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Morse Spectrum for Nonautonomous Differential Equations

MORSE SPECTRUM
FOR NONAUTONOMOUS DIFFERENTIAL EQUATIONS

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Dedicated to Ludwig Arnold on the occasion of his 70th birthday

The concept of a Morse decomposition consisting of nonautonomous sets is reviewed for
linear cocycle mappings w.r.t. to past, future and all-time convergences. In each case, the
set of accumulation points of the finite-time Lyapunov exponents corresponding to points
in a nonautonomous set is shown to be an interval. For a finest Morse decomposition, the
Morse spectrum is defined as the union of all of the above accumulation point intervals
over the different nonautonomous sets in such a finest Morse decomposition. In addition,
Morse spectrum is shown to be independent of which finest Morse decomposition is used,
when more than one exists.

1. Introduction

Exponential growth rates for solutions of linear nonautonomous differential equa-
tions are a classical topic in the theory of dynamical systems. There are basically
two approaches: Either one starts with exponential growth rates and constructs
corresponding (generalized) versions of eigenspaces from them; this leads, e.g., to
dichotomy spectra and to Oseledet's Theorem. Or one starts with an appropriate
generalized version of eigenspaces, and then considers the associated exponential
growth rates. In the context of linear flows on vector bundles with compact base
space, the latter approach can be based on a topological analysis of the induced

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system on projective space, where, following fundamental work by C. Conley, a finest Morse decomposition can be constructed for systems with a chain transitive base space. This leads to Selgrade's Theorem, proofs of which are given, e.g., by *Salamon & Zehnder*[11] and *Bronshteĭn*[1]. This theorem gives a decomposition of the vector bundle into linear subbundles, which, in the autonomous case, reduce to the sums of generalized eigenspaces corresponding to eigenvalues with coinciding real part. Then one can associate to each subbundle a spectral interval of generalized Lyapunov exponents, the Morse spectrum (*Colonius & Kliemann*[4], *Grüne*[6]). Recently, *San Martin & Seco*[12] have extended the Morse spectrum to flows on flag bundles in the context of semi-simple Lie groups.

For nonautonomous differential equations where no compact base space is present, these techniques are not applicable. However, nonautonomous versions of Morse decompositions have recently been derived based on generalizations of pullback attractors (see *Rasmussen*[9,10], and see *Palmer & Siegmund*[7] for attractor-repeller pairs). The present paper shows how to define an analogous notion of Morse spectrum in this nonautonomous context. Here the various time domains give rise to different notions, namely past, future and all-time objects.

For the readers' convenience, we have collected some basic notions from nonautonomous dynamics in the ensuing Section 2. Then, in Sections 3 and 4, basic results from *Rasmussen*[9,10] are recalled which yield appropriate notions of attractors and repellers leading to finest Morse decompositions for the induced systems on projective space. Section 5 presents the main results of the present paper: a nonautonomous version of Morse spectrum with some insight into its structure. Finally, an example is discussed.

Notation. Given a metric space (X, d), we write $U_\varepsilon(x_0) = \{x \in X : d(x, x_0) < \varepsilon\}$ for the ε-neighborhood of a point $x_0 \in X$. For arbitrary nonempty sets $A, B \subset X$ and $x \in X$, let $d(x, A) := \inf\{d(x, y) : y \in A\}$ be the *distance* of x to A and $d(A|B) := \sup\{d(x, B) : x \in A\}$ be the *Hausdorff semi-distance* of A and B.

We denote by $\mathbb{R}^{N \times N}$ the set of all real $N \times N$ matrices. The Euclidean space $\mathbb{R}^N$ is equipped with the Euclidean norm $\|\cdot\|$, which is induced by the scalar product $\langle \cdot, \cdot \rangle$, defined by $\langle x, y \rangle := \sum_{i=1}^N x_i y_i$. To introduce the real projective space $\mathbb{P}^{N-1}$ of $\mathbb{R}^N$, we say that two nonzero elements $x, y \in \mathbb{R}^N$ are equivalent if there exists a $c \in \mathbb{R}$ such that $x = cy$. The equivalence class of $x \in \mathbb{R}^N$ is denoted by $\mathbb{P}x$, and we call the set of all equivalent classes the *projective space* $\mathbb{P}^{N-1}$. Equipped with the metric $d_\mathbb{P} : \mathbb{P}^{N-1} \times \mathbb{P}^{N-1} \to [0, \sqrt{2}]$, given by

$$d_\mathbb{P}(\mathbb{P}v, \mathbb{P}w) = \min\left\{\left\|\frac{v}{\|v\|} - \frac{w}{\|w\|}\right\|, \left\|\frac{v}{\|v\|} + \frac{w}{\|w\|}\right\|\right\} \quad \text{for all } 0 \neq v, w \in \mathbb{R}^N,$$

the projective space is a compact metric space. For any $v \in \mathbb{P}^{N-1}$, we define $\mathbb{P}^{-1}v := \{x \in \mathbb{R}^N : \mathbb{P}x = v\} \cup \{0\}$.

2. Cocycles and Nonautonomous Sets

Throughout this paper, $\mathbb{I}$ denotes a real interval of the form $(-\infty, 0]$, $[0, \infty)$ or $\mathbb{R}$, respectively. Given a metric space (X, d), a *cocycle* is a mapping $\varphi : \mathbb{I} \times \mathbb{I} \times X \to X$ with

$$\varphi(\tau, \tau, \xi) = \xi \quad \text{and} \quad \varphi(t, \tau, \xi) = \varphi(t, s, \varphi(s, \tau, \xi))$$

for all $\tau, t, s \in \mathbb{I}$ and $\xi \in X$. The set X is called *phase space*, and $\mathbb{I} \times X$ is called *extended phase space*. The general solution of a nonautonomous differential equation $\dot{x} = f(t, x)$ is a cocycle if the *right hand side* $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies conditions guaranteeing global existence and uniqueness of solutions.

A subset M of the extended phase space $\mathbb{I} \times X$ is called *nonautonomous set*; we use the term *t-fiber of M* for the set $M(t) := \{x \in X : (t, x) \in M\}$ (which may be empty), $t \in \mathbb{I}$. We call M *closed* or *compact* if all t-fibers are closed or compact, respectively. Finally, a nonautonomous set M is called *invariant* (w.r.t. the cocycle φ) if $\varphi(t, \tau, M(\tau)) = M(\tau + t)$ for all $t, \tau \in \mathbb{I}$.

In case $X = \mathbb{R}^N$, a cocycle φ is called *linear* if for given $\alpha, \beta \in \mathbb{R}$, we have

$$\varphi(t, \tau, \alpha x + \beta y) = \alpha \varphi(t, \tau, x) + \beta \varphi(t, \tau, y) \quad \text{for all } t, \tau \in \mathbb{I} \text{ and } x, y \in \mathbb{R}^N . \quad (2.2)$$

For instance, a linear cocycle is generated by a linear nonautonomous differential equation $\dot{x} = B(t)x$, where $B : \mathbb{I} \to \mathbb{R}^{N \times N}$ is continuous. Given a linear cocycle φ, there exists a corresponding matrix-valued function $\Phi : \mathbb{I} \times \mathbb{I} \to \mathbb{R}^{N \times N}$ with $\Phi(t, \tau)x = \varphi(t, \tau, x)$ for all $t, \tau \in \mathbb{R}$ and $x \in \mathbb{R}^N$. We will also use the term *linear cocycle* for this function. Φ canonically induces a cocycle $\mathbb{P}\Phi$ on $\mathbb{P}^{N-1}$ by the definition

$$\mathbb{P}\Phi(t, \tau)\mathbb{P}x := \mathbb{P}(\Phi(t, \tau)x) \quad \text{for all } t, \tau \in \mathbb{R} \text{ and } x \in \mathbb{R}^N$$

(see *Colonius & Kliemann*[5], Lemma 5.2.1, p. 149).

3. Attractivity and Repulsivity

In this section, several notions of local attractivity and repulsivity are explained (see also *Rasmussen*[8]). The concepts are introduced for the past (past attractivity and repulsivity), the future (future attractivity and repulsivity) and the entire time (all-time attractivity and repulsivity).

Throughout this section, let (X, d) be a metric space and $\varphi : \mathbb{I} \times \mathbb{I} \times X \to X$ be a cocycle.

Note that the following notions of attractor are local forms of attractors which have been discussed since the 1990s. For instance, a past attractor is a local form of a pullback attractor (see, e.g., *Cheban, Kloeden & Schmalfuß*[2]), i.e., it attracts a neighborhood of itself in the sense of pullback attraction. Moreover, a future attractor is a local form of a forward attractor, and an all-time attractor is a local form of a uniform attractor as discussed, e.g., in *Chepyzhov & Vishik*[3].

Definition 3.1 (Attractors and repellers). Let A and R be invariant and compact nonautonomous sets.

(i) In case $\mathbb{I}$ is unbounded below, A is called a *past attractor* if there exists an $\eta > 0$ such that
$$\lim_{t\to\infty} d\big(\varphi(\tau,\tau-t,U_\eta(A(\tau-t)))\big|A(\tau)\big) = 0 \quad \text{for all } \tau \le 0, \qquad (3.1)$$

(ii) and R is called a *past repeller* if there exists an $\eta > 0$ such that
$$\lim_{t\to\infty} d\big(\varphi(\tau-t,\tau,U_\eta(R(\tau)))\big|R(\tau-t)\big) = 0 \quad \text{for all } \tau \le 0. \qquad (3.2)$$

(iii) In case $\mathbb{I}$ is unbounded above, A is called a *future attractor* if there exists an $\eta > 0$ such that
$$\lim_{t\to\infty} d\big(\varphi(\tau+t,\tau,U_\eta(A(\tau)))\big|A(\tau+t)\big) = 0 \quad \text{for all } \tau \ge 0, \qquad (3.3)$$

(iv) and R is called a *future repeller* if there exists an $\eta > 0$ such that
$$\lim_{t\to\infty} d\big(\varphi(\tau,\tau+t,U_\eta(R(\tau+t)))\big|R(\tau)\big) = 0 \quad \text{for all } \tau \ge 0. \qquad (3.4)$$

(v) In case $\mathbb{I} = \mathbb{R}$, A is called an *all-time attractor* if A is both a past and future attractor, i.e., there exists an $\eta > 0$ such that (3.1) and (3.3) hold,

(vi) and R is called an *all-time repeller* if R is both a past and future repeller, i.e., there exists an $\eta > 0$ such that (3.2) and (3.4) hold.

Remark 3.1.

(i) The notion of an all-time attractor and all-time repeller is slightly different from the concept used in *Rasmussen*[8,10], where uniform attraction and repulsion is required, respectively.

(ii) The notions of future attractivity and repulsivity can be derived from the concept of past attractivity and repulsivity via time reversal. A past attractor (repeller, respectively) corresponds to a future repeller (attractor, respectively) of the system under time reversal.

(iii) The Hausdorff semi-distance d in Definition 3.1 can equivalently be replaced by the Hausdorff distance d_H, which for nonempty sets $A, B \subset X$ is defined by $d_H(A,B) := \max\{d(A|B), d(B|A)\}$.

(iv) Every invariant and compact nonautonomous set of the differential equation $\dot{x} = x$ is a past repeller. Therefore, past repellers are not uniquely determined in general, in contrast to past attractors (see *Rasmussen*[8], Proposition 2.37).

Example 3.1. We consider the linear nonautonomous differential equation
$$\dot{x} = a(t)x$$
with a continuous function $a : \mathbb{R} \to \mathbb{R}$, . Then every invariant and compact nonautonomous set $M \subset \mathbb{R} \times \mathbb{R}$ is a

- past attractor if and only if $\lim_{t\to-\infty}\int_t^0 a(s)\,ds = -\infty\,,$
- past repeller if and only if $\lim_{t\to-\infty}\int_t^0 a(s)\,ds = \infty\,,$
- future attractor if and only if $\lim_{t\to\infty}\int_0^t a(s)\,ds = -\infty\,,$
- future repeller if and only if $\lim_{t\to\infty}\int_0^t a(s)\,ds = \infty\,,$
- all-time attractor if and only if $\lim_{t\to\infty}\max_{\tau\in\{0,-t\}}\int_\tau^{\tau+t} a(s)\,ds = -\infty\,,$
- all-time repeller if and only if $\lim_{t\to\infty}\min_{\tau\in\{0,-t\}}\int_\tau^{\tau+t} a(s)\,ds = \infty\,.$

All relations follow from the representation $\varphi(t,\tau,\xi) = \xi\exp\big(\int_\tau^t a(s)\,ds\big)$, which implies $d(\varphi(t,\tau,U_\eta(\{0\}))|\{0\}) = \eta\exp\big(\int_\tau^t a(s)\,ds\big)$. Then let t tend to ∞ in case of the future, or τ tend to $-\infty$ in case of the past, or both limits in case of the entire time.

4. Morse Decomposition

This section is devoted to a summary of the basic results from *Rasmussen*[9,10] concerning the existence of finest nonautonomous Morse decompositions of the projective flow associated with a linear cocycle. Note that concerning the case of all-time Morse decompositions, the results from *Rasmussen*[10] need to be adapted, since the definitions of all-time attractivity and repulsivity are not the same (cf. Remark 3.1 (i)). This will be done in a forthcoming paper.

The first step towards a Morse decomposition is the construction of attractor-repeller pairs.

Theorem 4.1 (Attractor-repeller pairs). *Let* $\Phi : \mathbb{I}\times\mathbb{I}\times\mathbb{R}^N \to \mathbb{R}^N$ *be a linear cocycle with induced cocycle* $\mathbb{P}\Phi$ *on projective space. Then the following statements are fulfilled:*

(i) *Let* $\mathbb{I}$ *be unbounded below and* R *be a past repeller of* $\mathbb{P}\Phi$*, i.e., there exists an* $\eta > 0$ *such that* (3.2) *holds. Then the nonautonomous set* R^**, defined by*

$$R^*(\tau) := \bigcap_{t^*\geq 0}\ \overline{\bigcup_{t\geq t^*}\mathbb{P}\Phi(\tau,\tau-t)\big(\mathbb{P}^{N-1}\setminus U_\eta(R(\tau-t))\big)}\quad\text{for all }\tau\in\mathbb{I}, \text{ (4.1)}$$

is a past attractor, which is maximal outside R *in the following sense: Any past attractor* $A\supsetneq R^*$ *has nonempty intersection with* R*. We call* (R^*,R) *a* past attractor-repeller pair.

(ii) *Let* $\mathbb{I}$ *be unbounded above and* A *be a future attractor of* $\mathbb{P}\Phi$*, i.e., there exists an* $\eta > 0$ *such that* (3.3) *holds. Then the nonautonomous set* A^**, defined by*

$$A^*(\tau) := \bigcap_{t^*\geq 0}\ \overline{\bigcup_{t\geq t^*}\mathbb{P}\Phi(\tau,\tau+t)\big(\mathbb{P}^{N-1}\setminus U_\eta(A(\tau+t))\big)}\quad\text{for all }\tau\in\mathbb{I}, \text{ (4.2)}$$

is a future repeller, which is maximal outside A *in the following sense: Any future repeller* $R\supsetneq A^*$ *has nonempty intersection with* A*. We call* (A,A^*) *a* future attractor-repeller pair.

(iii) *Let $\mathbb{I} = \mathbb{R}$ and A be an all-time attractor of $\mathbb{P}\Phi$. Then the nonautonomous set A^*, defined by (4.2) is an all-time repeller, which is maximal outside A in the following sense: Any all-time repeller $R \supsetneq A^*$ has nonempty intersection with A. We call (A, A^*) an* all-time attractor-repeller pair.

(iv) *Let $\mathbb{I} = \mathbb{R}$ and R be an all-time repeller of $\mathbb{P}\Phi$. Then the nonautonomous set R^*, defined by (4.1) is an all-time attractor, which is maximal outside R in the following sense: Any all-time attractor $A \supsetneq R^*$ has nonempty intersection with R. We call (R^*, R) an* all-time attractor-repeller pair.

Proof. See *Rasmussen*[9], Theorem 4.3, for the past and future case and a forthcoming paper for the entire time (cf. also *Rasmussen*[10], Theorem 3.2). □

Remark 4.1.

(i) In general, there is no formalism to obtain a past repeller from a past attractor and to get a future attractor from a future repeller (see *Rasmussen*[9], Example 4.4).

(ii) For an all-time attractor A, the relation $(A^*)^* = A$ is fulfilled, and an all-time repeller R fulfills $(R^*)^* = R$ (cf. a forthcoming paper or *Rasmussen*[10], Theorem 3.2).

(iii) In *Palmer & Siegmund*[7], so-called *generalized attractor-repeller pairs* are introduced, which consist of two invariant nonautonomous sets A and R of Φ whose fibers are linear subspaces of $\mathbb{R}^N$ fulfilling the following three conditions:

   (a) $A(t) \oplus R(t) = \mathbb{R}^N$ for all $t \in \mathbb{R}$,

   (b) given $\tau \in \mathbb{R}$, $0 \neq \xi \in A(\tau)$ and $0 \neq \eta \in R(\tau)$, we have

$$\frac{\|\Phi(t,\tau)\eta\|}{\|\Phi(t,\tau)\xi\|} \to 0 \text{ as } t \to \infty \quad \text{and} \quad \frac{\|\Phi(t,\tau)\xi\|}{\|\Phi(t,\tau)\eta\|} \to 0 \text{ as } t \to -\infty,$$

   (c) the angle between $A(t)$ and $R(t)$ is bounded below by a positive number.

   It is easy to see that each all-time attractor-repeller pair forms a generalized attractor-repeller pair.

The notion of an attractor-repeller pair is generalized by the following definition.

Definition 4.1 (Morse decompositions). A set $\{M_1, M_2, \dots, M_n\}$ of nonautonomous sets, the so-called *Morse sets*, is called *past (future, all-time, respectively) Morse decomposition* of $\mathbb{P}\Phi$ if the representation

$$M_i = A_i \cap R_{i-1} \quad \text{for all } i \in \{1, \dots, n\}$$

is fulfilled with past (future, all-time, respectively) attractor-repeller pairs (A_i, R_i), $i \in \{0, \dots, n\}$, fulfilling

$$\emptyset = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_n = \mathbb{I} \times \mathbb{P}^{N-1}$$

and

$$\mathbb{I} \times \mathbb{P}^{N-1} = R_0 \supsetneq R_1 \supsetneq \cdots \supsetneq R_n = \emptyset \,.$$

The following theorem shows that Morse decompositions are crucial for the dynamical behavior of the nonautonomous dynamical system.

Theorem 4.2 (Dynamical properties). *The following statements are fulfilled:*

(i) Convergence in forward time. *Let* $\{M_1, \ldots, M_n\}$ *be a future (all-time, respectively) Morse decomposition of* $\mathbb{P}\Phi$. *Then for all* $(\tau, x) \in \mathbb{I} \times \mathbb{P}^{N-1}$, *there exists an* $i \in \{1, \ldots, n\}$ *with*

$$\lim_{t \to \infty} d_{\mathbb{P}}\big(\mathbb{P}\Phi(\tau + t, \tau)x, M_i(\tau + t)\big) = 0 \,.$$

(ii) Convergence in backward time. *Let* $\{M_1, \ldots, M_n\}$ *be a past (all-time, respectively) Morse decomposition of* $\mathbb{P}\Phi$. *Then for all* $(\tau, x) \in \mathbb{I} \times \mathbb{P}^{N-1}$, *there exists an* $i \in \{1, \ldots, n\}$ *with*

$$\lim_{t \to \infty} d_{\mathbb{P}}\big(\mathbb{P}\Phi(\tau - t, \tau)x, M_i(\tau - t)\big) = 0 \,.$$

Proof. See *Rasmussen*[9], Theorem 8.5, and *Rasmussen*[10], Theorem 4.4.   □

Further convergence results for Morse decompositions can be found in *Rasmussen*[9], Theorem 5.6.

We conclude this section by stating a result concerning finest Morse decompositions, which is an analog to the Theorem of Selgrade (see *Selgrade*[13]).

Theorem 4.3 (Finest Morse decomposition). *There exists a* finest past (future, all-time, respectively) Morse decomposition $\{M_1, \ldots, M_n\}$ *of* $\mathbb{P}\Phi$, *i.e., the number of Morse sets of every past (future, all-time, respectively) Morse decomposition is bounded by* n. *Moreover, we have* $n \leq N$ *and the decomposition*

$$\mathbb{P}^{-1}M_1(t) \oplus \cdots \oplus \mathbb{P}^{-1}M_n(t) = \mathbb{R}^N \quad \text{for all } t \in \mathbb{I} \,.$$

Proof. See *Rasmussen*[9], Theorem 8.7, for the past and future case and a forthcoming paper for the entire time (cf. also *Rasmussen*[10], Theorem 5.1).   □

Remark 4.2. A finest Morse decomposition for the past and future is not uniquely determined, but one obtains that the Morse sets of two finest Morse decompositions are converging to each other in Hausdorff distance when time tends to the past or future, respectively.

5. Morse Spectrum

In this section, we introduce Morse spectra for a linear cocycle $\Phi : \mathbb{I} \times \mathbb{I} \to \mathbb{R}^{N \times N}$. Fundamental for what follows is the definition of a finite-time exponential growth rate.

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Definition 5.1 (Finite-time exponential growth rate). For any $(\tau, \xi) \in \mathbb{I} \times \mathbb{R}^N$, $\xi \neq 0$, and $T > 0$ with $\tau + T \in \mathbb{I}$, we define the *finite-time exponential growth rate* by

$$\lambda^T(\tau, \xi) := \frac{1}{T} \ln \frac{\|\Phi(\tau + T, \tau)\xi\|}{\|\xi\|}.$$

Using this concept, we study limits of growth rates which are attained from initial values (τ, ξ) within a fixed linear and invariant nonautonomous set.

Definition 5.2. Let $M \subset \mathbb{I} \times \mathbb{R}^N$ be an invariant nonautonomous set such that the fibers of M, i.e., the sets $M(t)$ for $t \in \mathbb{I}$, are linear subspaces of $\mathbb{R}^N$.

(i) If $\mathbb{I}$ is unbounded below, the *past spectrum of* Φ *over* M is defined by

$$\Sigma^-(M) := \Big\{ \mu \in \overline{\mathbb{R}} : \text{there exist sequences } T_k \to \infty \text{ and } (\tau_k, \xi_k) \in M$$
$$\text{with } \tau_k, \tau_k + T_k \leq 0 \text{ and } \lim_{k \to \infty} \lambda^{T_k}(\tau_k, \xi_k) = \mu \Big\}.$$

(ii) If $\mathbb{I}$ is unbounded above, the *future spectrum of* Φ *over* M is defined by

$$\Sigma^+(M) := \Big\{ \mu \in \overline{\mathbb{R}} : \text{there exist sequences } T_k \to \infty \text{ and } (\tau_k, \xi_k) \in M$$
$$\text{with } \tau_k, \tau_k + T_k \geq 0 \text{ and } \lim_{k \to \infty} \lambda^{T_k}(\tau_k, \xi_k) = \mu \Big\}.$$

(iii) If $\mathbb{I} = \mathbb{R}$, the *all-time spectrum of* Φ *over* M is defined by

$$\Sigma^{\pm}(M) := \Big\{ \mu \in \overline{\mathbb{R}} : \text{there exist sequences } T_k \to \infty \text{ and } (\tau_k, \xi_k) \in M$$
$$\text{with } \lim_{k \to \infty} \lambda^{T_k}(\tau_k, \xi_k) = \mu \Big\}.$$

Note that the conditions $\tau_k + T_k \leq 0$ and $T_k \to \infty$ in (i) imply that $\tau_k \to -\infty$ when $k \to \infty$. The limits in (i) can thus be seen as *pullback limits*, whereas we obtain *forward limits* in (ii) and *arbitrary limits* in (iii).

In the following, we allow $\pm\infty$ as boundary points of intervals, e.g., $[-\infty, \infty] := (-\infty, \infty) \cup \{-\infty, \infty\}$ and $[\infty, \infty] := \{\infty\}$.

With these definitions, the past (future, all-time, respectively) spectrum of a linear and invariant nonautonomous set turns out to be a closed interval.

Theorem 5.1. *Let* $M \subset \mathbb{I} \times \mathbb{R}^N$ *be an invariant nonautonomous set such that the fibers of* M *are linear subspaces of* $\mathbb{R}^N$, *and let* $\Sigma(M) := \Sigma^-(M), \Sigma^+(M), \Sigma^{\pm}(M)$ *be the past (future, all-time, respectively) spectrum of* Φ *over* M, *respectively. Then* $\Sigma(M)$ *is nonempty, closed and connected, i.e., there exist* $-\infty \leq \sigma_* \leq \sigma^* \leq \infty$ *such that*

$$\Sigma(M) = [\sigma_*, \sigma^*].$$

Proof. The closedness of the spectrum follows directly from the definition via the limits, and the spectrum is nonempty, since the limes superior of the sequence

$\lambda^{T_k}(\tau_k, \xi_k)$ for given arbitrary sequences $T_k \to \infty$ and $(\tau_k, \xi_k) \in M$ belongs to $\Sigma(M)$. We now choose $s^-, s^+ \in \Sigma(M)$ with $s^- < s^+$, and let $s \in (s^-, s^+)$. Then there exist sequences $T_k^-, T_k^+ \to \infty$ and $(\tau_k^-, \xi_k^-), (\tau_k^+, \xi_k^+) \in M$ such that

$$\lim_{k\to\infty} \lambda^{T_k^-}(\tau_k^-, \xi_k^-) = s^- \quad \text{and} \quad \lim_{k\to\infty} \lambda^{T_k^+}(\tau_k^+, \xi_k^+) = s^+ \,.$$

We can assume that both $\lambda^{T_k^+}(\tau_k^+, \xi_k^+) > s$ and $\lambda^{T_k^-}(\tau_k^-, \xi_k^-) < s$ is fulfilled for all $k \in \mathbb{N}$. Given $k \in \mathbb{N}$, the function

$$c \mapsto \lambda^{cT_k^+ + (1-c)T_k^-}(c\tau_k^+ + (1-c)\tau_k^-, c\xi_k^+ + (1-c)\xi_k^-), \quad c \in [0,1]\,,$$

is continuous, and hence, there exists a $c_k \in [0,1]$ such that

$$\lambda^{c_k T_k^+ + (1-c_k)T_k^-}(c_k \tau_k^+ + (1-c_k)\tau_k^-, c_k \xi_k^+ + (1-c_k)\xi_k^-) = s\,.$$

This implies $s \in \Sigma(M)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The following notion of Morse spectrum relies on finest Morse decompositions. In the case of the past and future, a finest Morse decomposition is not uniquely determined (see Remark 4.2), but the following lemma says that the kind of nonuniqueness does not affect the spectra over the Morse sets.

Lemma 5.1. *Let $\mathcal{M} := \{M_1, \dots, M_n\}$ and $\tilde{\mathcal{M}} := \{\tilde{M}_1, \dots, \tilde{M}_n\}$ be finest future Morse decompositions of $\mathbb{P}\Phi$. Then for all $i \in \{1, \dots, n\}$, we have*

$$\Sigma^+(\mathbb{P}^{-1}M_i) = \Sigma^+(\mathbb{P}^{-1}\tilde{M}_i)\,.$$

A similar statement is fulfilled for finest past Morse decompositions.

Proof. Let $\emptyset = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_n = \mathbb{I} \times \mathbb{P}^{N-1}$ and $\emptyset = \tilde{A}_0 \subsetneq \tilde{A}_1 \subsetneq \cdots \subsetneq \tilde{A}_n = \mathbb{I} \times \mathbb{P}^{N-1}$ be the future attractor sequences leading to $\mathcal{M}$ and $\tilde{\mathcal{M}}$, respectively. Note that we have $A_j^* = \tilde{A}_j^*$ for $j \in \{1, \dots, n\}$ (see *Rasmussen*[9], Theorem 8.7). Let $i \in \{1, \dots, n\}$ and $\mu \in \Sigma^+(\mathbb{P}^{-1}M_i)$, i.e., there exist sequences $T_k \to \infty$ and $(\tau_k, \xi_k) \in \mathbb{P}^{-1}M_i$ with $\tau_k, \tau_k + T_k \geq 0$ and $\lim_{k\to\infty} \lambda^{T_k}(\tau_k, \xi_k) = \mu$. We denote the projection on $\mathbb{P}^{-1}\tilde{M}_i(t)$ with null space $\mathbb{P}^{-1}\tilde{M}_1(t) \oplus \cdots \oplus \mathbb{P}^{-1}\tilde{M}_{i-1}(t) \oplus \mathbb{P}^{-1}\tilde{M}_{i+1}(t) \oplus \mathbb{P}^{-1}\tilde{M}_n(t)$ by $P(t)$, $t \in \mathbb{I}$, and we define

$$\tilde{\xi}_k := P(\tau_k)\xi_k \quad \text{and} \quad \eta_k := \xi_k - \tilde{\xi}_k \quad \text{for all } k \in \mathbb{N}\,.$$

We have $M_i(t), \tilde{M}_i(t) \subset A_{i-1}^*(t) = \tilde{A}_{i-1}^*(t) = \mathbb{P}(\mathbb{P}^{-1}\tilde{M}_i(t) \oplus \cdots \oplus \mathbb{P}^{-1}\tilde{M}_n(t))$ for $t \in \mathbb{I}$ (note that the last equality follows from the proof of *Rasmussen*[9], Theorem 8.7), and this implies

$$\eta_k \in \mathbb{P}^{-1}\tilde{M}_{i+1}(\tau_k) \oplus \cdots \oplus \mathbb{P}^{-1}\tilde{M}_n(\tau_k) \quad \text{for all } k \in \mathbb{N}, \qquad (5.8)$$

since $P(\tau_k)\eta_k = 0$. Let $Q(t)$, $t \in \mathbb{I}$, denote the projection on $\mathbb{P}^{-1}\tilde{M}_1(t) \oplus \cdots \oplus \mathbb{P}^{-1}\tilde{M}_i(t)$ with null space $\mathbb{P}^{-1}\tilde{M}_{i+1}(t) \oplus \cdots \oplus \mathbb{P}^{-1}\tilde{M}_n(t)$. By (5.8), we have

$$\tilde{\xi}_k = P(\tau_k)\xi_k = Q(\tau_k)\xi_k \quad \text{for all } k \in \mathbb{N}\,. \qquad (5.9)$$

Moreover, the projected null space of Q, i.e., the set $\mathbb{P}(\mathbb{P}^{-1}\tilde{M}_{i+1} \oplus \cdots \oplus \mathbb{P}^{-1}\tilde{M}_n) = \mathbb{P}(\mathbb{P}^{-1}M_{i+1} \oplus \cdots \oplus \mathbb{P}^{-1}M_n)$, is a future repeller, and the projected range of Q, i.e., the set $\mathbb{P}(\mathbb{P}^{-1}M_1 \oplus \cdots \oplus \mathbb{P}^{-1}M_i)$, is the corresponding future attractor. This means that the projected range and null space of $Q(t)$ are separated uniformly for all $t \in \mathbb{I}$ (see *Rasmussen*[8], Theorem 3.5 (i)), and hence, we have a $K_1 > 1$ such that $\|Q(\tau)\xi\| \leq K_1\|\xi\|$ for all $\tau \in \mathbb{I}$ and $\xi \in \mathbb{R}^N$. Furthermore, the set $\mathbb{P}^{-1}M_i(t)$ is also separated from the projected null space of Q uniformly in $t \in \mathbb{I}$, which is given by $\mathbb{P}(\mathbb{P}^{-1}\tilde{M}_{i+1} \oplus \cdots \oplus \mathbb{P}^{-1}\tilde{M}_n)(t) = \mathbb{P}(\mathbb{P}^{-1}M_{i+1} \oplus \cdots \oplus \mathbb{P}^{-1}M_n)(t)$ (note that $\mathbb{P}(\mathbb{P}^{-1}M_1 \oplus \cdots \oplus \mathbb{P}^{-1}M_i)$ and $\mathbb{P}(\mathbb{P}^{-1}\tilde{M}_{i+1} \oplus \cdots \oplus \mathbb{P}^{-1}\tilde{M}_n) = \mathbb{P}(\mathbb{P}^{-1}M_{i+1} \oplus \cdots \oplus \mathbb{P}^{-1}M_n)$ are a future attractor-repeller pair), and thus, there exists a constant $K_2 > 1$ with

$$\|P(\tau)\xi\| \overset{(*)}{=} \|Q(\tau)\xi\| \geq \frac{1}{K_2}\|\xi\| \quad \text{for all } (\tau, \xi) \in \mathbb{P}^{-1}M_i.$$

Note that $(*)$ follows as in (5.9), where $(\tau_k, \xi_k) \in \mathbb{P}^{-1}M_i$. We obtain

$$\begin{aligned}
\lambda^{T_k}(\tau_k, \tilde{\xi}_k) &= \frac{1}{T_k}\ln\left(\frac{\|\Phi(\tau_k + T_k, \tau_k)P(\tau_k)\xi_k\|}{\|P(\tau_k)\xi_k\|}\right) \\
&= \frac{1}{T_k}\ln\left(\frac{\|P(\tau_k + T_k)\Phi(\tau_k + T_k, \tau_k)\xi_k\|}{\|P(\tau_k)\xi_k\|}\right) \\
&\leq \frac{1}{T_k}\ln\left(K_1 K_2 \frac{\|\Phi(\tau_k + T_k, \tau_k)\xi_k\|}{\|\xi_k\|}\right)
\end{aligned}$$

(we again used the fact that $Q(t)$ and $P(t)$ coincide on $\mathbb{P}^{-1}M_i(t)$). In addition, we have

$$\begin{aligned}
\lambda^{T_k}(\tau_k, \tilde{\xi}_k) &= \frac{1}{T_k}\ln\left(\frac{\|P(\tau_k + T_k)\Phi(\tau_k + T_k, \tau_k)\xi_k\|}{\|P(\tau_k)\xi_k\|}\right) \\
&\geq \frac{1}{T_k}\ln\left(\frac{\|\Phi(\tau_k + T_k, \tau_k)\xi_k\|}{K_1 K_2\|\xi_k\|}\right),
\end{aligned}$$

and we obtain $\mu \in \Sigma^+(\tilde{M}_i)$. This finishes the proof of this lemma. $\qquad\square$

The following definition of a Morse spectrum builds upon finest Morse decompositions. In fact, the Morse spectrum is defined as the union of the spectra of the corresponding Morse sets.

Definition 5.3 (Morse spectrum). Let $\{M_1, \ldots, M_n\}$ be a finest past (future, all-time, respectively) Morse decomposition of $\mathbb{P}\Phi$ from Theorem 4.3. Then the *past (future, all-time, respectively) Morse spectrum* of Φ is defined by

$$\Sigma(\Phi) := \bigcup_{i=1}^{n} \Sigma\left(\mathbb{P}^{-1}M_i\right),$$

where $\Sigma(M_i)$ denotes the past (future, all-time) spectrum of Φ over M_i, $i = 1, \ldots, n$.

The following theorem says that the Morse spectra contain all Lyapunov exponents of the system.

Theorem 5.2 (Lyapunov exponents are contained in the Morse spectrum). *Let $(\tau, \xi) \in \mathbb{I} \times \mathbb{R}^N$ such that*

$$\mu = \lim_{k \to \infty} \frac{1}{T_k} \ln \|\Phi(\tau + T_k, \tau)\xi\|$$

for some $\mu \in \overline{\mathbb{R}}$ and some sequence $\{T_k\}_{k \in \mathbb{N}}$ with $\lim_{k \to \infty} T_k = \infty$. Then $\mu \in \Sigma^+(\Phi) \subset \Sigma^\pm(\Phi)$. Moreover, let $(\tau, \xi) \in \mathbb{I} \times \mathbb{R}^N$ such that

$$\mu = \lim_{k \to \infty} \frac{1}{T_k} \ln \|\Phi(\tau - T_k, \tau)\xi\|$$

for some $\mu \in \overline{\mathbb{R}}$ and some sequence $\{T_k\}_{k \in \mathbb{N}}$ with $\lim_{k \to \infty} T_k = \infty$. Then $-\mu \in \Sigma^-(\Phi) \subset \Sigma^\pm(\Phi)$.

Proof. Let $\{M_1, \ldots, M_n\}$ be some finest future Morse decomposition. We write $\xi = m_1 + \cdots + m_n$ with $m_i \in \mathbb{P}^{-1} M_i(\tau)$, $i \in \{1, \ldots, n\}$, and we choose $j \in \{1, \ldots, n\}$ minimal with $m_j \neq 0$. Then *Rasmussen*[8], Proposition 3.19, implies that

$$\lim_{k \to \infty} \frac{\|\Phi(\tau + T_k, \tau)m_i\|}{\|\Phi(\tau + T_k, \tau)m_j\|} = 0 \quad \text{for all } i \in \{j+1, \ldots, n\} \tag{5.14}$$

(note that $\mathbb{P}(\mathbb{P}^{-1} M_1 \oplus \cdots \oplus \mathbb{P}^{-1} M_j)$ is a future attractor). In addition, we obtain

$$\|\Phi(\tau + T_k, \tau)\xi\| \leq \sum_{i=j}^{n} \|\Phi(\tau + T_k, \tau)m_i\|$$

$$= \|\Phi(\tau + T_k, \tau)m_j\| \sum_{i=j}^{n} \frac{\|\Phi(\tau + T_k, \tau)m_i\|}{\|\Phi(\tau + T_k, \tau)m_j\|}$$

$$= \|\Phi(\tau + T_k, \tau)m_j\| \left(1 + \sum_{i=j+1}^{n} \frac{\|\Phi(\tau + T_k, \tau)m_i\|}{\|\Phi(\tau + T_k, \tau)m_j\|} \right)$$

and

$$\|\Phi(\tau + T_k, \tau)\xi\| \geq \left| \|\Phi(\tau + T_k, \tau)m_j\| - \left\| \sum_{i=j+1}^{n} \Phi(\tau + T_k, \tau)m_i \right\| \right|$$

$$= \|\Phi(\tau + T_k, \tau)m_j\| \left| 1 - \left\| \sum_{i=j+1}^{n} \frac{\Phi(\tau + T_k, \tau)m_i}{\|\Phi(\tau + T_k, \tau)m_j\|} \right\| \right|.$$

Hence, (5.14) implies that

$$\mu = \lim_{k \to \infty} \frac{1}{T_k} \ln \|\Phi(\tau + T_k, \tau)\xi\| = \lim_{k \to \infty} \frac{1}{T_k} \ln \|\Phi(\tau + T_k, \tau)m_j\|$$

$$= \lim_{k \to \infty} \frac{1}{T_k} \ln \frac{\|\Phi(\tau + T_k, \tau)m_j\|}{\|m_j\|}.$$

The second statement of this theorem can be proved similarly. $\square$

6. An Example

The following example shows that the intervals of the Morse spectrum need not be disjoint. It is a modification of *Salamon & Zehnder*[11], Example 2.14.

Example 6.1. Consider the differential equation

$$\dot{x} = \cos(\zeta(t))x, \quad \dot{y} = (\alpha + \cos(\zeta(t))y, \tag{6.1}$$

where $\alpha > 0$ is a parameter and $\zeta(t)$ is the solution of $\dot{\zeta}(t) = \sin \zeta(t)$, $\zeta(0) = \pi/2$. Then $\zeta(t), \dot{\zeta}(t) > 0$ for all $t \in \mathbb{R}$ and $\zeta(t) \to 0$ for $t \to -\infty$ and $\zeta(t) \to \pi$ for $t \to +\infty$. The sets

$$\mathcal{V}_1 := \mathbb{R} \times \mathbb{R} \times \{0\} \quad \text{and} \quad \mathcal{V}_2 := \mathbb{R} \times \{0\} \times \mathbb{R}$$

are invariant nonautonomous sets with linear fibers. For the projectivized flow, the set $M_1(t) := \mathbb{P}\mathcal{V}_1(t)$ is an all-time attractor and $M_2(t) := \mathbb{P}\mathcal{V}_2(t)$ is its complementary repeller. They form a finest all-time Morse decomposition. It follows from (6.1) that

$$\Sigma^{\pm}(M_1) = [-1, 1] \quad \text{and} \quad \Sigma^{\pm}(M_2) = [\alpha - 1, \alpha + 1].$$

Hence, for $\alpha \in (0, 2]$, these spectral intervals intersect.

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