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Existence and L-estimates for elliptic equations involving convolution

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In this article, with a fixed $p\in(1,+\infty)$ and a bounded domain $\Omega\subset\mathbb{R}^N$, $N\geq 2$, whose boundary $\partial\Omega$ fulfills the Lipschitz regularity, we study the following boundary value problem

$$\begin{aligned} -\operatorname{div}\mathcal{A}(x,u,\nabla u)+a|u|^{p-2}u &= \mathcal{B}(x,\rho * E(u),\nabla(\rho * E(u))) && \text{in } \Omega,\\ \mathcal{A}(x,u,\nabla u)\cdot\nu &= \mathcal{C}(x,u) && \text{on } \partial\Omega, \end{aligned} \qquad (P)$$

where $\mathcal{A}:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$, $\mathcal{B}:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$, $\mathcal{C}:\partial\Omega\times\mathbb{R}\to\mathbb{R}$ are Carathéodory functions, $a>0$ is a constant, $E:W^{1,p}(\Omega)\to W^{1,p}(\mathbb{R}^N)$ is an extension operator related to Ω, and ρ is an integrable function on $\mathbb{R}^N$. This is a novel problem that involves the nonlocal operator assigning to u the convolution $\rho * E(u)$ of ρ with $E(u)$. Under verifiable conditions, we prove the existence of a (weak) solution to problem (P) by using the surjectivity theorem for pseudomonotone operators. Moreover, through a modified version of Moser iteration up to the boundary, we show that (any) weak solution to (P) is bounded.

KEYWORDS

boundedness of solutions, convolution, critical growth on the boundary, elliptic operators of divergence type, Moser iteration

1 | INTRODUCTION

Let $\Omega\subset\mathbb{R}^N$, $N\geq 2$, be a bounded domain with a Lipschitz continuous boundary $\partial\Omega$ and let $p\in(1,+\infty)$ be a real number. It is well known that there exists an extension operator $E:W^{1,p}(\Omega)\to W^{1,p}(\mathbb{R}^N)$ meaning that E is a linear map satisfying

$$E(u)|_\Omega=u,\quad \forall\, u\in W^{1,p}(\Omega),$$

and for which there exists a constant $C=C(\Omega)>0$ depending only on Ω such that

$$\left.\begin{aligned} &\|E(u)\|_{L^p(\mathbb{R}^N)}\leq C(\Omega)\|u\|_{L^p(\Omega)}\\ \text{and}\quad &\|E(u)\|_{W^{1,p}(\mathbb{R}^N)}\leq C(\Omega)\|u\|_{W^{1,p}(\Omega)}, \end{aligned}\right.\quad \forall\, u\in W^{1,p}(\Omega),$$

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(see References 1,2). In the terminology of Reference 1 such a map E is called a (1,p)-extension operator for Ω. Generally, the extension operators are constructed by using reflection maps and partitions of unity. For the rest of the article, we fix an extension operator $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$.

We state the following boundary value problem

$$\begin{aligned} -\mathrm{div}\mathcal{A}(x,u,\nabla u) + a|u|^{p-2}u &= \mathcal{B}(x, \rho * E(u), \nabla(\rho * E(u))) && \text{in } \Omega, \\ \mathcal{A}(x,u,\nabla u)\cdot \nu &= \mathcal{C}(x,u) && \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $a > 0$ is a constant, $\nu(x)$ denotes the outer unit normal of Ω at $x \in \partial\Omega$, $\rho * E(u)$ stands for the convolution product of some integrable function ρ on $\mathbb{R}^N$ with $E(u)$, and $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are Carathéodory functions satisfying suitable p-structure growth conditions. Due to the presence of convolution, problem (1) is nonlocal. Furthermore, in the statement of problem (1), we have full dependence on the solution u and on its gradient ∇u, which makes the problem highly non-variational, so the variational methods are not applicable. The boundary condition in (1) is nonhomogeneous and includes the Robin boundary condition.

The starting point of this work has been the elliptic problem in Reference 3 with homogeneous Dirichlet boundary condition

$$\begin{aligned} -\Delta_p u - \mu\Delta_q u &= f(x, \rho * u, \nabla(\rho * u)) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{2}$$

involving the p-Laplacian Δ_p and the q-Laplacian Δ_q with $1 < q < p < +\infty$, where for the first time the boundary value problem with convolution for solution and its gradient was considered. Any solution $u \in W_0^{1,p}(\Omega)$ of (2) can be identified with $E(u) \in W^{1,p}(\mathbb{R}^N)$ obtained by extension with zero outside Ω. In this case, both ρ and u are integrable functions on $\mathbb{R}^N$ and the convolution $\rho * u$ in (2) makes sense. This is no longer possible for (1) because we have $u \in W^{1,p}(\Omega)$ and the extension by zero outside Ω generally does nor produce an element of $W^{1,p}(\mathbb{R}^N)$. Here is the essential point where the extension operator E is necessary in (1).

Finally, among papers involving quasilinear elliptic equations with convection term we can refer to Reference 4.

The aim of this article is two fold: to establish an existence result for (1) and to provide a priori estimates for the solutions to (1) up to the boundary showing their uniform boundedness. The proof of existence of solutions to (1) relies on the theory of pseudomonotone operators and properties of convolution and extension operator. In order to prove a priori estimates for problem (1) and show the boundedness of its solutions, we develop a modified version of Moser iteration originating in References 5 and 6.

First, we recall that the critical exponents corresponding to p in Ω and on $\partial\Omega$ are denoted by p^* and p_*, respectively (see Section 2).

For the existence result, our assumptions are as follows.

(A) The maps $\mathcal{A} : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$, $\mathcal{B} : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, and $\mathcal{C} : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions (i.e, they are measurable in the first variable and continuous in the others) satisfying the following conditions:

$$\begin{aligned} &(\mathrm{A1}) && |\mathcal{A}(x,s,\xi)| \le a_1|\xi|^{p-1} + a_2|s|^{p-1} + a_3 && \text{for a.e. } x \in \Omega, \\ &(\mathrm{A2}) && \mathcal{A}(x,s,\xi-\xi')\cdot(\xi-\xi') > 0 && \text{for a.e. } x \in \Omega, \\ &(\mathrm{A3}) && \mathcal{A}(x,s,\xi)\cdot\xi \ge a_4|\xi|^p - a_5 && \text{for a.e. } x \in \Omega, \\ &(\mathrm{A4}) && |\mathcal{B}(x,s,\xi)| \le f(x) + b_1|s|^{\alpha_1} + b_2|\xi|^{\alpha_2} && \text{for a.e. } x \in \Omega, \\ &(\mathrm{A5}) && |\mathcal{C}(x,s)| \le c_1|s|^{\alpha_3} + c_2 && \text{for a.e. } x \in \partial\Omega, \end{aligned}$$

for all $s \in \mathbb{R}$ and $\xi, \xi' \in \mathbb{R}^N$, $\xi \neq \xi'$, with positive constants a_i,b_j,c_k ($i \in \{1,\dots,5\}$, j, $k \in \{1,2\}$), with

$$\alpha_1, \alpha_2, \alpha_3 \in [0, p-1), \tag{3}$$

and a nonnegative function $f \in L^{r'}(\Omega)$ with $r \in [1,p^*)$.

Assumptions (A1)-(A2) are the Leray-Lions conditions, while (A3) is a coercivity condition. In problem (2), we have $\mathcal{A}(x,s,\xi)=|\xi|^{p-2}\xi+\mu|\xi|^{q-2}\xi$, with $1<q<p<+\infty$ and $\mu\geq 0$, which fulfills these assumptions. The maps $\mathcal{B}$ and $\mathcal{C}$ are only subject to the growth conditions (A4)-(A5).

By a (weak) solution to problem (1), we mean any function $u\in W^{1,p}(\Omega)$ verifying

$$\int_\Omega \mathcal{A}(x,u,\nabla u)\cdot\nabla\varphi dx + a\int_\Omega |u|^{p-2}u\varphi dx = \int_\Omega \mathcal{B}(x,\rho * E(u),\nabla(\rho * E(u)))\varphi dx + \int_{\partial\Omega}\mathcal{C}(x,u)\varphi d\sigma, \tag{4}$$

for all $\varphi\in W^{1,p}(\Omega)$. Under assumptions (A), all the integrals in (4) are finite for $u,\varphi\in W^{1,p}(\Omega)$, thus the definition of weak solution is meaningful. In the same spirit, $u\in W_0^{1,p}(\Omega)$ is a (weak) solution to (2) if

$$\int_\Omega \left(|\nabla u|^{p-2}+\mu|\nabla u|^{q-2}\right)\nabla u\cdot\nabla\varphi dx = \int_\Omega \mathcal{B}(x,\rho * u,\nabla(\rho * u))\varphi dx$$

holds for every $\varphi\in W_0^{1,p}(\Omega)$.

Theorem 1. *Let $\Omega\subset\mathbb{R}^N$ be a bounded domain with a Lipschitz continuous boundary $\partial\Omega$ endowed with the extension operator $E: W^{1,p}(\Omega)\to W^{1,p}(\mathbb{R}^N)$ and let $\rho\in L^1(\mathbb{R}^N)$. If hypotheses (A) are satisfied, then there exists a (weak) solution to problem (1).*

The proof of Theorem 1 is the object of Section 3.

Now we turn to the uniform boundedness of solutions to problem (1). We formulate the following hypotheses.

(H) The maps $\mathcal{A}:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$, $\mathcal{B}:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$, and $\mathcal{C}:\partial\Omega\times\mathbb{R}\to\mathbb{R}$ are Carathéodory functions satisfying the conditions

$$\begin{array}{lll}
\text{(H1)} & |\mathcal{A}(x,s,\xi)|\leq a_1|\xi|^{p-1}+a_2|s|^{p^*\frac{p-1}{p}}+a_3 & \text{for a.e. } x\in\Omega,\\
\text{(H2)} & \mathcal{A}(x,s,\xi)\cdot\xi\geq a_4|\xi|^p-a_5|s|^{p^*}-a_6 & \text{for a.e. } x\in\Omega,\\
\text{(H3)} & |\mathcal{B}(x,s,\xi)|\leq f(x)+b_1|s|^{\alpha_1}+b_2|\xi|^{\alpha_2} & \text{for a.e. } x\in\Omega,\\
\text{(H4)} & |\mathcal{C}(x,s)|\leq c_1|s|^{p_*-1}+c_2 & \text{for a.e. } x\in\partial\Omega,
\end{array}$$

for all $s\in\mathbb{R}$ and $\xi\in\mathbb{R}^N$, with nonnegative constants a_i,b_j,c_k ($i\in\{1,\dots,6\}$, $j,k\in\{1,2\}$) and α_1,α_2 such that

$$0\leq\alpha_1<p^*-p,\qquad 0\leq\alpha_2<\min\left\{p-1,\frac{p}{p^*}(p^*-p)\right\}, \tag{5}$$

and a nonnegative function $f\in L^{r'}(\Omega)$, with $r\in[1,p^*/p)$.

Theorem 2. *Let $\Omega\subset\mathbb{R}^N$ be a bounded domain with a Lipschitz continuous boundary $\partial\Omega$ endowed with the extension operator $E: W^{1,p}(\Omega)\to W^{1,p}(\mathbb{R}^N)$ and let $\rho\in L^1(\mathbb{R}^N)$. Assume that hypotheses (H) are satisfied. Then, every (weak) solution $u\in W^{1,p}(\Omega)$ to problem (1) belongs to $L^\infty(\Omega)$ with the trace $\gamma u\in L^\infty(\partial\Omega)$.*

The proof of Theorem 2 is given in Section 4.

Combining Theorems 1 and 2, we obtain the following existence result of bounded solutions to problem (1).

Corollary 1. *Let $\Omega\subset\mathbb{R}^N$ be a bounded domain with a Lipschitz continuous boundary $\partial\Omega$ endowed with the extension operator $E: W^{1,p}(\Omega)\to W^{1,p}(\mathbb{R}^N)$ and let $\rho\in L^1(\mathbb{R}^N)$. Assume that hypotheses (A1)-(A3), (A4) with α_2 as in (5), and (A5) are satisfied. Then, there exists a (weak) solution $u\in W^{1,p}(\Omega)$ to problem (1) which belongs to $L^\infty(\Omega)$ and whose trace γu is an element of $L^\infty(\partial\Omega)$.*

Corollary 1 is a direct consequence of Theorems 1 and 2 noticing that Theorems 1 and 2 can be simultaneously applied.

We illustrate the applicability of our results by an example using the extension operator constructed in p. 275 of Reference 2.

Example 1. Consider in $\mathbb{R}^2$ the rectangular domains $\Omega=(0,1)\times(0,1)$, $\Omega_1=(0,1)\times(-1,1)$, $\Omega_2=(-1,1)\times(-1,1)$, $\Omega_3=(-1,1)\times(-1,3)$, $\tilde{\Omega}=(-1,3)\times(-1,3)$. We introduce the maps $R_1: W^{1,p}(\Omega)\to W^{1,p}(\Omega_1)$, $R_2: W^{1,p}(\Omega_1)\to W^{1,p}(\Omega_2)$,

$R_3 : W^{1,p}(\Omega_2) \to W^{1,p}(\Omega_3)$, and $R_4 : W^{1,p}(\Omega_3) \to W^{1,p}(\tilde{\Omega})$, respectively, by

$$(R_1 u)(x_1, x_2) = \begin{cases} u(x_1, x_2) & \text{if } x_2 > 0, \\ u(x_1, -x_2) & \text{if } x_2 < 0 \end{cases}$$

for all $u \in W^{1,p}(\Omega)$ and $(x_1, x_2) \in \Omega$,

$$(R_2 u)(x_1, x_2) = \begin{cases} u(x_1, x_2) & \text{if } x_1 > 0, \\ u(-x_1, x_2) & \text{if } x_1 < 0 \end{cases}$$

for all $u \in W^{1,p}(\Omega_1)$ and $(x_1, x_2) \in \Omega_1$,

$$(R_3 u)(x_1, x_2) = \begin{cases} u(x_1, x_2) & \text{if } x_2 < 1, \\ u(x_1, 2 - x_2) & \text{if } x_2 > 1 \end{cases}$$

for all $u \in W^{1,p}(\Omega_2)$ and $(x_1, x_2) \in \Omega_2$, and

$$(R_4 u)(x_1, x_2) = \begin{cases} u(x_1, x_2) & \text{if } x_1 < 1, \\ u(2 - x_1, x_2) & \text{if } x_1 > 1. \end{cases}$$

for all $u \in W^{1,p}(\Omega_3)$ and $(x_1, x_2) \in \Omega_3$.

For a fixed $\psi \in C^1(\tilde{\Omega})$ with $\psi = 1$ on Ω and supp $\psi \subset \tilde{\Omega}$, the linear map $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^2)$ which carries each $u \in W^{1,p}(\Omega)$ to the function $Eu \in W^{1,p}(\mathbb{R}^2)$ obtained by extending $\psi(R_4 \circ R_3 \circ R_2 \circ R_1 u)$ with zero outside $\tilde{\Omega}$ is an extension operator. Accordingly, given a constant $a > 0$, a function $\rho \in L^1(\mathbb{R}^2)$, and a Carathéodory function $\mathcal{B} : \Omega \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ satisfying (H) and (5), we state the Neumann problem

$$\begin{aligned} -\Delta_p u + a|u|^{p-2}u = \mathcal{B}(x, \rho * E(u), \nabla(\rho * E(u))) \quad & \text{in } \Omega, \\ |\nabla u|^{p-2}\nabla u \cdot \nu = 0 \quad & \text{on } \partial\Omega. \end{aligned}$$

A frequent form of $\mathcal{B}$ is $\mathcal{B}(x, s, \xi) = g(s) + h(\xi)$. Our results apply to the stated problem.

The rest of the article is organized as follows. Section 2 contains preliminaries to be used in the sequel. In Section 3, we prove Theorem 1. In Section 4, we prove Theorem 2.

2 | PRELIMINARIES

The Euclidean norm of $\mathbb{R}^N$ is denoted by $|\cdot|$, while the notation $\cdot$ stands for the standard inner product on $\mathbb{R}^N$. By $|\cdot|$, we also denote the Lebesgue measure on $\mathbb{R}^N$. In the rest of the article, for every $r \in (1, +\infty)$, we denote by r' its Hölder conjugate, that is, $r' = \frac{r}{r-1}$.

For any $r \in [1, +\infty)$ and a domain $\Omega \subset \mathbb{R}^N$, we denote by $L^r(\Omega)$ and $W^{1,r}(\Omega)$ the usual Lebesgue and Sobolev spaces equipped with the norms

$$\begin{aligned} \|u\|_{L^r(\Omega)} &= \left(\int_\Omega |u|^r dx \right)^{\frac{1}{r}}, \\ \|u\|_{W^{1,r}(\Omega)} &= \left(\int_\Omega |\nabla u|^r dx \right)^{\frac{1}{r}} + \left(\int_\Omega |u|^r dx \right)^{\frac{1}{r}}. \end{aligned} \tag{6}$$

Recall that the norm of $L^\infty(\Omega)$ is

$$\|u\|_{L^\infty(\Omega)} = \text{ess sup}_\Omega |u|.$$

For any $u \in W^{1,r}(\Omega)$, set $u^{\pm} := \max\{\pm u, 0\}$, which yields

$$u^{\pm} \in W^{1,r}(\Omega), \quad |u| = u^{+} + u^{-}, \quad u = u^{+} - u^{-}. \tag{7}$$

By the Sobolev embedding theorem, there exists a linear continuous embedding $i : W^{1,r}(\Omega) \to L^{r^*}(\Omega)$, where the corresponding critical exponent r^* in the domain is given by

$$r^* = \begin{cases} \frac{Nr}{N-r} & \text{if } r < N, \\ +\infty & \text{if } r \geq N. \end{cases}$$

The boundary $\partial\Omega$ is endowed with the $(N-1)$-dimensional Hausdorff (surface) measure. The measure of $\partial\Omega$ is denoted by $|\partial\Omega|$. The Lebesgue spaces $L^s(\partial\Omega)$, with $1 \leq s \leq +\infty$, have the norms

$$\|u\|_{L^s(\partial\Omega)} = \left(\int_{\partial\Omega} |u|^s d\sigma\right)^{\frac{1}{s}} \quad (1 \leq s < +\infty), \qquad \|u\|_{L^\infty(\partial\Omega)} = \operatorname{ess\,sup}_{\partial\Omega}|u|.$$

There exists a unique linear continuous map $\gamma : W^{1,r}(\Omega) \to L^{r_*}(\partial\Omega)$, called the trace map, characterized by $\gamma(u) = u|_{\partial\Omega}$ whenever $u \in W^{1,r}(\Omega) \cap C(\overline{\Omega})$, where r_* is the corresponding critical exponent on the boundary defined as

$$r_* = \begin{cases} \frac{(N-1)r}{N-r} & \text{if } r < N, \\ +\infty & \text{if } r \geq N. \end{cases}$$

As usual, the subspace of $W^{1,r}(\Omega)$ consisting of zero trace elements is denoted $W_0^{1,r}(\Omega)$. For the sake of notational simplicity, we drop the use of the symbol γ writing simply u in place of γu. We refer to Reference 1 for the theory of Sobolev spaces.

The following propositions are useful in the proof of our boundedness result.

Proposition 1 (Proposition 2.2 of Reference 5). *Let $u \in L^p(\Omega)$, $1 < p < +\infty$, be nonnegative. If it holds*

$$\|u\|_{L^{\alpha_n}(\Omega)} \leq C,$$

for a constant $C > 0$ and a sequence $(\alpha_n) \subset \mathbb{R}_+$ such that $\alpha_n \to +\infty$ as $n \to \infty$, then $u \in L^\infty(\Omega)$.

Proposition 2 (Proposition 2.4 of Reference 5). *Let $1 < p < +\infty$ and let $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Then, $u \in L^\infty(\partial\Omega)$.*

Recall that for $\rho \in L^1(\mathbb{R}^N)$ and $u \in W^{1,p}(\mathbb{R}^N)$, with $1 < p < +\infty$, the convolution $\rho * u$ is defined by

$$(\rho * u)(x) := \int_{\mathbb{R}^N} \rho(x-y)u(y)dy \quad \text{for a.e. } x \in \mathbb{R}^N.$$

The weak partial derivatives of the convolution $\rho * u$ are expressed by

$$\frac{\partial}{\partial x_i}(\rho * u) = \rho * \frac{\partial u}{\partial x_i} \quad \text{for } i = 1, \ldots, N.$$

Thanks to Tonelli's and Fubini's theorems as well as Hölder's inequality, there hold

$$\|\rho * u\|_{L^r(\mathbb{R}^N)} \leq \|\rho\|_{L^1(\mathbb{R}^N)} \|u\|_{L^r(\mathbb{R}^N)},$$

for every $r \in [1, p^*]$ and

$$\left\|\rho * \frac{\partial u}{\partial x_i}\right\|_{L^p(\mathbb{R}^N)} \leq \|\rho\|_{L^1(\mathbb{R}^N)} \left\|\frac{\partial u}{\partial x_i}\right\|_{L^p(\mathbb{R}^N)} \quad \text{for } i = 1, \ldots, N \tag{8}$$

(see Theorem 4.15 of Reference 2). Taking into account the fact that the function $t \mapsto t^{1/2}$ is sublinear as well as the function $t \mapsto t^p$ is convex on $(0,+\infty)$ and (8), it follows that

$$\begin{aligned}\|\nabla(\rho * u)\|^p_{L^p(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} |\nabla(\rho * u)|^p dx = \int_{\mathbb{R}^N} \left(\sum_{i=1}^N \left(\rho * \frac{\partial u}{\partial x_i} \right)^2 \right)^{p/2} dx \\ &\le \int_{\mathbb{R}^N} \left(\sum_{i=1}^N \left| \rho * \frac{\partial u}{\partial x_i} \right| \right)^p dx \le N^{p-1} \int_{\mathbb{R}^N} \sum_{i=1}^N \left| \rho * \frac{\partial u}{\partial x_i} \right|^p dx \\ &\le N^p \|\rho\|^p_{L^1(\mathbb{R}^N)} \|\nabla u\|^p_{L^p(\mathbb{R}^N)}.\end{aligned}$$

Finally, we recall the main theorem on the pseudomonotone operators that will be used to prove our existence result. Let X be a reflexive Banach space endowed with the norm $\|\cdot\|$. The norm convergence is denoted by $\to$ and the weak convergence by $\rightharpoonup$. We denote by X^* the topological dual of X and by $\langle \cdot\, , \cdot \rangle$ the duality pairing between X and X^*. A map $A : X \to X^*$ is called bounded if it maps bounded sets to bounded sets. It is said to be coercive if there holds

$$\lim_{\|u\| \to +\infty} \frac{\langle Au, u \rangle}{\|u\|} = +\infty.$$

Finally, A is called pseudomonotone if $u_n \rightharpoonup u$ in X and

$$\limsup_{n \to +\infty} \langle Au_n, u_n - u \rangle \le 0$$

imply

$$\langle Au, u - w \rangle \le \liminf_{n \to +\infty} \langle Au_n, u_n - w \rangle, \quad \forall\, w \in X.$$

The surjectivity theorem for pseudomonotone operators reads as follows (see, e.g., Reference 7).

Theorem 3. *Let X be a reflexive Banach space, let $A : X \to X^*$ be a pseudomonotone, bounded, and coercive operator, and let $g \in X^*$. Then, there exists at least a solution $u \in X$ to the equation $Au = g$.*

3 | PROOF OF THEOREM 1

Throughout the proof of the theorem, we will denote by C_i, $i \in \mathbb{N}$, constants which depend on the given data.

With a fixed $\rho \in L^1(\mathbb{R}^N)$ and an extension operator $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$, we introduce the nonlinear operator $T : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ by

$$\begin{aligned}\langle Tu, \varphi \rangle &= \int_\Omega \mathcal{A}(x, u, \nabla u) \cdot \nabla \varphi dx + a \int_\Omega |u|^{p-2} u \varphi dx \\ &\quad - \int_\Omega \mathcal{B}(x, \rho * E(u), \nabla(\rho * E(u))) \varphi dx - \int_{\partial\Omega} C(x, u) \varphi d\sigma, \end{aligned} \tag{9}$$

for all $u, \varphi \in W^{1,p}(\Omega)$. Assumption (A) guarantees that T is well defined.

Let us show that T is also bounded. Indeed, fix $\varphi \in W^{1,p}(\Omega)$ such that $\|\varphi\|_{W^{1,p}(\Omega)} \le 1$. Then,

$$\begin{aligned}|\langle Tu, \varphi \rangle| &\le \int_\Omega |\mathcal{A}(x, u, \nabla u)||\nabla \varphi| dx + a \int_\Omega |u|^{p-1} |\varphi| dx \\ &\quad + \int_\Omega |\mathcal{B}(x, \rho * E(u), \nabla(\rho * E(u)))||\varphi| dx + \int_{\partial\Omega} |C(x, u)||\varphi| d\sigma. \end{aligned} \tag{10}$$

We estimate the terms of the inequality above separately. First, observe that

$$\begin{aligned}\int_\Omega |\mathcal{A}(x,u,\nabla u)||\nabla\varphi| dx &\le \int_\Omega \left(a_1|\nabla u|^{p-1} + a_2|u|^{p-1} + a_3\right)|\nabla\varphi| dx\\ &\le a_1\|\nabla u\|^{p-1}_{L^p(\Omega)}\|\nabla\varphi\|_{L^p(\Omega)} + a_2\|u\|^{p-1}_{L^p(\Omega)}\|\nabla\varphi\|_{L^p(\Omega)}\\ &\quad + a_3|\Omega|^{\frac{p-1}{p}}\|\nabla\varphi\|_{L^p(\Omega)}\\ &\le a_1\|\nabla u\|^{p-1}_{L^p(\Omega)} + a_2\|u\|^{p-1}_{L^p(\Omega)} + C_1,\end{aligned} \tag{11}$$

as well as

$$a\int_\Omega |u|^{p-1}|\varphi| dx \le a\|u\|^{p-1}_{L^p(\Omega)}\|\varphi\|_{L^p(\Omega)} \le a\|u\|^{p-1}_{L^p(\Omega)}. \tag{12}$$

Thanks to (A4), we also have

$$\begin{aligned}&\int_\Omega |\mathcal{B}(x,\rho * E(u),\nabla(\rho * E(u)))||\varphi| dx\\ &\le \int_\Omega (f(x) + b_1|\rho * E(u)|^{\alpha_1} + b_2|\nabla(\rho * E(u))|^{\alpha_2})|\varphi| dx.\end{aligned} \tag{13}$$

We consider the terms in (13) separately. First note that Hölder's inequality gives

$$\begin{aligned}\int_\Omega f(x)|\varphi| dx &\le \|f\|_{L^{r'}(\Omega)}\|\varphi\|_{L^r(\Omega)}\\ &\le \|f\|_{L^{r'}(\Omega)}\|\varphi\|_{L^p(\Omega)}|\Omega|^{\frac{p-r}{pr}}\\ &\le C_2.\end{aligned} \tag{14}$$

Moreover, exploiting the properties of E and of the convolution and the Sobolev embedding, we have

$$\begin{aligned}b_1\int_\Omega |\rho * E(u)|^{\alpha_1}|\varphi| dx &\le \|\rho * E(u)\|^{\alpha_1}_{L^{p^*}(\mathbb{R}^N)}\|\varphi\|_{L^{\frac{p^*}{p^*-\alpha_1}}(\Omega)}\\ &\le C_3\|\rho\|^{\alpha_1}_{L^1(\mathbb{R}^N)}\|E(u)\|^{\alpha_1}_{L^{p^*}(\mathbb{R}^N)}\|\varphi\|_{L^{p^*}(\Omega)}\\ &\le C_4\|\rho\|^{\alpha_1}_{L^1(\mathbb{R}^N)}\|u\|^{\alpha_1}_{L^{p^*}(\Omega)}\|\varphi\|_{W^{1,p}(\Omega)}\\ &\le C_5\|u\|^{\alpha_1}_{W^{1,p}(\Omega)},\end{aligned} \tag{15}$$

as well as

$$\begin{aligned}b_2\int_\Omega |\nabla(\rho * E(u))|^{\alpha_2}|\varphi| dx &\le b_2\|\nabla(\rho * E(u))\|^{\alpha_2}_{L^p(\mathbb{R}^N)}\|\varphi\|_{L^{\frac{p}{p-\alpha_2}}(\Omega)}\\ &\le C_6\|\rho\|^{\alpha_2}_{L^1(\mathbb{R}^N)}\|\nabla E(u)\|^{\alpha_2}_{L^p(\mathbb{R}^N)}\|\varphi\|_{W^{1,p}(\Omega)}\\ &\le C_7\|\nabla u\|^{\alpha_2}_{L^p(\mathbb{R}^N)} \le C_7\|u\|^{\alpha_2}_{W^{1,p}(\Omega)}.\end{aligned} \tag{16}$$

Finally, hypothesis (A5) gives the following estimate for the boundary term in (10)

$$\begin{aligned}\int_{\partial\Omega} |\mathcal{C}(x,u)||\varphi| d\sigma &\le \int_{\partial\Omega} (c_1|u|^{\alpha_3} + c_2)|\varphi| d\sigma\\ &\le c_1\|u\|^{\alpha_3}_{L^{p_*}(\partial\Omega)}\|\varphi\|_{L^{\frac{p_*}{p_*-\alpha_3}}(\partial\Omega)} + c_2|\partial\Omega|^{\frac{p-1}{p}}\|\varphi\|_{L^p(\partial\Omega)}\\ &\le c_1\|u\|^{\alpha_3}_{W^{1,p}(\Omega)} + C_8.\end{aligned} \tag{17}$$

Taking into account (11)-(17) and applying once again the Sobolev embedding, from (10) we derive

$$|\langle Tu, \varphi\rangle| \le C_9(\|u\|^{\beta}_{W^{1,p}(\Omega)} + 1),$$

for all $\|\varphi\|_{W^{1,p}(\Omega)} \le 1$, with $\beta := \max\{p-1, \alpha_1, \alpha_2, \alpha_3\}$. This in turn implies

$$\|Tu\|_{(W^{1,p}(\Omega))^*} \le C_9(\|u\|^{\beta}_{W^{1,p}(\Omega)} + 1),$$

which shows that T is bounded.

Now we prove that T is pseudomonotone. Toward this, let $(u_n)_{n\in\mathbb{N}} \subset W^{1,p}(\Omega)$ be a sequence satisfying $u_n \rightharpoonup u$ for some $u \in W^{1,p}(\Omega)$ and

$$\limsup_{n\to+\infty} \langle Tu_n, u_n - u\rangle \le 0. \tag{18}$$

By Hölder's inequality and Rellich-Kondrachov compact embedding theorem it follows that, passing to a subsequence if necessary,

$$\begin{aligned}\left|\int_\Omega |u_n|^{p-2}u_n(u_n-u)dx\right| &\le \int_\Omega |u_n|^{p-1}|u_n-u|dx \\ &\le \|u_n\|^{p-1}_{L^p(\Omega)}\|u_n-u\|_{L^p(\Omega)} \to 0 \quad \text{as } n\to\infty.\end{aligned} \tag{19}$$

With a similar argument already exploited in (14)-(16), we have

$$\begin{aligned}&\left|\int_\Omega \mathcal{B}(x, \rho * E(u_n), \nabla(\rho * E(u_n)))(u_n-u)dx\right| \\ &\le \int_\Omega |\mathcal{B}(x, \rho * E(u_n), \nabla(\rho * E(u_n)))|\,|u_n-u|dx \\ &\le C_{10}\|u_n-u\|_{L^r(\Omega)} + C_{11}\|u_n\|^{\alpha_1}_{W^{1,p}(\Omega)}\|u_n-u\|_{L^{\frac{p^*}{p^*-\alpha_1}}(\Omega)} \\ &+ C_{12}\|u_n\|^{\alpha_2}_{W^{1,p}(\Omega)}\|u_n-u\|_{L^{\frac{p}{p-\alpha_2}}(\Omega)}\end{aligned}$$

for all $u \in W^{1,p}(\Omega)$. Since

$$r, \frac{p^*}{p^*-\alpha_1}, \frac{p}{p-\alpha_2} < p^*,$$

we can apply the Rellich-Kondrachov compact embedding theorem to the previous estimate, which gives

$$\lim_{n\to+\infty}\int_\Omega \mathcal{B}(x, \rho * E(u_n), \nabla(\rho * E(u_n)))(u_n-u)dx = 0. \tag{20}$$

Finally, assumption (A), Hölder's inequality, and the compactness of the trace mappings due to the inequalities

$$p, \frac{p_*}{p_*-\alpha_3} < p_*,$$

give

$$\begin{aligned}\left|\int_{\partial\Omega} C(x,u_n)(u_n-u)d\sigma\right| &\le \int_{\partial\Omega}(c_1|u_n|^{\alpha_3}+c_2)|u_n-u|d\sigma \\ &\le c_1\|u_n\|^{\alpha_3}_{L^{p_*}(\partial\Omega)}\|u_n-u\|_{L^{\frac{p_*}{p_*-\alpha_3}}(\partial\Omega)} \\ &+ c_2|\partial\Omega|^{\frac{p-1}{p}}\|u_n-u\|_{L^p(\partial\Omega)} \to 0 \quad \text{as } n\to\infty.\end{aligned} \tag{21}$$

If we gather (19), (20), and (21), in view of (9), then inequality (18) becomes

$$\limsup_{n\to+\infty}\int_\Omega \mathcal{A}(x,u_n,\nabla u_n)(u_n-u)dx \le 0. \tag{22}$$

Thanks to assumptions (A1)-(A3), it is allowed to invoke Theorem 2.109 of Reference 7. Then (22) and the weak convergence $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$ ensure the strong convergence $u_n \to u$ in $W^{1,p}(\Omega)$. Once the strong convergence is achieved, it is straightforward to deduce from the continuity of the involved Nemytskii maps that the nonlinear operator T is pseudomonotone.

The next step is to show that T is coercive. To this end, first observe that

$$\begin{aligned}\langle Tv,v\rangle &= \int_\Omega \mathcal{A}(x,v,\nabla v)\cdot\nabla v dx + a\int_\Omega |v|^p dx\\ &\quad+\int_\Omega \mathcal{B}(x,\rho*E(v),\nabla(\rho*E(v)))v dx+\int_{\partial\Omega}\mathcal{C}(x,v)v d\sigma.\end{aligned} \tag{23}$$

We estimate the terms of the inequality above separately. First, thanks to assumption (A3), we have

$$\begin{aligned}\int_\Omega \mathcal{A}(x,v,\nabla v)\cdot\nabla v dx &\ge \int_\Omega (a_4|\nabla v|^p - a_5)dx\\ &= a_4\|\nabla v\|^p_{L^p(\Omega)} - a_5|\Omega|.\end{aligned}$$

Moreover, reasoning as in (14)-(16) we have

$$\begin{aligned}\int_\Omega \mathcal{B}(x,\rho*E(v),\nabla(\rho*E(v)))v &\ge -\int_\Omega \left[f(x)+b_1|\rho*E(v)|^{\alpha_1}+b_2|\nabla(\rho*E(v))|^{\alpha_2}\right]|v|dx\\ &\ge -C_{13}\|v\|_{L^{r}(\Omega)} - C_{14}\|v\|^{\alpha_1}_{W^{1,p}(\Omega)}\|v\|_{L^{\frac{p^*}{p^*-\alpha_1}}(\Omega)}\\ &\quad - C_{15}\|v\|^{\alpha_2}_{W^{1,p}(\Omega)}\|v\|_{L^{\frac{p}{p-\alpha_2}}(\Omega)},\end{aligned}$$

as well as

$$\begin{aligned}\int_{\partial\Omega}\mathcal{C}(x,v)v d\sigma &\ge -\int_{\partial\Omega}(c_1|v|^{\alpha_3}+c_2)|v|d\sigma\\ &\ge -c_1\|v\|^{\alpha_3+1}_{L^{\alpha_3+1}(\partial\Omega)} - C_{16}\|v\|_{L^p(\partial\Omega)}.\end{aligned}$$

From (23), we easily derive

$$\begin{aligned}\langle Tv,v\rangle &\ge a_4\||\nabla v\||^p_{L^p(\Omega)}\| + a\|v\|^p_{L^p(\Omega)}\\ &\quad - C_{17}\left(\|v\|^{\alpha_1+1}_{W^{1,p}(\Omega)}+\|v\|^{\alpha_2+1}_{W^{1,p}(\Omega)}+\|v\|^{\alpha_3+1}_{W^{1,p}(\Omega)}+\|v\|_{W^{1,p}(\Omega)}+1\right),\end{aligned}$$

for every $v\in W^{1,p}(\Omega)$. Then by virtue of hypothesis (3), we have

$$\lim_{\|v\|_{W^{1,p}(\Omega)}\to+\infty}\frac{\langle Tv,v\rangle}{\|v\|_{W^{1,p}(\Omega)}} = +\infty,$$

thus the coercivity of T ensues. We have already shown that the nonlinear operator T is bounded, pseudomonotone and coercive. Consequently, all the requirements of Theorem 3 are fulfilled. Therefore, there exists $u\in W^{1,p}(\Omega)$ verifying $T(u)=0$. Taking into account (9), it follows that u is a weak solution to problem (1), which completes the proof.

4 | PROOF OF THEOREM 2

Let $u \in W^{1,p}(\Omega)$ be a weak solution to (1) for which we can admit that $u \not\equiv 0$. First, we show that $u \in L^r(\Omega)$ for every $r \in [1,+\infty)$. According to (7) and due to the fact that, in the nonlocal terms, the operator E and the convolution with ρ are linear maps, we can suppose that $u \geq 0$, otherwise we work with u^+ and u^-. Moreover, throughout the proof, we will denote by M_i, $i \in \mathbb{N}$, constants which depend on the given data and possibly on the solution itself, and we will specify the dependance when it will be relevant.

Let $h > 0$ and set $u_h(x) := \min\{u(x), h\}$ for $x \in \Omega$. For every number $\kappa > 0$, choose $\varphi = uu_h^{\kappa p}$ as test function in (4). We note that

$$\nabla\varphi = u_h^{\kappa p}\nabla u + \kappa p u u_h^{\kappa p-1}\nabla u_h.$$

Inserting such a φ in (4) gives

$$\begin{aligned}&\int_\Omega (\mathcal{A}(x,u,\nabla u)\cdot\nabla u)u_h^{\kappa p}dx + \kappa p\int_\Omega(\mathcal{A}(x,u,\nabla u)\cdot\nabla u_h)u_h^{\kappa p-1}udx + a\int_\Omega u^p u_h^{\kappa p}dx\\ &= \int_\Omega \mathcal{B}(x,\rho * E(u),\nabla(\rho * E(u)))uu_h^{\kappa p}dx + \int_{\partial\Omega} C(x,u)uu_h^{\kappa p}d\sigma.\end{aligned} \tag{24}$$

Applying condition (H2) yields

$$\begin{aligned}&\int_\Omega (\mathcal{A}(x,u,\nabla u)\cdot\nabla u)u_h^{\kappa p}dx\\ &\geq \int_\Omega \left[a_4|\nabla u|^p - a_5 u^{p^*} - a_6\right]u_h^{\kappa p}dx\\ &\geq a_4\int_\Omega |\nabla u|^p u_h^{\kappa p}dx - (a_5+a_6)\int_\Omega u^{p^*}u_h^{\kappa p}dx - a_6|\Omega|,\end{aligned} \tag{25}$$

and

$$\begin{aligned}&\int_\Omega (\mathcal{A}(x,u,\nabla u)\cdot\nabla u_h)u_h^{\kappa p-1}udx\\ &= \int_{\{x\in\Omega:\ u(x)\leq h\}} (\mathcal{A}(x,u,\nabla u)\cdot\nabla u)u_h^{\kappa p}dx\\ &\geq \int_{\{x\in\Omega:\ u(x)\leq h\}} \left[a_4|\nabla u|^p - a_5 u^{p^*} - a_6\right]u_h^{\kappa p}dx\\ &\geq a_4\int_{\{x\in\Omega:\ u(x)\leq h\}} |\nabla u|^p u_h^{\kappa p}dx - (a_5+a_6)\int_\Omega u^{p^*}u_h^{\kappa p}dx - \kappa p a_6|\Omega|.\end{aligned} \tag{26}$$

Note that in the last passage of both (25) and (26), we use the following fact

$$u_h^{\kappa p} \leq u^{p^*}u_h^{\kappa p} + 1.$$

Indeed, if $u > 1$, then $u^{p^*} > 1$, which implies that

$$u_h^{\kappa p} \leq u^{p^*}u_h^{\kappa p} < u^{p^*}u_h^{\kappa p} + 1.$$

If $u \leq 1$, then we refer to the definition of $u_h := \min\{u(x), h\}$, and again distinguish among two cases.

If $h > 1$, then $u_h(x) = u(x) \leq 1$, and it follows that

$$u_h^{\kappa p} \leq 1 < 1 + u^{p^*}u_h^{\kappa p},$$

because $u^{p^*}\ u_h^{\kappa p}>0$. If $h\leq 1$, then $u_h(x)=h\leq 1$, and we have again

$$u_h^{\kappa p}\leq 1<1+u^{p^*}\ u_h^{\kappa p}.$$

By means of condition (H3), we have

$$\begin{aligned}&\int_\Omega \mathcal{B}(x,\rho*E(u),\nabla(\rho*E(u)))uu_h^{\kappa p}dx\\ &\leq \int_\Omega (f(x)+b_1|\rho*E(u)|^{\alpha_1}+b_2|\nabla(\rho*E(u))|^{\alpha_2})\,uu_h^{\kappa p}dx.\end{aligned} \tag{27}$$

We estimate the terms on the right-hand side of (27) separately. First, through Hölder's inequality, we have

$$\int_\Omega f(x)uu_h^{\kappa p}dx\leq \|f\|_{r'}\left(\int_\Omega (uu_h^{\kappa p})^r dx\right)^{1/r}\leq M_1(1+\|uu_h^\kappa\|^p_{L^{pr}(\Omega)}). \tag{28}$$

Moreover, we set $r_1:=\frac{p^*}{p^*-\alpha_1}$ and $r_2:=\frac{p}{p-\alpha_2}$. Making use of Hölder's inequality, with an argument similar as in (15)-(16), we find that

$$\begin{aligned}\int_\Omega |\rho*E(u)|^{\alpha_1}uu_h^{\kappa p}dx&\leq \|\rho*E(u)\|^{\alpha_1}_{L^{p^*}(\mathbb{R}^N)}\|uu_h^{\kappa p}\|_{L^{r_1}(\Omega)}\\ &\leq M_2\|\rho*E(u)\|^{\alpha_1}_{W^{1,p}(\mathbb{R}^N)}\|uu_h^{\kappa p}\|_{L^{r_1}(\Omega)}\\ &\leq M_3\|\rho\|^{\alpha_1}_{L^1(\mathbb{R}^N)}\|u\|^{\alpha_1}_{W^{1,p}(\Omega)}\|uu_h^{\kappa p}\|_{L^{r_1}(\Omega)}\\ &\leq M_4\left(1+\|uu_h^\kappa\|^p_{L^{pr_1}(\Omega)}\right),\end{aligned} \tag{29}$$

and

$$\begin{aligned}\int_\Omega |\nabla(\rho*E(u))|^{\alpha_2}uu_h^{\kappa p}dx&\leq M_5\||\nabla(\rho*E(u))|\|^{\alpha_2}_{L^p(\mathbb{R}^N)}\|uu_h^{\kappa p}\|_{L^{r_2}(\Omega)}\\ &\leq M_6\|\rho\|^{\alpha_2}_{L^1(\mathbb{R}^N)}\||\nabla u|\|^{\alpha_2}_{L^p(\Omega)}\|uu_h^{\kappa p}\|_{L^{r_2}(\Omega)}\\ &\leq M_7\left(1+\|uu_h^\kappa\|^p_{L^{pr_2}(\Omega)}\right),\end{aligned} \tag{30}$$

where the constants M_4 and M_7 depend on the solution u, precisely

$$M_4=M_4(\|u\|_{W^{1,p}(\Omega)})\quad\text{and}\quad M_7=M_7(\|\nabla u\|_{L^p(\Omega)}). \tag{31}$$

Via hypothesis (H4), we estimate

$$\begin{aligned}\int_{\partial\Omega} \mathcal{C}(x,u)uu_h^{\kappa p}d\sigma&\leq \int_{\partial\Omega}\left(c_1u^{p_*-1}+c_2\right)uu_h^{\kappa p}d\sigma\\ &\leq (c_1+c_2)\int_{\partial\Omega}u^{p_*}u_h^{\kappa p}d\sigma+c_2|\partial\Omega|.\end{aligned} \tag{32}$$

From (5) and the hypothesis on r, we see that

$$\tilde r:=\max\{r,r_1,r_2\}<\frac{p^*}{p}. \tag{33}$$

Combining (24)-(30), (32), (33) results in

$$\begin{aligned}&a_4\left(\int_\Omega |\nabla u|^p u_h^{\kappa p}dx+\kappa p\int_{\{x\in\Omega:\ u(x)\le h\}}|\nabla u|^p u_h^{\kappa p}dx\right)\\ &\le (\kappa p+1)(a_5+a_6)\int_\Omega u^{p^*}\ u_h^{\kappa p}dx+(c_1+c_2)\int_{\partial\Omega}u^{p_*}\ u_h^{\kappa p}d\sigma\\ &\quad +M_8\|uu_h^\kappa\|^p_{L^{p\tilde r}(\Omega)}+M_9(\kappa+1),\end{aligned}\tag{34}$$

with positive constants M_8 and M_9 independent on κ.

Note that

$$\begin{aligned}&\int_\Omega |\nabla u|^p u_h^{\kappa p}dx+\kappa p\int_{\{x\in\Omega:\ u(x)\le h\}}|\nabla u|^p u_h^{\kappa p}dx\\ &=\int_{\{x\in\Omega:\ u(x)>h\}}|\nabla u|^p u_h^{\kappa p}dx+(\kappa p+1)\int_{\{x\in\Omega:\ u(x)\le h\}}|\nabla u|^p u_h^{\kappa p}dx\\ &\ge \frac{\kappa p+1}{(\kappa+1)^p}\int_{\{x\in\Omega:\ u(x)>h\}}|\nabla u|^p u_h^{\kappa p}dx+(\kappa p+1)\int_{\{x\in\Omega:\ u(x)\le h\}}|\nabla u|^p u_h^{\kappa p}dx\\ &\ge \frac{\kappa p+1}{(\kappa+1)^p}\int_\Omega |\nabla(uu_h^\kappa)|^p dx,\end{aligned}$$

thanks to Bernoulli's inequality $(\kappa+1)^p\ge\kappa p+1$ and to the fact that $(\kappa+1)^p>1$. Therefore, (34) and (6) entail

$$\begin{aligned}\frac{\kappa p+1}{(\kappa+1)^p}\|uu_h^\kappa\|^p_{W^{1,p}(\Omega)}&\le \frac{\kappa p+1}{(\kappa+1)^p}\|uu_h^\kappa\|^p_{L^p(\Omega)}+M_{10}(\kappa p+1)\int_\Omega u^{p^*}u_h^{\kappa p}dx\\ &\quad+M_{11}\int_{\partial\Omega}u^{p_*}u_h^{\kappa p}d\sigma+M_8\|uu_h^\kappa\|^p_{L^{p\tilde r}(\Omega)}+M_9(\kappa+1)\\ &\le M_{10}(\kappa p+1)\int_\Omega u^{p^*}u_h^{\kappa p}dx+M_{11}\int_{\partial\Omega}u^{p_*}u_h^{\kappa p}d\sigma\\ &\quad+M_{12}\left(\frac{\kappa p+1}{(\kappa+1)^p}+1\right)\|uu_h^\kappa\|^p_{L^{p\tilde r}(\Omega)}+M_9(\kappa+1).\end{aligned}\tag{35}$$

We now aim to estimate the critical integrals on the right-hand side of (35). To this end, we set $A:=u^{p^*-p}$ and $B:=u^{p_*-p}$, and take $\Lambda,\Gamma>0$. Then Hölder's inequality and the Sobolev embedding give

$$\begin{aligned}\int_\Omega u^{p^*}u_h^{\kappa p}dx&=\int_{\{x\in\Omega:\ A(x)\le\Lambda\}}A(uu_h^\kappa)^p dx+\int_{\{x\in\Omega:\ A(x)>\Lambda\}}A(uu_h^\kappa)^p dx\\ &\le\Lambda\int_{\{x\in\Omega:\ A(x)\le\Lambda}(uu_h^\kappa)^p dx\\ &\quad+\left(\int_{\{x\in\Omega:\ A(x)>\Lambda\}}A^{\frac{p^*}{p^*-p}}dx\right)^{\frac{p^*-p}{p^*}}\left(\int_\Omega (uu_h^\kappa)^{p^*}dx\right)^{\frac{p}{p^*}}\\ &\le\Lambda\|uu_h^\kappa\|^p_{L^p(\Omega)}+\left(\int_{\{x\in\Omega:\ A(x)>\Lambda\}}A^{\frac{p^*}{p^*-p}}dx\right)^{\frac{p^*-p}{p^*}}C_\Omega^p\|uu_h^\kappa\|^p_{W^{1,p}(\Omega)},\end{aligned}\tag{36}$$

as well as

$$\begin{aligned}\int_{\partial\Omega}u^{p_*}\ u_h^{\kappa p}d\sigma&=\int_{\{x\in\partial\Omega:\ B(x)\le\Gamma\}}B(uu_h^\kappa)^p d\sigma+\int_{\{x\in\partial\Omega:\ B(x)>\Gamma\}}B(uu_h^\kappa)^p d\sigma\\ &\le\Gamma\int_{\{x\in\partial\Omega:\ B(x)\le\Gamma\}}(uu_h^\kappa)^p d\sigma\\ &\quad+\left(\int_{\{x\in\partial\Omega:\ B(x)>\Gamma\}}B^{\frac{p_*}{p_*-p}}d\sigma\right)^{\frac{p_*-p}{p_*}}\left(\int_{\partial\Omega}(uu_h^\kappa)^{p_*}\ d\sigma\right)^{\frac{p}{p_*}}\end{aligned}$$

$$\leq \Gamma \|uu_h^\kappa\|^p_{L^p(\partial\Omega)} + \left(\int_{\{x\in\partial\Omega:\ B(x)>\Gamma\}} B^{\frac{p_*}{p_*-p}}\,d\sigma\right)^{\frac{p_*-p}{p_*}} c^p_{\partial\Omega}\|uu_h^\kappa\|^p_{W^{1,p}(\Omega)}, \tag{37}$$

with the embedding constants C_Ω and $c_{\partial\Omega}$. Moreover, if we set

$$f_1(\Lambda) := \left(\int_{\{x\in\Omega:\ A(x)>\Lambda\}} A^{\frac{p^*}{p^*-p}}\,dx\right)^{\frac{p^*-p}{p^*}}$$
$$\text{as well as}\quad f_2(\Gamma) := \left(\int_{\{x\in\partial\Omega:\ B(x)>\Gamma\}} B^{\frac{p_*}{p_*-p}}\,d\sigma\right)^{\frac{p_*-p}{p_*}}, \tag{38}$$

we see that

$$f_1(\Lambda)\to 0 \quad \text{as } \Lambda\to 0 \quad \text{as well as} \quad f_2(\Gamma)\to 0 \quad \text{as } \Gamma\to 0. \tag{39}$$

From (35), taking into account (36)-(38) and applying Hölder's inequality, we have

$$\begin{aligned}\frac{\kappa p+1}{(\kappa+1)^p}\|uu_h^\kappa\|^p_{W^{1,p}(\Omega)} &\leq M_{13}\left((\kappa p+1)\Lambda + 1 + \frac{\kappa p+1}{(\kappa+1)^p}\right)\|uu_h^\kappa\|^p_{L^{p\tilde r}(\Omega)}\\ &\quad M_{10}(\kappa p+1)f_1(\Lambda)C^p_\Omega\|uu_h^\kappa\|^p_{W^{1,p}(\Omega)} + M_{11}\Gamma\|uu_h^\kappa\|^p_{L^p(\partial\Omega)}\\ &\quad + M_{11}f_2(\Gamma)c^p_{\partial\Omega}\|uu_h^\kappa\|^p_{W^{1,p}(\Omega)} + M_9(\kappa+1).\end{aligned} \tag{40}$$

Taking into account (39), we can choose $\Lambda = \Lambda(\kappa,u), \Gamma = \Gamma(\kappa,u) > 0$ large enough in order to have

$$M_{10}(\kappa p+1)f_1(\Lambda)C^p_\Omega = \frac{\kappa p+1}{4(\kappa+1)^p} \quad \text{as well as} \quad M_{11}f_2(\Gamma)c^p_{\partial\Omega} = \frac{\kappa p+1}{4(\kappa+1)^p}.$$

Then from (40), we have

$$\begin{aligned}\frac{\kappa p+1}{4(\kappa+1)^p}\|uu_h^\kappa\|^p_{W^{1,p}(\Omega)} &\leq M_{13}\left((\kappa p+1)\Lambda(\kappa,u) + 1 + \frac{\kappa p+1}{(\kappa+1)^p}\right)\|uu_h^\kappa\|^p_{L^{p\tilde r}(\Omega)}\\ &\quad + M_{11}\Gamma(\kappa,u)\|uu_h^\kappa\|^p_{L^p(\partial\Omega)} + M_9(\kappa+1),\end{aligned}$$

where both $\Lambda(\kappa,u)$, $\Gamma(\kappa,u)$ depend on κ and on the solution itself.

From this point, we proceed as in Theorem 3.1, Case I.1 of Reference 5, with $\|uu_h^\kappa\|_{L^p(\Omega)}$ replaced by $\|uu_h^\kappa\|_{L^{p\tilde r}(\Omega)}$, which gives us

$$\|u\|_{L^{(\kappa+1)p^*}(\Omega)} \leq M_{14}(\kappa,u),$$

for any $\kappa > 0$, where $M_{14}(\kappa,u)$ is a positive constant which depends on κ and on the solution u. Consequently, the claim that $u\in L^r(\Omega)$ for every $r\in[1,+\infty)$ follows.

Once the $L^r(\Omega)$-bound is reached, the proof of the $L^r(\partial\Omega)$-boundedness is straightforward (see Case I.2 of Reference 5).

We are now in a position to establish the L^∞-boundedness of u. Taking advantage of (33), we fix $q_1 \in (p\tilde r, p^*)$ and $q_2\in(p,p_*)$. By Hölder's inequality and the obtained L^r-bounds in Ω and on $\partial\Omega$, we can express (35) in the form

$$\begin{aligned}\frac{\kappa p+1}{(\kappa+1)^p}\|uu_h^\kappa\|^p_{W^{1,p}(\Omega)} &\leq M_{15}\left(\frac{\kappa p+1}{(\kappa+1)^p} + \kappa p + 2\right)\|uu_h^\kappa\|^p_{L^{q_1}(\Omega)}\\ &\quad + M_{16}\|uu_h^\kappa\|^p_{L^{q_2}(\partial\Omega)} + M_{17}(\kappa+1).\end{aligned}$$

Then, proceeding as in Case II.1 of Reference 5, arranging the constants and applying Hölder's inequality, the Sobolev embedding, and Fatou's lemma, we achieve

$$\|u\|_{L^{(\kappa_n+1)p^*}(\Omega)} \leq M_{18},$$

where M_{18} is independent on κ and $(\kappa_n+1)p^* \to \infty$ as $n \to \infty$.

Therefore, we can invoke Proposition 1, whence $u \in L^\infty(\Omega)$. Finally, by Proposition 2, it follows that $\gamma u \in L^\infty(\partial\Omega)$. The proof is thus complete.

Remark 1. Hypothesis (H1) is not needed in the proof of Theorem 2, but it is necessary in order to have a well-defined weak solution as formulated in (4).

Remark 2. The bounds obtained in Theorem 2 depend on the data in assumption (H) and on the solution itself. The proof shows that the following estimate is valid

$$\|u\|_{L^r(\Omega)} \le M(\|u\|_{L^{p^*}(\Omega)}), \quad \forall\, r \ge 1, \tag{41}$$

with a constant $M(\|u\|_{L^{p^*}(\Omega)})$ depending on $\|u\|_{p^*}$. The key step for proving estimate (41) is (31).

Remark 3. Once (41) is reached, an alternative reasoning to get the uniform boundedness of u can be carried out as follows. Let $0 < t < \|u\|_{L^\infty(\Omega)}$, where a priori one can have $\|u\|_{L^\infty(\Omega)} = +\infty$. Setting

$$\Omega_t = \{x \in \Omega : |u(x)| > t\},$$

it is clear that

$$\|u\|_{L^r(\Omega)} \ge \left(\int_{\Omega_t} |u|^r dx\right)^{\frac{1}{r}} \ge t|\Omega_t|^{\frac{1}{r}}, \quad \forall\, r \ge 1,$$

so

$$\liminf_{r\to\infty} \|u\|_{L^r(\Omega)} \ge t.$$

Since $t \in (0, \|u\|_\infty)$ is arbitrary, we deduce that

$$\liminf_{r\to\infty} \|u\|_{L^r(\Omega)} \ge \|u\|_{L^\infty(\Omega)}.$$

In view of estimate (41), the conclusion that $u \in L^\infty(\Omega)$ is achieved.

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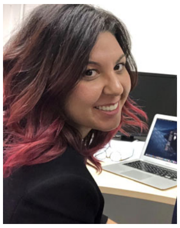

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