The curvature of SU (5)/(Sp (2)xS1)

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Angaben zur Veröffentlichung / Publication details:

Heintze, Ernst. 1971. "The curvature of SU (5)/(Sp (2)xS1)." *Inventiones Mathematicae* 13 (3): 205–12. https://doi.org/10.1007/bf01404630.

Inventiones math. 13, 205–212 (1971)

The Curvature of $SU(5)/(Sp(2)\times S^1)$

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1. Introduction

Berger ([1]) classified all normal homogeneous, simply connected Riemannian manifolds of positive curvature. He arrived at the conclusion that these manifolds are homeomorphic (diffeomorphic in fact if one uses the Proposition 4.3 of Helgason ([5])) to a sphere S^n or one of the projective spaces $\mathbb{C}P^n$, $\mathbb{H}P^n$ or $\mathbb{C}aP^2$, with two exceptions. These exceptional spaces V_1 and V_2, which by definition carry a normal homogeneous metric, have the dimension seven and thirteen, respectively. Berger denoted them by $Sp(2)/SU(2)$ and $SU(5)/(Sp(2)\times S^1)$.

Especially in connection with the sphere theorem ([4]) one is interested in the ratio k between the minima and maxima of their sectional curvatures. Elíasson ([3]) calculated this ratio for V_1 to $k_{V_1}=\frac{1}{37}$. We will show that $k_{V_2}=\frac{16}{29\cdot 37}\approx 0.0149$.

2. Preliminaries

(i) If G is a Lie group, we denote its Lie algebra by $\underline{G}$.

(ii) The curvature of a normal homogeneous space G/H is calculated in the following way ([6]): Let $\langle\, ,\rangle$ denote the bi-invariant scalar product of $\underline{G}$ which induces the metric, let $\|\ \|$ denote the associated norm and $\underline{G}=\underline{H}+\underline{M}$ the orthogonal vector space decomposition. Then we have for each two-dimensional subspace $\sigma\subset\underline{M}$, spanned by orthonormal vectors X and Y,

$$K(\sigma)=\tfrac{1}{4}\|[X,Y]_{\underline{M}}\|^2+\|[X,Y]_{\underline{H}}\|^2,$$

where the subscripts $\underline{M}$ and $\underline{H}$ denote the orthogonal projection on $\underline{M}$ and $\underline{H}$, respectively.

(iii) Up to a positive factor, a compact simple Lie algebra possesses exactly one bi-invariant scalar product, namely the negative of its Killing form. Therefore each automorphism leaves the bi-invariant scalar product invariant.

(iv) Remarks on the definition of V_2: Taking $V_2=G/H$, Berger ([1]) had shown that up to isomorphisms $\underline{G}=A_4$ and $\underline{H}=C_2+R$, where A_4 denotes the Lie algebra of $SU(5)$, C_2 the Lie algebra of $Sp(2)$ and R the centralizer of $A_3=\underline{SU(4)}$ in A_4. $Sp(2)\subset SU(4)\subset SU(5)$ may be imbedded canonically. Isomorphism means here: there is a Lie algebra isomorphism of $\underline{G}$ on A_4, mapping $\underline{H}$ on C_2+R. $\underline{H}$ generates the closed subgroup

$$H=\left\{\begin{pmatrix} zA & \\ & \bar{z}^4 \end{pmatrix} \Big/ A\in Sp(2),\ z\in\mathbb{C},\ |z|=1\right\}$$

in $G=SU(5)$. Therefore we have $V_2=SU(5)/H$ with H locally isomorphic to $Sp(2)\times S^1$.

$C_2\subset A_3\subset A_4$ and R may in the following denote the same Lie algebras as above.

(v) The bi-invariant scalar product of A_4 induces the bi-invariant scalar product on A_3 and C_2. Assuming that $A_4=(C_2+R)+\underline{M}$ and $A_3=C_2+\underline{P}$ are orthogonal decompositions of A_4 and A_3, respectively, we have $\underline{P}\subset\underline{M}$ and we may decompose $\underline{M}$ orthogonally in $\underline{M}=\underline{N}+\underline{P}$. The two decompositions $A_3=C_2+\underline{P}$ and $A_4=(A_3+R)+\underline{N}$ belong to the symmetric spaces $S^5=SO(6)/SO(5)$ and $\mathbb{C}P^4=SU(5)/U(4)$. To take advantage of the fact that S^5 is a space of constant curvature, we remark: There is an isomorphism ξ ([5]) of A_3 onto $\underline{SO(6)}$ with $\xi(C_2)=\underline{SO(5)}\subset\underline{SO(6)}$. Let $\xi(\underline{P})=\underline{P}'$. Then $\underline{SO(6)}=\underline{SO(5)}+\underline{P}'$ is an orthogonal decomposition of $\underline{SO(6)}$ and the adjoint representation of $SO(5)$ operates on the five dimensional vector space $\underline{P}'$ as $SO(5)$ on $\mathbb{R}^5$ ([2], p. 99), hence transitively on the orthonormal pairs of $\underline{P}'$. Therefore the subgroup, generated by the elements e^{adX}, $X\in C_2$, also operates transitively on the orthonormal pairs of $\underline{P}$. Since $[C_2,\underline{P}]\subset\underline{P}$, $[C_2,\underline{N}]\subset\underline{N}$ and $[C_2,\underline{H}]\subset\underline{H}$, this subgroup leaves the decomposition of $A_4=\underline{H}+\underline{N}+\underline{P}$ invariant and therefore the curvature too, i.e., for any such automorphism φ, $K(\varphi(\sigma))=K(\sigma)$.

(vi) We use the decomposition $A_4=\underline{H}+\underline{M}=(C_2+R)+\underline{N}+\underline{P}$. Let $H_1,\dots,H_{11}$; $M_1,\dots,M_8$ and $M_9,\dots,M_{13}$ orthonormal bases of $\underline{H}$, $\underline{N}$ and $\underline{P}$, respectively. Then we have to calculate the minimum and the maximum of the function $K(X,Y)$ for orthonormal $X=\sum_{i=1}^{13}a_iM_i$ and $Y=\sum_{i=1}^{13}b_iM_i$. Because of (v) we can restrict ourselves to $X=\sum_{i=1}^{9}a_iM_i$ and $Y=\sum_{i=1}^{10}b_iM_i$, so that we have to find the minimum and the maximum of the function

$$K(a_1,\dots,a_9,b_1,\dots,b_{10})$$
$$=K(X,Y)=\tfrac{1}{4}\Big\|\sum_{i,j}a_ib_j[M_i,M_j]_{\underline{M}}\Big\|^2+\Big\|\sum_{i,j}a_ib_j[M_i,M_j]_{\underline{H}}\Big\|^2$$

under the additional conditions

$$\sum_i a_i^2=\sum_j b_j^2=1 \quad\text{and}\quad \sum_{i=1}^{9}a_ib_i=0.$$

Let
$$[M_i, M_j]=\sum_{\rho=1}^{11} c_{ij}^{\rho} H_{\rho}+\sum_{\sigma=1}^{13} d_{ij}^{\sigma} M_{\sigma}.$$
Then
$$K(a_1,\dots,b_{10})=\tfrac{1}{4}\sum_{\sigma=1}^{13}\Big(\sum_i a_i\big(\sum_j b_j d_{ij}^{\sigma}\big)\Big)^2+\sum_{\rho=1}^{11}\Big(\sum_i a_i\big(\sum_j b_j c_{ij}^{\rho}\big)\Big)^2.$$
Our calculations are completely elementary as soon as we know the Lie algebra constants c_{ij}^{ρ} and d_{ij}^{σ}.

3. Orthogonal Decomposition of the Lie Algebra and Calculation of the Brackets

Up to isomorphisms we have:

$A_4=\underline{SU(5)}=\{X/X$ complex 5×5 matrix, ${}^tX+\overline{X}=0$, $\operatorname{Tr} X=0\}$,

$C_2=\{X=(x_{ij})_{1\leq i,j\leq 5}\in A_4/x_{i5}=x_{5i}=0$ for $i=1,\dots,5$, $JX+{}^tXJ=0\}$,

$$R=\left\{\lambda\sqrt{-1}\begin{pmatrix}1&&&&\\&1&&&\\&&1&&\\&&&1&\\&&&&-4\end{pmatrix}\Big/\lambda\in\mathbb{R}\right\}.$$

The bracket operation is given by $[X,Y]=XY-YX$. J denotes the matrix $\begin{pmatrix}0&I_2&0\\-I_2&0&\vdots\\0&\dots&0\end{pmatrix}$, where I_2 is the unit matrix of order two.

The bi-invariant scalar product of A_4 is given by $\langle X,Y\rangle=-\frac{1}{4}\operatorname{Tr} XY$ (up to a positive factor). This form is obviously bi-linear and bi-invariant and it is also positive definite as can be seen from the fact: for each $X\in A_4$ there is a matrix $u\in SU(5)$ such that uXu^{-1} has a diagonal form.

Let $E_{\mu\nu}$ denote the matrices $(\delta_{i\mu}\delta_{j\nu})_{1\leq i,j\leq 5}$ and $Q_{\mu\nu}=E_{\mu\nu}-E_{\nu\mu}$, $R_{\mu\nu}=\sqrt{-1}(E_{\mu\nu}+E_{\nu\mu})$, $P_{\mu}=\sqrt{-1}(E_{\mu\mu}-E_{55})$, $1\leq\mu,\nu\leq 5$.

Then the following vectors constitute an orthonormal basis of A_4:

$H_1=P_1+P_2-P_3-P_4$, $H_2=Q_{13}+Q_{24}$, $H_3=R_{13}+R_{24}$,

$H_4=P_1-P_2-P_3+P_4$, $H_5=Q_{13}-Q_{24}$, $H_6=R_{13}-R_{24}$,

$H_7=R_{12}-R_{34}$, $H_8=Q_{14}+Q_{23}$, $H_9=R_{14}+R_{23}$,

$H_{10}=Q_{12}+Q_{34}$, $H_{11}=1/\sqrt{5}\cdot(P_1+P_2+P_3+P_4)$,

$M_1=\sqrt{2}Q_{15}$, $M_2=\sqrt{2}Q_{25}$, $M_3=\sqrt{2}Q_{35}$, $M_4=\sqrt{2}Q_{45}$,

$M_5=\sqrt{2}R_{15}$, $M_6=\sqrt{2}R_{25}$, $M_7=\sqrt{2}R_{35}$, $M_8=\sqrt{2}R_{45}$,

$M_9=Q_{12}-Q_{34}$, $M_{10}=Q_{14}-Q_{23}$, $M_{11}=R_{12}+R_{34}$,

$M_{12}=R_{14}-R_{23}$, $M_{13}=P_1-P_2+P_3-P_4$.

$H_1, \ldots, H_{10}$; H_{11}; $M_1, \ldots, M_8$ and $M_9, \ldots, M_{13}$ constitute bases for C_2, R, $\underline{N}$ and $\underline{P}$, respectively.

The calculations of the brackets $[M_i, M_j] = -[M_j, M_i]$, $1 \leqq i < j \leqq 10$, yield:

$$
\begin{aligned}
[M_1, M_2] &= -M_9 - H_{10}\\
[M_1, M_3] &= -H_2 - H_5\\
[M_1, M_4] &= -M_{10} - H_8\\
[M_1, M_5] &= M_{13} + H_1 + H_4 + \sqrt{5}\, H_{11}\\
[M_1, M_6] &= M_{11} + H_7\\
[M_1, M_7] &= H_3 + H_6\\
[M_1, M_8] &= M_{12} + H_9\\
[M_1, M_9] &= M_2\\
[M_1, M_{10}] &= M_4
\end{aligned}
$$

$$
\begin{aligned}
[M_2, M_3] &= M_{10} - H_8\\
[M_2, M_4] &= -H_2 + H_5\\
[M_2, M_5] &= M_{11} + H_7\\
[M_2, M_6] &= -M_{13} + H_1 - H_4 + \sqrt{5}\, H_{11}\\
[M_2, M_7] &= -M_{12} + H_9\\
[M_2, M_8] &= H_3 - H_6\\
[M_2, M_9] &= -M_1\\
[M_2, M_{10}] &= -M_3
\end{aligned}
$$

$$
\begin{aligned}
[M_3, M_4] &= M_9 - H_{10}\\
[M_3, M_5] &= H_3 + H_6\\
[M_3, M_6] &= -M_{10} + H_9\\
[M_3, M_7] &= M_{13} - H_1 - H_4 + \sqrt{5}\, H_{11}\\
[M_3, M_8] &= M_{11} - H_7\\
[M_3, M_9] &= -M_4\\
[M_3, M_{10}] &= M_2
\end{aligned}
$$

$$
\begin{aligned}
[M_4, M_5] &= M_{12} + H_9\\
[M_4, M_6] &= H_3 - H_6\\
[M_4, M_7] &= M_{11} - H_7
\end{aligned}
$$

$$
\begin{aligned}
[M_4, M_8] &= -M_{13} - H_1 + H_4 + \sqrt{5}\,H_{11}\\
[M_4, M_9] &= M_3\\
[M_4, M_{10}] &= -M_1\\
[M_5, M_6] &= -M_9 - H_{10}\\
[M_5, M_7] &= -H_2 - H_5\\
[M_5, M_8] &= -M_{10} - H_8\\
[M_5, M_9] &= M_6\\
[M_5, M_{10}] &= M_8\\
[M_6, M_7] &= M_{10} - H_8\\
[M_6, M_8] &= -H_2 + H_5\\
[M_6, M_9] &= -M_5\\
[M_6, M_{10}] &= -M_7\\
[M_7, M_8] &= M_9 - H_{10}\\
[M_7, M_9] &= -M_8\\
[M_7, M_{10}] &= M_6\\
[M_8, M_9] &= M_7\\
[M_8, M_{10}] &= -M_5\\
[M_9, M_{10}] &= -2H_2.
\end{aligned}
$$

4. Maximum and Minimum of the Sectional Curvature

For the sake of simplicity we introduce the following notations:

$$
\begin{aligned}
&A=(a_1,\ldots,a_8), \qquad B=(b_1,\ldots,b_8),\\
&X_\rho=\Big(\sum_j b_j c^\rho_{1j},\ldots,\sum_j b_j c^\rho_{8j}\Big), \qquad \rho=1,\ldots,10,\\
&X_{11}=1/\sqrt{5}\Big(\sum_j b_j c^{11}_{1j},\ldots,\sum_j b_j c^{11}_{8j}\Big),\\
&Y_\sigma=\Big(\sum_j b_j d^\sigma_{1j},\ldots,\sum_j b_j d^\sigma_{8j}\Big), \qquad \sigma=1,\ldots,13,\\
&Z_1=(-b_3,b_4,b_1,-b_2,b_7,-b_8,-b_5,b_6),\\
&Z_2=(b_7,-b_8,-b_5,b_6,b_3,-b_4,-b_1,b_2),\\
&Z_3=(b_3,b_4,-b_1,-b_2,-b_7,-b_8,b_5,b_6),\\
&Z_4=(b_7,b_8,-b_5,-b_6,b_3,b_4,-b_1,-b_2).
\end{aligned}
$$

From the table of the brackets we obtain:

$$X_1 = (b_5, b_6, -b_7, -b_8, -b_1, -b_2, b_3, b_4),$$
$$X_2 = (-b_3, -b_4, b_1, b_2, -b_7, -b_8, b_5, b_6),$$
$$X_3 = (b_7, b_8, b_5, b_6, -b_3, -b_4, -b_1, -b_2),$$
$$X_4 = (b_5, -b_6, -b_7, b_8, -b_1, b_2, b_3, -b_4),$$
$$X_5 = (-b_3, b_4, b_1, -b_2, -b_7, b_8, b_5, -b_6),$$
$$X_6 = (b_7, -b_8, b_5, -b_6, -b_3, b_4, -b_1, b_2),$$
$$X_7 = (b_6, b_5, -b_8, -b_7, -b_2, -b_1, b_4, b_3),$$
$$X_8 = (-b_4, -b_3, b_2, b_1, -b_8, -b_7, b_6, b_5),$$
$$X_9 = (b_8, b_7, b_6, b_5, -b_4, -b_3, -b_2, -b_1),$$
$$X_{10} = (-b_2, b_1, -b_4, b_3, -b_6, b_5, -b_8, b_7),$$
$$X_{11} = (b_5, b_6, b_7, b_8, -b_1, -b_2, -b_3, -b_4),$$

$$Y_1 = (0, -b_9, 0, -b_{10}, 0, 0, 0, 0),$$
$$Y_2 = (b_9, 0, b_{10}, 0, 0, 0, 0, 0),$$
$$Y_3 = (0, -b_{10}, 0, b_9, 0, 0, 0, 0),$$
$$Y_4 = (b_{10}, 0, -b_9, 0, 0, 0, 0, 0),$$
$$Y_5 = (0, 0, 0, 0, 0, -b_9, 0, -b_{10}),$$
$$Y_6 = (0, 0, 0, 0, b_9, 0, b_{10}, 0),$$
$$Y_7 = (0, 0, 0, 0, 0, -b_{10}, 0, b_9),$$
$$Y_8 = (0, 0, 0, 0, b_{10}, 0, -b_9, 0),$$
$$Y_9 = (-b_2, b_1, b_4, -b_3, -b_6, b_5, b_8, -b_7),$$
$$Y_{10} = (-b_4, b_3, -b_2, b_1, -b_8, b_7, -b_6, b_5),$$
$$Y_{11} = (b_6, b_5, b_8, b_7, -b_2, -b_1, -b_4, -b_3),$$
$$Y_{12} = (b_8, -b_7, -b_6, b_5, -b_4, b_3, b_2, -b_1),$$
$$Y_{13} = (b_5, -b_6, b_7, -b_8, -b_1, b_2, -b_3, b_4).$$

With these notations we obtain for the curvature:

$$4K(a_1, \ldots, a_9, b_1, \ldots, b_{10})$$
$$= 4\left(\sum_{\rho=1}^{10} (AX_\rho)^2 + 5(AX_{11})^2 + 4a_9^2 b_{10}^2 - 4a_9 b_{10}(AX_2)\right)$$
$$+ \sum_{\sigma=1}^{13} (AY_\sigma)^2 + a_9^2 B^2 - 2a_9 b_9 AB - 2a_9 b_{10} AX_2),$$

and the additional conditions:

$$A^2+a_9^2=B^2+b_9^2+b_{10}^2=1, \qquad AB+a_9 b_9=0,$$

where we used the canonical scalar product of $\mathbb{R}^8$. We remark that the following vectors are pairwise orthogonal and of the same length:

a) $B, X_7, X_8, X_9, X_{10}, Y_{13}, Z_1, Z_2$ with length $|B|$,

b) $Y_1, \ldots, Y_8$ with length $(b_9^2+b_{10}^2)^{\frac{1}{2}}$,

c) $B, Y_9, \ldots, Y_{13}, Z_3, Z_4$ with length $|B|$.

The following lemma is the main step of our calculations.

Lemma 1. (i) $\sum_{\nu=1}^{8}(AY_\nu)^2=A^2(b_9^2+b_{10}^2)$,

(ii) $\sum_{\nu=9}^{13}(AY_\nu)^2=A^2B^2-(AB)^2-(AZ_3)^2-(AZ_4)^2$,

(iii) $\sum_{\nu=7}^{10}(AX_\nu)^2=A^2B^2-(AY_{13})^2-(AB)^2-(AZ_1)^2-(AZ_2)^2$,

(iv) $\sum_{\nu=1}^{6}(AX_\nu)^2=(AX_{11})^2+(AY_{13})^2+\sum_{\nu=1}^{4}(AZ_\nu)^2$.

Proof. (i) – (iii) are direct applications of the next lemma and the remark above. The identity (iv) is most easily checked, if one compares the squares and the mixed terms on both sides.

Lemma 2. *If $B_1, \ldots, B_r$ are pairwise orthogonal vectors of $\mathbb{R}^n$ and of the same length B^2, we have for each $A\in\mathbb{R}^n$:*

$$\sum_{\nu=1}^{r}(AB_\nu)^2\leqq A^2B^2.$$

If $r=n$, equality holds.

Proof. Linear algebra.

With Lemma 1 the curvature formula is reduced to:

$$\begin{aligned}4K(a_1,\ldots,b_{10})=1+4\big(A^2B^2-(AB)^2\big)+15a_9^2b_{10}^2-18a_9b_{10}AX_2\\ +3\big((AZ_3)^2+(AZ_4)^2+8(AX_{11})^2\big).\end{aligned}$$

Proposition 1. *The maximum of the curvature of V_2 is 29/4.*

Proof. For $X=M_1$ and $Y=M_5$, i.e. $a_1=b_5=1$, $a_i=b_i=0$ otherwise, $K(X,Y)=\frac{29}{4}$.

On the other hand,

$$\begin{aligned}4K(a_1,\ldots,b_{10})&\leqq 5+3\big(5(|a_9|\,|b_{10}|+|A|\,|B|)^2+3\big)\\ &\leqq 5+3\big(5(A^2+a_9^2)(B^2+b_{10}^2)+3\big)\\ &\leqq 29,\end{aligned}$$

since $|AX_2|\leqq|A|\,|B|$ and since Z_3, Z_4 and X_{11} are pairwise orthogonal and of the same length.

Proposition 2. *The minimum of the curvature of V_2 is 4/37.*

Proof. For

$$X=\sqrt{\tfrac{12}{27}}(M_1+M_6)+\sqrt{\tfrac{13}{37}}\,M_9,\qquad Y=-\sqrt{\tfrac{12}{27}}(M_3+M_8)+\sqrt{\tfrac{13}{37}}\,M_{10}$$

$$K(X,Y)=\tfrac{4}{37}.$$

On the other hand,

$$4K(a_1,\dots,b_{10})\geqq 1+4(A^2B^2-(AB)^2)-18\,|a_9b_{10}|\,|AX_2|+15a_9^2b_{10}^2.$$

If we put $x=(A^2B^2-(AB)^2)^{\frac{1}{2}}\leqq\frac{1}{2}(A^2+B^2)$, $y=|a_9b_{10}|\leqq\frac{1}{2}(a_9^2+b_{10}^2)$, it follows from the additional conditions: $x+y\leqq 1$. Since B and X_2 are orthogonal and of the same length, Lemma 2 yields $|AX_2|\leqq x$. Therefore $4K(a_1,\dots,b_{10})\geqq 1+4x^2-18xy+15y^2$, where x and y are at least restricted by $x,y\geqq 0$ and $x+y\leqq 1$. The minimum of $1+4x^2-18xy+15y^2$ for $x+y\leqq 1$ and $x,y\geqq 0$ is obviously attained for $x+y=1$. Thus $4K(a_1,\dots,b_{10})\geqq 37x^2-48x+16\geqq\frac{16}{37}$.

Theorem. *The ratio between the minimum and the maximum of the sectional curvature of $V_2=SU(5)/(Sp(2)\times S^1)$ amounts to $k_{V_2}=16/29\cdot 37$.*

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(Received February 9, 1971)