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Actions of Linearized Polynomials on the Algebraic Closure of a Finite Field

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Abstract. Let g and h be monic polynomials in $F[x]$, where F is the finite field of order q. We define a dynamical system by letting the q-linearized polynomial associated with g act on equivalence classes of a certain F-subspace of the algebraic closure $\bar{F}$ of F in which related elements of $\bar{F}$ lie in the same orbit under the action of the q-linearized polynomial associated with h. When $h = x$, this is equivalent to the system in which the dynamic polynomial g acts on irreducible polynomials over F as discussed in [CH], where a conjecture of Morton [M] was proved as regards linearized polynomials. A generalization of that result is proved here. This states that when g and h are non-constant relatively prime polynomials, then there are infinitely many classes with prescribed preperiod and primitive period in the (g, h)-dynamical system.

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1. Introduction

Let $F = \mathrm{GF}(q)$ and $\bar{F}$ be the algebraic closure of F. For any polynomial $f = \sum_i f_i x^i$ in $F[x]$, let $A_q(f)$ be the associated (additive) *q-linearized polynomial* (or simply *q-polynomial*) $\sum_i f_i x^{q^i}$ and set

$$x^f := A_q(f)(x). \tag{1.1}$$

By these means we obtain the set A_F of all q-polynomials. Moreover, A_F acquires a ring structure through addition and *polynomial composition.* Indeed, the association $f \to A_q(f)$ yields a ring isomorphism from $F[x]$ to A_F (see [O] or [LN]).

By the *dynamics* of a mapping γ on a set S is meant all that pertains to the orbits of elements of S under iterates of γ. For $i \geq 0$ let $\gamma^{(i)}$ denote the ith iterate of γ (with $\gamma^{(0)}$ being the identity on S). An element $s \in S$ is called *periodic*, if its orbit $\{\gamma^{(i)}(s) \mid i \geq 0\}$ is finite. If s is periodic and $k \geq 0$ and $n \geq 1$ are minimial such that $\gamma^{(k)}(s) = \gamma^{(k+n)}(s)$, then k is called the *preperiod of* s, while n is called the *primitive period* of s. A periodic element is called *purely periodic* if its preperiod is zero. The *backward orbit* of $s \in S$ is the set of all $t \in S$ for which there exists an $i \geq 0$ such that $\gamma^{(i)}(t) = s$, excluding the members of the orbit of s different from s, if s is purely periodic. We refer to (S, γ) as a *dynamical system.*

In this paper γ will be induced by a monic q-polynomial $A_q(g)$ in which event the above mentioned isomorphism reduces the study of iterates and composites of dynamical mappings to that of powers and products of ordinary polynomials, respectively. For the set S we may, in the first instance, choose $S = \bar{F}$. If $A_q(g)$ acts naturally on S (i.e., by evaluation), it is clear that every element is periodic. Moreover, if g is non-constant, then, given n, there are at most finitely many purely periodic elements of primitive period n: this is because $A_q(g^n - 1)$ is a nonzero polynomial and thus has only finitely many roots. Studies delineating those primitive periods (and preperiods) that can be realized in the dynamical system $(\bar{F}, A_q(g))$ and on other questions relating the dynamical structure to the polynomial and field structures have been undertaken by Batra and Morton [BM1, BM2] and Chou and Cohen [CC] and will not be discussed here in detail.

Nevertheless we elaborate on one aspect of the system just introduced. The subset V_g of $\bar{F}$ comprising the purely periodic elements of $(\bar{F}, A_q(g))$ can be partitioned into equivalence classes under the relation ρ_g defined by the rule that $(\alpha, \beta) \in \rho_g$ if and only if α and β lie in the same orbit under $A_q(g)$ (evidently, this can be done for every dynamical system (S, γ)). For example, if $g = x$ (so that $A_q(x)$ is the Frobenius automorphism $w \to w^q$

on $\bar{F}$), then $V_x = \bar{F}$ and $(\alpha, \beta) \in \rho_x$ if and only if α and β have the same minimal polynomial over F. In fact, if μ_α denotes the minimal polynomial of α over F, then $\alpha \to \mu_\alpha$ induces a bijection between the set V_x/ρ_x of equivalence classes of ρ_x and the set I_F of monic *irreducible* polynomials in $F[x]$. Observe that each $n \geq 1$ occurs as a primitive period. Another case in which $V_g = \bar{F}$ is the trivial one when $g = 1$, i.e., when $A_q(g)$ is the identity map: then V_1/ρ_1 essentially is the same as the set $\bar{F}$. Section 2 will include a more explicit description of the set V_g in general: it is always an F-subspace of $\bar{F}$ which is invariant under the Frobenius automorphism.

For a more general dynamical system, yet one that retains $A_q(g)$ as the dynamical polynomial, let h be another monic polynomial in $F[x]$ and take S to be the set V_h/ρ_h. Since $(\alpha^g)^h = (\alpha^h)^g$, there is a natural action of $A_q(g)$ on V_h/ρ_h: if $\rho_h(\alpha)$ denotes the equivalence class of α, then

$$A_q(g)(\rho_h(\alpha)) := \rho_h(A_q(g)(\alpha)) = \rho_h(\alpha^g) \tag{1.2}$$

is well-defined. In particular, if $h = 1$ we recover the situation discussed above and if $h = x$, then, from the previous paragraph we obtain a system equivalent to that in which the linearized polynomial $G := A_q(g)$ acts on I_F by defining

$$G(\mu_\alpha) := \mu_{\alpha^g}.$$

Such dynamical systems (I_F, G) (in a more general context in which the dynamic polynomial G need not be additive) were introduced by Vivaldi [V]. They were studied more intensively by Batra and Morton [BM1], [BM2], by Morton [M] and by Cohen and Hachenberger [CH]. For these systems the dynamics is richer because, potentially, there are *infinitely* many purely periodic elements of *any* given period. Indeed, in [CH], in establishing a conjecture of Morton [M] in the case of q-polynomials, it was shown that, for g not of the form x^l ($l \geq 0$ an integer), the system $(I_F, A_q(g))$, equivalently $(V_x/\rho_x, A_q(g))$, contains infinitely many elements having prescribed primitive period n (≥ 1) and preperiod k (≥ 0).

In the present paper we consider the general situation with arbitrary monic q-polynomials g and h. For simplicity, we refer to $(V_h/\rho_h, A_q(g))$ as the (g, h)-*(dynamical) system.* In order to extend the above mentioned result of [CH], necessarily, we except those polynomials satisfying a relation of the form $g^r = h^s$ where $r, s \geq 0$ are integers. In particular, we suppose that g and h are non-constant polynomials. In fact, since $A_q(h^i)$ ($i \geq 0$) induces the identity mapping on V_h/ρ_h, the (gh^i, h)-system has the same dynamics as the (g, h)-system. Consequently, we may assume that h does not divide g. Additionally, we impose the further constraint that g and h

are relatively prime, which however may not be altogether necessary. The result is as follows.

Theorem 1.1 *Let g and h be monic and relatively prime non-constant polynomials in $F[x]$. Let $n \geq 1$ and $k \geq 0$ be integers. Then there exist infinitely many classes $\lambda \in V_h/\rho_h$ which, with respect to the action of $A_q(g)$ definied in (1.2), have primitive period n and preperiod k.*

To complete this introduction we mention a simplification. If $g = g_0^l$ ($l \geq 2$), the existence of an element λ in V_h/ρ_h with primitive period ln and preperiod lk with respect to the (g_0, h)-system guarantees that λ has primitive period n and preperiod k in the (g, h)-system. Hence, we may assume that g is a *non-power*, i.e., g is different from g_0^l for $l \geq 2$, an integer.

2. Additive order and its uses

The notion of the additive order or F-order of an element $\alpha \in \bar{F}$ is fundamental to our study. References relevant to the present context are [H1], [CH].

By Section 1, $\bar{F}$ can be interpreted as an $F[x]$-module wherein the action of $f \in F[x]$ on $\alpha \in \bar{F}$ is given by $\alpha^f := A_q(f)(\alpha)$ (see (1.1)). Let $\mathcal{P}_F$ denote the set of all monic polynomials of $F[x]$ which are indivisible by x. The finite $F[x]$-submodules of $\bar{F}$ correspond bijectively to the members of $\mathcal{P}_F$: $f \in \mathcal{P}_F$ corresponds to the set of roots of $A_q(f)$ in $\bar{F}$. Moreover, every finite $F[x]$-submodule is cyclic, i.e., free on one generator. For any $\alpha \in \bar{F}$, the *F-order* or *additive order* of α (denoted by $\mathrm{Ord}_F(\alpha)$) is the polynomial $f \in \mathcal{P}_F$ of least degree for which $\alpha^f = 0$. In particular $\mathrm{Ord}_F(0) = 1$. The generators of the submodule corresponding to $f \in \mathcal{P}_F$ are exactly the elements $\alpha \in \bar{F}$ such that $\mathrm{Ord}_F(\alpha) = f$. There are precisely $\Phi_q(f)$ (> 0) such generators, where Φ_q denotes the finite field Euler function. For more details about $F[x]$-submodules, we refer to [H1, H2].

We state some simple properties of additive orders. For $\alpha, \beta \in \bar{F}$, $\mathrm{Ord}_F(\alpha + \beta)$ is a divisor of $\mathrm{Ord}_F(\alpha) \cdot \mathrm{Ord}_F(\beta)$ with equality if $\mathrm{Ord}_F(\alpha)$ and $\mathrm{Ord}_F(\beta)$ are coprime. A crucial result for the dynamics of linearized polynomials is the following.

Lemma 2.1 *Let $\alpha \in \bar{F}$. If $\mathrm{Ord}_F(\alpha) = f$ and $g \in F[x]$, then $\mathrm{Ord}_F(\alpha^g) = f/\gcd(g, f)$ (where* gcd *denotes the greatest common (monic) divisor).* □

Let $h \in F[x]$ be monic. We are now prepared to deduce a description of the subset V_h of $\bar{F}$ comprising exactly the purely periodic elements of the dynamical system $(\bar{F}, A_q(h))$ (see Section 1).

Proposition 2.2 *For any polynomial $h \in F[x]$,*

$$V_h = \{\alpha \in \bar{F} \mid \gcd(\mathrm{Ord}_F(\alpha), h) = 1\}. \tag{2.1}$$

Moreover, V_h is an $F[x]$- submodule of $\bar{F}$.

Proof. Let C_h be the right hand side of (2.1), i.e., the set of all elements $\alpha \in \bar{F}$ whose F-order is coprime to h. If $\alpha \in \bar{F}$ and $f \in F[x]$ then, by Lemma 2.1, the F-order of α^f is a divisor of $\mathrm{Ord}_F(\alpha)$. Thus, C_h is invariant under the action of $A_q(f)$ for all $f \in F[x]$. Now, if $\alpha \in V_h$, then $\alpha^{h^n} = \alpha$ for some integer $n \geq 1$. Thus, a further application of Lemma 2.1 shows that $\alpha \in C_h$. Conversely, if $\alpha \in C_h$, let l be the *multiplicative order of h modulo* $\mathrm{Ord}_F(\alpha)$, i.e., $l \geq 1$ is the least integer such that $h^l - 1$ is divisible by $\mathrm{Ord}_F(\alpha)$. Then $\alpha^{h^l} = \alpha$, whence $\alpha \in V_h$. □

For simplicity, throughout let $M_h := V_h/\rho_h$. As in the proof of Proposition 2.2, Lemma 2.1 shows that, for $\alpha, \beta \in V_h$, $\mathrm{Ord}_F(\alpha) = \mathrm{Ord}_F(\beta)$ whenever $\alpha \in \rho_h(\beta)$. Hence, for any $\lambda \in M_h$, we may define

$$\mathrm{Ord}_F(\lambda) := \mathrm{Ord}_F(\alpha), \text{ where } \alpha \in \lambda. \tag{2.2}$$

This shows that each member of M_h is periodic. We next proceed to demonstrate the pre-eminence of additive order for the preperiod and primitive periods of elements of M_h in the (g, h)-dynamical system for arbitrary polynomials g and h. Clearly, $\lambda = \rho_h(\alpha)$ $(\alpha \in V_h)$ is *purely periodic* if and only if $\alpha^{g^n} = \alpha^{h^l}$ for some integers $n \geq 1$ and $l \geq 0$. In this case the primitive period of λ is the minimal such value of n, denoted by $\pi_{g,h}\{\lambda\}$. Then, clearly,

$$\pi_{g,h}\{\lambda\} = \min\{n \mid \mathrm{Ord}_F(\lambda) \text{ divides } g^n - h^l \text{ for some } l \geq 0\}. \tag{2.3}$$

Combining (2.2) and Lemma 2.1, similarly to the proof of Proposition 2.2, we derive a description of the set $\mathcal{P}_{g,h}$ of purely periodic elements in the (g, h)-system.

Proposition 2.3 *Let $P_{g,h}$ be the set of purely periodic elements of M_h in the (g, h)-system and let $V_{g,h}$ be the union of all $\lambda \in P_{g,h}$ (regarding each such λ as a subset of V_h). Then $V_{g,h} = V_g \cap V_h = V_{gh}$, i.e., $V_{g,h}$ is the set of all $\alpha \in \bar{F}$ whose F-order is coprime to gh.* □

Following Proposition 2.3, for any $n \geq 1$, we define $P_{g,h}(n)$ as the subset of $P_{g,h}$ comprising all elements of primitive period n. From (2.3), if $\lambda \in$

(1) *If $\nu(g)$ divides h, then $\mathcal{P}_{g,h} = M_h$.*

(2) *If $\nu(g)$ does not divide h, then for each $\lambda \in \mathcal{P}_{g,h}$ and each integer $k \geq 0$, there exists an $\eta \in M_h$ which has preperiod k and satisfies $A_q(g)^{(k)}(\eta) = \lambda$.* □

From now on we shall only consider purely periodic elements and therefore assume that all additive orders f are co-prime to gh. Because of (2.4) we tend to work with F-orders rather than members of $P_{g,h}$.

We finally remark that for the $(g, 1)$-system (with $M_1 = \bar{F}$), Chou and Cohen [CC] further classify the preperiodic structure. Similar details can certainly be set down for the general (g, h)-system.

3. Infinitely many F-orders f with $\pi_{g,h}(f) = n$ from one

Consider a (g, h)-system, where g and h are non-constant monic polynomials over F. Following Section 2, define $\mathcal{P}_{g,h}(n)$ as the set of all F-orders $f \in \mathcal{P}_F$ prime to gh such that $\pi_{g,h}(f) = n$. The aim of this section is to show that, under the assumption of Theorem 1.1, $\mathcal{P}_{g,h}(n)$ is infinite provided it is non-empty.

Observe first that (2.3) and (2.4) can be recast to yield

$$\pi_{g,h}(f) = \min \{n | \, f \text{ divides } g^n - h^l \text{ for some } l \geq 0 \}. \tag{3.1}$$

Thus, $\pi_{g,h}(f)$ can be interpreted as the group order of $g + fF[x]$ in the group of units $U_f^\times$ modulo f factorized by the subgroup $[h]$ generated by $h + fF[x]$. In fact, our main problem concerning the primitive periods in the (g, h)-system can be formulated as follows.

Given polynomials g, h over F and $n \geq 1$ an integer, do there exist infinitely many $f \in \mathcal{P}_F$, relatively prime to gh, such that the group order of $g + fF[x]$ in $U_f^\times / [h]$ is equal to n?

Lemma 3.1 *Assume that $f, f^* \in \mathcal{P}_F$ are relatively prime to gh. Let $n \geq 1$ and $m \geq 0$ be integers. Then the following hold.*

(1) *f divides $g^n - h^l$ if and only if $\pi_{g,h}(f)$ divides n.*

(2) *If f divides f^*, then $\pi_{g,h}(f)$ divides $\pi_{g,h}(f^*)$.* □

We are now prepared to prove the main result of this section. Note that it is convenient to assume that g and h are coprime.

Theorem 3.2 *Let g, h be monic non-constant polynomials in $F[x]$ which are relatively prime. Assume that f is a polynomial in $\mathcal{P}_F$ relatively prime to gh. If $\pi_{g,h}(f) = n$, then $\mathcal{P}_{g,h}(n)$ is infinite.*

Proof. Because $\pi_{g,h}(f) = n$, by (2.3) there exists $m \geq 0$ such that f divides $f_0 := g^n - h^m$. Since g and h are relatively prime, f_0 is prime to gh and therefore a member of $\mathcal{P}_{g,h}$. Moreover, from Lemma 3.1, $\pi_{g,h}(f_0) = n$. Let d be the multiplicative order of h modulo f. For $l \geq 0$, let $f_l := g^n - h^{m+ld}$ (relatively prime to gh). Since f divides both $g^n - h^m$ and $h^d - 1$, it also divides $f_l = g^n - h^m - h^m(h^{ld} - 1)$. Thus, again by Lemma 3.1, $\pi_{g,h}(f_l) = n$ for all l. Since the f_i are all distinct ($i \geq 0$), we conclude that $\mathcal{P}_{g,h}(n)$ is infinite. □

We remark that it follows from Theorem 3.2 that $\mathcal{P}_{g,h}(1)$ is infinite because $\{0\}$ is a member.

4. Irreducible F-orders with primitive period coprime to p

Let p be the characteristic of $F = \mathrm{GF}(q)$. In this section we consider again the (g, h)-system with g and h non-constant and relatively prime as in the statement of Theorem 1.1. Further, without loss of generality (as noted at the end of Section 1), we suppose that g is a *non-power*. The polynomial h, however, may be a power and we define m to be the maximal integer indivisible by p such that $h = h_0^m$ for some $h_0 \in F[x]$. Note that, additionally, h may be a p^eth power for some $e \geq 0$. If $m = 1$ we shall say that h is *at most a p-power*.

Given any n (≥ 1) indivisibe by p we prove directly that $\mathcal{P}_{g,h}(n)$ is infinite (and so certainly non-empty, cf. Theorem 3.2). To accomplish this goal, we seek to enumerate those F-orders f with $\pi_{g,h}(f) = n$ for which f is an *irreducible* polynomial over F of degree d, where n is a divisor of $q^d - 1$ and f is coprime to gh. Let $N_n^*(d)$ be the number of such f. We can take d to be any integer such that the least common multiple of m and n is a divisor of $q^d - 1$.

Suppose f is irreducible of degree d and θ is a root of f. Then $F(\theta) = \mathrm{GF}(q^d) =: F_d$, say. Moreover, by (3.1) we have

$$\pi_{g,h}(f) = \min\{n |\, g^n(\theta) = h^l(\theta) \text{ for some } l \geq 0\}. \tag{4.1}$$

Now, for the next part, suppose that h is at most a p-power. We shall indicate later modifications which treat the general case. Introducing yet one further notion of order - this time the multiplicative order $\mathrm{ord}(w)$ of non-zero elements w of $\bar{F}$ - we consider the relationship to (4.1) of the following conditions involving an element θ of F_d; namely

$$\mathrm{ord}(h(\theta)) = \frac{q^d - 1}{n}, \ h(\theta) \neq 0 \tag{4.2}$$

$$\gcd(\frac{q^d-1}{\mathrm{ord}(g(\theta))}, n) = 1, \ g(\theta) \neq 0. \tag{4.3}$$

Assume that (4.2) and (4.3) hold for $\theta \in F_d$. Suppose, in fact, that $\theta \in F_{d_0}$, where d_0 divides d. Then $g(\theta), h(\theta) \in F_{d_0}$ and $\mathrm{ord}(g(\theta))$ and $\mathrm{ord}(h(\theta))$ are divisors of $q^{d_0}-1$. Hence, from (4.2), n is a multiple of $\frac{q^d-1}{q^{d_0}-1}$, whereas, from (4.3), n is relatively prime to this number. Hence $d = d_0$ and $F_d = F(\theta)$. Consequently, f is irreducible of degree d. Furthermore, (4.2) implies that $h(\theta)$ is a generator of the (cyclic) group of nth powers in F_d^*. Hence $g^n(\theta) = h^l(\theta)$ for some $l \geq 0$. Moreover, (4.3) guarantees that $g(\theta)$ is *not any kind of nth power* in F_d, i.e., $g(\theta) = \beta^e$ for e dividing n implies $e = 1$. Thus, (4.1) holds and $\pi_{g,h}(f) = n$. We conclude that if $N_n(d)$ denotes the cardinality of the subset of F_d satisfying (4.2) and (4.3) then, clearly,

$$N_n^*(d) \geq \frac{1}{d} \cdot N_n(d),$$

and it suffices to show that $N_n(d)$ is positive.

To state our results we repeat some notation from [CH]. Define n_1 as the part of n involving primes common to n and $\frac{q^d-1}{n}$. More precisely, write $n = n_1 n_2$, where n_1 and n_2 are relatively prime, the squarefree part $\nu(n_1)$ of n_1 is equal to the squarefree part of $\gcd(\frac{q^d-1}{n}, n)$ and $\gcd(\frac{q^d-1}{n}, n_2) = 1$.

In this section φ and μ are the regular Euler and Möbius functions, respectively, and, if $\omega(k)$ is the number of distinct prime factors of k, then $W(k) := 2^{\omega(k)}$ is the number of squarefree factors of k.

The crucial result is the following.

Proposition 4.1 *Let g and h be non-constant monic polynomials in $F[x]$, where $F = \mathrm{GF}(q)$. Assume that g and h are relatively prime and that g is a non-power and h at most a p-power. Then, for any integers n (≥ 1) indivisible by the characteristic p of F and d such that n divides $q^d - 1$, we have*

$$N_n(d) = \frac{\varphi(\frac{q^d-1}{n_1})\varphi(n_1)}{(q^d-1)n} \cdot (q^d + R), \tag{4.4}$$

where

$$|R| \leq nMq^{d/2}W(q^d-1)W(n_1) \tag{4.5}$$

and $M = \deg(g) + \deg(h) - 1$. (The trivial case $Mq^d = 2$ is excluded.)

Proof. Employing the characteristic functions E_1 and E_2 for the sets of elements of F_d satisfying (4.3) and (4.2), respectively, we obtain

$$N_n(d) = \sum_{\alpha \in F_d} E_1(\alpha) E_2(\alpha).$$

Here, see e.g., [Co],

$$E_1(\alpha) = \frac{\varphi(n)}{n} \sum_{r|n} \frac{\mu(r)}{\varphi(r)} \sum_{\mathrm{ord}(\chi)=r} \chi(g(\alpha)), \tag{4.6}$$

where the sum over χ is over all $\varphi(r)$ multiplicative characters of F_d of order r. The sum $E_2(\alpha)$ (associated with (4.2)) is rather more awkward but has the shape (taken from Lemma 2 of Carlitz [C]) given by

$$E_2(\alpha) = \frac{\varphi(\frac{q^d-1}{n})}{q^d-1} \sum_{s|q-1} \frac{\mu(s^*)}{\varphi(s^*)} \sum_{\mathrm{ord}(\eta)=s} \eta(h(\alpha)), \tag{4.7}$$

where the sum over η is over all multiplicative characters of order s and

$$s^* = \frac{s}{\gcd(s,n)}.$$

Accordingly,

$$N_n(d) = \frac{\varphi(n)}{n} \frac{\varphi(\frac{q^d-1}{n})}{(q^d-1)} \sum_{r|n} \sum_{s|q^d-1} \frac{\mu(r)}{\varphi(r)} \frac{\mu(s^*)}{\varphi(s^*)} \sum_{\mathrm{ord}(\chi)=r} \sum_{\mathrm{ord}(\eta)=s} S(\chi,\eta), \tag{4.8}$$

where $S(\chi,\eta)$ denotes the character sum

$$S(\chi,\eta) = \sum_{\alpha \in F_d} \chi(g(\alpha))\eta(h(\alpha)).$$

Because n_2 and $\frac{q^d-1}{n}$ are relatively prime, it is easy to see that, in (4.8), n_1 may replace n in $\varphi(n)\varphi(\frac{q^d-1}{n})$. But the important step is to estimate the character sums $S(\chi,\eta)$ for the characters which appear in (4.8). Clearly, if $\chi = \chi_0$ and $\eta = \eta_0$ are the trivial characters (of order 1), then

$$S(\chi,\eta) = q^d - M_0, \tag{4.9}$$

where $M_0 \leq M+1$ is the number of zeros of gh in F_d. Otherwise, by Weil's Theorem, see [L] (Chapter 6, Theorem 3, part (1)),

$$|S(\chi,\eta)| \leq M q^{d/2}. \tag{4.10}$$

At this point it must be emphasized that (4.10) need not be valid for all relevant characters χ, η (not both trivial) if the conditions g, h relatively

prime, or h at most a p-power, were to be relaxed and a discussion of the general situation has to overcome such difficulties.

We deduce from (4.8) to (4.10) that $N_n(d)$ has the form (4.4), where

$$q^{-d/2}|R| \le M \sum_{r|n} \sum_{s|q^d-1} \frac{\lambda(r)}{\varphi(r)} \frac{\lambda(s^*)}{\varphi(s^*)} \varphi(r)\varphi(s) =: T_1 \tag{4.11}$$

and $\lambda = \mu^2$ here denotes Liouville's function. Evidently,

$$T_1 = MW(n)T_2,$$

where

$$T_2 = \sum_{s|q^d-1} \frac{\lambda(s^*)\varphi(s)}{\varphi(s^*)}. \tag{4.12}$$

Now, let Q be the part of $q^d - 1$ prime to n. Then, by the multiplicativity of the functions involved, T_2 can be expressed as

$$T_2 = \sum_{t|Q} \lambda(t) \cdot \sum_{u|q^d-1,\, \nu(u)|\nu(n)} \frac{\lambda(u^*)\varphi(u)}{\varphi(u^*)} = W(Q)T_3,$$

where the definition of u^* is analogous to that of s^* and where

$$T_3 = \sum_{u|n\nu(n_1)} \frac{\lambda(u^*)\varphi(u)}{\varphi(u^*)},$$

since $\lambda(u^*) = 0$ unless u divides $n\nu(n_1)$. Somewhat surprisingly perhaps, T_3 can be evaluated exactly (see [CH]) as

$$T_3 = nW(n_1),$$

leading to a precise evaluation of T_1. Using the fact that $W(n)W(Q) = W(q^d - 1)$ we deduce the bound (4.5) for $|R|$. □

By means of Proposition 4.1 and the explicit bound $W(k) \le 5k^{1/4}$ (see Lemma 3.3 of [CH]) we obtain the following result which establishes Theorem 1.1 for n indivisible by p and h at most a p-power (because we can choose any value of d larger than the stated bound to guarantee that $\mathcal{P}_{g,h}(n)$ is infinite).

Theorem 4.2 *Let g and h be non-constant monic polynomials in $F[x]$, where $F = \mathrm{GF}(q)$. Assume that g and h are relatively prime and that g is a non-power and h at most a p-power. Then, for any integer n (≥ 1)*

indivisible by the characteristic p of F and any integer d such that n divides $q^d - 1$ and

$$d \geq \frac{4\log(25n^{\frac{5}{4}}M)}{\log(q)}$$

(where $M = \deg(g) + \deg(h) - 1$), we have that $N_n(d)$ and $N_n^*(d)$ are positive. □

To complete this section we outline the modifications to the above discussion when h is a power. Assume $h = h_0^m$ with m indivisible by p as described at the beginning of the section. The other assumed conditions remain in force. In particular, we suppose that $q^d - 1$ is divisible by the least common multiple L of m and n. Let $l := \gcd(n, m)$, then $n' := n/l$ and $m' := m/l$ are relatively prime. We claim that the following extensions of (4.2) and (4.3) guarantee that $\theta \in F_d$ is the root of an irreducible polynomial f of degree d such that (4.1) holds (so that $\pi_{g,h}(f) = n$), they are

$$\mathrm{ord}(h_0(\theta)) = \frac{q^d - 1}{n'}, \ h_0(\theta) \neq 0, \tag{4.13}$$

$$\mathrm{ord}(g(\theta)) \text{ divides } \tfrac{q^d-1}{m'}, \ \gcd(\tfrac{q^d-1}{m'\mathrm{ord}(g(\theta))}, m'n') = 1, \ g(\theta) \neq 0. \tag{4.14}$$

Observe that (4.13) means that $h(\theta)$ generates the Lth powers of F_d^*. Further, (4.14) implies that $g(\theta)$ is an m'th power but no higher power which is a divisor of L. Note that h_0 is at most a p-power and we could carry out a calculation similar to that of Proposition 4.1 to yield a satisfactory estimate for the cardinality of the subset of F_d satisfying (4.13) and (4.14).

An alternative to the above procedure is to replace (4.14) by the more stringent condition

$$\mathrm{ord}(g(\theta)) = \frac{q^d - 1}{m'} \tag{4.15}$$

and employ some of the estimates used in Proposition 4.1.

To illustrate the above, take $n = 12$ and $m = 8$; thus $q^d - 1$ is divisible by $L = 24$. Further $n' = 3$ and $m' = 2$. Also (4.13) means that $h_0(\theta)$ is the cube of a primitive element of F_d. On the other hand, (4.14) implies that $g(\theta)$ is a square but neither a cube nor a 4th power, whereas (4.15) simply means that $g(\theta)$ is the square of a primitive element.

Denote by $N_n'(d)$ the cardinality of the subset of F_d satisfying (4.13) and (4.15). Then, by following the proof of Proposition 4.1, but using a further analogue of (4.12) for T_1 as well as T_2, we obtain an expression for $N_n'(d)$ of the form

$$N'_n(d) = c(q^d + R),\ c > 0,$$

where

$$|R| \leq mnMq^{d/2}W(\frac{q^d-1}{n})W(\frac{q^d-1}{m}).$$

Though this is not the best possible lower bound for $N_n(d)$, it leads to a satisfactory extension of Theorem 4.2 that suffices to establish that $\mathcal{P}_{g,h}(n)$ is infinite for n indivisible by p.

5. Generating F-orders with primitive period divisible by p

Once more consider the (g,h)-system where g and h are non-constant and relatively prime. We know from Sections 3 and 4 that $\mathcal{P}_{g,h}(n)$ is infinite whenever n is indivisible by the characteristic p of $F = \mathrm{GF}(q)$. Given n not divisible by p, we shall show in this section that from any $f \in \mathcal{P}_{g,h}(n)$ can be derived a distinct F-order f_l in $\mathcal{P}_{g,h}(np^l)$ for each $l \geq 1$. As a consequence of this, Theorem 1.1 is completely proved. Assume throughout that f and gh are relatively prime.

First, some remarks are offered on where to look for F-orders with primitive periods divisible by p. In Section 4, for any n indivisible by p, we found *irreducible* polynomials f in $\mathcal{P}_{g,h}(n)$. Although this is far from a comprehensive treatment, it is the case that, in broad terms, such periods are associated with square-free F-orders f.

On the one hand, $\pi_{g,h}(f) = n$ is indivisible by p whenever f is square-free. To justify this, suppose p divides n. Let N be the multiplicative order of g modulo f. Then f divides $g^N - 1$ and so, by the definition of n and Lemma 3.1 (1), n divides N. Consequently, p divides N and f divides $g^{N/p} - 1$ (since f is square-free). This contradicts the definition of N.

On the other hand, if p does not divide n and $f \in \mathcal{P}_{g,h}(n)$, we claim that the square-free part $\nu(f)$ of f also lies in $\mathcal{P}_{g,h}(n)$. To justify this, let $\pi_{g,h}(\nu(f)) = k$ and let f divide $\nu(f)^{p^l}$, where $l \geq 0$. Then $\nu(f)$ divides $g^k - h^m$, say, and so $\nu(f)^{p^l}$ divides $g^{kp^l} - h^{mp^l}$. It follows from Lemma 3.1 that $n = \pi_{g,h}(f)$ divides $\pi_{g,h}(\nu(f)^{p^l})$ and the latter divides kp^l. Since n is indivisible by p we conclude that $k = n$.

The above argument also reveals that, if $f \in \mathcal{P}_{g,h}(n)$ and $j \geq 0$ is an integer, then $\pi_{g,h}(f^{p^j})$ is of the form np^{l_j} with $(l_j)_{j\geq 0}$ being an increasing sequence of nonnegative integers. Thus it is sensible to search for members of $\mathcal{P}_{g,h}(np^l)$ of the form f^{p^j}. The key result is as follows.

Proposition 5.1 *Let g and h be relatively prime and monic polynomials in $F[x]$ of degree at least 1. Let $n \geq 1$ be an integer and assume that $\pi_{g,h}(f)$ divides n where $f \in \mathcal{P}_f$ is relatively prime to gh. Then there exists a power $P > 1$ of the characteristic p of F such that $\pi_{g,h}(f^P)$ does not divide n.*

Proof. Observe first that by (1) of Lemma 3.1, if $k = \pi_{g,h}(f)$ divides n, then f divides $g^n - h^m$ for some $m \geq 0$. Now assume by way of contradiction that $\pi_{g,h}(f^P)$ divides n for each power $P \geq 1$ of p. Let $h^m = h_0^{m_0}$, where h_0 is a divisor of h which is not a pth power and analogously let $g^n = g_0^{n_0}$, where g_0 divides g and is not a pth power. Then $\pi_{g_0,h_0}(f^P)$ divides n_0 for each power $P \geq 1$ of p, and therefore the assumption of the proposition is satisfied for the (g_0, h_0)-system, f and $n_0 \geq 1$. From now on, we assume that g and h are not pth powers and shall derive a contradiction.

For a power $P \geq 1$ of p, let $m(P)$ be the unique nonnegative integer bounded by the multiplicative order of h modulo f^P such that $g^n - h^{m(P)}$ is divisible by f^P. Let $r = r(P)$ be the largest power of p dividing $\gcd(n, m(P))$ and write $N := N(P) = n/r$, $M(P) := m(P)/r$. Observe that r is bounded since n is fixed. Moreover, N or $M(P)$ is not divisible by p. We assume that r divides P and let $Q := P/r$. Then f^Q divides $g^N - h^{M(P)}$ as well as $g^{nQ} - h^{m(1)Q}$. Consequently, letting for simplicity $M = M(P)$ and $m = m(1)$, f^Q divides

$$A = A(P) := -(g^N - h^M)g^{nQ-N} + g^{nQ} - h^{mQ} = h^M g^{nQ-N} - h^{mQ}.$$

If $a \in F[x]$ is such that $A = af^Q$ and Q is larger than 1, then the formal derivative A' of A is equal to

$$A' = a'f^Q = h^{M-1}g^{nQ-N-1}(Mh'g - Ng'h).$$

If $B(P) := Mh'g - Ng'h \neq 0$ then $A' \neq 0$, whence the relative primeness of f and gh implies that f^Q divides $B(P)$. Since the degree of $B(P)$ is bounded for all P, this gives a contradiction for sufficiently large P (and Q). Thus, $B(P) = 0$ for large P, which we now assume. If $M \equiv 0 \bmod p$ then p does not divide N and therefore $g'h = 0$, whence $g' = 0$. This is a contradiction to the assumptions that $\deg(g) \geq 1$ and that g is not a pth power. Similarly, if $N \equiv 0 \bmod p$, then p does not divide M and therefore $h'g = 0$, whence $h' = 0$. Again, this is a contradiction. We deduce that p does not divide NM and therefore $h'g = \gamma g'h$ for some nonzero $\gamma \in F$. But this cannot happen, as g and h are assumed to be relatively prime and neither g' nor h' is zero. This completes the proof of Proposition 5.1. □

We now resume the discussion of the (g, h)-system described at the beginning of the section. Assume from now on that $f \in \mathcal{P}_{g,h}(n)$ for a given $n \geq 1$

(we know the existence of f when n is indivisible by p). An application of Proposition 5.1 shows that there exists an integer $j \geq 1$ such that $\pi_{g,h}(f^{p^j})$ does not divide n. In fact, $\pi_{g,h}(f^{p^j}) = np^l$ for some $l \geq 1$. Now let $\kappa(f)$ be the *p-index of* f, i.e., the least integer $k \geq 1$ such that np divides $\pi_{g,h}(f^{p^k})$. Then it is clear that $f_1 := f^{\kappa(f)} \in \mathcal{P}_{g,h}(np)$. If, by induction, $f_i \in \mathcal{P}_{g,h}(np^i)$ for some $i \geq 1$, then $f_{i+1} := f_i^{\kappa(f_i)} \in \mathcal{P}_{g,h}(np^{i+1})$. This finally completes the proof of Theorem 1.1, since $\mathcal{P}_{g,h}(n)$ is known to be nonempty (in fact infinite) if p does not divide n.

Nevertheless for h not a pth power, we give a final result representing a more precise version of the above. There is also a small restriction of f, namely that its degree be at least that of h.

Theorem 5.2 *Let g and h be monic non-constant polynomials over F which are relatively prime. Assume that h is not a pth power. Assume further that, for a given n, $f \in \mathcal{P}_{g,h}(n)$ and $\deg(f) \geq \deg(h)$. Let $\kappa := \kappa(f)$ be the p-index of f. Then*

$$\pi_{g,h}(f^{p^{\kappa+l}}) = np^{l+1} \text{ for all } l \geq 0. \tag{5.1}$$

Proof. The condition on h means that h' is non-zero. By the definition of κ, (5.1) is valid for $l = 0$. Assume by induction that the assertion holds for all $j \leq l$ and some $l \geq 0$. Assume further that, for some $c \in F[x]$ and some $m \geq 0$,

$$cf^{p^{\kappa+l+1}} = g^{np^{l+1}} - h^m.$$

Differentiating, we obtain that $f^{p^{\kappa+l}}$ divides $mh^{m-1}h'$. Using the facts that f and h are relatively prime and $\deg(f) \geq \deg(h)$, we deduce that m is divisible by p. Thus, $f^{p^{\kappa+l}}$ divides $g^{np^l} - h^{m/p}$, a contradiction to $\pi_{g,h}(f^{\kappa+l}) = np^{l+1}$. This completes the proof. □

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