Improving the theoretical upper bound for the expected number of shadow-vertices in the Rotation-Symmetry-Model*

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Abstract

In the papers [3] and [4] we had found an upper bound for the expected number of shadow vertices for linear optimization problems which are distributed according to the Rotation-Symmetry-Model, denoted by $E_{m,n}(S)$, with m inequality-restrictions and n variables. This bound was a function $m^{\frac{1}{n-1}} \cdot n^3 \cdot Const.$
It applies to all dimension-pairs ($m \geq n$) and all distributions, which guarantee independence, rotation-symmetry and identity of the distribution of the restriction vectors.
However, the bound is not sharp in dependency upon n. An asymptotical lower bound had been proven in the form $m^{\frac{1}{n-1}} \cdot n^2 \cdot Const.$, where asymptotic means n fixed, $m \to \infty$. At the same time we know from [2] and [4] that an asymptotical upper bound of the form $m^{\frac{1}{n-1}} \cdot n^2 \cdot Const.$ exists. We understand the discrepancy to the general bound, when we consider the real aim of the analysis. It should deliver a common upper bound for all rotation-symmetric distributions and all possible pairs of dimensions ($m \geq n$). This task leads to technical complications in the proof coming from the difficulty of exact calculations of space angles and from the requirement of dealing with any kind of rotation-symmetric distribution simultaneously.
In the paper in hand we increase the precision significantly by using an improved method for estimating spherical angles.
Finally we obtain a general new upper bound of size

$$m^{\frac{1}{n-1}} \cdot n^{\frac{5}{2}} \cdot Const.$$

*dedicated to Prof. Dr. Helmut Brakhage on the occasion of his retirement

1 Introduction and Notation

In this paper we try to sharpen and to simplify the polynomiality-proof from [4] for $E_{m,n}(S)$, which is the expected number of shadow-vertices in linear programming problems of the type

$$\text{maximize } v^T x \quad \text{subject to} \quad a_1^T x \le 1, \dots, a_m^T x \le 1 \quad \text{where} \quad x, v, a_1, \dots, a_m \in \mathbb{R}^n. \tag{1}$$

We assume a distribution of the linear programming problems corresponding to the stochastic model (RSM):

> The vectors $a_1, \dots, a_m, v$ and an auxiliary vector u are distributed on $\mathbb{R}^n \setminus \{0\}$ independently, identically and symmetrically under rotations. (2)

This assumption justifies concentration on the nondegeneracy-cases, which are almost sure.

> Nondegeneracy is valid, if any n vectors out of $\{a_1, \dots, a_m, v, u\}$ are linear independent, and if any $n+1$ vectors out of $\{a_1, \dots, a_m, v, u\}$ are in general position. (3)

For problems of type $E_{m,n}(S)$ our figure delivers the expected number (1) of pivot steps when m inequalities and n variables are present and when we project on $Span(u, v)$.

> Let X be the feasible region $\{x \mid a_1^T x \le 1, \dots, a_m^T x \le 1\}$. A vertex x_* is called shadow vertex of X with regard to u and v, if orthogonal projection on $Span(u, v)$ maps x_* on a vertex of the two-dimensional image of X in $Span(u, v)$. (4)

For S we could give a dual characterization in the space of the a_i in [4]. This characterization admits an integral representation for $E_{m,n}(S)$ by means of stochastic geometry. The latter describes the probability, that a certain basic solution of problem (1) actually is a shadow vertex, and it multiplies that probability with the number of candidates, i.e. $\binom{m}{n}$. The integral representation achieved by that way can be simplified significantly by two coordinate-transformations. The result is the following form:

$$\begin{aligned} E_{m,n}(S) \;=\; & \binom{m}{n} \cdot n \cdot \{(n-2)!\}^2 \cdot \lambda_{n-1}(\omega_n) \cdot \lambda_{n-2}(\omega_{n-1}) \cdot \\ & \int_0^1 G(h)^{m-n} \int_{\mathbb{R}^{n-1}} \int_0^{\sqrt{1-h^2}} |\Theta - c_n^{n-1}| \int_{\mathbb{R}^{n-2}} \cdots \int_{\mathbb{R}^{n-2}} |\lambda_{n-2}\{KH(c_1, \dots, c_{n-1})\}|^2 \cdot \\ & W(c_1, \dots, c_{n-1})\, f(c_1) \cdots f(c_{n-1})\, d\bar{c}_1 \dots d\bar{c}_{n-1}\, d\Theta f(c_n)\, d\bar{c}_n\, dh. \end{aligned} \tag{5}$$

That integral geometrically describes the expected number of those facets of the polytope $KH(a_1,\ldots,a_m)$ which are intersected by $Span(u,v)$. Very profitable is the comparison with a closely related figure $E_{m,n}(Z)$, which gives the expected number of facets being intersected by the ray $\mathbb{R}^+v$. The integral formula for that purpose is:

$$\begin{aligned} E_{m,n}(Z) \quad = \quad & \binom{m}{n} \cdot n \cdot \{(n-2)!\}^2 \cdot \lambda_{n-1}(\omega_n) \cdot \lambda_{n-2}(\omega_{n-1}) \cdot \\ & \int_0^1 G(h)^{m-n} \int_{\mathbb{R}^{n-1}} \int_0^{\sqrt{1-h^2}} |\Theta - c_n^{n-1}| \int_{\mathbb{R}^{n-2}} \cdots \int_{\mathbb{R}^{n-2}} |\lambda_{n-2}\{KH(c_1,\ldots,c_{n-1})\}|^2 \cdot \\ & V(c_1,\ldots,c_n)\, f(c_1)\cdots f(c_{n-1})\, d\overline{\overline{c}}_1 \cdots d\overline{\overline{c}}_{n-1}\, d\Theta f(c_n)\, d\overline{c}_n\, dh. \end{aligned} \tag{6}$$

This comparison turns out to be advantageous, since we know that

$$E_{m,n}(Z) \leq 1, \tag{7}$$

and therefore

$$E_{m,n}(S) \leq \frac{E_{m,n}(S)}{E_{m,n}(Z)}. \tag{8}$$

The similarity of the integrals in numerator and denominator simplifies the evaluation of the quotient. Its estimation becomes easier than that of (5).
But first we have to explain the notation.

We use the abbreviations *Span*, KH, KK for linear hull, convex hull, convex cone respectively.
Ω_k resp. ω_k denote the unit ball resp. the unit sphere in $\mathbb{R}^k$.

$$\Omega_k := \{x \mid \|x\| \leq 1,\ x \in \mathbb{R}^k\} \text{ and } \omega_k := \{x \mid \|x\| = 1,\ x \in \mathbb{R}^k\}. \tag{9}$$

λ_k stands for the k-dimensional Lebesgue-measure. Hence

$$\lambda_n(\Omega_n) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n+2}{2})} \text{ and } \lambda_{n-1}(\omega_n) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}. \tag{10}$$

$c_1,\ldots,c_n$ are (column-)vectors in $\mathbb{R}^n$, i.e. $c = (c^1,\ldots,c^n)^T$.
$\overline{c} = (c^1,\ldots,c^{n-1})^T$ and $\overline{\overline{c}} = (c^1,\ldots,c^{n-2})^T$ give the corresponding truncated vectors.
The result of our special coordinate-transformations will be:
$c_1^n = \ldots = c_n^n = h,\ c_1^{n-1} = \ldots = c_{n-1}^{n-1} = \Theta$ with $h \geq 0,\ \Theta \geq 0$, where $h, \Theta \in \mathbb{R}$.

Let f describe the density function of our distribution on $\mathbb{R}^n$
and let F do the same for the probability $F(r) := P(\|x\| \leq r)$ for $r \in [0,\infty]$,
($F(r)$ is the so-called radial distribution function). (11)

The marginal distribution function for the given distribution will be called G, hence

$$G(h) := P(x^n \leq h) \,\forall\, h \in \mathbb{R}. \tag{12}$$

$W(c_1, \ldots, c_{n-1})$ stands for the spherical angle generated by $KK(c_1, \ldots, c_{n-1})$, respectively for the share of the corresponding unit ball in the hyperplane $H(0, c_1, \ldots, c_{n-1})$ which belongs to that convex cone:

$$W(c_1, \ldots, c_{n-1}) := \frac{\lambda_{n-1}(\Omega_n \cap KK(c_1, \ldots, c_{n-1}))}{\lambda_{n-1}(\Omega_{n-1})} = \frac{\lambda_{n-2}(\omega_n \cap KK(c_1, \ldots, c_{n-1}))}{\lambda_{n-2}(\omega_{n-1})}. \tag{13}$$

Analogously we interpret the figure $V(c_1, \ldots, c_n)$ from formula (6) as the spherical angle resp. the intersection-share of the cone $KK(c_1, \ldots, c_n)$:

$$V(c_1, \ldots, c_n) := \frac{\lambda_n(\Omega_n \cap KK(c_1, \ldots, c_n))}{\lambda_n(\Omega_n)} = \frac{\lambda_{n-1}(\omega_n \cap KK(c_1, \ldots, c_n))}{\lambda_{n-1}(\omega_n)}. \tag{14}$$

Now we are going to exploit a consequent refinement of the principle of pointwise comparison in order to describe the relation between W and V more precisely. This enables us to find an improved estimation for the quotient. The new method turns out to be superior to the application of the principle of Cavalieri, which had been used so far. After that we can simulate the proof from [4] with improved parameters. So we can save a factor $O(\sqrt{n})$.

Theorem 1
For all distributions according to our rotation-symmetry-model (2) we have

$$E_{m,n}(S) \leq m^{\frac{1}{n-1}} \cdot n^{\frac{5}{2}} \cdot Const. \tag{15}$$

Corollary 1
Also this enables us to improve the upper bound (known from [4]) for the expected number of pivot steps s_t of the dimension-by-dimension algorithm for complete solution of (1) to

$$E_{m,n}(s_t) \leq m^{\frac{1}{n-1}} \cdot n^{\frac{7}{2}} \cdot Const. \tag{16}$$

2 Exploitation of the principle of pointwise comparison

According to our method used so far the key to the derivation of an upper-bound for $\frac{E_{m,n}(S)}{E_{m,n}(Z)}$ lies in the pointwise comparison for identical values of $t := \sqrt{h^2 + \Theta^2}$. This makes a significant simplification of the integral quotient possible, since several figures in the numerator- and the denominator-integral remain invariant as long as t is fixed. This holds because

- the internal distribution of the "random variables" $(\hat{\hat{c}}, \Theta, h)^T$ in $\{c \mid c^n = h,\ c^{n-1} = \Theta\}$ is identical for all pairs (h, Θ) with constant value of $t := \sqrt{h^2 + \Theta^2}$,

- the stochastic weight of those configurations with identical t is almost the same,
- the spherical measures $W(c_1, \ldots, c_{n-1})$ do not vary, when $\overline{\overline{c}}_1, \ldots, \overline{\overline{c}}_{n-1}$ and t remain fixed, but h and Θ vary simultaneously.

With the substitution $t := \sqrt{h^2 + \Theta^2}$, $T := \sqrt{t^2 - h^2}$ we obtain

$$\frac{E_{m,n}(S)}{E_{m,n}(Z)} = \frac{n\int\limits_0^1 t\int\limits_0^t G(h)^{m-n}T^{-1} \int\limits_{\mathbb{R}^{n-1}} |T - c_n^{n-1}| \int\limits_{\mathbb{R}^{n-2}} \cdots \int\limits_{\mathbb{R}^{n-2}} |\lambda_{n-2}\{KH(c_1, \ldots, c_{n-1})\}|^2}{\int\limits_0^1 t\int\limits_0^t G(h)^{m-n}T^{-1} \int\limits_{\mathbb{R}^{n-1}} |T - c_n^{n-1}| \int\limits_{\mathbb{R}^{n-2}} \cdots \int\limits_{\mathbb{R}^{n-2}} |\lambda_{n-2}\{KH(c_1, \ldots, c_{n-1})\}|^2 \cdot}$$
$$\frac{\cdot W(c_1, \ldots, c_{n-1})\, f(c_1) \cdots f(c_{n-1})\, d\overline{\overline{c}}_1 \cdots d\overline{\overline{c}}_{n-1}\, f(c_n)\, d\overline{c}_n\, dh\, dt}{\cdot V(c_1, \ldots, c_n)\, f(c_1) \cdots f(c_{n-1})\, d\overline{\overline{c}}_1 \cdots d\overline{\overline{c}}_{n-1}\, f(c_n)\, d\overline{c}_n\, dh\, dt} \le$$
$$\le \sup_{t \in [0,1]} \frac{n\int\limits_0^t G(h)^{m-n}T^{-1} \int\limits_{\mathbb{R}^{n-1}} |T - c_n^{n-1}| \int\limits_{\mathbb{R}^{n-2}} \cdots \int\limits_{\mathbb{R}^{n-2}} |\lambda_{n-2}\{KH(c_1, \ldots, c_{n-1})\}|^2}{\int\limits_0^t G(h)^{m-n}T^{-1} \int\limits_{\mathbb{R}^{n-1}} |T - c_n^{n-1}| \int\limits_{\mathbb{R}^{n-2}} \cdots \int\limits_{\mathbb{R}^{n-2}} |\lambda_{n-2}\{KH(c_1, \ldots, c_{n-1})\}|^2}$$
$$\frac{\cdot W(c_1, \ldots, c_{n-1})\, f(c_1) \cdots f(c_{n-1})\, d\overline{\overline{c}}_1 \cdots d\overline{\overline{c}}_{n-1}\, f(c_n)\, d\overline{c}_n\, dh}{\cdot V(c_1, \ldots, c_n)\, f(c_1) \cdots f(c_{n-1})\, d\overline{\overline{c}}_1 \cdots d\overline{\overline{c}}_{n-1}\, f(c_n)\, d\overline{c}_n\, dh}. \tag{17}$$

We take into regard that $W(c_1, \ldots, c_{n-1})$ is proportional to

$$\lambda_{n-2}(\omega_n \cap KK(c_1, .., c_{n-1})) = \int\limits_{KH(c_1,..,c_{n-1})} \frac{\sqrt{h^2 + \Theta^2}}{\sqrt{h^2 + \Theta^2 + \hat{\hat{c}}^T \hat{\hat{c}}}^{\,n-1}}\, d\hat{\hat{c}} = \int\limits_{KK(c_1,..,c_{n-1}) \cap \omega_n} \lambda(dw)$$

where $(\hat{\hat{c}}, T, h)^T$ is an element in the (T, h) area and $w = \frac{(\hat{\hat{c}},T,h)}{\|(\hat{\hat{c}},T,h)^T\|}$. By partitioning the (T, h) area in infinitesimally small surface-elements $d\hat{\hat{c}}$ we have also partitioned the surface of $\omega_n \cap KK(c_1, \ldots, c_{n-1})$ in infinitesimally small surface elements $M(w)$ implicitly. Their extensions may be different, but they all are extremely small. Also the figures $M(w)$ admit a characterization of $\lambda_{n-1}(\omega_n \cap KK(c_1, \ldots, c_n))$. This holds because

$$\omega_n \cap KK(c_1, \ldots, c_n) = \bigcup_{w \in \omega_n \cap KK(c_1,\ldots,c_n)} [KK\{c_n, M(w)\} \cap \omega_n].$$

Denote $KK\{c_n, M(w)\} \cap \omega_n$ by $\tilde{M}(c_n, w)$, then

$$\lambda_{n-2}(\omega_n \cap KK(c_1, \ldots, c_{n-1})) = \int\limits_{KK(c_1,\ldots,c_{n-1}) \cap \omega_n} \lambda(dw) \quad \text{and}$$

$$\lambda_{n-1}(\omega_n \cap KK(c_1,\dots,c_n)) = \int\limits_{KK(c_1,\dots,c_{n-1})\cap\omega_n} \frac{\lambda_{n-1}(\tilde{M}(c_n,w))}{\lambda_{n-2}(M(w))}\lambda(dw). \qquad (18)$$

Let us interpret $M(w)$ as the result of a projection of $d\hat{\hat{c}}$ on ω_n. The surface-element $M(w)$ is in any case dependent upon (T,h) and upon $\hat{\hat{c}}$. So we are allowed to cumulate over all w, which result from fixing t and $\hat{\hat{c}}$, but varying (T,h). This yields (by using I for the indicator of an event):

$$\frac{E_{m,n}(S)}{E_{m,n}(Z)} \le \sup_{t\in[0,1]} \frac{\int\limits_{\mathbb{R}^{n-2}} \lambda_{n-1}(\omega_n)\cdot n\cdot \int\limits_0^t G(h)^{m-n}\,T^{-1} \int\limits_{\mathbb{R}^{n-1}} |T-c_n^{n-1}|\cdot}{\int\limits_{\mathbb{R}^{n-2}} \lambda_{n-2}(\omega_{n-1}) \int\limits_0^t G(h)^{m-n}\,T^{-1} \int\limits_{\mathbb{R}^{n-1}} |T-c_n^{n-1}|\cdot}$$

$$\frac{\int\limits_{\mathbb{R}^{n-2}}\cdots\int\limits_{\mathbb{R}^{n-2}} |\lambda_{n-2}\{KH(c_1,\dots,c_{n-1})\}|^2\, I(\hat{\hat{c}} \in KH(\overline{\overline{c}}_1\dots\overline{\overline{c}}_{n-1}))\cdot}{\int\limits_{\mathbb{R}^{n-2}}\cdots\int\limits_{\mathbb{R}^{n-2}} |\lambda_{n-2}\{KH(c_1,\dots,c_{n-1})\}|^2\, I(\hat{\hat{c}} \in KH(\overline{\overline{c}}_1,\dots,\overline{\overline{c}}_{n-1}))\cdot}$$

$$\frac{\lambda_{n-2}(M(w(\hat{\hat{c}},T,h)))\, f(c_1)\cdots f(c_{n-1})\, d\overline{\overline{c}}_1\cdots d\overline{\overline{c}}_{n-1}\, f(c_n)\, d\overline{c}_n\, dh\, d\hat{\hat{c}}}{\lambda_{n-1}(\tilde{M}(c_n,w(\hat{\hat{c}},T,h)))\, f(c_1)\cdots f(c_{n-1})\, d\overline{\overline{c}}_1\cdots d\overline{\overline{c}}_{n-1}\, f(c_n)\, d\overline{c}_n\, dh\, d\hat{\hat{c}}} \le$$

$$\le \sup_{t\in[0,1]} \sup_{\hat{c}\in\mathbb{R}^{n-2}} \frac{\lambda_{n-1}(\omega_n)\cdot n\cdot \int\limits_0^t G(h)^{m-n}\,T^{-1} \int\limits_{\mathbb{R}^{n-1}} |T-c_n^{n-1}|\cdot}{\lambda_{n-2}(\omega_{n-1}) \int\limits_0^t G(h)^{m-n}\,T^{-1} \int\limits_{\mathbb{R}^{n-1}} |T-c_n^{n-1}|\cdot}$$

$$\frac{\int\limits_{\mathbb{R}^{n-2}}\cdots\int\limits_{\mathbb{R}^{n-2}} |\lambda_{n-2}\{KH(c_1,..,c_{n-1})\}|^2\, I(\hat{\hat{c}} \in KH(\overline{\overline{c}}_1,..,\overline{\overline{c}}_{n-1}))\lambda_{n-2}(M(w(\hat{\hat{c}},T,h)))}{\int\limits_{\mathbb{R}^{n-2}}\cdots\int\limits_{\mathbb{R}^{n-2}} |\lambda_{n-2}\{KH(c_1,..,c_{n-1})\}|^2\, I(\hat{\hat{c}} \in KH(\overline{\overline{c}}_1,..,\overline{\overline{c}}_{n-1}))\lambda_{n-1}(\tilde{M}\{c_n,w(\hat{\hat{c}},T,h)\})}$$

$$\frac{f(c_1)\cdots f(c_{n-1})\, d\overline{\overline{c}}_1\cdots d\overline{\overline{c}}_{n-1}\, f(c_n)\, d\overline{c}_n\, dh}{f(c_1)\cdots f(c_{n-1})\, d\overline{\overline{c}}_1\cdots d\overline{\overline{c}}_{n-1}\, f(c_n)\, d\overline{c}_n\, dh}. \qquad (19)$$

For further simplification we try to estimate $\lambda_{n-1}(\tilde{M}\{c_n, w(\hat{\hat{c}},.,.)\})$ by means of $\lambda_{n-2}(M(w))$.

3 A refined analysis of the spherical angle

In [4] the principle of Cavalieri had been applied in order to achieve an upper bound for $V(c_1,\dots,c_n)$ in terms of $W(c_1,\dots,c_{n-1})$. This time we use a more precise method.
Let w be a surface-point of ω_n, belonging to $KK(c_1,\dots,c_{n-1})$ simultaneously. That w is induced by a point $(\hat{\hat{c}},T,h)^T$ in $KH(c_1,\dots,c_{n-1})$, whose projection on ω_n is exactly w. Let $M(w)$ be an infinitesimally small area about w in ω_n and let $\hat{M}(w)$ be the intersection of $KK(M(w))$ and the tangential space at w to ω_n, both belonging to the hyperplane $H(0,c_1,\dots,c_{n-1})$, which holds all those points.

Then $Dim(\hat{M}(w)) = n-2$. $Span(w,z)$ is orthogonal to $\hat{M}(w)$, where z is the normal vector to $W(c_1,\ldots,c_{n-1})$ (orientied towards e_n and normalized). Hence $\hat{M}(w)) \subset w + Span(w,z)^{\perp}$. If $M(w)$ is small enough, then $\hat{M}(w)$ represents a sufficiently precise approximation for $M(w)$. We are interested in the spherical angle (with respect to ω_n), which is induced by $M(w)$ in cooperation with an arbitrary point $q \in \omega_n$, i.e.

$$\frac{\lambda_{n-1}(KK\{q, M(w)\} \cap \omega_n)}{\lambda_{n-1}(\omega_n)}. \tag{20}$$

We may suppose, that $q = e_n$ and $w \in Span(e_n, e_{n-1})$, i.e. $w = (0,\ldots,w^{n-1},w^n)^T$. For abbreviation we write $\tilde{M}(w) := \tilde{M}\{e_n, w\}$. Then the spherical angle of the cone under consideration is determined by the three formulas

$$\text{spherical angle} = \text{horizontal extension} \cdot \text{depth-extension}, \tag{21}$$

.

$$\text{horizontal extension of } e_n \text{ on } M(w) := \frac{\lambda_{n-2}\{x \mid \|x\| = 1, x^n = 0, Span(x, e_n) \cap M(w) \neq \emptyset\}}{\lambda_{n-2}\{x \mid \|x\| = 1,\ x^n = 0\}} \tag{22}$$

$$\text{depth-extension of } e_n \text{ on } M(w) \ = \frac{\lambda_{n-2}(\omega_{n-1})}{\lambda_{n-1}(\omega_n)} \cdot \int_{w^n}^{1} \sqrt{1-h^2}^{\,n-3} dh. \tag{23}$$

It is known that $\frac{\lambda_{n-2}(\omega_{n-1})}{\lambda_{n-1}(\omega_n)} \cdot \sqrt{1-h^2}^{\,n-3}$ is the marginal density of the surface-share of ω_n along the n-th coordinate. For all points x of the "equatorial set" in the numerator of (22) we move on $\omega_n \cap Span(e_n, x)$ starting from e_n in direction to the equator ($x^n = 0$) until $M(w)$ is reached. The infinitesimality of $M(w)$ confirms that this is approximately until $h = w^n$.

Remark 1
For the horizontal extension the following formula holds

$$\frac{\lambda_{n-2}(M(w)) \cdot \cos(\angle(z, \hat{w}^{\perp}))}{(1-(w^n)^2)^{\frac{n-2}{2}}}, \tag{24}$$

where $\hat{w}^{\perp}$ stands for $(0,\ldots,0,-w^n,w^{n-1})^T$. So it is the tangential vector, oriented towards e_n to the set $\omega_n \cap Span(e_n, e_{n-1})$ at w. $\angle$ denotes the respective angle between the corresponding vectors.
The total spherical angle is calculated by

$$\frac{\lambda_{n-2}(\omega_{n-1})}{\lambda_{n-1}(\omega_n)} \cdot \int_{w^n}^{1} \sqrt{1-h^2}^{\,n-3} dh \cdot \frac{\lambda_{n-2}(M(w)) \cdot \cos(\angle(z, \hat{w}^{\perp}))}{(1-(w^n)^2)^{\frac{n-2}{2}}}. \tag{25}$$

Of course, this formula holds for $w^n < 0$, too. In that case, the horizontal extension is the same (symmetry between e_n and $-e_n$). Different is the behaviour of the depth-extension, where the traversed angle between e_n and w now exceeds $\frac{\pi}{2}$.
Sometimes we will use the abbreviation $\eta := w^n$.

Lemma 1
The spherical angle under consideration admits the following transformations.

$$\frac{\lambda_{n-2}(\omega_{n-1})}{\lambda_{n-1}(\omega_n)} \cdot \int\limits_{w^n}^{1} \sqrt{1-h^2}^{\,n-3} dh \cdot \frac{\lambda_{n-2}(M(w)) \cdot \cos\left(\angle(z, \hat{w}^{\perp})\right)}{(1-\eta^2)^{\frac{n-2}{2}}} = \tag{26}$$

$$= \frac{\lambda_{n-2}(\omega_{n-1})}{\lambda_{n-1}(\omega_n)} \cdot \int\limits_{|w^n|}^{1} \sqrt{1-h^2}^{\,n-3} h\, dh \cdot \frac{\lambda_{n-2}(M(w)) \cdot \cos\left(\angle(z, \hat{w}^{\perp})\right)}{(1-\eta^2)^{\frac{n-2}{2}}} \frac{\int_{w^n}^{1} \sqrt{1-h^2}^{\,n-3} dh}{\int_{|w^n|}^{1} \sqrt{1-h^2}^{\,n-3} h\, dh} =$$

$$= \left\langle \frac{\lambda_{n-2}(\omega_{n-1})}{\lambda_{n-1}(\omega_n)} \cdot (1-\eta^2)^{\frac{1}{2}} \cdot \frac{1}{n-1} \cdot \lambda_{n-2}(M(w)) \cdot \cos\left(\angle(z, \hat{w}^{\perp})\right) \right\rangle \frac{\int_{w^n}^{1} \sqrt{1-h^2}^{\,n-3} dh}{\int_{|w^n|}^{1} \sqrt{1-h^2}^{\,n-3} h\, dh}$$

Here, the term in $\langle\rangle$ gives the Cavalieri-estimation for the spherical angle.

Remark 2
The expression $(1-\eta^2)^{\frac{1}{2}} \cdot \cos\left(\angle(z, \hat{w}^{\perp})\right)$ tells the distance of the point e_n to the hyperplane $H(0, c_1, \ldots, c_{n-1})$ (whose normal vector is z).

Proof

$$e_n = ({e_n}^T \hat{w}^{\perp})\hat{w}^{\perp} + ({e_n}^T w)w = (1-\eta^2)^{\frac{1}{2}} \hat{w}^{\perp} + \eta w$$
$$\Longrightarrow z^T e_n = (1-\eta^2)^{\frac{1}{2}} (z^T \hat{w}^{\perp}) = (1-\eta^2)^{\frac{1}{2}} \cos\left(\angle(z, \hat{w}^{\perp})\right). \quad \square$$

Remark 3
If we multiply the distance mentioned above with $\frac{1}{n-1} \cdot \lambda_{n-2}(M(w)) \cdot \frac{\lambda_{n-2}(\omega_{n-1})}{\lambda_{n-1}(\omega_n)}$, then we exactly obtain the estimation from [4] by means of the formula of Cavalieri. Here $M(w)$ has replaced $\omega_n \cap KK(c_1, \ldots, c_{n-1})$.

Proof
The Cavalieri-estimation works as follows:

$$\frac{\lambda_n(KH\{0, \tilde{M}(w)\})}{\lambda_{n-1}(KH\{0, M(w)\})} \frac{\lambda_{n-1}(\Omega_{n-1})}{\lambda_n(\Omega_n)} \geq \frac{\lambda_{n-1}(\Omega_{n-1})}{n \cdot \lambda_n(\Omega_n)} (\text{distance of } e_n \text{ to } H(0, c_1, \ldots, c_{n-1})) =$$
$$= \frac{\lambda_{n-2}(\omega_{n-1})}{(n-1) \cdot \lambda_{n-1}(\omega_n)} (1-\eta^2)^{\frac{1}{2}} \cdot \cos\left(\angle(z, \hat{w}^{\perp})\right) \square \tag{27}$$

Now look at Lemma 1. The factor obtained there outside $\langle\rangle$ tends to ∞ for $w^n \to -1$. Already at $w^n = 0$ it yields an enlargement of the denominator-integral by the factor $\frac{1}{\mu_n}$. Here we use (as often in the following) the notation $\mu_n := \frac{2\cdot\lambda_{n-2}(\omega_{n-1})}{(n-1)\lambda_{n-1}(\omega_n)}$.

Remark 4

For μ_n the following relations hold [4]

$$\sqrt{\frac{2(n-2)}{(n-1)^2\pi}} \le \mu_n = \frac{2\cdot\lambda_{n-2}(\omega_{n-1})}{(n-1)\lambda_{n-1}(\omega_n)} \le \sqrt{\frac{2}{(n-1)\pi}}. \tag{28}$$

Lemma 2

The term

$$\frac{\int\limits_{w^n}^{1} \sqrt{1-h^2}^{\,n-3} dh}{(n-1)\int\limits_{|w^n|}^{1} \sqrt{1-h^2}^{\,n-3} h dh} = \frac{\int\limits_{\eta}^{1} \sqrt{1-h^2}^{\,n-3} dh}{(n-1)\int\limits_{|\eta|}^{1} \sqrt{1-h^2}^{\,n-3} h dh} = \frac{\int\limits_{\eta}^{1} \sqrt{1-h^2}^{\,n-3} dh}{(1-\eta^2)^{\frac{n-1}{2}}} \tag{29}$$

represents a monotonously decreasing, convex function of $w^n = \eta$ in the interval $[-1,1]$.

Proof

The first derivative is of value

$$\frac{-(1-\eta^2)^{\frac{n-3}{2}}(1-\eta^2)^{\frac{n-1}{2}} + \int\limits_{\eta}^{1} \sqrt{1-h^2}^{\,n-3} dh \cdot \eta (1-\eta^2)^{\frac{n-3}{2}}(n-1)}{(1-\eta^2)^{n-1}} =$$

$$= \frac{-(1-\eta^2)^{\frac{n-1}{2}} + \int\limits_{\eta}^{1} \sqrt{1-h^2}^{\,n-3} dh \cdot \eta(n-1)}{(1-\eta^2)^{\frac{n-1}{2}+1}} \tag{30}$$

This shows that the value of the derivative at $\eta = 0$ is just -1.
For $\eta < 0$ convexity is obvious. this results from the numerator being positive and increasing towards -1. Its derivative is $\int\limits_{\eta}^{1} \sqrt{1-h^2}^{\,n-3} dh \cdot (n-1)$.
The denominator is positive and increases while η grows.
So, also the derivative of the total expression increases with η.
In order to assure this behaviour also for $\eta > 0$, we perform additional transformations, which are feasible only here.

$$(30) \quad = \quad \frac{-\int\limits_{\eta}^{1} \sqrt{1-h^2}^{\frac{n-3}{2}} h\, dh \cdot (n-1) + \int\limits_{\eta}^{1} \sqrt{1-h^2}^{\frac{n-3}{2}} dh \cdot \eta(n-1)}{(1-\eta^2) \int\limits_{\eta}^{1} \sqrt{1-h^2}^{\frac{n-3}{2}} h\, dh} =$$

$$= \quad -\frac{1}{1+\eta}\cdot\frac{\int_\eta^1 \sqrt{1-h^2}^{\frac{n-3}{2}}\cdot\frac{h-\eta}{1-\eta}\,dh}{\int_\eta^1 \sqrt{1-h^2}^{\frac{n-3}{2}}\cdot h\,dh} =$$

$$= \quad -\frac{1}{1+\eta}\cdot\left[\frac{\int_\eta^1 \sqrt{1-h^2}^{\frac{n-3}{2}}\cdot\frac{h-\eta}{1-\eta}\,dh}{\int_\eta^1 \sqrt{1-h^2}^{\frac{n-3}{2}}\,dh}\right]\cdot\left[\frac{\int_\eta^1 \sqrt{1-h^2}^{\frac{n-3}{2}}\cdot h\,dh}{\int_\eta^1 \sqrt{1-h^2}^{\frac{n-3}{2}}\,dh}\right]^{-1}.$$

The first quotient decreases while η grows. The rest gives a relation between two expectation values. For growing η we observe a transformation of weights in favour of higher values of h. Hence the expectation value in the last brackets grows. For the objective figure in the first expectation value $\frac{h-\eta}{1-\eta}$ the effect is just the opposite. The larger η becomes, the steeper is the relative descent of the density-function. So, the large values of the objective figure (variable) get less and less weight. So the quotient (relation) decreases. The negative total term cannot avoid increasing and also the derivative grows. □

Now, let us quantify the improvement. Difficult is the treatment of $\int_{w^n}^1 \sqrt{1-h^2}^{n-3}dh$. In the case $w^n >> 0$ a simple approximation is given by

$$\int_{w^n}^1 \sqrt{1-h^2}^{n-3}dh \sim \int_{w^n}^1 \sqrt{1-h^2}^{n-3}\cdot h\,dh = \frac{1}{n-1}\cdot(1-(w^n)^2)^{\frac{n-1}{2}}. \qquad (31)$$

In the interval $1 > h > w^n > Const.$ we have underestimated the integral at most by a factor $Const.$ Now we derive an estimation for all $w^n > 0$ and we set η for w^n.

Lemma 3

$$\frac{\int_\eta^1 \sqrt{1-h^2}^{n-3}\cdot h\,dh}{\int_\eta^1 \sqrt{1-h^2}^{n-3}dh} \le \eta + (1-\eta)\cdot\frac{2\cdot\lambda_{n-2}(\omega_{n-1})}{(n-1)\lambda_{n-1}(\omega_n)}\ \forall\,\eta > 0. \qquad (32)$$

Proof

$$\frac{\int_\eta^1 \sqrt{1-h^2}^{n-3}\cdot h\,dh}{\int_\eta^1 \sqrt{1-h^2}^{n-3}dh} = \eta + \frac{\int_0^1 \sqrt{1-[\eta+(1-\eta)x]^2}^{n-3}\cdot x\,dx}{\int_0^1 \sqrt{1-[\eta+(1-\eta)x]^2}^{n-3}dx}\cdot(1-\eta) \le^*$$

$$\le \quad \eta + \frac{\int_0^1 \sqrt{1-[x]^2}^{n-3}\cdot x\,dx}{\int_0^1 \sqrt{1-[x]^2}^{n-3}dx}\cdot(1-\eta) = \eta + (1-\eta)\cdot\frac{2\cdot\lambda_{n-2}(\omega_{n-1})}{(n-1)\lambda_{n-1}(\omega_n)}. \qquad (33)$$

* holds, because we know that for $0 \leq x_k \leq x_g \leq 1$:

$$\frac{1-x_k^2}{1-x_g^2} = \frac{(1-x_k)(1+x_k)}{(1-x_g)(1+x_g)} \leq \frac{1-x_k}{1-x_g}$$

$$\Longrightarrow \frac{1-x_k^2}{1-x_g^2} \leq \frac{(1-\eta)^2(1-x_k^2)+2\eta(1-\eta)(1-x_k)}{(1-\eta)^2(1-x_g^2)+2\eta(1-\eta)(1-x_g)} = \frac{1-[\eta+(1-\eta)x_k]^2}{1-[\eta+(1-\eta)x_g]^2}.$$

This means a transformation of weights in favour of higher x-values, as soon as η decreases. The extremal case is $\eta = 0$. There the expectation value of x is the largest. □

Corollary 2
For $\eta = w^n > 0$ the total spherical angle possesses the following lower bound:

$$\frac{\lambda_{n-1}(\tilde{M}(w))}{\lambda_{n-1}(\omega_n)} \geq \left[\frac{\lambda_{n-2}(\omega_{n-1})}{\lambda_{n-1}(\omega_n)} \frac{1}{n-1} \cdot (1-\eta^2)^{\frac{1}{2}} \cos\left(\angle(z, \hat{w}^{\perp})\right) \cdot \lambda_{n-2}(M(w))\right] \frac{1}{\eta + (1-\eta)\cdot\mu_n}$$

Remark 5
The correction derived above consists of the factor $\frac{1}{\eta+(1-\eta)\cdot\mu_n}$. It will contribute to a decrement of the total expression (19) particularly, when $\eta << 1$. In case of $\eta \to 0$ this factor will converge towards $\frac{(n-1)\lambda_{n-1}(\omega_n)}{2\cdot\lambda_{n-2}(\omega_{n-1})} = \frac{1}{\mu_n}$.

Remark 6
If we formulate corollary 2 for sections of the unit ball in the type known from [4], then we obtain:

$$\frac{\lambda_n(KK\{e_n, M(w)\} \cap \Omega_n)}{\lambda_n(\Omega_n)} \geq$$

$$\geq \frac{\lambda_{n-1}(\Omega_{n-1})}{n\cdot\lambda_n(\Omega_n)} \frac{\text{distance of } \frac{c_n}{\|c_n\|} \text{ to } H(0, c_1, \ldots, c_{n-1})}{\eta+(1-\eta)\mu_n} \lambda_{n-1}(KH\{0, M(w)\}) \quad (34)$$

4 Insertion into the old proof

w has been created by normalization of a vector $(\hat{\hat{c}}, T, h)^T$ such that $\hat{\hat{c}} \in \mathbb{R}^{n-2}$, where $T = \sqrt{t^2 - h^2}$. Now we try to fix $\hat{\hat{c}}$ and t. After that we vary h (and implicitly T) over all possible values. w is now a function of $(\hat{\hat{c}}, T, h)$, namely

$$w(\hat{\hat{c}}, T, h) = \frac{1}{\sqrt{h^2 + T^2 + \hat{\hat{c}}^T \hat{\hat{c}}}} \begin{pmatrix} \hat{\hat{c}} \\ T \\ h \end{pmatrix} = \frac{1}{\sqrt{t^2 + \hat{\hat{c}}^T \hat{\hat{c}}}} \begin{pmatrix} \hat{\hat{c}} \\ T \\ h \end{pmatrix}. \quad (35)$$

As we already know from §3, the combination of a point w with a point $q = \frac{c_n}{\|c_n\|}$ yields an improvement factor compared with the estimation used before. This factor will be called

$$\Phi(w^T \frac{c_n}{\|c_n\|}) = \Phi(w(\hat{\hat{c}},T,h)^T \frac{c_n}{\|c_n\|}) = \Phi(\eta) = \frac{\int\limits_{\eta}^{1} \sqrt{1-h^2}^{\,n-3} dh}{(1-\eta^2)^{\frac{n-1}{2}} \frac{1}{n-1}} \quad \text{with } \eta = w^T \frac{c_n}{\|c_n\|} \tag{36}$$

This factor Φ can also be regarded as a monotonously decreasing function of η such that $\Phi : [-1,+1] \to \mathbb{R}^+$. So we obtain the following estimation:

$$\frac{E_{m,n}(S)}{E_{m,n}(Z)} \leq \frac{\frac{\lambda_n(\Omega_n)}{\lambda_{n-1}(\Omega_{n-1})} \cdot n^2 \cdot \int\limits_0^t G(h)^{m-n}\, T^{-1} \int\limits_{\mathbb{R}^{n-1}} |T - c_n^{n-1}|\, f(c_n)\, d\overline{c}_n\, dh}{\int\limits_0^t G(h)^{m-n} \frac{h}{Tt} \int\limits_{\mathbb{R}^{n-1}} |T - c_n^{n-1}|^2 \frac{1}{\|c_n\|} \Phi(w(\hat{\hat{c}},T,h)^T \frac{c_n}{\|c_n\|}) f(c_n) d\overline{c}_n\, dh} \tag{37}$$

For simplification we try to exploit the convexity of Φ, which had been proven in Lemma 2. We consider a c_n and cumulate each time over a quadrupel of corresponding points c_{ni}, $i = 1,2,3,4$, resp. over the points $\xi_i := \frac{1}{\|c_{ni}\|} c_{ni}$. Let $c_{n1} := c_n$ induce the following four points:

$$\xi_1 := \frac{1}{\|c_n\|} c_{n1} := \frac{1}{\|c_n\|} \begin{pmatrix} \overline{\overline{c_n}} \\ c_n^{n-1} \\ h \end{pmatrix}, \quad \xi_2 := \frac{1}{\|c_n\|} c_{n2} := \frac{1}{\|c_n\|} \begin{pmatrix} -\overline{\overline{c_n}} \\ c_n^{n-1} \\ h \end{pmatrix},$$

$$\xi_3 := \frac{1}{\|c_n\|} c_{n3} := \frac{1}{\|c_n\|} \begin{pmatrix} \overline{\overline{c_n}} \\ -c_n^{n-1} \\ h \end{pmatrix}, \quad \xi_4 := \frac{1}{\|c_n\|} c_{n4} := \frac{1}{\|c_n\|} \begin{pmatrix} -\overline{\overline{c_n}} \\ -c_n^{n-1} \\ h \end{pmatrix}. \tag{38}$$

The barycenter of the four points c_{ni} lies in $\begin{pmatrix} \overline{\overline{0}} \\ 0 \\ h \end{pmatrix}$, the barycenter of the ξ_i is located in $\frac{h}{\|c_n\|} e_n = \frac{1}{\|c_n\|} \begin{pmatrix} \overline{\overline{0}} \\ 0 \\ h \end{pmatrix}$.

Lemma 4

We achieve a decrement of the denominator of (37), if we consistently use the point $\frac{h}{\|c_n\|} e_n$ *instead of* $\frac{c_n}{\|c_n\|}$ *in the argument of* Φ.

Proof

In the denominator of (37) all points c_{ni} have the same density f. c_{n1} and c_{n2} are even identically weighted (with exception of Φ), because here only the $(n-1)$th coordinate is relevant. The same holds for the two other points c_{n3} and c_{n4}.

Let w.l.o.g. $c_n^{n-1} \leq 0$ and $\hat{\hat{c}}^T \overline{\overline{c_n}} \leq 0$ as well as $w = \frac{1}{\sqrt{h^2+T^2+\hat{\hat{c}}^T \hat{\hat{c}}}} \begin{pmatrix} \hat{\hat{c}} \\ T \\ h \end{pmatrix}$ with $T > 0$.

Hence the weight of the two points ξ_1 and ξ_2 will be greater than that of the pair ξ_3 and ξ_4, since $|T - c_n^{n-1}| > |T + c_n^{n-1}|$. Now the convexity of Φ yields:

$$\frac{|T - c_n^{n-1}|^2 \cdot (\Phi(w^T \xi_1) + \Phi(w^T \xi_2)) + |T + c_n^{n-1}|^2 \cdot (\Phi(w^T \xi_3) + \Phi(w^T \xi_4))}{|T - c_n^{n-1}|^2 \cdot 2 + |T + c_n^{n-1}|^2 \cdot 2} \geq$$

$$\geq \Phi\left(w^T \frac{|T - c_n^{n-1}|^2 \cdot (\xi_1 + \xi_2) + |T + c_n^{n-1}|^2 \cdot (\xi_3 + \xi_4)}{|T - c_n^{n-1}|^2 \cdot 2 + |T + c_n^{n-1}|^2 \cdot 2} \right) =: \Phi(w^T \xi). \quad (39)$$

Here ξ is a vector with the properties $\bar{\bar{\xi}} = 0$, $\xi^{n-1} \leq 0$, $\xi^n = \frac{h}{\|c_n\|}$. Since Φ increases while $\eta = w^T x$ decreases, and because

$$w^T \xi = \frac{h^2 + T\xi^{n-1}}{\|c_n\| \sqrt{h^2 + T^2 + \hat{\hat{c}}^T \hat{\hat{c}}}} \leq \frac{h^2}{\|c_n\| \sqrt{h^2 + T^2 + \hat{\hat{c}}^T \hat{\hat{c}}}} = w^T \frac{h}{\|c_n\|} e_n, \quad (40)$$

the replacement mentioned above yields a smaller value of the denominator. □
Since in any case

$$w^T \frac{h}{\|c_n\|} e_n = \eta \geq 0, \quad (41)$$

we may make use of the following estimation:

Lemma 5
The denominator of (37) is decreased, if $\Phi(w(\hat{\hat{c}}, T, h)^T \frac{c_n}{\|c_n\|})$ is replaced by

$$\frac{1}{\eta + (1 - \eta)\mu_n} = \frac{1}{\frac{w^n h}{\|c_n\|} + (1 - \frac{w^n h}{\|c_n\|})\mu_n}. \quad (42)$$

Lemma 6
For given t and h and after fixing $\frac{h}{\|c_n\|} e_n$ as reference point the worst (smallest) improvement-factor (according to 42) is (where we varied over all values of $\hat{\hat{c}}$):

$$\Psi(h, t, r) := \frac{1}{\frac{h \cdot h}{\|c_n\| \cdot t} + (1 - \frac{h \cdot h}{\|c_n\| \cdot t})\mu_n}. \quad (43)$$

<u>Proof</u>
The figure (42) is monotonously decreasing for growing w^n. But the greatest value of w^n in (35) will be generated by

$$\hat{\hat{c}} = 0 \text{ and } w = \frac{1}{\sqrt{h^2 + T^2}} \begin{pmatrix} 0 \\ T \\ h \end{pmatrix} = \frac{1}{t} \begin{pmatrix} 0 \\ T \\ h \end{pmatrix}. \square \quad (44)$$

So one obtains

$$\frac{E_{m,n}(S)}{E_{m,n}(Z)} \leq \frac{\lambda_n(\Omega_n) \cdot n^2 \cdot \int\limits_0^t G(h)^{m-n}\, T^{-1} \int\limits_{\mathbb{R}^{n-1}} |T - c_n^{n-1}|\, f(c_n)\, d\overline{c}_n\, dh}{\lambda_{n-1}(\Omega_{n-1}) \int\limits_0^t G(h)^{m-n} \frac{h}{Tt} \int\limits_{\mathbb{R}^{n-1}} |T - c_n^{n-1}|^2 \frac{1}{\|c_n\|} \Psi(h,t,r) f(c_n) d\overline{c}_n\, dh}. \tag{45}$$

5 Calculation of the Expected Number

We change to polar coordinates and use $\|c_n\| = r = r(c_n), h, \gamma(c_n) \in \omega_{n-1}$, such that

$$c_n = \begin{pmatrix} \sqrt{r^2 - h^2}\, \gamma(c_n) \\ h \end{pmatrix} \text{ and } \overline{c_n} = \sqrt{r^2 - h^2}\, \gamma(c_n).$$

For abbreviation we set $R := R(r,h) = \sqrt{r^2 - h^2}$.
Analyzing (45) we exploit the fact that for fixed $r(c_n)$ we have

$$\frac{1}{r(c_n)} \left[\int\limits_{\omega_{n-1}(R)} |T - c_n^{n-1}|^2 d\gamma_R(\overline{c_n}) \right] = \frac{1}{r(c_n)} \left[T^2 + \frac{1}{n-1} R^2 \right] R^{n-2} \lambda_{n-2}(\omega_{n-1})\,. \tag{46}$$

For the numerator we analoguously obtain

$$\begin{aligned}
\int\limits_{\omega_{n-1}(R)} |T - c_n^{n-1}| d\gamma_R(\overline{c_n}) &= \int\limits_{c_n^{n-1} \leq -T \leq 0} T + |c_n^{n-1}| d\gamma_R(\overline{c_n}) + \int\limits_{0 \leq T \leq c_n^{n-1}} |c_n^{n-1}| - T d\gamma_R(\overline{c_n}) \\
&+ \int\limits_{-T \leq c_n^{n-1} \leq 0} T + |c_n^{n-1}| d\gamma_R(\overline{c_n}) + \int\limits_{0 \leq c_n^{n-1} \leq T} T - |c_n^{n-1}| d\gamma_R(\overline{c_n}) \\
&= \int\limits_{-T \leq c_n^{n-1} \leq T} T\, d\gamma_R(\overline{c_n}) + \int\limits_{|T| \leq |c_n^{n-1}|} |c_n^{n-1}|\, d\gamma_R(\overline{c_n}) \\
&= T \cdot \int\limits_{\omega_{n-1}(R)} d\gamma_R(\overline{c_n}) + \int\limits_{|T| \leq |c_n^{n-1}|} |c_n^{n-1}| - T\, d\gamma_R(\overline{c_n}) \\
&\leq \int\limits_{\omega_{n-1}(R)} \max\{R, T\}\, d\gamma_R(\overline{c_n}).
\end{aligned} \tag{47}$$

Now we have an upper bound for our quotient

$$\frac{E_{m,n}(S)}{E_{m,n}(Z)} \leq \frac{\lambda_n(\Omega_n) \cdot n^2 \cdot \int\limits_0^t G(h)^{m-n}\, T^{-1} \int\limits_h^1 R^{n-3} r^{-n+2} \max\{T, R\}\, dF(r)\, dh}{\lambda_{n-1}(\Omega_{n-1}) \int\limits_0^t G(h)^{m-n} \frac{h}{Tt} \int\limits_h^1 R^{n-3} r^{-n+1} (T^2 + \frac{1}{n-1} R^2) \Psi(h,t,r)\, dF(r)\, dh}. \tag{48}$$

Use of the very pessimistic estimation

$$T^2 + \frac{1}{n-1} R^2 \geq \frac{1}{n-1} \cdot \max\{T^2, R^2\} \tag{49}$$

yields

$$\frac{E_{m,n}(S)}{E_{m,n}(Z)} \leq \frac{\lambda_n(\Omega_n) n^2 \; (n-1) \int\limits_0^t G(h)^{m-n}\, T^{-1} \int\limits_h^1 R^{n-3}\, r^{-n+2} \; \max\{T, R\}\, dF(r)\, dh}{\lambda_{n-1}(\Omega_{n-1}) \int\limits_0^t G(h)^{m-n} \frac{h}{Tt} \int\limits_h^1 R^{n-3} r^{-n+1} \max\{T^2, R^2\} \Psi(h,t,r)\, dF(r)\, dh} \tag{50}$$

Now we are decided to partition the area of integration $(r, h) \in [h, 1] \times [0, t]$ in different subareas. For each part we will estimate the corresponding integral quotient from above. The very worst item of those upper bounds gives us - according to the principle of pointwise comparison - an upper bound for the complete integral-quotient.
Here a permutation of the order of integrations is recommended.

$$\begin{aligned} \frac{E_{m,n}(S)}{E_{m,n}(Z)} \;\; \leq \;\; & \frac{\lambda_n(\Omega_n) n^2 \; (n-1)}{\lambda_{n-1}(\Omega_{n-1})} \cdot \\ \cdot \;\; & \left\{ \frac{\int\limits_0^t \int\limits_0^r G(h)^{m-n}\, T^{-1} R^{n-3}\, r^{-n+2} T \, dh\, dF(r) +}{\int\limits_0^t \int\limits_0^r G(h)^{m-n} T^{-1} h t^{-1} R^{n-3} r^{-n+1} T^2 \Psi(h,t,r)\, dh\, dF(r) +} \right. \\ & \left. \frac{+ \int\limits_t^1 \int\limits_0^t G(h)^{m-n} T^{-1} R^{n-3} r^{-n+2} R \, dh\, dF(r)}{+ \int\limits_t^1 \int\limits_0^t G(h)^{m-n} T^{-1} h t^{-1} R^{n-3} r^{-n+1} R^2 \Psi(h,t,r)\, dh\, dF(r)} \right\}. \end{aligned} \tag{51}$$

The partition-subsets will be:

$$\begin{aligned} B_1 \;&:=\; \{(r,h) | 0 \leq r \leq t \wedge 0 \leq h \leq \mu_n r\} \cup \{(r,h) | t \leq r \leq 1 \wedge 0 \leq h \leq \mu_n t\} \\ B_2 \;&:=\; \{(r,h) | 0 \leq r \leq t \wedge \mu_n r \leq h \leq \mu_n t\} \\ B_3 \;&:=\; \{(r,h) | t \leq r \leq 1 \wedge \mu_n t \leq h \leq t\} \cup \{(r,h) | \mu_n t \leq r \leq t \wedge \mu_n t \leq h \leq t\} \end{aligned} \tag{52}$$

(The combination $r \leq \mu_n t \leq h$ cannot occur, because of $r \geq h$.)
For each subarea we are able to derive bounds for $\Psi(h, t, r)$.

Lemma 7
In B_1:

$$\Psi(h,t,r) = \frac{1}{\frac{h \cdot h}{r(c_n) \cdot t} + (1 - \frac{h \cdot h}{r(c_n) \cdot t}) \mu_n} \geq \frac{1}{\mu_n + (1-\mu_n)\mu_n} \geq \frac{1}{2\mu_n}. \tag{53}$$

In B_2:

$$\Psi(h,t,r) = \frac{1}{\frac{hh}{r(c_n)t} + (1-\frac{hh}{r(c_n)t})\mu_n} \geq \frac{1}{\frac{h}{r(c_n)}\mu_n + (1-\frac{h}{r(c_n)}\mu_n)\mu_n} \geq \frac{1}{\frac{h}{r(c_n)}\{1+\mu_n\}} \geq \frac{1}{2\frac{h}{r}} \tag{54}$$

In B_3 we have in case of $t \leq r$

$$\Psi(h,t,r) = \frac{1}{\frac{h \cdot h}{r(c_n)\cdot t} + (1-\frac{h\cdot h}{r(c_n)\cdot t})\mu_n} \geq \frac{1}{\frac{h}{t} + (1-\frac{h}{t})\mu_n} \geq \frac{1}{2\frac{h}{t}} \tag{55}$$

and in case of $r \leq t$

$$\Psi(h,t,r) = \frac{1}{\frac{h \cdot h}{r(c_n)\cdot t} + (1-\frac{h\cdot h}{r(c_n)\cdot t})\mu_n} \geq \frac{1}{\frac{h}{r} + (1-\frac{h}{r})\mu_n} \geq \frac{1}{2\frac{h}{r}}. \tag{56}$$

The rest of the paper deals with deriving upper bounds for (51) on the different subareas. The corresponding integral-quotients will be denoted by Q_1, Q_2, Q_3.

Proposition 1
In $B_1 = \{(r,h)|0 \leq r \leq t \wedge 0 \leq h \leq \mu_n r\} \cup \{(r,h)|t \leq r \leq 1 \wedge 0 \leq h \leq \mu_n t\}$ we have

$$Q_1 \leq \frac{4\lambda_n(\Omega_n)n^2(n-1)e^{\frac{1}{\pi}}}{\lambda_{n-1}(\Omega_{n-1})\sqrt{1-\mu_n{}^2}}. \tag{57}$$

Proof

$$\begin{aligned}
Q_1 \;&\leq \frac{2\lambda_n(\Omega_n)n^2(n-1)}{\lambda_{n-1}(\Omega_{n-1})} \cdot \\
&\cdot \left\{\frac{\int_0^t\int_0^{\mu_n r} G(h)^{m-n}\, R^{n-3}\, r^{-n+2}\, dh\, dF(r) + \int_t^1\int_0^{\mu_n t} G(h)^{m-n} T^{-1} R^{n-2} r^{-n+2}\, dh\, dF(r)}{\int_0^t\int_0^{\mu_n r} G(h)^{m-n} \frac{Th}{tr\mu_n} R^{n-3} r^{-n+2}\, dh\, dF(r) + \int_t^1\int_0^{\mu_n t} G(h)^{m-n} \frac{h}{Tt\mu_n} R^{n-1} r^{-n+1}\, dh\, dF(r)}\right\} \\
&\leq \frac{2\lambda_n(\Omega_n)n^2(n-1)\mu_n}{\lambda_{n-1}(\Omega_{n-1})\sqrt{1-\mu_n{}^2}} \cdot \\
&\cdot \left\{\frac{\int_0^t\int_0^{\mu_n r} G(h)^{m-n}\, R^{n-3}\, r^{-n+2}\, dh\, dF(r) + \int_t^1\int_0^{\mu_n t} G(h)^{m-n} T^{-1} R^{n-2} r^{-n+2}\, dh\, dF(r)}{\int_0^t\int_0^{\mu_n r} G(h)^{m-n} \frac{h}{r} R^{n-3} r^{-n+2}\, dh\, dF(r) + \int_t^1\int_0^{\mu_n t} G(h)^{m-n} \frac{h}{Tt} R^{n-2} r^{-n+2}\, dh\, dF(r)}\right\} \\
&\leq \frac{2\lambda_n(\Omega_n)n^2(n-1)\mu_n}{\lambda_{n-1}(\Omega_{n-1})\sqrt{1-\mu_n{}^2}} \cdot \\
&\max\left\{\max_{0\leq r\leq t} \frac{\int_0^{\mu_n r} G(h)^{m-n} R^{n-3} r^{-n+2}\, dh}{\int_0^{\mu_n r} G(h)^{m-n} \frac{h}{r} R^{n-3} r^{-n+2}\, dh}, \max_{t\leq r\leq 1} \frac{\int_0^{\mu_n t} G(h)^{m-n} T^{-1} R^{n-2} r^{-n+2}\, dh}{\int_0^{\mu_n t} G(h)^{m-n} \frac{h}{Tt} R^{n-2} r^{-n+2}\, dh}\right\}
\end{aligned}$$

$$\leq \quad \frac{2\lambda_n(\Omega_n)n^2(n-1)\mu_n}{\lambda_{n-1}(\Omega_{n-1})\sqrt{1-\mu_n{}^2}} \cdot \max\left\{ \max_{0\leq r\leq t} \frac{\int_0^{\mu_n r} R^{n-3}\, r^{-n+2}\, dh}{\int_0^{\mu_n r} \frac{h}{r} R^{n-3} r^{-n+2}\, dh},\ \max_{t\leq r\leq 1} \frac{\int_0^{\mu_n t} T^{-1} R^{n-2} r^{-n+2}\, dh}{\int_0^{\mu_n t} \frac{h}{Tt} R^{n-2} r^{-n+2}\, dh} \right\}$$

(Here we made use of the fact that $G(h)$ grows with $\frac{h}{r}$ resp. with $\frac{h}{t}$.)

The first quotient is constant with respect to variation of r.
In the second quotient $\frac{R}{r} \cdot \frac{t}{T}$ is increasing with $\frac{h}{t}$, hence there the maximal argument is $r = t$. We obtain

$$Q_1 \leq \frac{2\lambda_n(\Omega_n)n^2(n-1)\mu_n}{\lambda_{n-1}(\Omega_{n-1})\sqrt{1-\mu_n{}^2}} \cdot \frac{\int_0^{\mu_n t} T^{n-3} t^{-n+2}\, dh}{\int_0^{\mu_n t} h t^{-1} T^{n-3} t^{-n+2}\, dh}.$$

But on $[0, \mu_n t]$ the term $\frac{T^{n-3}}{t^{n-3}} = \sqrt{1 - \frac{h^2}{t^2}}^{\,n-3}$ is almost constant, because it is 1 at $h = 0$, it is monotonously decreasing in the interior of the interval, and at $h = \mu_n t$ we have

$$\sqrt{1-\mu_n^2}^{\,n-3} \geq \sqrt{1 - \frac{2}{(n-1)\pi}}^{\,n-3} = \left(1 - \frac{2}{\pi}\frac{1}{n-1}\right)^{\frac{1}{2}(n-3)} > e^{-\frac{1}{\pi}}.$$

So we can enlarge the density of the numerator by replacing it by $\frac{1}{t}$ and diminish the density of the denominator by replacing it by $\frac{1}{t}e^{-\frac{1}{\pi}}$. Now we have increased the quotient and we obtain

$$Q_1 \leq \frac{2\lambda_n(\Omega_n)n^2(n-1)\frac{1}{t}e^{\frac{1}{\pi}}\mu_n}{\lambda_{n-1}(\Omega_{n-1})\sqrt{1-\mu_n{}^2}\frac{1}{t}} \cdot \frac{\int_0^{\mu_n t} dh}{\int_0^{\mu_n t} h t^{-1}\, dh} = \frac{4\lambda_n(\Omega_n)n^2(n-1)e^{\frac{1}{\pi}}}{\lambda_{n-1}(\Omega_{n-1})\sqrt{1-\mu_n{}^2}}. \ \square$$

Proposition 2
For $B_2 = \{(r, h) | 0 \leq r \leq t \wedge \mu_n r \leq h \leq \mu_n t\}$ we obtain

$$Q_2 \leq \frac{2\lambda_n(\Omega_n)n^2(n-1)}{\lambda_{n-1}(\Omega_{n-1})\sqrt{1-\mu_n{}^2}}. \tag{58}$$

<u>Proof</u>

$$Q_2 \leq \frac{2\lambda_n(\Omega_n)n^2\,(n-1)}{\lambda_{n-1}(\Omega_{n-1})} \cdot \frac{\int_0^t \int_{\mu_n r}^{\mu_n t} G(h)^{m-n}\, R^{n-3}\, r^{-n+2}\, dh\, dF(r)}{\int_0^t \int_{\mu_n r}^{\mu_n t} G(h)^{m-n} T t^{-1} h r^{-1} r h^{-1} R^{n-3} r^{-n+2}\, dh\, dF(r)}$$

$$\leq \frac{2\lambda_n(\Omega_n)n^2(n-1)}{\lambda_{n-1}(\Omega_{n-1})\sqrt{1-\mu_n{}^2}}$$, because in B_2 it holds that $\frac{T}{t} \geq \sqrt{1-\mu_n{}^2}$. $\square$

Now only the rest of the integral ($h > \mu_n t$) remains for evaluation. Here we cannot ignore the influence of the monotonously increasing function $G(h)$, since it forces $\frac{R}{r}$ to become very small (it pushes h to the top, i.e. to t). Therefore we manage the growth of $G(h)$ in another way this time. Let us take into account the integral only after that $\hat{h}$, where $G(\hat{h}) = (1 - \frac{1}{m-n+1})$. The rest of the integral can be discussed and analyzed by simple arguments.
Consider the term for the third area B_3

$$Q_3 \leq \frac{2\lambda_n(\Omega_n)n^2\ (n-1)}{\lambda_{n-1}(\Omega_{n-1})} \cdot \tag{59}$$
$$\cdot \frac{\int\limits_{\mu_n t}^{t}\int\limits_{\mu_n t}^{r} G(h)^{m-n}\, R^{n-3}\, r^{-n+2}\, dh\, dF(r) + \int\limits_{t}^{1}\int\limits_{\mu_n t}^{t} G(h)^{m-n} T^{-1} R^{n-2} r^{-n+2}\, dh\, dF(r)}{\int\limits_{\mu_n t}^{t}\int\limits_{\mu_n t}^{r} G(h)^{m-n} \frac{T\,h\,r}{t\,r\,h} R^{n-3} r^{-n+2}\, dh\, dF(r) + \int\limits_{t}^{1}\int\limits_{\mu_n t}^{t} G(h)^{m-n} \frac{R\,h\,t}{T\,r\,t\,h} R^{n-2} r^{-n+2}\, dh\, dF(r)}.$$

In [4] it is explained in detail, how to achieve a greater value than Q_3 by changing to a distribution function $\overline{F}$, such that for a certain $\overline{r}$ the following holds:

$$\overline{F}(r) = \begin{cases} 0 & r \leq \overline{r} \\ F(r) & r > \overline{r} \end{cases} . \tag{60}$$

We have chosen $\overline{r}$ in such a way, that there the one-point-distribution-quotient (all weight on one radius)

$$\frac{\int\limits_{\mu_n t}^{t} G(h)^{m-n} T^{-1} R^{n-2} r^{-n+2}\, dh}{\int\limits_{\mu_n t}^{t} G(h)^{m-n} T^{-1} R r^{-1} R^{n-2} r^{-n+2}\, dh} \text{ becomes maximal over } r \in [t, 1]. \tag{61}$$

Values with $r \leq t$ yield smaller one-point-distribution-quotients in (59), because

$$\frac{\int\limits_{\mu_n t}^{r} G(h)^{m-n} R^{n-3} r^{-n+2}\, dh}{\int\limits_{\mu_n t}^{r} G(h)^{m-n} T t^{-1} R^{n-3} r^{-n+2}\, dh} \leq \frac{\int\limits_{\mu_n t}^{t} G(h)^{m-n} T^{n-3} t^{-n+2}\, dh}{\int\limits_{\mu_n t}^{t} G(h)^{m-n} T t^{-1} T^{n-3} t^{-n+2}\, dh}. \tag{62}$$

Notice that $\frac{T}{t}$ on $[\mu_n t, t]$ decreases monotonously. The smallest values of that "objective variable" will be attained at $h \approx t$. But now we perform a transformation of weights right here according to

$$\begin{cases} \frac{R^{n-3}}{T^{n-3}} & \text{for} \quad h \leq r \\ 0 & \text{for} \quad r \leq h \leq t \end{cases} \text{(monotonously decreasing with } h) \tag{63}$$

from the right to the left in (62). This transformation supports larger values of $\frac{T}{t}$ and decreases the right quotient from (62).

So we choose a $\overline{r} \geq t$ with maximal quotient and we concentrate all weight of the $r \leq \overline{r}$ on the one $\overline{r}$, i.e. we deal with the quotient

$$Q_3' = \frac{2\lambda_n(\Omega_n)n^2\ (n-1)}{\lambda_{n-1}(\Omega_{n-1})} \cdot \frac{\int\limits_t^1 \int\limits_{\mu_n t}^t G(h)^{m-n} T^{-1} R^{n-2} r^{-n+2}\, dh\, d\overline{F}(r)}{\int\limits_t^1 \int\limits_{\mu_n t}^t G(h)^{m-n} T^{-1} R r^{-1} R^{n-2} r^{-n+2}\, dh\, d\overline{F}(r)}. \tag{64}$$

Here, $Q_3 \leq Q_3'$. This has the following reason. Because simultaneously the figure

$$\int\limits_{\mu_n t}^{r} G(h)^{m-n} R^{n-3} r^{-n+2}\, dh = \int\limits_{\frac{\mu_n t}{r}}^{1} G(qr)^{m-n} \sqrt{1-q^2}^{\,n-3} dq \tag{65}$$

increases monotonously as a result of the monotony of G, we observe an increment of weight for growing r until t. Afterwards, $(r > t)$ the weight increases as a result of the monotony of $\frac{R}{r}$. Finally we know that more weight than had been before on the radii $r \in [\mu_n t, \overline{r}]$, is now relocated on the extremal radius $\overline{r}$. So, our integral quotient is smaller now.
We evaluate the Q_3'-quotient only on $[\max\{\mu_n t, \tilde{h}\}, t]$, where $\tilde{h}$ is chosen in such a way, that

$$G_F(\tilde{h}) := G(\tilde{h}) = 1 - \frac{1}{m-n+1}. \tag{66}$$

For the distribution induced by $\overline{F}$ it is sure that

$$G_F(\tilde{h}) = 1 - \frac{1}{m-n+1} \geq G_{\overline{F}}(\tilde{h}) \ \text{ because of } \ G_{\overline{F}}(h) \leq G_F(h)\ \forall\, h. \tag{67}$$

$$\text{So we have } \forall\, h \in [\mu_n t, t] \text{ such that } h < \tilde{h}:\ G_{\overline{F}}(h) \leq G_F(h) \leq 1 - \frac{1}{m-n+1}. \tag{68}$$

The pointwise integral quotient from Q_3' has for fixed h the form:

$$\frac{\int\limits_t^1 R^{n-2} r^{-n+2} d\overline{F}(r)}{\int\limits_t^1 R^{n-1} r^{-n+1} d\overline{F}(r)} \leq \left[\int\limits_t^1 R^{n-1} r^{-n+1} d\overline{F}(r)\right]^{-\frac{1}{n-1}} \quad \text{because of} \quad \int\limits_t^1 d\overline{F}(r) = 1. \tag{69}$$

Besides of that the following relation holds as a result of the definition of G:

$$\int\limits_h^1 R^{n-1} r^{-n+1} d\overline{F}(r) = (n-1) \int\limits_h^1 \int\limits_{\frac{h}{r}}^1 (1-\sigma^2)^{\frac{n-3}{2}} \sigma d\sigma d\overline{F}(r) \geq$$

$$\geq \ 2 \cdot \frac{\lambda_{n-1}(\omega_n)}{\lambda_{n-1}(\omega_n)} \int\limits_h^1 \int\limits_{\frac{h}{r}}^1 (1-\sigma^2)^{\frac{n-3}{2}} d\sigma d\overline{F}(r) = \ 2[1 - G_{\overline{F}}(h)]. \tag{70}$$

Hence we know for all pointwise integral-quotients

$$\frac{\int_t^1 R^{n-2} r^{-n+2} d\overline{F}(r)}{\int_t^1 R^{n-1} r^{-n+1} d\overline{F}(r)} \leq \{2[1-G_{\overline{F}}(h)]\}^{-\frac{1}{n-1}}. \tag{71}$$

So, for all $h < \tilde{h}$, it is guaranteed, that the pointwise quotient cannot become larger than

$$2^{-\frac{1}{n-1}} \cdot [m-n+1]^{\frac{1}{n-1}}. \tag{72}$$

Still $[\max\{\mu_n t, \tilde{h}\}, t]$ remains to be analyzed.
The integral quotient restricted on that region will be denoted by Q_3''. In the area under consideration it holds that

$$G(h) \geq 1 - \frac{1}{m-n+1} \quad \text{and hence} \quad [G(h)]^{m-n} \geq e^{-1}.$$

So, the nontrivial case $\tilde{h} < t$ yields

$$Q_3'' \leq \frac{2e\lambda_n(\Omega_n)n^2\ (n-1)}{\lambda_{n-1}(\Omega_{n-1})} \cdot \frac{\int_t^1 \int_{\max\{\tilde{h},\mu_n t\}}^t T^{-1} R^{n-2} r^{-n+2}\, dh\, d\overline{F}(r)}{\int_t^1 \int_{\max\{\tilde{h},\mu_n t\}}^t T^{-1} R^{n-1} r^{-n+1}\, dh\, d\overline{F}(r)}. \tag{73}$$

Let ζ be the lower bound for the inner integration area. For each r the following estimation is known.

Lemma 8

$$\frac{\int_\zeta^t T^{-1} R^{n-1} r^{-n+1}\, dh}{\int_\zeta^t T^{-1} R^{n-2} r^{-n+2}\, dh} \geq \frac{1}{r}\sqrt{\frac{1}{3}(r^2-\zeta^2)}. \tag{74}$$

Proof

$$\frac{\int_\zeta^t \frac{1}{\sqrt{t^2-h^2}} \frac{\sqrt{r^2-h^2}^{\,n-1}}{r^{n-1}}\, dh}{\int_\zeta^t \frac{1}{\sqrt{t^2-h^2}} \frac{\sqrt{r^2-h^2}^{\,n-2}}{r^{n-2}}\, dh} \geq \frac{\int_\zeta^t \frac{h}{\sqrt{t^2-h^2}} \frac{\sqrt{r^2-h^2}^{\,2}}{r^2}\, dh}{\int_\zeta^t \frac{h}{\sqrt{t^2-h^2}} \frac{\sqrt{r^2-h^2}}{r}\, dh} \geq \frac{1}{r}\left[\frac{\int_\zeta^t \frac{h}{\sqrt{t^2-h^2}}(r^2-h^2)\, dh}{\int_\zeta^t \frac{h}{\sqrt{t^2-h^2}}\, dh}\right]^{\frac{1}{2}} =$$

$$= \left[\frac{\int_0^{\sqrt{t^2-\zeta^2}} (r^2-t^2+u^2)du}{r^2 \int_0^{\sqrt{t^2-\zeta^2}} du}\right]^{\frac{1}{2}} = \frac{\left[\frac{1}{3}(t^2-\zeta^2)+(r^2-t^2)\right]^{\frac{1}{2}}}{r} = \frac{\left[\frac{1}{3}(r^2-\zeta^2)+\frac{2}{3}(r^2-t^2)\right]^{\frac{1}{2}}}{r}$$

In the first row it is exploited, that the expectation value of an objective-variable decreases, when the density is multiplied with a function, which is monotonously decreasing with the objective variable. □

Insertion into (73) yields

$$Q_3'' \leq \frac{2e\sqrt{3}\lambda_n(\Omega_n)n^2(n-1)}{\lambda_{n-1}(\Omega_{n-1})} \cdot \frac{\int_t^1 \int_\zeta^t T^{-1} R^{n-2} r^{-n+2}\, dh\, d\overline{F}(r)}{\int_t^1 \int_\zeta^t T^{-1} R^{n-2} r^{-n+2}\, dh\, \frac{\sqrt{r^2-\zeta^2}}{r} d\overline{F}(r)}. \tag{75}$$

If we replace t by τ in the upper limit of the integration interval and if we let move τ down to ζ_+, then the following term increases.

$$\frac{2e\sqrt{3}\lambda_n(\Omega_n)n^2(n-1)}{\lambda_{n-1}(\Omega_{n-1})} \cdot \frac{\int_t^1 \int_\zeta^\tau \frac{1}{\sqrt{t^2-\zeta^2}} R^{n-2} r^{-n+2}\, dh\, d\overline{F}(r)}{\int_t^1 \int_\zeta^\tau \frac{1}{\sqrt{t^2-\zeta^2}} R^{n-2} r^{-n+2}\, dh\, \frac{\sqrt{r^2-\zeta^2}}{r} d\overline{F}(r)} \tag{76}$$

This is true, because we have carried out a transformation of weights in favour of smaller values of r . We conclude

$$\begin{aligned} Q_3'' &\leq \frac{2e\sqrt{3}\lambda_n(\Omega_n)n^2(n-1)}{\lambda_{n-1}(\Omega_{n-1})} \cdot \frac{\int_t^1 \frac{1}{\sqrt{t^2-\zeta^2}} \frac{\sqrt{r^2-\zeta^2}^{\,n-2}}{r^{n-2}}\, d\overline{F}(r)}{\int_t^1 \frac{1}{\sqrt{t^2-\zeta^2}} \frac{\sqrt{r^2-\zeta^2}^{\,n-2}}{r^{n-2}} \frac{\sqrt{r^2-\zeta^2}}{r} d\overline{F}(r)} = \\ &= \frac{2e\sqrt{3}\lambda_n(\Omega_n)n^2(n-1)}{\lambda_{n-1}(\Omega_{n-1})} \cdot \frac{\int_t^1 \frac{\sqrt{r^2-\zeta^2}^{\,n-2}}{r^{n-2}}\, d\overline{F}(r)}{\int_t^1 \frac{\sqrt{r^2-\zeta^2}^{\,n-1}}{r^{n-1}}\, d\overline{F}(r)} \\ &\leq \frac{2e\sqrt{3}\lambda_n(\Omega_n)n^2(n-1)}{\lambda_{n-1}(\Omega_{n-1})} \cdot \max\left\{(m-n+1)^{\frac{1}{n-1}},\ \frac{1}{\sqrt{1-{\mu_n}^2}}\right\}. \end{aligned} \tag{77}$$

Proposition 3

In $B_3 = \{(r,h) | t \leq r \leq 1 \wedge \mu_n t \leq h \leq t\} \cup \{(r,h) | \mu_n t \leq r \leq t \wedge \mu_n t \leq h \leq t\}$ *it is true that*

$$Q_3 \leq \frac{2e\sqrt{3}\lambda_n(\Omega_n)n^2(n-1)}{\lambda_{n-1}(\Omega_{n-1})} \cdot \max\left\{(m-n+1)^{\frac{1}{n-1}},\ \frac{1}{\sqrt{1-{\mu_n}^2}}\right\}. \tag{78}$$

Theorem 2

$$E_{m,n}(S) \leq \sqrt{2\pi}\cdot 2\cdot e\cdot\sqrt{3}\cdot(n)^{\frac{3}{2}}\cdot(n-1)\cdot\max\left\{(m-n+1)^{\frac{1}{n-1}},\ \frac{1}{\sqrt{1-\frac{2}{(n-1)\pi}}}\right\}. \tag{79}$$

Proof

$$
\begin{aligned}
E_{m,n}(S) &\leq \frac{2e\sqrt{3}\lambda_n(\Omega_n)n^2(n-1)}{\lambda_{n-1}(\Omega_{n-1})} \cdot \max\left\{(m-n+1)^{\frac{1}{n-1}},\ \frac{1}{\sqrt{1-{\mu_n}^2}}\right\} \\
&\leq \sqrt{\frac{2\pi}{n}} \cdot n^2 \cdot (n-1) \cdot 2 \cdot e \cdot \sqrt{3} \cdot \max\left\{(m-n+1)^{\frac{1}{n-1}},\ \frac{1}{\sqrt{1-\frac{2}{(n-1)\pi}}}\right\}.
\end{aligned}
$$

This holds, because the constants in proposition 3 turn out to be maximal. □

References

[1] Borgwardt, K. H. : Untersuchungen zur Asymptotik der mittleren Schrittzahl von Simplexverfahren in der linearen Optimierung, Dissertation Kaiserslautern, 1977

[2] Borgwardt, K. H. : Some Distribution-Independent Results About the Asymptotic Order of the Average Number of Pivot Steps of the Simplex Method, Mathematics of Operations Research 7, 1982, 441–462

[3] Borgwardt, K. H. : The Average Number of Pivot Steps Required by the Simplex-Method is Polynomial, Zeitschrift f'ur Operations Research 26, 1982, 157–177

[4] Borgwardt, K. H. : The Simplex Method, A Probabilistic Analysis, Springer Verlag, Heidelberg,1987