A space-time adaptive discontinuous Galerkin method for the numerical solution of the dynamic quasi-static von Kármán equations ☆

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1. Introduction

The dynamic quasi-static von Kármán equations are a coupled system of PDEs consisting of a nonlinear fourth order parabolic equation and a nonlinear fourth order elliptic equation and describe the bending of shells (cf. the monograph [12] and the references therein). They have been studied analytically in [10,19,31]. However, in contrast to their stationary counterpart (cf., e.g., [4,7,8,28,30]), for their numerical solution not much work has been published (see [31] for a finite difference approximation).

In this paper, we consider an adaptive space-time C^0 Interior Penalty Discontinuous Galerkin (C^0IPDG) approximation of the dynamic quasi-static von Kármán equations with homogeneous Dirichlet boundary conditions and an equilibrated a posteriori error estimator. The discretization in time is taken care of by the backward Euler scheme with respect of a partitioning of the time interval. On the other hand, for each time step the C^0IPDG method can be derived from a six-field formulation of the finite element discretized equations and will be shown to admit a unique solution for triangulations of sufficiently small mesh size. The equilibrated a posteriori error estimator consists of easily computable local residual-type contributions. It can be derived from a more general result [32] on convex minimization problems and provides an upper bound for the discretization error in the broken $W^{2,2}$ norm in terms of the associated primal and dual energy functionals. It requires the construction of equilibrated fluxes and equilibrated moment tensors which can be computed on local patches around interior nodal points of the triangulations. Moreover, we study its relationship with a residual-type a posteriori error estimator. The fully discretized quasi-static von Kármán equations are solved by a predictor-corrector continuation strategy featuring constant continuation as a predictor and Newton's method as a corrector which allows an adaptive choice of the time steps. Numerical results illustrate the performance of the suggested approach.

The paper is organized as follows: In section 2, we will introduce the dynamic quasi-static von Kármán equations with homogeneous Dirichlet boundary conditions and, under some restrictions on the data, show the existence and uniqueness of a weak solution (Theorem 2.1). The discretization in time is dealt with in Section 3. For each time step, we have to solve a coupled system of two nonlinear fourth order elliptic equations. The existence and uniqueness will be established (Theorem 3.1).

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Section 4 is devoted to the derivation of the C^0IPDG approximation by means of a six-field formulation of the finite element discretized equations. We will show the existence and uniqueness of a solution for triangulations of sufficiently small mesh size (Theorem 4.2). In section 5, we will derive an equilibrated a posteriori error estimator providing an upper bound for the discretization error in the broken $W^{2,2}$ norm in terms of the associated primal and dual energy functionals. The construction of such an equilibrated a posteriori error estimator is dealt with in section 6. In particular, it requires the computation of equilibrated fluxes and equilibrated moment tensors on local patches around interior nodal points of the triangulations. Section 7 is devoted to the relationship with a residual-type a posteriori error estimator (Theorem 7.1) and Section 9 deals with the predictor-corrector continuation strategy and adaptive time stepping. Finally, in section 9 we provide a documentation of numerical results illustrating the performance of the suggested approach.

2. The dynamic quasi-static von Kármán equations

We use standard notation from Lebesgue and Sobolev space theory (cf., e.g., [35]). In particular, for a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d, d \in \mathbb{N}$, with boundary $\Gamma = \partial\Omega$ we refer to $L^p(\Omega;\mathbb{R}^d)$ and $L^p(\Omega;\mathbb{R}^{d\times d}), 1 < p < \infty$, as the Banach spaces of p-th power Lebesgue integrable functions and tensors on Ω with norms $\|\cdot\|_{L^p(\Omega;\mathbb{R}^d)}$ and $\|\cdot\|_{L^p(\Omega;\mathbb{R}^{d\times d})}$. In case $d=1$ we will write $L^p(\Omega)$ instead of $L^p(\Omega;\mathbb{R})$. Matrix-valued functions in $L^p(\Omega;\mathbb{R}^{d\times d})$ will be denoted by $\underline{\underline{\mathbf{p}}} = (p_{ij})_{i,j=1}^d$ and for $\underline{\underline{\mathbf{p}}} \in L^p(\Omega;\mathbb{R}^{d\times d}), \underline{\underline{\mathbf{q}}} \in L^q(\Omega;\mathbb{R}^{d\times d}), 1/p+1/q=1$, we use the notation $\underline{\underline{\mathbf{p}}} : \underline{\underline{\mathbf{q}}}$ for $\underline{\underline{\mathbf{p}}} : \underline{\underline{\mathbf{q}}} := \sum_{i,j=1}^d p_{ij}q_{ij}$. Further, for $u \in W^{2,p}(\Omega)$, we refer to $D^2u := (\partial^2 u/\partial x_i \partial x_j)_{i,j=1}^2$ as the matrix of second partial derivatives.
We denote by $W^{s,2}(\Omega), s \in \mathbb{R}_+$, the Sobolev spaces with norms $\|\cdot\|_{W^{s,2}(\Omega)}$ and by $W_0^{s,2}(\Omega)$ the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{s,2}(\Omega)}$. Functions $u \in W^{2,2}(\Omega)$ have a trace $u|_\Gamma$ on the boundary $\Gamma = \partial\Omega$ with $u|_\Gamma \in W^{3/2,2}(\Gamma)$. Further, we define $\underline{\mathbf{H}}(\text{div},\Omega)$ and $\underline{\underline{\mathbf{H}}}(\text{div}^2,\Omega)$, as the Banach spaces

$$\underline{\mathbf{H}}(\text{div},\Omega) = \{\underline{\boldsymbol{\tau}} \in L^2(\Omega;\mathbb{R}^d) \mid \nabla\cdot\underline{\boldsymbol{\tau}} \in L^2(\Omega)\},$$
$$\underline{\underline{\mathbf{H}}}(\text{div}^2,\Omega) = \{\underline{\underline{\boldsymbol{\tau}}} \in L^2(\Omega;\mathbb{R}^{d\times d}) \mid \nabla\cdot\underline{\underline{\boldsymbol{\tau}}} \in \underline{\mathbf{H}}(\text{div},\Omega)\}$$

with the graph norms

$$\|\underline{\boldsymbol{\tau}}\|_{\underline{\mathbf{H}}(\text{div},\Omega)} := \Big(\|\underline{\boldsymbol{\tau}}\|^2_{L^2(\Omega;\mathbb{R}^d)} + \|\nabla\cdot\underline{\boldsymbol{\tau}}\|^2_{L^2(\Omega)}\Big)^{1/2},$$
$$\|\underline{\underline{\boldsymbol{\tau}}}\|_{\underline{\underline{\mathbf{H}}}(\text{div}^2,\Omega)} := \Big(\|\underline{\underline{\boldsymbol{\tau}}}\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})} + \|\nabla\cdot\underline{\underline{\boldsymbol{\tau}}}\|^2_{L^2(\Omega;\mathbb{R}^d)} + \|\nabla\cdot\nabla\cdot\underline{\underline{\boldsymbol{\tau}}}\|^2_{L^2(\Omega)}\Big)^{1/2}.$$

For later use we recall Young's inequality

$$\prod_{i=1}^2 a_i \le \frac{\varepsilon}{p} a_1^p + \frac{\varepsilon^{-q/p}}{q} a_2^q \tag{2.1}$$

for $a_i > 0, 1 \le i \le 2$, and $1 < p, q < \infty, 1/p+1/q=1$, and any $\varepsilon > 0$, as well as the following inequality:
Let $w_i \in \mathbb{R}, 1 \le i \le 2$, and $0 \le r < \infty$. Then it holds (cf., e.g., [34], page 136)

$$(|w_1|+|w_2|)^r \le 2^r\Big(|w_1|^r + |w_2|^r\Big). \tag{2.2}$$

Given a bounded polygonal domain $\Omega \subset \mathbb{R}^2$ with boundary $\Gamma = \partial\Omega$ and exterior unit normal vector $\mathbf{n}_\Gamma$ as well as $T > 0$, the dynamic quasi-static von Kármán equations with homogeneous Dirichlet boundary conditions are given by a coupled system of fourth order equations

$$\frac{\partial u_1}{\partial t} + \Delta^2 u_1 - [u_1+\Theta, u_2 + f_{int}] + g\, u_1 = f_{ext} \text{ in } \Sigma := \Omega \times [0,T], \tag{2.3a}$$
$$\Delta^2 u_2 + [u_1 + 2\Theta, u_1] = 0 \text{ in } \Omega, \tag{2.3b}$$
$$u_i(\cdot,t) = \mathbf{n}_\Gamma \cdot \nabla u_i(\cdot,t) = 0 \text{ on } \Gamma,\ t \in [0,T],\ 1 \le i \le 2, \tag{2.3c}$$
$$u_1(\cdot,0) = u_1^0 \text{ in } \Omega, \tag{2.3d}$$

where u_1 is the vertical displacement of the shell, u_2 is the Airy stress, f_{int} is the internal force, Θ is the mapping measuring the deviation of the middle surface of the reference configuration of the shell from the plane, g stands for non-conservative loads, f_{ext} is an external force, and $[v,w]$ stands for the von Kármán bracket

$$[v,w] := \text{cof}(D^2v) : D^2w \tag{2.4}$$

with the cofactor matrix $\text{cof}(D^2v)$ of D^2v.
The weak formulation of (2.3) requires the computation of $u_1 \in L^2([0,T], W_0^{2,2}(\Omega))$, $\frac{\partial u_1}{\partial t} \in L^2([0,T], L^2(\Omega))$ and $u_2 \in W_0^{2,2}(\Omega)$ such that for all $v \in W_0^{2,2}(\Omega)$ the following system of variational equations is satisfied

$$\int_0^T \Big(\int_\Omega \frac{\partial u_1}{\partial t}\, v\, dx + \int_\Omega D^2 u_1 : D^2 v\, dx - \tag{2.5a}$$
$$\int_\Omega ([u_1+\Theta, u_2+f_{int}] - g\, u_1)\, v\, dx \Big) dt = \int_0^T \int_\Omega f_{ext}\, v\, dx\, dt,$$
$$\int_\Omega D^2 u_2 : D^2 v\, dx + \int_\Omega [u_1 + 2\Theta, u_1] v\, dx = 0. \tag{2.5b}$$

The existence of a weak solution has been shown in [10,26]. The following existence result has been provided by Theorem 2 in [26].

Theorem 2.1. *Consider the quasi-static von Kármán equations* (2.3a), (2.3b) *under the nonhomogeneous boundary conditions*

$$u_i(\cdot,t) - c_i = \mathbf{n}_\gamma \cdot \nabla(u_i(\cdot,t) - c_i) = 0,\ t \in [0,T],\ c_i \in H^2(\Omega),\ 1 \le i \le 2,$$

and the initial condition (2.3d). *Under the condition*

$$\int_\Omega \Big(E(c_1) + E(c_2)\Big)\, dx \le (27\ meas\ \Omega)^{-1},$$
$$E(c_i) := \sum_{|\alpha|=2} \frac{2}{\alpha!} |D^\alpha c_i|^2, 1 \le i \le 2,$$

the nonhomogeneous initial-boundary value admits a weak solution.

Remark 2.2. Since in this paper, we assume homogeneous boundary conditions (cf. (2.3c)), the existence result does not assume smallness of the data.

3. Discretization in time

We consider a discretization in time with respect to a partition of the time interval $[0,T]$ into subintervals $[t_{m-1}, t_m], 1 \le m \le M, M \in \mathbb{N}$, of length $\tau_m := t_m - t_{m-1}$. We denote by u_1^m and u_2^m approximations of u_1 and u_2 at time t_m and discretize the time derivative in (2.3a) by the backward difference quotient: Given $u_1^{m-1}, 1 \le m < M$, compute u_1^m and u_2^m such that

$$u_1^m - u_1^{m-1} + \tau_m \Delta^2 u_1^m - \tau_m([u_1^m + \Theta, u_2^m + f_{int}^m] - g^m u_1^m) = \tau_m f_{ext}^m \text{ in } \Omega, \tag{3.1a}$$
$$\Delta^2 u_2^m + [u_1^m + 2\Theta, u_1^m] = 0, \tag{3.1b}$$
$$u_i^m(\cdot,t) = \mathbf{n}_\Gamma \cdot \nabla u_i^m(\cdot,t) = 0 \text{ on } \Gamma,\ t \in [0,T],\ 1 \le i \le 2, \tag{3.1c}$$

where $f_{int}^m = f(\cdot,t_m)_{int}$, $f_{ext}^m = f(\cdot,t_m)_{ext}$, and $g^m = g(\cdot,t_m)$.
The weak solution of (3.1) amounts to the computation of $\mathbf{u}^m = (u_1^m, u_2^m)^T \in \underline{\mathbf{V}} := W_0^{2,2}(\Omega) \times W_0^{2,2}(\Omega)$ such that for $v \in W_0^{2,2}(\Omega)$ it holds

$$\int_\Omega (u_1^m - u_1^{m-1}) v\, dx + \tau_m \int_\Omega \Delta u_1^m : \Delta v\, dx - \tag{3.2a}$$
$$\tau_m \int_\Omega ([u_1^m + \Theta, u_2^m + f_{int}^m] - g^m u_1^m)\, v\, dx = \tau_m \int_\Omega f_{ext}^m v\, dx \quad \text{in } \Omega,$$

$$\int_\Omega \Delta u_2^m : \Delta v\, dx + \int_\Omega ([u_1^m + 2\Theta, u_1^m])\, v\, dx = 0. \tag{3.2b}$$

The coupled system (3.2) constitutes the necessary and sufficient optimality condition for the minimization of the primal energy functional

$$J_P^m(u_1,u_2) = \frac{1}{2}\int_\Omega \left(|u_1^m - u_1^{m-1}|^2 + \tau_m |D^2 u_1^m|^2\right) dx + \tag{3.3}$$

$$\frac{1}{2}\int_\Omega |D^2 u_2^m|^2\, dx - \frac{\tau_m}{2}\int_\Omega ([u_1^m + \Theta, u_2^m + f_{int}^m]\, u_1^m - g^m\, |u_1^m|^2)\, dx +$$

$$\int_\Omega [u_1^m + 2\Theta, u_1^m]\, u_2^m\, dx - \tau_m \int_\Omega f_{ext}^m u_1^m\, dx - \frac{1}{2}\int_\Omega |u_1^{m-1}|^2\, dx.$$

We introduce a bilinear form $A^m : \underline{\mathbf{V}} \times \underline{\mathbf{V}} \to \mathbb{R}, \underline{\mathbf{V}} := W_0^{2,2}(\Omega) \times W_0^{2,2}(\Omega)$, and a semilinear form $B^m : \underline{\mathbf{V}} \times \underline{\mathbf{V}} \times \underline{\mathbf{V}} \to \mathbb{R}$ according to

$$A^m(\mathbf{u}^m, \mathbf{v}^m) := a^m(u_1^m, v_1^m) + a^m(u_2^m, v_2^m), \tag{3.4a}$$

$$B^m(\mathbf{u}^m, \mathbf{v}^m, \mathbf{w}^m) := b^m(u_1^m, v_2^m, w_1^m) + b^m(u_2^m, v_1^m, w_1^m) - b^m(u_1^m, v_1^m, w_2^m), \tag{3.4b}$$

where $\mathbf{u}^m = (u_1^m, u_2^m)^T, \mathbf{v}^m = (v_1^m, v_2^m)^T, \mathbf{w}^m = (w_1^m, w_2^m)^T$, and the forms $a^m(\cdot,\cdot) : V \times V \to \mathbb{R}$ and $b^m(\cdot,\cdot,\cdot) : V \times V \times V \to \mathbb{R}$ are given by

$$a^m(\mathbf{u}^m, \mathbf{v}^m) := \sum_{i=1}^{2} \int_\Omega \left((u_i^m - u_i^{m-1})\, v^m + D^2 u_i^m : D^2 v^m\right) dx, \tag{3.5a}$$

$$\begin{aligned} b^m(\mathbf{u}^m, \mathbf{v}^m, \mathbf{w}^m) \\ := -\tau_m \int_\Omega \left((\mathrm{cof}(D^2 u_1^m + \Theta)\nabla(v_2^m + f_{int}^m) \cdot \nabla w_1^m - g^m u_1^m)\, w_1^m\right) dx \\ + \int_\Omega (\mathrm{cof}(D^2 u_1^m + 2\Theta)\nabla v_1^m \cdot \nabla w_2^m\, dx, \quad \mathbf{u}^m, \mathbf{v}^m, \mathbf{w}^m \in \mathbf{V}. \end{aligned} \tag{3.5b}$$

Within this setting the weak formulation amounts to the computation of $\mathbf{u}^m \in \underline{\mathbf{V}}$ such that for all $\mathbf{v} \in \underline{\mathbf{V}}$ it holds

$$A^m(\mathbf{u}^m, \mathbf{v}) + B^m(\mathbf{u}^m, \mathbf{u}^m, \mathbf{v}) = (\mathbf{f}_{ext}^m, \mathbf{v})_{L^2(\Omega)}, \quad \mathbf{f}_{ext}^m := (f_{ext}^m, 0)^T. \tag{3.6}$$

We define operators $\mathcal{A}^m : \underline{\mathbf{V}} \to \underline{\mathbf{V}}^*$ and $\mathcal{B}^m : \underline{\mathbf{V}} \to \underline{\mathbf{V}}^*$ by means of

$$< \mathcal{A}^m \mathbf{u}^m, \mathbf{v} >_{\underline{\mathbf{V}}^*, \underline{\mathbf{V}}} := A^m(\mathbf{u}^m, \mathbf{v}), \tag{3.7a}$$

$$< \mathcal{B}^m \mathbf{u}^m, \mathbf{v} >_{\underline{\mathbf{V}}^*, \underline{\mathbf{V}}} := B^m(\mathbf{u}^m, \mathbf{u}^m, \mathbf{v}). \tag{3.7b}$$

Then the operator form of (3.6) reads

$$\mathcal{A}^m \mathbf{u}^m + \mathcal{B}^m \mathbf{u}^m = \mathbf{f}_{ext}. \tag{3.8}$$

The operator $\mathcal{B}^m$ is Fréchet differentiable at $\mathbf{u}^m$ in the direction of $\mathbf{v}$ with the Fréchet derivative given by

$$< (\mathcal{B}^m)'(\mathbf{u}^m)\mathbf{v}^m, \mathbf{w}^m >_{\underline{\mathbf{V}}^*, \underline{\mathbf{V}}} := 2\mathcal{B}^m(\mathbf{u}^m, \mathbf{v}^m, \mathbf{w}^m), \quad \mathbf{u}^m, \mathbf{v}^m, \mathbf{w}^m \in \underline{\mathbf{V}}. \tag{3.9}$$

The existence of a weak solution has been shown in [11,27].

A weak solution $\mathbf{u}^m \in \underline{\mathbf{V}}$ is said to be a regular solution, if the linearized operator $\mathcal{L}^m : \underline{\mathbf{V}} \to \underline{\mathbf{V}}^*$ given by

$$\mathcal{L}^m \mathbf{v} := \mathcal{A}^m \mathbf{v} + (\mathcal{B}^m)'(\mathbf{u})\mathbf{v} \tag{3.10}$$

is nonsingular. The following result has been shown in [29].

Theorem 3.1. *If $\mathbf{u}^m \in \underline{\mathbf{V}}$ is a regular weak solution, then there exists an open ball $B(\mathbf{u}^m) \subset L^2(\Omega)$ such that $\mathcal{A}^m + (\mathcal{B}^m)'(\mathbf{u}^m)$ is an isomorphism from $\underline{\mathbf{V}}$ into $\underline{\mathbf{V}}^*$ for all $\mathbf{v} \in B(\mathbf{u}^m)$. In particular, there exist constants $C_i > 0, 1 \le i \le 2$, such that*

$$\|\mathcal{A}^m + (\mathcal{B}^m)'(\mathbf{u}^m)\|_{\mathbb{L}(\underline{\mathbf{V}}, \underline{\mathbf{V}}^*)} \le C_1, \quad \|(\mathcal{A}^m + (\mathcal{B}^m)'(\mathbf{u}^m))^{-1}\|_{\mathbb{L}(\underline{\mathbf{V}}^*, \underline{\mathbf{V}})} \le C_2, \tag{3.11}$$

where $\mathbb{L}(\underline{\mathbf{V}}, \underline{\mathbf{V}}^*)$ stands for the space of bounded linear mappings from $\underline{\mathbf{V}}$ into $\underline{\mathbf{V}}^*$.

4. C^0IPDG approximation of the von Kármán equations

Let $\mathcal{T}_h$ be a geometrically conforming, locally quasi-uniform, and shape-regular, simplicial triangulation of the computational domain Ω. Given $D \subset \bar{\Omega}$, we denote by $\mathcal{N}_h(D)$ and $\mathcal{E}_h(D)$ the set of vertices and edges of $\mathcal{T}_h$ in D, and we refer to $P_k(D), k \in \mathbb{N}$, as the set of polynomials of degree $\le k$ on D. Moreover, $h_K, K \in \mathcal{T}_h$, and $h_E, E \in \mathcal{E}_h$, stand for the diameter of K and the length of E, respectively. We define $h := \min\{h_K \mid K \in \mathcal{T}_h\}$. Given any $0 < \delta < 1$, we denote by $\mathbb{T}(\delta)$ the set of all triangulations $\mathcal{T}_h$ with mesh size $h_T \le \delta$ for all $T \in \mathcal{T}_h \in \mathbb{T}(\delta)$. For two quantities $a, b \in \mathbb{R}$ we will write $a \lesssim b$, if there exists a constant $C > 0$, independent of h, such that $a \le Cb$.
Due to the local quasi-uniformity and shape regularity of the triangulation there exist constants $0 < c_Q \le C_Q, 0 < c_R \le C_R, 0 < c_S \le C_S$, such that for all $K \in \mathcal{T}_h$ it holds

$$c_Q h_K \le h \le C_Q h_K, \tag{4.1a}$$

$$c_R h_K \le h_E \le C_R h_K, \quad E \in \mathcal{E}_h(\partial K), \tag{4.1b}$$

$$c_S |K| \le h_K^2 \le C_S |K|. \tag{4.1c}$$

We will use the following inverse inequality (cf., e.g., Theorem. 3.2.6 in [13]): For $1 \le p \le \infty$ there exists a constant $C_{inv} > 0$, only depending on p, the polynomial degree k, and the local geometry of the triangulation, such that for $v_h \in P_k(K)$ it holds

$$\|\nabla v_h\|_{L^p(K;\mathbb{R}^2)} \le C_{inv} h_K^{-1} \|v_h\|_{L^p(K)}. \tag{4.2}$$

We will also use the following trace inequality (cf., e.g., [16]): For $1 \le p \le \infty$ there exists a constant $C_{tr} > 0$, only depending on p, the polynomial degree k, and the local geometry of the triangulation, such that for $v_h \in P_k(K)$ and $K \in \mathcal{T}_h$ it holds

$$\|v_h\|_{L^p(\partial K)} \le C_{tr} h_K^{-1/p} \|v_h\|_{L^p(K)}. \tag{4.3}$$

For $E \in \mathcal{E}_h(\Omega)$, $E = K_+ \cap K_-$, $K_\pm \in \mathcal{T}_h(\Omega)$, and $v_h \in V_h$, we denote the average and jump of v_h across E by $\{v_h\}_E$ and $[v_h]_E$, i.e.,

$$\{v_h\}_E := \frac{1}{2}\left(v_h|_{E\cap K_+} + v_h|_{E\cap K_-}\right), \quad [v_h]_E := v_h|_{E\cap K_+} - v_h|_{E\cap K_-},$$

whereas for $E \in \mathcal{E}_h(\Gamma)$ we set

$$\{v_h\}_E := v_h|_E, \quad [v_h]_E := v_h|_E.$$

The averages $\{\nabla v_h\}_E, \{\underline{\tau}_h\}_E$ and jumps $[\nabla v_h]_E, [\underline{\tau}_h]_E$ of vector-valued functions ∇v_h and $\underline{\tau}_h$ as well as the averages $\{D^2 v_h\}_E, \{\underline{\underline{\tau}}_h\}_E$ and jumps $[D^2 v_h]_E, [\underline{\underline{\tau}}_h]_E$ of matrix-valued functions $D^2 v_h$ and $\underline{\underline{\tau}}_h$ are defined analogously. For $E \in \mathcal{E}_h(\Omega)$ it holds

$$\int_E [u_h\, v_h]_E\, ds = \int_E \left(\{u_h\}_E\, [v_h]_E + [u_h]_E\, \{v_h\}_E\right) ds. \tag{4.4}$$

We further denote by $\mathbf{n}_E, E \in \mathcal{E}_h(\Omega)$, with $E = K_+ \cap K_-$ the unit normal on E pointing from K_+ to K_- and by $\mathbf{n}_E, E \in \mathcal{E}_h(\Gamma)$, the exterior unit normal on E.

We define the broken $W^{2,2}$-space $W^{2,2}(\Omega; \mathcal{T}_h)$ by

$$W^{2,2}(\Omega; \mathcal{T}_h) := \{v_h \in L^2(\Omega) \mid v_h|_K \in W^{2,2}(K), K \in \mathcal{T}_h\}, \tag{4.5}$$

equipped with the norm

$$\|v_h\|_{W^{2,2}(\Omega;\mathcal{T}_h)} := \left(\sum_{K\in\mathcal{T}_h} \|v_h\|_{W^{2,2}(K)}^2\right)^{1/2}, \tag{4.6}$$

and the broken spaces $\underline{\mathbf{H}}(\mathrm{div}, \Omega; \mathcal{T}_h)$ and $\underline{\underline{\mathbf{H}}}(\mathrm{div}^2, \Omega; \mathcal{T}_h)$ by

$$\underline{\mathbf{H}}(\mathrm{div}, \Omega; \mathcal{T}_h) := \{\underline{\mathbf{q}}_h \in L^2(\Omega; \mathbb{R}^2) \mid \underline{\mathbf{q}}_h|_K \in \underline{\mathbf{H}}(\mathrm{div}; K), K \in \mathcal{T}_h\}, \tag{4.7a}$$

$$\underline{\underline{\mathbf{H}}}(\mathrm{div}^2, \Omega; \mathcal{T}_h) := \{\underline{\underline{\mathbf{q}}}_h \in L^2(\Omega; \mathbb{R}^{2\times 2}) \mid \underline{\underline{\mathbf{q}}}_h|_K \in \underline{\underline{\mathbf{H}}}(\mathrm{div}^2; K), K \in \mathcal{T}_h\}, \tag{4.7b}$$

equipped with the norms

$$\|\underline{\mathbf{q}}_h\|_{\underline{\mathbf{H}}(\mathrm{div},\Omega;\mathcal{T}_h)} := \Big(\sum_{K\in\mathcal{T}_h} \|\underline{\mathbf{q}}_h\|^2_{\underline{\mathbf{H}}(\mathrm{div};K)}\Big)^{1/2}, \tag{4.8a}$$

$$\|\underline{\underline{\mathbf{q}}}_h\|_{\underline{\underline{\mathbf{H}}}(\mathrm{div}^2,\Omega;\mathcal{T}_h)} := \Big(\sum_{K\in\mathcal{T}_h} \|\underline{\underline{\mathbf{q}}}_h\|^2_{\underline{\underline{\mathbf{H}}}(\mathrm{div}^2;K)}\Big)^{1/2}. \tag{4.8b}$$

We denote by Π_k the orthogonal L^2 projection of $L^2(\Omega)$ onto V_h, which can be defined elementwise by

$$\int_\Omega \Pi_k(v)v_h\,dx = \sum_{K\in\mathcal{T}_h}\int_K \Pi_{K,k}(v)v_h\,dx,\ v\in L^2(\Omega), \tag{4.9}$$

$$\int_K \Pi_{K,k}(v)p_k\,dx = \int_K vp_k\,dx,\quad p_k\in P_k(K), K\in\mathcal{T}_h.$$

We note that Π_k can be extended to $L^p(\Omega)$ for $p\in[1,2)$ and $p\in[2,\infty]$ (cf., e.g., [14]).
We further denote by $\underline{\underline{\mathbf{\Pi}}}_k$ and $\underline{\mathbf{\Pi}}_k$ the L^2 projections of $L^2(\Omega;\mathbb{R}^{2\times 2})$ onto $\underline{\underline{\mathbf{V}}}_h$ and of $L^2(\Omega;\mathbb{R}^2)$ onto $\underline{\mathbf{V}}_h$ which can also be defined elementwise similar to (4.9) involving $\underline{\underline{\mathbf{\Pi}}}_{K,k}$ and $\underline{\mathbf{\Pi}}_{K,k}, K\in\mathcal{T}_h$. The L^2 projections of $L^2(\Gamma)$ onto $\{v_h\in L^2(\Gamma)\mid v_h|_E\in P_k(E), E\in\mathcal{E}_h(\Gamma)\}$ and of $L^2(\Gamma;\mathbb{R}^{2\times2})$ onto $\{\underline{\underline{\mathbf{q}}}_h \in L^2(\Gamma;\mathbb{R}^{2\times2}) \mid \underline{\underline{\mathbf{q}}}_h|_E \in P_k(E)^{2\times2}, E\in\mathcal{E}_h(\Gamma)\}$, will be denoted by $\Pi_{\Gamma,k}$ and $\underline{\underline{\mathbf{\Pi}}}_{\Gamma,k}$, respectively.
For $\mathbf{v}\in W^{2,2}(\Omega;\mathcal{T}_h)^2$ we redefine the primal energy functional (3.3) according to

$$J_P(\mathbf{v}) := \frac12\sum_{K\in\mathcal{T}_h}\Big(\int_K |v_1-u_1^{m-1}|^2\,dx + \tau_m\int_K |D^2v_1|^2dx + \frac12\sum_{K\in\mathcal{T}_h}\int_K |D^2v_2|^2dx\Big)$$
$$-\frac{\tau_m}{2}\sum_{K\in\mathcal{T}_h}\int_K (\mathrm{cof}(D^2v_1+\Theta) : D^2(v_2+f^m_{int})\,v_1 - g^m|v_1|^2)\,dx +$$
$$\sum_{K\in\mathcal{T}_h}\int_K \mathrm{cof}(D^2v_1+2\Theta) : D^2v_1\,v_2\,dx - \tau_m\sum_{K\in\mathcal{T}_h}\int_K f^m_{ext}v_1\,dx -$$
$$\frac12\sum_{K\in\mathcal{T}_h}\Big(\int_K |u_1^{m-1}|^2\,dx, \tag{4.10}$$

and note that it reduces to (3.3) for $\mathbf{v}\in W^{2,2}(\Omega)^2$.
We consider the finite element approximation with the DG spaces

$$V_h := \{v_h\in C(\bar\Omega)\mid v_h|_K\in P_k(K),\ K\in\mathcal{T}_h\}, \tag{4.11a}$$

$$\underline{\mathbf{V}}_h := \{\underline{\mathbf{q}}_h : \bar\Omega\to\mathbb{R}^2 \mid \underline{\mathbf{q}}_h|_K\in P_{k-1}(K)^2,\ K\in\mathcal{T}_h\}, \tag{4.11b}$$

$$\underline{\underline{\mathbf{V}}}_h := \{\underline{\underline{\mathbf{q}}}_h : \bar\Omega\to\mathbb{R}^{2\times2}) \mid \underline{\underline{\mathbf{q}}}_h|_K\in P_k(K)^{2\times2},\ K\in\mathcal{T}_h\}. \tag{4.11c}$$

We note that for $k\ge2$ we have $V_h\subset W^{2,2}(\Omega;\mathcal{T}_h)$. Moreover, for $\underline{\underline{\mathbf{q}}}_h\in\underline{\underline{\mathbf{V}}}_h$, we have $\nabla\cdot\underline{\underline{\mathbf{q}}}_h|_K\in P_{k-1}(K)^2$ and $\nabla\cdot\nabla\cdot\underline{\underline{\mathbf{q}}}_h|_K\in P_{k-2}(K), K\in\mathcal{T}_h$.

For $u_h\in V_h$ we define the broken gradient $\nabla_h u_h$ and the broken Hessian $D_h^2u_h$ by means of

$$\nabla_h u_h|_K := \nabla u_h|_K,\quad K\in\mathcal{T}_h, \tag{4.12a}$$

$$D_h^2u_h|_K := D^2u_h|_K,\quad K\in\mathcal{T}_h. \tag{4.12b}$$

Setting $\mathbf{u}_h=(u_{h,1},u_{h,2})^T$, we consider a six-field formulation which will lead us to the C^0IPDG approximation:

$$\underline{\mathbf{p}}^m_{h,i} = \nabla_h u_{h,i},\ i\in\{1,2\}, \tag{4.13a}$$

$$\underline{\underline{\mathbf{p}}}^m_{h,i} = D_h^2 u_{h,i},\ i\in\{1,2\}, \tag{4.13b}$$

$$\Pi_{k-2}(u^m_{h,1}) + \tau_m\Big(\nabla\cdot\nabla\cdot\underline{\underline{\mathbf{p}}}^m_{h,1} - \Pi_{k-2}(\mathrm{cof}(\underline{\underline{\mathbf{p}}}^m_{h,1}+\Theta_h) : (\underline{\underline{\mathbf{p}}}^m_{h,2} + f^m_{h,int}) - g^m_h u^m_{h,1})\Big) \tag{4.13c}$$
$$= \Pi_{k-2}(u^{m-1}_{h,1}) + \tau_m f^m_{h,ext}\quad \text{in each } K\in\mathcal{T}_h,$$

where $f^m_{h,int}, f^m_{h,ext}$, and Θ_h, g^m_h are given such that $f^m_{h,int}|_K, f^m_{h,ext}|_K$, and $\Theta_h, g^m_h|_K$ are the L^2 projections of f^m_{int}, f^m_{ext}, and Θ, g^m onto $P_{k-2}(K)$ for each $K\in\mathcal{T}_h$,

$$\nabla\cdot\nabla\cdot\underline{\underline{\mathbf{p}}}^m_{h,2} + \Pi_{k-2}(\mathrm{cof}(\underline{\underline{\mathbf{p}}}^m_{h,1}+2\Theta_h) : \underline{\underline{\mathbf{p}}}^m_{h,1}) = 0\quad \text{in each } K\in\mathcal{T}_h. \tag{4.13d}$$

We multiply (4.13a) by $\underline{\mathbf{q}}_h\in\underline{\mathbf{V}}_h$ and (4.13b) by $\underline{\underline{\mathbf{q}}}_h\in\underline{\underline{\mathbf{V}}}_h$, integrate and sum over all $K\in\mathcal{T}_h$.

$$\sum_{K\in\mathcal{T}_h}\int_K \underline{\mathbf{p}}^m_{h,i}\cdot\underline{\mathbf{q}}_h\,dx = \sum_{K\in\mathcal{T}_h}\int_K \nabla u^m_{h,i}\cdot\underline{\mathbf{q}}_h\,dx,\ 1\le i\le2, \tag{4.14}$$

$$\sum_{K\in\mathcal{T}_h}\int_K \underline{\underline{\mathbf{p}}}^m_{h,i} : \underline{\underline{\mathbf{q}}}_h\,dx = \sum_{K\in\mathcal{T}_h}\int_K D^2u^m_{h,i} : \underline{\underline{\mathbf{q}}}_h\,dx,\ 1\le i\le2. \tag{4.15}$$

We multiply (4.13c) by $v_h\in V_h$, integrate and sum over all $K\in\mathcal{T}_h$, and apply Green's formula twice. We thus obtain

$$\sum_{K\in\mathcal{T}_h}\Big(\int_K \Pi_{k-2}(u^m_{h,1})\,v_h\,dx + \tau_m(\int_K \nabla\cdot\nabla\cdot\underline{\underline{\mathbf{p}}}^m_{h,1}\,v_h\,dx - \tag{4.16}$$
$$\int_K \Pi_{k-2}(\mathrm{cof}(\underline{\underline{\mathbf{p}}}^m_{h,1}+\Theta_h) : (\underline{\underline{\mathbf{p}}}^m_{h,2}+f^m_{h,int}) - g^m_h u^m_{h,1}))\,v_h\,dx\Big) =$$
$$\sum_{K\in\mathcal{T}_h}\Big(\int_K \Pi_{k-2}(u^m_{h,1})\,v_h\,dx + \tau_m(\int_K \underline{\underline{\mathbf{p}}}^m_{h,1} : D^2v_h\,dx -$$
$$\int_K \Pi_{k-2}(\mathrm{cof}(\underline{\underline{\mathbf{p}}}^m_{h,1}+\Theta_h) : (\underline{\underline{\mathbf{p}}}^m_{h,2}+f^m_{h,int}) - g^m_h u^m_{h,1})\,v_h\,dx)\Big) +$$
$$\tau_m\Big(\sum_{K\in\mathcal{T}_h}\int_{\partial K} \mathbf{n}_{\partial K}\cdot\nabla\cdot\underline{\underline{\mathbf{p}}}^m_{h,1}\,v_h\,ds - \sum_{K\in\mathcal{T}_h}\int_{\partial K} (\underline{\underline{\mathbf{p}}}^m_{h,1}\mathbf{n}_{\partial K})\cdot\nabla v_h\,ds\Big) =$$
$$\sum_{K\in\mathcal{T}_h}\Big(\int_K \Pi_{k-2}(u^{m-1}_{h,1})\,v_h\,dx + \tau_m\int_K f^m_{h,ext}v_h\,dx\Big).$$

We proceed in the same way with (4.13d) which yields

$$\sum_{K\in\mathcal{T}_h}\Big(\int_K (\nabla\cdot\nabla\cdot\underline{\underline{\mathbf{p}}}^m_{h,2}\,v_h\,dx + \int_K \Pi_{k-2}(\mathrm{cof}(\underline{\underline{\mathbf{p}}}^m_{h,1}+2\Theta_h) : \underline{\underline{\mathbf{p}}}^m_{h,1})\,v_h)\,dx\Big) = \tag{4.17}$$
$$\sum_{K\in\mathcal{T}_h}\Big(\int_K \underline{\underline{\mathbf{p}}}^m_{h,2} : D^2v_h\,dx + \int_K \Pi_{k-2}(\mathrm{cof}(\underline{\underline{\mathbf{p}}}^m_{h,1}+2\Theta_h) : \underline{\underline{\mathbf{p}}}^m_{h,1})\,v_h\,dx\Big) +$$
$$\sum_{K\in\mathcal{T}_h}\int_{\partial K} \mathbf{n}_{\partial K}\cdot\nabla\cdot\underline{\underline{\mathbf{p}}}^m_{h,2}\,v_h\,ds - \sum_{K\in\mathcal{T}_h}\int_{\partial K} (\underline{\underline{\mathbf{p}}}^m_{h,2}\mathbf{n}_{\partial K})\cdot\nabla v_h\,ds = 0.$$

We replace $\underline{\underline{\mathbf{p}}}^m_{h,1}|_{\partial K}\,\mathbf{n}_{\partial K}$ and $\nabla\cdot\underline{\underline{\mathbf{p}}}^m_{h,1}|_{\partial K}$ in (4.19) by $\hat{\underline{\mathbf{p}}}^{m,(1,1)}_{\partial K}$ and $\hat{\underline{\mathbf{p}}}^{m,(1,2)}_{\partial K}$, where $\hat{\underline{\mathbf{p}}}^{m,(1,i)}_{\partial K}, 1\le i\le2$, are numerical flux functions. Likewise, we replace $\underline{\underline{\mathbf{p}}}^m_{h,2}|_{\partial K}\,\mathbf{n}_{\partial K}$ and $\nabla\cdot\underline{\underline{\mathbf{p}}}^m_{h,2}|_{\partial K}$ in (4.17) by $\hat{\underline{\mathbf{p}}}^{m,(2,1)}_{\partial K}$ and $\hat{\underline{\mathbf{p}}}^{m,(2,2)}_{\partial K}$, where $\hat{\underline{\mathbf{p}}}^{m,(2,i)}_{\partial K}, 1\le i\le2$, are also suitably chosen numerical flux functions. We thus obtain the following system of discrete variational equations:
Find $(\underline{\underline{\mathbf{p}}}^m_{h,i}, \underline{\mathbf{p}}_{h,i}, u_{h,i})\in\underline{\underline{\mathbf{V}}}_h\times\underline{\mathbf{V}}_h\times V_h, 1\le i\le2$, such that for all $(\underline{\underline{\mathbf{q}}}_h, \underline{\boldsymbol{\varphi}}_h, v_h)\in\underline{\underline{\mathbf{V}}}_h\times\underline{\mathbf{V}}_h\times V_h$ it holds

$$\sum_{K\in\mathcal{T}_h}\int_K \underline{\underline{\mathbf{p}}}^m_{h,1} : \underline{\underline{\mathbf{q}}}_h\,dx = \sum_{K\in\mathcal{T}_h}\int_K D^2u^m_{h,1} : \underline{\underline{\mathbf{q}}}_h\,dx, \tag{4.18a}$$

$$\sum_{K\in\mathcal{T}_h}\int_K \underline{\mathbf{p}}^m_{h,1}\cdot\underline{\boldsymbol{\varphi}}_h\,dx = -\sum_{K\in\mathcal{T}_h}\int_K \underline{\underline{\mathbf{p}}}^m_{h,1} : \nabla\underline{\boldsymbol{\varphi}}_h\,dx + \sum_{K\in\mathcal{T}_h}\int_{\partial K} \hat{\underline{\mathbf{p}}}^{m,(1,1)}_{\partial K}\cdot\underline{\boldsymbol{\varphi}}_h\,ds, \tag{4.18b}$$

$$\sum_{K\in\mathcal{T}_h}\Big(\int_K \Pi_{k-2}(u^m_{h,1})\,v_h\,dx + \tau_m\int_K \nabla\cdot\underline{\mathbf{p}}^m_{h,1}\,v_h\,dx\Big) =$$
$$\sum_{K\in\mathcal{T}_h}\Big(\int_K \Pi_{k-2}(u^m_{h,1})\,v_h\,dx - \tau_m\int_K \underline{\mathbf{p}}^m_{h,1}\cdot\nabla v_h\,dx\Big) \tag{4.18c}$$

$$+\tau_m \sum_{K\in\mathcal{T}_h}\int_{\partial K} \mathbf{n}_{\partial K}\cdot \hat{\underline{\mathbf{p}}}^{m,(1,2)}_{\partial K}\, v_h\, ds =$$

$$\tau_m \sum_{K\in\mathcal{T}_h}\int_K \Pi_{k-2}(\mathrm{cof}(D^2u^m_{h,1}+\Theta_h):(D^2u^m_{h,2}+f^m_{h,int})-g^m_h u^m_{h,1})\, v_h\, dx +$$

$$\sum_{K\in\mathcal{T}_h}\Big(\int_K \Pi_{k-2}(u^{m-1}_{h,1})\, v_h\, dx + \tau_m \int_K f^m_{h,ext}\, v_h\, dx\Big)$$

and

$$\sum_{K\in\mathcal{T}_h}\int_K \underline{\underline{\mathbf{p}}}^m_{h,2} : \underline{\underline{\mathbf{q}}}_h\, dx = \sum_{K\in\mathcal{T}_h}\int_K D^2u^m_{h,2} : \underline{\underline{\mathbf{q}}}_h\, dx, \tag{4.19a}$$

$$\sum_{K\in\mathcal{T}_h}\int_K \underline{\mathbf{p}}^m_{h,2}\cdot \underline{\boldsymbol{\varphi}}_h\, dx = -\sum_{K\in\mathcal{T}_h}\int_K \underline{\underline{\mathbf{p}}}^m_{h,2} : \nabla\underline{\boldsymbol{\varphi}}_h\, dx + \sum_{K\in\mathcal{T}_h}\int_{\partial K} \hat{\underline{\mathbf{p}}}^{m,(2,1)}_{\partial K}\cdot \underline{\boldsymbol{\varphi}}_h\, ds, \tag{4.19b}$$

$$\sum_{K\in\mathcal{T}_h}\int_K \nabla\cdot \underline{\mathbf{p}}^m_{h,2}\, v_h\, dx =$$

$$-\sum_{K\in\mathcal{T}_h}\int_K \underline{\mathbf{p}}^m_{h,2}\cdot\nabla v_h\, dx + \sum_{K\in\mathcal{T}_h}\int_{\partial K} \mathbf{n}_{\partial K}\cdot \hat{\underline{\mathbf{p}}}^{m,(2,2)}_{\partial K}\, v_h\, ds = \tag{4.19c}$$

$$-\sum_{K\in\mathcal{T}_h}\int_K \Pi_{k-2}(\mathrm{cof}(D^2u^m_{h,1}+2\Theta_h):D^2u^m_{h,1})\, v_h\, dx.$$

In particular, for the six-field formulation of the C^0IPDG approximation (4.22) the numerical flux functions $\hat{\underline{\mathbf{p}}}^{m,(i,j)}_{\partial K}, 1\le i,j\le 2$, are chosen as follows:

$$\hat{\underline{\mathbf{p}}}^{m,(1,1)}_{\partial K}|_E := \Big(\{\underline{\underline{\boldsymbol{\Pi}}}_{k-2}(\underline{\underline{\mathbf{z}}}^m_{h,1})\}_E - \alpha_1 h_E^{-1}\{\underline{\underline{\mathbf{w}}}^m_h\}_E\Big)\, \mathbf{n}_E,\ E\in\mathcal{E}_h(\Gamma), \tag{4.20a}$$

$$\hat{\underline{\mathbf{p}}}^{m,(1,2)}_{\partial K}|_E := \begin{cases} \mathbf{0},\ E\in\mathcal{E}_h(\Omega) \\ \nabla\cdot\underline{\underline{\boldsymbol{\Pi}}}_{k-2}(\underline{\underline{\mathbf{z}}}^m_{h,1}) + \alpha_2\, h_E^{-3} z^m_{h,1}\, \mathbf{n}_E,\ E\in\mathcal{E}_h(\Gamma)\end{cases} \tag{4.20b}$$

and

$$\hat{\underline{\mathbf{p}}}^{m,(2,1)}_{\partial K}|_E := \Big(\{\underline{\underline{\mathbf{z}}}^m_{h,2}\}_E - \frac{\alpha_1}{\tau_m} h_E^{-1}\{\underline{\underline{\mathbf{w}}}^m_{h,2}\}_E\Big)\, \mathbf{n}_E,\ E\in\mathcal{E}_h(\Gamma), \tag{4.21a}$$

$$\hat{\underline{\mathbf{p}}}^{m,(2,2)}_{\partial K}|_E := \begin{cases} \mathbf{0},\ E\in\mathcal{E}_h(\Omega) \\ \nabla\cdot\underline{\underline{\mathbf{z}}}^m_{h,2} + \frac{\alpha_2}{\tau_m}\, h_E^{-3} z^m_{h,2}\, \mathbf{n}_E,\ E\in\mathcal{E}_h(\Gamma)\end{cases}, \tag{4.21b}$$

where $\underline{\underline{\mathbf{z}}}^m_{h,i} := D^2_h u^m_{h,i}, \underline{\underline{\mathbf{w}}}^m_{h,i} := \nabla u^m_{h,i}\otimes\mathbf{n}_E$, and $z^m_{h,i} := u^m_{h,i}, 1\le i\le 2$.
The particular choice (4.20), (4.21) of the numerical flux functions allows to eliminate $\underline{\underline{\mathbf{p}}}^m_{h,i}, 1\le i\le 2$, from (4.18) and (4.19). We thus obtain the following C^0IPDG approximation of the von Kármán equations: Find $\mathbf{u}^m_h\in\mathbf{V}_h$ such that for all $\mathbf{v}_h\in\mathbf{V}_h$ it holds

$$a_h^{DG}(\mathbf{u}^m_h,\mathbf{v}_h) = \ell(\mathbf{v}_h), \tag{4.22}$$

where the semilinear form $a_h^{DG}(\cdot,\cdot):\mathbf{V}_h\times\mathbf{V}_h\to\mathbb{R}$ and the functional $\ell_h:\mathbf{V}_h\to\mathbb{R}$ are given by

$$a_h^{DG}(\mathbf{u}^m_h,\mathbf{v}_h) := \sum_{K\in\mathcal{T}_h}\Big(\int_K \Pi_{k-2}(u^m_{h,1})\, v_{h,1}\, dx + \tau_m\int_K D^2u^m_{h,1} : D^2v_{h,1}\, dx - \tag{4.23a}$$

$$\tau_m \int_K \Pi_{k-2}(\mathrm{cof}(D^2u^m_{h,1}+\Theta_h):(D^2u^m_{h,2}+f^m_{h,int})-g^m_h u^m_{h,1})\, v_{h,1}\, dx +$$

$$\int_K D^2u^m_{h,2} : D^2v_{h,2}\, dx + \int_K \Pi_{k-2}(\mathrm{cof}(D^2u^m_{h,1}+2\Theta_h):D^2u^m_{h,1})\, v_{h,2}\, dx\Big) -$$

$$\sum_{i=1}^2\Big(\sum_{E\in\mathcal{E}_h(\Omega)}\int_E \{D^2u^m_{h,i}\}_E : [\nabla v_{h,i}\otimes\mathbf{n}_E]_E ds +$$

$$\sum_{E\in\mathcal{E}_h(\Gamma)}\int_E \mathbf{n}_E\cdot\{\nabla\cdot D^2u^m_{h,i}\}_E v_{h,i} ds\Big)$$

$$+\sum_{i=1}^2\Big(\alpha_1\sum_{E\in\mathcal{E}_h(\Gamma)} h_E^{-1}\int_E (\nabla u^m_{h,1}\otimes\mathbf{n}_E)\,\mathbf{n}_E\cdot\nabla v_{h,i} ds$$

$$+\alpha_2\sum_{E\in\mathcal{E}_h(\Gamma)} h_E^{-3}\int_E u^m_{h,i} v_{h,i} ds\Big), \tag{4.23b}$$

$$\ell_h(\mathbf{v}^m_h) := \sum_{K\in\mathcal{T}_h}\Big(\int_K \underline{\underline{\boldsymbol{\Pi}}}_{k-2}(u^{m-1}_{h,1})\, v_{h,1}\, dx + \tau_m\int_K f^m_{h,ext}\, v_{h,1}\, dx.$$

Theorem 4.1. *The six-field formulation (4.18), (4.19) with the numerical flux functions given by (4.20) and (4.21) is equivalent with (4.22). In particular, if $u_{h,1}\in V_h$ and $u_{h,2}\in V_h$ is the solution of (4.22), there exists pairs $(\underline{\underline{\mathbf{p}}}_{h,i}, \underline{\mathbf{p}}_{h,i})\in\underline{\underline{\mathbf{V}}}_h\times\underline{\mathbf{V}}_h, 1\le i\le 2$ such that the triples $(\underline{\underline{\mathbf{p}}}_{h,i}, \underline{\mathbf{p}}_{h,i}, u_{h,i})\in\underline{\underline{\mathbf{V}}}_h\times\underline{\mathbf{V}}_h\times V_h, 1\le i\le 2$, satisfy (4.18), (4.19). Conversely, if the triples $(\underline{\underline{\mathbf{p}}}_{h,i}, \underline{\mathbf{p}}_{h,i}, u_{h,i})\in\underline{\underline{\mathbf{V}}}_h\times\underline{\mathbf{V}}_h\times V_h, 1\le i\le 2$, satisfy (4.18), (4.19), then $u_{h,i}\in V_h, 1\le i\le 2$, solve (4.22).*

Proof. Let $u_{h,i}\in V_h, 1\le i\le 2$, be solutions of (4.22). We then define $\underline{\underline{\mathbf{p}}}_{h,i}\in\underline{\underline{\mathbf{V}}}_h, 1\le i\le 2$, by means of (4.18a), (4.19a) and afterwards $\underline{\mathbf{p}}_{h,i}\in\underline{\mathbf{V}}_h, 1\le i\le 2$, according to (4.18b), (4.19b). We choose $\underline{\underline{\mathbf{q}}}_h = D^2 v_h$ in (4.18a), (4.19a) and $\underline{\boldsymbol{\varphi}}_h = \nabla v_h$ in (4.18b), (4.19b) and insert the resulting expressions into (4.18c), (4.19c) observing (4.20), (4.21). It follows that

$$\sum_{K\in\mathcal{T}_h}\int_K \nabla\cdot\underline{\mathbf{p}}^m_{h,1}\, v_h\, dx = \sum_{K\in\mathcal{T}_h}\int_K \underline{\underline{\mathbf{p}}}^m_{h,1} : D^2v_h\, dx - \sum_{K\in\mathcal{T}_h}\int_{\partial K} \hat{\underline{\mathbf{p}}}^{m,(1,1)}_{\partial K}\cdot\nabla v_h\, ds +$$

$$\sum_{K\in\mathcal{T}_h}\int_{\partial K} \mathbf{n}_{\partial K}\cdot\hat{\underline{\mathbf{p}}}^{m,(1,2)}_{\partial K}\, v_h\, ds = \sum_{K\in\mathcal{T}_h}\int_K D^2u^m_{h,1} : D^2v_h\, dx -$$

$$\tau_m\sum_{E\in\mathcal{E}_h(\Omega)}\int_E \{D^2u^m_{h,1}\}_E : [\nabla v_h\otimes\mathbf{n}_E]_E\, ds +$$

$$\tau_m\sum_{E\in\mathcal{E}_h(\Gamma}\int_E \mathbf{n}_E\cdot\{\nabla\cdot D^2u^m_{h,1}\}_E\, v_h\, ds +$$

$$\alpha_1\sum_{E\in\mathcal{E}_h(\Omega)} h_E^{-1}\int_E [\nabla u^m_{h,1}\otimes\mathbf{n}_E]_E : [\nabla v_h\otimes\mathbf{n}_E]_E\, ds +$$

$$\alpha_2\sum_{E\in\mathcal{E}_h(\Gamma)} h_E^{-3}\int_E u^m_{h,1}\, v_h\, ds,$$

and

$$\sum_{K\in\mathcal{T}_h}\int_K \nabla\cdot\underline{\mathbf{p}}^m_{h,2}\, v_h\, dx = \sum_{K\in\mathcal{T}_h}\int_K \underline{\underline{\mathbf{p}}}^m_{h,2} : D^2v_h\, dx - \sum_{K\in\mathcal{T}_h}\int_{\partial K} \hat{\underline{\mathbf{p}}}^{m,(2,1)}_{\partial K}\cdot\nabla v_h\, ds +$$

$$\sum_{K\in\mathcal{T}_h}\int_{\partial K} \mathbf{n}_{\partial K}\cdot\hat{\underline{\mathbf{p}}}^{m,(2,2}_{\partial K}\, v_h\, ds = \sum_{K\in\mathcal{T}_h}\int_K D^2u^m_{h,2} : D^2v_h\, dx -$$

$$\sum_{E\in\mathcal{E}_h(\Omega)}\int_E \{D^2u^m_{h,2}\}_E : [\nabla v_h\otimes\mathbf{n}_E]_E\, ds +$$

$$\sum_{E\in\mathcal{E}_h(\Gamma)}\int_E \mathbf{n}_E\cdot\{\nabla\cdot D^2u^m_{h,2}\}_E\, v_h\, ds +$$

$$\alpha_1\sum_{E\in\mathcal{E}_h(\Omega)} h_E^{-1}\int_E [\nabla u^m_{h,2}\otimes\mathbf{n}_E]_E : [\nabla v_h\otimes\mathbf{n}_E]_E\, ds +$$

$$\alpha_2\sum_{E\in\mathcal{E}_h(\Gamma)} h_E^{-3}\int_E u^m_{h,2}\, v_h\, ds.$$

In view of (4.22) and (4.23) we deduce that the last equation in (4.18) and (4.19) is satisfied.
Conversely, if the triples $(\underline{\underline{\mathbf{p}}}^m_{h,i}, \underline{\mathbf{p}}^m_{h,i}, u^m_{h,i})\in\underline{\underline{\mathbf{V}}}_h\times\underline{\mathbf{V}}_h\times V_h, 1\le i\le 2$, satisfy (4.18), (4.19), we choose $\underline{\underline{\mathbf{q}}}_h = D^2v_h$ in (4.18a), (4.19a), and $\underline{\boldsymbol{\varphi}}_h = \nabla v_h$ in (4.18b), (4.19b), and insert them into (4.18c), (4.19c). Taking (4.20) and (4.21) into account this shows that $\mathbf{u}_h\in\mathbf{V}_h$ satisfies (4.22). □

As in section 2 we define a bilinear form $A_{DG}(\cdot,\cdot):(\mathbf{V}_h+\mathbf{V})\times(\mathbf{V}_h+\mathbf{V})\to\mathbb{R}$ and a semilinear form $B_{DG}(\cdot,\cdot):(\mathbf{V}_h+\mathbf{V})\times(\mathbf{V}_h+\mathbf{V})\times(\mathbf{V}_h+\mathbf{V})\to\mathbb{R}$ by means of

$$A_{DG}(\mathbf{u}_h^m,\mathbf{v}_h):=\sum_{K\in\mathcal{T}_h}\Big(\Big(\int_K \Pi_{k-2}(u_{h,1}^m)v_{h,1}\,dx+\tau_m\int_K D^2u_{h,1}^m : D^2v_{h,1}\,dx+ \tag{4.24a}$$

$$\int_K D^2u_{h,2}^m : D^2v_{h,2}\,dx\Big)-\sum_{E\in\mathcal{E}_h(\Omega)}\Big(\tau_m\int_E\{\underline{\underline{\mathbf{\Pi}}}_{k-2}(D^2u_{h,1}^m)\}_E : [\nabla v_{h,1}\otimes\mathbf{n}_E]_E+$$

$$\int_E\{\underline{\underline{\mathbf{\Pi}}}_{k-2}(D^2u_{h,2}^m)\}_E : [\nabla v_{h,2}\otimes\mathbf{n}_E]_E\Big)\,ds+$$

$$\sum_{E\in\mathcal{E}_h(\Gamma)}\Big(\tau_m\int_E \mathbf{n}_E\cdot\{\nabla\cdot\underline{\underline{\mathbf{\Pi}}}_{k-2}(D^2u_{h,1}^m)\}_E\, v_{h,1}\,ds+$$

$$\int_E \mathbf{n}_E\cdot\{\nabla\cdot\underline{\underline{\mathbf{\Pi}}}_{k-2}(D^2u_{h,2}^m)\}_E\, v_{h,2}\,ds\Big)+$$

$$\sum_{i=1}^2\Big(\alpha_1\sum_{E\in\mathcal{E}_h(\Gamma)}h_E^{-1}\int_E(\nabla u_{h,i}^m\otimes\mathbf{n}_E)\,\mathbf{n}_E\cdot\nabla v_{h,i}ds$$

$$+\alpha_2\sum_{E\in\mathcal{E}_h(\Gamma)}h_E^{-3}\int_E u_{h,i}^m\,v_{h,i}\,ds\Big),$$

$$B_{DG}(\mathbf{u}_h^m,\mathbf{v}_h,\mathbf{w}_h):=-\frac{\tau_m}{2}\Big(\sum_{K\in\mathcal{T}_h}\int_K\Pi_{k-2}(\mathrm{cof}(D^2u_{h,1}^m+\Theta_h):(D^2v_{h,2}+ \tag{4.24b}$$

$$f_{h,int}^m)-g_h^m u_{h,1}^m)\,w_{h,1}\,dx+\sum_{K\in\mathcal{T}_h}\int_K\Pi_{k-2}(\mathrm{cof}(D^2u_{h,2}^m+f_{h,int}^m):(D^2v_{h,1}+\Theta_h)-$$

$$g_h^m u_{h,1}^m)\,w_{h,1}\,dx\Big)+\sum_{K\in\mathcal{T}_h}\int_K\Pi_{k-2}(\mathrm{cof}(D^2u_{h,1}+2\Theta_h):D^2v_{h,1})\,w_{h,2}\,dx,$$

where $\mathbf{u}_h^m=(u_{h,1}^m,u_{h,2}^m)^T$, $\mathbf{v}_h=(v_{h,1},v_{h,2})^T$, $\mathbf{w}_h=(w_{h,1},w_{h,2})^T$, $u_{h,i},v_{h,i},w_{h,i}\in V_h+V, 1\le i\le 2$.
Then the C^0IPDG approximation (4.22) can be written as: Find $\mathbf{u}_h^m\in\mathbf{V}_h$ such that for all $\mathbf{v}_h\in\mathbf{V}_h$ it holds

$$A_{DG}(\mathbf{u}_h^m,\mathbf{v}_h)+B_{DG}(\mathbf{u}_h^m,\mathbf{u}_h^m,\mathbf{v}_h)=\tau_m\sum_{K\in\mathcal{T}_h}(f_{h,ext}^m,v_{h,1})_{L^2(K)}. \tag{4.25}$$

We introduce operators $\mathcal{A}_{DG}:(\mathbf{V}_h+\mathbf{V})\to(\mathbf{V}_h^*+\mathbf{V}^*)$ and $\mathcal{B}_{DG}:(\mathbf{V}_h+\mathbf{V})\to(\mathbf{V}_h^*+\mathbf{V}^*)$ according to

$$<\mathcal{A}_{DG}\mathbf{u}_h^m,\mathbf{v}_h>_{\mathbf{V}_h^*+\mathbf{V}^*,\mathbf{V}_h+\mathbf{V}}:=A_{DG}(\mathbf{u}_h^m,\mathbf{v}_h), \tag{4.26a}$$

$$<\mathcal{B}_{DG}\mathbf{u}_h^m,\mathbf{v}_h>_{\mathbf{V}_h^*+\mathbf{V}^*,\mathbf{V}_h+\mathbf{V}}:=B_{DG}(\mathbf{u}_h^m,\mathbf{v}_h), \tag{4.26b}$$

so that (4.25) can be written as

$$\mathcal{A}_{DG}\mathbf{u}_h^m+\mathcal{B}_{DG}\mathbf{u}_h^m=\mathbf{f}_{h,ext}^m,\quad \mathbf{f}_{h,ext}^m=(f_{h,ext}^m,0)^T. \tag{4.27}$$

A slight variation of Theorem 2.1 in [9] yields the following existence and uniqueness result.

Theorem 4.2. *Given $f_{ext}^m\in L^2(\Omega)$, let $\mathbf{u}^m\in\underline{\mathbf{V}}$ be a regular weak solution of the von Kármán equations. Then there exist $\delta_0,\varepsilon_0>0$, such that for any triangulation $\mathcal{T}_h\in\mathbb{T}(\delta_0)$ there exists a unique solution $\mathbf{u}_h^m\in\underline{\mathbf{V}}_h$ of the C^0IPDG approximation (4.22) satisfying*

$$\sum_{i=1}^2\sum_{K\in\mathcal{T}_h}\int_K\|D^2u_{h,i}^m\|^2\,dx+\sum_{K\in\mathcal{T}_h}\int_K|f_{h,ext}^m-f_{ext}^m|^2\,dx<\varepsilon_0. \tag{4.28}$$

We note that $u_{h,i}^m\notin W^{2,2}(\Omega), 1\le i\le 2$, but conforming finite element functions $u_{h,i}^{m,c}\in V_h^c:=V_h\cap W^{2,2}(\Omega)$ can be obtained from $u_h^m\in\mathbf{V}_h$ by postprocessing. In particular, let V_h^c be the generalized version of the Hsieh-Clough-Tocher C [1] conforming finite element space as constructed in [18] and let $u_{h,i}^{m,c}=E_h(u_{h,i}^m)$ be the extensions of $u_{h,i}^m$ to V_h^c as constructed in [20]. By a result from [21] there exist constants $C_{c,|\nu|}>0, |\nu|\le 2$, and $C_{ext}>0$, depending only on the local geometry of the triangulation and on the penalty parameters $\alpha_i, 1\le i\le 2$, such that for $1\le i\le 2$ it holds

$$\Big(\sum_{K\in\mathcal{T}_h}\|D^\nu u_{h,i}^{m,c}\|_{L^p(K)}^p\Big)^{1/p}\le C_{c,|\nu|}\Big(\sum_{K\in\mathcal{T}_h}\|D^\nu u_{h,i}^m\|_{L^p(K)}^p\Big)^{1/p},\quad |\nu|\le 2, \tag{4.29a}$$

$$\sum_{i=1}^2\|u_{h,i}^m-u_{h,i}^{m,c}\|_{W^{2,p}(\Omega;\mathcal{T}_h)}^p\le C_c\sum_{i=1}^2\Big(\sum_{E\in\mathcal{E}_h(\Omega)}h_E^{-p/q}\int_E|[\nabla u_{h,i}^m\otimes\mathbf{n}_E]_E|^p\,ds+ \tag{4.29b}$$

$$\sum_{E\in\mathcal{E}_h(\Gamma)}h_E^{-p(q+1)/q}\int_E|u_{h,i}^m|^p\,ds\Big).$$

5. An a posteriori error estimator for the global discretization error

Given reflexive Banach spaces V,Q with norms $\|\cdot\|_V,\|\cdot\|_Q$, convex and coercive objective functionals $C:V\to\mathbb{R}$, $D:Q\to\mathbb{R}$, and a bounded linear operator $\Lambda:V\to Q$, we consider the minimization problem

$$\inf_{u\in V}J(u) \tag{5.1}$$

for the objective functional

$$J(u):=C(u)+D(\Lambda u). \tag{5.2}$$

An abstract approach to the a posteriori error control for (5.1) has been provided in [32] (cf. also [33] and the references therein). The a posteriori error control relies on the dual formulation of (5.1)

$$\sup_{q^*\in Q}J^*(q^*)\quad or\quad \inf_{q^*\in Q^*}(-J^*(q^*)), \tag{5.3}$$

in terms of the Fenchel conjugate J^* of J as given by

$$J^*(q^*)=-C^*(-\Lambda^*q^*)-D^*(q^*), \tag{5.4}$$

where C^* and D^* are the Fenchel conjugates of C and D and Λ^* stands for the adjoint of Λ.
Given some approximation $u_h\in V$ of the minimizer u of (5.1), the a posteriori error estimate Theorem 2.2 from [32] (cf. also Section 3 in [1] and [33]) states that for any admissible function $q^*\in Q^*$ it holds

$$\Phi_\delta(\Lambda(u_h-u))\le M_C(\Lambda^*q^*,u_h)+M_D(q^*,\Lambda u_h), \tag{5.5}$$

where $\Phi_\delta:Q\to\mathbb{R}_+$ is a continuous functional such that $\Phi_\delta(0)=0$ and for all $q_i\in B(0,\delta):=\{q\in Q\mid ||q||_Q<\delta\}$, $\delta>0$, $1\le i\le 2$, it holds

$$D((q_1+q_2)/2)+\Phi_\delta(q_2-q_1)\le(D(q_1)+D(q_2))/2$$

and

$$M_C(\Lambda^*q^*,u_h):=\frac12\Big(C(u_h)+C^*(\Lambda^*q^*)-<\Lambda^*q^*,u_h>_{V^*,V}\Big),$$

$$M_D(q^*,\Lambda u_h):=\frac12\Big(D(\Lambda v)+D^*(-q^*)-<q^*,\Lambda u_h>_{Q^*,Q}\Big).$$

We apply the above result for $\mathbf{V}=W_0^{2,2}(\Omega)^2$, $Q:=L^2(\Omega;\mathbb{R}^{2\times2})^2$, $\Lambda=D^2$, and

$$C(u_{h,1}^{m,c},u_{h,2}^{m,c}):=-\int_\Omega f_{ext}^m u_{h,1}^{m,c}\,dx, \tag{5.6a}$$

$$D(D^2u_{h,1}^{m,c},D^2u_{h,2}^{m,c}):=\frac12\sum_{K\in\mathcal{T}_h}\int_K\Big(|\Pi_{k-2}(u_{h,1}^{m,c}-u_{h,1}^{m-1})|^2+\tau_m|D^2u_{h,1}^{m,c}|^2\,dx\Big)- \tag{5.6b}$$

$$\frac{\tau_m}{2}\sum_{K\in\mathcal{T}_h}\int_K\Pi_{k-2}(\mathrm{cof}(D^2u_{h,1}^{m,c}+\Theta_h):D^2(u_{h,2}^{m,c}+f_{h,int}^m)-g_h^m u_{h,1}^m)u_{h,1}^{m,c}\,dx+$$

$$\sum_{K\in\mathcal{T}_h}\int_K \Pi_{k-2}(\text{cof}(D^2u_{h,1}^{m,c}+2\Theta_h):D^2u_{h,1}^{m,c})u_{h,2}^{m,c}\,dx + I_{K_1}(u_{h,1}^{m,c},u_{h,2}^{m,c}) -$$
$$\frac{1}{2}\sum_{K\in\mathcal{T}_h}\int_K |\Pi_{k-2}(u_{h,1}^{m-1})|^2\,dx,$$

where I_{K_1} is the indicator function of $K_1 := W_0^{2,2}(\Omega)^2$. We obtain:

$$C^*(-\Lambda^*\underline{\underline{\mathbf{q}}}_1^*,-\Lambda^*\underline{\underline{\mathbf{q}}}_2^*) := I_{K_2}(\underline{\underline{\mathbf{q}}}_1^*,\underline{\underline{\mathbf{q}}}_2^*),\quad \underline{\underline{\mathbf{q}}}_i^*\in\underline{\underline{\mathbf{H}}}(\text{div}^2;\Omega),\ 1\le i\le 2, \tag{5.7a}$$
$$D^*(\underline{\underline{\mathbf{q}}}_1^*,\underline{\underline{\mathbf{q}}}_2^*) := \frac{1}{2}\sum_{i=1}^2\int_\Omega |\underline{\underline{\mathbf{q}}}_i^*|^2\,dx,\quad \underline{\underline{\mathbf{q}}}_i^*\in\underline{\underline{\mathbf{H}}}(\text{div}^2;\Omega),\ 1\le i\le 2, \tag{5.7b}$$

where I_{K_2} is the indicator function of the closed convex set

$$K_2 := \{(\underline{\underline{\mathbf{q}}}_1^*,\underline{\underline{\mathbf{q}}}_1^*)\in\underline{\underline{\mathbf{H}}}(\text{div}^2;\Omega)^2 \mid \Pi_{k-2}(u_{h,1}^m+\tau_m\nabla\cdot\nabla\cdot\underline{\underline{\mathbf{q}}}_1^* - \tag{5.7c}$$
$$\tau_m\Pi_{k-2}(\text{cof}(D^2u_{h,1}^m+\Theta_h):(D^2u_{h,2}^m+f_{h,int}^m)-g_h^m u_{h,1}^m) = \tau_m f_{h,ext}^m\ ,$$
$$\nabla\cdot\nabla\cdot\underline{\underline{\mathbf{q}}}_2^* + \Pi_{k-2}(\text{cof}(D^2u_{h,1}^m+2\Theta_h):D^2u_{h,1}^m) = 0 \text{ in } \Omega\}.$$

We call $\underline{\underline{\mathbf{p}}}_{h,i}^{m,eq}\in\underline{\underline{\mathbf{V}}}_h$, $1\le i\le 2$, equilibrated moment tensors, if

$$\underline{\underline{\mathbf{p}}}_{h,i}^{m,eq}\in\underline{\underline{\mathbf{H}}}(\text{div}^2;\Omega) \tag{5.8a}$$

and $\underline{\underline{\mathbf{p}}}_{h,i}^{m,eq}$ satisfy the equilibrium conditions

$$\Pi_{k-2}(u_{h,1}^m+\tau_m\nabla\cdot\nabla\cdot\underline{\underline{\mathbf{p}}}_1^{m,eq} - \tag{5.8b}$$
$$\tau_m\Pi_{k-2}(\text{cof}(D^2u_{h,1}^m+\Theta_h):(D^2u_{h,2}^m+f_{h,int}^m)-g_h^m u_{h,1}^m) = \tau_m f_{h,ext}^m \text{ in } \Omega,$$
$$\nabla\cdot\nabla\cdot\underline{\underline{\mathbf{p}}}_2^{m,eq} + \Pi_{k-2}(\text{cof}(D^2u_{h,1}^m+2\Theta_h):D^2u_{h,1}^m) = 0 \text{ in } \Omega. \tag{5.8c}$$

Moreover, we choose $\underline{\underline{\mathbf{p}}}_{c,1}^m\in\underline{\underline{\mathbf{H}}}_{0,\Gamma}(\text{div}^2,\Omega)$ such that

$$\nabla\cdot\nabla\cdot\underline{\underline{\mathbf{p}}}_{c,1}^m = f_{ext}^m - f_{h,ext}^m, \tag{5.9}$$

and set $\underline{\underline{\mathbf{p}}}_{c,2}^m = 0$. It follows that $(\underline{\underline{\mathbf{p}}}_{h,1}^{m,eq}+\underline{\underline{\mathbf{p}}}_{c,1},\underline{\underline{\mathbf{p}}}_{h,2}^{m,eq}+\underline{\underline{\mathbf{p}}}_{c,2})\in K_2$, i.e.,

$$I_{K_2}(\underline{\underline{\mathbf{p}}}_{h,1}^{m,eq}+\underline{\underline{\mathbf{p}}}_{c,1},\underline{\underline{\mathbf{p}}}_{h,2}^{m,eq}+\underline{\underline{\mathbf{p}}}_{c,2}) = 0.$$

Similar to (3.8) in Example 2 (p-Laplace problem) of [32], the estimate (5.5) leads to:

$$\|u_1^m-u_{h,1}^{m,c}\|_{L^2(\Omega)}^2 + \tau_m\|u_1^m-u_{h,1}^{m,c}\|_{W^{2,2}(\Omega)}^2 + \|u_2^m-u_{h,2}^{m,c}\|_{W^{2,2}(\Omega)}^2 \le \tag{5.10}$$
$$2\Big(J_P(u_{h,1}^{m,c},u_{h,2}^{m,c}) + I_{K_1}(u_{h,1}^{m,c},u_{h,2}^{m,c}) + J_D(\underline{\underline{\mathbf{p}}}_{h,1}^{m,eq}+\underline{\underline{\mathbf{p}}}_{c,1}^m,\underline{\underline{\mathbf{p}}}_{h,2}^{m,eq}+\underline{\underline{\mathbf{p}}}_{c,2}^m)\Big).$$

In view (5.7) we have

$$J_D(\underline{\underline{\mathbf{p}}}_{h,1}^{m,eq}+\underline{\underline{\mathbf{p}}}_{c,1}^m,\underline{\underline{\mathbf{p}}}_{h,2}^{m,eq}+\underline{\underline{\mathbf{p}}}_{c,2}^m) = \sum_{K\in\mathcal{T}_h}\Big(\frac{1}{2}\int_K |\Pi_{k-2}(u_{h,1}^{m,c}-u_{h,1}^{m-1})|^2\,dx + \tag{5.11}$$
$$\frac{\tau_m}{2}\int_K |\underline{\underline{\mathbf{p}}}_{h,1}^{m,eq}+\underline{\underline{\mathbf{p}}}_{c,1}^m|^2\,dx + \frac{1}{2}\int_K |\underline{\underline{\mathbf{p}}}_{h,2}^{m,eq}+\underline{\underline{\mathbf{p}}}_{c,2}^m|^2\,dx - \frac{1}{2}\int_K |\Pi_{k-2}(u_{h,1}^{m-1})|^2\,dx.$$

Using (2.2), we find

$$\frac{1}{2}\sum_{i=1}^2\sum_{K\in\mathcal{T}_h}\int_K |\underline{\underline{\mathbf{p}}}_{h,i}^{m,eq}+\underline{\underline{\mathbf{p}}}_{c,i}^m|^2\,dx \le 2\Big(\sum_{K\in\mathcal{T}_h}\int_K |\underline{\underline{\mathbf{p}}}_{h,i}^{m,eq}|^2\,dx + \sum_{K\in\mathcal{T}_h}\int_K |\underline{\underline{\mathbf{p}}}_{c,i}|^2\,dx\Big). \tag{5.12}$$

In order to estimate the second term on the right-hand side of (5.12) we use the Poincaré-Friedrichs inequalities

$$\|v-|K|^{-1}\int_K v\,dx\|_{L^2(K)} \le C_{PF}^{(1)}\ h_K\|\nabla v\|_{L^2(K)},\quad v\in W^{1,2}(K),\ K\in\mathcal{T}_h, \tag{5.13a}$$

$$\|v-|E|^{-1}\int_E v\,ds\|_{L^2(E)} \le C_{PF}^{(2)}\ h_E\|\nabla v\|_{L^2(E)},\quad v\in W^{1,2}(E),\ E\in\mathcal{E}_h(\Gamma_N), \tag{5.13b}$$

where $C_{PF}^{(i)}$, $1\le i\le 2$, are positive constants (cf., e.g., [13]).

Lemma 5.1. *Suppose that the following regularity assumption is satisfied: For* $\underline{\underline{\tau}}\in\underline{\underline{\mathbf{H}}}_{0,\Gamma}(div^2,\Omega)$ *and the weak solution* $z\in W_0^{2,2}(\Omega)$ *of the elliptic boundary value problem*

$$\nabla\cdot\nabla\cdot D^2z = \nabla\cdot\nabla\cdot\underline{\underline{\tau}}\quad \text{in } \Omega, \tag{5.14a}$$
$$z = \mathbf{n}_\Gamma\cdot\nabla z = 0\quad \text{on } \Gamma, \tag{5.14b}$$

there exists a constant $C_z^{(1)}>0$ *such that*

$$D^2z|_\Gamma\in L^2(\Gamma;\mathbb{R}^{2\times 2}),\quad \|D^2z\|_{L^2(\Gamma;\mathbb{R}^{2\times 2})}\le C_z^{(1)}. \tag{5.15}$$

Moreover, there exists a constant $C_z^{(2)}>0$ *such that*

$$\|\nabla z\|_{L^2(\Omega;\mathbb{R}^2)}\le C_z^{(2)}. \tag{5.16}$$

Then for $\underline{\underline{\mathbf{p}}}_{c,1}^m\in\underline{\underline{\mathbf{H}}}_0(div^2,\Omega)$ *as given by* (5.9) *there exists a constant* $C_U>0$, *depending on* C_1, C_2 *and* $C_z^{(i)}, C_{PF}^{(i)}$, $1\le i\le 2$, *such that it holds*

$$\|\underline{\underline{\mathbf{p}}}_{c,1}^m\|_{L^2(\Omega;\mathbb{R}^{2\times 2})}^2 \le C_U\ osc_{h,1}, \tag{5.17}$$

where $osc_{h,1}$ *refers to the data oscillation*

$$osc_{h,1} := \sum_{K\in\mathcal{T}_h}\tau_m osc_{K,1},\quad osc_{K,1} := \begin{cases} h_K^2\int_K |f_{ext}^m-f_{h,ext}^m|^2\,dx, & k=2\\ h_K^4\int_K |f_{ext}^m-f_{h,ext}^m|^2\,dx, & k\ge 3\end{cases}. \tag{5.18}$$

Proof. We have

$$\|\underline{\underline{\mathbf{p}}}_{c,1}^m\|_{L^2(\Omega;\mathbb{R}^{2\times 2})} =$$
$$\sup\{\int_\Omega \underline{\underline{\mathbf{p}}}_{c,1}^m:\underline{\underline{\tau}}\,dx \mid \underline{\underline{\tau}}\in\underline{\underline{\mathbf{H}}}_{0,\Gamma_D}(\text{div}^2,\Omega),\ \|\underline{\underline{\tau}}\|_{L^2(\Omega;\mathbb{R}^{2\times 2})}\le 1\}.$$

For $\underline{\underline{\tau}}\in\underline{\underline{\mathbf{H}}}_{0,\Gamma_D}(\text{div}^2,\Omega)$ there exists $v\in W_0^{2,2}(\Omega)$ such that $\underline{\underline{\tau}} = D^2v$. In fact, v can be chosen as the weak solution of the boundary value problem (5.14). Hence, we have

$$\|\underline{\underline{\mathbf{p}}}_{c,1}^m\|_{L^2(\Omega,\mathbb{R}^{2\times 2})} \le \sup_{\|D^2v\|_{L^2(\Omega;\mathbb{R}^{2\times 2})}\le 1}\int_\Omega \underline{\underline{\mathbf{p}}}_{c,1}^m : D^2v\,dx. \tag{5.19}$$

Applying Green's formula twice locally on each $K\in\mathcal{T}_h$ and observing (5.9), we get

$$\int_\Omega \underline{\underline{\mathbf{p}}}_{c,1}^m : D^2z\,dx = \sum_{K\in\mathcal{T}_h}\int_K \nabla\cdot\nabla\cdot\underline{\underline{\mathbf{p}}}_{c,1}^m\ z\,dx +$$
$$\sum_{E\in\mathcal{E}_h(\Gamma)}\int_E \underline{\underline{\mathbf{p}}}_{c,1}^m\ \mathbf{n}_\Gamma\cdot\nabla z\,ds = \sum_{K\in\mathcal{T}_h}\int_K (f_{ext}^m-f_{h,ext}^m)\ z\,dx. \tag{5.20}$$

In order to estimate the first term on the right-hand side of (5.20) we first consider the case $k=2$. In view of the choice of f_h we have

$$\sum_{K\in\mathcal{T}_h}\int_K (f_{ext}^m-f_{h,ext}^m)\ z\,dx = \sum_{K\in\mathcal{T}_h}\int_K (f_{ext}^m-f_{h,ext}^m)\ (z-p_0)\,dx,$$

where $p_0 := |K|^{-1}\int_K z\,dx$, and hence, an application of (5.13a) and (5.16) yields

$$\Big|\sum_{K\in\mathcal{T}_h}\int_K (f_{ext}^m-f_{h,ext}^m)\ z\,dx\Big| \le \tag{5.21}$$

$$\sum_{K\in\mathcal{T}_h}\Big(\int_K |f_{ext}^m - f_{h,ext}^m|^2\,dx\Big)^{1/q}\Big(\int_K |z-p_0|^2\,dx\Big)^{1/2} \le$$
$$C_{PF}^{(1)}\Big(\sum_{K\in\mathcal{T}_h} h_K^q \int_K |f_{ext}^m - f_{h,ext}^m|^2\,dx\Big)^{1/2}\Big(\sum_{K\in\mathcal{T}_h}\int_K |\nabla z|^2\,dx\Big)^{1/2} \le$$
$$C_z^{(2)}C_{PF}^{(1)}\ osc_{h,1}^{1/2}.$$

In case $k\ge 3$ we have

$$\sum_{K\in\mathcal{T}_h}\int_K (f_{ext}^m - f_{h,ext}^m)\ z\,dx = \sum_{K\in\mathcal{T}_h}\int_K (f-f_h)\,(z-p_1)\,dx, \quad p_1\in P_1(K).$$

We fix $p_1\in P_1(K)$ by the interpolation conditions $\int_K p_1 dx = |K|^{-1}\int_K z\,dx$ and $\int_K \nabla p_1 dx = |K|^{-1}\int_K \nabla z\,dx$. An application of (5.13a) gives

$$\Big|\sum_{K\in\mathcal{T}_h}\int_K (f_{ext}^m - f_{h,ext}^m)\ z\,dx\Big| \le \tag{5.22}$$
$$\sum_{K\in\mathcal{T}_h}\Big(\int_K |f_{ext}^m - f_{h,ext}^m|^2\,dx\Big)^{1/q}\Big(\int_K |z-p_1|^2\,dx\Big)^{1/2} \le$$
$$C_{PF}^{(1)}\sum_{K\in\mathcal{T}_h}\Big(h_K^q\int_K |f_{ext}^m - f_{h,ext}^m|^2\,dx\Big)^{1/q}\Big(\int_K |\nabla(z-p_1)|^2\,dx\Big)^{1/2}.$$

Setting $\nabla p_1 = (p_{11},p_{12})^T$, another application of (5.13a) yields

$$\|\frac{\partial z}{\partial x_i} - p_{1i}\|_{L^2(K)} \le C_{PF}^{(1)}\ h_K\ \|\nabla\frac{\partial z}{\partial x_i}\|_{L^2(K)},\ 1\le i\le 2.$$

Hence, using (2.2), we obtain

$$\Big(\int_K |\nabla z - |K|^{-1}\int_K |\nabla(z-p_1)|^2\,dx\Big)^{1/2} \le \tag{5.23}$$
$$4\Big(\Big(\int_K |\frac{\partial z}{\partial x_1} - p_{11}|^p\,dx\Big)^{1/2} + \Big(\int_K |\frac{\partial z}{\partial x_2} - p_{12}|^2|\Big)\,dx\Big)^{1/2}\Big) \le$$
$$4C_{PF}^{(1)}h_K\Big(\int_K |D^2 z|^2\,dx\Big)^{1/2}.$$

Using (5.23) in (5.22) and observing (5.15) it follows that

$$\Big|\sum_{K\in\mathcal{T}_h}\int_K (f_{ext}^m - f_{h,ext}^m)\ z\,dx\Big| \le \tag{5.24}$$
$$4(C_{PF}^{(1)})^2\Big(\sum_{K\in\mathcal{T}_h} h_K^4\int_K |f_{ext}^m - f_{h,ext}^m|^2\,dx\Big)^{1/2}\Big(\sum_{K\in\mathcal{T}_h}\int_K |D^2 z|^2\,dx\Big)^{1/2} \le$$
$$4(C_{PF}^{(1)})^2 C_z^{(1)}\Big(\sum_{K\in\mathcal{T}_h} h_K^4\int_K |f_{ext}^m - f_{h,ext}^m|^2\,dx\Big)^{1/2}.$$

The assertion now follows from (5.21) and (5.24). □

Moreover, as far as $J_P(u_{h,1}^{m,c},u_{h,2}^{m,c})$ is concerned, we have

$$J_P(u_{h,1}^{m,c},u_{h,2}^{m,c}) = J_P(u_{h,1}^m,u_{h,2}^m) + \frac{1}{2}\sum_{K\in\mathcal{T}_h}\int_K \Big(|\Pi_{k-2}(u_{h,1}^{m,c} - u_{h,1}^{m-1})|^2 - \tag{5.25}$$
$$|\Pi_{k-2}(u_{h,1}^m) - u_{h,1}^{m-1}|^2\Big)\,dx + \frac{\tau_m}{2}\sum_{K\in\mathcal{T}_h}\int_K \Big(|D^2 u_{h,1}^{m,c}|^2 -$$
$$|D^2 u_{h,1}^{m,c}|^2\Big)\,dx + \frac{1}{2}\sum_{K\in\mathcal{T}_h}\int_K \Big(|D^2 u_{h,2}^{m,c}|^2 - |D^2 u_{h,2}^{m,c}|^2\Big)\,dx$$
$$-\frac{\tau_m}{2}\sum_{K\in\mathcal{T}_h}\int_K \Big(\Pi_{k-2}(\mathrm{cof}(D^2 u_{h,1}^m + D^2\Theta_h) : (D^2 u_{h,2}^m + f_{h,int}^m) -$$
$$g_h^m u_{h,1}^m)\ u_{h,1}^m + \Pi_{k-2}(\mathrm{cof}(D^2 u_{h,1}^{m,c} + D^2\Theta_h) : (D^2 u_{h,2}^{m,c} + f_{h,int}^m) -$$
$$g_h^m u_{h,1}^m)\ u_{h,1}^{m,c}\Big)dx + \sum_{K\in\mathcal{T}_h}\int_K \Pi_{k-2}(\mathrm{cof}(D^2 u_{h,1}^m + 2\Theta_h) : D^2 u_{h,1}^m)\ u_{h,2}^m\,dx -$$
$$\tau_m\sum_{K\in\mathcal{T}_h}\int_K f_{ext}^m(u_{h,1}^m - u_{h,1}^{m,c})\,dx.$$

Lemma 5.2. *Let $\mathbf{u}_h^m \in V_h\times V_h$ be the solution of (4.22) and let $(u_{h,1}^{m,c},u_{h,2}^{m,c})\in V_h^c\times V_h^c$ be its postprocessed finite element function. Then it holds*

$$\Big|J_P(u_{h,1}^{m,c},u_{h,2}^{m,c}) - J_P(u_{h,1}^m,u_{h,2}^m))\Big| \lesssim \sum_{K\in\mathcal{T}_h}\kappa_K^{eq}, \tag{5.26}$$

where

$$\kappa_K^{eq} := \frac{1}{\sqrt{2}}((1+C_{2,0})\|u_{h,1}^m\|_{L^2(K)}^2 + \|u_{h,1}^{m-1}\|_{L^2(K)}^2)^{1/2}\|u_{h,1}^m - u_{h,1}^{m,c}\|_{L^2(K)} +$$
$$\tau_m\Big(\|D^2 u_{h,1}^m\|_{L^2(K;\mathbb{R}^{2\times2})}\|D^2 u_{h,1}^m - D^2 u_{h,1}^{m,c}\|_{L^2(K;\mathbb{R}^{2\times2})} +$$
$$\|D^2 u_{h,1}^m - D^2 u_{h,1}^{m,c}\|_{L^2(K;\mathbb{R}^{2\times2})}^2\Big) + \|D^2 u_{h,2}^m\|_{L^2(K;\mathbb{R}^{2\times2})}\|D^2 u_{h,2}^m - D^2 u_{h,2}^{m,c}\|_{L^2(K;\mathbb{R}^{2\times2})}$$
$$+\|D^2 u_{h,2}^m - D^2 u_{h,2}^{m,c}\|_{L^2(K;\mathbb{R}^{2\times2})}^2 + \frac{\tau_m}{2}\Big(\Big(\|D^2 u_{h,1}^m\|_{L^3(K;\mathbb{R}^{2\times2})}\|D^2 u_{h,2}^m\|_{L^3(K;\mathbb{R}^{2\times2})} +$$
$$\|D^2\Theta_h\|_{L^3(K;\mathbb{R}^{2\times2})}\|D^2 u_{h,2}^m\|_{L^3(K;\mathbb{R}^{2\times2})} + C_{3,0}\|D^2 f_{h,int}^m\|_{L^3(K;\mathbb{R}^{2\times2})}$$
$$\|D^2 u_{h,1}^m\|_{L^3(K;\mathbb{R}^{2\times2})} + \|D^2\Theta_h\|_{L^3(K;\mathbb{R}^{2\times2})}\|D^2 f_{h,int}^m\|_{L^3(K;\mathbb{R}^{2\times2})}\Big)\|u_{h,1}^m - u_{h,1}^{m,c}\|_{L^3(K)}\Big)$$
$$+\Big(C_{3,0}\|u_{h,1}^m\|_{L^3(K)}\|D^2 u_{h,2}^m\|_{L^3(K;\mathbb{R}^{2\times2})} + \|D^2 f_{h,int}^m\|_{L^3(K;\mathbb{R}^{2\times2})}\|u_{h,1}^m\|_{L^3(K)}\Big) \tag{5.27}$$

$$\|D^2 u_{h,1}^m - D^2 u_{h,1}^{m,c}\|_{L^3(K;\mathbb{R}^{2\times2})} + \Big(C_{3,0}C_{3,2}\|u_{h,1}^m\|_{L^3(K)}\|D^2 u_{h,1}^m\|_{L^3(K;\mathbb{R}^{2\times2})} +$$
$$C_{3,0}\|D^2\Theta_h\|_{L^3(K;\mathbb{R}^{2\times2})}\|u_{h,1}^m\|_{L^3(K)}\Big)\|D^2 u_{h,2}^m - D^2 u_{h,2}^{m,c}\|_{L^3(K;\mathbb{R}^{2\times2})} +$$
$$\frac{1}{2}\Big((1+C_{3,2})\|u_{h,2}^m\|_{L^3(K)}\|D^2 u_{h,1}^m\|_{L^3(K;\mathbb{R}^{2\times2})}\|D^2 u_{h,1}^m - D^2 u_{h,1}^{m,c}\|_{L^3(K;\mathbb{R}^{2\times2})} +$$
$$C_{3,2}^2\|D^2 u_{h,1}^m\|_{L^3(K;\mathbb{R}^{2\times2})}^2\|u_{h,2}^m - u_{h,2}^{m,c}\|_{L^3(K)}\Big) + 2\Big(\|D^2\Theta_h\|_{L^3(K;\mathbb{R}^{2\times2})}\Big(\|u_{h,2}^m\|_{L^3(K)}$$
$$\|D^2 u_{h,1}^m - D^2 u_{h,1}^{m,c}\|_{L^3(K;\mathbb{R}^{2\times2})} + C_{3,2}\|D^2 u_{h,1}^m\|_{L^3(K;\mathbb{R}^{2\times2})} + \|u_{h,2}^m - u_{h,2}^{m,c}\|_{L^3(K)}\Big)\Big) +$$
$$\tau_m\|g_h^m\|_{L^2(K)}\|u_{h,1}^m - u_{h,1}^{m,c}\|_{L^2(K)} + \tau_m\|f_{ext}^m\|_{L^2(K)}\|u_{h,1}^m - u_{h,1}^{m,c}\|_{L^2(K)}. \tag{5.28}$$

Proof. For the second term on the right-hand side of (5.25), using the Cauchy-Schwarz inequality, Young's inequality, and (4.29a), we find

$$|\frac{1}{2}\int_K \Big(|\Pi_{k-2}(u_{h,1}^{m,c} - u_{h,1}^{m-1})|^2 - |\Pi_{k-2}(u_{h,1}^m - u_{h,1}^{m-1})|^2\Big)\,dx| \le \tag{5.29}$$
$$\frac{1}{2}\int_K |u_{h,1}^m + u_{h,1}^{m,c} - 2u_{h,1}^{m-1}||u_{h,1}^m) - u_{h,1}^{m,c}|\,dx \le$$
$$\frac{1}{\sqrt{2}}\Big((1+C_{0,2})\int_K |u_{h,1}^m|^2\,dx + \int_K |u_{h,1}^{m-1}|^2\,dx\Big)^{1/2}\Big(\int_K |u_{h,1}^m - u_{h,1}^{m,c}|^2\,dx\Big)^{1/2}.$$

Moreover, by Taylor expansion and using (2.2) as well as Hölder's inequality we get

$$\Big|\frac{1}{2}\int_K \Big(|D^2 u_{h,i}^{m,c}|^2 - |D^2 u_{h,i}^m|^2\Big)\Big| = \tag{5.30}$$
$$\Big|\int_K\int_0^1 (D^2 u_{h,i}^m + \lambda D^2(u_{h,i}^{m,c} - u_{h,i}^m)\,d\lambda : D^2(u_{h,i}^{m,c} - u_{h,i}^m)\,dx\Big| \le$$
$$\int_K\int_0^1 |D^2 u_{h,i}^m + \lambda D^2(u_{h,i}^{m,c} - u_{h,i}^m)||D^2(u_{h,i}^{m,c} - u_{h,i}^m)|\,d\lambda\,dx \le$$
$$2\int_K\int_0^1 \Big(|D^2 u_{h,i}^m| + \lambda|D^2(u_{h,i}^m - u_{h,i}^{m,c})||D^2(u_{h,i}^m - u_{h,i}^{m,c})|\,d\lambda\,dx \le$$
$$2\Big(\int_K |D^2 u_{h,i}^m|\,dx\Big)^{1/2}\Big(\int_K |D^2(u_{h,i}^m - u_{h,i}^{m,c})|^2\,dx\Big)^{1/2} +$$
$$\int_K |D^2(u_{h,i}^m - u_{h,i}^{m,c})|^2\,dx,\ 1\le i\le 2.$$

Further, for the fifth term on the right-hand side of (5.25) it follows that

$$
\Big|\int_K \Big(\Pi_{k-2}(\mathrm{cof}(D^2u_{h,1}^m + D^2\Theta_h) : (D^2u_{h,2}^m + f_{h,int}^m) - g_h^m u_{h,1}^m)\, u_{h,1}^m -
$$
$$
\Pi_{k-2}(\mathrm{cof}(D^2u_{h,1}^{m,c} + D^2\Theta_h) : (D^2u_{h,2}^{m,c} + f_{h,int}^m) - g_h^m u_{h,1}^{m,c})\, u_{h,1}^{m,c}\Big)\, dx\Big| \le
$$
$$
\Big(\|D^2u_{h,1}^m\|_{L^3(K;\mathbb{R}^{2\times2})}\|D^2u_{h,2}^m\|_{L^3(K;\mathbb{R}^{2\times2})} + \|D^2\Theta_h\|_{L^3(K;\mathbb{R}^{2\times2})}\|D^2u_{h,2}^m\|_{L^3(K;\mathbb{R}^{2\times2})} +
$$
$$
C_{3,0}\|D^2 f_{h,int}^m\|_{L^3(K;\mathbb{R}^{2\times2})}\|D^2u_{h,1}^m\|_{L^3(K;\mathbb{R}^{2\times2})} + \|D^2\Theta_h\|_{L^3(K;\mathbb{R}^{2\times2})}
$$
$$
\|D^2 f_{h,int}^m\|_{L^3(K;\mathbb{R}^{2\times2})}\Big)\|u_{h,1}^m - u_{h,1}^{m,c}\|_{L^3(K)} + \Big(C_{3,0}\|u_{h,1}^m\|_{L^3(K)}\|D^2u_{h,2}^m\|_{L^3(K;\mathbb{R}^{2\times2})} +
$$
$$
\|D^2 f_{h,int}^m\|_{L^3(K;\mathbb{R}^{2\times2})}\|u_{h,1}^m\|_{L^3(K)}\Big)\|D^2u_{h,1}^m - D^2u_{h,1}^{m,c}\|_{L^3(K;\mathbb{R}^{2\times2})} +
$$
$$
\Big(C_{3,0}C_{3,2}\|u_{h,1}^m\|_{L^3(K)}\|D^2u_{h,1}^m\|_{L^3(K;\mathbb{R}^{2\times2})} + C_{3,0}\|D^2\Theta_h\|_{L^3(K;\mathbb{R}^{2\times2})}\|u_{h,1}^m\|_{L^3(K)}\Big)
$$
$$
\|D^2u_{h,2}^m - D^2u_{h,2}^{m,c}\|_{L^3(K;\mathbb{R}^{2\times2})}\Big) + \frac{1}{2}\Big((1 + C_{3,2})\|u_{h,2}^m\|_{L^3(K)}\|D^2u_{h,1}^m\|_{L^3(K;\mathbb{R}^{2\times2})}
$$
$$
\|D^2u_{h,1}^m - D^2u_{h,1}^{m,c}\|_{L^3(K;\mathbb{R}^{2\times2})} + \|g_h^m\|_{L^2(K)}\|u_{h,1}^m - u_{h,1}^{m,c}\|_{L^2(K)}. \tag{5.31}
$$

The sixth term on the right-hand side of (5.25) can be estimated from above similarly. Finally, we have

$$
\Big|\sum_{K\in\mathcal{T}_h}\int_K f_{ext}^m(u_{h,1} - u_{h,1}^c)\, dx\Big| \le
$$
$$
\sum_{K\in\mathcal{T}_h}\Big(\int_K |f_{ext}^m|^2\, dx\Big)^{1/2}\Big(\int_K |u_{h,1} - u_{h,1}^c|^2\, dx\Big)^{1/2}. \tag{5.32}
$$

The assertion now follows from (5.25) and (5.29)-(5.32). □

For practical purposes, we further replace the indicator function $I_{K_1}(u_{h,1}^{m,c}, u_{h,2}^{m,c})$ (cf. (5.6)) by the penalty term

$$
\sum_{i=1}^2\Big(\alpha_1\sum_{E\in\mathcal{E}_h(\Gamma)} h_E^{-1}\int_E |\mathbf{n}_E\cdot\nabla u_{h,i}^{m,c}|^2\, ds + \alpha_2\sum_{E\in\mathcal{E}_h(\Gamma)} h_E^{-3}\int_E |u_{h,i}^{m,c}|^2\, ds\Big). \tag{5.33}
$$

In view of the construction of $u_{h,i}^{m,c}$ we have $u_{h,i}^{m,c}|_E = u_{h,i}^m$ on $E\in\mathcal{E}_h(\Gamma)$ and hence, (5.33) gives rise to the data oscillations

$$
\alpha_1\, osc_{h,2}^m + \alpha_2\, osc_{h,3}^m := \alpha_1\sum_{K\in\mathcal{T}_h} osc_{K,2}^m + \alpha_2\sum_{K\in\mathcal{T}_h} osc_{K,3}^m, \tag{5.34a}
$$
$$
osc_{K,2}^m := \sum_{i=1}^2\sum_{E\in\mathcal{E}_h(\partial K\cap\Gamma)} h_E^{-1}\int_E |\mathbf{n}_E\cdot\nabla u_{h,i}^m|^2\, ds, \tag{5.34b}
$$
$$
osc_{K,3}^m := \sum_{i=1}^2\sum_{E\in\mathcal{E}_h(\partial K\cap\Gamma)} h_E^{-3}\int_E |u_{h,i}^m|^2\, ds. \tag{5.34c}
$$

Using Lemma 5.1 and Lemma 5.2 in (5.10) yields

$$
\sum_{i=1}^2 \|u^m - u_{h,i}^m\|_{W^{2,2}(\Omega;\mathcal{T}_h)}^2 \lesssim \sum_{i=1}^3 \eta_{h,i}^{m,eq}. \tag{5.35a}
$$

Here, $\eta_{h,1}^{m,eq}$ and $\eta_{h,2}^{m,eq}$ are given by

$$
\eta_{h,i}^{m,eq} := \sum_{K\in\mathcal{T}_h}\eta_{K,i}^{m,eq}, \quad 1\le i\le 3, \tag{5.35b}
$$

where $\eta_{K,i}^{m,eq}, 1\le i\le 3$, read as follows:

$$
\eta_{K,1}^{eq} := \frac{1}{2}\int_K |\Pi_{k-2}(u_{h,1}^m - u_{h,1}^{m-1})|^2\, dx + \frac{\tau_m}{2}\int_K |D^2u_{h,1}^m|^2\, dx - \tag{5.35c}
$$
$$
\tau_m\int_K \Pi_{k-2}(\mathrm{cof}(D^2u_{h,1}^m + \Theta_h) : (D^2u_{h,2}^m + f_{h,int}^m) - g_h^m u_{h,1}^m)\, u_{h,1}^m\, dx -
$$
$$
\tau_m\int_K f_{ext}^m u_{h,1}^m\, dx + \frac{1}{2}\int_K |\underline{\mathbf{p}}_{h,1}^{eq}|^2\, dx - \frac{1}{2}\int_K |\Pi_{k-2}(u_{h,1}^{m-1}|^2\, dx\ ,
$$
$$
\eta_{K,2}^{eq} := \frac{1}{2}\int_K |D^2u_{h,2}^m|^2\, dx + \int_K \Pi_{k-2}(\mathrm{cof}(D^2u_{h,1}^m + 2\Theta_h) : D^2u_{h,1}^m)\, u_{h,2}^m\, dx +
$$
$$
\frac{1}{2}\int_K |\underline{\mathbf{p}}_{h,2}^{eq}|^2\, dx\ , \tag{5.35d}
$$
$$
\eta_{K,3}^{eq} := \|u_{h,1}^m - u_{h,1}^{m,c}\|_{L^2(K)}^2 + \tau_m\|u_{h,1}^m - u_{h,1}^{m,c}\|_{W^{2,2}(K)}^2 + \tag{5.35e}
$$
$$
\|u_{h,2}^m - u_{h,2}^{m,c}\|_{W^{2,2}(K)}^2 + \kappa_K^{eq} + \sum_{i=1}^3 osc_{K,i}^m.
$$

The right-hand side in (5.35) is then a computable and localizable quantity for the a posteriori estimation of the global discretization error. It gives rise to the following equilibrated a posteriori error estimator

$$
\eta_h^{m,eq} := \sum_{i=1}^3 \eta_{h,i}^{m,eq}, \quad \eta_{h,i}^{m,eq} := \sum_{K\in\mathcal{T}_h(\Omega)}\eta_{K,i}^{m,eq},\ 1\le i\le 3. \tag{5.36}
$$

The construction of an equilibrated flux will be dealt with in the subsequent section.

Remark 5.3. The efficiency of the equilibrated a posteriori error estimator $\eta_h^{m,eq}$ can be shown locally on patches around interior nodal points by using techniques from [36] involving suitably chosen bubble functions.

6. Construction of an equilibrated flux

We construct equilibrated fluxes $\underline{\mathbf{p}}_{h,i}^{m,eq}\in\underline{\mathbf{V}}(h)\cap\underline{\mathbf{H}}(\mathrm{div};\Omega)$ and equilibrated moment tensors $\underline{\underline{\mathbf{p}}}_{h,i}^{m,eq}\in\underline{\underline{\mathbf{V}}}_h\cap\underline{\underline{\mathbf{H}}}(\mathrm{div}^2,\Omega), 1\le i\le 2$, by an interpolation on each element. Thus it is a local procedure. In particular, denoting by $\mathbf{BDM}_k(K), k\in\mathbb{N}$, the Brezzi-Douglas-Marini finite element of order k (cf., e.g., [5]), we first construct auxiliary vector fields $\underline{\mathbf{p}}_{h,i}^{m,eq}\in\underline{\mathbf{H}}(\mathrm{div},\Omega), \underline{\mathbf{p}}_{h,i}^{m,eq}|_K\in\mathbf{BDM}_{k-1}(K), K\in\mathcal{T}_h(\Omega), 1\le i\le 2$, satisfying

$$
\Pi_{k-2}(u_h^m) + \tau_m\nabla\cdot\underline{\mathbf{p}}_{h,1}^{m,eq} - \tau_m\Pi_{k-2}(\mathrm{cof}(D^2u^m_{h,1+D^2\Theta_h)} : (D^2u_{h,2}^m + D^2 f_{h,int}^m) - \tag{6.1a}
$$
$$
g_h^m u_{h,1}^m) = \Pi_{k-2}(u_h^{m-1}) + \tau_m f_{h,ext}^m,
$$
$$
\nabla\cdot\underline{\mathbf{p}}_{h,2}^{m,eq} + \Pi_{k-2}(\mathrm{cof}(D^2u_{h,1}^m + 2\Theta_h) : D^2u_{h,1}^m) = 0, \tag{6.1b}
$$

in each $K\in\mathcal{T}_h$ and then equilibrated moment tensors $\underline{\underline{\mathbf{p}}}_{h,i}^{m,eq}\in\underline{\underline{\mathbf{V}}}_h, 1\le i\le 2$, satisfying (5.8).

For the construction of the auxiliary vector fields we recall the following result:

Lemma 6.1. *Any vector field $\underline{\mathbf{q}}\in P_k(K)^2, k\in\mathbb{N}$, is uniquely defined by the following degrees of freedom*

$$
\int_E \mathbf{n}_E\cdot\underline{\mathbf{q}}\, p_k\, ds, \quad p_k\in P_k(E),\ E\in\mathcal{E}_h(\partial K), \tag{6.2a}
$$
$$
\int_K \underline{\mathbf{q}}\cdot\nabla p_{k-1}\, dx, \qquad p_{k-1}\in P_{k-1}(K), \tag{6.2b}
$$
$$
\int_K \underline{\mathbf{q}}\cdot\mathbf{curl}(b_K p_{k-2})\, dx, p_{k-2}\in P_{k-2}(K), \tag{6.2c}
$$

where b_K in (6.2c) is the element bubble function on K given by $b_K = \prod_{i=1}^3\lambda_i^K$ and $\lambda_i^K, 1\le i\le 3$, are the barycentric coordinates of K. Moreover, there exists a positive constant $C_E^{(1)}$, depending only on k and the local geometry of the triangulation $\mathcal{T}_h$, such that

$$
\int_K |\underline{\mathbf{q}}|^2\, dx \le C_E^{(1)}\Big(\sum_{E\in\mathcal{E}_h(\partial K)} h_E\int_E |\mathbf{n}_E\cdot\underline{\mathbf{q}}|^2\, ds + h_K^2\int_K |\nabla\cdot\underline{\mathbf{q}}|^2\, dx + \tag{6.3}
$$
$$
h_K^2\max\Big\{\int_K |\underline{\mathbf{q}}\cdot\boldsymbol{curl}(b_K p_{k-2})|^2\, dx \mid p_{k-2}\in P_{k-2}(K),\ \max_{x\in K}|p_{k-2}(x)|\le 1\Big\}\Big).
$$

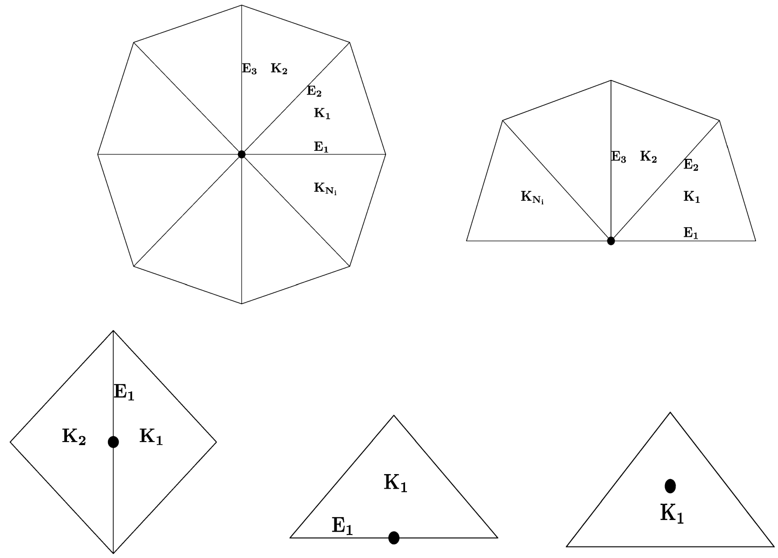

Fig. 1. Patch ω_i associated with nodal point $x_i \in \mathcal{N}_h(\bar{\Omega})$ featuring N_i triangles $K_\ell, 1 \le \ell \le N_i$ ($x_i \in \mathcal{N}_h(\partial E \cap \Omega)$ (Case 1, top left), $x_i \in \mathcal{N}_h(\partial E \cap \Gamma)$ (Case 2, top right), $x_i \in \mathcal{N}_h(\text{int}E \cap \Omega)$ (Case 3, bottom left), $x_i \in \mathcal{N}_h(\text{int}E \cap \Gamma)$ (Case 4, bottom middle), and $x_i \in \mathcal{N}_h(\text{int}K \cap \Omega)$ (Case 5, bottom right)).

Proof. For the uniqueness result we refer to [5]. The estimate (6.3) can be derived by standard scaling arguments (cf. Lemma 3.1 and Remark 3.3 in [2]). □

The construction of the auxiliary vector fields $\underline{\mathbf{p}}_{h,i}^{m,eq}, 1 \le i \le 2$, will be done on patches ω_i consisting of all triangles $K \in \mathcal{T}_h$ that have an interior nodal point $x_i \in \mathcal{N}_h(\Omega)$ in common. We assume that ω_i consists of N_i triangles $T_\ell, 1 \le \ell \le N_i$. We enumerate the interior edges $E_m, 1 \le m \le M_i$, counterclockwise and distinguish five cases (cf. Fig. 1).

We follow the techniques from [3] and construct the auxiliary vector fields $\underline{\mathbf{p}}_{h,\ell}^{m,eq}, 1 \le \ell \le 2$, patchwise:

$$\underline{\mathbf{p}}_{h,\ell}^{m,eq} = \sum_{i=1}^{n_h} \underline{\mathbf{p}}_{h,\ell}^{m,\omega_i}. \tag{6.4}$$

For a patch ω_i, we construct $\underline{\mathbf{p}}_{h,\ell}^{m,\omega_i}, 1 \le \ell \le 2$, such that

$$\underline{\mathbf{p}}_{h,\ell}^{m,\omega_i}|_{K_\ell} \in \mathbf{BDM}_{k-1}(K_\ell)\,, \tag{6.5}$$

$$\Pi_{k-2}(u_{h,1}^m) + \tau_m(\nabla \cdot \underline{\mathbf{p}}_{h,1}^{m,\omega_i} - \Pi_{k-2}(\text{cof}(D^2 u_{h,1}^m + D^2\Theta_h) : (D^2 u_{h,2}^m + f_{h,int}^m) - g_h^m u_{h,1}^m) = u_{h,1}^{m-1} + \tau_m f_{h,ext}^m \text{ in } \omega_i\,,$$

$$\nabla \cdot \underline{\mathbf{p}}_{h,2}^{m,\omega_i} + \Pi_{k-2}(\text{cof}(D^2 u_{h,1}^m + 2\Theta_h) : D^2 u_{h,1}^m) = 0 \text{ in } \omega_i\,,$$

$$\underline{\mathbf{p}}_{h,\ell}^{m,\omega_i} = \hat{\underline{\mathbf{p}}}_{\ell,\partial K_\ell}^{m,\omega_i}|_E,\ E \in \mathcal{E}_h(\text{int}\ \omega_i),$$

where, denoting by $\varphi_h^{(x_i)} \in V_h^{(k)}$ the nodal basis function associated with x_i, $\underline{\mathbf{p}}_{h,\ell}^{m,\omega_i}$ and $\hat{\underline{\mathbf{p}}}_{\ell,\partial K}^{m,\omega_i}|_E$ are given by

$$\underline{\mathbf{p}}_{h,\ell}^{m,\omega_i} := \varphi_h^{(x_i)} \underline{\mathbf{p}}_{h,\ell}^{m,eq}, \quad \hat{\underline{\mathbf{p}}}_{\ell,\partial K}^{m,\omega_i}|_E := \varphi_h^{(x_i)} \hat{\underline{\mathbf{p}}}_{\partial K}^{m,(1,\ell)}|_E\ , 1 \le i \le n_h. \tag{6.6}$$

Case 1 ($x_i \in \mathcal{E}_h(\partial E \cap \Omega)$): For $\ell = 1, 2, \cdots, N_i$ we compute $\underline{\mathbf{p}}_{h,1}^{(m,\omega_i)}|_{K_\ell} \in \mathbf{BDM}_{k-1}(K_\ell)$ according to

$$\int_{E_\ell} \underline{\mathbf{p}}_{h,1}^{(m,\omega_i)}|_{K_\ell} \mathbf{n}_{E_\ell \cap K_\ell}\ p_{k-1}\ ds\ = \tag{6.7a}$$

$$\begin{cases} \int_{E_\ell} \left(\mathbf{n}_{E_\ell} \cdot \hat{\underline{\mathbf{p}}}_{\partial K_\ell}^{m,(1,1)}\right)|_{E_\ell},\ \ell = 1 \\ \int_{E_\ell} \mathbf{n}_{E_\ell \cap K_\ell} \cdot \underline{\mathbf{p}}_{h,1}^{(m,\omega_i)}|_{K_\ell}\ p_{k-1}\ ds,\ \ell = 2, 3, \cdots, N_i \end{cases},\ p_{k-1} \in P_{k-1}(E_\ell)\,,$$

$$\int_{E_{\ell+1}} \mathbf{n}_{E_{\ell+1} \cap K_\ell} \cdot \underline{\mathbf{p}}_{h,1}^{(m,\omega_i)}|_{K_\ell}\ p_{k-1}\ ds\ = \tag{6.7b}$$

$$\begin{cases} \int_{E_{\ell+1}} \mathbf{n}_{E_{\ell+1}} \cdot \hat{\underline{\mathbf{p}}}_{\partial K_\ell}^{m,(1,1)}|_{E_{\ell+1}}\ p_{k-1}\ ds,\ \ell = N_i \\ \int_{E_\ell} \mathbf{n}_{E_\ell} \cdot \hat{\underline{\mathbf{p}}}_{\partial K}^{m,(1,1)} \mathbf{n}_{E_\ell}\ p_{k-1}\ ds\ ,\ \ell = 1, 2, \cdots, N_i \end{cases},\ p_{k-1} \in P_{k-1}(E_{\ell+1}),$$

$p_{k-1} \in P_{k-1}(E_{\ell+1})$,

$$\mathbf{n}_E \cdot \underline{\mathbf{p}}_{h,1}^{(m,\omega_i)} = 0,\ E \in \mathcal{E}_h(K_\ell \cap \partial\omega_i)\,, \tag{6.7c}$$

$$\tau_m \int_{K_\ell} \underline{\mathbf{p}}_{h,1}^{(m,\omega_i)}|_{K_\ell} \cdot \nabla p_{k-2}\ dx = -\tau_m \int_{K_\ell} \nabla \cdot \underline{\mathbf{p}}_{h,1}^{(m,\omega_i)}\ p_{k-2}\ dx\ = \tag{6.7d}$$

$$- \int_K \Pi_{k-2}(u_{h,1}^m - u_{h,1}^{m-1})\ p_{k-2}\ dx\ +$$

$$\tau_m \int_K \Pi_{k-2}(\text{cof}(D^2 u_{h,1}^m + \Theta_h) : D^2(u_{h,2}^m + f_{h,int}^m) - g_h^m u_{h,1}^m)\ p_{k-2}\ dx\ +$$

$$\tau_m \int_K f_{h,ext}^m\ p_{k-2}\ dx,\ p_{k-2} \in P_{k-2}(K_\ell),$$

$$\int_{K_\ell} \underline{\mathbf{p}}_{h,1}^{(m,\omega_i)}|_{K_\ell} \cdot \mathbf{curl}(b_{K_\ell} p_{k-3})\ dx = \int_{K_\ell} \nabla \cdot D^2 u_{h,1}^m \cdot \mathbf{curl}(b_{K_\ell} p_{k-3})\ dx\ , \tag{6.7e}$$

$p_{k-3} \in P_{k-3}(K_\ell)$.

The construction of the equilibrated flux $\underline{\mathbf{p}}_{h,2}^{m,eq}$ can be done in a similar way.

For the construction of the equilibrated moment tensors $\underline{\underline{\mathbf{p}}}_{h,i}^{eq}, 1 \le i \le 2$, we begin with the specification of the degrees of freedom for tensors $\underline{\underline{\mathbf{p}}} = (p_{ij})_{i,j=1}^d \in \underline{\underline{\mathbf{V}}}_h$. We note that

$$\dim\ P_k(K)^{2\times 2} = 2(k+1)(k+2). \tag{6.8}$$

Lemma 6.2. *Any* $\underline{\underline{\mathbf{p}}} \in P_k(K)^{2\times 2}$ *with* $\underline{\mathbf{p}}^{(i)} = (p_{i1}, p_{i2})^T, 1 \le i \le 2$, *is uniquely determined by the following degrees of freedom (DOF)*

$$\int_E \underline{\underline{\mathbf{p}}}\ \mathbf{n}_E \cdot \underline{\mathbf{p}}_k\ ds, \quad \underline{\mathbf{p}}_k \in P_k(E)^2,\ E \in \mathcal{E}_h(\partial K), \tag{6.9a}$$

$$\int_K \underline{\underline{\mathbf{p}}} : \nabla \underline{\mathbf{p}}_{k-1}\ dx, \quad \underline{\mathbf{p}}_{k-1} \in P_{k-1}(K)^2 \backslash P_0(K)^2, \tag{6.9b}$$

$$\int_K \underline{\mathbf{p}}^{(i)} \cdot \mathbf{curl}(b_K p_{k-2})\ dx, \quad p_{k-2} \in P_{k-2}(K),\ 1 \le i \le 2. \tag{6.9c}$$

The numbers of degrees of freedom (DOF) associated with (6.9a)-(6.9c) are as follows

DOF (6.9a) $= 6(k+1)$,

DOF (6.9b) $= k(k+1) - 2$,

DOF (6.9c) $= (k-1)k$

and sum up to the right-hand side in (6.8).

Proof. The interpolation conditions for $\underline{\mathbf{p}}^{(1)}$ and $\underline{\mathbf{p}}^{(2)}$ are separated. The vector field $\underline{\mathbf{p}}^{(i)}$ (for $1 \le i \le 2$) is determined by the degrees of freedom

$$\int_E \mathbf{n}_E \cdot \underline{\mathbf{p}}^{(i)} p_k \, ds, \quad p_k \in P_k(E),\ E \in \mathcal{E}_h(\partial K),$$

$$\int_K \underline{\mathbf{p}}^{(i)} \cdot \nabla p_{k-1} \, dx, \quad p_{k-1} \in P_{k-1}(K) \backslash P_0(K),$$

$$\int_K \underline{\mathbf{p}}^{(i)} \cdot \mathrm{curl}(b_K p_{k-2}) \, dx, \quad p_{k-2} \in P_{k-2}(K) .$$

By applying Lemma 6.1 we conclude that there is a unique solution. □

Lemma 6.3. *Let* $\underline{\underline{\mathbf{q}}} = (\underline{\mathbf{q}}^{(1)}, \underline{\mathbf{q}}^{(2)}) \in P_k(K)^{2\times 2}$. *Then there exists a positive constant* $C_E^{(2)}$, *depending only on the polynomial degree* k *and the local geometry of the triangulation* $\mathcal{T}_h$, *such that*

$$\int_K |\underline{\underline{\mathbf{q}}}|^2 \, dx \le C_E^{(2)} \Big(\sum_{E \in \mathcal{E}_h(\partial K)} h_E \int_E |\underline{\underline{\mathbf{q}}}\, \mathbf{n}_E|^2 \, ds + h_K^2 \int_K |\nabla \cdot \underline{\underline{\mathbf{q}}}|^2 \, dx + \tag{6.10}$$

$$h_K^2 \sum_{i=1}^{2} max \Big\{ \int_K |\underline{\mathbf{q}}^{(i)} \cdot \mathbf{curl}(b_K p_{k-2})|^2 \, dx;\ p_{k-2} \in P_{k-2}, \max_{x \in K} |p_{k-2}(x)| \le 1 \Big\} \Big).$$

Proof. As in the proof of Lemma 6.1, the estimate (6.10) follows by standard scaling arguments. □

Now, for the construction of the equilibrated moment tensor $\underline{\underline{\mathbf{p}}}_{h,1}^{m,eq}$ we set

$$\underline{\mathbf{z}}_h^{(m,1)} := \Big(\frac{\partial^2 u_{h,1}^m}{\partial x_1^2}, \frac{\partial^2 u_{h,1}^m}{\partial x_1 \partial x_2}\Big)^T, \quad \underline{\mathbf{z}}_h^{(m,2)} := \Big(\frac{\partial^2 u_{h,1}^m}{\partial x_1 \partial x_2}, \frac{\partial^2 u_{h,1}^m}{\partial x_2^2}\Big)^T.$$

We construct $\underline{\underline{\mathbf{p}}}_{h,1}^{m,eq} = (p_{ij}^{h,m,eq})_{i,j=1}^2, 1 \le m \le 2$, with

$$\underline{\mathbf{p}}_{h,m,eq}^{(i)} = (p_{i1}^{h,m,eq}, p_{i2}^{h,m,eq})^T, 1 \le i \le 2,$$

patchwise:

$$\underline{\underline{\mathbf{p}}}_{h,1}^{m,eq} = \sum_{i=1}^{n_h} \underline{\underline{\mathbf{p}}}_{h,1}^{(m,\omega_i)}. \tag{6.11}$$

For a patch ω_i, we construct $\underline{\underline{\mathbf{p}}}_{h,1}^{(m,\omega_i)}$ such that

$$\underline{\underline{\mathbf{p}}}_{h,1}^{(m,\omega_i)}|_{K_\ell} \in \mathbf{BDM}_k(K_\ell) , \tag{6.12}$$

$$\nabla \cdot \underline{\underline{\mathbf{p}}}_{h,1}^{(m,\omega_i)} = \underline{\mathbf{p}}_{h,1}^{(m,\omega_i)} \text{ in } \omega_i, \quad \underline{\underline{\mathbf{p}}}_{h,1}^{(m,\omega_i)} \mathbf{n}_E = \underline{\hat{\mathbf{p}}}_{\partial K}^{m,(1,1)}|_E,\ E \in \mathcal{E}_h(\text{int } \omega_i),\ 1 \le \ell \le N_i,$$

where, denoting by $\varphi_h^{(x_i)} \in V_h^{(k)}$ the nodal basis function associated with x_i, $\underline{\mathbf{p}}_{h,m}^{\omega_i}$ and $\underline{\hat{\mathbf{p}}}_{\partial K}^{m,\omega_i}|_E$ are given by

$$\underline{\mathbf{p}}_{h,1}^{(m,\omega_i)} := \varphi_h^{(x_i)} \underline{\underline{\mathbf{p}}}_{h,1}^{m,eq}, \quad \underline{\underline{\hat{\mathbf{p}}}}_{\partial K}^{m,\omega_i}|_E \mathbf{n}_E := \varphi_h^{(x_i)} (\underline{\hat{\mathbf{p}}}_{\partial K}^{m,(1,1)})|_E ,1 \le i \le n_h. \tag{6.13}$$

Moreover, we define $\underline{\mathbf{z}}_{h,\nu}^{(m,\omega_i)}$ according to

$$\underline{\mathbf{z}}_{h,\nu}^{(m,\omega_i)} := \varphi_h^{(x_i)} \underline{\mathbf{z}}_h^{(m,\nu)},\ 1 \le \nu \le 2. \tag{6.14}$$

Case 1 ($x_i \in \mathcal{E}_h(\partial E \cap \Omega)$): For $\ell = 1, 2, \cdots, N_i$ we compute $\underline{\underline{\mathbf{p}}}_{h,1}^{(m,\omega_i)}|_{K_\ell}$ with $\underline{\underline{\mathbf{p}}}_{h,1}^{(m,\omega_i)}|_{K_\ell} \in \mathbf{BDM}_k(K_\ell)$ according to

$$\int_{E_\ell} \underline{\underline{\mathbf{p}}}_{h,1}^{(m,\omega_i)}|_{K_\ell} \mathbf{n}_{E_\ell \cap K_\ell} \cdot \underline{\mathbf{p}}_k \, ds = \tag{6.15a}$$

$$\begin{cases} \int_{E_\ell} \Big(\underline{\hat{\mathbf{p}}}_{\partial K_\ell,1}^{(m,\omega_i)}\Big)|_{E_\ell},\ \ell = 1 \\ \int_{E_\ell} \underline{\underline{\mathbf{p}}}_{h,1}^{(m,\omega_i)}|_{K_\ell} \mathbf{n}_{E_\ell \cap K_\ell} \cdot \underline{\mathbf{p}}_k \, ds,\ \ell = 2, 3, \cdots, N_i \end{cases}, \underline{\mathbf{p}}_k \in P_k(E_\ell)^2 ,$$

$$\int_{E_{\ell+1}} \underline{\underline{\mathbf{p}}}_{h,1}^{(m,\omega_i)}|_{K_\ell} \mathbf{n}_{E_{\ell+1} \cap K_\ell} \cdot \underline{\mathbf{p}}_k \, ds = \tag{6.15b}$$

$$\begin{cases} \int_{E_{\ell+1}} \cdot \underline{\hat{\mathbf{p}}}_{\partial K_\ell,1}^{(m,\omega_i)}|_{E_{\ell+1}} \cdot \underline{\mathbf{p}}_k \, ds,\ \ell = N_i \\ \int_{E_\ell} \underline{\underline{\hat{\mathbf{p}}}}_{\partial K,1}^{(m,\omega_i)} \mathbf{n}_{E_\ell} \cdot \underline{\mathbf{p}}_k \, ds, \underline{\mathbf{p}}_k \in P_k(E_{\ell+1})^2, \ell = 1, 2, \cdots, N_i \end{cases}, \quad \underline{\mathbf{p}}_k \in P_k(E_{\ell+1})^2,$$

$$\underline{\underline{\mathbf{p}}}_{h,1}^{(m,\omega_i)}\ \mathbf{n}_E = \mathbf{0},\ E \in \mathcal{E}_h(K_\ell \cap \partial\omega_i) , \tag{6.15c}$$

$$\int_{K_\ell} \underline{\underline{\mathbf{p}}}_{h,1}^{(m,\omega_i)}|_{K_\ell} : \nabla \underline{\mathbf{p}}_{k-1} \, dx = - \int_{K_\ell} \underline{\mathbf{p}}_{h,1}^{(m,\omega_i)} \cdot \underline{\mathbf{p}}_{k-1} \, dx + \tag{6.15d}$$

$$\int_{\partial K_\ell} \underline{\underline{\mathbf{p}}}_{h,1}^{(m,\omega_i)}|_{K_\ell} \mathbf{n}_{\partial K_\ell} \cdot \underline{\mathbf{p}}_{k-1} \, ds \ ,\ \underline{\mathbf{p}}_{k-1} \in P_{k-1}(K_\ell)^2,$$

$$\int_{K_\ell} \underline{\mathbf{p}}_{h,m,eq}^{(\nu)}|_{K_\ell} \cdot \mathbf{curl}(b_{K_\ell} p_{k-2}) \, dx = \int_{K_\ell} \underline{\mathbf{z}}_h^{(m,\nu)} \cdot \mathbf{curl}(b_{K_\ell} p_{k-2}) \, dx , \tag{6.15e}$$

$1 \le \nu \le 2,\ p_{k-2} \in P_{k-2}(K_\ell)$.

The construction of the equilibrated moment tensor $\underline{\underline{\mathbf{p}}}_{h,2}^{m,eq}$ can be done in a similar way.

7. Relationship with a residual-type a posteriori error estimator

Using the techniques from [36], the residual-type a posteriori error estimator for the dynamic quasi-static von Kármán equations with homogeneous Dirichlet boundary conditions reads as follows

$$\eta_h^{m,res} := \sum_{i=1}^{8} \sum_{K \in \mathcal{T}_h} \eta_{K,i}^{m,res} + \sum_{i=1}^{6} \sum_{K \in \mathcal{T}_h} \widetilde{\eta}_{K,i}^{m,res}. \tag{7.1}$$

For $1 \le m \le M$ the element residuals $\eta_{K,i}^{m,res}, 1 \le i \le 8$, and $\widetilde{\eta}_{K,i}^{m,res}, 1 \le i \le 6$, are given by

$$\eta_{K,1}^{m,res} := h_K^2 \int_K |\Pi_{k-2}(u_{h,1}^m) + \tau_m \Delta^2 u_{h,1}^m - \tau_m \Pi_{k-2}(\mathrm{cof}(\Delta^2 u_{h,1}^m + \tag{7.2a}$$

$$D^2\Theta_h) : (D^2 u_{h,2}^m + f_{h,int}^m) - g_h^m u_{h,1}^m) - \Pi_{k-2}(u_{h,1}^{m-1}) - \tau_m f_{h,ext}^m|^2 \, dx,$$

$$\eta_{K,2}^{m,res} := h_K^2 \int_K |(\Delta^2 u_{h,2}^m + \Pi_{k-2}(\mathrm{cof}((D^2 u_{h,1}^m + 2\Theta_h) : D^2 u_{h,1}^m))|^2 \, dx, \tag{7.2b}$$

$$\eta_{K,i+2}^{m,res} := \kappa_E \sum_{E \in \mathcal{E}_h(K)} h_E \int_E |\mathbf{n}_E \cdot [\nabla \cdot D^2 u_{h,i}^m|^2 \, ds,\ 1 \le i \le 2, \tag{7.2c}$$

$$\eta_{K,i+4}^{m,res} := \kappa_E \sum_{E \in \mathcal{E}_h(K)} h_E \int_E |[D^2 u_{h,i}^m]_E \mathbf{n}_E|^2 \, ds,\ 1 \le i \le 2, \tag{7.2d}$$

$$\eta_{K,i+6}^{m,res} := \kappa_E \sum_{E \in \mathcal{E}_h(K)} \int_E |[\nabla u_{h,i}^m \otimes \mathbf{n}_E]_E|^2 \, ds,\ 1 \le i \le 2, \tag{7.2e}$$

$$\widetilde{\eta}_{K,i}^{m,res} := (\eta_{K,i}^m)^{1/2} |D^2 u_{h,1}^m|_{DG,\Omega},\ i = 1, 3, 5, \tag{7.2f}$$

$$\widetilde{\eta}_{K,i}^{m,res} := (\eta_{K,i}^m)^{1/2} |D^2 u_{h,2}^m|_{DG,\Omega},\ i = 2, 4, 6, \tag{7.2g}$$

where

$$\kappa_E := \begin{cases} \frac{1}{2}, \ E \in \mathcal{E}_h(\Omega) \\ 1, \ E \in \mathcal{E}_h(\Gamma) \end{cases}, \tag{7.2h}$$

and

$$|D^2 u_{h,i}^m|_{DG,\Omega} := \Big(\sum_{K \in \mathcal{T}_h} \int_K |D^2 u_{h,i}^m|^2 \, dx \Big)^{1/2}, \ 1 \le i \le 2. \tag{7.2i}$$

We further define data oscillations $\widetilde{osc}_{h,i}^m, 1 \le i \le 2$, according to

$$\widetilde{osc}_{h,1}^m := \begin{cases} (\mathrm{osc}_{h,1}^m)^{1/2}(|\nabla u_{h,1}^m|_{DG,\Omega} + |\nabla u_{h,2}^m|_{DG,\Omega}), \ k = 2 \\ (\mathrm{osc}_{h,1}^m)^{1/2}(|D^2 u_{h,1}^m|_{DG,\Omega} + |D^2 u_{h,2}^m|_{DG,\Omega}), \ k \ge 3 \end{cases}, \tag{7.2j}$$

$$\widetilde{osc}_{h,2}^m := (\mathrm{osc}_{h,2}^m)^{1/2}(|D^2 u_{h,1}^m|_{DG,\Gamma} + |D^2 u_{h,2}^m|_{DG,\Gamma}), \tag{7.2k}$$

where $|\nabla u_{h,i}^m|_{DG,\Omega}$ and $|D^2 u_{h,i}^m|_{DG,\Gamma}, 1 \le i \le 2$, are given by

$$|\nabla u_{h,i}^m|_{DG,\Omega} := \Big(\sum_{K \in \mathcal{T}_h} |\nabla u_{h,i}^m|^2 \, dx \Big)^{1/2}, \tag{7.2l}$$

$$|D^2 u_{h,i}^m|_{DG,\Gamma} := \Big(\sum_{E \in \mathcal{E}_h} (\Gamma) |D^2 u_{h,i}^m|^2 \, ds \Big)^{1/2}. \tag{7.2m}$$

The following result establishes the relationship between the equilibrated and the residual a posteriori error estimator.

Theorem 7.1. *Let $\mathbf{u}_h \in \underline{\mathbf{V}}_h$ be the C^0IPDG approximation as given by (4.22) and let $\eta_h^{eq}, \eta_{h,i}^{res}, 1 \le i \le 8, \tilde{\eta}_{h,i}^{res}, 1 \le i \le 6$, and $osc_{h,i}, 1 \le i \le 3, \widetilde{osc}_{h,i}, 1 \le i \le 2$, be the equilibrated and the residual a posteriori error estimators as well as the data oscillations as given by (5.36), (7.2), and (5.27). Then there exists a constant $C_{res} > 0$, depending on $c_R, C_{rec}, \alpha_i, C_E^{(i)}, C_{PF}^{(i)}, 1 \le i \le 2$, such that*

$$\eta_{h,1}^{m,eq} \le C_{res} \Big(\sum_{i=1}^{8} \eta_{h,i}^{m,res} + \sum_{i=1}^{6} \tilde{\eta}_{h,i}^{m,res} + osc_{h,1}^m + osc_{h,3}^m + \widetilde{osc}_{h,1}^m + \widetilde{osc}_{h,2}^m \Big). \tag{7.3}$$

Moreover, if we use (4.29) in (5.35b), then κ_K^{eq} can be estimated from above in terms of residuals and data oscillations.

The assertion can be established by standard means.

Remark 7.2. Using techniques from [36], it can be shown that the residual-based error estimator is efficient.

8. Predictor-corrector continuation method

The time adaptivity used in this paper is dictated by the convergence of Newton's method for the numerical solution of the nonlinear IPDG approximation (4.22) and not by an upper bound for the discretization error in time, because the time steps predicted by the latter are much larger than those by the former.
Setting $\mathbf{x}^{m,i} := (x_1^{m,i}, \cdots, x_{N_h}^{m,i})^T, N_h := \dim V_h, 1 \le i \le 2$, the algebraic formulation of (4.22) leads to a nonlinear system of the form

$$\mathbf{F}(\mathbf{x}^{m,1}, \mathbf{x}^{m,2}, t_m) = \mathbf{0}, \tag{8.1}$$

with a continuously differentiable nonlinear mapping $\mathbf{F} : \mathbb{R}^{N_h} \times \mathbb{R}^{N_h} \times \mathbb{R}_+ \to \mathbb{R}^{N_h} \times \mathbb{R}^{N_h}$ Hence, the nonlinear system (8.1) can be solved by Newton's method. The problem is the appropriate choice of the time step sizes $\tau_m, 1 \le m \le M$, in order to guarantee convergence. In fact, a uniform choice $\tau_m = T/M$ only works, if M is chosen sufficiently large which would require an unnecessary huge amount of time steps. An appropriate way to overcome this difficulty is to consider (8.1) as a parameter dependent nonlinear system with the time as a parameter and to apply a predictor-corrector continuation strategy with an adaptive choice of the time steps (cf., e.g., [15,22–25]). Given $\mathbf{x}^{m-1,i}, 1 \le i \le 2$, the time step size $\tau_{m-1,0} = \tau_{m-1}$, and setting $\nu = 0$, where ν is a counter for the predictor-corrector steps, the predictor step for (8.1) consists of constant continuation leading to the initial guesses

$$\mathbf{x}^{(m,i,\nu)} = \mathbf{x}^{m-1,i}, \quad t_m = t_{m-1} + \tau_{m-1,\nu}, \ 1 \le i \le 2. \tag{8.2}$$

Setting $\ell_1 = 0$ and $\mathbf{x}^{(m,i,\nu,\ell_1)} = \mathbf{x}^{(m,i,\nu)}$, for $\ell_1 \le \ell_{max}$, where $\ell_{max} > 0$ is a pre-specified maximal number, the Newton iteration

$$\partial_{\mathbf{x}^m} \mathbf{F}(\mathbf{x}^{(m,1,\nu,\ell_1)}, \mathbf{x}^{(m,2,\nu,\ell_1)}, t_m) \Delta \mathbf{x}^{(m,\nu,\ell_1)} = -\mathbf{F}(\mathbf{x}^{(m,1,\nu,\ell_1)}, \mathbf{x}^{(m,2,\nu,\ell_1)}, t_m), \tag{8.3}$$
$$\mathbf{x}^{(m,\nu,\ell_1+1)} = \mathbf{x}^{(m,\nu,\ell_1)} + \Delta \mathbf{x}^{(m,\ell,\ell_1)}, \quad \ell_1 \ge 0,$$

serves as a corrector whose convergence is monitored by the contraction factor

$$\Lambda_{\mathbf{x}}^{(m,\nu,\ell_1)} = \frac{\|\overline{\Delta \mathbf{x}^{(m,\nu,\ell_1)}}\|}{\|\Delta \mathbf{x}^{(m,\nu,\ell_1)}\|}, \tag{8.4}$$

where $\overline{\Delta \mathbf{x}^{(m,\nu,\ell_1)}}$ is the solution of the auxiliary Newton step

$$\partial_{\mathbf{x}} \mathbf{F}(\mathbf{x}^{(m,1,\nu,\ell_1)}, \mathbf{x}^{m,2,\nu,\ell-1}, t_m) \overline{\Delta \mathbf{x}^{(m,\nu,\ell_1)}} = -\mathbf{F}(\mathbf{x}^{(m,1,\nu,\ell_1+1)}, \mathbf{x}^{m,2,\nu,\ell_1+1}, t_m). \tag{8.5}$$

If the contraction factor satisfies

$$\Lambda_{\mathbf{x}}^{(m,\nu,\ell_1)} < \frac{1}{2}, \tag{8.6}$$

we set $\ell_1 = \ell_1 + 1$. If $\ell_1 > \ell_{max}$, both the Newton iteration and the predictor-corrector continuation strategy are terminated indicating non-convergence. Otherwise, we continue the Newton iteration (8.3). If (8.6) does not hold true, we set $\nu = \nu + 1$ and the time step is reduced according to

$$\tau_{m,\nu} = \max\Big(\frac{\sqrt{2} - 1}{\sqrt{4\Lambda_{\mathbf{x}}^{(m,\nu,\ell_1)} + 1} - 1} \ \tau_{m,\nu-1}, \tau_{min} \Big), \tag{8.7}$$

where $\tau_{min} > 0$ is some pre-specified minimal time step. If $\tau_{m,\nu} > \tau_{min}$, we go back to the prediction step (8.2). Otherwise, the predictor-corrector strategy is stopped indicating non-convergence. The Newton iteration is terminated successfully, if for some $\ell_1^* > 0$ the relative error of two subsequent Newton iterates satisfies

$$\frac{\|\mathbf{x}^{(m,\nu,\ell_1^*)} - \mathbf{x}^{(m,\nu,\ell_1^*-1)}\|}{\|\mathbf{x}^{(m,\nu,\ell_1^*)}\|} < \varepsilon_T \tag{8.8}$$

for some pre-specified accuracy $\varepsilon_T > 0$.
In this case, we set

$$\mathbf{x}^m = \mathbf{x}^{(m,\nu,\ell_1^*)} \tag{8.9}$$

and predict a new time step according to

$$\tau_m = \frac{(\sqrt{2} - 1) \ \|\Delta \mathbf{x}^{(m,\nu,0)}\|}{2\Lambda_{\mathbf{x}}^{(m,\nu,0)} \ \|\mathbf{x}^{(m,\nu,0)} - \mathbf{x}^m\|} \ \tau_{m,\nu}, \tag{8.10}$$

where amp > 1 is a pre-specified amplification factor for the time step sizes. We set $m = m + 1$ and begin new predictor-corrector iterations for the time interval $[t_m, t_{m+1}]$.

Remark 8.1. The adaptivity in time is based on the predictor-corrector continuation strategy which guarantees convergence of the Newton iteration. We have also implemented the time error estimator from [36] of the reference list. It turned out that the time steps predicted by the predictor-corrector continuation strategy were always much smaller than the time steps predicted by the estimator from [36]. Also, the accumulation of the error in time by the predictor-corrector continuation strategy was much smaller than for the estimator from [36].
In [6], a conditional a posteriori error bound for semilinear convection-diffusion problems has been derived where the condition can be implemented in practice by the convergence of Newton's method. It would be interesting to see whether this also holds true for the quasi-static von Kármán equations. However, a rigorous analysis would enlarge the size of the paper significantly and should thus be left for future research.

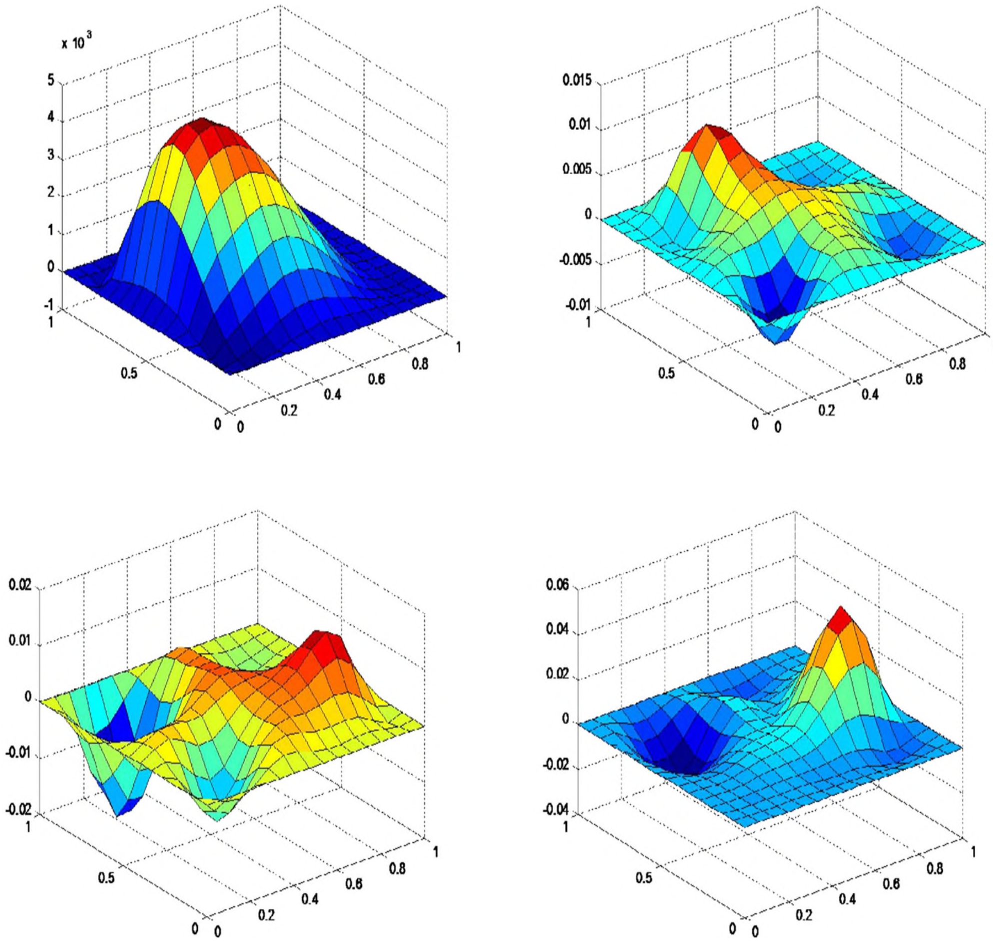

Fig. 2. Computed solution component $u_{h,1}$ at time $t = 0.1$ sec (top left), $t = 2.5$ sec (top right), $t = 6.5$ sec (bottom left), $t = 80.0$ sec (bottom right).

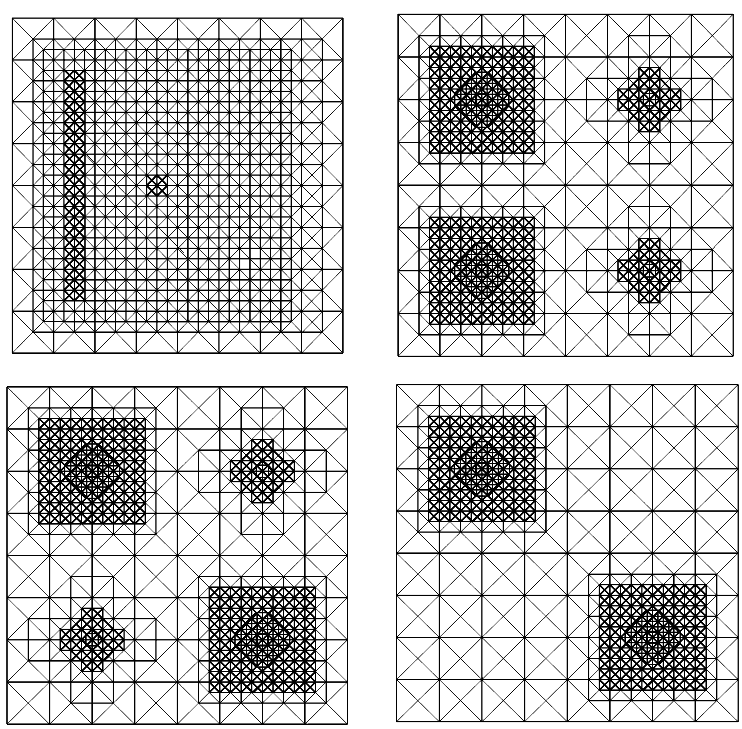

Fig. 3. Adaptively generated meshes for polynomial degree $k = 2$ at time $t = 0.1$ sec (top left), $t = 2.5$ sec (top right), $t = 6.5$ sec (bottom left), $t = 80.0$ sec (bottom right).

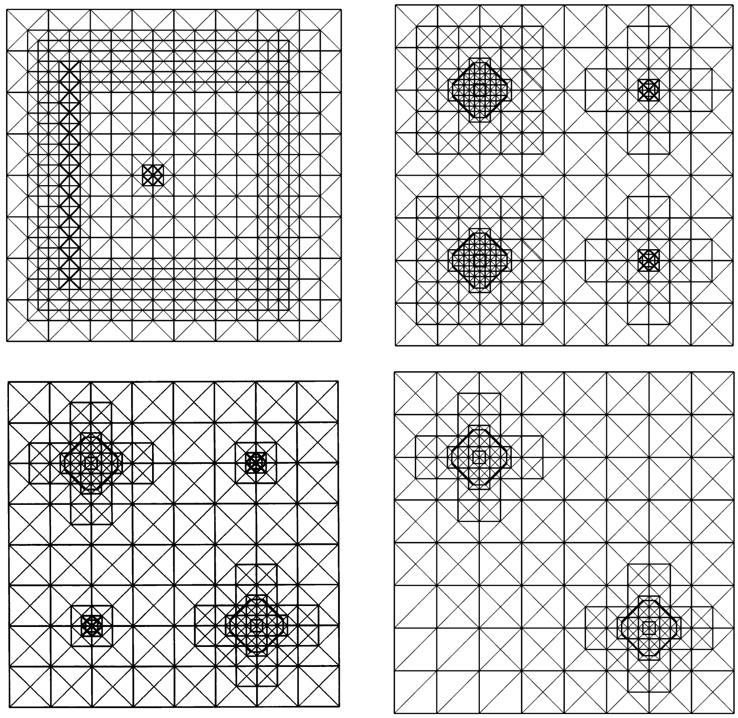

Fig. 4. Adaptively generated meshes for polynomial degree $k = 3$ at time $t = 0.1$ sec (top left), $t = 2.5$ sec (top right), $t = 6.5$ sec (bottom left), $t = 80.0$ sec (bottom right).

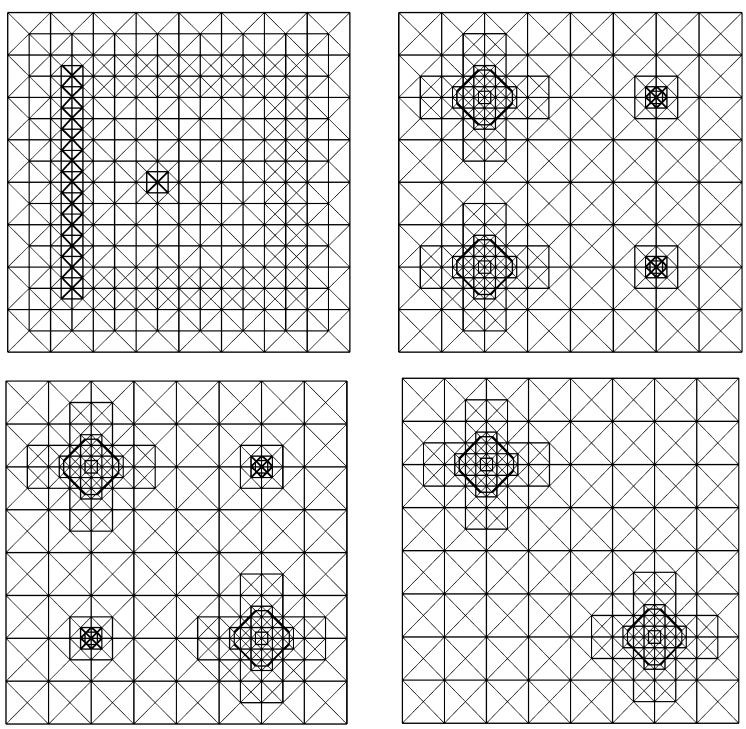

Fig. 5. Adaptively generated meshes for polynomial degree $k = 4$ at time $t = 0.1$ sec (top left), $t = 2.5$ sec (top right), $t = 6.5$ sec (bottom left), $t = 80.0$ sec (bottom right).

9. Numerical results

As a numerical example, we have chosen the example from [31] with Ω as the square $\Omega := (-1, +1)^2$ and final time $T = 100$ sec. The data for the problem are given as follows:

$$\Theta = -0.5 x_1 x_2 (x_1 - 1)(x_2 - 1)\exp(-x_1^2 - x_2^2), \quad f_{int} = 0.8 x_1 \exp(-x_1^2 - x_2^2),$$

$$g = 0.8 x_1 (\exp(-x_1^2) - \exp(-x_1^2)), \quad f_{ext} = \sin^2(\pi x_1)\cos^2(\pi x_1).$$

At each time step, we have implemented the C^0IPDG approximation (4.22) with the penalty parameters $\alpha_i, 1 \le i \le 2$, chosen as $\alpha_1 = 12.0\ k^2$ and $\alpha_2 = 2.5\ k^6$ for the polynomial degrees $k = 2, k = 3$, and $k = 4$. Further, we have implemented the adaptive algorithm based on the equilibrated error estimator η_h^{eq} by Dörfler marking [17], i.e., given a bulk parameter $\gamma \in (0, 1)$, we have selected a set $\mathcal{M}_h \subset \mathcal{T}_h$ according to

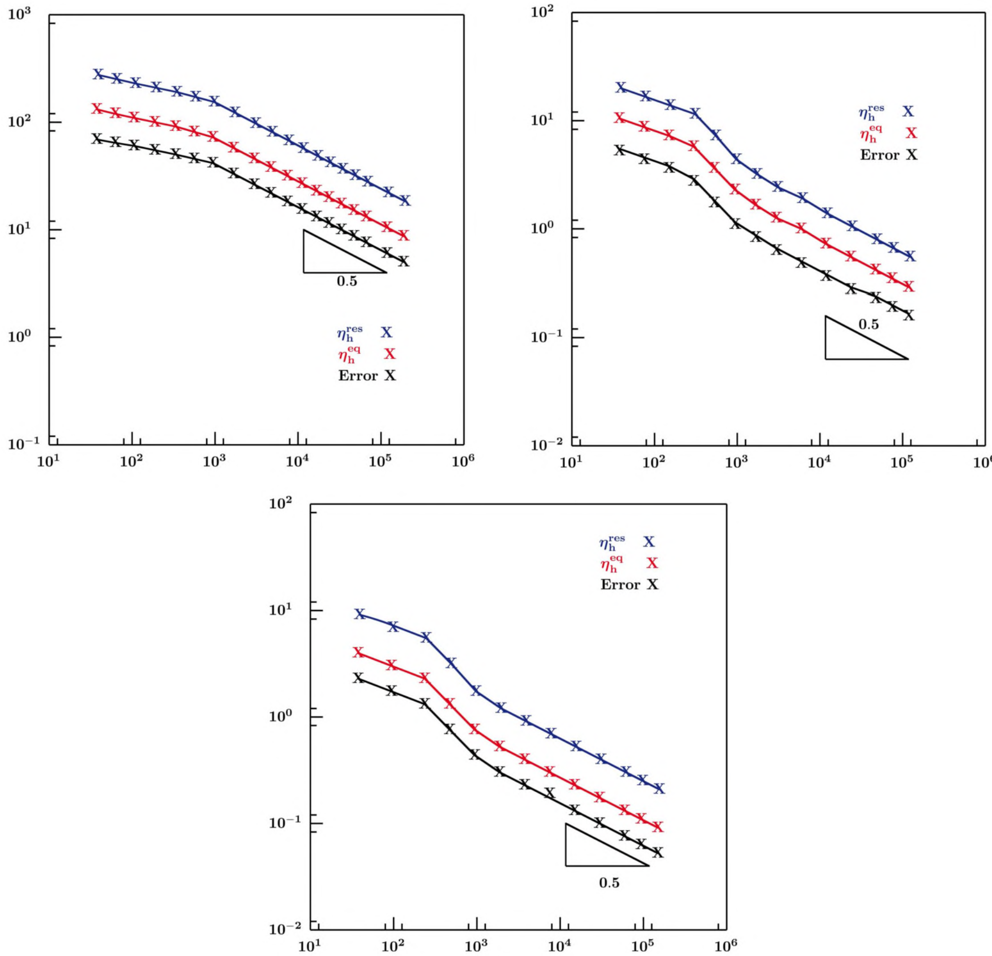

Fig. 6. Discretization error, equilibrated a posteriori error estimator, and residual-type a posteriori error estimator for polynomial degree $k = 2$ (top left), $k = 3$ (top right), and $k = 4$ (bottom) at time $t = 0.1$ sec.

$$\gamma \sum_{K \in \mathcal{T}_h} \eta_K^{eq} \leq \sum_{K \in \mathcal{M}_h} \eta_K^{eq}$$

and we have refined elements $K \in \mathcal{M}_h$ by newest vertex bisection. In case of the residual-based error estimator η_h^{res} we have implemented the adaptive refinement likewise. For $k = 4$ we have chosen the initial triangulation from a partitioning of Ω into squares of width $1/8$ and then drawing diagonals from bottom left to top right in order to obtain a simplicial triangulation with right isosceles. To reduce computational work, for $k = 3$ the initial triangulation has been chosen as the final triangulation for $k = 4$, and likewise, for $k = 2$ the initial triangulation has been chosen as the final triangulation for $k = 3$. The data for the predictor-corrector continuation strategy have been chosen as follows:

$$\nu_{max} = 50, \quad \tau_{min} = 1.0 \cdot 10^{-6}, \quad \varepsilon = 1.0 \cdot 10^{-3}, \quad \text{amp} = 1.2.$$

Fig. 2 displays the numerically computed solution component $u_{h,1}$ of the dynamic quasi-static Kármán equations at time $t = 0.1$ sec (top left), $t = 2.5$ sec (top right), $t = 6.5$ sec (bottom left), $t = 8.0$ sec (bottom right).

The adaptively generated meshes based on the equilibrated a posteriori error estimator with $\gamma = 0.4$ for the polynomial degree $k = 2$ at time $t = 0.1$ sec, $t = 2.5$ sec, $t = 6.5$ sec, and $t = 80.0$ sec are shown in Fig. 3. Those for the polynomial degrees $k = 3$ and $k = 4$ at those times are displayed in Fig. 4 and Fig. 5. We observe a significant refinement in regions with steep gradients.

The adaptively generated meshes based on the residual-type a posteriori error estimator look similarly and are therefore omitted.

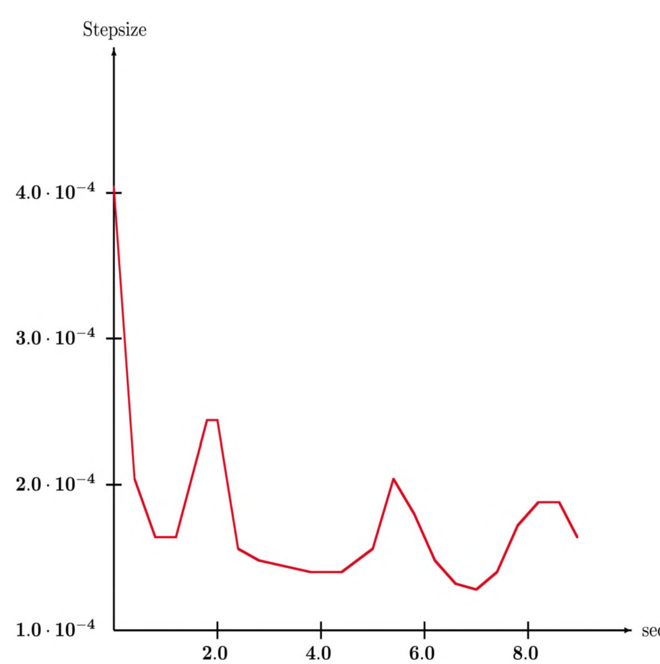

Fig. 7. Adaptively chosen time-steps from $t = 0.0$ sec to $t = 9.0$ sec.

Since the exact solution of the initial-boundary value problem for the dynamic quasi-static von Kármán equations under consideration is not known, in order to compute the global discretization error, at each respective time instant we have chosen the last generated adaptive mesh, refined uniformly twice, computed an approximate solution by the C^0IPDG method with respect to that mesh, and taken this approximation as a substitute for the exact solution. Fig. 6 shows the discretization error, the equilibrated a posteriori error estimator, and the residual-type a posteriori error estimator for polynomial degree $k = 2$ (top left), $k = 3$ (top right), and $k = 4$ (bottom) at time $t = 0.1$ sec. In all cases we observe the optimal convergence rate of 0.5. For $k = 3$ and $k = 4$ the decay is faster than 0.5 in the pre-asymptotic regime, but approaches 0.5 asymptotically. The equilibrated error estimator is smaller than the residual-based error estimator by approximately $1/2$ of an order of magnitude. The results at times $t = 2.5$ sec, $t = 6.5$ sec, and $t = 80.0$ sec are similar and are therefore omitted.

Finally, Fig. 7 displays the adaptively chosen time-steps, obtained by the predictor-corrector continuation strategy, from $t = 0.0$ sec to $t = 6.5$ sec. We observe a drop in the time-steps when there is a significant change in the location of the maxima and minima of the solution.

Data availability

Data will be made available on request.

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