} \rangle_{\alpha} \approx 1 + V_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3\mathbf{r}_4} (G_{\mathbf{r}_1\mathbf{r}_2}G_{\mathbf{r}_3\mathbf{r}_4} - G_{\mathbf{r}_1\mathbf{r}_4}G_{\mathbf{r}_3\mathbf{r}_2}) = 1. \quad (\text{B.5})$$

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