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ORIGINAL ARTICLE

Contributions to Plasma Physics

Chiral Josephson effect in double layers: The role of particle–hole duality

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Abstract

The Josephson effect of inter-layer s-wave pairing in a double layer of two chiral metals with Josephson junctions along the y direction is considered. Starting from the Bogoliubov de Gennes equations of electron–electron and electron–hole double layers, we employ the duality transformation between the two systems to determine the relation of the zero-energy quasiparticle modes at a Josephson junction in both systems. The appearance of an exceptional point at zero energy is observed, where the four-dimensional eigenspace coalesces to a two-dimensional eigenspace. In the second part of the article, the coupling of the quasiparticle currents and the supercurrent is studied. The coupling is based on the quasiparticle charge conservation in the form of a continuity equation. Although the quasiparticle modes differ between the electron–electron and the electron–hole double layers, their corresponding currents are the same.

KEYWORDS

chiral double layer, Josephson effect

1 | INTRODUCTION

One of the most fascinating observations in condensed matter physics is the pairing effect, leading to phenomena such as superconductivity and superfluidity. Although the pairing effect is of quantum nature, its theoretical description in terms of a macroscopic order parameter field is given by a mean-field (or Ginzburg-Landau) theory. Excitations in the form of quasiparticles, on the other hand, are represented by the Bogoliubov de Gennes (BdG) equation that describes a single-particle quantum wave function.[1] A more comprehensive discussion of this equation can be found in Ref. [2]. Recent research on layered chiral materials has revived the interest in this approach, in particular, for the special properties of zero-energy (or mid-gap) modes.[3–9]

An interesting phenomenon related to pairing is the Josephson effect[10] that originates in the coupling between the macroscopic superfluid or superconducting order parameter with the quasiparticle modes induced by a Josephson junction.[11–16] It has been discussed for different systems, including electron–electron graphene bilayers,[17,18] systems with spin-orbit coupling,[19] electron–hole bilayers,[20,21] electron–hole double layers[22–24] and electron–hole double-bilayers.[25,26] An important aspect of the quasiparticles is their sensitivity to the underlying spatial structure in terms of geometry and topology, in particular, for zero-energy modes.[5,18] The interplay of the Josephson effect with the topological properties of the quasiparticle modes was recently discussed for an electron–electron double layer (EEDL)[27] and for an electron–hole double layer (EHDL)[24] separately. One purpose of the present paper is to study the connection of these two systems through the particle–hole duality and how their corresponding zero-energy modes at a Josephson junction are related. Another purpose is to investigate the coupling of the quasiparticle currents to the supercurrent in the EEDL and in the EHDL, a feature that is directly linked to the Josephson effect.

The article is organized as follows. In Section 2, the BdG equations for the chiral EEDL (Section 2.1) and for the EHDL (Section 2.2) are separately introduced and their relation through the duality transformation is explained at the end of Section 2.2. The Josephson current, induced by a Josephson junction along the y direction of the layers, is investigated in Section 3 on the basis of quasiparticle charge conservation. Finally, in Section 4 we discuss and interpret the results of Sections 2 and 3. Appendix A provides details of the quasiparticle modes and Appendix B presents some details of the current calculation.

2 | MODEL: BOGOLIUBOV DE GENNES EQUATION/HAMILTONIAN

The EEDL and the EHDL are dual to each other.[28] In the following, we discuss these two cases separately and compare the resulting Josephson currents in Section 4. Both systems are treated within a BCS-like mean-field approach. This leads to an order parameter Δ that characterizes the superconducting state of the EEDL and the superfluid state of the EHDL. Excitations in the form of quasiparticles are obtained from the corresponding BdG Hamiltonian, where the latter describes the quantum fluctuations about the mean-field approximation. In the following discussion, we consider the inter-layer pairing but ignore the intra-layer pairing. This is a simplification that is plausible for a small distance of the layers and due to screening inside the layers but has been debated in the literature.[29,30] Moreover, inter-layer tunnelling is suppressed by a dielectric between the layers.

2.1 | Chiral electron–electron double layer

The EEDL comprises two electronic layers with a positively charged extra layer. The latter can either be an external gate (as visualized in Figure 1a) or is provided by the positive charges inside the metallic layers. In both cases the entire system preserves charge neutrality. The electrons in the two layers repel each other due to the Coulomb interaction. The geometric constraint enables the electrons at a fixed density to form inter-layer Cooper pairs. This is formally supported by the duality transformation to the EHDL, in which the electron–hole pairs are subject to an attractive Coulomb interaction. In other words, the formation of inter-layer electron–hole pairs in the EHDL[22] is transformed into inter-layer electron–electron pairs by the duality transformation.[28] Then the related quasiparticles are described by the BdG Hamiltonian of two layers with opposite chiralities, as follows.[27]

$$H_{\rm EEDL} = \begin{pmatrix} h_1\sigma_1 + h_2\sigma_2 & \Delta\sigma_2 \\ \Delta\sigma_2 & h_1\sigma_1 - h_2\sigma_2 \end{pmatrix}, \tag{1}$$

where σ_j are Pauli matrices, h_j are tight-binding hopping matrices and Δ is the real pairing order parameter. For the subsequent discussion, we assume a honeycomb lattice for the underlying spatial structure of the tight-binding model, such that the quasiparticle Hamiltonian describes graphene-like materials. Assuming translational invariance in y direction, the low-energy BdG Hamiltonian becomes with $h_1 \sim i\hbar v_F\partial_x$, $h_2 \sim \hbar v_F k_y$

$$H_{\rm EEDL} \sim \begin{pmatrix} i\hbar v_F\partial_x\sigma_1 + \hbar v_F k_y\sigma_2 & \Delta(x)\sigma_2 \\ \Delta(x)\sigma_2 & i\hbar v_F\partial_x\sigma_1 - \hbar v_F k_y\sigma_2 \end{pmatrix}, \tag{2}$$

where v_F is the Fermi velocity.

Now we consider a domain wall in y direction at $x = 0$, as sketched in Figure 2a: $\Delta(x) = {\rm sgn}(x)|\Delta|$. The resulting eigenvalue problem can be solved. At zero energy there is an exceptional point for the Hamiltonian (2), where the four-fold degeneracy coalesces to a two-dimensional eigenspace with two independent zero-energy modes (cf. Appendix A):

$$\Psi_1 = \frac{1}{\mathcal{N}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}e^{-|\Delta||x|/\hbar v_F}, \quad \Psi_2 = \frac{1}{\mathcal{N}}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}e^{-|\Delta||x|/\hbar v_F} \tag{3}$$

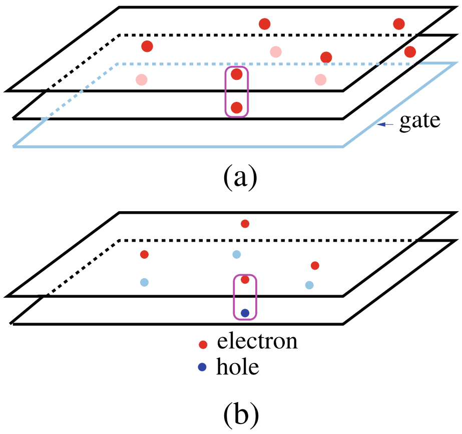


FIGURE 1 An electron–electron double layer (a) and an electron–hole double layer (b) with inter-layer pairing due to Coulomb interaction, where the schematic gate in (a) is positively charged and guarantees charge neutrality. Inter-layer tunnelling is suppressed by a dielectric medium and inter-layer pairing requires a small distance of the layers to make the Coulomb interaction sufficiently strong.

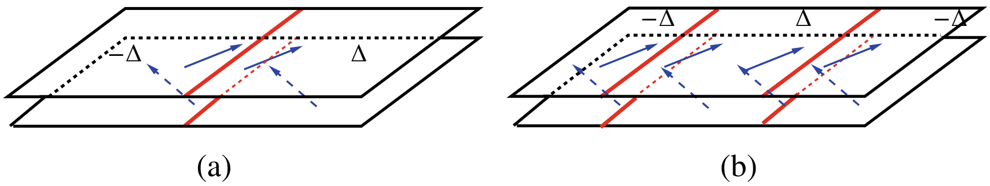


FIGURE 2 Electronic double layer with domain walls, which is given by a sign jump of the pairing order parameter. The local currents (blue arrows) flow in the same (opposite) direction in the two layers parallel (perpendicular) to the domain wall.

with the normalization $\mathcal{N} = \sqrt{2v_F\hbar/|\Delta|}$. Any superposition of the two zero-energy modes $\Phi = a_1\Psi_1 + a_2\Psi_2$ with complex coefficients $a_j = |a_j|e^{i\varphi_j}$ (and normalization $|a_1|^2 + |a_2|^2 = 1$) is also a zero-energy mode. Thus, the zero-energy eigenmodes are complex in general and only real for a special choice of the coefficients. These two modes are expressed separately for the top and for the bottom layer as

$$\Phi_\uparrow = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \frac{e^{-|\Delta||x|/\hbar v_F}}{\mathcal{N}}, \quad \Phi_\downarrow = \begin{pmatrix} a_1 \\ -a_2 \end{pmatrix} \frac{e^{-|\Delta||x|/\hbar v_F}}{\mathcal{N}}, \tag{4}$$

which will be used for the calculation of the Josephson currents in Section 3.

2.2 | Chiral electron–hole double layer

The BdG Hamiltonian of the EHDL reads.[24]

$$H_{\mathrm{EHDL}} = \begin{pmatrix} h_1\sigma_1 + h_2\sigma_2 & \Delta\sigma_3 \\ \Delta^*\sigma_3 & h_1\sigma_1 + h_2\sigma_2 \end{pmatrix}, \tag{5}$$

where the chirality of the two layers is the same now and the complex pairing order parameter appears with a Pauli matrix σ_3. This means that there is a coupling between the same metallic bands of the two layers with opposite signs though. The BdG Hamiltonian is dual to the BdG Hamiltonian of the EEDL in Equation (1), and the duality transformation reads

$$H_{\mathrm{EHDL}}(i\Delta) = VH_{\mathrm{EEDL}}(\Delta)V, \quad V = \begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \tag{6}$$

where the order parameter acquires a global imaginary unit. This implies for the eigenvalue equation $H_{\rm EEDL}(\Delta)\Psi_E = E\Psi_E$

$$H_{\rm EHDL}(i\Delta)V\Psi_E = VH_{\rm EEDL}(\Delta)VV\Psi_E = EV\Psi_E, \quad (7)$$

that is, $V\Psi_E$ is eigenmode of $H_{\rm EHDL}(\Delta)$ with eigenvalue E. Thus, the spectrum is invariant under the duality transformation, whereas the zero-energy eigenmodes are not. In particular, from Equation (3) for the domain wall $\Delta(x) = i{\rm sgn}(x)|\Delta|$ an exceptional point at zero energy and a two-dimensional eigenspace of zero-energy modes

$$V\Psi_1 = \frac{1}{\mathcal{N}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}e^{-|\Delta||x|/\hbar v_F}, \quad V\Psi_2 = \frac{1}{\mathcal{N}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}e^{-|\Delta||x|/\hbar v_F}. \quad (8)$$

From these modes, we can construct again the zero-energy modes of the individual layers as

$$V\Phi = \begin{pmatrix}\Phi'_\uparrow\\ \Phi'_\downarrow\end{pmatrix} \quad \text{with} \quad \Phi'_\uparrow = \begin{pmatrix}a_1\\a_2\end{pmatrix}\frac{e^{-|\Delta||x|/\hbar v_F}}{\mathcal{N}}, \quad \Phi'_\downarrow = \begin{pmatrix}-a_2\\a_1\end{pmatrix}\frac{e^{-|\Delta||x|/\hbar v_F}}{\mathcal{N}}. \quad (9)$$

3 | JOSEPHSON CURRENTS

Currents in a superconductor are either carried by paired particles (supercurrent j^s) or by quasiparticles (quasiparticle current j). These two types of currents are not independent but coupled through the continuity equation. This was demonstrated, for instance, in a paper by Blonder et al. who obtained this equation from the conservation of quasiparticle charge as.[11]

$$\partial_t\Phi\cdot\Phi + \partial_x I_x + \partial_y I_y = 0, \qquad I_{x,y} = j_{x,y} + j^s_{x,y} \quad (10)$$

with the time-dependent quasiparticle density $\Phi\cdot\Phi$. In the case of double layers, there are only intra-layer currents because the layers are separated by a dielectric interlayer. In this case, we replace Equation (10) with the two equations for the ↑ and ↓ layer, respectively

$$\partial_t\Phi_\sigma\cdot\Phi_\sigma + \partial_x I_{x\sigma} = 0 \quad (\sigma = \uparrow,\downarrow). \quad (11)$$

Here we have assumed that $\partial_y I_{y\sigma} = 0$ due to translational invariance in the y-direction.

The quasiparticle currents $j_{x\uparrow}$ and $j_{x\downarrow}$ are directly calculated from the commutator, with the BdG Hamiltonian as $j_x = \frac{i}{\hbar}[H_{\rm EEDL}, x]$; for the zero-energy quasiparticle modes of Equation (4) we get

$$j_{x\uparrow} = -j_{x\downarrow} = v_F\Phi_\uparrow\cdot\sigma_1\Phi_\uparrow = \frac{|\Delta|}{\hbar}Re\left(a_1a_2^*\right)e^{-2|\Delta||x|/\hbar v_F}. \quad (12)$$

These current components decay exponentially from the domain wall at $x = 0$ on the scale $\hbar v_F/2|\Delta|$. They flow in opposite directions in the two layers, as indicated in Figure 2a.

Returning to the continuity equations, we get from integrating along the x direction the x-components of the supercurrents as (cf. Appendix B)

$$j^s_{x\uparrow,\downarrow}(x) = \mp\frac{|\Delta|}{\hbar}Re\left(a_1a_2^*\right)\left(1 - e^{-2|\Delta||x|/\hbar v_F}\right) = \mp\frac{|\Delta|}{\hbar}Re\left(a_1a_2^*\right) - j_{x\uparrow,\downarrow}(x). \quad (13)$$

The supercurrent component $j^s_{x\uparrow,\downarrow}(x)$ vanishes at the domain wall $x = 0$ and becomes $\mp\frac{|\Delta|}{\hbar}Re\left(a_1a_2^*\right)$ for $|x| \gg \hbar v_F/|\Delta|$ with opposite sign for the two layers. Thus, at the domain wall, there is only a quasiparticle current. Moreover, from $j_y = \frac{i}{\hbar}[H_{\rm EEDL}, y]$, we get the y-components of the quasiparticle currents as

$$j_{y\uparrow} = j_{y\downarrow} = v_F\Phi_\uparrow\cdot\sigma_2\Phi_\uparrow = -\frac{|\Delta|}{\hbar}Im\left(a_1a_2^*\right)e^{-2|\Delta||x|/\hbar v_F}. \quad (14)$$

4 | DISCUSSION AND CONCLUSIONS

The main result of our calculation is the relation between the supercurrent and the quasiparticle current in the two layers, which is based on the continuity Equation (11). The total current is uniform and reads

$$I_{x\uparrow,\downarrow} = j^{s}_{x\uparrow,\downarrow}(x) + j_{x\uparrow,\downarrow}(x) = \mp\frac{|\Delta|}{\hbar} Re\left(a_1 a_2^*\right). \tag{15}$$

This result implies also a relation between the zero-energy modes of Equation (4) at the Fermi level and their currents in Equations (12), (13) and (14): The fact that the coefficients a_1 and a_2 of the zero-energy eigenmodes appear in the current indicates a coupling between those modes and the currents. This relation is remarkable because it means that we can tune the coefficients of the zero-energy mode by varying the current in the double layer. The latter can be induced by an external current, which couples to the current inside the superconductor. The creation and the measurement of currents in the EHDL was discussed in Ref. [31].

The quasiparticle current is created at the domain wall at $x = 0$ and decays according to Equations (12) and (14) exponentially from $x = 0$ on the scale $\hbar v_F/2|\Delta|$. This means that the decay depends on the material through its Fermi velocity v_F. For instance, in graphene and silicon carbide, we have $v_F \approx 8.3\times 10^5$ m/s,[32] and $v_F \approx 3.5\times 10^5$ m/s for black phosphorus.[33]

The BdG Hamiltonians $H_{\rm EDDL}$ of the EDDL and $H_{\rm EHDL}$ of the EHDL lead to similar results due to their duality relations in Equations (6), (7) and (8). In particular, the Josephson currents are the same for both cases, whereas the zero-energy quasiparticle modes are different. The latter is demonstrated by the transformation of Equation (8). Moreover, both Hamiltonians get a sign change under the following transformation

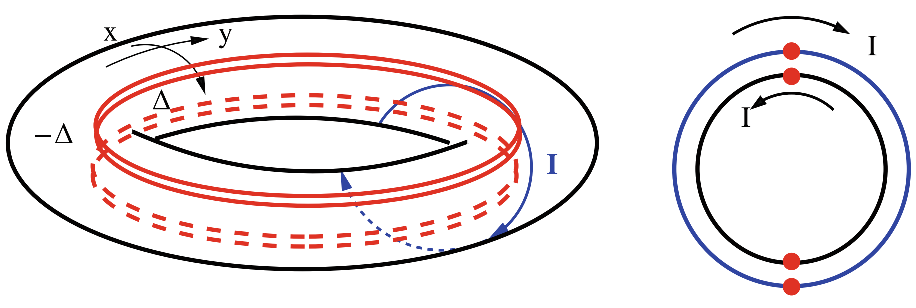
$$H_{\rm EEDL} \to T H_{\rm EEDL} T = -H_{\rm EEDL}, \quad T = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \tag{16}$$

and

$$H_{\rm EHDL} \to T' H_{\rm EHDL} T' = -H_{\rm EHDL}, \quad T' = VTV = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \tag{17}$$

which reflects the chirality of both quasiparticle Hamiltonians and the fact that their chirality transformations are not identical.

The above results indicate that a domain wall affects the current distribution in the system. Thus, for the general case, we must take into account all edges and domain walls, where the order parameter changes. On the other hand, we can avoid edges by choosing proper boundary conditions. In y direction, we have already assumed periodic boundary conditions to create a uniform mode in this direction. Assuming two domain walls (cf. Figure 2b) and periodic boundary conditions in x direction for both layers individually, the resulting system is a double torus with one inside the other. This geometry has no edges except for the domain walls, as visualized in Figure 3. Then the coefficients a_1 and a_2 of the zero-energy modes are fixed by the matching condition of Equation (15).



(a) (b)

FIGURE 3 After gluing both layers in Figure 2b) individually, we obtain a double torus with a cross-section visualized in b), where each torus has two domain walls. Then the currents wind along the two domain walls clockwise and counterclockwise around each torus, respectively.

While these considerations give us an idea about the role of the Josephson effect in chiral double layers, a complete description requires a solution of the entire microscopic model through a self-consistent approach. Then the supercurrent is induced by an external current or an external magnetic field, which is represented by a vector potential in the BdG Hamiltonian. This external field also affects the order parameter field Δ. Such a calculation is beyond the framework of the present approach but might be considered in the future.

In conclusion, we studied pairing in the EEDL and in the EHDL on the basis of the BdG equation. Both systems are connected by the duality relation (6). Although the systems have different quasiparticle modes in the presence of a domain wall, they have the same Josephson currents. The duality could be useful to connect two different branches in recent experiments, namely those with EEDL and with EHDL. Moreover, the Josephson current is related to the coefficients of the zero-energy mode through Equation (15). This reflects an intimate relationship between the Josephson current and the quasiparticle mode at zero energy. The coupling of the zero-energy modes and the current due to the Josephson effect has the potential for the development of new technologies. For instance, it was already used to create and manipulate qubits in quantum computational devices.[34–36] Finally, an extension of our analysis might be useful for future studies of more complex, multi-band systems with electron–electron and electron–hole pairing.

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DATA AVAILABILITY STATEMENT

Data sharing not applicable-no new data generated, the article describes entirely theoretical research.

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APPENDIX A. COALESCENT EIGENMODES

For the eigenmodes of the BdG Hamiltonian H_{EEDL}, we make the ansatz $\Psi(x) = \psi e^{-bx}$, where ψ is a four-component spinor and b depends on x. This gives for $k_y = 0$ with $\bar{b} = v_F \hbar b$ the four-dimensional eigenvalue equation

$$H_{\text{EEDL}}\psi = \begin{pmatrix} 0 & -i\bar{b} & 0 & -i\Delta \\ -i\bar{b} & 0 & i\Delta & 0 \\ 0 & -i\Delta & 0 & -i\bar{b} \\ i\Delta & 0 & -i\bar{b} & 0 \end{pmatrix} \psi = E\psi, \tag{A1}$$

which has the eigenvalue $E_- = -\sqrt{|\Delta|^2 - \bar{b}^2}$ with the pair of eigenspinors

$$\psi_{1-} = \begin{pmatrix} 1 \\ 0 \\ \bar{b}/\Delta \\ -i\sqrt{|\Delta|^2 - \bar{b}^2}/\Delta \end{pmatrix}, \quad \psi_{2-} = \begin{pmatrix} 0 \\ 1 \\ i\sqrt{|\Delta|^2 - \bar{b}^2}/\Delta \\ -\bar{b}/\Delta \end{pmatrix} \tag{A2}$$

and the eigenvalue $E_+ = \sqrt{|\Delta|^2 - \bar{b}^2}$ with the pair of eigenspinors

$$\psi_{1+} = \begin{pmatrix} 1 \\ 0 \\ \bar{b}/\Delta \\ i\sqrt{|\Delta|^2 - \bar{b}^2}/\Delta \end{pmatrix}, \quad \psi_{2+} = \begin{pmatrix} 0 \\ 1 \\ -i\sqrt{|\Delta|^2 - \bar{b}^2}/\Delta \\ -\bar{b}/\Delta \end{pmatrix}. \tag{A3}$$

This indicates a two-fold degeneracy of the eigenvalues $E_\pm$, respectively. The limit $\bar{b} \to \Delta$ yields $E = 0$ and the pairwise coalescent eigenspinors as

$$\psi_{1-} \to \psi_{1+} \to \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \psi_{2-} \to \psi_{2+} \to \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}. \tag{A4}$$

Thus, the eigenspace at $E = 0$ has only two dimensions, which represents an exceptional point.[37] This effect is known for solutions of the BdG Hamiltonian with point and line defects.[38]

APPENDIX B. JOSEPHSON CURRENTS IN A DOUBLE LAYER

With the help of the BdG equation, we derive the continuity equation for the two layers separately as.[11]

$$\partial_t \Phi_\sigma \cdot \Phi_\sigma + \partial_x I_{x\sigma} = 0 \quad (\sigma = \uparrow, \downarrow), \tag{B1}$$

where the total current $I = j + j^s$ is the sum of the quasiparticle current j and the supercurrent j^s. For the y component, we have $\partial_y I_{y\sigma} = 0$ due to the uniform mode in the y direction. The quasiparticle current operator of a BdG Hamiltonian $H_{\rm BdG}$ reads $j_x = \frac{i}{\hbar}[H_{\rm BdG}, x]$.

The BdG equation of the EEDL yields the continuity Equation (B1) in the top layer

$$\partial_t \Phi_\uparrow \cdot \Phi_\uparrow + \partial_x j_{x\uparrow} = i\frac{\Delta}{\hbar}\Psi_\downarrow^* \sigma_2 \Psi_\uparrow - i\frac{\Delta^*}{\hbar}\Psi_\uparrow^* \sigma_2 \Psi_\downarrow = 2\mathrm{sgn}(x)\frac{|\Delta|^2}{v_F \hbar^2} Re\left(a_1 a_2^*\right) e^{-2|\Delta||x|/\hbar v_F} \tag{B2}$$

and in the bottom layer

$$\partial_t \Phi_\downarrow \cdot \Phi_\downarrow + \partial_x j_{x\downarrow} = i\frac{\Delta^*}{\hbar}\Psi_\uparrow^* \sigma_2 \Psi_\downarrow - i\frac{\Delta}{\hbar}\Psi_\downarrow^* \sigma_2 \Psi_\uparrow = -2\mathrm{sgn}(x)\frac{|\Delta|^2}{v_F \hbar^2} Re\left(a_1 a_2^*\right) e^{-2|\Delta||x|/\hbar v_F}. \tag{B3}$$

The expressions on the right-hand side of the equations are equal up to a minus sign. The quasiparticle currents $j_{x\uparrow}$ and $j_{x\downarrow}$ are directly calculated from the commutator $j_x = \frac{i}{\hbar}[H_{\rm EEDL}, x]$, which gives

$$j_{x\uparrow} = -j_{x\downarrow} = -\frac{|\Delta|}{\hbar} Re\left(a_1 a_2^*\right) e^{-2|\Delta||x|/\hbar v_F}, \tag{B4}$$

such that we obtain from the continuity equations after integration along the x direction the x-components of the supercurrents as

$$j^s_{x\uparrow,\downarrow}(x) = \mp\frac{|\Delta|}{\hbar} Re\left(a_1 a_2^*\right)\left(1 - e^{-2|\Delta||x|/\hbar v_F}\right). \tag{B5}$$

The supercurrent component $j^s_{x\uparrow,\downarrow}(x)$ vanishes at the domain wall $x = 0$ and becomes $\mp\frac{|\Delta|}{\hbar} Re\left(a_1 a_2^*\right)$ for $|x| \gg \hbar v_F/|\Delta|$. Finally, from $j_y = \frac{i}{\hbar}[H_{\rm EEDL}, y]$ we get the y-components of the quasiparticle currents

$$j_{y\uparrow} = j_{y\downarrow} = -v_F \Phi_\downarrow \cdot \sigma_2 \Phi_\downarrow = -\frac{|\Delta|}{\hbar} Im\left(a_1 a_2^*\right) e^{-2|\Delta||x|/\hbar v_F}. \tag{B6}$$

The corresponding continuity equation of the EHDL reads for the top layer.[24]

$$\partial_t \Phi_\uparrow \cdot \Phi_\uparrow + \partial_x j_{x\uparrow} = i\frac{\Delta}{\hbar}\Psi_\downarrow^* \sigma_3 \Psi_\uparrow - i\frac{\Delta^*}{\hbar}\Psi_\uparrow^* \sigma_3 \Psi_\downarrow = 2\mathrm{sgn}(x)\frac{|\Delta|^2}{v_F \hbar^2} Re\left(a_1 a_2^*\right) e^{-2|\Delta||x|/\hbar v_F} \tag{B7}$$

and for the bottom layer

$$\partial_t \Phi_\downarrow \cdot \Phi_\downarrow + \partial_x j_{x\downarrow} = i\frac{\Delta^*}{\hbar}\Psi_\uparrow^* \sigma_3 \Psi_\downarrow - i\frac{\Delta}{\hbar}\Psi_\downarrow^* \sigma_3 \Psi_\uparrow = -2\mathrm{sgn}(x)\frac{|\Delta|^2}{v_F \hbar^2} Re\left(a_1 a_2^*\right) e^{-2|\Delta||x|/\hbar v_F}, \tag{B8}$$

since $i(\Delta - \Delta^*) = -2sgn(x)|\Delta|$. Together with the commutators $j_x = \frac{i}{\hbar}[H_{\rm EHDL}, x]$ and $j_y = \frac{i}{\hbar}[H_{\rm EHDL}, y]$, we obtain for the current components of the EHDL the same expression as given in Equations (12)–(14). This agreement of the currents is a consequence of the duality relation between the EEDL and the EHDL.