An adaptive C^0 Interior Penalty Discontinuous Galerkin method and an equilibrated a posteriori error estimator for the von Kármán equations

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1. Introduction

The von Kármán equations are a coupled system of two nonlinear fourth order elliptic equations describing the bending of thin elastic plates. They have been studied analytically by Berger [2], Berger and Fife [3,4], and Knightly [24] (cf. also the monograph [16] and the references therein). For their numerical solution both conforming and nonconforming as well as mixed finite elements have been proposed (cf., e.g., [8,11,12,25,27–29]). In particular, Discontinuous Galerkin (DG) methods with emphasis on C^0 Interior Penalty Discontinuous Galerkin (C^0IPDG) methods have been developed and analyzed both for the von Kármán equations themselves as well as for associated optimal control problems (see [7,10,13,15,26]). Moreover, adaptive mesh refinement based on residual-type a posteriori error estimators has been studied in [10,11,13,14].

In this paper, we consider an adaptive C^0 Interior Penalty Discontinuous Galerkin (C^0IPDG) approximation of the fourth order von Kármán equations with homogeneous Dirichlet boundary conditions and an equilibrated a posteriori error estimator. The C^0IPDG method can be derived from a six-field formulation of the finite element discretized von Kármán equations and will be shown to admit a unique solution for triangulations of sufficiently small mesh size. The novel contribution of this paper is an equilibrated a posteriori error estimator. It consists of easily computable local residual-type contributions. It can be derived from a more general result [30] on convex minimization problems and provides an upper bound for the discretization error in the broken $W_0^{2,2}$ norm in terms of the associated primal and dual energy functionals. Moreover, we

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study its relationship with a residual-type a posteriori error estimator. Numerical results illustrate the performance of the suggested approach.

The paper is organized as follows: In section 2, we will introduce the fourth order von Kármán equations with homogeneous Dirichlet boundary conditions and show the existence of a regular weak solution (Theorem 2.2). Section 3 is devoted to the derivation of the C^0IPDG approximation by means of a six-field formulation of the finite element discretized von Kármán equations. We will establish the existence and uniqueness of a solution for triangulations of sufficiently small mesh size (Theorem 3.2). In section 4, we will derive an equilibrated a posteriori error estimator providing an upper bound for the discretization error in the broken $W_0^{2,2}$ norm in terms of the associated primal and dual energy functionals (Lemma 4.2). The construction of such an equilibrated a posteriori error estimator is dealt with in section 5. In particular, it requires the computation of equilibrated fluxes and equilibrated moment tensors on local patches around interior nodal points of the triangulations. Section 6 is devoted to the relationship with a residual-type a posteriori error estimator (Theorem 6.1). Finally, in section 7 we provide a documentation of numerical results illustrating the performance of the suggested approach.

2. The von Kármán equations

We use standard notation from Lebesgue and Sobolev space theory (cf., e.g., [33]). In particular, for a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d, d \in \mathbb{N}$, with boundary $\Gamma = \partial\Omega$ we refer to $L^p(\Omega;\mathbb{R}^d)$ and $L^p(\Omega;\mathbb{R}^{d\times d})$, $1 < p < \infty$, as the Banach spaces of p-th power Lebesgue integrable functions and tensors on Ω with norms $\|\cdot\|_{L^p(\Omega;\mathbb{R}^d)}$ and $\|\cdot\|_{L^p(\Omega;\mathbb{R}^{d\times d})}$. In case $d = 1$ we will write $L^p(\Omega)$ instead of $L^p(\Omega;\mathbb{R})$. Matrix-valued functions in $L^p(\Omega;\mathbb{R}^{d\times d})$ will be denoted by $\underline{\underline{\mathbf{p}}} = (p_{ij})_{i,j=1}^d$ and for $\underline{\underline{\mathbf{p}}} \in L^p(\Omega;\mathbb{R}^{d\times d})$, $\underline{\underline{\mathbf{q}}} \in L^q(\Omega;\mathbb{R}^{d\times d})$, $1/p + 1/q = 1$, we use the notation $\underline{\underline{\mathbf{p}}} : \underline{\underline{\mathbf{q}}}$ for $\underline{\underline{\mathbf{p}}} : \underline{\underline{\mathbf{q}}} := \sum_{i,j=1}^d p_{ij}q_{ij}$. Further, for $u \in W^{2,p}(\Omega)$, we refer to $D^2u := (\partial^2 u/\partial x_i \partial x_j)_{i,j=1}^2$ as the matrix of second partial derivatives.
We denote by $W^{s,2}(\Omega), s \in \mathbb{R}_+$, the Sobolev spaces with norms $\|\cdot\|_{W^{s,2}(\Omega)}$ and by $W_0^{s,2}(\Omega)$ the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{s,2}(\Omega)}$. Functions $u \in W^{2,2}(\Omega)$ have a trace $u|_\Gamma$ on the boundary $\Gamma = \partial\Omega$ with $u|_\Gamma \in W^{3/2,2}(\Gamma)$. Further, we define $\underline{\mathbf{H}}(\text{div},\Omega)$ and $\underline{\underline{\mathbf{H}}}(\text{div}^2,\Omega)$, as the Banach spaces

$$\underline{\mathbf{H}}(\text{div},\Omega) = \{\underline{\boldsymbol{\tau}} \in L^2(\Omega;\mathbb{R}^d) \mid \nabla\cdot\underline{\boldsymbol{\tau}} \in L^2(\Omega)\},$$
$$\underline{\underline{\mathbf{H}}}(\text{div}^2,\Omega) = \{\underline{\underline{\boldsymbol{\tau}}} \in L^2(\Omega;\mathbb{R}^{d\times d}) \mid \nabla\cdot\underline{\underline{\boldsymbol{\tau}}} \in \underline{\mathbf{H}}(\text{div},\Omega)\}$$

with the graph norms

$$\|\underline{\boldsymbol{\tau}}\|_{\underline{\mathbf{H}}(\text{div},\Omega)} := \left(\|\underline{\boldsymbol{\tau}}\|^2_{L^2(\Omega;\mathbb{R}^d)} + \|\nabla\cdot\underline{\boldsymbol{\tau}}\|^2_{L^2(\Omega)}\right)^{1/2},$$
$$\|\underline{\underline{\boldsymbol{\tau}}}\|_{\underline{\underline{\mathbf{H}}}(\text{div}^2,\Omega)} := \left(\|\underline{\underline{\boldsymbol{\tau}}}\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})} + \|\nabla\cdot\underline{\underline{\boldsymbol{\tau}}}\|^2_{L^2(\Omega;\mathbb{R}^d)} + \|\nabla\cdot\nabla\cdot\underline{\underline{\boldsymbol{\tau}}}\|^2_{L^2(\Omega)}\right)^{1/2}.$$

For later use we recall Young's inequality

$$\prod_{i=1}^2 a_i \le \frac{\varepsilon}{p} a_1^p + \frac{\varepsilon^{-q/p}}{q} a_2^q \tag{2.1}$$

for $a_i > 0, 1 \le i \le 2$, and $1 < p, q < \infty, 1/p + 1/q = 1$, and any $\varepsilon > 0$, as well as the following inequality:
Let $w_i \in \mathbb{R}, 1 \le i \le 2$, and $0 \le r < \infty$. Then it holds (cf. [32], page 136)

$$(|w_1| + |w_2|)^r \le 2^r \Big(|w_1|^r + |w_2|^r\Big). \tag{2.2}$$

Given a bounded polygonal domain $\Omega \subset \mathbb{R}^2$ with boundary $\Gamma = \partial\Omega$ and exterior unit normal vector $\mathbf{n}_\Gamma$ as well as $f \in L^2(\Omega)$, the von Kármán equations with homogeneous Dirichlet boundary conditions are given by a coupled system of fourth order elliptic equations

$$\Delta^2 u_1 - [u_1, u_2] = f \quad \text{in } \Omega, \tag{2.3a}$$
$$\Delta^2 u_2 + \frac{1}{2}[u_1, u_1] = 0 \quad \text{in } \Omega, \tag{2.3b}$$
$$u_i = \mathbf{n}_\Gamma \cdot \nabla u_i = 0 \quad \text{on } \Gamma, 1 \le i \le 2, \tag{2.3c}$$

where u_1 and u_2 are the displacements of the plate and $[u, v]$ stands for the von Kármán bracket

$$[u, v] := \text{cof}(D^2u) : D^2 v \tag{2.4}$$

with the cofactor matrix $\text{cof}(D^2u)$ of D^2u.

The weak formulation of (2.3) requires the computation of $u_i \in V := W_0^{2,2}(\Omega), 1 \le i \le 2$, such that for all $v \in V$ the following system of variational equations is satisfied

$$\int_\Omega D^2 u_1 : D^2 v \, dx - \int_\Omega [u_1, u_2] v \, dx = \int_\Omega f v \, dx, \tag{2.5a}$$

$$\int_\Omega D^2 u_2 : D^2 v \, dx + \frac{1}{2} \int_\Omega [u_1, u_1] v \, dx = 0, \tag{2.5b}$$

which constitute the necessary and sufficient optimality condition for the minimization of the primal energy functional

$$J_P(u_1, u_2) = \frac{1}{2} \sum_{i=1}^{2} \int_\Omega |D^2 u_i|^2 \, dx - \frac{1}{2} \int_\Omega [u_1, u_2] u_1 \, dx + \frac{1}{2} \int_\Omega [u_1, u_2] u_2 \, dx - \int_\Omega f u_1 \, dx. \tag{2.6}$$

We introduce a bilinear form $A : \underline{\mathbf{V}} \times \underline{\mathbf{V}} \to \mathbb{R}, \underline{\mathbf{V}} := W_0^{2,2}(\Omega) \times W_0^{2,2}(\Omega)$, and a semilinear form $B : \underline{\mathbf{V}} \times \underline{\mathbf{V}} \times \underline{\mathbf{V}} \to \mathbb{R}$ according to

$$A(\mathbf{u}, \mathbf{v}) := a(u_1, v_1) + a(u_2, v_2), \tag{2.7a}$$

$$B(\mathbf{u}, \mathbf{v}, \mathbf{w}) := b(u_1, v_2, w_1) + b(u_2, v_1, w_1) - b(u_1, v_1, w_2), \tag{2.7b}$$

where $\mathbf{u} = (u_1, u_2)^T, \mathbf{v} = (v_1, v_2)^T, \mathbf{w} = (w_1, w_2)^T$, and the forms $a(\cdot, \cdot) : V \times V \to \mathbb{R}$ and $b(\cdot, \cdot, \cdot) : V \times V \times V \to \mathbb{R}$ are given by

$$a(u, v) := \int_\Omega D^2 u : D^2 v \, dx, \tag{2.8a}$$

$$b(u, v, w) := \frac{1}{2} \int_\Omega (\mathrm{cof}(D^2 u) \nabla v) \cdot \nabla w \, dx, \quad u, v, w \in V. \tag{2.8b}$$

Within this setting the weak formulation amounts to the computation of $\mathbf{u} \in \underline{\mathbf{V}}$ such that for all $\mathbf{v} \in \underline{\mathbf{V}}$ it holds

$$A(\mathbf{u}, \mathbf{v}) + B(\mathbf{u}, \mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})_{L^2(\Omega)}, \quad \mathbf{f} := (f, 0)^T. \tag{2.9}$$

We define operators $\mathcal{A} : \underline{\mathbf{V}} \to \underline{\mathbf{V}}^*$ and $\mathcal{B} : \underline{\mathbf{V}} \to \underline{\mathbf{V}}^*$ by means of

$$< \mathcal{A}\mathbf{u}, \mathbf{v} >_{\underline{\mathbf{V}}^*, \underline{\mathbf{V}}} := A(\mathbf{u}, \mathbf{v}), \tag{2.10a}$$

$$< \mathcal{B}\mathbf{u}, \mathbf{v} >_{\underline{\mathbf{V}}^*, \underline{\mathbf{V}}} := B(\mathbf{u}, \mathbf{u}, \mathbf{v}), \tag{2.10b}$$

where $< \cdot, \cdot >_{\underline{\mathbf{V}}^*, \underline{\mathbf{V}}}$ is the dual pairing between $\underline{\mathbf{V}}^*$ and $\underline{\mathbf{V}}$. Then the operator form of (2.9) reads

$$\mathcal{A}\mathbf{u} + \mathcal{B}\mathbf{u} = \mathbf{f}. \tag{2.11}$$

The operator $\mathcal{B}$ is Fréchet differentiable at $\mathbf{u}$ in the direction of $\mathbf{v}$ with the Fréchet derivative given by

$$< \mathcal{B}'(\mathbf{u})\mathbf{v}, \mathbf{w} >_{\underline{\mathbf{V}}^*, \underline{\mathbf{V}}} := 2\mathcal{B}(\mathbf{u}, \mathbf{v}, \mathbf{w}), \quad \mathbf{u}, \mathbf{v}, \mathbf{w} \in \underline{\mathbf{V}}. \tag{2.12}$$

The existence of a weak solution has been shown in [16,24].

Theorem 2.1. *For $f \in L^2(\Omega)$ the von Kármán equations admit a weak solution $\mathbf{u} \in \underline{\mathbf{V}}$.*

A weak solution $\mathbf{u} \in \underline{\mathbf{V}}$ is said to be a regular solution, if the linearized operator $\mathcal{L} : \underline{\mathbf{V}} \to \underline{\mathbf{V}}^*$ given by

$$\mathcal{L}\mathbf{v} := \mathcal{A}\mathbf{v} + \mathcal{B}'(\mathbf{u})\mathbf{v} \tag{2.13}$$

is nonsingular. The following result has been shown in [26].

Theorem 2.2. *If $\mathbf{u}^* \in \underline{\mathbf{V}}$ is a regular weak solution of the von Kármán equations, then there exists an open ball $B(\mathbf{u}^*) \subset L^2(\Omega)$ such that $\mathcal{A} + \mathcal{B}'(\mathbf{u}^*)$ is an isomorphism from $\underline{\mathbf{V}}$ into $\underline{\mathbf{V}}^*$ for all $\mathbf{u} \in B(\mathbf{u}^*)$. In particular, there exist constants $C_i > 0, 1 \le i \le 2$, such that*

$$\|\mathcal{A} + \mathcal{B}'(\mathbf{u}^*)\|_{\mathbb{L}(\underline{\mathbf{V}}, \underline{\mathbf{V}}^*)} \le C_1, \quad \|(\mathcal{A} + \mathcal{B}'(\mathbf{u}^*))^{-1}\|_{\mathbb{L}(\underline{\mathbf{V}}^*, \underline{\mathbf{V}})} \le C_2, \tag{2.14}$$

where $\mathbb{L}(\underline{\mathbf{V}}, \underline{\mathbf{V}}^*)$ stands for the space of bounded linear mappings from $\underline{\mathbf{V}}$ into $\underline{\mathbf{V}}^*$.

3. C^{0}IPDG approximation of the von Kármán equations

Let $\mathcal{T}_h$ be a geometrically conforming, locally quasi-uniform, simplicial triangulation of the computational domain Ω. Given $D \subset \bar{\Omega}$, we denote by $\mathcal{N}_h(D)$ and $\mathcal{E}_h(D)$ the set of vertices and edges of $\mathcal{T}_h$ in D, and we refer to $P_k(D)$, $k \in \mathbb{N}$, as the set of polynomials of degree $\le k$ on D. Moreover, h_K, $K \in \mathcal{T}_h$, and h_E, $E \in \mathcal{E}_h$, stand for the diameter of K and the length of E, respectively. We define $h := \min\,\{h_K \mid K \in \mathcal{T}_h\}$. Given any $0 < \delta < 1$, we denote by $\mathbb{T}(\delta)$ the set of all triangulations $\mathcal{T}_h$ with mesh size $h_T \le \delta$ for all $T \in \mathcal{T}_h \in \mathbb{T}(\delta)$. For two quantities $a, b \in \mathbb{R}$ we will write $a \lesssim b$, if there exists a constant $C > 0$, independent of h, such that $a \le Cb$.
Due to the local quasi-uniformity of the triangulation there exist constants $0 < c_Q \le C_Q$, $0 < c_R \le C_R$, $0 < c_S \le C_S$, such that for all $K \in \mathcal{T}_h$ it holds

$$c_Q\, h_K \le h \le C_Q\, h_K, \tag{3.1a}$$

$$c_R h_K \le h_E \le C_R h_K, \quad E \in \mathcal{E}_h(\partial K), \tag{3.1b}$$

$$c_S |K| \le h_K^2 \le C_S |K|. \tag{3.1c}$$

We will use the following inverse inequality (cf., e.g., Theorem. 3.2.6 in [17]): For $1 \le p \le \infty$ there exists a constant $C_{inv} > 0$, only depending on p, the polynomial degree k, and the local geometry of the triangulation, such that for $v_h \in P_k(K)$ it holds

$$\|\nabla v_h\|_{L^p(K;\mathbb{R}^2)} \le C_{inv} h_K^{-1} \|v_h\|_{L^p(K)}. \tag{3.2}$$

We will also use the following trace inequality (cf., e.g., [19]): For $1 \le p \le \infty$ there exists a constant $C_{tr} > 0$, only depending on p, the polynomial degree k, and the local geometry of the triangulation, such that for $v_h \in P_k(K)$ and $K \in \mathcal{T}_h$ it holds

$$\|v_h\|_{L^p(\partial K)} \le C_{tr} h_K^{-1/p} \|v_h\|_{L^p(K)}. \tag{3.3}$$

For $E \in \mathcal{E}_h(\Omega)$, $E = K_+ \cap K_-$, $K_\pm \in \mathcal{T}_h(\Omega)$, and $v_h \in V_h$, we denote the average and jump of v_h across E by $\{v_h\}_E$ and $[v_h]_E$, i.e.,

$$\{v_h\}_E := \frac{1}{2}\Big(v_h|_{E\cap K_+} + v_h|_{E\cap K_-}\Big), \quad [v_h]_E := v_h|_{E\cap K_+} - v_h|_{E\cap K_-},$$

whereas for $E \in \mathcal{E}_h(\Gamma)$ we set

$$\{v_h\}_E := v_h|_E, \quad [v_h]_E := v_h|_E.$$

The averages $\{\nabla v_h\}_E, \{\underline{\boldsymbol{\tau}}_h\}_E$ and jumps $[\nabla v_h]_E, [\underline{\boldsymbol{\tau}}_h]_E$ of vector-valued functions ∇v_h and $\underline{\boldsymbol{\tau}}_h$ as well as the averages $\{D^2 v_h\}_E, \{\underline{\underline{\boldsymbol{\tau}}}_h\}_E$ and jumps $[D^2 v_h]_E, [\underline{\underline{\boldsymbol{\tau}}}_h]_E$ of matrix-valued functions $D^2 v_h$ and $\underline{\underline{\boldsymbol{\tau}}}_h$ are defined analogously. For $E \in \mathcal{E}_h(\Omega)$ it holds

$$\int_E [u_h\; v_h]_E\, ds = \int_E \Big(\{u_h\}_E\, [v_h]_E + [u_h]_E\, \{v_h\}_E\Big)\, ds. \tag{3.4}$$

We further denote by $\mathbf{n}_E$, $E \in \mathcal{E}_h(\Omega)$, with $E = K_+ \cap K_-$ the unit normal on E pointing from K_+ to K_- and by $\mathbf{n}_E$, $E \in \mathcal{E}_h(\Gamma)$, the exterior unit normal on E.

We define the broken $W^{2,2}$-space $W^{2,2}(\Omega; \mathcal{T}_h)$ by

$$W^{2,2}(\Omega; \mathcal{T}_h) := \{v_h \in L^2(\Omega) \mid v_h|_K \in W^{2,2}(K), K \in \mathcal{T}_h\}, \tag{3.5}$$

where $W^{2,2}(K)$ is the Sobolev space $W^{2,2}$ for the simplex K with norm $\|\cdot\|_{W^{2,2}(K)}$. $W^{2,2}(\Omega; \mathcal{T}_h)$ is equipped with the norm

$$\|v_h\|_{W^{2,2}(\Omega;\mathcal{T}_h)} := \Big(\sum_{K\in\mathcal{T}_h} \|v_h\|^2_{W^{2,2}(K)}\Big)^{1/2}, \tag{3.6}$$

and the broken spaces $\underline{\mathbf{H}}(\mathrm{div}, \Omega; \mathcal{T}_h)$ and $\underline{\underline{\mathbf{H}}}(\mathrm{div}^2, \Omega; \mathcal{T}_h)$ by

$$\underline{\mathbf{H}}(\mathrm{div}, \Omega; \mathcal{T}_h) := \{\underline{\mathbf{q}}_h \in L^2(\Omega; \mathbb{R}^2) \mid \underline{\mathbf{q}}_h|_K \in \underline{\mathbf{H}}(\mathrm{div}; K), K \in \mathcal{T}_h\}, \tag{3.7a}$$

$$\underline{\underline{\mathbf{H}}}(\mathrm{div}^2, \Omega; \mathcal{T}_h) := \{\underline{\underline{\mathbf{q}}}_h \in L^2(\Omega; \mathbb{R}^{2\times 2}) \mid \underline{\underline{\mathbf{q}}}_h|_K \in \underline{\underline{\mathbf{H}}}(\mathrm{div}^2; K), K \in \mathcal{T}_h\}, \tag{3.7b}$$

equipped with the norms

$$\|\underline{\mathbf{q}}_h\|_{\underline{\underline{\mathbf{H}}}(\mathrm{div},\Omega;\mathcal{T}_h)} := \Big(\sum_{K\in\mathcal{T}_h} \|\underline{\mathbf{q}}_h\|^2_{\underline{\mathbf{H}}(\mathrm{div};K)}\Big)^{1/2}, \tag{3.8a}$$

$$\|\underline{\underline{\mathbf{q}}}_h\|_{\underline{\underline{\mathbf{H}}}(\mathrm{div}^2,\Omega;\mathcal{T}_h)} := \Big(\sum_{K\in\mathcal{T}_h} \|\underline{\underline{\mathbf{q}}}_h\|^2_{\underline{\underline{\mathbf{H}}}(\mathrm{div}^2;K)}\Big)^{1/2}. \tag{3.8b}$$

For $v \in W^{2,2}(\Omega;\mathcal{T}_h)$ we redefine the primal energy functional (2.6) according to

$$J_P(\mathbf{v}) := \frac{1}{2}\sum_{i=1}^{2}\sum_{K\in\mathcal{T}_h}\int_K |D^2 v_i|^2\,dx + \frac{1}{2}\sum_{K\in\mathcal{T}_h}\int_K \mathrm{cof}(D^2 v_1) : D^2 v_2\, v_1\,dx - \tag{3.9}$$
$$\frac{1}{2}\sum_{K\in\mathcal{T}_h}\int_K \mathrm{cof}(D^2 v_1) : D^2 v_1\, v_2\,dx - \sum_{K\in\mathcal{T}_h}\int_K f\,v_1\,dx,$$

and note that it reduces to (2.6) for $\mathbf{v} \in W^{2,2}(\Omega)^2$.
We consider the finite element approximation with the DG spaces

$$V_h := \{v_h : \bar{\Omega} \to \mathbb{R} \mid v_h|_K \in P_k(K),\ K \in \mathcal{T}_h\}, \tag{3.10a}$$

$$\underline{\mathbf{V}}_h := \{\underline{\mathbf{q}}_h : \bar{\Omega} \to \mathbb{R}^2 \mid \underline{\mathbf{q}}_h|_K \in P_{k-1}(K)^2,\ K \in \mathcal{T}_h\}, \tag{3.10b}$$

$$\underline{\underline{\mathbf{V}}}_h := \{\underline{\underline{\mathbf{q}}}_h : \bar{\Omega} \to \mathbb{R}^{2\times 2}) \mid \underline{\underline{\mathbf{q}}}_h|_K \in P_k(K)^{2\times 2},\ K \in \mathcal{T}_h\}. \tag{3.10c}$$

We note that for $k \geq 2$ we have $V_h \subset W^{2,2}(\Omega;\mathcal{T}_h)$. Moreover, for $\underline{\underline{\mathbf{q}}}_h \in \underline{\underline{\mathbf{V}}}_h$, we have $\nabla\cdot\underline{\underline{\mathbf{q}}}_h|_K \in P_{k-1}(K)^2$ and $\nabla\cdot\nabla\cdot\underline{\underline{\mathbf{q}}}_h|_K \in P_{k-2}(K), K \in \mathcal{T}_h$.

We denote by Π_k the orthogonal L^2 projection of $L^2(\Omega)$ onto V_h, which can be defined elementwise by

$$\int_\Omega \Pi_k(v)v_h\,dx = \sum_{K\in\mathcal{T}_h}\int_K \Pi_{K,k}(v)v_h\,dx,\ \ v \in L^2(\Omega), \tag{3.11}$$
$$\int_K \Pi_{K,k}(v)p_k\,dx = \int_K v p_k\,dx,\quad p_k \in P_k(K), K \in \mathcal{T}_h.$$

We note that Π_k can be extended to $L^p(\Omega)$ for $p \in [1,2)$ and $p \in [2,\infty]$ (cf., e.g., [18]).
We further denote by $\underline{\underline{\mathbf{\Pi}}}_k$ and $\underline{\mathbf{\Pi}}_k$ the L^2 projections of $L^2(\Omega;\mathbb{R}^{2\times 2})$ onto $\underline{\underline{\mathbf{V}}}_h$ and of $L^2(\Omega;\mathbb{R}^2)$ onto $\underline{\mathbf{V}}_h$ which can also be defined elementwise similar to (3.11) involving $\underline{\underline{\mathbf{\Pi}}}_{K,k}$ and $\underline{\mathbf{\Pi}}_{K,k}$, $K \in \mathcal{T}_h$. The L^2 projections of $L^2(\Gamma)$ onto $\{v_h \in L^2(\Gamma) \mid v_h|_E \in P_k(E), E \in \mathcal{E}_h(\Gamma)\}$ and of $L^2(\Gamma;\mathbb{R}^{2\times 2})$ onto $\{\underline{\underline{\mathbf{q}}}_h \in L^2(\Gamma;\mathbb{R}^{2\times 2}) \mid \underline{\underline{\mathbf{q}}}_h|_E \in P_k(E)^{2\times 2}, E \in \mathcal{E}_h(\Gamma)\}$, will be denoted by $\Pi_{\Gamma,k}$ and $\underline{\underline{\mathbf{\Pi}}}_{\Gamma,k}$, respectively.

For $u_h \in V_h$ we define the broken gradient $\nabla_h u_h$ and the broken Hessian $D_h^2 u_h$ by means of

$$\nabla_h u_h|_K := \nabla u_h|_K,\quad K \in \mathcal{T}_h, \tag{3.12a}$$

$$D_h^2 u_h|_K := D^2 u_h|_K,\quad K \in \mathcal{T}_h. \tag{3.12b}$$

We consider a six-field formulation which will lead us to the C^0 IPDG approximation. To this end, we set

$$\underline{\mathbf{p}}_{h,i} = \nabla_h u_{h,i},\ \ 1 \leq i \leq 2, \tag{3.13a}$$

$$\underline{\underline{\mathbf{p}}}_{h,i} = D_h^2 u_{h,i},\ \ 1 \leq i \leq 2, \tag{3.13b}$$

$$\nabla\cdot\nabla\cdot\underline{\underline{\mathbf{p}}}_{h,1} - \mathrm{cof}(\underline{\underline{\mathbf{p}}}_{h,1}) : \underline{\underline{\mathbf{p}}}_{h,2} = f_h \quad \text{in each } K \in \mathcal{T}_h, \tag{3.13c}$$

$$\nabla\cdot\nabla\cdot\underline{\underline{\mathbf{p}}}_{h,2} + \frac{1}{2}\mathrm{cof}(\underline{\underline{\mathbf{p}}}_{h,1}) : \underline{\underline{\mathbf{p}}}_{h,1} = 0 \quad \text{in each } K \in \mathcal{T}_h, \tag{3.13d}$$

where f_h is given such that $f_h|_K$ is the L^2 projection of f onto $P_{k-2}(K)$ for each $K \in \mathcal{T}_h$. We multiply (3.13a) by $\underline{\mathbf{q}}_h \in \underline{\mathbf{V}}_h$ and (3.13b) by $\underline{\underline{\mathbf{q}}}_h \in \underline{\underline{\mathbf{V}}}_h$, integrate and sum over all $K \in \mathcal{T}_h$.

$$\sum_{K\in\mathcal{T}_h}\int_K \underline{\mathbf{p}}_{h,i}\cdot\underline{\mathbf{q}}_h\,dx = \sum_{K\in\mathcal{T}_h}\int_K \nabla u_{h,i}\cdot\underline{\mathbf{q}}_h\,dx,\ \ 1 \leq i \leq 2, \tag{3.14}$$

$$\sum_{K\in\mathcal{T}_h}\int_K \underline{\underline{\mathbf{p}}}_{h,i}:\underline{\underline{\mathbf{q}}}_h\,dx = \sum_{K\in\mathcal{T}_h}\int_K D^2u_{h,i}:\underline{\underline{\mathbf{q}}}_h\,dx,\ 1\le i\le 2. \tag{3.15}$$

We multiply (3.13c) by $v_h\in V_h$, integrate and sum over all $K\in\mathcal{T}_h$, and apply Green's formula twice. We thus obtain

$$\begin{aligned}&\sum_{K\in\mathcal{T}_h}\int_K \nabla\cdot\nabla\cdot\underline{\underline{\mathbf{p}}}_{h,1}v_h\,dx - \sum_{K\in\mathcal{T}_h}\int_K \mathrm{cof}(\underline{\underline{\mathbf{p}}}_{h,1}):\underline{\underline{\mathbf{p}}}_{h,2}v_h\,dx \;=\\ &\sum_{K\in\mathcal{T}_h}\int_K \underline{\underline{\mathbf{p}}}_{h,1}:D^2v_h\,dx - \sum_{K\in\mathcal{T}_h}\int_K \mathrm{cof}(\underline{\underline{\mathbf{p}}}_{h,1}):\underline{\underline{\mathbf{p}}}_{h,2}v_h\,dx\,+\\ &\sum_{K\in\mathcal{T}_h}\int_{\partial K}\mathbf{n}_{\partial k}\cdot\nabla\cdot\underline{\underline{\mathbf{p}}}_{h,1}v_h\,ds - \sum_{K\in\mathcal{T}_h}\int_{\partial K}(\underline{\underline{\mathbf{p}}}_{h,1}\mathbf{n}_{\partial k})\cdot\nabla v_h\,ds = \sum_{K\in\mathcal{T}_h}\int_K f_hv_h\,dx.\end{aligned} \tag{3.16}$$

We proceed in the same way with (3.13d) which yields

$$\begin{aligned}&\sum_{K\in\mathcal{T}_h}\int_K \nabla\cdot\nabla\cdot\underline{\underline{\mathbf{p}}}_{h,2}v_h\,dx + \frac{1}{2}\sum_{K\in\mathcal{T}_h}\int_K \mathrm{cof}(\underline{\underline{\mathbf{p}}}_{h,1}):\underline{\underline{\mathbf{p}}}_{h,1}v_h\,dx \;=\\ &\sum_{K\in\mathcal{T}_h}\int_K \underline{\underline{\mathbf{p}}}_{h,2}:D^2v_h\,dx + \frac{1}{2}\sum_{K\in\mathcal{T}_h}\int_K \mathrm{cof}(\underline{\underline{\mathbf{p}}}_{h,1}):\underline{\underline{\mathbf{p}}}_{h,1}v_h\,dx\,+\\ &\sum_{K\in\mathcal{T}_h}\int_{\partial K}\mathbf{n}_{\partial k}\cdot\nabla\cdot\underline{\underline{\mathbf{p}}}_{h,2}v_h\,ds - \sum_{K\in\mathcal{T}_h}\int_{\partial K}(\underline{\underline{\mathbf{p}}}_{h,2}\mathbf{n}_{\partial k})\cdot\nabla v_h\,ds = 0.\end{aligned} \tag{3.17}$$

We replace $\underline{\underline{\mathbf{p}}}_{h,1}|_{\partial K}\ \mathbf{n}_{\partial K}$ and $\nabla\cdot\underline{\underline{\mathbf{p}}}_{h,1}|_{\partial K}$ in (3.16) by $\hat{\underline{\mathbf{p}}}^{(1,1)}_{\partial K}$ and $\hat{\underline{\mathbf{p}}}^{(1,2)}_{\partial K}$, where $\hat{\underline{\mathbf{p}}}^{(1,i)}_{\partial K}$, $1\le i\le 2$, are numerical flux functions. Likewise, we replace $\underline{\underline{\mathbf{p}}}_{h,2}|_{\partial K}\ \mathbf{n}_{\partial K}$ and $\nabla\cdot\underline{\underline{\mathbf{p}}}_{h,2}|_{\partial K}$ in (3.17) by $\hat{\underline{\mathbf{p}}}^{(2,1)}_{\partial K}$ and $\hat{\underline{\mathbf{p}}}^{(2,2)}_{\partial K}$, where $\hat{\underline{\mathbf{p}}}^{(2,i)}_{\partial K}$, $1\le i\le 2$, are also suitably chosen numerical flux functions. We thus obtain the following system of discrete variational equations:
Find $(\underline{\underline{\mathbf{p}}}_{h,i},\underline{\mathbf{p}}_{h,i},u_{h,i})\in\underline{\underline{\mathbf{V}}}_h\times\underline{\mathbf{V}}_h\times V_h$, $1\le i\le 2$, such that for all $(\underline{\underline{\mathbf{q}}}_h,\underline{\boldsymbol{\varphi}}_h,v_h)\in\underline{\underline{\mathbf{V}}}_h\times\underline{\mathbf{V}}_h\times V_h$ it holds

$$\sum_{K\in\mathcal{T}_h}\int_K \underline{\underline{\mathbf{p}}}_{h,1}:\underline{\underline{\mathbf{q}}}_h\,dx = \sum_{K\in\mathcal{T}_h}\int_K D^2u_{h,1}:\underline{\underline{\mathbf{q}}}_h\,dx, \tag{3.18a}$$

$$\sum_{K\in\mathcal{T}_h}\int_K \underline{\mathbf{p}}_{h,1}\cdot\underline{\boldsymbol{\varphi}}_h\,dx = -\sum_{K\in\mathcal{T}_h}\int_K \underline{\underline{\mathbf{p}}}_{h,1}:\nabla\underline{\boldsymbol{\varphi}}_h\,dx + \sum_{K\in\mathcal{T}_h}\int_{\partial K}\hat{\underline{\mathbf{p}}}^{(1,1)}_{\partial K}\cdot\underline{\boldsymbol{\varphi}}_h\,ds, \tag{3.18b}$$

$$\begin{aligned}&\sum_{K\in\mathcal{T}_h}\int_K \nabla\cdot\underline{\mathbf{p}}_{h,1}\ v_h\,dx = -\sum_{K\in\mathcal{T}_h}\int_K \underline{\mathbf{p}}_{h,1}\cdot\nabla v_h\,dx\,+\\ &\sum_{K\in\mathcal{T}_h}\int_{\partial K}\mathbf{n}_{\partial K}\cdot\hat{\underline{\mathbf{p}}}^{(1,2)}_{\partial K}\ v_h\,ds = \sum_{K\in\mathcal{T}_h}\int_K \mathrm{cof}(D^2u_{h,1}):D^2u_{h,2}\ v_h\,dx + \sum_{K\in\mathcal{T}_h}\int_K f_h\ v_h\,dx\end{aligned} \tag{3.18c}$$

and

$$\sum_{K\in\mathcal{T}_h}\int_K \underline{\underline{\mathbf{p}}}_{h,2}:\underline{\underline{\mathbf{q}}}_h\,dx = \sum_{K\in\mathcal{T}_h}\int_K D^2u_{h,2}:\underline{\underline{\mathbf{q}}}_h\,dx, \tag{3.19a}$$

$$\sum_{K\in\mathcal{T}_h}\int_K \underline{\mathbf{p}}_{h,2}\cdot\underline{\boldsymbol{\varphi}}_h\,dx = -\sum_{K\in\mathcal{T}_h}\int_K \underline{\underline{\mathbf{p}}}_{h,2}:\nabla\underline{\boldsymbol{\varphi}}_h\,dx + \sum_{K\in\mathcal{T}_h}\int_{\partial K}\hat{\underline{\mathbf{p}}}^{(2,1)}_{\partial K}\cdot\underline{\boldsymbol{\varphi}}_h\,ds, \tag{3.19b}$$

$$\begin{aligned}&\sum_{K\in\mathcal{T}_h}\int_K \nabla\cdot\underline{\mathbf{p}}_{h,2}\ v_h\,dx = -\sum_{K\in\mathcal{T}_h}\int_K \underline{\mathbf{p}}_{h,2}\cdot\nabla v_h\,dx\,+\\ &\sum_{K\in\mathcal{T}_h}\int_{\partial K}\mathbf{n}_{\partial K}\cdot\hat{\underline{\mathbf{p}}}^{(2,2)}_{\partial K}\ v_h\,ds = -\frac{1}{2}\sum_{K\in\mathcal{T}_h}\int_K \mathrm{cof}(D^2u_{h,1}):D^2u_{h,1}\ v_h\,dx.\end{aligned} \tag{3.19c}$$

In particular, for the six-field formulation of the C^0IPDG approximation (3.22) the numerical flux functions $\hat{\underline{\mathbf{p}}}^{(i,j)}_{\partial K}$, $1\le i,j\le 2$, are chosen as follows:

$$\hat{\underline{\mathbf{p}}}^{(1,1)}_{\partial K}|_E := \left(\{\underline{\underline{\mathbf{z}}}_{h,1}\}_E - \alpha_1 h_E^{-1}\{\underline{\underline{\mathbf{w}}}_{h,1}\}_E\right)\mathbf{n}_E, \ E \in \mathcal{E}_h(\Gamma), \tag{3.20a}$$

$$\hat{\underline{\mathbf{p}}}^{(1,2)}_{\partial K}|_E := \begin{cases} \mathbf{0} & , E \in \mathcal{E}_h(\Omega) \\ \nabla \cdot \underline{\underline{\mathbf{\Pi}}}_k(\underline{\underline{\mathbf{z}}}_{h,1}) + \alpha_2\, h_E^{-3} z_{h,1}\, \mathbf{n}_E & , E \in \mathcal{E}_h(\Gamma) \end{cases} \tag{3.20b}$$

and

$$\hat{\underline{\mathbf{p}}}^{(2,1)}_{\partial K}|_E := \left(\{\underline{\underline{\mathbf{z}}}_{h,2}\}_E - \alpha_1 h_E^{-1}\{\underline{\underline{\mathbf{w}}}_{h,2}\}_E\right)\mathbf{n}_E, \ E \in \mathcal{E}_h(\Gamma), \tag{3.21a}$$

$$\hat{\underline{\mathbf{p}}}^{(2,2)}_{\partial K}|_E := \begin{cases} \mathbf{0} & , E \in \mathcal{E}_h(\Omega) \\ \nabla \cdot \underline{\underline{\mathbf{\Pi}}}_k(\underline{\underline{\mathbf{z}}}_{h,2}) + \alpha_2\, h_E^{-3} z_{h,2}\, \mathbf{n}_E & , E \in \mathcal{E}_h(\Gamma) \end{cases}, \tag{3.21b}$$

where $\underline{\underline{\mathbf{z}}}_{h,i} := D_h^2 u_{h,i}, \underline{\underline{\mathbf{w}}}_{h,i} := \nabla u_{h,i} \otimes \mathbf{n}_E, z_{h,i} := u_{h,i}, 1 \le i \le 2$, and $\alpha_i > 0, 1 \le i \le 2$, are suitably chosen penalty parameters. The particular choice (3.20), (3.21) of the numerical flux functions allows to eliminate $\underline{\underline{\mathbf{p}}}_{h,i}, \underline{\mathbf{p}}_{h,i}, 1 \le i \le 2$, from (3.18) and (3.19). We thus obtain the following C^0IPDG approximation of the von Kármán equations: Find $\mathbf{u}_h \in V_h \times V_h$ such that for all $\mathbf{v}_h \in V_h \times V_h$ it holds

$$a_h^{DG}(\mathbf{u}_h, \mathbf{v}_h) = \ell(\mathbf{v}_h), \tag{3.22}$$

where the semilinear form $a_h^{DG}(\cdot,\cdot) : V_h^2 \times V_h^2 \to \mathbb{R}$ and the functional $\ell_h : V_h^2 \to \mathbb{R}$ are given by

$$\begin{aligned} a_h^{DG}(\mathbf{u}_h, \mathbf{v}_h) &:= \sum_{i=1}^{2} \sum_{K\in\mathcal{T}_h} \int_K D^2 u_{h,i} : D^2 v_{h,i}\, dx + \\ &\frac{1}{2}\sum_{K\in\mathcal{T}_h}\int_K \Big(\mathrm{cof}(D^2u_{h,1}) : D^2u_{h,2}\, v_{h,1} + \mathrm{cof}(D^2u_{h,2}) : D^2u_{h,1}\, v_{h,1} - \\ &\mathrm{cof}(D^2u_{h,1}) : D^2u_{h,1}\, v_{h,2}\Big)\, dx - \sum_{i=1}^{2}\Big(\sum_{E\in\mathcal{E}_h(\Omega)}\int_E \{D^2u_{h,i}\}_E : [\nabla v_{h,i}\otimes \mathbf{n}_E]_E\, ds + \\ &\sum_{E\in\mathcal{E}_h(\Omega)}\int_E \mathbf{n}_E\cdot\{\nabla\cdot\underline{\underline{\mathbf{\Pi}}}_k(D^2u_{h,i})\}_E\, v_{h,i}\, ds\Big) + \\ &\sum_{i=1}^{2}\Big(\alpha_1\sum_{E\in\mathcal{E}_h(\Gamma)} h_E^{-1}\int_E (\nabla u_{h,i}\otimes\mathbf{n}_E)\,\mathbf{n}_E\cdot\nabla v_{h,i} ds + \alpha_2 \sum_{E\in\mathcal{E}_h(\Gamma)} h_E^{-3}\int_E u_{h,i}v_{h,i} ds\Big), \end{aligned} \tag{3.23a}$$

$$\ell_h(\mathbf{v}_h) := \sum_{K\in\mathcal{T}_h}\int_K f_h\, v_{h,1}\, dx. \tag{3.23b}$$

Theorem 3.1. *The six-field formulation (3.18), (3.19) with the numerical flux functions given by (3.20) and (3.21) is equivalent with (3.22). In particular, if* $\mathbf{u}_h = (u_{h,1}, u_{h,2})^T \in V_h \times V_h$ *is the solution of (3.22), there exists pairs* $(\underline{\underline{\mathbf{p}}}_{h,i}, \underline{\mathbf{p}}_{h,i}) \in \underline{\underline{\mathbf{V}}}_h \times \underline{\mathbf{V}}_h, 1 \le i \le 2$ *such that the triples* $(\underline{\underline{\mathbf{p}}}_{h,i}, \underline{\mathbf{p}}_{h,i}, u_{h,i}) \in \underline{\underline{\mathbf{V}}}_h \times \underline{\mathbf{V}}_h \times V_h, 1 \le i \le 2$*, satisfy (3.18), (3.19). Conversely, if the triples* $(\underline{\underline{\mathbf{p}}}_{h,i}, \underline{\mathbf{p}}_{h,i}, u_{h,i}) \in \underline{\underline{\mathbf{V}}}_h \times \underline{\mathbf{V}}_h \times V_h, 1 \le i \le 2$*, satisfy (3.18), (3.19), then* $\mathbf{u}_h$ *solves (3.22).*

The proof of 3.1 will be given in Appendix A.

As in section 2 we define a bilinear form $A_{DG}(\cdot,\cdot) : (V_h + V)^2 \times (V_h + V)^2 \to \mathbb{R}$ and a semilinear form $B_{DG}(\cdot,\cdot) : (V_h + V)^2 \times (V_h + V)^2 \times (V_h + V)^2 \to \mathbb{R}$ by means of

$$\begin{aligned} A_{DG}(\mathbf{u}_h, \mathbf{v}_h) &:= \sum_{i=1}^{2}\Big(\sum_{K\in\mathcal{T}_h}\int_K D^2u_{h,i} : D^2 v_{h,i}\, dx - \\ &\sum_{E\in\mathcal{E}_h(\Omega)}\int_E \{D^2u_{h,i}\}_E : [\nabla v_{h,i}\otimes\mathbf{n}_E]_E\, ds + \\ &\sum_{E\in\mathcal{E}_h(\Omega)}\int_E \mathbf{n}_E\cdot\{\nabla\cdot\underline{\underline{\mathbf{\Pi}}}_k(D^2u_{h,i})\}_E\, v_{h,i}\, ds + \end{aligned} \tag{3.24a}$$

$$\alpha_1 \sum_{E\in\mathcal{E}_h(\Gamma)} h_E^{-1} \int_E (\nabla u_{h,1}\otimes \mathbf{n}_E)\,\mathbf{n}_E\cdot\nabla v_{h,i}\,ds + \alpha_2 \sum_{E\in\mathcal{E}_h(\Gamma)} h_E^{-3}\int_E u_{h,i}\,v_{h,i}\,ds\Big),$$

$$B_{DG}(\mathbf{u}_h,\mathbf{v}_h,\mathbf{w}_h) := \frac{1}{2}\Big(\sum_{K\in\mathcal{T}_h}\int_K \mathrm{cof}(D^2u_{h,1}) : D^2v_{h,2}\,w_{h,1}\,dx + \tag{3.24b}$$

$$\sum_{K\in\mathcal{T}_h}\int_K \mathrm{cof}(D^2u_{h,2}) : D^2v_{h,1}\,w_{h,1}\,dx - \sum_{K\in\mathcal{T}_h}\int_K \mathrm{cof}(D^2u_{h,1}) : D^2v_{h,1}\,w_{h,2}\,dx\Big),$$

where $\mathbf{u}_h = (u_{h,1},u_{h,2})^T$, $\mathbf{v}_h = (v_{h,1},v_{h,2})^T$, $\mathbf{w}_h = (w_{h,1},w_{h,2})^T$, $u_{h,i},v_{h,i},w_{h,i}\in V_h+V, 1\le i\le 2$.
Then the C^0IPDG approximation (3.22) can be written as: Find $\mathbf{u}_h\in V_h\times V_h$ such that for all $\mathbf{v}_h\in V_h\times V_h$ it holds

$$A_{DG}(\mathbf{u}_h,\mathbf{v}_h) + B_{DG}(\mathbf{u}_h,\mathbf{u}_h,\mathbf{v}_h) = \sum_{K\in\mathcal{T}_h}(f_h,v_{h,1})_{L^2(K)}. \tag{3.25}$$

We introduce operators $\mathcal{A}_{DG} : (V_h+V)^2\to(V_h^*+V^*)^2$ and $\mathcal{B}_{DG} : (V_h+V)^2\to(V_h^*+V^*)^2$ according to

$$<\mathcal{A}_{DG}\mathbf{u}_h,\mathbf{v}_h>_{(V_h^*+V^*)^2,(V_h+V)^2} := A_{DG}(\mathbf{u}_h,\mathbf{v}_h), \tag{3.26a}$$
$$<\mathcal{B}_{DG}\mathbf{u}_h,\mathbf{v}_h>_{(V_h^*+V^*)^2,(V_h+V)^2} := B_{DG}(\mathbf{u}_h,\mathbf{u}_h,\mathbf{v}_h), \tag{3.26b}$$

so that (3.25) can be written as

$$\mathcal{A}_{DG}\mathbf{u}_h + \mathcal{B}_{DG}\mathbf{u}_h = \mathbf{f}_h, \quad \mathbf{f}_h = (f_h,0)^T. \tag{3.27}$$

A slight variation of Theorem 2.1 in [13] yields the following existence and uniqueness result.

Theorem 3.2. *Given $f\in L^2(\Omega)$, let $\mathbf{u}\in\underline{\mathbf{V}}$ be a regular weak solution of the von Kármán equations. Then there exist $\delta_0,\varepsilon_0>0$, such that for any triangulation $\mathcal{T}_h\in\mathbb{T}(\delta_0)$ there exists a unique solution $\mathbf{u}_h\in\underline{\mathbf{V}}_h$ of the C^0IPDG approximation (3.22) satisfying*

$$\sum_{i=1}^{2}\sum_{K\in\mathcal{T}_h}\int_K \|D^2u_{h,i}\|^2\,dx + \sum_{K\in\mathcal{T}_h}\int_K |f_h-f|^2\,dx < \varepsilon_0. \tag{3.28}$$

We note that $u_{h,i}\notin W^{2,2}(\Omega), 1\le i\le 2$, but conforming finite element functions $u_{h,i}^c\in V_h^c := V_h\cap W^{2,2}(\Omega)$ can be obtained from $u_h\in V_h$ by postprocessing. In particular, let V_h^c be the generalized version of the Hsieh-Clough-Tocher C^1 conforming finite element space as constructed in [21] and let $u_h^c = E_h(u_h)$ be the extension of u_h to V_h^c as constructed in [22]. By a result from [23] there exist constants $C_{c,|\nu|}>0, |\nu|\le 2$, and $C_{ext}>0$, depending only on the local geometry of the triangulation and on the penalty parameters $\alpha_i, 1\le i\le 2$, such that for $1\le i\le 2$ it holds

$$\Big(\sum_{K\in\mathcal{T}_h}\|D^\nu u_{h,i}^c\|_{L^2(K)}^2\Big)^{1/2} \le C_{c,|\nu|}\Big(\sum_{K\in\mathcal{T}_h}\|D^\nu u_{h,i}\|_{L^2(K)}^2\Big)^{1/2}, \quad |\nu|\le 2\,, \tag{3.29a}$$

$$\|u_h-u_h^c\|_{W^{2,2}(\Omega;\mathcal{T}_h)}^2 \le \tag{3.29b}$$
$$C_{ext}\Big(\sum_{E\in\mathcal{E}_h(\Omega)} h_E^{-1}\int_E |[\nabla u_{h,i}\otimes\mathbf{n}_E]_E|^2\,ds + \sum_{E\in\mathcal{E}_h(\Gamma)} h_E^{-3}\int_E |u_{h,i}|^2\,ds\Big)\,.$$

4. An a posteriori error estimator for the global discretization error

Given reflexive Banach spaces V,Q with norms $\|\cdot\|_V,\|\cdot\|_Q$, convex and coercive objective functionals $C:V\to\mathbb{R}$, $D:Q\to\mathbb{R}$, and a bounded linear operator $\Lambda:V\to Q$, we consider the minimization problem

$$\inf_{u\in V} J(u) \tag{4.1}$$

for the objective functional

$$J(u) := C(u) + D(\Lambda u). \tag{4.2}$$

An abstract approach to the a posteriori error control for (4.1) has been provided in [30] (see also [31]). The a posteriori error control relies on the dual formulation of (4.1)

$$\sup_{q^* \in Q^*} J^*(q^*) \quad \text{or} \quad \inf_{q^* \in Q^*} (-J^*(q^*)), \tag{4.3}$$

in terms of the Fenchel conjugate J^* of J as given by

$$J^*(q^*) = -C^*(-\Lambda^* q^*) - D^*(q^*), \tag{4.4}$$

where C^* and D^* are the Fenchel conjugates of C and D and Λ^* stands for the adjoint of Λ.
Given some approximation $u_h \in V$ of the minimizer u of (4.1), the a posteriori error estimate Theorem 2.2 from [30] (cf. also Section 3 in [1] and [31]) states that for any admissible function $q^* \in Q^*$ it holds

$$\Phi_\delta(\Lambda(u_h - u)) \le M_C(\Lambda^* q^*, u_h) + M_D(q^*, \Lambda u_h), \tag{4.5}$$

where $\Phi_\delta : Q \to \mathbb{R}_+$ is a continuous functional such that $\Phi_\delta(0) = 0$ and for all $q_i \in B(0,\delta) := \{q \in Q \mid ||q||_Q < \delta\}$, $\delta > 0$, $1 \le i \le 2$, it holds

$$D((q_1 + q_2)/2) + \Phi_\delta(q_2 - q_1) \le (D(q_1) + D(q_2))/2$$

and

$$M_C(\Lambda^* q^*, u_h) := \frac{1}{2}\Big(C(u_h) + C^*(\Lambda^* q^*) - < \Lambda^* q^*, u_h >_{V^*,V}\Big),$$
$$M_D(q^*, \Lambda u_h) := \frac{1}{2}\Big(D(\Lambda v) + D^*(-q^*) - < q^*, \Lambda u_h >_{Q^*,Q}\Big).$$

We apply the above result for $V = W_0^{2,2}(\Omega)^2$, $Q := L^2(\Omega; \mathbb{R}^{2\times 2})^2$, $\Lambda = D^2$ (D^2 being the Hessian), and

$$C(u_{h,1}^c, u_{h,2}^c) := -\int_\Omega f u_{h,1}^c \, dx, \tag{4.6a}$$

$$D(D^2 u_{h,1}^c, D^2 u_{h,2}^c) := \frac{1}{2} \sum_{i=1}^{2} \sum_{K \in \mathcal{T}_h} \int_K |D^2 u_{h,i}^c|^2 \, dx + I_{K_1}(u_{h,1}^c, u_{h,2}^c), \tag{4.6b}$$

where I_{K_1} is the indicator function of $K_1 := W_0^{2,2}(\Omega)^2$. We obtain:

$$C^*(-\Lambda^* \underline{\underline{\mathbf{q}}}_1^*, -\Lambda^* \underline{\underline{\mathbf{q}}}_2^*) := I_{K_2}(\underline{\underline{\mathbf{q}}}_1^*, \underline{\underline{\mathbf{q}}}_2^*), \quad \underline{\underline{\mathbf{q}}}_i^* \in \underline{\underline{\mathbf{H}}}(\mathrm{div}^2; \Omega), 1 \le i \le 2, \tag{4.7a}$$

$$D^*(\underline{\underline{\mathbf{q}}}_1^*, \underline{\underline{\mathbf{q}}}_2^*) := \frac{1}{2} \sum_{i=1}^{2} \int_\Omega |\underline{\underline{\mathbf{q}}}_i^*|^2 \, dx, \quad \underline{\underline{\mathbf{q}}}_i^* \in \underline{\underline{\mathbf{H}}}(\mathrm{div}^2; \Omega), 1 \le i \le 2, \tag{4.7b}$$

where I_{K_2} is the indicator function of the closed convex set

$$\begin{aligned} K_2 := \{(\underline{\underline{\mathbf{q}}}_1^*, \underline{\underline{\mathbf{q}}}_2^*) \in \underline{\underline{\mathbf{H}}}(\mathrm{div}^2; \Omega)^2 \mid \nabla \cdot \nabla \cdot \underline{\underline{\mathbf{q}}}_1^* - \Pi_{k-2}([u_{h,1}, u_{h,2}]) = f_h \,, \\ \nabla \cdot \nabla \cdot \underline{\underline{\mathbf{q}}}_2^* + \frac{1}{2}\Pi_{k-2}([u_{h,1}, u_{h,1}]) = 0 \text{ in } \Omega\}. \end{aligned} \tag{4.7c}$$

Similar to (3.8) in Example 2 (p-Laplace problem) of [30], the estimate (4.5) leads to

$$\|u - u_h\|_V^p \le C_{est}\Big(C(u_h) + C^*(-\Lambda^* q^*) + D(\Lambda u_h) + D^*(q^*)\Big), \tag{4.8}$$

where $C_{est} := 2^p p/2$.
We call $\underline{\underline{\mathbf{p}}}_{h,i}^{eq} \in \underline{\underline{\mathbf{V}}}_h$, $1 \le i \le 2$, equilibrated moment tensors, if

$$\underline{\underline{\mathbf{p}}}_{h,i}^{eq} \in \underline{\underline{\mathbf{H}}}(\mathrm{div}^2; \Omega) \tag{4.9a}$$

and $\underline{\underline{\mathbf{p}}}_{h,i}^{eq}$ satisfy the equilibrium conditions

$$\nabla \cdot \nabla \cdot \underline{\underline{\mathbf{p}}}_{h,1}^{eq} - \Pi_{k-2}([u_{h,1}, u_{h,2}]) = f_h \text{ in } \Omega, \tag{4.9b}$$

$$\nabla \cdot \nabla \cdot \underline{\underline{\mathbf{p}}}_{h,2}^{eq} + \frac{1}{2}\Pi_{k-2}([u_{h,1}, u_{h,1}]) = 0 \text{ in } \Omega. \tag{4.9c}$$

Moreover, we choose $\underline{\underline{\mathbf{p}}}_{c,1} \in \underline{\underline{\mathbf{H}}}(\mathrm{div}^2, \Omega)$ such that

$$\nabla \cdot \nabla \cdot \underline{\underline{\mathbf{p}}}_{c,1} = f - f_h \tag{4.10}$$

and set $\underline{\underline{\mathbf{p}}}_{c,2} = \mathbf{0}$. It follows that $(\underline{\underline{\mathbf{p}}}^{eq}_{h,1} + \underline{\underline{\mathbf{p}}}_{c,1}, \underline{\underline{\mathbf{p}}}^{eq}_{h,2} + \underline{\underline{\mathbf{p}}}_{c,2}) \in K_2$, i.e., $I_{K_2}(\underline{\underline{\mathbf{p}}}^{eq}_{h,1} + \underline{\underline{\mathbf{p}}}_{c,1}, \underline{\underline{\mathbf{p}}}^{eq}_{h,2} + \underline{\underline{\mathbf{p}}}_{c,2}) = 0$, and hence, (4.8) reads as follows:

$$\sum_{i=1}^{2} \|u_i - \mathbf{u}^c_{h,i}\|_{W^{2,2}(\Omega)} \lesssim J_P(u^c_{h,1}, u^c_{h,2}) + I_{K_1}(u^c_{h,1}, u^c_{h,2}) + J_D(\underline{\underline{\mathbf{p}}}^{eq}_{h,1} + \underline{\underline{\mathbf{p}}}_{c,1}, \underline{\underline{\mathbf{p}}}^{eq}_{h,2} + \underline{\underline{\mathbf{p}}}_{c,2}). \tag{4.11}$$

In view of (4.7) we have

$$J_D(\underline{\underline{\mathbf{p}}}^{eq}_{h,1} + \underline{\underline{\mathbf{p}}}_{c,1}, \underline{\underline{\mathbf{p}}}^{eq}_{h,2} + \underline{\underline{\mathbf{p}}}_{c,2}) = \frac{1}{2} \sum_{i=1}^{2} \sum_{K \in \mathcal{T}_h} \int_K |\underline{\underline{\mathbf{p}}}^{eq}_{h,i} + \underline{\underline{\mathbf{p}}}_{c,i}|^2 \, dx. \tag{4.12}$$

Using (2.2), we find

$$\frac{1}{2} \sum_{i=1}^{2} \sum_{K \in \mathcal{T}_h} \int_K |\underline{\underline{\mathbf{p}}}^{eq}_{h,i} + \underline{\underline{\mathbf{p}}}_{c,i}|^2 \, dx \le 2\Big(\sum_{i=1}^{2} \sum_{K \in \mathcal{T}_h} \int_K |\underline{\underline{\mathbf{p}}}^{eq}_{h,i}|^2 \, dx + \sum_{K \in \mathcal{T}_h} \int_K |\underline{\underline{\mathbf{p}}}_{c,1}|^2 \, dx \Big). \tag{4.13}$$

In order to estimate the second term on the right-hand side of (4.13) we use the Poincaré-Friedrichs inequalities

$$\|v - |K|^{-1} \int_K v \, dx\|_{L^2(K)} \le C^{(1)}_{PF} \, h_K \|\nabla v\|_{L^2(K)}, \quad v \in W^{1,2}(K), \ K \in \mathcal{T}_h, \tag{4.14a}$$

$$\|v - |E|^{-1} \int_E v \, ds\|_{L^2(E)} \le C^{(2)}_{PF} \, h_E \|\nabla v\|_{L^2(E)}, \quad v \in W^{1,2}(E), \ E \in \mathcal{E}_h(\Gamma), \tag{4.14b}$$

where $C^{(i)}_{PF}, 1 \le i \le 2$, are positive constants (cf., e.g., [17]).

Lemma 4.1. *Suppose that the following regularity assumption is satisfied: For $\underline{\underline{\boldsymbol{\tau}}} \in \underline{\underline{\mathbf{H}}}(div, \Omega)$ and the weak solution $z \in W^{2,2}_0(\Omega)$ of the elliptic boundary value problem*

$$\nabla \cdot \nabla \cdot D^2 z = \nabla \cdot \nabla \cdot \underline{\underline{\boldsymbol{\tau}}} \quad in\ \Omega, \tag{4.15a}$$

$$z = \mathbf{n}_\Gamma \cdot \nabla z = 0 \quad on\ \Gamma, \tag{4.15b}$$

there exists a constant $C^{(1)}_z > 0$ such that

$$D^2 z|_\Gamma \in L^2(\Gamma; \mathbb{R}^{2\times 2}), \quad \|D^2 z\|_{L^2(\Gamma;\mathbb{R}^{2\times 2})} \le C^{(1)}_z. \tag{4.16}$$

Moreover, there exists a constant $C^{(2)}_z > 0$ such that

$$\|\nabla z\|_{L^2(\Omega;\mathbb{R}^2)} \le C^{(2)}_z. \tag{4.17}$$

Then for $\underline{\underline{\mathbf{p}}}_{c,1} \in \underline{\underline{\mathbf{H}}}(div^2, \Omega)$ as given by (4.10) there exists a constant $C_U > 0$, depending on $C^{(i)}_z, C^{(i)}_{PF}, 1 \le i \le 2$, such that it holds

$$\|\underline{\underline{\mathbf{p}}}_{c,1}\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2})} \le C_U \ osc_{h,1}, \tag{4.18}$$

where $osc_{h,1}$ refers to the data oscillation

$$osc_{h,1} := \sum_{K \in \mathcal{T}_h} osc_{K,1}, \quad osc_{K,1} := \begin{cases} h_K^2 \int_K |f - f_h|^2 \, dx & , k = 2 \\ h_K^4 \int_K |f - f_h|^2 \, dx & , k \ge 3 \end{cases}. \tag{4.19}$$

The proof of Lemma 4.1 will be given in Appendix B.

Moreover, as far as $J_P(u^c_{h,1}, u^c_{h,2})$ is concerned, we have

$$J_P(u^c_{h,1}, u^c_{h,2}) = J_P(u_{h,1}, u_{h,2}) + \frac{1}{2} \sum_{K \in \mathcal{T}_h} \sum_{i=1}^{2} \int_K \Big(|D^2 u^c_{h,i}|^2 - |D^2 u_{h,i}|^2 \Big) \, dx + \tag{4.20}$$

$$\sum_{K\in\mathcal{T}_h}\int_K \Big(\mathrm{cof}(D^2u_{h,1}):D^2u_{h,2}\,u_{h,1}-\mathrm{cof}(D^2u^c_{h,1}):D^2u^c_{h,2}\,u^c_{h,1}\Big)\,dx -$$
$$\frac{1}{2}\sum_{K\in\mathcal{T}_h}\int_K \Big(\mathrm{cof}(D^2u_{h,1}):D^2u_{h,1}\,u_{h,2}-\mathrm{cof}(D^2u^c_{h,1}):D^2u^c_{h,1}\,u^c_{h,2}\Big)\,dx +$$
$$\sum_{K\in\mathcal{T}_h}\int_K f\,(u_{h,1}-u^c_{h,1})\,dx.$$

Lemma 4.2. *Let $\mathbf{u}_h\in V_h^2$ be the solution of (3.22) and let $(u^c_{h,1},u^c_{h,2})\in V^c_h\times V^c_h$ be its postprocessed finite element function. Then it holds*

$$\Big|J_P(u^c_{h,1},u^c_{h,2})-J_P(u_{h,1},u_{h,2})\Big|\lesssim\sum_{K\in\mathcal{T}_h}\kappa_K^{eq}, \tag{4.21}$$

where

$$\kappa_K^{eq}:=\sum_{i=1}^2\|u_{h,i}-u^c_{h,i}\|^2_{W^{2,2}(K)}+\sum_{i=1}^2\|D^2u_{h,i}\|_{L^2(K)}+ \tag{4.22}$$
$$\Big(\frac{1}{2}(C_{c,2}+1)\|u_{h,2}\|_{L^3(K)}\|D^2u_{h,1}\|_{L^3(K)}+C_{c,2}\|u_{h,1}\|_{L^3(K)}\|D^2u_{h,2}\|_{L^3(K)}+$$
$$C^2_{c,2}\|D^2u_{h,1}\|_{L^3(K)}\|D^2u_{h,2}\|_{L^3(K)}\Big)\|u_{h,1}-u^c_{h,1}\|_{W^{2,3}(K)}+$$
$$\Big(\|u_{h,1}\|_{L^3(K)}\|D^2u_{h,1}\|_{L^3(K)}+\frac{1}{2}C^2_{c,2}\|D^2u_{h,1}\|^2_{L^3(K)}\Big)\|u_{h,2}-u^c_{h,2}\|_{W^{2,3}(K)}+$$
$$\|f\|_{L^2(K)}\sum_{i=1}^2\|u_{h,i}-u^c_{h,i}\|_{W^{2,2}(K)}.$$

Proof. By Taylor expansion and using (2.2) as well as Hölder's inequality we find

$$\Big|\frac{1}{2}\sum_{K\in\mathcal{T}_h}\int_K\Big(|D^2u^c_{h,i}|^2-|D^2u_{h,i}|^2\Big)\Big| = \tag{4.23}$$
$$\Big|\sum_{K\in\mathcal{T}_h}\int_K\int_0^1(D^2u_{h,i}+\lambda D^2(u^c_{h,i}-u_{h,i})\,d\lambda:D^2(u^c_{h,i}-u_{h,i})\,dx\Big|$$
$$\le\sum_{K\in\mathcal{T}_h}\int_K\int_0^1|D^2u_{h,i}+\lambda D^2(u^c_{h,i}-u_{h,i})||D^2(u^c_{h,i}-u_{h,i})|\,d\lambda\,dx\ \le$$
$$2\sum_{K\in\mathcal{T}_h}\int_K\int_0^1\Big(|D^2u_{h,i}|+\lambda|D^2(u_{h,i}-u^c_{h,i})||D^2(u_{h,i}-u^c_{h,i})|\,d\lambda\,dx\ \le$$
$$2\sum_{K\in\mathcal{T}_h}\Big(\int_K|D^2u_{h,i}|\,dx\Big)^{1/2}\Big(\int_K|D^2(u_{h,i}-u^c_{h,i})|^2\,dx\Big)^{1/2}+$$
$$\sum_{K\in\mathcal{T}_h}\int_K|D^2(u_{h,i}-u^c_{h,i})|^2\,dx.$$

Moreover, using Hölder's inequality and (3.29a) we have

$$\Big|\sum_{K\in\mathcal{T}_h}\int_K\Big(\mathrm{cof}(D^2u_{h,1}):D^2u_{h,2}\,u_{h,1}-\mathrm{cof}(D^2u^c_{h,1}):D^2u^c_{h,2}\,u^c_{h,1}\Big)\,dx\Big|\ \le \tag{4.24}$$
$$\sum_{K\in\mathcal{T}_h}\Big(\|u_{h,1}\|_{L^3(K)}\|D^2u_{h,1}\|_{L^3(K)}\|D^2u_{h,2}-D^2u^c_{h,2}\|_{L^3(K)}+$$

$$C_{c,2}\|u_{h,1}\|_{L^3(K)}\|D^2u_{h,2}\|_{L^3(K)}\|D^2u_{h,1}-D^2u^c_{h,1}\|_{L^3(K)}+$$
$$C^2_{c,2}\|D^2u_{h,1}\|_{L^3(K)}\|D^2u_{h,2}\|_{L^3(K)}\|u_{h,1}-u^c_{h,1}\|_{L^3(K)}\Big).$$

Likewise, we obtain

$$\Big|\frac{1}{2}\sum_{K\in\mathcal{T}_h}\int_K\Big(\mathrm{cof}(D^2u_{h,1}):D^2u_{h,1}\,u_{h,2}-\mathrm{cof}(D^2u^c_{h,1}):D^2u^c_{h,1}\,u^c_{h,2}\Big)\,dx\Big|\le \tag{4.25}$$
$$\frac{1}{2}\sum_{K\in\mathcal{T}_h}\Big((1+C_{c,2})\|u_{h,2}\|_{L^3(K)}\|D^2u_{h,1}\|_{L^3(K)}\|D^2u_{h,1}-D^2u^c_{h,1}\|_{L^3(K)}+$$
$$C^2_{c,2}\|D^2u_{h,1}\|^2_{L^3(K)}\|D^2u_{h,2}-D^2u^c_{h,2}\|_{L^3(K)}\Big).$$

Finally, for the last term on the right-hand side of (4.20) we find

$$\Big|\sum_{K\in\mathcal{T}_h}\int_K f(u_{h,i}-u^c_{h,i})\,dx\le\sum_{K\in\mathcal{T}_h}\Big(\int_K|f|^2\,dx\Big)^{1/2}\Big(\int_K|u_{h,i}-u^c_{h,i}|^2\,dx\Big)^{1/2}. \tag{4.26}$$

The assertion now follows from (4.20) and (4.23) - (4.26). □

For practical purposes, we further replace $I_{K_1}(u^c_{h,1},u^c_{h,2})$ by the penalty terms

$$\sum_{i=1}^{2}\Big(\alpha_1\sum_{E\in\mathcal{E}_h(\Gamma)}h_E^{-1}\int_E|\mathbf{n}_E\cdot\nabla u^c_{h,i}|^2\,ds+\alpha_2\sum_{E\in\mathcal{E}_h(\Gamma)}h_E^{-3}\int_E|u^c_{h,i}|^2\,ds\Big). \tag{4.27}$$

In view of the construction of $u^c_{h,i}$ we have $u^c_{h,i}|_E=u_{h,i}$ on $E\in\mathcal{E}_h(\Gamma)$ and hence, (4.27) gives rise to the data oscillations

$$\alpha_1\,osc_{h,2}+\alpha_2\,osc_{h,3}:=\alpha_1\sum_{K\in\mathcal{T}_h}osc_{K,2}+\alpha_2\sum_{K\in\mathcal{T}_h}osc_{K,3}, \tag{4.28a}$$
$$osc_{K,2}:=\sum_{i=1}^{2}\sum_{E\in\mathcal{E}_h(\partial K\cap\Gamma)}h_E^{-1}\int_E|\mathbf{n}_E\cdot\nabla u_{h,i}|^2\,ds, \tag{4.28b}$$
$$osc_{K,3}:=\sum_{i=1}^{2}\sum_{E\in\mathcal{E}_h(\partial K\cap\Gamma)}h_E^{-3}\int_E|u_{h,i}|^2\,ds. \tag{4.28c}$$

Using Lemma 4.1 and Lemma 4.2 in (4.11) yields

$$\|\mathbf{u}-\mathbf{u}_h\|^2_{W^{2,2}(\Omega;\mathcal{T}_h)^2}\lesssim\sum_{i=1}^{3}\eta^{eq}_{h,i}. \tag{4.29a}$$

Here, $\eta^{eq}_{h,1}$ and $\eta^{eq}_{h,2}$ are given by

$$\eta^{eq}_{h,i}:=\sum_{K\in\mathcal{T}_h}\eta^{eq}_{K,i},\quad 1\le i\le 3, \tag{4.29b}$$

where $\eta^{eq}_{K,i}$, $1\le i\le 3$, read as follows:

$$\eta^{eq}_{K,1}:=\frac{1}{2}\int_K|D^2u_{h,1}|^2\,dx-\frac{1}{2}\int_K f u_{h,1}\,dx+\frac{1}{2}\int_K|\underline{\underline{\mathbf{p}}}^{eq}_{h,1}|^2\,dx, \tag{4.29c}$$
$$\eta^{eq}_{K,2}:=\frac{1}{2}\int_K|D^2u_{h,2}|^2\,dx+\frac{1}{2}\int_K|\underline{\underline{\mathbf{p}}}^{eq}_{h,2}|^2\,dx, \tag{4.29d}$$
$$\eta^{eq}_{K,3}:=\sum_{i=1}^{2}\|u_{h,i}-u^c_{h,i}\|^2_{W^{2,2}(K)}+\kappa^{eq}_K+\sum_{i=1}^{3}osc_{K,i}. \tag{4.29e}$$

The right-hand side in (4.29) is then a computable and localizable quantity for the a posteriori estimation of the global discretization error. It gives rise to the following equilibrated a posteriori error estimator

$$\eta_h^{eq} := \sum_{i=1}^{3} \eta_{h,i}^{eq}, \quad \eta_{h,i}^{eq} := \sum_{K\in\mathcal{T}_h(\Omega)} \eta_{K,i}^{eq}, \ 1 \le i \le 3. \tag{4.30}$$

The construction of an equilibrated flux will be dealt with in the subsequent section.

5. Construction of an equilibrated flux

We construct equilibrated fluxes $\underline{\mathbf{p}}_{h,i}^{eq} \in \underline{\mathbf{V}}_h \cap \underline{\mathbf{H}}(\text{div}, \Omega)$ and equilibrated moment tensors $\underline{\underline{\mathbf{p}}}_{h,i}^{eq} \in \underline{\underline{\mathbf{V}}}_h \cap \underline{\underline{\mathbf{H}}}(\text{div}^2, \Omega), 1 \le i \le 2$, by an interpolation on each element. Thus it is a local procedure. In particular, denoting by $\mathbf{BDM}_k(K), k \in \mathbb{N}$, the Brezzi-Douglas-Marini finite element of order k (cf., e.g., [9]), we first construct auxiliary vector fields $\underline{\mathbf{p}}_{h,i}^{eq} \in \underline{\mathbf{H}}(\text{div}, \Omega), \underline{\mathbf{p}}_{h,i}^{eq}|_K \in \mathbf{BDM}_{k-1}(K), K \in \mathcal{T}_h(\Omega), 1 \le i \le 2$, satisfying

$$\nabla \cdot \underline{\mathbf{p}}_{h,1}^{eq} - \Pi_{k-2}(\text{cof}(D^2 u_{h,1} : D^2 u_{h,2}) = f_h, \tag{5.1a}$$

$$\nabla \cdot \underline{\mathbf{p}}_{h,2}^{eq} + \frac{1}{2}\Pi_{k-2}(\text{cof}(D^2 u_{h,1} : D^2 u_{h,1}) = 0, \tag{5.1b}$$

in each $K \in \mathcal{T}_h$ and then equilibrated moment tensors $\underline{\underline{\mathbf{p}}}_{h,i}^{eq} \in \underline{\underline{\mathbf{V}}}_h, 1 \le i \le 2$, satisfying (4.9).

For the construction of the auxiliary vector fields we recall the following result:

Lemma 5.1. *Any vector field $\underline{\mathbf{q}} \in P_k(K)^2, k \in \mathbb{N}$, is uniquely defined by the following degrees of freedom*

$$\int_E \mathbf{n}_E \cdot \underline{\mathbf{q}}\, p_k\, ds, \quad p_k \in P_k(E), \ E \in \mathcal{E}_h(\partial K), \tag{5.2a}$$

$$\int_K \underline{\mathbf{q}} \cdot \nabla p_{k-1}\, dx, \qquad p_{k-1} \in P_{k-1}(K), \tag{5.2b}$$

$$\int_K \underline{\mathbf{q}} \cdot \mathbf{curl}(b_K\, p_{k-2})\, dx, \tag{5.2c}$$

where b_K in (5.2c) is the element bubble function on K given by $b_K = \prod_{i=1}^{3} \lambda_i^K$ and $\lambda_i^K, 1 \le i \le 3$, are the barycentric coordinates of K. Moreover, there exists a positive constant $C_E^{(1)}$, depending only on k and the local geometry of the triangulation $\mathcal{T}_h$, such that

$$\int_K |\underline{\mathbf{q}}|^2\, dx \le C_E^{(1)} \Big(\sum_{E\in\mathcal{E}_h(\partial K)} h_E \int_E |\mathbf{n}_E \cdot \underline{\mathbf{q}}|^2\, ds + h_K^2 \int_K |\nabla \cdot \underline{\mathbf{q}}|^2\, dx + \\ h_K^2 \max\Big\{ \int_K |\underline{\mathbf{q}} \cdot \mathbf{curl}(b_K\, p_{k-2})|^2\, dx \mid p_{k-2} \in P_{k-2}(K), \ \max_{x\in K} |p_{k-2}(x)| \le 1 \Big\}\Big). \tag{5.3}$$

Proof. For the uniqueness result we refer to [9]. The estimate (5.3) can be derived by standard scaling arguments (cf. Lemma 3.1 and Remark 3.3 in [5]). □

The construction of the auxiliary vector field $\underline{\mathbf{p}}_{h,i}^{eq}, 1 \le i \le 2$, will be done on patches ω_i consisting of all triangles $K \in \mathcal{T}_h$ that have an interior nodal point $x_i \in \mathcal{N}_h(\Omega)$ in common. We assume that ω_i consists of N_i triangles $T_\ell, 1 \le \ell \le N_i$. We enumerate the interior edges $E_m, 1 \le m \le M_i$, counterclockwise and distinguish five cases (cf. Fig. 1).

We follow the techniques from [6] and construct the auxiliary vector fields $\underline{\mathbf{p}}_{h,m}^{eq}, 1 \le m \le 2$, patchwise:

$$\underline{\mathbf{p}}_{h,m}^{eq} = \sum_{i=1}^{n_h} \underline{\mathbf{p}}_{h,m}^{\omega_i}. \tag{5.4}$$

For a patch ω_i, we construct $\underline{\mathbf{p}}_{h,m}^{\omega_i}$ such that

$$\begin{aligned} &\underline{\mathbf{p}}_{h,m}^{\omega_i}|_{K_\ell} \in \mathbf{BDM}_{k-1}(K_\ell)\ , \\ &\nabla \cdot \underline{\mathbf{p}}_{h,1}^{\omega_i} - \Pi_{k-2}(\text{cof}(D^2 u_{h,1}) : D^2 u_{h,2}) = f_h \text{ in } \omega_i\ , \\ &\nabla \cdot \underline{\mathbf{p}}_{h,2}^{\omega_i} + \frac{1}{2}\Pi_{k-2}(\text{cof}(D^2 u_{h,1}) : D^2 u_{h,1}) = 0 \text{ in } \omega_i\ , \end{aligned} \tag{5.5}$$

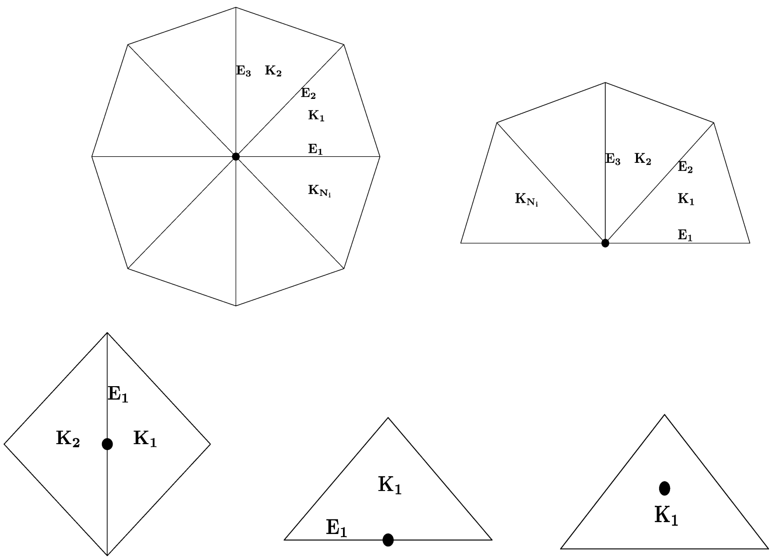

Fig. 1. Patch ω_i associated with nodal point $x_i \in \mathcal{N}_h(\bar{\Omega})$ featuring N_i triangles $K_\ell, 1 \le \ell \le N_i$ ($x_i \in \mathcal{N}_h(\partial E \cap \Omega)$ (Case 1, top left), $x_i \in \mathcal{N}_h(\partial E \cap \Gamma)$ (Case 2, top right), $x_i \in \mathcal{N}_h(\text{int}E \cap \Omega)$ (Case 3, bottom left), $x_i \in \mathcal{N}_h(\text{int}E \cap \Gamma)$ (Case 4, bottom middle), and $x_i \in \mathcal{N}_h(\text{int}K \cap \Omega)$ (Case 5, bottom right)).

$$\underline{\mathbf{p}}_{h,m}^{\omega_i} = \hat{\underline{\mathbf{p}}}_{\partial K}^{m,\omega_i}|_E,\ E \in \mathcal{E}_h(\text{int}\,\omega_i),\ 1 \le \ell \le N_i,$$

where, denoting by $\varphi_h^{(x_i)} \in V_h^{(k)}$ the nodal basis function associated with x_i, $\underline{\mathbf{p}}_{h,m}^{\omega_i}$ and $\hat{\underline{\mathbf{p}}}_{\partial K}^{m,\omega_i}|_E$ are given by

$$\underline{\mathbf{p}}_{h,m}^{\omega_i} := \varphi_h^{(x_i)} \underline{\mathbf{p}}_{h,m}^{eq}, \quad \hat{\underline{\mathbf{p}}}_{\partial K}^{m,\omega_i}|_E := \varphi_h^{(x_i)} \hat{\underline{\mathbf{p}}}_{\partial K}^{(m,1)}|_E\ , 1 \le i \le n_h. \tag{5.6}$$

For the construction of the auxiliary vector field $\underline{\mathbf{p}}_{h,1}^{eq}$ we set

$$\underline{\mathbf{z}}_h^{(1)} := (\frac{\partial^2 u_{h,1}}{\partial x_1^2}, \frac{\partial^2 u_{h,1}}{\partial x_1 \partial x_2})^T, \quad \underline{\mathbf{z}}_h^{(2)} := (\frac{\partial^2 u_{h,1}}{\partial x_1 \partial x_2}, \frac{\partial^2 u_{h,1}}{\partial x_2^2})^T.$$

Moreover, we define $\underline{\mathbf{z}}_h^{(1,\omega_i)}$ according to

$$\underline{\mathbf{z}}_h^{1,\omega_i} := \varphi_h^{(x_i)} \underline{\mathbf{z}}_h^{(1)}. \tag{5.7}$$

Case 1 ($x_i \in \mathcal{E}_h(\partial E \cap \Omega)$): For $\ell = 1, 2, \cdots, N_i$ we compute $\underline{\mathbf{p}}_{h,m}^{\omega_i}|_{K_\ell} \in \mathbf{BDM}_{k-1}(K_\ell)$ according to

$$\int_{E_\ell} \underline{\mathbf{p}}_{h,1}^{\omega_i}|_{K_\ell} \mathbf{n}_{E_\ell \cap K_\ell}\ p_{k-1}\, ds\ = \tag{5.8a}$$

$$\begin{cases} \int_{E_\ell} \left(\mathbf{n}_{E_\ell} \cdot \hat{\underline{\mathbf{p}}}_{\partial K_\ell}^{m,\omega_i}\right)|_{E_\ell}\ , \ell = 1 \\ \int_{E_\ell} \mathbf{n}_{E_\ell \cap K_\ell} \cdot \underline{\mathbf{p}}_{h,1}^{\omega_i}|_{K_\ell}\ p_{k-1}\, ds\ , \ell = 2, 3, \cdots, N_i \end{cases} p_{k-1} \in P_{k-1}(E_\ell)\ ,$$

$$\int_{E_{\ell+1}} \mathbf{n}_{E_{\ell+1} \cap K_\ell} \cdot \underline{\mathbf{p}}_{h,1}^{\omega_i}|_{K_\ell}\ p_{k-1}\, ds\ = \tag{5.8b}$$

$$\begin{cases} \int_{E_{\ell+1}} \mathbf{n}_{E_{\ell+1}} \cdot \hat{\underline{\mathbf{p}}}_{\partial K_\ell}^{1,\omega_i}|_{E_{\ell+1}}\ p_{k-1}\, ds\ , \ell = N_i \\ \int_{E_\ell} \mathbf{n}_{E_\ell} \cdot \hat{\underline{\mathbf{p}}}_{\partial K}^{m,\omega_i} \mathbf{n}_{E_\ell}\ p_{k-1}\, ds\ , \ell = 1, 2, \cdots, N_i \end{cases} \quad p_{k-1} \in P_{k-1}(E_{\ell+1})\ ,$$

$$p_{k-1} \in P_{k-1}(E_{\ell+1}),$$

$$\mathbf{n}_E \cdot \underline{\mathbf{p}}_{h,1}^{\omega_i} = 0,\ E \in \mathcal{E}_h(K_\ell \cap \partial\omega_i)\ , \tag{5.8c}$$

$$\int_{K_\ell} \underline{\mathbf{p}}_{h,1}^{\omega_i}|_{K_\ell} \cdot \nabla p_{k-2}\, dx = -\int_{K_\ell} \left(\nabla \cdot \underline{\mathbf{p}}_{h,1}^{\omega_i} + \text{cof}(D^2 u_{h,1}) : D^2 u_{h,2} - f_h\right) p_{k-2}\, dx + \tag{5.8d}$$

$$\int_{\partial K_\ell} \mathbf{n}_{\partial K_\ell} \cdot \underline{\mathbf{p}}^{\omega_i}_{h,m}|_{K_\ell}\, p_{k-2}\, ds\ ,\ \ p_{k-2} \in P_{k-2}(K_\ell),$$

$$\int_{K_\ell} \underline{\mathbf{p}}^{\omega_i}_{h,2}|_{K_\ell} \cdot \nabla p_{k-2}\, dx = -\int_{K_\ell} \left(\nabla \cdot \underline{\mathbf{p}}^{\omega_i}_{h,2} + \mathrm{cof}(D^2 u_{h,1}) : D^2 u_{h,1}\right) p_{k-2}\, dx + \tag{5.8e}$$

$$\int_{\partial K_\ell} \mathbf{n}_{\partial K_\ell} \cdot \underline{\mathbf{p}}^{\omega_i}_{h,2}|_{K_\ell}\, p_{k-2}\, ds\ ,\ \ p_{k-2} \in P_{k-2}(K_\ell),$$

$$\int_{K_\ell} \underline{\mathbf{p}}^{\omega_i}_{h,m}|_{K_\ell} \cdot \mathbf{curl}(b_{K_\ell} p_{k-3})\, dx = \int_{K_\ell} \underline{\mathbf{z}}_h^{(m,\omega_i)} \cdot \mathbf{curl}(b_{K_\ell} p_{k-3})\, dx\ ,\ 1 \le m \le 2, \tag{5.8f}$$

$$p_{k-3} \in P_{k-3}(K_\ell).$$

The cases Case 2 - Case 5 can be treated in a similar way.

For the construction of the equilibrated moment tensors $\underline{\underline{\mathbf{p}}}^{eq}_{h,i}$, $1 \le i \le 2$, we begin with the specification of the degrees of freedom for tensors $\underline{\underline{\mathbf{p}}} = (p_{ij})_{i,j=1}^d \in \underline{\underline{\mathbf{V}}}_h$. We note that

$$\dim P_k(K)^{2\times 2} = 2(k+1)(k+2). \tag{5.9}$$

Lemma 5.2. *Any $\underline{\underline{\mathbf{p}}} \in P_k(K)^{2\times 2}$ with $\underline{\mathbf{p}}^{(i)} = (p_{i1}, p_{i2})^T$, $1 \le i \le 2$, is uniquely determined by the following degrees of freedom (DOF)*

$$\int_E \underline{\underline{\mathbf{p}}}\, \mathbf{n}_E \cdot \underline{\mathbf{p}}_k\, ds, \quad \underline{\mathbf{p}}_k \in P_k(E)^2,\ E \in \mathcal{E}_h(\partial K), \tag{5.10a}$$

$$\int_K \underline{\underline{\mathbf{p}}} : \nabla \underline{\mathbf{p}}_{k-1}\, dx, \quad \underline{\mathbf{p}}_{k-1} \in P_{k-1}(K)^2 \backslash P_0(K)^2, \tag{5.10b}$$

$$\int_K \underline{\mathbf{p}}^{(i)} \cdot \mathbf{curl}(b_K\, p_{k-2})\, dx, \quad p_{k-2} \in P_{k-2}(K),\ 1 \le i \le 2. \tag{5.10c}$$

The numbers of degrees of freedom (DOF) associated with (5.10a)-(5.10c) are as follows

$$DOF\,(5.10a) = 6(k+1),$$
$$DOF\,(5.10b) = k(k+1) - 2,$$
$$DOF\,(5.10c) = (k-1)k$$

and sum up to the right-hand side in (5.9).

Proof. The interpolation conditions for $\underline{\mathbf{p}}^{(1)}$ and $\underline{\mathbf{p}}^{(2)}$ are separated. The vector field $\underline{\mathbf{p}}^{(i)}$ (for $1 \le i \le 2$) is determined by the degrees of freedom

$$\int_E \mathbf{n}_E \cdot \underline{\mathbf{p}}^{(i)} p_k\, ds, \quad p_k \in P_k(E),\ E \in \mathcal{E}_h(\partial K),$$

$$\int_K \underline{\mathbf{p}}^{(i)} \cdot \nabla p_{k-1}\, dx, \quad p_{k-1} \in P_{k-1}(K) \backslash P_0(K),$$

$$\int_K \underline{\mathbf{p}}^{(i)} \cdot \mathrm{curl}(b_K\, p_{k-2})\, dx, \quad p_{k-2} \in P_{k-2}(K)\ .$$

By applying Lemma 5.1 we conclude that there is a unique solution. □

Lemma 5.3. *Let $\underline{\underline{\mathbf{q}}} = (\underline{\underline{\mathbf{q}}}^{(1)}, \underline{\underline{\mathbf{q}}}^{(2)}) \in P_k(K)^{2\times 2}$. Then there exists a positive constant $C_E^{(2)}$, depending only on the polynomial degree k and the local geometry of the triangulation $\mathcal{T}_h$, such that*

$$\int_K |\underline{\underline{\mathbf{q}}}|^2\,dx \le C_E^{(2)} \Big(\sum_{E\in\mathcal{E}_h(\partial K)} h_E \int_E |\underline{\underline{\mathbf{q}}}\,\mathbf{n}_E|^2\,ds + h_K^2 \int_K |\nabla\cdot\underline{\underline{\mathbf{q}}}|^2\,dx + \tag{5.11}$$

$$h_K^2 \sum_{i=1}^{2} max\Big\{ \int_K |\underline{\mathbf{q}}^{(i)}\cdot\mathbf{curl}(b_K p_{k-2})|^2\,dx;\; p_{k-2}\in P_{k-2}, \max_{x\in K}|p_{k-2}(x)|\le 1\Big\}\Big).$$

Proof. As in the proof of Lemma 5.1, the estimate (5.11) follows by standard scaling arguments. □

Now, for the construction of the equilibrated moment tensor $\underline{\underline{\mathbf{p}}}_{h,1}^{eq}$ we set

$$\underline{\mathbf{z}}_h^{(1)} := (\frac{\partial^2 u_{h,1}}{\partial x_1^2}, \frac{\partial^2 u_{h,1}}{\partial x_1\partial x_2})^T, \quad \underline{\mathbf{z}}_h^{(2)} := (\frac{\partial^2 u_{h,1}}{\partial x_1\partial x_2}, \frac{\partial^2 u_{h,1}}{\partial x_2^2})^T.$$

We construct $\underline{\underline{\mathbf{p}}}_{h,m}^{eq} = (p_{ij}^{h,m,eq})_{i,j=1}^2$, $1\le m\le 2$, with $\underline{\mathbf{p}}_{h,m,eq}^{(i)} = (p_{i1}^{h,m,eq}, p_{i2}^{h,m,eq})^T$, $1\le i\le 2$, patchwise:

$$\underline{\underline{\mathbf{p}}}_{h,m}^{eq} = \sum_{i=1}^{n_h} \underline{\underline{\mathbf{p}}}_{h,m}^{\omega_i}. \tag{5.12}$$

For a patch ω_i, we construct $\underline{\underline{\mathbf{p}}}_{h,m}^{\omega_i}$ such that

$$\underline{\underline{\mathbf{p}}}_{h,m}^{\omega_i}|_{K_\ell} \in \mathbf{BDM}_k(K_\ell)\,, \tag{5.13}$$

$$\nabla\cdot\underline{\underline{\mathbf{p}}}_{h,m}^{\omega_i} = \underline{\mathbf{p}}_{h,m}^{\omega_i} \text{ in } \omega_i, \quad \underline{\underline{\mathbf{p}}}_{h,m}^{\omega_i}\mathbf{n}_E = \hat{\underline{\mathbf{p}}}_{\partial K}^{m,\omega_i}|_E,\; E\in\mathcal{E}_h(\text{int}\,\omega_i),\; 1\le\ell\le N_i,$$

where, denoting by $\varphi_h^{(x_i)}\in V_h^{(k)}$ the nodal basis function associated with x_i, $\underline{\mathbf{p}}_{h,m}^{\omega_i}$ and $\hat{\underline{\mathbf{p}}}_{\partial K}^{m,\omega_i}|_E$ are given by

$$\underline{\mathbf{p}}_{h,m}^{\omega_i} := \varphi_h^{(x_i)}\underline{\mathbf{p}}_{h,m}^{eq}, \quad \hat{\underline{\underline{\mathbf{p}}}}_{\partial K}^{m,\omega_i}|_E\mathbf{n}_E := \varphi_h^{(x_i)}\hat{\underline{\mathbf{p}}}_{\partial K}^{(m,1)}|_E\;,\; 1\le i\le n_h. \tag{5.14}$$

Moreover, we define $\underline{\mathbf{z}}_h^{(\ell,\omega_i)}$ according to

$$\underline{\mathbf{z}}_h^{\ell,\omega_i} := \varphi_h^{(x_i)}\underline{\mathbf{z}}_h^{(\ell)},\; 1\le\ell\le 2. \tag{5.15}$$

Case 1 ($x_i\in\mathcal{E}_h(\partial E\cap\Omega)$): For $\ell=1,2,\cdots,N_i$ we compute $\underline{\underline{\mathbf{p}}}_{h,m}^{\omega_i}|_{K_\ell}$ with $\underline{\underline{\mathbf{p}}}_{h,m}^{\omega_i}|_{K_\ell}\in\mathbf{BDM}_k(K_\ell)$, $1\le m\le 2$, according to

$$\int_{E_\ell} \underline{\underline{\mathbf{p}}}_{h,m}^{\omega_i}|_{K_\ell}\mathbf{n}_{E_\ell\cap K_\ell}\cdot\underline{\mathbf{p}}_k\,ds = \tag{5.16a}$$

$$\begin{cases} \int_{E_\ell} \Big(\hat{\underline{\mathbf{p}}}_{\partial K_\ell}^{m,\omega_i}\Big)|_{E_\ell} & ,\ell=1 \\ \int_{E_\ell} \underline{\underline{\mathbf{p}}}_{h,m}^{\omega_i}|_{K_\ell}\mathbf{n}_{E_\ell\cap K_\ell}\cdot\underline{\mathbf{p}}_k\,ds & ,\ell=2,3,\cdots,N_i \end{cases} \underline{\mathbf{p}}_k\in P_k(E_\ell)^2\,,$$

$$\int_{E_{\ell+1}} \underline{\underline{\mathbf{p}}}_{h,m}^{\omega_i}|_{K_\ell}\mathbf{n}_{E_{\ell+1}\cap K_\ell}\cdot\underline{\mathbf{p}}_k\,ds = \tag{5.16b}$$

$$\begin{cases} \int_{E_{\ell+1}} \cdot\hat{\underline{\mathbf{p}}}_{\partial K_\ell}^{m,\omega_i}|_{E_{\ell+1}}\cdot\underline{\mathbf{p}}_k\,ds & ,\ell=N_i \\ \int_{E_\ell} \hat{\underline{\underline{\mathbf{p}}}}_{\partial K}^{m,\omega_i}\mathbf{n}_{E_\ell}\cdot\underline{\mathbf{p}}_k\,ds, \underline{\mathbf{p}}_k\in P_k(E_{\ell+1})^2, \ell=1,2,\cdots,N_i, & \end{cases} \underline{\mathbf{p}}_k\in P_k(E_{\ell+1})^2,$$

$$\underline{\underline{\mathbf{p}}}_{h,m}^{\omega_i}\,\mathbf{n}_E = \mathbf{0},\; E\in\mathcal{E}_h(K_\ell\cap\partial\omega_i)\,, \tag{5.16c}$$

$$\int_{K_\ell} \underline{\underline{\mathbf{p}}}_{h,m}^{\omega_i}|_{K_\ell} : \nabla\underline{\mathbf{p}}_{k-1}\,dx = -\int_{K_\ell} \underline{\mathbf{p}}_{h,m}^{\omega_i}\cdot\underline{\mathbf{p}}_{k-1}\,dx + \tag{5.16d}$$

$$\int_{\partial K_\ell} \underline{\underline{\mathbf{p}}}_{h,m}^{\omega_i}|_{K_\ell}\mathbf{n}_{\partial K_\ell}\cdot\underline{\mathbf{p}}_{k-1}\,ds\;,\; \underline{\mathbf{p}}_{k-1}\in P_{k-1}(K_\ell)^2,$$

$$\int_{K_\ell} \underline{\underline{\mathbf{p}}}_{h,m}^{(m,\omega_i)}|_{K_\ell} \cdot \mathbf{curl}(b_{K_\ell} p_{k-2})\, dx = \int_{K_\ell} \underline{\mathbf{z}}_{h,m}^{(j,\omega_i)} \cdot \mathbf{curl}(b_{K_\ell} p_{k-2})\, dx\,, \tag{5.16e}$$

$1 \le j \le 2,\ p_{k-2} \in P_{k-2}(K_\ell)$.

Again, the cases Case 2 - Case 5 can be done analogously.

6. Relationship with a residual-type a posteriori error estimator

The residual-type a posteriori error estimator for the von Kármán equations with homogeneous Dirichlet boundary conditions reads as follows

$$\eta_h^{res} := \sum_{i=1}^{8} \sum_{K\in\mathcal{T}_h} \eta_{K,i}^{res} + \sum_{i=1}^{6} \sum_{K\in\mathcal{T}_h} \widetilde{\eta}_{K,i}^{res}. \tag{6.1}$$

The element residuals $\eta_{K,i}^{res}, 1 \le i \le 8$, and $\tilde{\eta}_{K,i}^{res}, 1 \le i \le 6$, are given by

$$\eta_{K,1}^{res} := \int_K |\Delta^2 u_{h,1} - \mathrm{cof}(D^2 u_{h,1} : D^2 u_{h,2} - f_h|^2\, dx, \tag{6.2a}$$

$$\eta_{K,2}^{res} := \sum_{K\in\mathcal{T}_h} h_K^2 \int_K |\Delta^2 u_{h,2} + \frac{1}{2}\mathrm{cof}(D^2 u_{h,1} : D^2 u_{h,1}|^2\, dx, \tag{6.2b}$$

$$\eta_{K,i+2}^{res} := \kappa_E \sum_{E\in\mathcal{E}_h(K)} h_E \int_E |\mathbf{n}_E \cdot [\nabla \cdot D^2 u_{h,i}|^2\, ds,\ 1 \le i \le 2, \tag{6.2c}$$

$$\eta_{K,i+4}^{res} := \kappa_E \sum_{E\in\mathcal{E}_h(K)} h_E \int_E |[D^2 u_{h,i}]_E \mathbf{n}_E|^2\, ds,\ 1 \le i \le 2, \tag{6.2d}$$

$$\eta_{K,i+6}^{res} := \kappa_E \sum_{E\in\mathcal{E}_h(K)} \int_E |[\nabla u_{h,i} \otimes \mathbf{n}_E]_E|^2\, ds,\ 1 \le i \le 2, \tag{6.2e}$$

$$\widetilde{\eta}_{K,i}^{res} := (\eta_{K,i})^{1/2} |D^2 u_h, 1|_{DG,\Omega},\ i = 1, 3, 5, \tag{6.2f}$$

$$\widetilde{\eta}_{K,i}^{res} := (\eta_{K,i})^{1/2} |D^2 u_h, 2|_{DG,\Omega},\ i = 2, 4, 6, \tag{6.2g}$$

where

$$\kappa_E := \begin{cases} \frac{1}{2} & , E \in \mathcal{E}_h(\Omega) \\ 1 & , E \in \mathcal{E}_h(\Gamma) \end{cases}, \tag{6.2h}$$

and

$$|D^2 u_{h,i}|_{DG,\Omega} := \Big(\sum_{K\in\mathcal{T}_h} \int_K |D^2 u_{h,i}|^2\, dx \Big)^{1/2},\ 1 \le i \le 2. \tag{6.2i}$$

We further define data oscillations $\widetilde{\mathrm{osc}}_{h,i}, 1 \le i \le 2$, according to

$$\widetilde{\mathrm{osc}}_{h,1} := \begin{cases} (\mathrm{osc}_{h,1})^{1/2} (|\nabla u_{h,1}|_{DG,\Omega} + |\nabla u_{h,2}|_{DG,\Omega}) & , k = 2 \\ (\mathrm{osc}_{h,1})^{1/2} (|D^2 u_{h,1}|_{DG,\Omega} + |D^2 u_{h,2}|_{DG,\Omega}) & , k \ge 3 \end{cases}, \tag{6.2j}$$

$$\widetilde{\mathrm{osc}}_{h,2} := (\mathrm{osc}_{h,2})^{1/2} (|D^2 u_{h,1}|_{DG,\Gamma} + |D^2 u_{h,2}|_{DG,\Gamma}), \tag{6.2k}$$

where $|\nabla u_{h,i}|_{DG,\Omega}$ and $|D^2 u_{h,i}|_{DG,\Gamma}, 1 \le i \le 2$, are given by

$$|\nabla u_{h,i}|_{DG,\Omega} := \Big(\sum_{K\in\mathcal{T}_h} \int_K |\nabla u_{h,i}|^2\, dx \Big)^{1/2}, \tag{6.2l}$$

$$|D^2 u_{h,i}|_{DG,\Gamma} := \Big(\sum_{E\in\mathcal{E}_h(\Gamma)} \int_E |D^2 u_{h,i}|^2\, ds \Big)^{1/2}. \tag{6.2m}$$

The following result establishes the relationship between the equilibrated and the residual a posteriori error estimator.

Theorem 6.1. *Let $\mathbf{u}_h \in \underline{\mathbf{V}}_h$ be the C^0IPDG approximation as given by (3.22) and let $\eta_h^{eq}, \eta_{h,i}^{res}, 1 \le i \le 8, \widetilde{\eta}_{h,i}^{res}, 1 \le i \le 6$, and $osc_{h,i}, 1 \le i \le 3$, and $\widetilde{osc}_{h,i}, 1 \le i \le 2$, be the equilibrated and the residual a posteriori error estimators as well as the data oscillations as given by (4.30), (6.2), and (4.22). Then there exists a constant $C_{res} > 0$, depending on $c_R, C_{rec}, \alpha_i, C_E^{(i)}, C_{PF}^{(i)}, 1 \le i \le 2$, such that*

$$\eta_{h,1}^{eq} \le C_{res}\Big(\sum_{i=1}^{8} \eta_{h,i}^{res} + \sum_{i=1}^{6} \widetilde{\eta}_{h,i}^{res} + \sum_{i=1}^{3} osc_{h,i} + \sum_{i=1}^{2} \widetilde{osc}_{h,i}\Big). \tag{6.3}$$

Moreover, if we use (3.29b) in (4.29b), then $\eta_{h,2}^{eq}$ can be estimated from above in terms of residuals and data oscillations.

The proof of Theorem 6.1 can be done by standard means.

Remark 6.2. Using techniques from [34], it can be shown that the residual-based error estimator is efficient.

7. Numerical results

We have implemented the C^0IPDG approximation (3.22) with the penalty parameters $\alpha_i, 1 \le i \le 2$, chosen as $\alpha_1 = 12.0\ k^2$ and $\alpha_2 = 2.5\ k^6$. Further, we have implemented the adaptive algorithm based on the equilibrated error estimator η_h^{eq} by Dörfler marking [20], i.e., given a bulk parameter $\Theta \in (0,1)$, we have selected a set $\mathcal{M}_h \subset \mathcal{T}_h$ according to

$$\Theta \sum_{K \in \mathcal{T}_h} \eta_K^{eq} \le \sum_{K \in \mathcal{M}_h} \eta_K^{eq}$$

and we have refined elements $K \in \mathcal{M}_h$ by newest vertex bisection. In case of the residual-based error estimator η_h^{res} we have implemented the adaptive refinement likewise.

As numerical example, we have chosen Ω as the L-shaped domain $\Omega := (-1,+1)^2 \setminus ([0,1) \times (-1,0])$. We have chosen $f \in L^2(\Omega)$ such that the solution given by (cf. Example 6.2 in [13]

$$\begin{aligned}
&u_1(r,\varphi) = u_2(r,\varphi) = r^{1+\gamma}\ (r^2\cos^2(\varphi) - 1)^2\ (r^2\sin^2(\varphi) - 1)^2\ g(\varphi),\\
&g(\varphi) := \Big(\frac{1}{\gamma - 1}\ \sin(3(\gamma - 1)/2\pi) - \frac{1}{\gamma + 1}\ \sin(3(\gamma + 1)/2\pi)\cos((\gamma - 1)\varphi) -\\
&\cos((\gamma - 1)\varphi)\Big) - \Big(\frac{1}{\gamma - 1}\ \sin((\gamma - 1)\varphi) - \frac{1}{\gamma + 1}\ \sin((\gamma + 1)\varphi)\\
&(\cos(3(\gamma - 1)/2\pi) - \cos(3(\gamma + 1)/2\pi))\Big),
\end{aligned}$$

where $\gamma \approx 0.5444837367$ is a non-characteristic root of $\gamma^2\ \sin^2(3\pi/2)$.
We note that the solution belongs to $W^{2+\gamma-\varepsilon,2}(\Omega)$ for any $\varepsilon > 0$. A similar regularity applies to the solution z of the boundary value problem (4.15).

We have performed computations for the polynomial degrees $k = 2, k = 3$, and $k = 4$. The numerical solution of the nonlinear C^0IPDG approximation (3.22) has been done by Newton's method with a relative tolerance of $\text{tol} = 10^{-3}$ as termination criterion for the Newton iterates. The expected convergence rate for the discretization error in the broken $W_0^{2,2}$ norm is 0.5.

Fig. 2 shows the adaptively generated meshes (bulk parameter $\Theta = 0.4$) for polynomial degree $k = 2$ (top left), $k = 3$ (top right), and $k = 4$ (bottom), where the adaptive mesh refinements were based on the equilibrated error estimator. As expected, we observe a pronounced refinement around the reentrant corner at the origin and substantially less refinement off the singularity for the higher polynomial degrees $k \ge 3$. The meshes obtained by the residual-based error estimator look similarly and are therefore omitted.

For $\Theta = 0.4$, Fig. 3 displays the discretization error in the broken $W_0^{2,2}$ norm, the equilibrated error estimator η_h^{eq}, and the residual-based error estimator η_h^{res} as a function of the total number of degrees of freedom (DOFs) on a logarithmic scale. The result for the polynomial degree $k = 2$ is depicted top left, those for the polynomial degrees $k = 3$ and $k = 4$ top right and bottom. In all cases we observe the optimal convergence of 0.5. For $k = 3$ and $k = 4$ the decay is faster than 0.5 in the pre-asymptotic regime, but approaches 0.5 asymptotically. The equilibrated error estimator is smaller than the residual-based error estimator by approximately 3/4 of an order of magnitude.

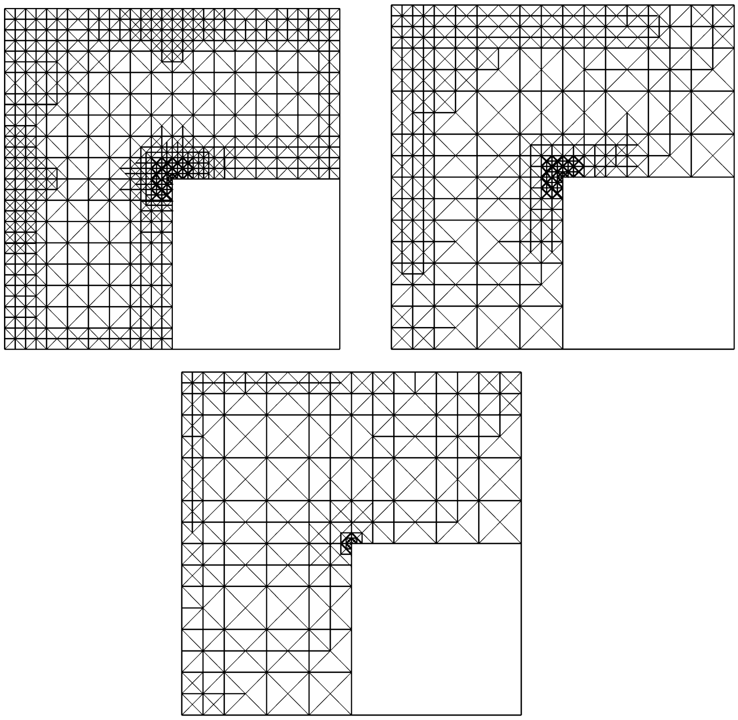

Fig. 2. Adaptively generated meshes (equilibrated error estimator) for polynomial degree $k = 2$ (top left), $k = 3$ (top right), and $k = 4$ (bottom).

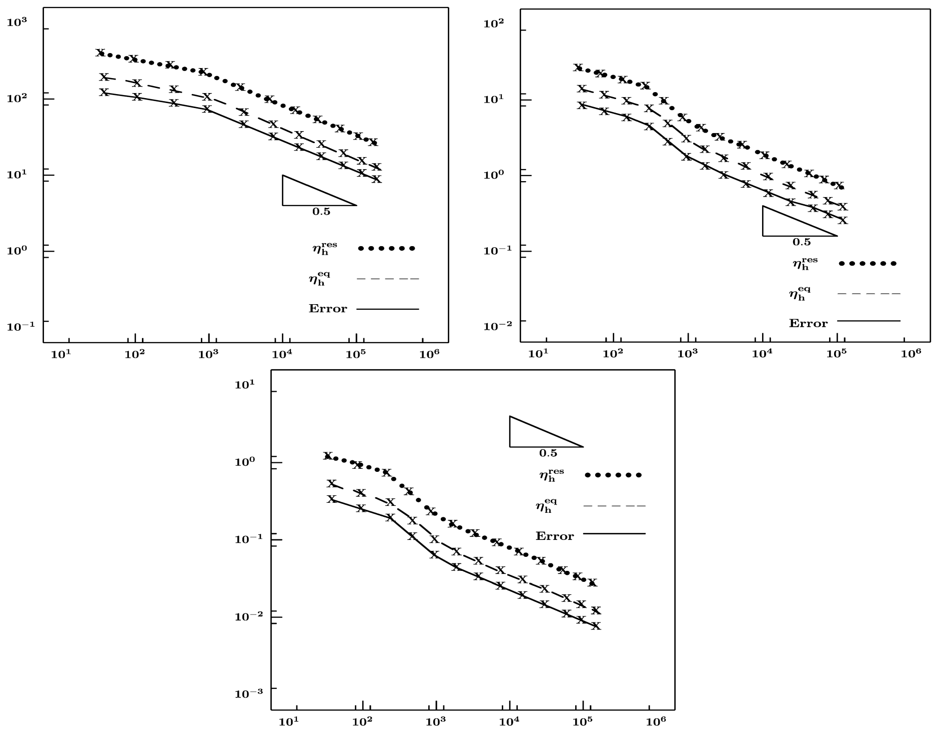

Fig. 3. The error in the broken $W_0^{2,2}$ norm (straight line), the equilibrated error estimator η_h^{eq} (dashed), and the residual-based error estimator η_h^{res} (dotted line) for polynomial degree $k = 2$ (top left), $k = 3$ (top right), and $k = 4$ (bottom).

Appendix A

Proof of Theorem 3.1.
Let $\mathbf{u}_h$ be the solution of (3.22). We then define $\underline{\underline{\mathbf{p}}}_{h,i} \in \underline{\underline{\mathbf{V}}}_h$, $1 \le i \le 2$, by means of (3.18a), (3.19a) and afterwards $\underline{\mathbf{p}}_{h,i} \in \underline{\mathbf{V}}_h$, $1 \le i \le 2$, according to (3.18b), (3.19b) We choose $\underline{\underline{\mathbf{q}}}_h = D^2 v_h$ in (3.18a), (3.19a) and $\underline{\boldsymbol{\varphi}}_h = \nabla v_h$ in (3.18b), (3.19b) and insert the resulting expressions into (3.18c), (3.19c) observing (3.20), (3.21). It follows that

$$\begin{aligned}
&\sum_{K\in\mathcal{T}_h}\int_K \nabla\cdot\underline{\mathbf{p}}_{h,1}\, v_h\,dx = \sum_{K\in\mathcal{T}_h}\int_K \underline{\underline{\mathbf{p}}}_{h,1} : D^2 v_h\,dx - \sum_{K\in\mathcal{T}_h}\int_{\partial K} \underline{\hat{\mathbf{p}}}_{\partial K}^{(1,1)}\cdot\nabla v_h\,ds + \\
&\sum_{K\in\mathcal{T}_h}\int_{\partial K} \mathbf{n}_{\partial K}\cdot\underline{\hat{\mathbf{p}}}_{\partial K}^{(1,2)}\, v_h\,ds = \sum_{K\in\mathcal{T}_h}\int_K D^2 u_{h,1} : D^2 v_h\,dx - \\
&\sum_{E\in\mathcal{E}_h(\Omega)}\int_E \{D^2 u_{h,1}\}_E : [\nabla v_h\otimes\mathbf{n}_E]_E\,ds + \\
&\sum_{E\in\mathcal{E}_h(\bar\Omega)}\int_E \mathbf{n}_E\cdot\{\nabla\cdot\underline{\underline{\boldsymbol{\Pi}}}_k(D^2 u_{h,1})\}_E\, v_h\,ds + \\
&\alpha_1\sum_{E\in\mathcal{E}_h(\Omega)} h_E^{-1}\int_E [\nabla u_{h,1}\otimes\mathbf{n}_E]_E : [\nabla v_h\otimes\mathbf{n}_E]_E\,ds + \\
&\alpha_2\sum_{E\in\mathcal{E}_h(\Gamma)} h_E^{-3}\int_E u_{h,1}\, v_h\,ds
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{K\in\mathcal{T}_h}\int_K \nabla\cdot\underline{\mathbf{p}}_{h,2}\, v_h\,dx = \sum_{K\in\mathcal{T}_h}\int_K \underline{\underline{\mathbf{p}}}_{h,2} : D^2 v_h\,dx - \sum_{K\in\mathcal{T}_h}\int_{\partial K} \underline{\hat{\mathbf{p}}}_{\partial K}^{(2,1)}\cdot\nabla v_h\,ds + \\
&\sum_{K\in\mathcal{T}_h}\int_{\partial K} \mathbf{n}_{\partial K}\cdot\underline{\hat{\mathbf{p}}}_{\partial K}^{(2,2}\, v_h\,ds = \sum_{K\in\mathcal{T}_h}\int_K D^2 u_{h,2} : D^2 v_h\,dx - \\
&\sum_{E\in\mathcal{E}_h(\Omega)}\int_E \{D^2 u_{h,2}\}_E : [\nabla v_h\otimes\mathbf{n}_E]_E\,ds + \\
&\sum_{E\in\mathcal{E}_h(\bar\Omega)}\int_E \mathbf{n}_E\cdot\{\nabla\cdot\underline{\underline{\boldsymbol{\Pi}}}_k(D^2 u_{h,2})\}_E\, v_h\,ds + \\
&\alpha_1\sum_{E\in\mathcal{E}_h(\Omega)} h_E^{-1}\int_E [\nabla u_{h,2}\otimes\mathbf{n}_E]_E : [\nabla v_h\otimes\mathbf{n}_E]_E\,ds + \\
&\alpha_2\sum_{E\in\mathcal{E}_h(\Gamma)} h_E^{-3}\int_E u_{h,2}\, v_h\,ds.
\end{aligned}$$

In view of (3.22) and (3.23) we deduce that the last equation in (3.18) and (3.19) is satisfied. Conversely, if the triples $(\underline{\underline{\mathbf{p}}}_{h,i}, \underline{\mathbf{p}}_{h,i}, u_{h,i}) \in \underline{\underline{\mathbf{V}}}_h \times \underline{\mathbf{V}}_h \times V_h$, $1 \le i \le 2$, satisfy (3.18), (3.19), we choose $\underline{\underline{\mathbf{q}}}_h = D^2 v_h$ in (3.18a), (3.19a), and $\underline{\boldsymbol{\varphi}}_h = \nabla v_h$ in (3.18b), (3.19b), and insert them into (3.18c), (3.19c). Taking (3.20) and (3.21) into account this shows that $\mathbf{u}_h$ satisfies (3.22).

Appendix B

Proof of Lemma 4.1.
We have

$$\|\underline{\underline{\mathbf{p}}}_{c,1}\|_{L^2(\Omega;\mathbb{R}^{2\times 2})} = \sup\{\int_\Omega \underline{\underline{\mathbf{p}}}_{c,1} : \underline{\underline{\boldsymbol{\tau}}}\,dx \mid \underline{\underline{\boldsymbol{\tau}}} \in \underline{\underline{\mathbf{H}}}(\mathrm{div}^2,\Omega),\ \|\underline{\underline{\boldsymbol{\tau}}}\|_{L^2(\Omega;\mathbb{R}^{2\times 2})} \le 1\}.$$

For $\underline{\underline{\boldsymbol{\tau}}} \in \underline{\underline{\mathbf{H}}}(\mathrm{div}^2,\Omega)$ there exists $v \in W_0^{2,2}(\Omega)$ such that $\underline{\underline{\boldsymbol{\tau}}} = D^2 v$. In fact, v can be chosen as the weak solution of the boundary value problem (4.15). Hence, we have

$$\|\underline{\underline{\mathbf{p}}}_{c,1}\|_{L^2(\Omega,\mathbb{R}^{2\times 2})} \le \sup_{\|D^2 v\|_{L^2(\Omega;\mathbb{R}^{2\times 2})}\le 1} \int_\Omega \underline{\underline{\mathbf{p}}}_{c,1} : D^2 v \, dx. \tag{B.1}$$

Applying Green's formula twice locally on each $K \in \mathcal{T}_h$ and observing (4.10), we get

$$\int_\Omega \underline{\underline{\mathbf{p}}}_{c,1} : D^2 z \, dx = \sum_{K\in\mathcal{T}_h} \int_K \nabla\cdot\nabla\cdot \underline{\underline{\mathbf{p}}}_{c,1} \, z \, dx + \sum_{E\in\mathcal{E}_h(\Gamma)} \int_E \underline{\underline{\mathbf{p}}}_{c,1} \mathbf{n}_\Gamma \cdot \nabla z \, ds = \sum_{K\in\mathcal{T}_h} \int_K (f-f_h)\, z \, dx. \tag{B.2}$$

In order to estimate the first term on the right-hand side of (B.2) we first consider the case $k=2$. In view of the choice of f_h we have

$$\sum_{K\in\mathcal{T}_h} \int_K (f-f_h)\, z\, dx = \sum_{K\in\mathcal{T}_h} \int_K (f-f_h)\,(z-p_0)\, dx,$$

where $p_0 := |K|^{-1} \int_K z \, dx$, and hence, an application of (4.14a) and (4.17) yields

$$\Big|\sum_{K\in\mathcal{T}_h} \int_K (f-f_h)\, z\, dx\Big| \le \sum_{K\in\mathcal{T}_h} \Big(\int_K |f-f_h|^2\, dx\Big)^{1/q} \Big(\int_K |z-p_0|^2\, dx\Big)^{1/2} \le C_{PF}^{(1)} \Big(\sum_{K\in\mathcal{T}_h} h_K^2 \int_K |f-f_h|^2\, dx\Big)^{1/2} \Big(\sum_{K\in\mathcal{T}_h} \int_K |\nabla z|^2\, dx\Big)^{1/2} \le C_z^{(2)} C_{PF}^{(1)}\, osc_{h,1}^{1/2}. \tag{B.3}$$

In case $k \ge 3$ we have

$$\sum_{K\in\mathcal{T}_h} \int_K (f-f_h)\, z\, dx = \sum_{K\in\mathcal{T}_h} \int_K (f-f_h)\,(z-p_1)\, dx, \quad p_1 \in P_1(K).$$

We fix $p_1 \in P_1(K)$ by the interpolation conditions $\int_K p_1 dx = |K|^{-1}\int_K z\, dx$ and $\int_K \nabla p_1 dx = |K|^{-1}\int_K \nabla z\, dx$. An application of (4.14a) gives

$$\Big|\sum_{K\in\mathcal{T}_h} \int_K (f-f_h)\, z\, dx\Big| \le \sum_{K\in\mathcal{T}_h} \Big(\int_K |f-f_h|^2\, dx\Big)^{1/q} \Big(\int_K |z-p_1|^2\, dx\Big)^{1/2} \le C_{PF}^{(1)} \sum_{K\in\mathcal{T}_h} \Big(h_K^2 \int_K |f-f_h|^2\, dx\Big)^{1/q} \Big(\int_K |\nabla(z-p_1)|^2\, dx\Big)^{1/2}. \tag{B.4}$$

Setting $\nabla p_1 = (p_{11}, p_{12})^T$, another application of (4.14a) yields

$$\|\frac{\partial z}{\partial x_i} - p_{1i}\|_{L^2(K)} \le C_{PF}^{(1)}\, h_K\, \|\nabla \frac{\partial z}{\partial x_i}\|_{L^2(K)}, \; 1\le i\le 2.$$

Hence, using (2.2), we obtain

$$\Big(\int_K |\nabla z - |K|^{-1} \int_K |\nabla(z-p_1)|^2\, dx\Big)^{1/2} \le 4\Big(\Big(\int_K |\frac{\partial z}{\partial x_1} - p_{11}|^p\, dx\Big)^{1/2} + \Big(\int_K |\frac{\partial z}{\partial x_2} - p_{12}|^2|\Big)\, dx\Big)^{1/2}\Big) \le \tag{B.5}$$

$$4C^{(1)}_{PF} h_K \Big(\int_K |D^2 z|^2 \, dx \Big)^{1/2}.$$

Using (B.5) in (B.4) and observing $\|D^2 z\|_{L^2(\Omega;\mathbb{R}^{2\times 2})} \le 1$ it follows that

$$\Big| \sum_{K\in\mathcal{T}_h} \int_K (f - f_h)\, z \, dx \Big| \le \tag{B.6}$$

$$4(C^{(1)}_{PF})^2 \Big(\sum_{K\in\mathcal{T}_h} h_K^4 \int_K |f - f_h|^2 \, dx \Big)^{1/2} \Big(\sum_{K\in\mathcal{T}_h} \int_K |D^2 z|^2 \, dx \Big)^{1/2} \le$$

$$4(C^{(1)}_{PF})^2 \Big(\sum_{K\in\mathcal{T}_h} h_K^4 \int_K |f - f_h|^2 \, dx \Big)^{1/2}.$$

The assertion now follows from (B.3) and (B.6).

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Further reading

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