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Persistence of autoregressive sequences with logarithmic tails*

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Abstract

We consider autoregressive sequences $X_n = aX_{n-1} + \xi_n$ and $M_n = \max\{aM_{n-1}, \xi_n\}$ with a constant $a \in (0,1)$ and with positive, independent and identically distributed innovations $\{\xi_k\}$. It is known that if $\mathbf{P}(\xi_1 > x) \sim \frac{d}{\log x}$ with some $d \in (0, -\log a)$ then the chains $\{X_n\}$ and $\{M_n\}$ are null recurrent. We investigate the tail behaviour of recurrence times in this case of logarithmically decaying tails. More precisely, we show that the tails of recurrence times are regularly varying of index $-1 - d/\log a$. We also prove limit theorems for $\{X_n\}$ and $\{M_n\}$ conditioned to stay over a fixed level x_0.

Furthermore, we study tail asymptotics for recurrence times of $\{X_n\}$ and $\{M_n\}$ in the case when these chains are positive recurrent and the tail of $\log \xi_1$ is subexponential.

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1 Introduction

Let $\{\xi_n\}_{n\ge 1}$ be a sequence of independent and identically distributed random variables. Let $a \in (0,1)$ be a constant. The corresponding AR(1)-sequence $X = \{X_n\}_{n\ge 0}$ is defined by

$$X_n := aX_{n-1} + \xi_n, \quad n \ge 1,$$

where the starting position X_0 can be either random or deterministic.

Besides the Markov chain X we shall consider the so-called *maximal autoregressive sequence* $M = \{M_n\}_{n\ge 0}$, where

$$M_n = \max\{aM_{n-1}, \xi_n\}, \quad n \ge 1.$$

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The Markov chains X and M have rather similar properties. If, for example, the innovations are non-negative then these two chains are recurrent, positive recurrent or transient at the same time. More precisely, according to Theorem 3.1 in Zerner [17], the chains $\{X_n\}$ and $\{M_n\}$ are recurrent if and only if

$$\sum_{n=0}^{\infty}\prod_{m=0}^{n}\mathbf{P}(|\xi_1|\le ta^{-m})=\infty \tag{1.1}$$

for every t satisfying $\mathbf{P}(|\xi_1|\le t)>0$. Furthermore, X and M are positive recurrent if and only if $\mathbf{E}[\log(1+|\xi_1|)]$ is finite.

In the sequel we will assume that the innovations satisfy

$$\mathbf{P}(\xi_n\ge 0)=1. \tag{1.2}$$

Under this standing assumption we may define

$$\eta_n:=\log_A\xi_n \quad\text{and}\quad R_n:=\log_A M_n,$$

where $A=a^{-1}$. Then the sequence $R=\{R_n\}_{n\ge 0}$ satisfies the recursive relation

$$R_n=\max\{R_{n-1}-1,\eta_n\},\quad n\ge 1.$$

This Markov chain is a special random exchange process, see Helland and Nilsen [11] for the definition of this class of processes.

In this paper we shall consider the case when the tail of innovations decreases logarithmically. More precisely, in the main part of the paper we will deal with the situation when

$$\mathbf{P}(\xi_1>x)\sim\frac{d}{\log x}\quad\text{as } x\to\infty \tag{1.3}$$

with some constant $d>0$. This is equivalent to

$$\mathbf{P}(\eta_1>y)\sim\frac{c}{y},\quad\text{where } c:=\frac{d}{\log A}. \tag{1.4}$$

(We shall explicitly mention one of these two conditions every time we need it.)

Notice also that if $d>0$ then $\mathbf{E}\log(1+\xi_1)=\infty$ and, consequently, the chains $\{X_n\}_{n\ge0}$ and $\{M_n\}_{n\ge0}$ are not positive recurrent. If (1.3) holds then, using the criterion (1.1), we conclude that

- $d>\log A$ $(c>1)$ $\Rightarrow$ $\{X_n\}_{n\ge0}$ and $\{M_n\}_{n\ge0}$ are transient;
- $d<\log A$ $(c<1)$ $\Rightarrow$ $\{X_n\}_{n\ge0}$ and $\{M_n\}_{n\ge0}$ are null-recurrent.

In the critical case $d=\log A$ $(c=1)$ one has to consider further terms in the asymptotic representation for the tails $\mathbf{P}(\xi_1>x)$ and $\mathbf{P}(\eta_1>y)$. Assume that, for some $k\ge 0$,

$$\mathbf{P}(\eta_1>y)=\frac{1}{y}\sum_{j=0}^{k}\prod_{l=1}^{j}\frac{1}{\log_{(l)}y}+(r_k+o(1))\frac{1}{y}\prod_{l=1}^{k+1}\frac{1}{\log_{(l)}y},\quad y\to\infty,$$

where $\log_{(l)}x$ is the l-th iteration of the logarithm. Then, applying (1.1) once again, we obtain

- $r_k>1$ $\Rightarrow$ $\{X_n\}$ and $\{M_n\}$ are transient;
- $r_k<1$ $\Rightarrow$ $\{X_n\}$ and $\{M_n\}$ are null-recurrent.

A further similarity between the chains $\{X_n\}_{n\geq 0}$ and $\{M_n\}_{n\geq 0}$ consists in the joint scaling behaviour of these chains. More precisely, Buraczewski and Iksanov [7] have shown that if (1.3) is valid then

$$\left(\frac{\log_A X_{[nt]}}{n}\right)_{t\geq 0} \Rightarrow Z = (Z_t)_{t\geq 0} \tag{1.5}$$

in the Skorohod J_1-topology on the space D. The limiting process Z is a self-similar Markov process. Convergence of one-dimensional marginals was previously established in [15]. In [7] it is described with the help of an appropriate Poisson point process. One can describe this limiting process also via the transition probabilities:

$$\mathbf{P}_x((x-t)^+ \leq Z_t \leq y) = \left(\frac{y}{y+t}\right)^c, \quad y \geq (x-t)^+,\ x \geq 0, \tag{1.6}$$

where c is defined in (1.4). When the limiting process starts at the origin this formula can be deduced from Remark 1.3 in [7]. For the case of a general non-negative starting point the transition probabilities can be deduced using (1.8) below. It is easy to see that if $X_0 = M_0$ then

$$M_k \leq X_k \leq (k+1)M_k \quad \text{for all } k \geq 1.$$

This implies that (1.5) is equivalent to

$$\left(\frac{\log_A M_{nt}}{n}\right)_{t\geq 0} \Rightarrow Z. \tag{1.7}$$

In its turn, (1.7) is equivalent to

$$\left(\frac{R_{nt}}{n}\right)_{t\geq 0} \Rightarrow Z. \tag{1.8}$$

The main purpose of this paper is to study the asymptotic behaviour of the recurrence times

$$\begin{aligned} T_x^{(X)} &:= \inf\{k \geq 1 :\ X_k \leq x\}, \\ T_x^{(M)} &:= \inf\{k \geq 1 :\ M_k \leq x\}, \\ T_x^{(R)} &:= \inf\{k \geq 1 :\ R_k \leq x\}. \end{aligned}$$

Persistence of auto-regressive processes has attracted a significant attention of many researchers in the recent past, but almost all results known in the literature deal with the case when some power moments of the innovations ξ_k are finite. Under this assumption it is known that the tail of $T_x^{(X)}$ has an exponential decay, i.e.

$$-\frac{1}{n}\log \mathbf{P}(T_x^{(X)} > n) \to \lambda \in (0,\infty). \tag{1.9}$$

For further details in this case we refer to [4], [12] and references therein. If all power moments of innovations are finite then $\mathbf{P}(T_x^{(X)} > n) \sim Ce^{-\lambda n}$ and the conditional distribution $\mathbf{P}(X_n \in \cdot | T_x^{(X)} > n)$ converges towards the corresponding quasi-stationary distribution, see [12]. It is worth mentioning that one can compute the persistence exponent λ only in some special cases. Some examples of autoregressive processes, for which there exist closed form expressions for λ, can be found in [1] and [4]. The authors of [3] have found a series expansion for λ in the case of normally distributed innovations.

In to contrast to the above mentioned contributions we focus on innovations with fat tail when all power moments of innovations are infinite. This will imply that (1.9) does not hold anymore and instead we will prove that the tails of the first hitting times $T_x^{(X)}$ exhibit subexponential decay.

We start with the null-recurrent case. More precisely we consider first the innovations that satisfy (1.3). As we have mentioned before, the chains $\{X_n\}_{n\geq 0}$, $\{M_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ have the same scaling limit Z in this case. For that reason we first collect some crucial for us properties of the process Z.

Theorem 1.1. *a) If $c\leq 1$ then the process Z is recurrent. If $c<1$ then the stopping time $T_0^{(Z)}:=\inf\{s>0: Z_s=0 \text{ or } Z_{s-}=0\}$ is almost surely finite and, furthermore, for every $z>0$,*

$$\mathbf{P}\left(T_0^{(Z)}>t \,\middle|\, Z_0=z\right)=\begin{cases} 1, & t<z,\\ \frac{1}{B(c,1-c)}\int_0^{z/t}(1-u)^{c-1}u^{-c}du, & t\geq z,\end{cases} \tag{1.10}$$

where $B(a,b)$ denotes the usual Euler Beta function evaluated at the points a and b.

b) The function $u(z)=z^{1-c}$ is harmonic for Z killed at $T_0^{(Z)}$, that is

$$u(z)=\mathbf{E}_z[u(Z_t);T_0^{(Z)}>t],\quad t,z>0.$$

c) The sequence of distributions $\mathbf{P}_z\left(Z\in\cdot|T_0^{(Z)}>1\right)$ on $D[0,1]$ converges weakly, as $z\to 0$, towards a non-degenerate distribution $\mathbf{Q}$.

We now turn to the recurrence times of the chains $\{M_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$. Since $R_n=\log_A M_n$,

$$T_{x_0}^{(R)}=\inf\{n\geq 1: R_n\leq x_0\}=\inf\{n\geq 1: M_n\leq A^{x_0}\}=T_{A^{x_0}}^{(M)}.$$

Thus, it suffices to formulate the results for one of these processes.

Set

$$u_0(x):=\int_0^x \mathbf{P}(\eta_1>y)dy,\quad x\geq 0$$

and

$$U_0(x):=\int_0^x e^{-u_0(y)}dy,\quad x\geq 0. \tag{1.11}$$

If (1.4) holds then $u_0(x)\sim c\log x$ as $x\to\infty$ and $e^{-u_0(x)}$ is regularly varying of index $-c$. Consequently, the function $U_0(x)$ is regularly varying of index $1-c$.

Theorem 1.2. *Assume that x_0 is such that $\mathbf{P}(\eta_1\leq x_0)\mathbf{P}(\eta_1>x_0)>0$. Then the equation*

$$G(x)=\mathbf{E}_x[G(R_1);T_{x_0}^{(R)}>1],\quad x>x_0$$

has a non-trivial non-negative solution if and only if $\mathbf{E}\eta_1^+=\infty$. In the latter case

$$G(x)=C\left(1+\sum_{j=1}^{\infty}\prod_{k=0}^{j-1}\mathbf{P}(\eta_1\leq x_0+k)1_{(x_0+j,\infty)}(x)\right)$$

for every $C>0$.

If (1.4) holds with some $c\in(0,1)$ then

(i) $G(x)\sim\gamma U_0(x)$ for some $\gamma\in(0,\infty)$;

(ii) there exists a constant $C>0$ such that such that for $x>x_0$

$$\frac{1}{C}\frac{G(x\wedge n)}{G(n)}\leq\mathbf{P}_x(T_{x_0}^{(R)}>n)\leq C\frac{G(x)}{G(n)},\quad n\geq 1;$$

(iii) there exists a positive constant $\varkappa = \varkappa(c)$ *such that for* $x > x_0$

$$\mathbf{P}_x(T_{x_0}^{(R)} > n) \sim \varkappa \frac{G(x)}{G(n)}, \quad n \to \infty.$$

and the sequence of conditional distributions $\mathbf{P}_x\left(\frac{R_{[nt]}}{n} \in \cdot \,\middle|\, T_{x_0}^{(R)} > n\right)$ *on* $D[0,1]$ *converges weakly to* $\mathbf{Q}$ *defined in Theorem 1.1.*

We now state our main result for the chain $\{X_n\}_{n\geq 0}$. We emphasize, that the construction of the harmonic function G in Theorem 1.2 is established in a rather explicit way. In contrast to this the proof for the existence of the harmonic function for autoregressive processes $\{X_n\}_{n\geq 0}$ – formulated in the following theorem – requires more powerful analytical and probabilistic tools.

Theorem 1.3. *Assume that* (1.4) *holds with some* $c \in (0,1)$*. (This is equivalent to* (1.3) *with* $0 < d < \log A$*.) For every* x_0 *satisfying* $\mathbf{P}(ax_0 + \xi_1 \leq x_0) > 0$ *we have:*

(i) There exists a strictly positive on (x_0, ∞) *function* V *such that*

$$V(x) = \mathbf{E}_x[V(X_1); T_{x_0}^{(X)} > 1], \quad x > x_0.$$

In other words, V *is harmonic for the chain* $\{X_n\}$ *killed at leaving* (x_0, ∞)*. Furthermore,* $V(A^x) \sim U_0(x)$*, where* U_0 *is defined in* (1.11)*.*

(ii) There exists a constant C *such that*

$$\frac{1}{C}\frac{V(x \wedge A^n)}{V(A^n)} \leq \mathbf{P}_x(T_{x_0}^{(X)} > n) \leq C\frac{V(x)}{V(A^n)} \tag{1.12}$$

for all $n \geq 1$ *and all* $x > x_0$*.*

(iii) There exists a positive constant $\varkappa = \varkappa(c)$ *such that, for every* $x > x_0$*,*

$$\mathbf{P}_x(T_{x_0}^{(X)} > n) \sim \varkappa \frac{V(x)}{V(A^n)}, \quad n \to \infty. \tag{1.13}$$

Furthermore, the sequence of conditional distributions

$$\mathbf{P}_x\left(\frac{\log_A X_{[nt]}}{n} \in \cdot \,\middle|\, T_{x_0}^{(X)} > n\right)$$

on $D[0,1]$ *converges weakly to* $\mathbf{Q}$ *defined in Theorem 1.1.*

We now turn to the positive recurrent case: $\mathbf{E}[\eta_1] < \infty$. To determine the tail behaviour of recurrence times we shall assume that $\overline{F}(y) := \mathbf{P}(\eta_1 > y)$ is subexponential. We make use of the following class introduced in [13].

Definition 1.4. *A distribution function* F *with finite* $\mu_+ = \int_0^\infty \overline{F}(y)dy < \infty$ *belongs to the class* $\mathcal{S}^*$ *of strong subexponential distributions if* $\overline{F}(x) > 0$ *for all* x *and*

$$\frac{\int_0^x \overline{F}(x-y)\overline{F}(y)dy}{\overline{F}(x)} \to 2\mu_+, \quad \text{as } x \to \infty.$$

This class is a proper subclass of class $\mathcal{S}$ of subexponential distributions. It is shown in [13] that the Pareto, lognormal and Weibull distributions belong to the class $\mathcal{S}^*$. An example of a subexponential distribution with finite mean which does not belong to $\mathcal{S}^*$ can be found in [9].

Theorem 1.5. *Assume that x_0 is such that $\mathbf{P}(\eta_1 \le x_0)\mathbf{P}(\eta_1 > x_0) > 0$. Assume also that $\mathbf{E}\eta_1 < \infty$ and that $F \in \mathcal{S}^*$. Then, for any $x > x_0$*

$$\mathbf{P}_x(T^{(R)}_{x_0} > n) \sim \mathbf{E}_x[T^{(R)}_{x_0}]\mathbf{P}(\eta_1 > n). \tag{1.14}$$

The expectation $\mathbf{E}_x[T^{(R)}_{x_0}]$ can be computed explicitly: for every $n \ge 0$ and every $x \in (x_0+n, x_0+n+1]$ one has

$$\mathbf{E}_x[T^{(R)}_{x_0}] = \frac{1}{\prod_{k=0}^{\infty}\mathbf{P}(\eta_1 \le x_0+k)}\left(1+\sum_{j=1}^{n}\prod_{k=0}^{j-1}\mathbf{P}(\eta_1 \le x_0+k)\right). \tag{1.15}$$

Our approach to the proof of this theorem is based on a recursive equation for the tail of $T^{(R)}_{x_0}$, see Proposition 6.1 below. In the case of the chain $\{X_n\}_{n\ge 0}$ we do not have such an equation and we have to work with upper and lower estimates. This leads to more restrictive assumptions on the tail of innovations η_k.

Theorem 1.6. *Assume that x_0 is such that $\mathbf{P}(ax_0 + \xi_1 \le x_0) > 0$. Assume also that $\mathbf{E}\eta < \infty$,that $F \in \mathcal{S}^*$ and that*

$$\mathbf{P}(\eta > x) \sim \mathbf{P}(\eta > x - \log x), \quad \text{as } x \to \infty. \tag{1.16}$$

Then, for any $x > x_0$,

$$\mathbf{P}_x(T^{(X)}_{x_0} > n) \sim \mathbf{E}_x[T^{(X)}_{x_0}]\mathbf{P}(\eta > n). \tag{1.17}$$

The rest of the paper is organised as follows. In Section 2 we discuss properties of Z and prove Theorem 1.1. In Section 3 we construct harmonic functions for processes under consideration proving corresponding parts of Theorem 1.2 and Theorem 1.3. In Section 4 we derive lower and upper bounds for recurrence times $T^{(R)}_{x_0}$ and $T^{(X)}_{x_0}$ proving part (ii) of Theorem 1.2 and Theorem 1.3. In Section 5 we obtain the asymptotics for tails of recurrence times given in part (iii) of Theorem 1.2 and Theorem 1.3. In Section 6 we prove Theorem 1.5 and in Section 7 we prove Theorem 1.6.

2 Properties of the limiting process Z: proof of Theorem 1.1

In this section we analyze the limit process $(Z_t)_{t\ge 0}$ and prove Theorem 1.1. We split the proof into several steps. Each of these steps corresponds to one subsection below.

2.1 Lamperti representation of Z

First we observe, that the explicit formula (1.6) for the transition probabilities demonstrates that the process Z is self-similar of index 1. One of the standard approaches used in the analysis of these processes is the Lamperti transformation, which connects self-similar Markov processes with Lévy processes. The main goal of this subsection consists in the derivation of such a representation.

First, it follows from (1.6) that if $t < x$ then

$$\mathbf{P}_x(Z_t = x-t) = \left(\frac{x-t}{x}\right)^c \text{ and } \frac{\mathbf{P}_x(Z_t \in dy)}{dy} = \frac{cty^{c-1}}{(t+y)^{c+1}},\ y > x-t. \tag{2.1}$$

If $t \ge x$ then

$$\frac{\mathbf{P}_x(Z_t \in dy)}{dy} = \frac{cty^{c-1}}{(t+y)^{c+1}},\ y > 0. \tag{2.2}$$

It is immediate from (1.6) that if $c \le 1$ then

$$\int_0^\infty \mathbf{P}_x(Z_t \le y)dt = \infty$$

for all $x, y > 0$. Therefore, the process Z is recurrent: it spends infinite amount of time in every interval $[0, y]$.

We next show that the state 0 is recurrent in the case $c < 1$. More precisely, we show that $\mathbf{P}_z(T_0^{(Z)} < \infty) = 1$ for every $z > 0$. For that reason we compute first the generator of Z. Fix some $x > 0$ and a continuously differentiable bounded function f. It follows then from (2.1) that

$$\mathbf{E}_x[f(Z_t)] = f(x-t)\left(\frac{x-t}{x}\right)^c + ct\int_{x-t}^{\infty}\frac{y^{c-1}}{(y+t)^{c+1}}f(y)dy, \quad t < x.$$

Therefore,

$$\frac{\mathbf{E}_x[f(Z_t)] - f(x)}{t} = \frac{f(x-t)-f(x)}{t} + f(x-t)\frac{\left(\frac{x-t}{x}\right)^c - 1}{t} + c\int_{x-t}^{\infty}\frac{y^{c-1}}{(y+t)^{c+1}}f(y)dy.$$

Letting now $t \to 0$, we conclude that

$$\begin{aligned}\mathcal{L}f(x) &= -f'(x) - c\frac{f(x)}{x} + c\int_x^{\infty}\frac{f(y)}{y^2}dy \\ &= -f'(x) + c\int_x^{\infty}\frac{f(y)-f(x)}{y^2}dy, \quad x > 0.\end{aligned} \tag{2.3}$$

It is easy to see that this generator can be represented as follows

$$\begin{aligned}\mathcal{L}f(x) &= -\left(1 - c\int_1^{\infty}\frac{\log u}{1+\log^2 u}\frac{du}{u^2}\right)f'(x) \\ &\quad + \frac{c}{x}\int_1^{\infty}\left(f(ux) - f(x) - \frac{\log u}{1+\log^2 u}xf'(x)\right)\frac{du}{u^2} \\ &= -\left(1 - c\int_1^{\infty}\frac{\log u}{1+\log^2 u}\frac{du}{u^2}\right)f'(x) + \frac{1}{x}\int_1^{\infty}h^*(x,u)\frac{c\log^2 u}{u^2(1+\log^2 u)}du,\end{aligned}$$

where

$$h^*(x,u) = \left(f(ux) - f(x) - \frac{\log u}{1+\log^2 u}xf'(x)\right)\frac{1+\log^2 u}{\log^2 u}. \tag{2.4}$$

Then, according to Theorem 6.1 in Lamperti [14], $\{Z_t, t < T_0^{(Z)}\}$ can be represented as the exponential functional of a time-changed Lévy process with the following Lévy-Khintchine exponent:

$$\Psi(\lambda) = -i\lambda\left(1 - c\int_1^{\infty}\frac{\log u}{1+\log^2 u}\frac{du}{u^2}\right) + \int_0^{\infty}\left(e^{i\lambda y} - 1 - \frac{i\lambda y}{1+y^2}\right)ce^{-y}dy.$$

Simplifying this expression, we get

$$\Psi(\lambda) = -i\lambda + c\int_0^{\infty}(e^{i\lambda y} - 1)e^{-y}dy.$$

This corresponds to the process $\zeta_t - t$, where $(\zeta_t)_{t\geq 0}$ is a compound Poisson process with intensity c and with exponentially distributed jumps. In particular, $\zeta_t - t \to -\infty$ a.s. as $t \to \infty$ in the case $c < 1$ and $\zeta_t - t$ is oscillating in the case $c = 1$. Then $T_0^{(Z)}$ is finite almost surely iff $c < 1$.

2.2 Analysis of the tails of T_0^Z

We continue the proof of part a) of Theorem 1.1 and establish the formula (1.10) for the tail of the first hitting time. Let

$$g(t,z) := \mathbf{P}_z(T_0^{(Z)} > t).$$

To find g we will now derive a differential equation for g and then solve it explicitly. It is clear that $g(t,z)=1$ for all $t\le z$. Using (2.2), we see that g solves the equation

$$g(t,z)=g(t-s,z-s)\left(\frac{z-s}{z}\right)^c+cs\int_{z-s}^\infty\frac{y^{c-1}}{(y+s)^{c+1}}g(t-s,y)dy,\quad s<z.$$

Letting $s\to 0$ we obtain the following decomposition for the expression on the right hand side:

$$g(t-s,z-s)\left(1-\frac{cs}{z}\right)+cs\int_z^\infty\frac{g(t,y)}{y^2}dy+o(s).$$

Therefore,

$$\frac{g(t,z)-g(t-s,z-s)}{s}=-\frac{c}{z}g(t,z)+c\int_z^\infty\frac{g(t,y)}{y^2}dy+o(1). \tag{2.5}$$

Since the process Z is self-similar with index 1,

$$g(t,z)=\mathbf{P}(T_0^{(Z)}>t\mid Z_0=z)=\mathbf{P}(T_0^{(Z)}>t/z\mid Z_0=1)=g\left(\frac{t}{z},1\right)=:h\left(\frac{t}{z}\right).$$

Thus, the relation (2.5) can be written in the following way:

$$\frac{h(t/z)-h((t-s)/(z-s))}{s}=-\frac{c}{z}h(t/z)+c\int_z^\infty\frac{h(t/y)}{y^2}dy+o(1).$$

Set now

$$\Delta=\frac{t}{z}-\frac{t-s}{z-s}.$$

Then

$$s=\frac{z^2}{z-t}\Delta+o(\Delta)$$

and, consequently,

$$\left(\frac{z-t}{z^2}\right)\frac{h(t/z)-h(t/z-\Delta)}{\Delta}=-\frac{c}{z}h(t/z)+c\int_z^\infty\frac{h(t/y)}{y^2}dy+o(1).$$

Letting here $\Delta\to 0$, we conclude that the function h satisfies

$$\left(\frac{1}{z}-\frac{t}{z^2}\right)h'\left(\frac{t}{z}\right)=-\frac{c}{z}h\left(\frac{t}{z}\right)+c\int_z^\infty\frac{h(t/y)}{y^2}dy,\ t>z.$$

Noting that $h(r)=1$ for all $r\le 1$ and substituting $t/y=x$, we get

$$\begin{aligned}\int_z^\infty\frac{h(t/y)}{y^2}dy&=\int_z^t\frac{h(t/y)}{y^2}dy+\frac{1}{t}\\&=\frac{1}{t}\int_1^{t/z}h(x)dx+\frac{1}{t}.\end{aligned}$$

Therefore,

$$(1-y)h'(y)=-ch(y)+\frac{c}{y}\left(1+\int_1^y h(x)dx\right),\quad y>1.$$

Differentiating this equation, we get

$$\begin{aligned}(1-y)h''(y)-h'(y)&=-ch'(y)+\frac{c}{y}h(y)-\frac{c}{y^2}\left(1+\int_1^y h(x)dx\right)\\&=-ch'(y)+\frac{c}{y}h(y)-\frac{1}{y}((1-y)h'(y)+ch(y)).\end{aligned}$$

Rearranging the terms, we arrive at the equation

$$(1-y)h''(y) = \left(1 - c - \frac{1-y}{y}\right) h'(y).$$

This is equivalent to

$$(\log h'(y))' = \frac{h''(y)}{h'(y)} = \frac{c-1}{y-1} - \frac{1}{y}.$$

Consequently,

$$h'(y) = C(y-1)^{c-1}y^{-1} \quad \text{and} \quad h(x) = C\int_x^\infty (y-1)^{c-1}y^{-1}dy.$$

The boundary condition $h(1) = 1$ leads to the equality

$$h(x) = \frac{\int_x^\infty (y-1)^{c-1}y^{-1}dy}{\int_1^\infty (y-1)^{c-1}y^{-1}dy}, \quad x \ge 1.$$

Substituting in these integrals $y = 1/z$, we finally get

$$h(x) = \frac{1}{B(c, 1-c)} \int_0^{1/x} (1-z)^{c-1}z^{-c}dz, \quad x \ge 1.$$

As a result we have (1.10). This formula can be also obtained via the Lamperti transformation mentioned above. If $Z_0 = 1$ then $T_0^{(Z)}$ has the same distribution as $I := \int_0^\infty e^{\zeta_t}dt$ and $1/I$ has the beta distribution with parameters c and $1-c$, see Bertoin and Yor [6].

2.3 Harmonic function for the killed process

We now turn to the proof of part (b). Recall, that this in the part we claim that $u(z) = z^{1-c}$ is harmonic for the process Z killed at the origin.

We start by showing that

$$\mathbf{E}_x[Z_t^{1-c}] = (\max\{t, x\})^{1-c}. \tag{2.6}$$

When $t \le x$, in view of (2.1),

$$\begin{aligned}\mathbf{E}_x[Z_t^{1-c}] &= \int_{x-t}^\infty y^{1-c}\mathbf{P}_x(Z_t \in dy) \\ &= (x-t)^{1-c}\left(\frac{x-t}{x}\right)^c + \int_{x-t}^\infty \frac{ct}{(t+y)^{c+1}}dy \\ &= \frac{x-t}{x^c} + ct\int_x^\infty \frac{dy}{y^{c+1}} = x^{1-c}.\end{aligned}$$

When $t > x$, by (2.2),

$$\mathbf{E}_x[Z_t^{1-c}] = \int_0^\infty \frac{ct}{(t+y)^{c+1}}dy = t^{1-c}.$$

Using (2.6), we obtain

$$\begin{aligned}\mathbf{E}_x[Z_t^{1-c}; T_0^{(Z)} > t] &= \mathbf{E}_x[Z_t^{1-c}] - \mathbf{E}_x[Z_t^{1-c}; T_0^{(Z)} \le t] \\ &= (\max\{t,x\})^{1-c} - \int_0^t \mathbf{P}_x(T_0^{(Z)} \in ds)\mathbf{E}_0[Z_{t-s}^{1-c}] \\ &= (\max\{t,x\})^{1-c} - \int_0^t (t-s)^{1-c}\mathbf{P}_x(T_0^{(Z)} \in ds).\end{aligned} \tag{2.7}$$

It follows from (1.10) that the integral in (2.7) is zero for $t \le x$, and that for $t > x$ one has

$$\begin{aligned}
&\int_0^t (t-s)^{1-c}\mathbf{P}_x(T_0^{(Z)} \in ds) \\
&\qquad = \int_x^t (t-s)^{1-c}\mathbf{P}_x(T_0^{(Z)} \in ds) \\
&\qquad = \frac{1}{B(c,1-c)} \int_x^t (t-s)^{1-c} \left(1-\frac{x}{s}\right)^{c-1} \left(\frac{x}{s}\right)^{-c} \frac{x}{s^2} ds \\
&\qquad = \frac{1}{B(c,1-c)} \int_x^t (t-s)^{1-c} \left(1-\frac{x}{s}\right)^{c-1} \left(\frac{s}{x}\right)^{c-1} \frac{1}{s} ds \\
&\qquad = \frac{x^{1-c}}{B(c,1-c)} \int_x^t (t-s)^{1-c} (s-x)^{c-1} \frac{1}{s} ds.
\end{aligned}$$

With the help of the substitution $v = \left(\frac{s-x}{t-s}\right)$ we get

$$\begin{aligned}
\int_x^t (t-s)^{1-c} (s-x)^{c-1} \frac{1}{s} ds &= \int_0^\infty v^{c-1} \frac{1+v}{x+tv} \left(\frac{t}{1+v} - \frac{x+tv}{(1+v)^2}\right) dv \\
&= t\int_0^\infty \frac{v^{c-1}}{x+tv} dv - \int_0^\infty \frac{v^{c-1}}{1+v} dv \\
&= \left(\left(\frac{t}{x}\right)^{1-c} - 1\right) \int_0^\infty \frac{v^{c-1}}{1+v} dv.
\end{aligned}$$

Noting now that $\int_0^\infty \frac{v^{c-1}}{1+v} dv = B(c,1-c)$, we conclude that

$$\int_0^t (t-s)^{1-c}\mathbf{P}_x(T_0^{(Z)} \in ds) = \max\{t^{1-c} - x^{1-c}, 0\}.$$

Plugging this into (2.7), we conclude that

$$\mathbf{E}_x[Z_t^{1-c}; T_0^{(Z)} > t] = x^{1-c}$$

for all $x, t > 0$. Thus, (b) is proven.

2.4 Convergence of one-dimensional conditioned marginals

As a first step towards assertion c) of Theorem 1.1 we first consider one-dimensional marginals. For $t \le x$ one has

$$\mathbf{P}_x(Z_t \le y; T_0^{(Z)} > t) = \mathbf{P}_x(Z_t \le y),\ y > 0.$$

If $t > x$ then

$$\begin{aligned}
\mathbf{P}_x(Z_t \le y; T_0^{(Z)} > t) &= \mathbf{P}_x(Z_t \le y) - \mathbf{P}_x(Z_t \le y; T_0^{(Z)} \le t) \\
&= \mathbf{P}_x(Z_t \le y) - \int_x^t \mathbf{P}_x(T_0^{(Z)} \in ds)\mathbf{P}_0(Z_{t-s} \le y).
\end{aligned}$$

Using now (1.6) and (1.10), we get

$$\begin{aligned}
&\mathbf{P}_x(Z_t \le y; T_0^{(Z)} > t) \\
&\qquad = \left(\frac{y}{y+t}\right)^c - \frac{1}{B(c,1-c)} \int_x^t \left(\frac{y}{y+t-s}\right)^c \left(1-\frac{x}{s}\right)^{c-1} \left(\frac{x}{s}\right)^{-c} \frac{x}{s^2} ds \\
&\qquad = \left(\frac{y}{y+t}\right)^c - \frac{1}{B(c,1-c)} \int_x^t \left(\frac{y}{y+t-s}\right)^c \left(\frac{s}{x} - 1\right)^{c-1} \frac{1}{s} ds.
\end{aligned}$$

This representation can be used to obtain an exact formula for the transition kernel $\mathbf{P}_x(Z_t \le y; T_0^{(Z)} > t)$ in terms of the hypergeometric function of two variables. Instead of doing that we shall determine the asymptotic, as $x \to 0$, behaviour of the distribution function $\mathbf{P}_x(Z_t \le y; T_0^{(Z)} > t)$. We start by noting that

$$\begin{aligned}
&\mathbf{P}_x(Z_t \le y; T_0^{(Z)} > t) \\
&\qquad = \left(\frac{y}{y+t}\right)^c \mathbf{P}_x(T_0^{(Z)} > t) - \frac{1}{B(c,1-c)} \int_x^t \Delta_{y,t}(s) \left(\frac{s}{x} - 1\right)^{c-1} \frac{1}{s} ds,
\end{aligned} \tag{2.8}$$

where

$$\Delta_{y,t}(s) = \left(\frac{y}{y+t-s}\right)^c - \left(\frac{y}{y+t}\right)^c.$$

Fix some $\varepsilon > 0$. It is easy to see that

$$\Delta_{y,t}(s) = \frac{y^c}{(y+t)^c}\left[\left(1 + \frac{s}{t+y-s}\right)^c - 1\right] \le \frac{cy^c}{(y+t)^c}\frac{s}{y+t-\varepsilon}$$

for all $s \le \varepsilon$. Therefore, for all $x < \varepsilon$,

$$\begin{aligned}
\int_x^\varepsilon \Delta_{y,t}(s) \left(\frac{s}{x} - 1\right)^{c-1} \frac{1}{s} ds &\le \frac{cy^c}{(y+t-\varepsilon)(y+t)^c} \int_x^\varepsilon \left(\frac{s}{x} - 1\right)^{c-1} ds \\
&\le \frac{y^c}{(y+t-\varepsilon)(y+t)^c} x^{1-c}\varepsilon^c.
\end{aligned} \tag{2.9}$$

Furthermore, as $x \to 0$,

$$\begin{aligned}
\int_\varepsilon^t \Delta_{y,t}(s) \left(\frac{s}{x} - 1\right)^{c-1} \frac{1}{s} ds &= x^{1-c} \int_\varepsilon^t \Delta_{y,t}(s) \left(s - x\right)^{c-1} \frac{1}{s} ds \\
&= x^{1-c}(1 + o(1)) \int_\varepsilon^t \Delta_{y,t}(s) s^{c-2} ds.
\end{aligned}$$

Combining this with (2.9) and letting $\varepsilon \to 0$, we conclude that

$$\lim_{x \to 0} x^{c-1} \int_x^t \Delta_{y,t}(s) \left(\frac{s}{x} - 1\right)^{c-1} \frac{1}{s} ds = \int_0^t \Delta_{y,t}(s) s^{c-2} ds. \tag{2.10}$$

Using the equality

$$\Delta_{y,t}(s) = \int_{y/(t+y)}^{y/(t+y-s)} cu^{c-1} du$$

and the Fubini theorem, we have

$$\begin{aligned}
\int_0^t \Delta_{y,t}(s) s^{c-2} ds &= \int_0^t \left(\int_{y/(t+y)}^{y/(t+y-s)} cu^{c-1} du\right) s^{c-2} ds \\
&= \int_{y/(y+t)}^1 cu^{c-1} \left(\int_{y+t-y/u}^t s^{c-2} ds\right) du \\
&= \frac{c}{1-c} \int_{y/(y+t)}^1 cu^{c-1} \left((y+t-y/u)^{c-1} - t^{c-1}\right) du \\
&= \frac{c}{1-c} \int_{y/(y+t)}^1 ((y+t)u - y)^{c-1} du - \frac{c}{1-c} \int_{y/(y+t)}^1 u^{c-1} du \\
&= \frac{1}{1-c} \frac{t^c}{y+t} - \frac{1}{1-c} t^{c-1} \left(1 - \left(\frac{y}{y+t}\right)^c\right).
\end{aligned}$$

Combining this with (2.10) and noting that

$$\mathbf{P}_x(T_0^{(Z)} > t) \sim \frac{x^{1-c}}{(1-c)B(c,1-c)} t^{c-1}, \quad x \to 0, \tag{2.11}$$

we conclude that

$$\lim_{x\to 0} \frac{\int_x^t \Delta_{y,t}(s)\left(\frac{s}{x}-1\right)^{c-1}\frac{1}{s}ds}{\mathbf{P}_x(T_0^{(Z)} > t)} = B(c,1-c)\left[\left(\frac{y}{y+t}\right)^c - \frac{y}{y+t}\right].$$

Combining this with (2.8), we finally obtain

$$\lim_{x\to 0} \mathbf{P}_x(Z_t \le y \mid T_0^{(Z)} > t) = \frac{y}{y+t}, \quad y > 0. \tag{2.12}$$

2.5 Functional convergence of the h-transformed process

Recall that we have to show, that the law $\mathbf{P}_z\left(Z \in \cdot \mid T_0^{(Z)} > 1\right)$ converges in the Skorokhod space. In order to establish this, we first analyze the Doob h- transform using the harmonic function $u(x) = x^{1-c}$ from part b) of Theorem (1.1) and prove functional convergence for the h-transformed process. As is well-known the Doob h-transform of $\mathcal{L}$ using this harmonic function is given by:

$$\widehat{\mathcal{L}}f(x) := \frac{1}{u(x)}\mathcal{L}(uf)(x), \quad x > 0.$$

The corresponding probability measure is given by

$$\widehat{\mathbf{E}}_x[g(Z)] := \frac{1}{u(x)}\mathbf{E}_x[g(Z)u(Z_t); \tau_0^{(Z)} > t]$$

for every bounded measurable functional g on $D[0,t]$.

From (2.3) we infer that

$$\begin{aligned}
\widehat{\mathcal{L}}f(x) &= \frac{1}{u(x)}\left[-u(x)f'(x) - u'(x)f(x) - c\frac{u(x)f(x)}{x} + c\int_x^\infty \frac{u(y)f(y)}{y^2}dy\right] \\
&= -f'(x) - \frac{f(x)}{x} + \frac{c}{x^{1-c}}\int_x^\infty \frac{f(y)}{y^{1+c}}dy \\
&= -f'(x) + \frac{c}{x^{1-c}}\int_x^\infty \frac{f(y)-f(x)}{y^{1+c}}dy.
\end{aligned} \tag{2.13}$$

As a result we have the following representation:

$$\begin{aligned}
\widehat{\mathcal{L}}f(x) &= -f'(x) + \frac{c}{x}\int_1^\infty \frac{f(ux)-f(x)}{u^{1+c}}du \\
&= -\left(1 - c\int_1^\infty \frac{\log u}{1+\log^2 u}\frac{du}{u^{1+c}}\right)f'(x) + \frac{1}{x}\int_1^\infty h^*(x,u)\frac{c\log^2 u}{u^{1+c}(1+\log^2 u)}du,
\end{aligned}$$

where h^* is given as in equation (2.4). This implies that, under $\widehat{\mathbf{P}}$, Z is self-similar and can be expressed via a Lévy process with the characteristic exponent

$$\widehat{\Psi}(\lambda) = -i\lambda + c\int_0^\infty (e^{i\lambda y} - 1)e^{-cy}dy.$$

This corresponds to $\widehat{\zeta}_t - t$, where $(\widehat{\zeta}_t)_{t\ge 0}$ is a compound Poisson process with intensity c and with positive jumps, which have exponential with parameter c distribution. This Lévy process is clearly oscillating. Consequently, using Theorem 1 in [6] we conclude that every exponential functional is almost surely infinite and there by the results of section 5.1 in [6] we arrive at

$$\widehat{\mathbf{P}}_x(T_0^{(Z)} = \infty) = 1, \quad x > 0.$$

According to Theorem 2 in Caballero and Chaumont [8], the sequence of measures $\widehat{\mathbf{P}}_x$ converges weakly on $D[0,1]$, as $x \to 0$, to a non-degenerate probabilistic measure $\widehat{\mathbf{P}}_0$.

2.6 Functional convergence of the conditioned process

Making use of the functional limit theorem for the h-transformed process we now show that $\mathbf{P}_x\left(Z \in \cdot \,|T_0^{(Z)} > 1\right)$ also converges weakly on $D[0,1]$.

It follows from the definition of $\widehat{\mathbf{P}}_x$ that

$$\begin{aligned}\widehat{\mathbf{P}}_0(Z_1 \le y) &= \lim_{x\to 0} \widehat{\mathbf{P}}_x(Z_1 \le y) \\ &= \lim_{x\to 0} \frac{\mathbf{P}_x(T_0^{(Z)} > 1)}{u(x)} \mathbf{E}_x[u(Z_1)1\{Z_1 \le y\} \mid T_0^{(Z)} > 1].\end{aligned}$$

Applying now (2.11) and (2.12), we obtain

$$\widehat{\mathbf{P}}_0(Z_1 \le y) = \frac{1}{(1-c)B(c,1-c)} \int_0^y \frac{z^{1-c}}{(1+z)^2} dz.$$

Consequently, the density of Z_1 under $\widehat{\mathbf{P}}_0$ is proportional to $\frac{z^{1-c}}{(1+z)^2}$. Let g be a bounded and continuous functional on $D[0,1]$ and let ε be a fixed positive number. Since $\widehat{\mathbf{P}}_0(Z_1 = \varepsilon) = 0$, the weak convergence $\widehat{\mathbf{P}}_x \Rightarrow \widehat{\mathbf{P}}_0$ implies that

$$\lim_{x\to 0} \widehat{\mathbf{E}}_x \left[\frac{g(Z)}{u(Z_1)}; Z_1 > \varepsilon\right] = \widehat{\mathbf{E}}_0 \left[\frac{g(Z)}{u(Z_1)}; Z_1 > \varepsilon\right]. \tag{2.14}$$

Since g is bounded,

$$\begin{aligned}\left|\widehat{\mathbf{E}}_x \left[\frac{g(Z)}{u(Z_1)}; Z_1 \le \varepsilon\right]\right| &\le C_g \widehat{\mathbf{E}}_x \left[\frac{1}{u(Z_1)}; Z_1 \le \varepsilon\right] \\ &\le C_g \frac{\mathbf{P}_x(T_0^{(Z)} > 1)}{u(x)} \mathbf{P}_x(Z_1 \le \varepsilon \mid T_0^{(Z)} > 1).\end{aligned}$$

Using (2.11) and (2.12), we conclude that

$$\limsup_{x\to 0} \left|\widehat{\mathbf{E}}_x \left[\frac{g(Z)}{u(Z_1)}; Z_1 \le \varepsilon\right]\right| \le \frac{C_g}{(1-c)B(c,1-c)} \varepsilon. \tag{2.15}$$

Finally, recalling that the density of Z_1 under $\widehat{\mathbf{P}}_0$ is proportional to $\frac{z^{1-c}}{(1+z)^2}$, we get

$$\begin{aligned}\left|\widehat{\mathbf{E}}_0 \left[\frac{g(Z)}{u(Z_1)}; Z_1 \le \varepsilon\right]\right| &\le C_g \widehat{\mathbf{E}}_0 \left[\frac{1}{u(Z_1)}; Z_1 \le \varepsilon\right] \\ &= C_g \int_0^\varepsilon (1+z)^{-2} dz \le C_g \varepsilon.\end{aligned} \tag{2.16}$$

Combining (2.14)—(2.16) and letting $\varepsilon \to 0$, we conclude that

$$\lim_{x\to 0} \widehat{\mathbf{E}}_x \left[\frac{g(Z)}{u(Z_1)}\right] = \widehat{\mathbf{E}}_0 \left[\frac{g(Z)}{u(Z_1)}\right].$$

Noting now that

$$\mathbf{E}_x[g(Z) \mid T_0^{(Z)} > 1] = \frac{u(x)}{\mathbf{P}_x(T_0^{(Z)} > 1)} \widehat{\mathbf{E}}_x \left[\frac{g(Z)}{u(Z_1)} \right]$$

and taking into account (2.11), we obtain

$$\lim_{x\to\infty} \mathbf{E}_x[g(Z) \mid T_0^{(Z)} > 1] = (1-c)B(c, 1-c)\widehat{\mathbf{E}}_0 \left[\frac{g(Z)}{u(Z_1)} \right].$$

This completes the proof of Theorem 1.1.

3 Analysis of harmonic functions in discrete time

In this section we construct and analyze harmonic functions of the involved discrete time Markov chains killed at the first hitting time.

3.1 Harmonic function for the random exchange process and for the maximal autoregressive process

Surprisingly, it will turn out, that in the case of the process $\{R_n\}n \geq 0$ it is possible to explicitly calculate the harmonic function. We will prove the following proposition, which coincides with the first part of Theorem 1.2. Observe that in the following proposition we do not need to assume the condition (1.4).

Proposition 3.1. *Assume that x_0 is such that $\mathbf{P}(\eta_1 \leq x_0)\mathbf{P}(\eta_1 > x_0) > 0$. Then the equation*

$$G(x) := \mathbf{E}_x[G(R_1); T_{x_0}^{(R)} > 1] = \mathbf{E}_x[G(R_1); R_1 > x_0], \quad x > x_0. \tag{3.1}$$

has a non-trivial non-negative solution if and only if $\mathbf{E}\eta_1^+ = \infty$. In the latter case

$$G(x) = C\left(1 + \sum_{j=1}^{\infty} \prod_{k=0}^{j-1} \mathbf{P}(\eta_1 \leq x_0 + k) 1_{(x_0+j+1,\infty)}(x)\right)$$

for every $C > 0$.

Proof. We first consider the equation (3.1) for $x \in (x_0, x_0 + 1]$. In this case one has

$$\{R_1 > x_0\} = \{R_1 = \eta_1 > x_0\}.$$

Therefore,

$$G(x) = \mathbf{E}[G(\eta_1); \eta_1 > x_0] \quad \text{for all } x \in (x_0, x_0 + 1].$$

For all $x > x_0 + 1$ on the other hand one has $\mathbf{P}_x(T_{x_0}^{(R)} > 1) = 1$. This implies that (3.1) reduces to

$$\begin{aligned} G(x) &= \mathbf{E}_x[G(R_1)] \\ &= G(x-1)\mathbf{P}(\eta_1 \leq x-1) + \mathbf{E}[G(\eta_1); \eta_1 > x-1], \quad x > x_0 + 1. \end{aligned} \tag{3.2}$$

If $x \in (x_0 + 1, x_0 + 2]$ then $x - 1 \in (x_0, x_0 + 1]$ and, consequently, $G(x-1) = G(x_0 + 1)$ for all $x \in (x_0 + 1, x_0 + 2]$. From this observation and from (3.2) we have

$$\begin{aligned} &G(x) \\ &= G(x_0 + 1)\mathbf{P}(\eta_1 \leq x - 1) + \mathbf{E}[G(\eta_1); \eta_1 > x - 1] \\ &= G(x_0 + 1)\mathbf{P}(\eta_1 \leq x - 1) + \mathbf{E}[G(\eta_1); \eta_1 \in (x - 1, x_0 + 1]] + \mathbf{E}[G(\eta_1); \eta_1 > x_0 + 1] \\ &= G(x_0 + 1)\mathbf{P}(\eta_1 \leq x_0 + 1) + \mathbf{E}[G(\eta_1); \eta_1 > x_0 + 1]. \end{aligned} \tag{3.3}$$

This equality implies that $G(x) = G(x_0 + 2)$ for all $x \in (x_0 + 1, x_0 + 2]$. Note also that

$$\begin{aligned} G(x_0+1) &= \mathbf{E}[G(\eta_1); \eta_1 > x_0] \\ &= G(x_0+1)\mathbf{P}(\eta_1 \in (x_0, x_0+1])) + \mathbf{E}[G(\eta_1); \eta_1 > x_0+1]. \end{aligned}$$

Combining this with (3.3), we conclude that

$$G(x_0+2) = G(x_0+1)\left(1 + \mathbf{P}(\eta_1 \le x_0)\right).$$

Fix now an integer n and consider the case $x \in (x_0 + n, x_0 + n + 1]$. Assume that we have already shown that $G(y) = G(x_0 + n)$ for all $y \in (x_0 + n - 1, x_0 + n]$. Then we have from (3.2)

$$\begin{aligned} G(x) &= G(x-1)\mathbf{P}(\eta_1 \le x-1) + \mathbf{E}[G(\eta_1); \eta_1 > x-1] \\ &= G(x_0+n)\mathbf{P}(\eta_1 \le x_0+n) + \mathbf{E}[G(\eta_1); \eta_1 > x_0+n]. \end{aligned}$$

Therefore, $G(x) = G(x_0 + n + 1)$ for all $x \in (x_0 + n, x_0 + n + 1]$. This means that this property is valid for all n.

One has also equalities

$$G(x_0+n+1) = G(x_0+n)\mathbf{P}(\eta_1 \le x_0+n) + \mathbf{E}[G(\eta_1); \eta_1 > x_0+n]$$

and

$$\begin{aligned} G(x_0+n) &= G(x_0+n-1)\mathbf{P}(\eta_1 \le x_0+n-1) + \mathbf{E}[G(\eta_1); \eta_1 > x_0+n-1] \\ &= G(x_0+n-1)\mathbf{P}(\eta_1 \le x_0+n-1) \\ &\quad + G(x_0+n)\mathbf{P}(\eta_1 \in (x_0+n-1, x_0+n]) + \mathbf{E}[G(\eta_1); \eta_1 > x_0+n]. \end{aligned}$$

Taking the difference we obtain

$$\begin{aligned} &G(x_0+n+1) - G(x_0+n) \\ &= G(x_0+n)\mathbf{P}(\eta_1 \le x_0+n) - G(x_0+n-1)\mathbf{P}(\eta_1 \le x_0+n-1) \\ &\qquad - G(x_0+n)\mathbf{P}(\eta_1 \in (x_0+n-1, x_0+n]) \\ &= \mathbf{P}(\eta_1 \le x_0+n-1)\left(G(x_0+n) - G(x_0+n-1)\right). \end{aligned}$$

Consequently,

$$G(x_0+n+1) - G(x_0+n) = G(x_0+1)\prod_{k=0}^{n-1}\mathbf{P}(\eta_1 \le x_0+k), \quad n \ge 1.$$

As a result we have

$$G(x) = G(x_0+1)\left(1 + \sum_{j=1}^{n}\prod_{k=0}^{j-1}\mathbf{P}(\eta_1 \le x_0+k)\right), \ x \in (x_0+n, x_0+n+1]. \tag{3.4}$$

Finally, in order to get a non-trivial solution we have to show that the equation

$$G(x_0+1) = \mathbf{E}[G(\eta_1); \eta_1 > x_0]$$

is solvable. In view of (3.4), the previous equation is equivalent to

$$G(x_0+1) = G(x_0+1)\sum_{n=0}^{\infty}\left(1 + \sum_{j=1}^{n}\prod_{k=0}^{j-1}\mathbf{P}(\eta_1 \le x_0+k)\right)\mathbf{P}(\eta_1 \in (x_0+n, x_0+n+1]).$$

Now we infer that (3.1) has a non-trivial solution if and only if

$$1=\sum_{n=0}^{\infty}\left(1+\sum_{j=1}^{n}\prod_{k=0}^{j-1}\mathbf{P}(\eta_1\le x_0+k)\right)\mathbf{P}(\eta_1\in(x_0+n,x_0+n+1]).$$

Clearly,

$$\begin{aligned}&\sum_{n=0}^{\infty}\left(1+\sum_{j=1}^{n}\prod_{k=0}^{j-1}\mathbf{P}(\eta_1\le x_0+k)\right)\mathbf{P}(\eta_1\in(x_0+n,x_0+n+1])\\&=\mathbf{P}(\eta_1>x_0)+\sum_{j=1}^{\infty}\prod_{k=0}^{j-1}\mathbf{P}(\eta_1\le x_0+k)\sum_{n=j}^{\infty}\mathbf{P}(\eta_1\in(x_0+n,x_0+n+1])\\&=\mathbf{P}(\eta_1>x_0)+\sum_{j=1}^{\infty}(1-\mathbf{P}(\eta_1\le x_0+j))\prod_{k=0}^{j-1}\mathbf{P}(\eta_1\le x_0+k).\end{aligned}$$

Furthermore, for every $N\ge 1$,

$$\begin{aligned}&\sum_{j=1}^{N}(1-\mathbf{P}(\eta_1\le x_0+j))\prod_{k=0}^{j-1}\mathbf{P}(\eta_1\le x_0+k)\\&\qquad=\sum_{j=1}^{N}\prod_{k=0}^{j-1}\mathbf{P}(\eta_1\le x_0+k)-\sum_{j=1}^{N}\prod_{k=0}^{j}\mathbf{P}(\eta_1\le x_0+k)\\&\qquad=\mathbf{P}(\eta_1\le x_0)-\prod_{k=0}^{N}\mathbf{P}(\eta_1\le x_0+k).\end{aligned}$$

This implies that

$$\begin{aligned}&\sum_{n=0}^{\infty}\left(1+\sum_{j=1}^{n}\prod_{k=0}^{j-1}\mathbf{P}(\eta_1\le x_0+k)\right)\mathbf{P}(\eta_1\in(x_0+n,x_0+n+1])\\&\qquad=1-\lim_{N\to\infty}\prod_{k=0}^{N}\mathbf{P}(\eta_1\le x_0+k).\end{aligned}$$

Thus, there is a non trivial solution $G(x)$ if and only if

$$\lim_{N\to\infty}\prod_{k=0}^{N}\mathbf{P}(\eta_1\le x_0+k)=0.$$

Noting that this is equivalent to $\mathbf{E}\eta_1^+=\infty$, we finish the proof of the proposition. □

We notice also that $\mathbf{E}\eta_1^+=\infty$, which ensures the existence of a harmonic function, in fact means that $\{R_n\}$ is either null recurrent or transient. If $\{R_n\}$ is recurrent and $\mathbf{P}(\eta_1\le x_0)>0$ then, according to (1.1), the function $G(x)$ grows unboundedly.

We proceed with proving part (i) of Theorem 1.2.

Lemma 3.2. *Assume, that if* (1.4) *holds with some* $c\in(0,1)$ *then there exists* $\gamma\in(0,\infty)$ *such that*

$$G(x)\sim\gamma U_0(x).$$

Proof. Observe that

$$\prod_{k=0}^{j-1}\mathbf{P}(\eta_1\le x_0+k)=\prod_{k=0}^{j-1}\big(1-\mathbf{P}(\eta_1>x_0+k)\big)$$
$$=\exp\Big(\sum_{k=0}^{j-1}\log\big(1-\mathbf{P}(\eta_1>x_0+k)\big)\Big)$$

Using Taylor's formula for the logarithm and taking into account (1.4) we conclude that

$$\prod_{k=0}^{j-1}\mathbf{P}(\eta_1\le x_0+k)\sim c_1\exp\big(-\sum_{k=0}^{j-1}\mathbf{P}(\eta_1>x_0+k)\big)$$
$$\sim c_2\exp\Big(\int_0^j\mathbf{P}(\eta_1>y)\,dy\Big)=c_2e^{-u_0(j)}.$$

Recalling that u_0 is regularly varying of index $-c$ we conclude that G is asymptotically equivalent to a multiple of U_0. This finishes the proof of the lemma. ☐

We conclude this paragraph with the following remark concerning the cases which are excluded from Theorem 1.2.

Remark 3.3. a) If $c=1$ then one has to take into account the asymptotic behaviour of the difference $\mathbf{P}(\eta_1>y)-1/y$. Assume, for example, that

$$\mathbf{P}(\eta_1>y)=\frac{1}{y}+\frac{\theta+o(1)}{y\log y}$$

for some $\theta\in(0,1)$. Then $\{R_n\}$ is null recurrent and there exists a slowly varying function L_1 such that $e^{-u_0(x)}\sim(\log x)^{-\theta}L_1(\log x)$. This implies that

$$G(x)\sim\frac{1}{1-\theta}(\log x)^{1-\theta}L_1(\log x)\quad\text{if }\theta<1.$$

b) If $\{R_n\}$ is transient then the function $x\mapsto\mathbf{P}_x(T^{(R)}_{x_0}=\infty)$ is harmonic and its limit, as $x\to\infty$, is equal to one. Then, according to (3.4),

$$\mathbf{P}_x(T^{(R)}_{x_0}=\infty)=\frac{1+\sum_{j\in[1,x-x_0)}\prod_{k=0}^{j-1}\mathbf{P}(\eta_1\le x_0+k)}{1+\sum_{j=1}^{\infty}\prod_{k=0}^{j-1}\mathbf{P}(\eta_1\le x_0+k)},\quad x>x_0. \tag{3.5}$$

If (1.4) holds with some positive $c>1$ then the chain is transient and, using the Karamata representation theorem, we obtain

$$\mathbf{P}_x(T^{(R)}_{x_0}<\infty)\sim\frac{1}{(c-1)}\frac{L(x)}{x^{c-1}},\quad x\to\infty.$$

3.2 Harmonic function for the autoregressive process: proof of Theorem 1.3(i)

In contrast to the previous subsection it does not seem possible to find explicit representations for the harmonic function in the autoregressive case. Instead we construct a harmonic function via appropriate supermartingales. First we prove assertions needed in the construction of the supermartingales.

Lemma 3.4. *Let W be an increasing, regularly varying of index $r\in(0,1)$ function. We assume also that $W'(x)=O\left(\frac{W(x)}{x}\right)$. If (1.4) holds then, as $z\to\infty$,*

$$\mathbf{E}[W(\log_A(A^{z-1}+A^{\eta_1}))]$$
$$=W(z-1)\mathbf{P}(\eta_1\le z-1)+\mathbf{E}[W(\eta_1);\eta_1>z-1]+o\left(\frac{W(z)}{z^2}\right).$$

Proof. We start by decomposing the expectation into two parts:

$$\begin{aligned}
&\mathbf{E}[W(\log_A(A^{z-1}+A^{\eta_1}))] \\
&= \mathbf{E}[W(\log_A(A^{z-1}+A^{\eta_1}));\eta_1 \le z-1] + \mathbf{E}[W(\log_A(A^{z-1}+A^{\eta_1}));\eta_1 > z-1] \\
&= \mathbf{E}[W(z-1+\log_A(1+A^{\eta_1-z+1}));\eta_1 \le z-1] \\
&\qquad\qquad + \mathbf{E}[W(\eta_1+\log_A(1+A^{z-1-\eta_1}));\eta_1 > z-1].
\end{aligned}$$

By the mean value theorem,

$$\begin{aligned}
&\mathbf{E}[W(z-1+\log_A(1+A^{\eta_1-z+1}));\eta_1 \le z-1] \\
&= W(z-1)\mathbf{P}(\eta_1 \le z-1) + \mathbf{E}[W'(z-1+\theta_1)\log_A(1+A^{\eta_1-z+1});\eta_1 \le z-1],
\end{aligned}$$

where $\theta_1 = \theta_1(z,\eta_1) \in (0,\log_A 2)$. Using now the assumption $W'(x) = O\left(\frac{W(x)}{x}\right)$, we obtain

$$\begin{aligned}
&\mathbf{E}[W(z-1+\log_A(+A^{\eta_1-z+1}));\eta_1 \le z-1] \\
&= W(z-1)\mathbf{P}(\eta_1 \le z-1) + O\left(\frac{W(z)}{z}\right)\mathbf{E}[\log_A(1+A^{\eta_1-z+1});\eta_1 \le z-1].
\end{aligned}$$

It is easy to see that

$$\log_A(1+A^{\eta_1-z+1}) = O\left(\frac{1}{z^2}\right)$$

if $\eta_1 \le z-1-2\log_A z$. Furthermore, (1.4) implies that

$$\mathbf{P}(z-1-2\log_A z < \eta_1 \le z-1) = o\left(\frac{1}{z}\right).$$

Combining these relations, we infer that

$$\mathbf{E}[\log_A(1+A^{\eta_1-z+1});\eta_1 \le z-1] = o\left(\frac{1}{z}\right).$$

As a result we have

$$\begin{aligned}
&\mathbf{E}[W(z-1+\log_A(+A^{\eta_1-z+1}));\eta_1 \le z-1] \\
&\qquad\qquad = W(z-1)\mathbf{P}(\eta_1 \le z-1) + o\left(\frac{W(z)}{z^2}\right). \qquad (3.6)
\end{aligned}$$

Using the mean value theorem and the assumption $W'(x) = O\left(\frac{W(x)}{x}\right)$ once again, we get

$$\begin{aligned}
&\mathbf{E}[W(\eta_1+\log_A(1+A^{z-1-\eta_1}));\eta_1 > z-1] \\
&= \mathbf{E}[W(\eta_1);\eta_1 > z-1] + O\left(\frac{W(z)}{z}\right)\mathbf{E}[\log_A(1+A^{z-1-\eta_1});\eta_1 > z-1].
\end{aligned}$$

Similar to the first part of the proof,

$$\mathbf{E}[\log_A(1+A^{z-1-\eta_1});\eta_1 > z-1] = o\left(\frac{1}{z}\right).$$

This leads to the equality

$$\begin{aligned}
&\mathbf{E}[W(\eta_1+\log_A(1+A^{z-1-\eta_1}));\eta_1 > z-1] \\
&\qquad\qquad = \mathbf{E}[W(\eta_1);\eta_1 > z-1] + o\left(\frac{W(z)}{z^2}\right).
\end{aligned}$$

Combining this with (3.6), we obtain the desired equality. □

For every $\varepsilon \ge 0$ we define

$$u_\varepsilon(x) = (1+\varepsilon)\int_0^x \mathbf{P}(\eta_1 > y)dy, \quad x \ge 0$$

and

$$U_\varepsilon(x) = \begin{cases} 0, & x \le 0 \\ \int_0^x e^{-u_\varepsilon(y)}dy, & x > 0. \end{cases}$$

Lemma 3.5. *For every $\varepsilon \in [0, \frac{1-c}{c})$ one has*

$$\mathbf{E}[U_\varepsilon(\log_A(A^{z-1} + A^{\eta_1}))] = U_\varepsilon(z) - \frac{\varepsilon}{1+\varepsilon}e^{-u_\varepsilon(z)} + O\left(\frac{U_\varepsilon(z)}{z^2}\right).$$

Proof. (1.4) yields

$$u_\varepsilon(x) \sim (1+\varepsilon)c\log x \quad \text{as } x \to \infty.$$

Furthermore, $U_\varepsilon(x)$ is regularly varying of index $1 - c(1+\varepsilon)$ and that

$$U'_\varepsilon(x) = e^{-u_\varepsilon(x)} \sim (1 - c(1+\varepsilon))\frac{U_\varepsilon(x)}{x}.$$

Therefore, we may apply Lemma 3.4 to the function U_ε:

$$\begin{aligned} &\mathbf{E}[U_\varepsilon(\log_A(A^{z-1} + A^{\eta_1}))] \\ &\qquad = U_\varepsilon(z-1)\mathbf{P}(\eta_1 \le z-1) + \mathbf{E}[U_\varepsilon(\eta_1); \eta_1 > z-1] + o\left(\frac{U_\varepsilon(z)}{z^2}\right). \end{aligned}$$

Integrating by parts, we have

$$\begin{aligned} \mathbf{E}[U_\varepsilon(\eta_1); \eta_1 > z-1] &= U_\varepsilon(z-1)\mathbf{P}(\eta_1 > z-1) + \int_{z-1}^\infty e^{-u_\varepsilon(y)}\mathbf{P}(\eta_1 > y)dy \\ &= U_\varepsilon(z-1)\mathbf{P}(\eta_1 > z-1) + \frac{1}{1+\varepsilon}\int_{z-1}^\infty e^{-u_\varepsilon(y)}u'_\varepsilon(y)dy \\ &= U_\varepsilon(z-1)\mathbf{P}(\eta_1 > z-1) + \frac{1}{1+\varepsilon}e^{-u_\varepsilon(z-1)}. \end{aligned}$$

Consequently,

$$\mathbf{E}[U_\varepsilon(\log_A(A^{z-1} + A^{\eta_1}))] = U_\varepsilon(z-1) + \frac{1}{1+\varepsilon}e^{-u_\varepsilon(z-1)} + o\left(\frac{U_\varepsilon(z)}{z^2}\right). \tag{3.7}$$

It remains now to notice that, by the Taylor formula,

$$\begin{aligned} U_\varepsilon(z) &= U_\varepsilon(z-1) + U'_\varepsilon(z-1) + \frac{1}{2}U''_\varepsilon(z-1+\theta) \\ &= U_\varepsilon(z-1) + e^{-u_\varepsilon(z-1)} + O\left(\frac{U_\varepsilon(z)}{z^2}\right). \end{aligned}$$

Using this in Equation (3.7) finishes the proof. □

Combining the functions U_0 and U_ε we can now easily construct supermartingales. To this end we notice that applying Lemma 3.5 and using $U_\varepsilon(x) = o(U_0(x))$, we get

$$\begin{aligned} &\mathbf{E}_x[U_0(\log_A X_1) + U_\varepsilon(\log_A X_1)] \\ &\quad = \mathbf{E}[U_0(\log_A(A^{x-1} + A^{\eta_1})) + U_\varepsilon(\log_A(A^{x-1} + A^{\eta_1}))] \\ &\quad = U_0(\log_A x) + U_\varepsilon(\log_A x) - \frac{\varepsilon}{1+\varepsilon}e^{-u_\varepsilon(\log_A x)} + O\left(\frac{U_0(\log_A x)}{(\log_A x)^2}\right). \end{aligned}$$

We know that $e^{-u_\varepsilon(z)}$ is regularly varying of index $-(1+\varepsilon)c$ and that $\frac{U_0(z)}{z^2}$ is regularly varying of index $-c-1$. Thus, for every $\varepsilon < \frac{1-c}{c}$ there exists x^* such that

$$\mathbf{E}_x[U_0(\log_A X_1) + U_\varepsilon(\log_A X_1)] \le U_0(\log_A x) + U_\varepsilon(\log_A x), \quad x \ge x^*.$$

This inequality implies that if $x_0 \ge x^*$ then the sequence

$$Z_n := U_0(\log_A X_{n\wedge T_{x_0}^{(X)}}) + U_\varepsilon(\log_A X_{n\wedge T_{x_0}^{(X)}})$$

is a supermartingale. We next notice that

$$\begin{aligned} Z_{n+1}1\{T_{x_0}^{(X)} > n+1\} - Z_n 1\{T_{x_0}^{(X)} > n\} \\ &= (Z_{n+1} - Z_n)1\{T_{x_0}^{(X)} > n\} - Z_{n+1}1\{T_{x_0}^{(X)} = n+1\} \\ &\le (Z_{n+1} - Z_n)1\{T_{x_0}^{(X)} > n\}. \end{aligned}$$

This implies that $Z_n 1\{T_{x_0}^{(X)} > n\}$ is also a supermartingale.

Proposition 3.6. *If $x_0 \ge x_*$ then the function*

$$V(x) := \lim_{n\to\infty} \mathbf{E}_x[U_0(\log_A X_n); T_{x_0}^{(X)} > n]$$

is well-defined, strictly positive on (x_0,∞) and harmonic for $\{X_n\}_{n\ge0}$ killed when exiting (x_0,∞). Furthermore, we have

$$V(x) \sim U_0(\log_A x) \quad \text{as } x\to\infty. \tag{3.8}$$

Proof. Using the supermartingale property of $Z_n 1\{T_{x_0}^{(X)} > n\}$ we conclude by the supermartingale convergence theorem that the function

$$V_\varepsilon(x) := \lim_{n\to\infty} \mathbf{E}_x[U_0(\log_A X_n) + U_\varepsilon(\log_A X_n); T_{x_0}^{(X)} > n]$$

is well-defined and finite. Furthermore,

$$V_\varepsilon(x) \le U_0(\log_A x) + U_\varepsilon(\log_A x) \le CU_0(\log_A x), \quad x > x_0.$$

We now recall that $U_\varepsilon(z) = o(U_0(z))$. Thus, for every $\delta > 0$ there exists B such that $U_\varepsilon(z) \le \delta U_0(z)$ for all $z \ge B$. Therefore,

$$\begin{aligned} \mathbf{E}_x[U_\varepsilon(\log_A X_n); T_{x_0}^{(X)} > n] \\ &= \mathbf{E}_x[U_\varepsilon(\log_A X_n); \log_A X_n \le B, T_{x_0}^{(X)} > n] \\ &\qquad + \mathbf{E}_x[U_\varepsilon(\log_A X_n); \log_A X_n > B, T_{x_0}^{(X)} > n] \\ &\le U_\varepsilon(B)\mathbf{P}_x(T_{x_0}^{(X)} > n) + \delta \mathbf{E}_x[U_0(\log_A X_n); T_{x_0}^{(X)} > n]. \end{aligned}$$

Recalling that $\mathbf{P}_X(T_{x_0}^{(X)} > n) \to 0$, we get

$$\limsup_{n\to\infty} \mathbf{E}_x[U_\varepsilon(\log_A X_n); T_{x_0}^{(X)} > n] \le \delta V_\varepsilon(x).$$

Letting now $\delta \to 0$ we conclude that

$$\lim_{n\to\infty} \mathbf{E}_x[U_\varepsilon(\log_A X_n); T_{x_0}^{(X)} > n] = 0$$

This means that V_ε does not depend on ε. Thus we may set

$$V(x) := \lim_{n\to\infty} \mathbf{E}_x[U_0(\log_A X_n); T_{x_0}^{(X)} > n].$$

Since U_0 and the chain $\{X_n\}$ are increasing, we infer that the function $V(x)$ is increasing as well.

By the Markov property,

$$\mathbf{E}_x[U_0(\log_A X_{n+1}); T_{x_0}^{(X)} > n+1] = \int_{x_0}^{\infty} \mathbf{P}_x(X_1 \in dy)\mathbf{E}_y[U_0(\log_A X_n); T_{x_0}^{(X)} > n].$$

It follows from the supermartinale property of $U_0(\log_A X_n) + U_\varepsilon(\log_A X_n)$ that

$$\begin{aligned}\mathbf{E}_y[U_0(\log_A X_n); T_{x_0}^{(X)} > n] &\le \mathbf{E}_y[U_0(\log_A X_n) + U_\varepsilon(\log_A X_n); T_{x_0}^{(X)} > n] \\ &\le U_0(\log_A y) + U_\varepsilon(\log_A y), \quad n \ge 1\end{aligned}$$

This allows one to apply the dominated convergence theorem and to conclude that

$$V(x) = \mathbf{E}_x[V(X_1); T_{x_0}^{(X)} > 1], \quad x > x_0.$$

In other words, $V(x)$ is harmonic for X_n killed at $T_{x_0}^{(X)}$. It is also clear that

$$V(x) \le U_0(\log_A x) + U_\varepsilon(\log_A x) \le CU_0(\log_A x).$$

To show that this function is strictly positive we notice that

$$\mathbf{E}[U_0(\log_A(A^{z-1} + A^{\eta_1}))] \ge U_0(z-1)\mathbf{P}(\eta_1 \le z-1) + \mathbf{E}[U_0(\eta_1); \eta_1 > z-1].$$

Using now the integration by parts, we get

$$\mathbf{E}[U_0(\log_A(A^{z-1} + A^{\eta_1}))] \ge U_0(z-1) + e^{-u_0(z-1)} \ge U_0(z).$$

In other words, the sequence $U_0(\log_A X_n)$ is a submartingale. Then, by the optional stopping theorem,

$$\mathbf{E}_x[U_0(\log_A X_n); T_{x_0}^{(X)} > n] \ge U_0(\log_A x) - \mathbf{E}_x[U_0(\log_A X_{T_{x_0}^{(X)}}); T_{x_0}^{(X)} \le n].$$

Letting here $n \to \infty$, we conclude that

$$V(x) \ge U_0(\log_A x) - \mathbf{E}[U_0(\log_A X_{T_{x_0}^{(X)}})] \ge U_0(\log_A x) - U_0(\log_A x_0).$$

Thus, $V(x) > 0$ for every $x > x_0$. Furthermore, one has the relation

$$V(x) \sim U_0(\log_A x) \quad \text{as } x \to \infty.$$

Thus, the proof is complete. $\square$

It remains to construct a harmonic function for the case $x_0 \le x_*$. Let V_* be the function corresponding to the stopping time $T_{x_*}^{(X)}$, i.e.

$$V_*(x) = \mathbf{E}_x[V_*(X_1); X_1 > x_*],\ x > x_*.$$

Define

$$V(x) = V_*(x)1\{x > x_*\} + \sum_{j=0}^{\infty} \int_{x_0}^{x_*} \mathbf{P}_x(X_j \in dz, T_{x_0}^{(X)} > j)g(z), \tag{3.9}$$

where

$$g(z) := \mathbf{E}_z[V_*(X_1); X_1 > x_*].$$

Then one has

$$\begin{aligned}
&\mathbf{E}_x[V(X_1); X_1 > x_0] \\
&\quad = \mathbf{E}_x[V_*(X_1); X_1 > x_*] + \int_{x_0}^{\infty} \mathbf{P}_x(X_1 \in dy) \sum_{j=0}^{\infty} \int_{x_0}^{x_*} \mathbf{P}_y(X_j \in dz, T_{x_0}^{(X)} > j) g(z) \\
&\quad = \mathbf{E}_x[V_*(X_1); X_1 > x_*] + \sum_{j=1}^{\infty} \int_{x_0}^{x_*} \mathbf{P}_x(X_j \in dz, T_{x_0}^{(X)} > j) g(z).
\end{aligned}$$

If $x > x_*$ then

$$\mathbf{E}_x[V_*(X_1); X_1 > x_*] = V_*(x).$$

Moreover, for $x \in (x_0, x_*]$ we have

$$\mathbf{E}_x[V_*(X_1); X_1 > x_*] = \int_{x_0}^{x_*} \mathbf{P}_x(X_0 \in dz, T_{x_0}^{(X)} > 0) g(z).$$

As a result,

$$\mathbf{E}_x[V(X_1); X_1 > x_0] = V(x), \quad x > x_0,$$

i.e. V is harmonic.

Since $V(x)$ is strictly positive on the half-line (x_0, ∞), we may perform the corresponding Doob h-transform via the transition probabilities:

$$\widehat{\mathbf{P}}_x^{(V)}(X_1 \in dy) = \frac{V(y)}{V(x)} \mathbf{P}_x(X_1 \in dy), \quad x, y > x_0. \tag{3.10}$$

The chain X_n becomes transient under this new measure. To see this we consider the sequence $(V(X_n))^{-1/2}$. It is immediate from the definition of $\widehat{\mathbf{P}}$ that

$$\begin{aligned}
\widehat{\mathbf{E}}_x^{(V)}\left[\frac{1}{\sqrt{V(X_1)}}\right] &= \frac{1}{V(x)} \mathbf{E}_x\left[\frac{V(X_1)}{\sqrt{V(X_1)}} 1\{T_{x_0}^{(X)} > 1\}\right] \\
&= \frac{1}{V(x)} \mathbf{E}_x\left[\sqrt{V(X_1)} 1\{T_{x_0}^{(X)} > 1\}\right].
\end{aligned}$$

Applying now the Jensen inequality, we obtain

$$\widehat{\mathbf{E}}_x^{(V)}\left[\frac{1}{\sqrt{V(X_1)}}\right] \le \frac{1}{V(x)} \sqrt{\mathbf{E}_x\left[V(X_1) 1\{T_{x_0}^{(X)} > 1\}\right]} = \frac{1}{\sqrt{V(x)}}.$$

In other words, the sequence $(V(X_n))^{-1/2}$ is a positive supermartingale. Due to the Doob convergence theorem, this sequence converges almost surely. Noticing that $\widehat{\mathbf{P}}_x(\limsup X_n = \infty) = 1$ implies that this limit of $(V(X_n))^{-1/2}$ is zero. This means that

$$X_n \to \infty \quad \widehat{\mathbf{P}}^{(V)} - \text{a.s.}$$

We now show that (3.8) holds also in the case when the harmonic function is defined by (3.9). Since the function $g(z)$ is increasing,

$$\begin{aligned}
&\sum_{j=0}^{\infty} \int_{x_0}^{x_*} \mathbf{P}_x(X_j \in dz, T_{x_0}^{(X)} > j) g(z) \\
&\qquad \le g(x_*) \sum_{j=0}^{\infty} \mathbf{P}_x(X_j \le x_*, T_{x_0}^{(X)} > j) \\
&\qquad \le g(x_*) \frac{V(x)}{\inf_{y \in (x_0, x_*)} V(y)} \sum_{j=0}^{\infty} \widehat{\mathbf{P}}_x(X_j \le x_*) \\
&\qquad = C(x_0, x_*) V(x) \sum_{j=0}^{\infty} \widehat{\mathbf{P}}_x(X_j \le x_*).
\end{aligned}$$

Due to the transience of $\{X_n\}$ under $\widehat{\mathbf{P}}$,

$$\sum_{j=0}^{\infty}\int_{x_0}^{x_*}\widehat{\mathbf{P}}_x(X_j\le x_*)\to 0\quad\text{as } x\to\infty.$$

This implies that

$$\sum_{j=0}^{\infty}\int_{x_0}^{x_*}\mathbf{P}_x(X_j\in dz, T^{(X)}_{x_0}>j)g(z)=o(V(x))$$

and, consequently,

$$V(x)\sim V_*(x)\sim U_0(\log_A x).$$

Thus, the proof of Theorem 1.3(i) is complete.

4 Lower and upper bounds for tails of recurrence times

We shall consider the chain $\{X_n\}_{n\ge 0}$ only and prove the bounds formulated in Theorem 1.3(ii). The proofs of corresponding estimates for chains $\{M_n\}_{n\ge 0}$ and $\{R_n\}_{n\ge 0}$ are simpler.

4.1 A lower bound for the tail of $T^{(X)}_{x_0}$

The main result of this subsection consists in the lower bound for the tails of the first hitting time.

Lemma 4.1. *Under the conditions of Theorem 1.3 there exists a constant $c>0$ such that*

$$\mathbf{P}(T^{(X)}_{x_0}>n)\ge c\frac{V(x\wedge A^n)}{V(A^n)}.$$

Proof. We first consider the case $x_0\ge x_*$. As we have seen in the previous section the harmonic function $V(x)$ is increasing in this case. Again let $\widehat{\mathbf{P}}$ denote the Doob h-transform of $\mathbf{P}$ via the harmonic function V, for its definition see (3.10). We start by showing that

$$\liminf_{n\to\infty}\widehat{\mathbf{P}}(X_n\le A^{Bn})>0 \tag{4.1}$$

for an appropriate constant B. For this we define

$$\sigma_y:=\inf\{n\ge 1: X_n\ge A^y\}.$$

By the total probability formula, for $x<A^{2n}$ and $B>2$,

$$\begin{aligned}
&\widehat{\mathbf{P}}_x(X_{\sigma_{2n}}\le A^{Bn},\sigma_{2n}\le n)\\
&=\sum_{k=1}^{n}\int_{x_0}^{A^{2n}}\widehat{\mathbf{P}}_x\left(\sigma_{2n}>k-1,X_{k-1}\in dz\right)\widehat{\mathbf{P}}_z(X_1\in(A^{2n},A^{Bn}])\\
&\ge\inf_{z<A^{2n}}\frac{\widehat{\mathbf{P}}_z(X_1\in(A^{2n},A^{Bn}])}{\widehat{\mathbf{P}}_z(X_1>A^{2n})}\sum_{k=1}^{n}\int_{x_0}^{A^{2n}}\widehat{\mathbf{P}}_x\left(\sigma_{2n}>k-1,X_{k-1}\in dz\right)\widehat{\mathbf{P}}_z(X_1>A^{2n})\\
&=\inf_{z<A^{2n}}\frac{\widehat{\mathbf{P}}_z(X_1\in(A^{2n},A^{Bn}])}{\widehat{\mathbf{P}}_z(X_1>A^{2n})}\widehat{\mathbf{P}}_x(\sigma_{2n}\le n).
\end{aligned}$$

For every $r \ge 2$, using the integration by parts, we get

$$\begin{aligned}
&\widehat{\mathbf{P}}_z(X_1 > A^{rn}) \\
&\quad = \frac{1}{V(z)} \int_{A^{rn}}^{\infty} V(y) \mathbf{P}_z(X_1 \in dy) \\
&\quad = \frac{1+o(1)}{V(z)} \int_{A^{rn}}^{\infty} U_0(\log_A y) \mathbf{P}_z(X_1 \in dy) \\
&\quad = \frac{1+o(1)}{V(z)} \left(U_0(rn) \mathbf{P}_z(X_1 > A^{rn}) + \int_{A^{rn}}^{\infty} U_0'(\log_A y)(\log_A y)' \mathbf{P}_z(X_1 > y) dy \right).
\end{aligned}$$

According to (1.4),

$$\mathbf{P}_z(X_1 > y) = \mathbf{P}(\eta_1 > \log_A(y - az)) \sim \frac{c}{\log_A y}$$

uniformly in $z \le A^{2n}$, $y \ge A^{2n}$. Therefore,

$$\begin{aligned}
\widehat{\mathbf{P}}_z(X_1 > A^{rn}) &= \frac{1+o(1)}{V(z)} \left(U_0(rn) \frac{c}{rn} + \int_{rn}^{\infty} e^{-u_0(t)} \mathbf{P}(\eta_1 > t) dt \right) \\
&= \frac{1+o(1)}{V(z)} \left(U_0(rn) \frac{c}{rn} + e^{-u_0(rn)} \right) = \frac{1+o(1)}{V(z)} r^{-c} \frac{U_0(n)}{n}.
\end{aligned}$$

This implies that

$$\inf_{z < A^{2n}} \frac{\widehat{\mathbf{P}}_z(X_1 \in (A^{2n}, A^{Bn}])}{\widehat{\mathbf{P}}_z(X_1 > A^{2n})} = 1 - \left(\frac{2}{B}\right)^c + o(1).$$

Taking $B = 2^{1+2/c}$, we conclude that

$$\widehat{\mathbf{P}}_x(X_{\sigma_{2n}} \le A^{2^{1+2/c} n}, \sigma_{2n} \le n) \ge \frac{1}{2} \widehat{\mathbf{P}}_x(\sigma_{2n} \le n), \quad x \le A^{2n}$$

for all n large enough. Using this bound, we obtain

$$\begin{aligned}
&\widehat{\mathbf{P}}_x(X_n \le A^{Bn}) \\
&\qquad \ge \widehat{\mathbf{P}}_x(\sigma_{2n} > n) + \widehat{\mathbf{P}}_x(X_n \le A^{Bn}, X_{\sigma_{2n}} \le A^{Bn}, \sigma_{2n} \le n) \\
&\qquad \ge \widehat{\mathbf{P}}_x(\sigma_{2n} > n) + \frac{1}{2} \widehat{\mathbf{P}}(\sigma_{2n} \le n) \widehat{\mathbf{P}}_x(X_n \le A^{Bn} | X_{\sigma_{2n}} \le A^{Bn}, \sigma_{2n} \le n).
\end{aligned}$$

By the strong Markov property,

$$\begin{aligned}
&\widehat{\mathbf{P}}_x(X_n \le A^{Bn}, X_{\sigma_{2n}} \le A^{Bn}, \sigma_{2n} \le n) \\
&\quad = \sum_{k=1}^{n} \int_{A^{2n}}^{A^{Bn}} \widehat{\mathbf{P}}_x(\sigma_{2n} = k, X_k \in dz) \widehat{\mathbf{P}}_z(X_{n-k} \le A^{Bn}) \\
&\quad \ge \sum_{k=1}^{n} \int_{A^{2n}}^{A^{Bn}} \widehat{\mathbf{P}}_x(\sigma_{2n} = k, X_k \in dz) \widehat{\mathbf{P}}_z(X_j \le X_{j-1} \text{ for all } j \le n-k) \\
&\quad \ge \widehat{\mathbf{P}}_x(\sigma_{2n} \le n, X_{\sigma_{2n}} \le A^{Bn}) \inf_{z \in (A^{2n}, A^{Bn})} \widehat{\mathbf{P}}_z(X_j \le X_{j-1} \text{ for all } j \le n).
\end{aligned}$$

Therefore,

$$\widehat{\mathbf{P}}_x(X_n \le A^{Bn} | X_{\sigma_{2n}} \le A^{Bn}, \sigma_{2n} \le n) \ge \inf_{z \in (A^{2n}, A^{Bn}]} \widehat{\mathbf{P}}_z(X_j \le X_{j-1} \text{ for all } j \le n).$$

If $X_0 = z \geq A^{2n}$ then $X_j \geq A^{2n-j}$ for every $j \geq 1$. If $A^n \geq x_0$ then $\mathbf{P}_z(T_{x_0}^{(X)} > n) = 1$ and, consequently,

$$\widehat{\mathbf{P}}_z(X_j \leq X_{j-1} \text{ for all } j \leq n) \geq \frac{V(A^n)}{V(z)} \mathbf{P}_z(X_j \leq X_{j-1} \text{ for all } j \leq n)$$

For every y we have

$$\mathbf{P}_y(X_1 \leq y) = \mathbf{P}(\eta_1 \leq \log_A y + \log_A(1-a)).$$

Thus, by the Markov property,

$$\begin{aligned}
&\mathbf{P}_z(X_j \leq X_{j-1} \text{ for all } j \leq n) \\
&\qquad \geq \mathbf{P}_z(X_j \leq X_{j-1} \text{ for all } j \leq n-1)\mathbf{P}(\eta_1 \leq n+1+\log_A(1-a)) \\
&\qquad \geq \ldots \geq \prod_{j=0}^{n-1} \mathbf{P}(\eta_1 \leq 2n-j+\log_A(1-a)) \\
&\qquad \geq (\mathbf{P}(\eta_1 \leq n+1+\log_A(1-a)))^n \sim e^{-c}.
\end{aligned}$$

This implies that

$$\inf_{z \in (A^{2n}, A^{Bn}]} \widehat{\mathbf{P}}_z(X_j \leq X_{j-1} \text{ for all } j \leq n) \geq \frac{V(A^n)}{V(A^{Bn})} \frac{e^{-c}}{2} \geq C_0.$$

Therefore,

$$\widehat{\mathbf{P}}_x(X_n \leq A^{Bn}) \geq \widehat{\mathbf{P}}_x(\sigma_{2n} > n) + \frac{C_0}{2}\widehat{\mathbf{P}}_x(\sigma_{2n} \leq n) \geq \frac{C_0}{2}$$

for all sufficiently large n. This finishes the proof of equation (4.1).

Recalling that V is increasing, we obtain the bound

$$\begin{aligned}
\mathbf{P}_x(T_{x_0}^{(X)} > n) &= V(x)\widehat{\mathbf{E}}_x^{(V)}\left[\frac{1}{V(X_n)}\right] \\
&\geq \frac{V(x)}{V(A^{Bn})}\widehat{\mathbf{P}}_x^{(V)}(X_n \leq A^{Bn}) \geq \frac{e^{-c}}{4}\frac{V(x)}{V(A^{Bn})}, \quad x \leq A^{2n}.
\end{aligned}$$

Consequently,

$$\mathbf{P}_x(T_{x_0}^{(X)} > n) \geq C_1 \frac{V(x \wedge A^n)}{V(A^n)} \tag{4.2}$$

for all $x > x_0 \geq x_*$.

Thus, it remains to consider the case $x_0 \leq x_*$. If $x > x_* + 1$ then $\mathbf{P}_x(T_{x_0}^{(X)} > n) \geq \mathbf{P}_x(T_{x_*}^{(X)} > n)$. Applying (4.2) with $x_0 = x_*$, we get

$$\mathbf{P}_x(T_{x_0}^{(X)} > n) \geq C_1 \frac{V_*(x \wedge A^n)}{V_*(A^n)} \geq C_2 \frac{V(x \wedge A^n)}{V(A^n)}.$$

If $x \leq x_* + 1$ then

$$\mathbf{P}_x(T_{x_0}^{(X)} > n) \geq \mathbf{P}(\xi_1 > x_* + 1)\mathbf{P}_{x_*+1}(T_{x_0}^{(X)} > n-1) \geq C_3 \frac{V(x \wedge A^n)}{V(A^n)}.$$

Together with (4.2) the proof is complete. □

4.2 Upper bounds for $\mathbf{P}(T_{x_0}^{(X)} > n)$

Lemma 4.2. *If $V(x)$ is increasing on (x_0, ∞) then*

$$\mathbf{P}_x(T_{x_0}^{(X)} > n) \le C\frac{V(x)}{V(A^n)}, \quad x > x_0,\ n \ge 1.$$

Proof. Since $\mathbf{P}_x(X_n > y)$ is monotonically increasing in x,

$$\mathbf{P}_x(X_n > y \mid T_{x_0}^{(X)} > n) \ge \mathbf{P}_x(X_n > y) \quad \text{for all } x, y > x_0.$$

Consequently,

$$\mathbf{E}_x[W(X_n) \mid T_{x_0}^{(X)} > n] \ge \mathbf{E}_x[W(X_n)] \ge \mathbf{E}_x[W(X_n)1\{X_n > x_0\}]$$

for every nonnegative increasing function W. (To prove this, one approximates W by functions of the form $\sum_k c_k 1_{(y_k,\infty)}$.) In particular, for $W = V$ one gets

$$\begin{aligned} V(x) &= \mathbf{E}_x[V(X_n),\, T_{x_0}^{(X)} > n] \\ &= \mathbf{P}_x(T_{x_0}^{(X)} > n)\mathbf{E}_x[V(X_n) \mid T_{x_0}^{(X)} > n] \\ &\ge \mathbf{P}_x(T_{x_0}^{(X)} > n)\mathbf{E}_x[V(X_n)1\{X_n > x_0\}] \end{aligned}$$

and can conclude

$$\mathbf{P}_x(T_{x_0}^{(X)} > n) \le \frac{V(x)}{\mathbf{E}_x[V(X_n)1\{X_n > x_0\}]} \le \frac{V(x)}{\mathbf{E}_0[V(X_n)1\{X_n > x_0\}]}\,. \tag{4.3}$$

As we already know, $\frac{\log_A X_n}{n}$ converges weakly to the distribution with density

$$\frac{cy^{c-1}}{(y+1)^{c+1}}\mathbf{1}_{\mathbb{R}^+}(y)\,.$$

The asymptotic behaviour in (4.3) is obtained most conveniently if one assumes that $\frac{\log_A X_n}{n}$ converges almost everywhere to some Z with this distribution. (On a suitable probability space, the sequence can always be constructed in such a way.) Then, as $v(x) := V(\log_A x)$ varies regularly with index $1-c$,

$$\frac{V(X_n)}{V(A^n)} = \frac{v(\log_A X_n)}{v(n)} = \frac{v\left(\frac{\log_A X_n}{n}n\right)}{v(n)} \sim \frac{v(Zn)}{v(n)} \to Z^{1-c}\,.$$

(More precisely, due to the monotonicity of V, one first gets for every fixed $N \in \mathbb{N}$

$$\limsup_{n\to\infty} \frac{v\left(\frac{\log_A X_n}{n}n\right)}{v(n)} \le \limsup_{n\to\infty} \frac{v\left(\sup_{k:k\ge N} \frac{\log_A X_k}{k}n\right)}{v(n)} = \left(\sup_{k:k\ge N} \frac{\log_A X_k}{k}\right)^{1-c}\,.$$

$N \to \infty$ shows $\limsup_{n\to\infty} \frac{v(\log_A X_n)}{v(n)} \le Z^{1-c}$ and likewise one checks that the lower limit has at least this value.)

Now one can apply the Fatou lemma:

$$\liminf_{n\to\infty} \frac{\mathbf{E}_0[V(X_n)1\{X_n > x_0\}]}{V(A^n)} \ge \mathbf{E}[Z^{1-c}] = \int_0^\infty y^{1-c}\frac{cy^{c-1}}{(y+1)^{c+1}} = 1\,,$$

so (4.3) yields the desired bound. □

Remark 4.3. We know from the construction of V that this function is increasing for $x_0 \ge x_*$. We now notice that, using Theorem 1.3(iii), one can infer that V is increasing for all x_0. Indeed, by the monotonicity of the chain $\{X_n\}$,

$$\mathbf{P}_x(T_{x_0}^{(X)} > n) \le \mathbf{P}_y(T_{x_0}^{(X)} > n)$$

for all n and for all $x \le y$. Combining this with the asymptotic relation $\mathbf{P}_x(T_{x_0}^{(X)} > n) \sim \varkappa(c)\frac{V(x)}{V(A^n)}$, we conclude that $V(x) \le V(y)$. Thus, the bound in Lemma 4.2 holds for each x_0 and, consequently, the upper bound in Theorem 1.3(ii) is valid.

Since the we do not have the monotonicity property of V in the case $x_0 \le x_*$ we next prove an alternative upper bound, avoiding the use of monotonicity arguments.

Lemma 4.4. *Assume that there exist x_1 and a subexponential distribution F such that*

$$\mathbf{P}_x(T_{x_1}^{(X)} > n) \le C(x)\overline{F}(n), \quad n \ge 0,\ x > x_1.$$

If $x_0 < x_1$ is such that $\mathbf{P}(ax_1 + \xi_1 < x_0) > 0$ then there exists $C(x_0, x)$ such that

$$\mathbf{P}_x(T_{x_0}^{(X)} > n) \le C(x, x_0)\overline{F}(n), \quad n \ge 0,\ x > x_0.$$

Proof. The assumption $\mathbf{P}(ax_1 + \xi_1 < x_0) > 0$ implies that

$$p := \mathbf{P}_{x_1}(X_{T_{x_1}^{(X)}} \le x_0) > 0.$$

Then we can represent the law of $T_{x_1}^{(X)}$ as a mixture of two distributions:

$$\begin{aligned}
&\mathbf{P}_{x_1}(T_{x_1}^{(X)} \in B) \\
&\qquad = p\mathbf{P}_{x_1}(T_{x_1}^{(X)} \in B | X_{T_{x_1}^{(X)}} \le x_0) + (1-p)\mathbf{P}_{x_1}(T_{x_1}^{(X)} \in B | X_{T_{x_1}^{(X)}} > x_0) \\
&\qquad =: p\mathbf{P}(\theta \in B) + (1-p)\mathbf{P}(\zeta \in B).
\end{aligned}$$

Noting that $\{X_n\}$ may visit $(x_0, x_1]$ several times before $T_{x_0}^{(X)}$ and using the monotonicity of the chain, we get

$$\mathbf{P}_{x_1}(T_{x_0}^{(X)} > n) \le p\sum_{k=0}^{\infty}(1-p)^k \mathbf{P}(\zeta_1 + \zeta_2 + \ldots + \zeta_k + \theta > n),$$

where $\{\zeta_k\}$ are independent copies of ζ. Under the assumptions of the lemma we have

$$\mathbf{P}(\zeta > n) \le C_1\overline{F}(n) \quad \text{and} \quad \mathbf{P}(\theta > n) \le C_2\overline{F}(n).$$

Then, by Proposition 4 in [5],

$$\mathbf{P}_{x_1}(T_{x_0}^{(X)} > n) \le C\overline{F}(n).$$

If the starting point x is smaller than x_1 then

$$\mathbf{P}_x(T_{x_0}^{(X)} > n) \le \mathbf{P}_{x_1}(T_{x_0}^{(X)} > n) \le C\overline{F}(n).$$

If the starting point x is bigger than x_1 then $\mathbf{P}_x(T_{x_0}^{(X)} > n)$ is bounded by the tail of the convolution of $\mathbf{P}_x(T_{x_1}^{(X)} \in \cdot)$ and $\mathbf{P}_{x_1}(T_{x_0}^{(X)} \in \cdot)$. Since the tails of these two distributions are $O(\overline{F}(n))$, the tail of their convolution is also $O(\overline{F}(n))$. This completes the proof of the lemma. $\square$

Corollary 4.5. *If* (1.4) *holds then*

$$\mathbf{P}_x(T^{(X)}_{x_0} > n) \le C\frac{V(x)}{V(A^n)}$$

for all $x > x_0$ *and all* $n \ge 1$.

Proof. It suffices to consider the case $x_0 < x_*$.

Since $V_*(A^n)$ is regularly varying then, in view of Lemma 4.2, the conditions of Lemma 4.4 are valid for $x_1 = x_*$ and $\overline{F}(n) \sim CU_0(n)$. Combining now Lemmata 4.2 and 4.4, we have, for $x > x_*$,

$$\mathbf{P}_x(T^{(X)}_{x_0} > n) \le \mathbf{P}_x(T^{(X)}_{x_*} > n/2) + \mathbf{P}_{x_*}(T^{(X)}_{x_0} > n/2) \le \frac{C_1 V_*(x) + C_2}{V_*(A^{n/2})}$$

Recalling that $V_*(A^n)$ is regularly varying and that $V(x) \ge V_*(x)$ in the case $x_0 < x_*$, we have the desired estimate for $x > x_*$. In the case $x \le x_*$ it suffices to apply Lemma 4.4. □

5 Proof of asymptotic relations

In this section we shall prove asymptotic relations in Theorem 1.3(iii). Exact asymptotics in Theorem 1.2 can be derived by exactly the same arguments, and we omit their proof.

We are going to apply Theorem 3.10 from Durrett [10] to the sequence of Markov processes

$$v^{(n)}_t := \frac{\log_A X_{[nt]}}{n}, \quad t \ge 0.$$

Since this sequence converges weakly to the process Z, which is non-degenerate and $\mathbf{P}_x(T^{(Z)}_0 > t)$ is strictly positive for all $x, t > 0$, we conclude that the conditions (i)-(iii) from [10] are fulfilled. Moreover, we have already shown that $\mathbf{P}_x\left(\cdot\ |T^{(Z)}_0 > 1\right)$ converges, as $x \to 0$, to a non-degenerate limit. Thus, it remains to check that

- $\mathbf{P}_{A^{nx_n}}(T^{(X)}_{x_0} > nt_n) \to \mathbf{P}_x(T^{(Z)}_0 > t)$ if $x_n \to x > 0$ and $t_n \to t > 0$;
- $\mathbf{P}_{A^{nx_n}}(T^{(X)}_{x_0} > nt_n) \to 0$ whenever $x_n \to 0$ and $t_n \to t > 0$;
- the sequence $v^{(n)}$ is tight; and
- $\lim_{h\to 0} \liminf_{n\to\infty} \mathbf{P}_{A^x}(v^{(n)}_t > h \mid T^{(X)}_{x_0} > n) = 1$ for every $t > 0$.

We start with the first condition.

Lemma 5.1. *If* $x_n \to x > 0$ *and* $t_n \to t > 0$ *then*

$$\mathbf{P}_{A^{nx_n}}(T^{(X)}_{x_0} > nt_n) \to \mathbf{P}_x(T^{(Z)}_0 > t).$$

Proof. Since $\mathbf{P}_y(T^{(X)}_{x_0} > m)$ is increasing in y and decreasing in m, it suffices to prove the lemma in the special case $x_n = x$ and $t_n = t$. We are going to apply Theorem 2.1 from [10]. We set

$$A_0 := \left\{f \in D[0,t] : \inf_{s\le t} f(s) > 0\right\}$$

and

$$A_n := \left\{f \in D[0,t] : \inf_{s\le t} f(s) > \frac{x_0}{n}\right\}, \quad n \ge 1.$$

Furthermore, for every $\varepsilon > 0$ we define

$$G_\varepsilon := \left\{f \in D[0,t] : \inf_{s\le t} f(s) > \varepsilon\right\}.$$

Then we have

$$\mathbf{P}_x(Z \in \partial G_\varepsilon) = 0 \quad \text{for all } \varepsilon, t > 0. \tag{5.1}$$

It is clear that $G_\varepsilon \subset A_n$ for all $n > x_0/\varepsilon$ and $G_{1/n} \uparrow A_0$. Thus, in order to apply Theorem 2.1 from [10] we have only to show that

$$\limsup_{n\to\infty} \mathbf{P}_{A^{xn}}(T_{x_0}^{(X)} > nt) \le \mathbf{P}_x(T_0^{(Z)} > t). \tag{5.2}$$

Fix some $\varepsilon < x$ and $\delta < t$. Then, using the monotonicity of the chain $\{X_n\}$, we get

$$\mathbf{P}_{A^{xn}}(T_{x_0}^{(X)} > nt) \le \mathbf{P}_{A^{xn}}\left(\inf_{s\le t-\delta} X_{[sn]} > A^{\varepsilon n}\right) + \mathbf{P}_{A^{\varepsilon n}}(T_{x_0}^{(X)} > \delta n).$$

According to the upper bound in (1.12),

$$\mathbf{P}_{A^{\varepsilon n}}(T_{x_0}^{(X)} > \delta n) \le C\frac{V(A^{\varepsilon n})}{V(A^{\delta n})}.$$

Recalling that $V(A^x)$ is regularly varying of index $1-c$, we conclude

$$\limsup_{n\to\infty} \mathbf{P}_{A^{\varepsilon n}}(T_{x_0}^{(X)} > \delta n) \le C\left(\frac{\varepsilon}{\delta}\right)^{1-c}.$$

Furthermore, combining (1.5) and (5.1), we get

$$\mathbf{P}_{A^{xn}}\left(\inf_{s\le t-\delta} X_{[sn]} > A^{\varepsilon n}\right) \to \mathbf{P}_x\left(\inf_{s\le t-\delta} Z_s > \varepsilon\right).$$

Consequently,

$$\begin{aligned}\limsup_{n\to\infty} \mathbf{P}_{A^{xn}}(T_{x_0}^{(X)} > nt) &\le \mathbf{P}_x\left(\inf_{s\le t-\delta} Z_s > \varepsilon\right) + C\left(\frac{\varepsilon}{\delta}\right)^{1-c}\\ &\le \mathbf{P}_x(T_0^{(Z)} > t-\delta) + C\left(\frac{\varepsilon}{\delta}\right)^{1-c}.\end{aligned}$$

Letting here first $\varepsilon \to 0$ and then $\delta \to 0$, we arrive at (5.2). Thus, the proof is complete. □

Lemma 5.2. *If $t_n \to t > 0$ and $x_n \to 0$ then*

$$\mathbf{P}_{A^{x_n n}}(T_{x_0}^{(X)} > nt) \to 0.$$

This is a simple consequence of the upper bound in (1.12) and we omit its proof.

Lemma 5.3. *For all $x > x_0$ and all $t > 0$ one has*

$$\lim_{h\to 0}\limsup_{n\to\infty} \mathbf{P}_x(v_t^{(n)} > h \mid T_{x_0}^{(X)} > n) = 1.$$

Proof. By the definition of $v^{(n)}$,

$$\mathbf{P}_x(v_t^{(n)} \le h \mid T_{x_0}^{(X)} > n) = \frac{\mathbf{P}_x(X_{[nt]} \le A^{hn}, T_{x_0}^{(X)} > n)}{\mathbf{P}_x(T_{x_0}^{(X)} > n)}.$$

Set $s = \min\{1,t\}/2$. Then, by the monotonicity of X_n,

$$\mathbf{P}_x(X_{[nt]} \le A^{hn}, T_{x_0}^{(X)} > n) \le \mathbf{P}_x(T_{x_0}^{(X)} > ns)\mathbf{P}_0(X_{[n(t-s)]} \le A^{hn}).$$

Therefore,

$$\mathbf{P}_x(v_t^{(n)} > h \mid T_{x_0}^{(X)} > n) \le \frac{\mathbf{P}_x(T_{x_0}^{(X)} > ns)}{\mathbf{P}_x(T_{x_0}^{(X)} > n)}\mathbf{P}_0(X_{[n(t-s)]} \le A^{hn}).$$

Taking into account (1.5), (1.12) and (1.6), we get

$$\limsup_{n\to\infty} \mathbf{P}_x(v_t^{(n)} > h | T_{x_0}^{(X)} > n) \le C s^{c-1} \mathbf{P}_0(Z_{t-s} \le h) \le C s^{c-1} \left(\frac{h}{h+t-s} \right)^c .$$

This yields the desired relation. □

To show the tightness we shall use the following upper bound for the conditional distribution of X_n.

Lemma 5.4. *There exists a constant C such that*

$$\mathbf{P}_x(X_n \ge A^y | T_{x_0}^{(X)} > n) \le C \frac{n}{y}, \quad y \ge 2 \log_A \left(x + \frac{1}{1-a} \right).$$

Proof. If $\xi_k < A^{y/2}$ for all $k \le n$ then

$$\begin{aligned} X_n &= a^n x + a^{n-1}\xi_1 + a^{n-2}\xi_2 + \ldots + \xi_n \\ &\le x + A^{y/2} \sum_{j=0}^{n-1} a^j \le x + A^{y/2} \frac{1}{1-a} \le A^y \end{aligned}$$

for all $y \ge 2\log_A \left(x + \frac{1}{1-a} \right)$. Therefore,

$$\begin{aligned} \mathbf{P}_x(X_n \ge A^y, T_{x_0}^{(X)} > n) &\le \sum_{k=1}^{n} \mathbf{P}_x(\xi_k \ge A^{y/2}, T_{x_0}^{(X)} > n) \\ &\le \sum_{k=1}^{n} \mathbf{P}_x(\xi_k \ge A^{y/2}, T_{x_0}^{(X)} > k-1) \\ &\le \mathbf{P}(\xi_1 \ge A^{y/2}) \sum_{k=1}^{n} \mathbf{P}_x(T_{x_0}^{(X)} > k-1) \end{aligned}$$

Using the upper bound in (1.12) and recalling that $V(A^x)$ is regularly varying with index $1-c$, we conclude that

$$\sum_{k=1}^{n} \mathbf{P}_x(T_{x_0}^{(X)} > k-1) \le 1 + C \sum_{j=1}^{n-1} \frac{V(x)}{V(A^j)} \le C \frac{nV(x)}{V(A^n)}.$$

Consequently,

$$\mathbf{P}_x(X_n \ge A^y, T_{x_0}^{(X)} > n) \le C \frac{nV(x)}{V(A^n)} \mathbf{P}(\eta_1 \ge y/2).$$

Combining this with (1.4) and with the lower bound in (1.12), we obtain the desired estimate. □

Lemma 5.5. *The sequence $v^{(n)}$ is tight.*

Proof. According to Theorem 3.6 in [10], it suffices show that

$$\lim_{K\to\infty} \limsup_{n\to\infty} \mathbf{P}_x(X_n > A^{nK} | T_{x_0}^{(X)} > n) = 0 \tag{5.3}$$

and

$$\lim_{t\to 0} \limsup_{n\to\infty} \mathbf{P}_x(X_{[nt]} > A^{nh} | T_{x_0}^{(X)} > n) = 0, \quad h > 0. \tag{5.4}$$

(5.3) is immediate from Lemma 5.4. To show (5.4) we first notice that, for every $t < 1$,

$$\mathbf{P}_x(X_{[nt]} > A^{nh} | T_{x_0}^{(X)} > n) \le \mathbf{P}_x(X_{[nt]} > A^{nh} | T_{x_0}^{(X)} > nt) \frac{\mathbf{P}_x(T_{x_0}^{(X)} > nt)}{\mathbf{P}_x(T_{x_0}^{(X)} > n)}.$$

Applying Lemma 5.4 to the first probability term on the right hand side, we get

$$\mathbf{P}_x(X_{[nt]} > A^{nh} | T_{x_0}^{(X)} > n) \le C \frac{t}{h} \frac{\mathbf{P}_x(T_{x_0}^{(X)} > nt)}{\mathbf{P}_x(T_{x_0}^{(X)} > n)}.$$

Using again (1.12), we have

$$\limsup_{n\to\infty} \frac{\mathbf{P}_x(T_{x_0}^{(X)} > nt)}{\mathbf{P}_x(T_{x_0}^{(X)} > n)} \le C t^{c-1}.$$

As a result we have the estimate

$$\limsup_{n\to\infty} \mathbf{P}_x(X_{[nt]} > A^{nh} | T_{x_0}^{(X)} > n) \le C \frac{t^c}{h},$$

which implies (5.4). □

We have checked all the conditions in Theorem 3.10 in [10]. Therefore, the sequence of distributions $\mathbf{P}_x\left(v^{(n)} \in \cdot | T_{x_0}^{(X)} > n\right)$ on $D[0,1]$ converges weakly towards the distribution Q introduced in Theorem 1.1.

5.1 Tail asymptotics of the first hitting time

In this section we aim to prove (1.13), which is a part of Theorem 1.1 (iii). Since V is harmonic,

$$\begin{aligned} V(x) &= \mathbf{E}_x[V(X_n); T_{x_0}^{(X)} > n] \\ &= \mathbf{E}_x[V(X_n); T_{x_0}^{(X)} > n, X_n \le A^{Kn}] + \mathbf{E}_x[V(X_n); T_{x_0}^{(X)} > n, X_n > A^{Kn}] \end{aligned} \tag{5.5}$$

for every $K > 0$.

We know that $V(x) \le C U_0(\log_A x)$. Therefore,

$$\begin{aligned} &\mathbf{E}_x[V(X_n); T_{x_0}^{(X)} > n, X_n > A^{Kn}] \\ &\le C\mathbf{E}_x[U_0(\log_A X_n); T_{x_0}^{(X)} > n, X_n > A^{Kn}] \\ &= CU_0(Kn)\mathbf{P}_x(\log_A X_n \ge Kn; T_{x_0}^{(X)} > n) \\ &\qquad + C\int_{Kn}^{\infty} U_0'(y)\mathbf{P}_x(\log_A X_n \ge y; T_{x_0}^{(X)} > n)dy. \end{aligned}$$

Combining Lemma 5.4 and (1.12), we have

$$\mathbf{P}_x(\log_A X_n \ge y; T_{x_0}^{(X)} > n) \le C \frac{nV(x)}{yV(A^n)}.$$

Consequently,

$$\begin{aligned} &\mathbf{E}_x[V(X_n); T_{x_0}^{(X)} > n, X_n > A^{Kn}] \\ &\le C_1 \frac{V(x)}{V(A^n)} \left(\frac{U_0(Kn)}{K} + n\int_{Kn}^{\infty} \frac{U_0'(y)}{y} dy \right) \\ &\le C_2 \frac{V(x)}{V(A^n)} \left(\frac{U_0(Kn)}{K} + n\int_{Kn}^{\infty} \frac{U_0(y)}{y^2} dy \right) \le C_3 \frac{V(x)}{V(A^n)} \frac{U_0(Kn)}{K}. \end{aligned}$$

Recalling that $V(A^n) \sim U_0(n)$ and that U_0 is regularly varying, we finally get

$$\limsup_{n\to\infty} \mathbf{E}_x[V(X_n); T_{x_0}^{(X)} > n, X_n > A^{Kn}] \le \frac{C}{K^c} V(x). \tag{5.6}$$

For the first summand on the right hand side of (6.8) we have

$$\begin{aligned}
&\mathbf{E}_x[V(X_n); T_{x_0}^{(X)} > n, X_n \le A^{Kn}] \\
&\quad = \mathbf{P}_x(T_{x_0}^{(X)} > n)\mathbf{E}_x[V(X_n)1\{X_n \le A^{Kn}\}|T_{x_0}^{(X)} > n] \\
&\quad = V(A^n)\mathbf{P}_x(T_{x_0}^{(X)} > n)\mathbf{E}_x\left[\frac{V(X_n)}{V(A^n)}1\{X_n \le A^{Kn}\}\Big| T_{x_0}^{(X)} > n\right].
\end{aligned}$$

It follows from the already proven conditional limit theorem and from (2.12) that

$$\lim_{n\to\infty} \mathbf{P}_x\left(\frac{\log_A X_n}{n} \le y \Big| T_{x_0}^{(X)} > n\right) = \frac{y}{y+1}, \quad y > 0.$$

Combining this with the regular variation property of V, we obtain

$$\begin{aligned}
&\mathbf{E}_x\left[\frac{V(X_n)}{V(A^n)}1\{X_n \le A^{Kn}\}\Big| T_{x_0}^{(X)} > n\right] \\
&\quad = (1+o(1))\mathbf{E}_x\left[\left(\frac{\log_A X_n}{n}\right)^{1-c} 1\{X_n \le A^{Kn}\}\Big| T_{x_0}^{(X)} > n\right] \\
&\quad = (1+o(1)) \int_0^K \frac{y^{c-1}}{(1+y)^2} dy.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\mathbf{E}_x[V(X_n); T_{x_0}^{(X)} > n, X_n \le A^{Kn}] \\
&\qquad = (1+o(1))V(A^n)\mathbf{P}_x(T_{x_0}^{(X)} > n) \int_0^K \frac{y^{c-1}}{(1+y)^2} dy.
\end{aligned} \tag{5.7}$$

Plugging (5.6) and (5.7) into (5.5) and letting $K \to \infty$, we obtain

$$\mathbf{P}_x(T_{x_0}^{(X)} > n) \sim \left(\int_0^\infty \frac{y^{c-1}}{(1+y)^2} dy\right)^{-1} \frac{V(x)}{V(A^n)}.$$

Thus, (1.13) holds with

$$\varkappa(c) = \left(\int_0^\infty \frac{y^{c-1}}{(1+y)^2} dy\right)^{-1} = \frac{1}{(1-c)B(c, 1-c)}.$$

6 Proof of Theorem 1.5

As we have already seen in the analysis of the harmonic function explicit calculations are often possible in the case of the Markov chain $\{R_n\}_{n\ge 0}$. This will be also the case in the following subsections.

6.1 Expectation of hitting times for the maximal autoregressive process

Put $u(x) = \mathbf{E}_x[T_{x_0}^{(R)}]$. In this subsection we will derive representation (1.15) for $u(x)$. Observe that the Markov property implies that $u(x)$ satisfies the following equality

$$\begin{aligned}
u(x) &= \mathbf{P}_x(T_{x_0}^{(R)} = 1) + \mathbf{E}_x[1 + u(R_1); T_{x_0}^{(R)} > 1] \\
&= \mathbf{P}_x(R_1 \le x_0) + \mathbf{E}_x[1 + u(R_1); R_1 > x_0], \quad x > x_0.
\end{aligned} \tag{6.1}$$

Assume first that $x \in (x_0, x_0+1]$. In this case one has

$$\{R_1 > x_0\} = \{R_1 = \eta_1 > x_0\}.$$

Therefore,

$$\begin{aligned} u(x) &= \mathbf{P}(\eta_1 \le x_0) + \mathbf{E}[1 + u(\eta_1); \eta_1 > x_0] \\ &= 1 + \mathbf{E}[u(\eta_1); \eta_1 > x_0] \quad \text{for all } x \in (x_0, x_0+1]. \end{aligned}$$

For all $x > x_0 + 1$ one has $\mathbf{P}_x(T^{(R)}_{x_0} = 1) = 0$. This implies that (6.1) reduces to

$$\begin{aligned} u(x) &= \mathbf{E}_x[1 + u(R_1)] \\ &= 1 + u(x-1)\mathbf{P}(\eta_1 \le x-1) + \mathbf{E}[u(\eta_1); \eta_1 > x-1], \quad x > x_0 + 1. \end{aligned} \tag{6.2}$$

If $x \in (x_0+1, x_0+2]$ then $x - 1 \in (x_0, x_0+1]$ and, consequently, $u(x-1) = u(x_0+1)$ for all $x \in (x_0+1, x_0+2]$. From this observation and from (6.2) we have

$$\begin{aligned} u(x) &= 1 + u(x_0+1)\mathbf{P}(\eta_1 \le x-1) + \mathbf{E}[u(\eta_1); \eta_1 > x-1] \\ &= 1 + u(x_0+1)\mathbf{P}(\eta_1 \le x-1) \\ &\qquad + \mathbf{E}[u(\eta_1); \eta_1 \in (x-1, x_0+1]] + \mathbf{E}[u(\eta_1); \eta_1 > x_0+1] \\ &= 1 + u(x_0+1)\mathbf{P}(\eta_1 \le x_0+1) + \mathbf{E}[u(\eta_1); \eta_1 > x_0+1]. \end{aligned} \tag{6.3}$$

This equality implies that $u(x) = u(x_0+2)$ for all $x \in (x_0+1, x_0+2]$. Note also that

$$\begin{aligned} u(x_0+1) &= 1 + \mathbf{E}[u(\eta_1); \eta_1 > x_0] \\ &= 1 + u(x_0+1)\mathbf{P}(\eta_1 \in (x_0, x_0+1])) + \mathbf{E}[u(\eta_1); \eta_1 > x_0+1]. \end{aligned}$$

Combining this with (6.3), we conclude that

$$u(x_0+2) = u(x_0+1)\left(1 + \mathbf{P}(\eta_1 \le x_0)\right).$$

Fix now an integer n and consider the case $x \in (x_0+n, x_0+n+1]$. Assume that we have already shown that $u(y) = u(x_0+n)$ for all $y \in (x_0+n-1, x_0+n]$. Then we have from (6.2)

$$\begin{aligned} u(x) &= 1 + u(x_0+n)\mathbf{P}(\eta_1 \le x-1) + \mathbf{E}[u(\eta_1); \eta_1 > x-1] \\ &= 1 + u(x_0+n)\mathbf{P}(\eta_1 \le x_0+n) + \mathbf{E}[u(\eta_1); \eta_1 > x_0+n]. \end{aligned}$$

Therefore, $u(x) = u(x_0+n+1)$ for all $x \in (x_0+n, x_0+n+1]$. This means that this property is valid for all n.

One has also equalities

$$u(x_0+n+1) = 1 + u(x_0+n)\mathbf{P}(\eta_1 \le x_0+n) + \mathbf{E}[u(\eta_1); \eta_1 > x_0+n]$$

and

$$\begin{aligned} u(x_0+n) &= 1 + u(x_0+n-1)\mathbf{P}(\eta_1 \le x_0+n-1) + \mathbf{E}[u(\eta_1); \eta_1 > x_0+n-1] \\ &= 1 + u(x_0+n-1)\mathbf{P}(\eta_1 \le x_0+n-1) \\ &\qquad + u(x_0+n)\mathbf{P}(\eta_1 \in (x_0+n-1, x_0+n]) + \mathbf{E}[u(\eta_1); \eta_1 > x_0+n]. \end{aligned}$$

Taking the difference we obtain

$$\begin{aligned} &u(x_0+n+1) - u(x_0+n) \\ &\quad = u(x_0+n)\mathbf{P}(\eta_1 \le x_0+n) - u(x_0+n-1)\mathbf{P}(\eta_1 \le x_0+n-1) \\ &\qquad\qquad - u(x_0+n)\mathbf{P}(\eta_1 \in (x_0+n-1, x_0+n]) \\ &\quad = \mathbf{P}(\eta_1 \le x_0+n-1)\left(u(x_0+n) - u(x_0+n-1)\right). \end{aligned}$$

Consequently,

$$u(x_0+n+1)-u(x_0+n)=u(x_0+1)\prod_{k=0}^{n-1}\mathbf{P}(\eta_1\le x_0+k),\quad n\ge 1.$$

As a result, we have the following expression for the expectation of the hitting time

$$u(x)=u(x_0+1)\left(1+\sum_{j=1}^{n}\prod_{k=0}^{j-1}\mathbf{P}(\eta_1\le x_0+k)\right),\ x\in(x_0+n,x_0+n+1]. \tag{6.4}$$

Finally, in order to get a finite solution we have to show that the equation

$$u(x_0+1)=1+\mathbf{E}[u(\eta_1);\eta_1>x_0]$$

is solvable. In view of (6.4), the previous equation is equivalent to

$$u(x_0+1)=1+u(x_0+1)\sum_{n=0}^{\infty}\left(1+\sum_{j=1}^{n}\prod_{k=0}^{j-1}\mathbf{P}(\eta_1\le x_0+k)\right)\mathbf{P}(\eta_1\in(x_0+n,x_0+n+1]).$$

Now we conclude that (6.1) has a finite solution if and only if

$$\sum_{n=0}^{\infty}\left(1+\sum_{j=1}^{n}\prod_{k=0}^{j-1}\mathbf{P}(\eta_1\le x_0+k)\right)\mathbf{P}(\eta_1\in(x_0+n,x_0+n+1])<1.$$

Clearly,

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(1+\sum_{j=1}^{n}\prod_{k=0}^{j-1}\mathbf{P}(\eta_1\le x_0+k)\right)\mathbf{P}(\eta_1\in(x_0+n,x_0+n+1])\\
&=\mathbf{P}(\eta_1>x_0)+\sum_{j=1}^{\infty}\prod_{k=0}^{j-1}\mathbf{P}(\eta_1\le x_0+k)\sum_{n=j}^{\infty}\mathbf{P}(\eta_1\in(x_0+n,x_0+n+1])\\
&=\mathbf{P}(\eta_1>x_0)+\sum_{j=1}^{\infty}(1-\mathbf{P}(\eta_1\le x_0+j))\prod_{k=0}^{j-1}\mathbf{P}(\eta_1\le x_0+k).
\end{aligned}$$

Furthermore, for every $N\ge 1$,

$$\begin{aligned}
&\sum_{j=1}^{N}(1-\mathbf{P}(\eta_1\le x_0+j))\prod_{k=0}^{j-1}\mathbf{P}(\eta_1\le x_0+k)\\
&\qquad=\sum_{j=1}^{N}\prod_{k=0}^{j-1}\mathbf{P}(\eta_1\le x_0+k)-\sum_{j=1}^{N}\prod_{k=0}^{j}\mathbf{P}(\eta_1\le x_0+k)\\
&\qquad=\mathbf{P}(\eta_1\le x_0)-\prod_{k=0}^{N}\mathbf{P}(\eta_1\le x_0+k).
\end{aligned}$$

This implies that

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(1+\sum_{j=1}^{n}\prod_{k=0}^{j-1}\mathbf{P}(\eta_1\le x_0+k)\right)\mathbf{P}(\eta_1\in(x_0+n,x_0+n+1])\\
&\qquad=1-\lim_{N\to\infty}\prod_{k=0}^{N}\mathbf{P}(\eta_1\le x_0+k).
\end{aligned}$$

Thus, there is a finite solution $u(x)$ if and only if

$$\lim_{N\to\infty}\prod_{k=0}^{N}\mathbf{P}(\eta_1\le x_0+k)>0.$$

Note that this is equivalent to $\mathbf{E}\eta_1^+<\infty$. Then,

$$u(x_0+1)=\frac{1}{\prod_{k=0}^{\infty}\mathbf{P}(\eta_1\le x_0+k)}.$$

Plugging this result into (6.4) gives us an explicit formula for the expected hitting time.

6.2 Recursion for tails of exit times

We will consider now $\mathbf{P}_x(T_{x_0}^{(R)}>n)$. Define

$$v(n,k)=\mathbf{P}_{x_0+k+1}(T_{x_0}^{(R)}>n),\quad n,k\ge 0$$

and

$$v_n=v(n,0).$$

Then the following result holds.

Proposition 6.1. *Assume that* $0<\mathbf{P}(\eta_1<x_0)<1$. *Then, for integer* $n,k\ge 0$,

$$\mathbf{P}_x(T_{x_0}^{(R)}>n)=v(n,k),\quad x\in(x_0+k,x_0+k+1]. \tag{6.5}$$

For $n\le k$ *we have* $v(n,k)=1$ *and for* $n>k$ *the following recursive equality holds*

$$v(n,k)=v(n,0)+\sum_{m=1}^{k}v(n-m,0)\prod_{j=0}^{m-1}\mathbf{P}(\eta_1\le x_0+j). \tag{6.6}$$

Furthermore,

$$\begin{aligned}v_n&=\mathbf{P}(\eta_1>x_0+n-1)+v_{n-1}\mathbf{P}(\eta_1\in(x_0,x_0+n-1])\\&\quad+\sum_{m=1}^{n-2}v_{n-m-1}\mathbf{P}(\eta_1\in(x_0+m,x_0+n-1])\prod_{j=0}^{m-1}\mathbf{P}(\eta_1\le x_0+j)\end{aligned} \tag{6.7}$$

and hence (6.6) *and* (6.7) *allow us to find* $\mathbf{P}_x(T_{x_0}^{(R)}>n)$ *recursively.*

Proof. It is clear that for $n\le k$ it holds $\mathbf{P}_x(T_{x_0}^{(R)}>n)=v(n,k)=1$. Hence, in the rest of the proof we will assume that $n>k$.

Let $x\in(x_0,x_0+1]$. Then, for $n>0$ we have,

$$\begin{aligned}\mathbf{P}_x(T_{x_0}^{(R)}>n)&=\int_{x_0}^{\infty}\mathbf{P}_x(R_1\in dy)\mathbf{P}_y(T_{x_0}^{(R)}>n-1)\\&=\int_{x_0}^{\infty}\mathbf{P}(\eta_1\in dy)\mathbf{P}_y(T_{x_0}^{(R)}>n-1).\end{aligned} \tag{6.8}$$

Clearly this probability is the same for each $x\in(x_0,x_0+1]$ and hence (6.5) holds for $k=0$.

Next consider $x \in (x_0+1, x_0+2]$. For every $n > 1$ we have

$$\begin{aligned}
&\mathbf{P}_x(T^{(R)}_{x_0} > n) \\
&= \mathbf{P}_{x-1}(T^{(R)}_{x_0} > n-1)\mathbf{P}(\eta_1 \le x-1) + \int_{x-1}^{\infty} \mathbf{P}(\eta_1 \in dy)\mathbf{P}_y(T^{(R)}_{x_0} > n-1) \\
&= \mathbf{P}_{x_0+1}(T^{(R)}_{x_0} > n-1)\mathbf{P}(\eta_1 \le x-1) + \int_{x-1}^{x_0+1} \mathbf{P}(\eta_1 \in dy)\mathbf{P}_{x_0+1}(T^{(R)}_{x_0} > n-1) \\
&\qquad + \int_{x_0+1}^{\infty} \mathbf{P}(\eta_1 \in dy)\mathbf{P}_y(T^{(R)}_{x_0} > n-1) \\
&= v(n-1,0)\mathbf{P}(\eta_1 \le x_0+1) + \int_{x_0+1}^{\infty} \mathbf{P}(\eta_1 \in dy)\mathbf{P}_y(T^{(R)}_{x_0} > n-1). \qquad (6.9)
\end{aligned}$$

This expression is constant for $x \in (x_0+1, x_0+2]$ and hence (6.5) holds for $k = 1$. Note also that it follows from (6.8) that

$$v(n,0) = v(n-1,0)\mathbf{P}(\eta_1 \in (x_0, x_0+1]) + \int_{x_0+1}^{\infty} \mathbf{P}(\eta_1 \in dy)\mathbf{P}_y(T^{(R)}_{x_0} > n-1).$$

Subtracting this expression from (6.9) we obtain

$$v(n,1) - v(n,0) = v(n-1,0)\mathbf{P}(\eta_1 \le x_0). \qquad (6.10)$$

We will now prove by induction that for $x \in (x_0+k, x_0+k]$ the tail $\mathbf{P}_x(T^{(R)}_{x_0} > n)$ is constant and will simultaneously show that for $k \ge 2$ that

$$v(n,k) - v(n,k-1) = (v(n-1,k-1) - v(n-1,k-2))\mathbf{P}(\eta_1 \le x_0+k-1). \qquad (6.11)$$

First consider the base of induction $k = 2$. In this case, for $n > 2$ and for $x \in (x_0+2, x_0+3]$, we have

$$\begin{aligned}
&\mathbf{P}_x(T^{(R)}_{x_0} > n) \\
&= \mathbf{P}_{x-1}(T^{(R)}_{x_0} > n-1)\mathbf{P}(\eta_1 \le x-1) + \int_{x-1}^{\infty} \mathbf{P}(\eta_1 \in dy)\mathbf{P}_y(T^{(R)}_{x_0} > n-1) \\
&= \mathbf{P}_{x_0+2}(T^{(R)}_{x_0} > n-1) + \int_{x-1}^{x_0+2} \mathbf{P}(\eta_1 \in dy)\mathbf{P}_{x_0+2}(T^{(R)}_{x_0} > n-1) \\
&\qquad + \int_{x_0+2}^{\infty} \mathbf{P}(\eta_1 \in dy)\mathbf{P}_y(T^{(R)}_{x_0} > n-1) \\
&= v(n-1,1)\mathbf{P}(\eta_1 \le x_0+2) + \int_{x_0+2}^{\infty} \mathbf{P}(\eta_1 \in dy)\mathbf{P}_y(T^{(R)}_{x_0} > n-1).
\end{aligned}$$

This expression clearly does not depend on x. Thus,

$$v(n,2) = v(n-1,1)\mathbf{P}(\eta_1 \le x_0+2) + \int_{x_0+2}^{\infty} \mathbf{P}(\eta_1 \in dy)\mathbf{P}_y(T^{(R)}_{x_0} > n-1). \qquad (6.12)$$

It also follows from (6.9) that

$$\begin{aligned}
v(n,1) &= v(n-1,0)\mathbf{P}(\eta_1 \le x_0+1) + v(n-1,1)\mathbf{P}(\eta_1 \in (x_0+1, x_0+2]) \\
&\qquad + \int_{x_0+2}^{\infty} \mathbf{P}(\eta_1 \in dy)\mathbf{P}_y(T^{(R)}_{x_0} > n-1).
\end{aligned}$$

Subtracting this equation from (6.12) we obtain

$$v(n,2) - v(n,1) = (v(n-1,1) - v(n-1,0))\mathbf{P}(\eta_1 \le x_0+1).$$

This is exactly (6.11) with $k = 2$. Thus, the base case is true.

We will now prove the induction step. Consider $x \in (x_0 + k, x_0 + k + 1]$. For $n > k$ we obtain, using the induction hypothesis,

$$\begin{aligned}
&\mathbf{P}_x(T_{x_0}^{(R)} > n) \\
&= \mathbf{P}_{x-1}(T_{x_0}^{(R)} > n-1)\mathbf{P}(\eta_1 \le x-1) + \int_{x-1}^{\infty} \mathbf{P}(\eta_1 \in dy)\mathbf{P}_y(T_{x_0}^{(R)} > n-1) \\
&= \mathbf{P}_{x_0+k}(T_{x_0}^{(R)} > n-1) + \int_{x-1}^{x_0+k} \mathbf{P}(\eta_1 \in dy)\mathbf{P}_{x_0+k}(T_{x_0}^{(R)} > n-1) \\
&\qquad + \int_{x_0+k}^{\infty} \mathbf{P}(\eta_1 \in dy)\mathbf{P}_y(T_{x_0}^{(R)} > n-1) \\
&= v(n-1, k-1)\mathbf{P}(\eta_1 \le x_0 + k) + \int_{x_0+k}^{\infty} \mathbf{P}(\eta_1 \in dy)\mathbf{P}_y(T_{x_0}^{(R)} > n-1).
\end{aligned}$$

This expression clearly does not depend on x and hence (6.5) holds. Thus,

$$v(n, k) = v(n-1, k-1)\mathbf{P}(\eta_1 \le x_0 + k) + \int_{x_0+k}^{\infty} \mathbf{P}(\eta_1 \in dy)\mathbf{P}_y(T_{x_0}^{(R)} > n-1). \tag{6.13}$$

The same expression is true for $k - 1$ by the induction hypothesis. Hence,

$$\begin{aligned}
v(n, k-1) = v(n-1, k-2)&\mathbf{P}(\eta_1 \le x_0 + k - 1) \\
&+ v(n-1, k-1)\mathbf{P}(\eta_1 \in (x_0 + k - 1, x_0 + k]) \\
&+ \int_{x_0+k}^{\infty} \mathbf{P}(\eta_1 \in dy)\mathbf{P}_y(T_{x_0}^{(R)} > n-1).
\end{aligned}$$

Subtracting this expression from (6.13) we obtain (6.11).

Now it follows from (6.10) and (6.11) that

$$v(n, k) - v(n, k-1) = v(n-k, 0) \prod_{j=0}^{k-1} \mathbf{P}(\eta_1 \le x_0 + j). \tag{6.14}$$

for $n > k$. Then the standard telescoping argument gives (6.6). Plugging (6.14) into (6.8) we obtain

$$\begin{aligned}
v_n &= \int_{x_0+1}^{\infty} \mathbf{P}(\eta_1 \in dy)\mathbf{P}_y(T_{x_0}^{(R)} > n-1) + v_{n-1}\mathbf{P}(\eta_1 \in (x_0, x_0+1]) \\
&= \sum_{l=1}^{n-2} \mathbf{P}(\eta_1 \in (x_0 + l, x_0 + l + 1])v(n-1, l) + \mathbf{P}(\eta_1 > x_0 + n - 1) \\
&\qquad + v_{n-1}\mathbf{P}(\eta_1 \in (x_0, x_0+1]) \\
&= \sum_{l=1}^{n-2} \mathbf{P}(\eta_1 \in (x_0 + l, x_0 + l + 1]) \left(v_{n-1} + \sum_{m=1}^{l} v_{n-m} \prod_{j=0}^{m-1} \mathbf{P}(\eta_1 \le x_0 + j) \right) \\
&\qquad + \mathbf{P}(\eta_1 > x_0 + n - 1) + v_{n-1}\mathbf{P}(\eta_1 \in (x_0, x_0+1]).
\end{aligned}$$

Swapping the order of summation we obtain (6.7). □

6.3 Heavy tails

To analyse the heavy-tailed case we need first the following definition. We say that a non-negative sequence $(a_n)_{n\geq 0}$ is subexponential if

$$\lim_{n\to\infty}\frac{a_n}{a_{n-1}}=1,\quad a_\infty:=\sum_{n=0}^{\infty}a_n<\infty$$
$$\sum_{k=0}^{n}a_k a_{n-k}\sim 2a_\infty a_n,\quad n\to\infty.$$

We start by deriving an upper bound for v_n.

Lemma 6.2. *Assume that $F\in\mathcal{S}^*$, where $F(x)=\mathbf{P}(\eta_1\leq x)$. Then there exists a constant C such that*

$$v_n\leq C\mathbf{P}(\eta_1>n),\quad n\geq 0.$$

Proof. Note that it follows from (6.7) that $v_n\leq w_n$, where the sequence $\{w_n\}$ is given by $w_1=\mathbf{P}(\eta_1>x_0)$ and for $n\geq 2$,

$$\begin{aligned}w_n&=\mathbf{P}(\eta_1>x_0+n-1)+w_{n-1}\mathbf{P}(\eta_1>x_0)\\&\quad+\sum_{m=1}^{n-2}w_{n-m-1}\mathbf{P}(\eta_1>x_0+m)\prod_{j=0}^{m-1}\mathbf{P}(\eta_1\leq x_0+j).\end{aligned}$$

Set $d_0=\mathbf{P}(\eta_1>x_0)$ and

$$d_m:=\mathbf{P}(\eta_1>x_0+m)\prod_{j=0}^{m-1}\mathbf{P}(\eta_1\leq x_0+j),\quad m\geq 1.$$

Set also $c_n=\mathbf{P}(\eta_1>x_0+n-1)$, $n\geq 1$ Then we have $w_1=c_1$ and

$$w_n=c_n+\sum_{m=0}^{n-2}w_{n-m-1}d_m,\quad n\geq 2. \tag{6.15}$$

Using (6.15), we obtain the following equality for generating functions:

$$\begin{aligned}\sum_{n=1}^{\infty}w_n s^n&=\sum_{n=1}^{\infty}c_n s^n+\sum_{n=2}^{\infty}s^n\sum_{m=0}^{n-2}w_{n-m-1}d_m\\&=\sum_{n=1}^{\infty}c_n s^n+s\sum_{m=0}^{\infty}d_m s^m\sum_{n=m+2}^{\infty}w_{n-m-1}s^{n-m-1}\\&=\sum_{n=1}^{\infty}c_n s^n+s\sum_{m=0}^{\infty}d_m s^m\left(\sum_{n=1}^{\infty}w_n s^n\right).\end{aligned}$$

Set, for brevity,

$$\widehat{w}(s)=\sum_{n=1}^{\infty}w_n s^n,\ \widehat{c}(s)=\sum_{n=1}^{\infty}c_n s^n,\text{and }\widehat{d}(s)=\sum_{n=0}^{\infty}d_n s^n.$$

Then we have

$$\widehat{w}(s)=\widehat{c}(s)+s\widehat{d}(s)\widehat{w}(s).$$

Solving this equality we obtain

$$\widehat{w}(s)=\frac{\widehat{c}(s)}{1-s\widehat{d}(s)}.$$

Noting that

$$
\begin{aligned}
d_m &= (1 - \mathbf{P}(\eta_1 \le x_0 + m)) \prod_{j=0}^{m-1} \mathbf{P}(\eta_1 \le x_0 + j)) \\
&= \prod_{j=0}^{m-1} \mathbf{P}(\eta_1 \le x_0 + j)) - \prod_{j=0}^{m} \mathbf{P}(\eta_1 \le x_0 + j)),
\end{aligned}
$$

we get

$$
\sum_{m=0}^{\infty} d_m = 1 - \prod_{j=0}^{\infty} \mathbf{P}(\eta_1 \le x_0 + j)) < 1,
$$

where the last inequality follows from the assumption $\mathbf{E}\eta < \infty$. Also it is clear that

$$
\frac{d_{n+1}}{d_n} = \mathbf{P}(\eta_1 \le n + x_0) \to 1, \quad n \to \infty
$$

and

$$
d_n \sim \mathbf{P}(\eta_1 > n) \prod_{j=0}^{\infty} \mathbf{P}(\eta_1 \le x_0 + j).
$$

Since $F \in \mathcal{S}^*$ we can see that $(d_n)_{n \ge 0}$ is a subexponential sequence.

Then, it follows from the results in the theory of locally subexponential distributions (see Corollary 2 and Proposition 4 in [5]) that $\frac{1}{1-s\widehat{d}(s)}$ is a generating function of subexponential sequence behaving like $C_2\mathbf{P}(\eta_1 > n)$. The same statement holds for $\widehat{c}(s)$. Hence w_n is obtained as a convolution of two subexponential sequences asymptotically equivalent to $C_1\mathbf{P}(\eta_1 > n)$ and $C_2\mathbf{P}(\eta_1 > n)$ and therefore behaves as $C_3\mathbf{P}(\eta_1 > n)$ for some C_3. This implies the statement of the lemma. □

In the following lemma we complete the proof of Theorem 1.5.

Lemma 6.3. *Assume that $F \in \mathcal{S}^*$, where $F(x) = \mathbf{P}(\eta_1 \le x)$. Then, for any $x > x_0$,*

$$
\mathbf{P}_x(T_{x_0}^{(R)} > n) \sim u(x)\mathbf{P}(\eta_1 > n), \quad n \to \infty,
$$

where the function $u(x) = \mathbf{E}_x[T_{x_0}^{(R)}]$ has been computed in (6.4)*.*

Proof. First we derive a lower bound. For every $N \ge 1$ one has

$$
\begin{aligned}
\{T_{x_0}^{(R)} > n\} &\supseteq \bigcup_{k=1}^{N} \{T_{x_0}^{(R)} > n, \eta_k > x_0 + n\} \\
&= \bigcup_{k=1}^{N} \{T_{x_0}^{(R)} > k-1, \eta_k > x_0 + n\}.
\end{aligned}
$$

Therefore, by the inclusion-exclusion argument,

$$
\begin{aligned}
\mathbf{P}_x(T_{x_0}^{(R)} > n) \ge{}& \sum_{k=1}^{N} \mathbf{P}_x(T_{x_0}^{(R)} > k-1)\mathbf{P}(\eta_k > x_0 + n) \\
&- \sum_{k=1}^{N-1} \mathbf{P}_x(T_{x_0}^{(R)} > k-1)\mathbf{P}(\eta_k > x_0 + n) \sum_{j=k+1}^{N} \mathbf{P}(\eta_j > x_0 + n) \\
\ge{}& (1 - N\mathbf{P}(\eta_1 > x_0 + n))\mathbf{P}(\eta_1 > x_0 + n) \sum_{k=1}^{N} \mathbf{P}_x(T_{x_0}^{(R)} > k-1).
\end{aligned}
$$

This implies that

$$\liminf_{n\to\infty} \frac{\mathbf{P}_x(T_{x_0}^{(R)} > n)}{\mathbf{P}(\eta_1 > x_0 + n)} \ge \sum_{k=1}^{N} \mathbf{P}_x(T_{x_0}^{(R)} > k-1).$$

Letting N to infinity we obtain

$$\liminf_{n\to\infty} \frac{\mathbf{P}_x(T_{x_0}^{(R)} > n)}{\mathbf{P}(\eta_1 > x_0 + n)} \ge \sum_{k=1}^{\infty} \mathbf{P}_x(T_{x_0}^{(R)} > k-1) = \mathbf{E}_x[T_{x_0}^{(R)}] = u(x). \tag{6.16}$$

We next derive the corresponding asymptotic precise upper bound for v_n. Fix $\varepsilon > 0$. From Lemma 6.2 and from the subexponentiality of $\mathbf{P}(\eta_1 > x_0 + n)$ we conclude that there exists N such that

$$\sum_{m=N+1}^{n-N} v_{n-m-1}\mathbf{P}(\eta_1 \in (x_0+m, x_0+n-1]) \le C \sum_{m=N}^{n-N} \overline{F}(n-m)\overline{F}(m) \le \frac{\varepsilon}{2}\overline{F}(x_0+n)$$

for all $n \ge 2N$. Also, since $F \in \mathcal{S}^*$ for any fixed i,

$$\mathbf{P}(\eta_1 \in (x_0+n-i, x_0+n]) = o(\overline{F}(n))$$

and therefore for all $n \ge 2N$,

$$\sum_{m=n-N}^{n-2} v_{n-m-1}\mathbf{P}(\eta_1 \in (x_0+m, x_0+n-1]) \le \frac{\varepsilon}{2}\overline{F}(x_0+n).$$

Combining these estimates with the representation (6.7), we get

$$v_n \le (1+\varepsilon)c_n + \sum_{m=0}^{N} d_m v_{n-m-1}, \quad n \ge 2N,$$

where the sequences $\{c_n\}$ and d_n are defined in the proof of Lemma 6.2.

Set now $w_n^{(N)} = v_n$ for $n < 2N$ and

$$w_n^{(N)} = (1+\varepsilon)\mathbf{P}(\eta_1 > x_0 + n - 1) + \sum_{m=0}^{N} d_m w_{n-m-1}^{(N)}.$$

Clearly $v_n \le w_n^{(N)}$ for all n.

Set also $\widehat{w}^{(N)}(s) = \sum_{n=2N}^{\infty} w_n^{(N)} s^n$ and $\widehat{c}^{(N)}(s) = \sum_{n=2N}^{\infty} c_n s^n$. Then one has

$$\begin{aligned}
\widehat{w}^{(N)}(s) &= (1+\varepsilon)\widehat{c}^{(N)}(s) + s\sum_{m=0}^{N} d_m s^m \sum_{n=2N}^{\infty} w_{n-m-1}^{(N)} s^{n-m-1} \\
&= (1+\varepsilon)\widehat{c}^{(N)}(s) + s\sum_{m=0}^{N} d_m s^m \sum_{n=2N-m-1}^{2N-1} w_{n-m-1}^{(N)} s^{n-m-1} \\
&\quad + s\widehat{w}^{(N)}(s)\sum_{m=0}^{N} d_m s^m.
\end{aligned}$$

Therefore,

$$\widehat{w}^{(N)}(s) = \frac{(1+\varepsilon)\widehat{c}^{(N)}(s) + s\sum_{m=0}^{N} d_m s^m \sum_{n=2N-m-1}^{2N-1} w_{n-m-1}^{(N)} s^{n-m-1}}{1 - s\sum_{m=0}^{N} d_m s^m}.$$

This implies that

$$w_n^{(N)} \sim \left(\frac{1+\varepsilon}{1-\sum_{m=0}^{N} d_m} \right) c_n.$$

Consequently,

$$\limsup_{n\to\infty} \frac{v_n}{c_n} \le \frac{1+\varepsilon}{1-\sum_{m=0}^{N} d_m} \le \frac{1+\varepsilon}{1-\sum_{m=0}^{\infty} d_m} = (1+\varepsilon)u(x_0+1).$$

Letting $\varepsilon \to 0$, we get

$$\limsup_{n\to\infty} \frac{v_n}{c_n} \le u(x_0+1).$$

Combining this with the lower bound (6.16), we conclude that

$$v_n \sim u(x_0+1)\mathbf{P}(\eta_1 > n), \quad n \to \infty. \tag{6.17}$$

Assume now that $x \in (x_0+k, x_0+k+1]$ for some $k \ge 1$. Then, by Proposition 6.1,

$$\mathbf{P}(T_{x_0}^{(R)} > n) = v_n + \sum_{m=1}^{k} v_{n-m} \prod_{j=0}^{m-1} \mathbf{P}(\eta_1 \le x_0 + j).$$

Using now (6.17), we conclude that

$$\mathbf{P}(T_{x_0}^{(R)} > n) \sim u(x_0+1) \left(1 + \sum_{m=1}^{k} \prod_{j=0}^{m-1} \mathbf{P}(\eta_1 \le x_0 + j) \right) \mathbf{P}(\eta > n).$$

Noting that $u(x_0+1) \left(1 + \sum_{m=1}^{k} \prod_{j=0}^{m-1} \mathbf{P}(\eta_1 \le x_0 + j) \right) = u(x)$ we complete the proof. □

7 Proof of Theorem 1.6

The lower bound for the tail of $T_{x_0}^{(X)}$ can be obtained by exactly the same arguments as the lower bound for $T_{x_0}^{(R)}$ in Lemma 6.3.

We turn to the corresponding upper bound. Set $c = \frac{1}{2\sum_{j=1}^{\infty} j^{-2}}$. For every $y \ge x_0$ we define the events

$$\left\{ \xi_k \le \frac{A^{n-k}}{(n-k+1)^2} cy \right\}, \quad k \le n.$$

On the intersection of these sets one has

$$\begin{aligned} X_n &= a^n X_0 + \sum_{k=1} a^{n-k} \xi_k \\ &\le a^n X_0 + \sum_{k=1}^{n} \frac{cy}{(n-k+1)^2} \le a^n X_0 + y/2. \end{aligned}$$

If n is sufficiently large, say $n \ge n_0 = n_0(X_0)$ then we infer that $X_n \le y$. Therefore,

$$\mathbf{P}_x(X_n > y, T_{x_0}^{(X)} > n) \le \sum_{k=1}^{n} \mathbf{P}_x(T_{x_0}^{(X)} > k-1) c_{n-k}(y), \quad n \ge n_0, \tag{7.1}$$

where

$$c_j(y) := \mathbf{P}\left(\xi_1 > \frac{A^j}{(j+1)^2} cy \right), \quad j \ge 0.$$

We first use this estimate with $y = x_0$. In this case we have

$$\mathbf{P}_x(T_{x_0}^{(X)} > n) \le \sum_{k=1}^{n} \mathbf{P}_x(T_{x_0}^{(X)} > k-1) c_{n-k}(x_0), \quad n \ge n_0.$$

Consider the sequence $\{w_n\}$ which is defined via the recursion

$$w_n = \sum_{k=1}^{n} w_{k-1} c_{n-k}(x_0)$$

with initial condition $w_0 = w_1 = \ldots = w_{n_0-1} = 1$. Then clearly

$$\mathbf{P}_x(T_{x_0}^{(X)} > n) \le w_n, \quad n \ge 0. \tag{7.2}$$

It is immediate from the definition of $\{w_n\}$ that

$$\begin{aligned}\sum_{n=n_0}^{\infty} w_n s^n &= \sum_{n=n_0}^{\infty} s^n \sum_{k=1}^{n} w_{k-1} c_{n-k}(x_0) \\ &= \sum_{n=n_0}^{\infty} s^n \sum_{k=1}^{n_0-1} w_{k-1} c_{n-k}(x_0) + \sum_{n=n_0}^{\infty} s^n \sum_{k=n_0}^{n} w_{k-1} c_{n-k}(x_0)\end{aligned}$$

Setting

$$d_n(x_0) := \sum_{k=1}^{n_0-1} w_{k-1} c_{n-k}(x_0)$$

and interchanging the order of summation in the second series, we conclude that

$$\sum_{n=n_0}^{\infty} w_n s^n = \frac{\sum_{n=n_0}^{\infty} d_n(x_0) s^n}{1 - s\sum_{j=0}^{\infty} c_j(x_0) s^j}.$$

Using once again the results from [5], we infer that

$$w_n \sim C\mathbf{P}(\eta_1 > n)$$

provided that $\sum_{j=0}^{\infty} c_j(x_0) < 1$. Combining this with (7.2), we obtain

$$\mathbf{P}_x(T_{x_0}^{(X)} > n) \le C\mathbf{P}(\eta_1 > n), \quad n \ge 0. \tag{7.3}$$

Using Lemma 4.4, we conclude that (7.3) is valid for all x_0 such that $\mathbf{P}(ax_0 + \xi_1 < x_0)$ is strictly positive.

Combining now (7.1), (7.3) and recalling that the sequences $\mathbf{P}(\eta_1 > n)$ and $c_n(y)$ are subexponential, we conclude that

$$\limsup_{n\to\infty} \frac{\mathbf{P}_x(X_n > y, T_{x_0}^{(X)} > n)}{\mathbf{P}(\eta_1 > n)} \le \mathbf{E}_x[T_{x_0}^{(X)}] + C(y), \tag{7.4}$$

where

$$C(y) := \sum_{k=0}^{\infty} c_k(y).$$

This quantity is finite due to the assumption $\mathbf{E}\eta_1 < \infty$. Furthermore, $C(y) \to 0$ as $y \to \infty$.

Fix now a integer-valued sequence $N_n \to \infty$ such that $\mathbf{P}(\eta_1 > n) \sim \mathbf{P}(\eta_1 > n - N_n)$. By the monotonicity of the chain $\{X_n\}$,

$$\begin{aligned}&\mathbf{P}_x(T_{x_0}^{(X)} > n) \\ &= \mathbf{P}_x(X_{n-N_n} > y, T_{x_0}^{(X)} > n) + \mathbf{P}_x(X_{n-N_n} \le y, T_{x_0}^{(X)} > n) \\ &\le \mathbf{P}_x(X_{n-N_n} > y, T_{x_0}^{(X)} > n - N_n) + \mathbf{P}_x(T_{x_0}^{(X)} > n - N_n)\mathbf{P}_y(T_{x_0}^{(X)} > N_n).\end{aligned}$$

Applying (7.3) and (7.4), we get

$$\begin{aligned}\limsup_{n\to\infty}\frac{\mathbf{P}_x(T_{x_0}^{(X)}>n)}{\mathbf{P}(\eta_1>n)} &\le \mathbf{E}_x[T_{x_0}^{(X)}]+C(y)+C\lim_{n\to\infty}\mathbf{P}_y(T_{x_0}^{(X)}>N_n)\\ &=\mathbf{E}_x[T_{x_0}^{(X)}]+C(y).\end{aligned}$$

Letting now $y\to\infty$ and recalling that $\lim_{y\to\infty}C(y)=0$, we finally obtain

$$\limsup_{n\to\infty}\frac{\mathbf{P}_x(T_{x_0}^{(X)}>n)}{\mathbf{P}(\eta_1>n)}\le \mathbf{E}_x[T_{x_0}^{(X)}].$$

Thus, the proof is complete.

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