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The Gromov–Lawson–Chernysh surgery theorem

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Abstract
This is an expository article without any claim of originality. We give a complete
and self-contained account of the Gromov–Lawson–Chernysh surgery theorem for
positive scalar curvature metrics.

Keywords Positive scalar curvature · Surgery · Homotopy equivalence

1 Introduction

A famous result by Gromov–Lawson [8] and Schoen–Yau [22] states that if M^d is a
closed manifold with a metric of positive scalar curvature and $\varphi : S^{d-k} \times \mathbb{R}^k \to M$ a
surgery datum of codimension $k \geq 3$, then the manifold $M_\varphi := M \backslash (S^{d-k} \times D^k) \cup_{S^{d-k} \times S^{k-1}} D^{d-k+1} \times S^{k-1}$ does have a metric of positive scalar curvature as
well. This has been the basis for virtually all existence results for psc metrics on
high-dimensional manifolds, the most prominent of which is [23].

A strengthening of the surgery theorem has been proven by Chernysh [3], based
on Gromov–Lawson's proof. His result implies that the two spaces $\mathcal{R}^+(M)$ and
$\mathcal{R}^+(M_\varphi)$ of psc metrics have the same homotopy type if in addition to $k \geq 3$ the
condition $d - k + 1 \geq 3$ is also satisfied.

To state Chernysh's theorems in full generality, some preliminaries are needed. To
keep the length of this introduction at bay, we state the results somewhat informally
and refer to the main body of the paper for precise definitions. We consider

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Riemannian metrics on compact manifolds with boundary (it is always assumed that the boundaries are equipped with collars). Let $\mathcal{R}(M)$ be the space of all Riemannian metrics h on M such that $h = g + \mathrm{d}t^2$ near ∂M, for some metric g on ∂M, and with respect to the given collar. Let $\mathcal{R}^+(M) \subset \mathcal{R}(M)$ be the subspace of metrics of positive scalar curvature. If $h \in \mathcal{R}^+(M)$ is of the form $g + \mathrm{d}t^2$ near ∂M, then g has positive scalar curvature as well, and hence mapping h to g defines a continuous restriction map

$$\mathrm{res} : \mathcal{R}^+(M) \to \mathcal{R}^+(\partial M).$$

We define

$$\mathcal{R}^+(M)_g := \mathrm{res}^{-1}(g),$$

the space of all Riemannian metrics of positive scalar curvature on M which near ∂M are equal to $g + \mathrm{d}t^2$.

Theorem 1.1 (*Chernysh* [4]) *The restriction map* $\mathrm{res} : \mathcal{R}^+(M) \to \mathcal{R}^+(\partial M)$ *is a Serre fibration.*

In fact, this is a slight improvement of the main result of [4], where it is only shown that res is a quasifibration. The proof of Theorem 1.1 is given in Sect. 5.1 and follows largely the idea of [4].

Now let N be a compact manifold with collared boundary and let $\varphi : N \times \mathbb{R}^k \to M$ be an open embedding such that $\varphi^{-1}(\partial M) = \partial N \times \mathbb{R}^k$ and such that φ is compatible with the chosen collars of M and N. Let $g_N \in \mathcal{R}(N)$ be a Riemannian metric on N, not necessarily of positive scalar curvature.

Let g_{tor}^k be a torpedo metric on $\mathbb{R}^k$ such that $\mathrm{scal}(g_{\mathrm{tor}}^k) + \mathrm{scal}(g_N) = \mathrm{scal}(g_N + g_{\mathrm{tor}}^k) > 0$. The precise definition of a torpedo metric will be given in (2.9) below, and for the time being, let us only list the most important features. First, g_{tor}^k is an $O(k)$-invariant metric on $\mathbb{R}^k$. Second, let $\psi : (0, \infty) \times S^{k-1} \to \mathbb{R}^k$ be the polar coordinate map and $\mathrm{d}\xi^2$ be the round metric on S^{k-1}. We require that $\psi^* g_{\mathrm{tor}}^k = \mathrm{d}t^2 + \delta \mathrm{d}\xi^2$ on $[R, \infty) \times S^{k-1}$ for some $R > 0$ and $\delta > 0$. Third, $\mathrm{scal}(g_{\mathrm{tor}}^k) \geq \frac{1}{\delta^2}(k-1)(k-2)$. We define the subspace

$$\mathcal{R}^+(M, \varphi) := \left\{ h \in \mathcal{R}^+(M) | \varphi^* h|_{N \times B_R^k} = (g_N + g_{\mathrm{tor}}^k)|_{N \times B_R^k} \right\} \subset \mathcal{R}^+(M).$$

Theorem 1.2 (*Chernysh* [3], *see also Walsh* [29]) *Let* $\varphi : N \times \mathbb{R}^k \to M$ *be an open embedding as before with* $k \geq 3$. *Let* $g_N \in \mathcal{R}(M)$ *be a Riemannian metric on N, not necessarily of positive scalar curvature. Let* g_{tor}^k *be a torpedo metric on* $\mathbb{R}^k$ *so that the product metric* $g_N + g_{\mathrm{tor}}^k$ *on* $N \times \mathbb{R}^k$ *has positive scalar curvature. Then the inclusion map*

$$\mathcal{R}^+(M, \varphi) \to \mathcal{R}^+(M)$$

is a weak homotopy equivalence.

The main bulk of this paper is devoted to a detailed discussion of the proof of Theorem 1.2.

Remark 1.3 What Gromov and Lawson proved is that under the hypotheses of Theorem 1.2 and for closed N, $\mathcal{R}^+(M, \varphi) \neq \emptyset$, provided that $\mathcal{R}^+(M) \neq \emptyset$. Later, Gajer [6] improved their result and proved that the inclusion map $\mathcal{R}^+(M, \varphi) \to \mathcal{R}^+(M)$ is 0-connected.

Remark 1.4 If $N = S^{d-k}$ and g_N is the round metric, one obtains a zig-zag

$$\mathcal{R}^+(M) \overset{\simeq}{\leftarrow} \mathcal{R}^+(M, \varphi) \cong \mathcal{R}^+(M_\varphi, \varphi') \to \mathcal{R}^+(M_\varphi),$$

where $\varphi' : S^{k-1} \times \mathbb{R}^{d-k+1} \to M_\varphi$ is the opposite surgery datum. It follows that $\mathcal{R}^+(M_\varphi) \neq \emptyset$ and $\mathcal{R}^+(M) \simeq \mathcal{R}^+(M_\varphi)$ if $3 \leq k \leq n - 2$.

More generally, Theorem 1.2 implies the following cobordism invariance result.

Theorem 1.5 *Let $\theta : B \to BO(d)$ be a fibration, $d \geq 6$. Assume that M_i, $i = 0, 1$, are two closed $(d-1)$-dimensional θ-manifolds which are θ-cobordant. Then*

1. *if the structure map $M_1 \to B$ is 2-connected, then there is a map $\mathcal{R}^+(M_0) \to \mathcal{R}^+(M_1)$ (in particular, if $\mathcal{R}^+(M_0) \neq \emptyset$, then $\mathcal{R}^+(M_1) \neq \emptyset$).*
2. *If in addition the structure map $M_0 \to B$ is 2-connected as well, then $\mathcal{R}^+(M_0) \simeq \mathcal{R}^+(M_1)$.*

The best-known special case is $B = B\mathrm{Spin}(d)$. In that case, the hypothesis that $M_i \to B\mathrm{Spin}(d)$ is 2-connected just means that M_i is simply connected. For such manifolds, Theorem 1.5 follows in a straightforward manner from Theorem 1.2 and the proof of the h-cobordism theorem (see, e.g. [15, Theorem VIII.4.1]), as explained in [29, §4]. The general case requires techniques from surgery and handlebody theory which are not so well known, which is why we include the proof in Sect. 6. Chernysh also proved a version of Theorem 1.2 for a fixed boundary condition, which is used in an essential way in [5]. To state it, let $\partial \varphi : \partial N \times \mathbb{R}^k \to \partial M$ be the induced embedding and let $g \in \mathcal{R}^+(\partial M, \partial \varphi)$ be a fixed boundary condition. We let

$$\mathcal{R}^+(M, \varphi)_g := \mathcal{R}^+(M, \varphi) \cap \mathcal{R}^+(M)_g.$$

Theorem 1.6 (*Chernysh* [4]) *Under the hypotheses of Theorem* 1.2, *the inclusion map*

$$\mathcal{R}^+(M, \varphi)_g \to \mathcal{R}^+(M)_g$$

is a weak homotopy equivalence.

The proof of Theorem 1.6 is only sketched in [4]. We give a proof, somewhat different from the proof envisioned in [4], in Sect. 5.2. Besides Theorems 1.2 and 1.1, the proof uses the (elementary) corner smoothing technique which was developed in [5, §2].

When [3] appeared, the result was apparently perceived as a curiosity and drew little attention. This has changed in recent years: Theorem 1.2 is an irreplaceable ingredient in the papers [2, 5]. Important parts of [3] are written in a fairly obscure

way, and the paper has never been published. Later Walsh published a paper [29] containing a proof of Theorem 1.2, but many relevant details are not addressed in [29]. Because of the importance of the result for [2] and [5], the first named author wanted to make sure that the result is correct and that he understands the proof properly. He suggested checking [3] and [29] as a project for the second author's Master's thesis. The present paper is the result of this checking process. Let us summarize our findings.

1. One half of the proof of Theorem 1.2 is virtually identical to the proof of the original Gromov–Lawson result. We found one small computational error, which is reproduced in various expositions of the result ([21, 28]). This error looks harmless at first sight, but enforces an alternative argument at one key juncture of the proof. Therefore, we decided to give a self-contained treatment of the whole proof.
2. All other arguments in Chernysh's paper are essentially correct and complete, albeit some parts of his paper are very intransparent and hard to decipher.
3. [29] leaves many questions open. In particular, it remains unclear to us how to fill in the details of the proof of Lemma 3.3 loc.cit., without using the quite technical computations of [3, §3] or computations of a similar delicacy.

2 Preliminary material

2.1 Spaces of psc metrics on manifolds with boundary

Let M be a compact manifold with boundary ∂M. We assume that the boundary of M comes equipped with a collar $\partial M \times [0, 1) \to M$. The collar identifies $\partial M \times [0, 1)$ with an open subset of M and we usually use this identification without further mentioning.

We only consider Riemannian metrics on M which have a simple structure near ∂M. More precisely, for $c \in (0, 1)$, we denote by $\mathcal{R}(M)^c$ the space of all Riemannian metrics h on M such that $h = g + \mathrm{d}t^2$ on $\partial M \times [0, c]$ for some metric g on ∂M.

We topologize $\mathcal{R}(M)^c$ as a subspace of the Fréchet space of smooth symmetric $(2, 0)$-tensor fields on M, with the usual C^∞-topology.

Now let $\mathcal{R}^+(M)^c \subset \mathcal{R}(M)^c$ be the subspace of all Riemannian metrics with positive scalar curvature (this is an open subspace). It follows from [18, Theorem 13] and [9, Proposition A.11] that $\mathcal{R}^+(M)^c$ has the homotopy type of a CW complex.

If $h \in \mathcal{R}^+(M)^c$ and $h = g + \mathrm{d}t^2$ on $\partial M \times [0, c]$, then $\mathrm{scal}(g + \mathrm{d}t^2) = \mathrm{scal}(g)$ and so the metric g on ∂M necessarily has positive scalar curvature. This defines a restriction map

$$\mathrm{res}^c : \mathcal{R}^+(M)^c \to \mathcal{R}^+(\partial M),$$

which is continuous. We define

$$\mathcal{R}^+(M)_g^c := (\mathrm{res}^c)^{-1}(g),$$

the space of all psc metrics on M which on $\partial M \times [0, c]$ are equal to $g + \mathrm{d}t^2$. Moreover, we define

$$\mathcal{R}^+(M) := \mathrm{colim}_{c \to 0} \mathcal{R}^+(M)^c$$

and

$$\mathcal{R}^+(M)_g := \mathrm{colim}_{c \to 0} \mathcal{R}^+(M)_g^c.$$

If $b > c$, then $\mathcal{R}^+(M)^b \subset \mathcal{R}^+(M)^c$ and $\mathcal{R}^+(M)_g^b \subset \mathcal{R}^+(M)_g^c$, and it is elementary to see that the inclusion maps are homotopy equivalences [2, Lemma 2.1].

The restriction maps induce a restriction map

$$\mathrm{res} : \mathcal{R}^+(M) \to \mathcal{R}^+(\partial M)$$

on the colimit, and there is a continuous bijection

$$\mathcal{R}^+(M)_g \to \mathrm{res}^{-1}(g). \tag{2.1}$$

There is no a priori reason why (2.1) should be a homeomorphism. However,

Lemma 2.2 *The map* (2.1) *is a weak homotopy equivalence.*

Proof The inclusion maps $\mathcal{R}^+(M)^b \to \mathcal{R}^+(M)^c$ and $\mathcal{R}^+(M)_g^b \to \mathcal{R}^+(M)_g^c$ are closed embeddings. Hence the Lemma then follows from the next one, which is a general fact. $\qquad\square$

Lemma 2.3 *Let $X_0 \to X_1 \to X_2 \to X_3 \to \dots$ be a sequence of closed embeddings of Hausdorff spaces and let $f_n : X_n \to Y$ be a compatible sequence of maps. Then the continuous bijection $\psi : \mathrm{colim}_n(f_n^{-1}(y)) \to (\mathrm{colim}_n f_n)^{-1}(y)$ is a weak homotopy equivalence.*

Proof It is enough to prove that if K is compact Hausdorff and $h : K \to \mathrm{colim}_n(f_n^{-1}(y))$ is a map (of sets), then h is continuous if and only if $\psi \circ h$ is continuous. If $g := \psi \circ h$ is continuous, then we can consider g as a map to $\mathrm{colim}_n X_n$. By [24, Lemma 3.6], there is n and $k : K \to X_n$ (continuous), so that $g := i_n \circ k$ ($i_n : X_n \to \mathrm{colim}_n X_n$ is the natural map). Now k maps into $f_n^{-1}(y)$, and so h can be written as the composition $K \xrightarrow{k} f_n^{-1}(y) \to \mathrm{colim}_n(f_n^{-1}(y))$ of continuous maps. $\qquad\square$

2.2 The trace construction

For the proof of both, Theorems 1.2 and 1.1, the following tedious but straightforward calculation using the standard formulas of Riemannian geometry is needed. We include the proof in Appendix A because we do not know an explicit reference.

Lemma 2.4 *Let $g : \mathbb{R} \to \mathcal{R}(M)$ be a smooth path. Let $h := \mathrm{d}t^2 + g(t)$ be the induced metric on $\mathbb{R} \times M$. Then the scalar curvature of h is given by the formula*

$$\mathrm{scal}(h) = \mathrm{scal}(g(t)) + \frac{3}{4} g^{ik} g^{jl} g_{ij,0} g_{kl,0} - g^{kl} g_{kl,00} - \frac{1}{4} g^{ik} g^{jl} g_{ik,0} g_{jl,0}.$$

Here we use a local coordinate system in M and the Einstein summation convention. Moreover, g_{ij} are the components of the metric tensor of g, g^{ij} the components of its inverse. A symbol as $g_{ij,k}$ denotes the derivative of g_{ij} with respect to the kth coordinate and similarly for higher derivatives. The 0th direction is the $\mathbb{R}$-direction.

We will use the following consequence of Lemma 2.4 in later sections.

Lemma 2.5 [6] *Let M be a compact manifold, P a compact space and let $G : P \times [0, 1] \to \mathcal{R}^+(M)$ be continuous and assume that $G(p, \_)$ is smooth for all p, and that all derivatives of G in the $[0, 1]$-direction depend continuously on (p, t). Assume that for each $p \in P$, there is $B_p \in \mathbb{R}$ such that $\mathrm{scal}(G(p, t)) \geq B_p$ for all $t \in [0, 1]$. Then for each $\eta > 0$, there is $\Lambda > 0$, such that if $f : \mathbb{R} \to [0, 1]$ is a smooth function with $\|f'\|_{C^0}, \|f''\|_{C^0} \leq \Lambda$, then the metric $G(p, f(t)) + \mathrm{d}t^2$ on $M \times \mathbb{R}$ satisfies $\mathrm{scal}(G(p, f(t)) + \mathrm{d}t^2) \geq B_p - \eta$.*

Proof Lemma 2.4 shows that there is $C > 0$ so that

$$\mathrm{scal}\big(G(p, f(t) + \mathrm{d}t^2\big) \geq B_g - C\big(\|f'\|_{C^0} + \|f''\|_{C^0}\big),$$

which immediately implies the claim.   □

2.3 Rotationally symmetric metrics

Let $\psi : (0, \infty) \times S^{k-1} \to \mathbb{R}^k \setminus \{0\}, (t, v) \mapsto tv$ be the polar coordinate map. We denote by $\mathrm{d}\xi^2$ the round metric on S^{k-1}. Furthermore, $S_r^{k-1} \subset \mathbb{R}^k$ denotes the sphere of radius r.

Lemma 2.6 *Let g be an $O(k)$-invariant Riemannian metric on B_R^k, i.e. for all $A \in O(k)$, we have $A^* g = g$.*

(1) *There exist smooth functions $a, f : (0, R) \to (0, \infty)$, such that $\psi^* g = a(t)^2 \mathrm{d}t^2 + f(t)^2 \mathrm{d}\xi^2$.*
(2) *$a(t) \equiv 1$ holds if and only if the rays $t \mapsto tv$ are unit speed geodesics for all $v \in S^{k-1}$. In this case we call g a normalized rotationally symmetric metric.*
(3) *Under the hypothesis of (2), f is the restriction of an odd smooth function $f : \mathbb{R} \to \mathbb{R}$ with $f'(0) = 1$. We call f the warping function of g.*
(4) *In that situation, the scalar curvature of g is given by*

$$\mathrm{scal}(g) = (k - 1)\left((k - 2)\frac{1 - f'^2}{f^2} - 2\frac{f''}{f} \right). \tag{2.7}$$

Proof For part (1), one uses that for each $v \in S^{k-1}$, there is an $A \in O(k)$ such that $Av = v$ and $A|_{v^\perp} = -\mathrm{id}$. It follows that at each point $0 \neq x \in B_R^k$, the spaces $\mathrm{span}\{x\}$ and $T_x(S_{\|x\|}^{k-1})$ are orthogonal with respect to g. Since $\mathrm{d}\xi^2$ is, up to a constant multiple, the only $O(k)$-invariant metric on S^{k-1}, the claim follows. Part (2) is clear. Part (3) can be found in [19, §3.4], and the computation for part (4) in [19, p. 69]. $\square$

We denote the scalar curvature of the metric $\mathrm{d}t^2 + f(t)^2 \mathrm{d}\xi^2$ by

$$\sigma(f) := \mathrm{scal}(\mathrm{d}t^2 + f(t)^2 \mathrm{d}\xi^2) = (k-1)\left((k-2)\frac{1-f'^2}{f^2} - 2\frac{f''}{f}\right). \tag{2.8}$$

The function $f(t) = \sin(t)$ on $[0, \pi)$ gives a metric which is isometric to the usual round metric on S^k. It has $\sigma(f) = k(k-1)$. Let us now give the precise definition of the torpedo metrics.

Definition 2.9 A *torpedo function of radius $\delta > 0$* is a function $f : [0, \infty) \to \mathbb{R}$ which is the restriction of a smooth odd function with $f'(0) = 1$, such that

(1) $0 \leq f' \leq 1$,
(2) $f'' \leq 0$,
(3) there is $R > 0$ so that $f \equiv \delta$ near $[R, \infty)$,
(4) $\sigma(f) \geq \frac{1}{\delta^2}(k-1)(k-2)$.

The metric $\mathrm{d}t^2 + f(t)^2 d\xi^2$ on $\mathbb{R}^k$ is called a *torpedo metric of radius δ*.

Let us give a concrete construction of a torpedo function. Let $\epsilon > 0$ be small and let $u : [0, \infty) \to \mathbb{R}$ be a function satisfying

- $u \equiv \mathrm{id}$ on $[0, \frac{\pi}{2} - \epsilon]$,
- $u \equiv \frac{\pi}{2}$ on $[\frac{\pi}{2} + \epsilon, \infty)$,
- $u'' \leq 0$ (together with the previous conditions, this implies $0 \leq u' \leq 1$).

We define $h_1(t) := \sin(u(t))$. By (2.7) we have

$$\sigma(h_1) = u'(t)^2 k(k-1) + (1 - u'(t)^2)\frac{(k-1)(k-2)}{\sin(u(t))^2} - 2(k-1)\frac{\cos(u(t))}{\sin(u(t))}u'' \geq$$

$$u'(t)^2 k(k-1) + (1 - u'(t)^2)(k-1)(k-2) \geq (k-1)(k-2),$$

so that h_1 is indeed a torpedo function of radius 1 (with $R \geq \frac{\pi}{2} + \epsilon$). For $\delta > 0$, the function

$$h_\delta(t) := \delta h_1\left(\frac{t}{\delta}\right) \tag{2.10}$$

is a torpedo function of radius δ. *For the rest of this paper, we fix a torpedo function h_1 of radius 1, and define $h_\delta(t) := \delta h_1(\frac{t}{\delta})$.*

3 The parametrized Gromov–Lawson construction

In this and the following section, we prove Theorem 1.2, and we begin with the precise statement. Let N and M be compact manifolds with collared boundary and let $\varphi : N \times \mathbb{R}^k \to M$ be an open embedding with $k \geq 3$. We assume that $\varphi^{-1}(\partial M) = (\partial N) \times \mathbb{R}^k$ and let $\partial \varphi : \partial N \times \mathbb{R}^k \to \partial M$ be the induced embedding. Furthermore, we assume that φ is compatible with the chosen collars, that is, if $(x, t, v) \in (\partial N) \times [0, 1) \times \mathbb{R}^k \subset N \times \mathbb{R}^k$, then

$$\varphi(x, t, v) = (\partial \varphi(x, v), t) \in \partial M \times [0, 1) \subset M.$$

From now on, we usually identify $N \times \mathbb{R}^k$ with an open subset of M via φ.

Let g_N be a Riemannian metric on N which is of the form $g_{\partial N} + \mathrm{d}t^2$ on $\partial N \times [0, 1)$. *It is not required that* $\mathrm{scal}(g_N) > 0$. Let

$$A := \inf(\mathrm{scal}(g_N)) \in \mathbb{R}$$

and pick $\delta > 0$ so that

$$\frac{1}{\delta^2}(k-1)(k-2) + A > 0.$$

Let g_{tor}^k be a torpedo metric on $\mathbb{R}^k$ of radius δ, and let $R > 0$ be as in Definition 2.9. For $c > 0$, define

$$\mathcal{R}^+(M, \varphi)^c := \{g \in \mathcal{R}^+(M)^c \mid g|_{N \times B_R^k} = (g_N + g_{\mathrm{tor}}^k)|_{N \times B_R^k}\}$$

and $\mathcal{R}^+(M, \varphi) := \mathrm{colim}_{c \to 0} \mathcal{R}^+(M, \varphi)^c$.

Theorem 3.1 *The inclusion maps*

$$\mathcal{R}^+(M, \varphi)^c \to \mathcal{R}^+(M)^c \text{ and } \mathcal{R}^+(M, \varphi) \to \mathcal{R}^+(M)$$

are weak homotopy equivalences.

The proof that we give will apply simultaneously to both cases, and for notational simplicity, we deal only with $\mathcal{R}^+(M, \varphi) \to \mathcal{R}^+(M)$. The proof is in two steps. We first introduce an intermediate space $\mathcal{R}^+(M, \varphi) \subset \mathcal{R}_{\mathrm{rot}}^+(M) \subset \mathcal{R}^+(M)$.

Definition 3.2 If M, N and g_N is as above, we define $\mathcal{R}_{\mathrm{rot}}^+(M) \subset \mathcal{R}^+(M)$ as the subspace of those g, such that there exists a rotationally symmetric normalized psc metric $g_{B_R^k}$ such that

$$g|_{N \times B_R^k} = g_N + g_{B_R^k}.$$

In this section, we show

Proposition 3.3 *The inclusion map*

$$\mathcal{R}_{\mathrm{rot}}^{+}(M) \to \mathcal{R}^{+}(M)$$

is a weak homotopy equivalence.

The proof of Proposition 3.3 is essentially the same as the original argument by Gromov and Lawson [8] (but note that Rosenberg–Stolz [21] corrected mistakes in [8]).

3.1 Adapting tubular neighbourhoods

In the proof of Theorem 1.2, we shall use several devices to change a Riemannian metric. One such device (which plays a minor, more technical role) is by suitable isotopies.

Definition 3.4 A Riemannian metric g on M is *normalized on the r_0-tube around N* if for each $p \in N$ and $v \in S^{k-1}$, the curve $[0, r_0] \to N \times \mathbb{R}^k \subset M$, $t \mapsto (p, tv)$, is a unit speed geodesic.

For example, each $g \in \mathcal{R}_{\mathrm{rot}}^{+}(M)$ is, by definition, normalized on the R-tube around N.

Note that the condition defined above relates the metric g to the embedding φ, and therefore it would be more appropriate to call it *normalized on the r_0-tube around N with respect to φ*. Since the embedding φ will be fixed all the time and since we exclusively consider the normalization condition with respect to φ, we decided to use the less precise wording.

Proposition 3.5 *(Adapting tubular neighbourhoods) Let (K, L) be a finite CW-pair and let $G : K \to \mathcal{R}(M)$ be continuous, so that $G(x)$ is normalized on the r-tube around N when $x \in L$. Then there exists $r_0 \in (0, r]$ and a continuous map $F : [0, 1] \times K \to \mathrm{Diff}(M)$ such that $F(t, x) = \mathrm{id}$ if $(t, x) \in (\{0\} \times K) \cup ([0, 1] \times L)$, $F(t, x)|_N = \mathrm{id}$ and such that $F(1, x)^* G(x)$ is normalized on the r_0-tube around N, for all $x \in K$.*

Proof The embedding $\varphi : N \times \mathbb{R}^k \to M$ identifies the normal bundle v_N^M with the trivial vector bundle $N \times \mathbb{R}^k$. For each Riemannian metric g on M, there are maps

$$\varphi_g : N \times \mathbb{R}^k \xrightarrow{\cong} \nu_N^M \xrightarrow{\gamma_g} TN^\perp \xdashrightarrow{\exp_g} M.$$

The first is the fixed isomorphism, the second is induced by the bundle metric g and the third is the Riemannian exponential map of g (and is only partially defined). The metric g is normalized on the r_0-tube around N if and only if φ_g is defined on $N \times B_{r_0}^k$ and agrees with φ there. Since N and K are compact, there is $r_0 > 0$ so that $\varphi_{G(x)}$ is defined on $N \times B_{r_0}^k$, injective and has image in $N \times \mathbb{R}^k \subset M$. There is an isotopy

$$H : [0, 1] \times K \times (N \times B_{r_0}^k) \to M$$

of embeddings such that $H(t, x, \_) = \varphi_{G(x)}$ for all $(t, x) \in (\{0\} \times K) \cup ([0, 1] \times L)$

and such that $H(1, x, \_) = \varphi$ for all $x \in K$. In the case $K = *$, this follows from the well-known result that tubular neighborhoods are unique up to isotopy [11, Theorem 4.5.3, 4.6.5]. The proof given in loc.cit. carries over to the parametrized and relative case without change.

An instance of the parametrized isotopy extension theorem [27, Theorem 6.1.1] shows that there exists $F : [0, 1] \times K \to \mathrm{Diff}(M)$ with $F(t, x) = \mathrm{id}$ when $(t, x) \in (\{0\} \times K) \cup ([0, 1] \times L)$ and $F(t, x)|_{N \times B_{r_0}^k} = H(t, x, \_)$. The Riemannian metric $F(1, t)^* G(x)$ is normalized on the r_0-tube around N. $\square$

3.2 Gromov–Lawson curves

One important step in the proof of Theorem 1.2 (well explained in, e.g. [21, 28]) is to obtain a deformation of a psc metric g on M by a deformation of M inside $M \times \mathbb{R}$ and to take the metric induced by $g + \mathrm{d}t^2$.

Definition 3.6 A *Gromov–Lawson curve* Γ is a smooth map $\Gamma : [0, 1] \times [0, \infty) \to \mathbb{R}^2$ such that

(1) $\Gamma(0, s) = (0, s)$ for all s,
(2) each curve $\Gamma_\lambda := \Gamma(\lambda, \_)$ is an embedding,
(3) there exists $\rho > r_0 > 0$ such that for all $\lambda \in [0, 1]$, $\Gamma_\lambda(s)$ lies on the positive r-axis for all $s \geq r_0$ and $\Gamma_\lambda(s) = (0, s)$ for all $s \geq \rho$. We call r_0 the *outer width* of Γ.
(4) Γ_1 has a horizontal line segment of height r_∞, i.e. there exists $0 < y_4 < y_5 \in \mathbb{R}$ such that the line segment between the points (y_4, r_∞) and (y_5, r_∞) lies in the image of Γ_1. We call r_∞ the *inner width* of Γ, and $\ell := |y_4 - y_5|$ is the *length of* Γ.
(5) $\Gamma_\lambda(0)$ lies on the y-axis, and this is the only point where Γ_λ meets the y-axis. Moreover, it does so at a right angle and follows the arc of a circle (of possibly infinite radius) in the region where $r \leq \frac{1}{2} r_\infty$.

A typical Gromov-Lawson curve is shown in Fig. 1. The indicated points (y_i, r_i) are important for the construction of these curves.

A Gromov–Lawson curve determines an isotopy of embeddings $E_\Gamma : [0, 1] \times M \to M \times \mathbb{R}$. Write Γ_λ^i, $i = 1, 2$, for the components of Γ_λ. First, we define $E_\Gamma : [0, 1] \times N \times \mathbb{R}^k \to N \times \mathbb{R}^k \times \mathbb{R}$ by

$$E_\Gamma(\lambda, p, v) := (p, \Gamma_\lambda^2(\|v\|) \frac{v}{\|v\|}, \Gamma_\lambda^1(\|v\|)) \tag{3.7}$$

(which is smooth by the condition on Γ near the x-axis). For $\|v\| \geq \rho$, $E_\Gamma(\lambda, p, v) = (p, v)$, and so we can extend $E_\Gamma(\lambda, \_)$ as the identity over all of M. Note that $E_\Gamma(0, \_)$ is just the inclusion $x \mapsto (x, 0)$. Let g_{Γ_λ} be the Riemannian metric

$$g_{\Gamma_\lambda} := E_\Gamma(\lambda, \_)^*(g + \mathrm{d}t^2)$$

on M obtained by restricting the product metric on $M \times \mathbb{R}$ to the image of $E_\Gamma(\lambda, \_)$
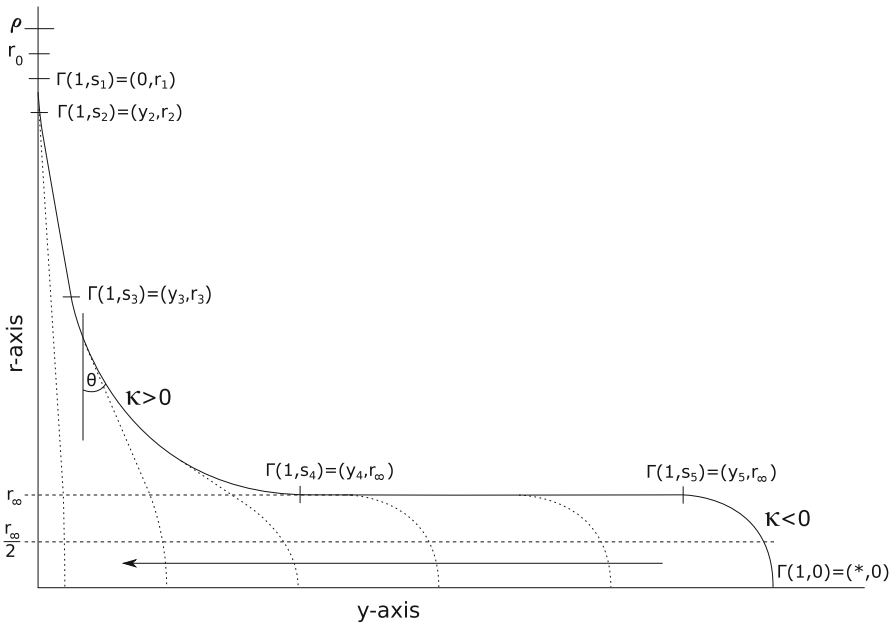
Fig. 1 A Gromov–Lawson curve (γ, H)

and pulling back to M. The key argument for the proof of Proposition 3.3 is the following result.

Proposition 3.8 *Let $K \subset \mathcal{R}(M)$ be compact. Suppose that each $g \in K$ is normalized on the r_0-tube around N and that $\mathrm{scal}(g) \geq B_g$ for some B_g. For every $\epsilon_0 > 0$, $\eta > 0$, there exists a Gromov–Lawson curve Γ such that*

(1) *the outer width of Γ is at most r_0,*
(2) *the inner width of Γ is at most ϵ_0,*
(3) $\mathrm{scal}(g_{\Gamma_\lambda}) \geq B_g - \eta$ *for all $g \in K$ and all $\lambda \in [0, 1]$.*
(4) *Moreover, for each $\ell > 0$, we can arrange the length to be at least ℓ.*

3.3 Construction of the Gromov–Lawson curve

In this subsection, we prove Proposition 3.8. We need a formula for the scalar curvature of the metric g_{Γ_λ}.

Let $I \subset \mathbb{R}$ be an interval and let $\gamma : I \to \mathbb{R}_+^2 := \{(y, r) | r \geq 0\}$ be a smooth embedded curve. We assume that whenever $\gamma(t)$ lies on the y-axis, then near t, γ follows a circle of possibly infinite radius perpendicular to the y-axis. Consider the hypersurface

$$Q_\gamma := \left\{(p, v, r) \in N \times \mathbb{R}^k \times \mathbb{R} \mid (r, \|v\|) \in \mathrm{Im}(\gamma)\right\} \subset N \times \mathbb{R}^k \times \mathbb{R} \subset M \times \mathbb{R}$$

(which is smooth because of the condition on γ near the y-axis). Let us recall some

formulas from the geometry of plane curves. In the situation we consider, the derivative vector of γ will lie in the fourth quadrant. We let θ be the angle between γ and the negative r-axis and let κ be the signed curvature of γ. If γ is *parametrized by arc-length*, the curvature is given by

$$\ddot{\gamma}(s) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \dot{\gamma}(s) \cdot \kappa(s) \tag{3.9}$$

or

$$\kappa(s) = \langle \ddot{\gamma}(s), \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \dot{\gamma}(s) \rangle.$$

The angle is given by the formula

$$\sin(\theta) = \langle \dot{\gamma}, e_1 \rangle; \quad \cos(\theta) = -\langle \dot{\gamma}, e_2 \rangle.$$

Note that

$$\frac{d}{ds} \theta(s) = \kappa(s). \tag{3.10}$$

If γ meets the y-axis in a circle of radius $\rho < \infty$, then $\kappa = -\frac{1}{\rho}$, $r = \sin(\theta)\rho$ near that point.

For a given Riemannian metric g on $N \times \mathbb{R}^k$, we get the Riemannian metric g_γ on Q_γ, obtained by restricting the product metric $g + dt^2$ on $N \times \mathbb{R}^{k+1}$ to Q_γ.

Lemma 3.11 *Let $K \subset \mathcal{R}(M)$ be compact. Assume that all $g \in K$ are normalized on the r_0-tube around N. Then there exists $0 < r_1 \leq r_0$ and $C > 0$ such that for all $g \in K$, and for all immersed curves in the region $\{(y,t) \in \mathbb{R}^2 | 0 < y \leq r_1\}$, we have*

$$\mathrm{scal}(g_\gamma) \geq \mathrm{scal}(g) + |\kappa| \left(-\mathrm{sign}(\kappa) \frac{2(k-1)\sin(\theta)}{r} - C\sin(\theta) \right)$$
$$+ \frac{(k-1)(k-2)\sin^2(\theta)}{r^2} - C\frac{\sin(\theta)^2}{r}.$$

The proof of the Lemma is given in Sect. 1.

Remark 3.12 This estimate originates from the curvature formula computed in [3, 8, 21] or [28]. These papers, however, contain a small computational error: There the formula has either $\kappa \sin(\theta)\frac{k-1}{r}$ instead of $\kappa \sin(\theta)\frac{2(k-1)}{r}$ or $2\frac{(k-1)(k-2)}{r^2}$ instead of $\frac{(k-1)(k-2)}{r^2}$. We will point out in the proof of Lemma 3.11 where the error occurs and in Remark 3.23 below, we discuss what impact this has on the proof of Proposition 3.3.

Corollary 3.13 *Let $K \subset \mathcal{R}(M)$ be compact. Assume that all $g \in K$ are normalized on the r_0-tube around N. Then for each $B \in \mathbb{R}$, there exists $\epsilon_0 > 0$ such that for*

$\epsilon \in (0, \epsilon_0)$, *the restriction of g to* $N \times S_\epsilon^{k-1}$ *has scalar curvature at least B.*

Proof Consider the curve $\gamma(s) := (s, \epsilon)$. The restriction of the product metric $g + dt^2$ to Q_γ is the product of $g|_{N \times S_\epsilon^{k-1}}$ and dt^2. Hence, $\mathrm{scal}(g_\gamma) = \mathrm{scal}(g|_{N \times S_\epsilon^{k-1}})$. In the case at hand, the curvature of γ is $\kappa = 0$, and $\theta = \frac{\pi}{2}$. Hence, from Lemma 3.11, we get

$$\mathrm{scal}(g_\gamma) \geq \mathrm{scal}(g) + \frac{(k-1)(k-2)}{\epsilon^2} - \frac{C}{\epsilon}.$$

If ϵ is small enough, the term $\frac{(k-1)(k-2)}{\epsilon^2}$ dominates all other terms. $\qquad\square$

Another auxiliary result is needed for the proof of Proposition 3.8.

Lemma 3.14 *Let $a > 0$ and consider the ordinary differential equation*

$$h'' = \frac{1 + h'^2}{a \cdot h}. \tag{3.15}$$

For any choice of initial values $h(t_0) > 0$ and $h'(t_0) < 0$, there is $T > t_0$ and a solution $h : [t_0, T] \to (0, \infty)$ such that $h' \leq 0$ and $h'(T) = 0$.

Proof Let $h : [t_0, T_1) \to \mathbb{R}$, $T_1 \in (t_0, \infty]$ be a maximal solution. We do not want to decide whether $T_1 = \infty$ or $T_1 < \infty$ and show that both cases lead to the desired conclusion. The quantity

$$C(t) := h^{\frac{1}{a}}(t) \cdot \frac{1}{\sqrt{1 + h'(t)^2}}$$

is constant as can be seen by differentiating. Also $C(t) > 0$ because of the initial conditions, and h is bounded from below by $C(t_0)^a$.

Suppose first $T_1 = \infty$. If $h'(t) < 0$ for all $t \geq t_0$, then h is decreasing and

$$h''(t) = \frac{1 + h'(t)^2}{a \cdot h(t)} \geq \frac{1}{a \cdot h(t)} \geq \frac{1}{a \cdot h(t_0)} =: b > 0$$

implies $h'(t) \geq b(t - t_0) + h'(t_0)$, which is a contradiction. Hence there is $T > t_0$ with $h'(T) = 0$.

If $T_1 < \infty$, we consider the trajectory of $(h(t), h'(t))$ in the phase diagram. Since $C(t)$ is constant, this trajectory lies on the level set $C^{-1}(C(t_0))$. Because $T_1 < \infty$, this trajectory leaves every compact subset of $\mathbb{R}^2$. The shape of the level set is so that this implies $\lim_{t \to T_1} h(t) = +\infty$. Hence by Rolle's theorem, $h'(T) = 0$ for some (minimal) $T > t_0$. $\qquad\square$

Remark 3.16 One can solve (3.15) explicitly, using that C is conserved. The above proof seems more efficient to us, though.

Proof of Proposition 3.8 We first construct a piecewise C^2 curve α and a homotopy α_λ of such curves. By a smoothing procedure, we obtain a homotopy β_λ of smooth

curves which will yield Γ_λ by a suitable reparametrization. We begin with the curve $\alpha = \alpha_1$. Let us pick some constants first.

(1) Choose $a > \frac{2}{k-2}$ (note that $-\frac{2}{a} + k - 2 > 0$).
(2) Choose $\rho > r_0$ arbitrarily.
(3) $0 < r_1 \leq r_0$ is chosen, so that the curvature estimate from Lemma 3.11 is valid for $r \leq r_1$, with a constant $C > 0$.
(4) Next, we choose $0 < r_2 \leq r_1$ so that

$$r_2 \leq \frac{k-1}{C}$$

(5) and pick $r_3 \in (0, r_2)$ arbitrarily.

Let us explain the choice of r_2. $\square$

Claim 3.17 *If γ is an immersed curve in the region $\{(y, r) | r \in (0, r_2]\}$ whose signed curvature κ is nonpositive, then $\mathrm{scal}(g_\gamma) \geq B_g$.*

To see this, estimate

$$\frac{(k-1)(k-2)\sin^2(\theta)}{r^2} - C\frac{\sin(\theta)^2}{r} = \frac{\sin(\theta)^2}{r}\left(\frac{(k-1)(k-2)}{r} - C\right)$$
$$\geq \frac{\sin(\theta)^2}{r}(k-3)C \geq 0, \tag{3.18}$$

using $r \leq r_2$. If $\kappa \leq 0$, then

$$|\kappa|\left(-\mathrm{sign}(\kappa)\frac{2(k-1)\sin(\theta)}{r} - C\sin(\theta)\right) = \sin(\theta)|\kappa|\left(\frac{2(k-1)}{r} - C\right) \geq 0. \tag{3.19}$$

Together with Lemma 3.11, these two inequalities establish Claim 3.17.

Let us now construct the first part of α. One device to construct a (unit speed) curve is by prescribing its curvature function. More precisely, let $J \subset \mathbb{R}$ be an interval and $s_0 \in J$. If a function $\kappa : J \to \mathbb{R}$ and initial values $\gamma(s_0)$ and $\dot\gamma(s_0)$ (the latter of unit length) are given, then the solution to the differential equation

$$\ddot\gamma(s) = \kappa(s)\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\dot\gamma(s)$$

is a unit speed curve with curvature function κ. If κ is piecewise continuous, then γ is piecewise C^2. We write $\theta(s)$ for the angle of the curve $\gamma(s)$.

(1) Choose $0 < \delta < \frac{r_2 - r_3}{3}$. Consider the function[1] $\kappa(s) = q\chi_{[\delta, 2\delta]}(s)$, for some $q > 0$, and the unit speed curve α on $[0, \infty)$ with initial values $\alpha(0) = (y_2, r_2) := (0, r_2)$ and $\dot\alpha(0) = (0, -1)$ and curvature function κ. If $q\delta < \frac{\pi}{2}$, the

---

[1] $\chi_S : X \to \{0, 1\}$ denotes the characteristic function of a subset $S \subset X$.

angle of the curve α will always be less than $\frac{\pi}{2}$, and so it crosses the horizontal line of height r_3 in some point (y_3, r_3), with an angle

$$\theta_0 = q\delta < \frac{\pi}{2}.$$

If q satisfies

$$q\left(\frac{2(k-1)}{r_3} + C\right) \leq \frac{1}{2}\eta,$$

we claim that $\mathrm{scal}(g_\alpha) \geq B_g - \frac{1}{2}\eta$. This follows from (3.18), Lemma 3.11, and the estimate

$$-\frac{2(k-1)\kappa \cdot \sin(\theta)}{r} - C|\kappa|\sin(\theta) \geq -\frac{2(k-1)q}{r_3} - Cq \geq -\frac{1}{2}\eta. \quad (3.20)$$

(2)   Now we pick $r_4 > 0$ so that $r_4 \leq \epsilon_0$, $r_4 < r_3$ and

$$r_4 \leq \frac{(k-1)\sin(\theta_0)^2(-\frac{2}{a} + k - 2)}{C(1 + \frac{1}{a})}.$$

Between height r_3 and r_4, the curve α follows the straight line of slope θ_0 (there is no problem with the psc condition, by Claim 3.17). It crosses the horizontal line of height r_4 at a certain point (y_4, r_4). If after that point, the curve α satisfies

$$\theta_0 \leq \theta \leq \frac{\pi}{2}, \ 0 \leq \kappa \leq \frac{\sin(\theta)}{ar}, \ 0 < r \leq r_4, \quad (3.21)$$

we estimate, using that $a > \frac{2}{k-2}$,

$$|\kappa|\left(-\mathrm{sign}(\kappa)\frac{2(k-1)\sin(\theta)}{r} - C\sin(\theta)\right) + \left(\frac{(k-1)(k-2)\sin(\theta)^2}{r^2} - C\frac{\sin(\theta)^2}{r}\right) \geq$$

$$\frac{\sin(\theta)}{ar}\left(-\frac{2(k-1)\sin(\theta)}{r} - C\sin(\theta)\right) + \left(\frac{(k-1)(k-2)\sin(\theta)^2}{r^2} - C\frac{\sin(\theta)^2}{r}\right) =$$

$$\frac{\sin(\theta)^2(k-1)}{r^2}\left(-\frac{2}{a} + k - 2\right) - \frac{C\sin(\theta)^2}{r}\left(1 + \frac{1}{a}\right) \geq$$

$$\frac{\sin(\theta_0)^2(k-1)}{rr_4}\left(-\frac{2}{a} + k - 2\right) - \frac{C}{r}\left(1 + \frac{1}{a}\right) =$$

$$\frac{1}{r}\left(\frac{\sin(\theta_0)^2(k-1)(-\frac{2}{a} + k - 2)}{r_4} - C(1 + \frac{1}{a})\right) \geq 0$$

by the definition of r_4. Altogether, Lemma 3.11 shows that (3.21) implies

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[1] $\chi_S : X \to \{0, 1\}$ denotes the characteristic function of a subset $S \subset X$.

$$\mathrm{scal}(g_\gamma) \geq B_g \tag{3.22}$$

in this region. We construct the curve α satisfying (3.21) as the graph of a function $f : [y_4, y_5] \to \mathbb{R}$. For curves of the form $t \mapsto (t, f(t))$, we have

$$\kappa = \frac{f''}{(\sqrt{1 + f'^2})^3} ; \; \sin(\theta) = \frac{1}{\sqrt{1 + f'^2}},$$

see, e.g. [1, p. 41]. Hence if we take f as the solution of the ordinary differential equation

$$f'' = \frac{1 + f'^2}{af}$$

with initial values

$$f(y_4) = r_4; \; f'(y_4) = -\frac{\cos(\theta_0)}{\sin(\theta_0)} < 0,$$

then the curve $\alpha(t) = (t, f(t))$ satisfies (3.21). By Lemma 3.14, there is a solution $f : [y_4, y_5] \to (0, \infty)$ with $f' \leq 0$ and $f'(y_5) = 0$. Let $r_5 := f(y_5) > 0$, and we let α be the graph of f in this region.

(3)   From the point (y_5, r_5) on, the curve α follows a straight horizontal line, of length 2ℓ (a little more than ℓ would suffice), until it reaches the point $(y_6, r_6) := (y_5 + 2\ell, r_5)$. Since $\kappa \equiv 0$, there is no problem with the psc condition here, by Claim 3.17. We let $r_\infty := r_5 = r_6$. The last piece of the curve (until it hits the y-axis) will be constructed at the end of the proof.

Now we parametrize the curve α by arclength, beginning at the point $\alpha(s_2) = (0, r_2)$ and call the reparametrized curve also α. Let $s_6 > s_5 > s_4 > s_3 > s_2$ be the points with $\alpha(s_i) = (y_i, r_i)$. The curve α is entirely determined by its curvature function κ. The function κ is zero outside the intervals $[s_0 + \delta, s_0 + 2\delta]$ and $[r_4, r_5]$. We have

$$\kappa|_{[s_0+\delta, s_0+2\delta]} = q > 0; \; \kappa_{[r_4, r_5]} > 0.$$

Now we pick $0 < \omega \ll \min(\frac{r_5}{2}, \delta, \ell)$. Let us now construct a homotopy of piecewise C^2-curves, which are defined on intervals of varying length $[s_2, s_6(\lambda)]$. Let $s_6(1) := s_6$ and $\alpha_1 := \alpha$. During the homotopy interval $[\frac{1}{2}, 1]$, we shrink down the size of the horizontal piece until it is ω (so that $s_6(\frac{1}{2}) = s_5 + \omega$), and $s_i(\lambda) = s_i$, for $i = 2, 3, 4, 5$, $\lambda \in [\frac{1}{2}, 1]$.

For $\lambda \in [0, \frac{1}{2}]$, consider the curvature function $\kappa_\lambda := \chi_{[s_2, s_2 + 2\lambda(s_5 - s_2)]} \kappa$ and let $s_5(\lambda)$ be the point where the curve α_λ with curvature function κ_λ reaches the horizontal line of height r_5 (it is always the case that $s_5(\lambda) \geq 2\lambda(s_5 - s_2)$) and put $s_6(\lambda) := s_5(\lambda) + \omega$.

By construction, the curves α_λ satisfy the psc condition $\mathrm{scal}(g_{\alpha_\lambda}) \geq B_g - \frac{\eta}{2}$, α_0 is the straight line on the r-axis, and $\alpha_1 = \alpha$. The curves α_λ are C^1 and piecewise C^2, and we need to smoothen them.

To that end, pick an even, smooth, nonnegative bump function ξ with support in $(-\frac{1}{4}, \frac{1}{4})$ and integral 1 and let $\xi_u(t) := \frac{1}{u}\xi(ut)$. For $u \in (0, \omega]$, we define the smooth curve

$$\beta_{\lambda,u} := \xi_u * \alpha_\lambda$$

using convolution. If $u \leq \omega$, then $\beta_{\lambda,u} \equiv \beta_\lambda$ near $s_6(\lambda)$ and near s_2, and $\beta_{0,u}$ lies on the r-axis. This holds because $\xi_u * f(t) = f(t)$ if f is linear near t.

For small enough u, the curve $\beta_{\lambda,u}$ satisfies the positive scalar curvature condition, namely $\mathrm{scal}(g_{\beta_{\lambda,u}}) \geq B_g - \eta$. This is no issue at points near which κ_λ is continuous. Near the discontinuity points, the angle and height of $\beta_{\lambda,u}$ is close to that for α_λ, while the curvature of $\beta_{\lambda,u}$ oscillates between the minimum and maximum value of κ_λ. The decisive estimates (3.18), (3.20) and (3.22) all hold if κ lies between 0 and the allowed maximum value. Note, however, that we might loose a bit scalar curvature.

To construct the last piece of the curves $\beta_{\lambda,u}$ on an interval $[s_6(\lambda), s_7(\lambda)]$, we take a smooth family of curves $\gamma_\lambda : [s_6(\lambda), s_7(\lambda)] \to \mathbb{R}^2$, such that

- γ_λ begins at the point $\beta_{\lambda,u}(s_6(\lambda))$, as a straight line with the same angle as $\beta_{\lambda,u}$,
- $\gamma_\lambda(s_7(\lambda))$ lies on the y-axis, and except on the interval $[s_6(\lambda), s_6(\lambda) + \omega]$, it is a circle,
- the curvature of γ_λ is ≤ 0.

These conditions enforce that γ_0 lies on the r-axis. The construction of such curves is easy and left to the reader. By Claim 3.17, there is no problem with the psc condition.

Finally, the curve Γ_λ is obtained by reparametrization (of the form $\Gamma_\lambda(s) := \beta_{\lambda,u}(s_7(\lambda) - s)$). It is extended to all of $[0, \infty)$, so that above ρ, it is just the curve $s \mapsto (0, s)$. □

Remark 3.23 The above proof is almost the same as that of the corresponding result in [21] or [28]. The difference is that in loc.cit., the slightly incorrect version of the curvature formula (3.11) is used. This allows the choice $a = 2$ in the quoted papers. In that case, the differential equation (3.15) has a simple explicit solution. We can pick $a = 2$ if $k > 4$, but if $k = 3$, we need $a > 2$, and the argument in loc.cit. does not work as stated there.

3.4 Completion of the proof of Proposition 3.3

We now give the proof of Proposition 3.3. We shall use the following well-known criterion for a map to be a weak equivalence.

Proposition 3.24 *Let $j : X \to Y$ be the inclusion of a subspace. Then the following are equivalent:*

(1) *j is a weak homotopy equivalence,*

(2)   for every $n \geq 0$ and every map $G_0 : D^n \to Y$ such that $G_0(S^{n-1}) \subset X$, there exists a homotopy G_s starting with G_0 such that $G_1(D^n) \subset X$ and $G_s(S^{n-1}) \subset X$ for all $s \in [0,1]$.

So we let

$$
\begin{array}{ccc}
S^{n-1} & \xrightarrow{\ G_0\ } & \mathcal{R}^+_{\mathrm{rot}}(M) \\
\downarrow & & \downarrow \\
D^n & \xrightarrow{\ G_0\ } & \mathcal{R}^+(M)
\end{array}
\tag{3.25}
$$

be a commutative diagram, and we have to produce a homotopy $G : [0,1] \times D^n \to \mathcal{R}^+(M)$ such that $G(0,x) = G_0(x)$ for all $x \in D^n$ and $G(s,x) \in \mathcal{R}^+_{\mathrm{rot}}(M)$ if $(s,x) \in (\{1\} \times D^n) \cup ([0,1] \times S^{n-1})$. Recall that $\mathcal{R}^+_{\mathrm{rot}}(M)$ denotes the space of psc metrics which are of the form $g_N + g_0$ on $N \times B^k_R$, for some normalized rotationally symmetric metric g_0 on B^k_R with $\mathrm{scal}(g_0) > 0$.

The first step is an application of Proposition 3.5.

Lemma 3.26 *There is a family $G'(s,x)$, $(s,x) \in [0,1] \times D^n$, of Riemannian metrics on M and $r_0 \in (0,R)$ such that*

(1)   *the metric $G'(0,x)$ has positive scalar curvature,*
(2) $G'(0,x) \in \mathcal{R}^+_{\mathrm{rot}}(M)$ *for all $x \in S^{n-1}$,*
(3)   *the map $G'(0,\_) : (D^n, S^{n-1}) \to (\mathcal{R}^+(M), \mathcal{R}^+_{\mathrm{rot}}(M))$ is homotopic to G_0 (as a map of space pairs),*
(4)   *for all $(s,x) \in [0,1] \times D^n$, the metric $G'(s,x)$ is normalized on the r_0-tube around N,*
(5)   *for all $(s,x) \in ([0,1] \times S^{n-1}) \cup (\{1\} \times D^n)$, the metric $G'(s,x)$ is rotationally symmetric on the r_0-tube around N, i.e. $G'(s,x) = g_N + g(s,x)$ for some rotationally symmetric $g(s,x)$.*

In short, we make the metrics $G(0,x)$ normalized on some tube, but in addition, we also take a crude interpolation of $G(0,x)$ to some rotationally symmetric metric, without taking the psc condition into account.

Proof Choose a Riemannian metric g on M such that $g|_{N \times B^k_R} = g_N + g'$, where g' is a rotationally symmetric normalized metric on B^k_R. For example, we can take g' to be the Euclidean metric. Let $\tilde{G}(s,x) := (1-s)G_0(x) + sg$, for $(s,x) \in [0,1] \times D^n$. We apply Proposition 3.5 to the map $\tilde{G}$ with $K = [0,1] \times D^n$ and $L = \{0\} \times S^{n-1} \cup \{1\} \times D^n$ and let F be the isotopy from that Proposition. Put $G'(s,x) := F(1,s,x)^* \tilde{G}(s,x)$. This has all the desired properties. $\square$

Now we replace the map G_0 in (3.25) by the map $G'(0,\_)$.
For $x \in S^{n-1}$, we write $G'(0,x) = g_N + g_0(x)$ on $N \times B^k_R$. Let

$$A := \inf \mathrm{scal}(g_N) \in \mathbb{R}.$$

Choose $\eta > 0$ so that

$$\forall x \in D^n : \inf \mathrm{scal}(G'(0,x)) - 2\eta \geq 0$$

and

$$\forall x \in S^{n-1} : \inf \mathrm{scal}(G'(0,x)|_{N \times B_R^k}) - 2\eta \geq A.$$

The second condition is implied by the first one if $A \leq 0$. If $A > 0$, then for each point $g \in \mathcal{R}_{\mathrm{rot}}^+(M)$ which is of the form $g_N + g_0$ on $N \times B_R^k$, we have $\mathrm{scal}(g|_{N \times B_R^k}) > A$ since otherwise g_0 will not be a psc metric.

Therefore, if we can produce a homotopy $G : [0,1] \times D^n \to \mathcal{R}^+(M)$, so that

(1) $G(0, \_) = G'(0, \_)$,
(2) $\inf \mathrm{scal}(G(\lambda, x)) \geq \inf \mathrm{scal}(G'(0,x)) - \eta$ and
(3) $G(\lambda, x)|_{N \times B_R^k}$ is   of   the   form $g_N + g_0(\lambda, x)$ for
    $(\lambda, x) \in ([0,1] \times S^{n-1}) \cup (\{1\} \times D^n)$, with some rotationally symmetric normalized metric $g_0(\lambda, x)$,

then $g_0(\lambda, x)$ will have positive scalar curvature for all $(\lambda, x) \in [0,1] \times S^{n-1}$, and G is a relative homotopy, and so we have finished the proof of Proposition 3.3.

Next, we determine the parameters ϵ_0 and ℓ for the Gromov–Lawson curve.

- Let $\epsilon_1 > 0$ be small enough, so that for all $\epsilon \in (0, \epsilon_1)$ and for all (s, x), the restriction of $G'(s, x)$ to the sphere $N \times S_\epsilon^{k-1}$ has scalar curvature $> \max(0, A)$. This is possible by Corollary 3.13.

Lemma 3.27 *For all $B \in \mathbb{R}$, there exists $\epsilon_0 \in (0, \epsilon_1]$ such that for all $(s, x) \in [0,1] \times D^n$, the metric $(G'(s, x))_\gamma$ has scalar curvature at least B, where γ is a curve in the plane with the following properties.*

(1) *There is $\epsilon \in (0, \epsilon_0]$ such that $0 \leq r \leq \epsilon$, $-\frac{1}{\epsilon} \leq \kappa \leq 0$ and $\theta \in [0, \frac{\pi}{2}]$,*
(2) *if $r \leq \frac{\epsilon}{\sqrt{2}}$, then γ is a circle of radius ϵ, and if $r \geq \frac{\epsilon}{\sqrt{2}}$, then $\theta \geq \frac{\pi}{4}$.*

Proof This is an application of Lemma 3.11, but much easier than Proposition 3.8. Pick r_1 so that the curvature estimate of Lemma 3.11 holds for all $G(s, x)$, with some constant C. At the points where γ is a circle, we have $\kappa = -\frac{1}{\epsilon}$ and $\sin(\theta)\epsilon = r$. Therefore, Lemma 3.11 yields

$$\mathrm{scal}(G'(s,x)_\gamma) \geq \mathrm{scal}(G'(s,x)) + \frac{1}{\epsilon^2}k(k-1) - \frac{2C}{\epsilon}.$$

In the region $\frac{\epsilon}{\sqrt{2}} \leq r \leq \epsilon$, Lemma 3.11 yields

$$\mathrm{scal}(G'(s,x)_\gamma) \geq \mathrm{scal}(G'(s,x)) - \frac{2C}{\epsilon} + \frac{(k-1)(k-2)}{2\epsilon^2}.$$

$\square$

- Now we choose $\epsilon_0 > 0$ so that $\epsilon_0 < r_0$ and that the conclusion of Lemma 3.27 holds with $B = A + 2\eta$. According to Proposition 3.8, there exists a Gromov–Lawson curve Γ with parameters η and ϵ_0, which has an inner width $r_\infty \leq \epsilon_0$.

Let $g_{s,x} := G'(s,x)|_{N \times S^{k-1}_{r_\infty}}$. This metric on $N \times S^{k-1}_{r_\infty}$ has scalar curvature at least $A + 2\eta$ by Lemma 3.27. We get a continuous map $[0,1] \times D^n \to \mathcal{R}^+(N \times S^{k-1}_{r_\infty})$, $(s,x) \mapsto g_{s,x}$.

Next let $f : \mathbb{R} \to [0,1]$ be a smooth function such that $f \equiv 0$ near $(-\infty, 0]$ and $f \equiv 1$ near $[1, \infty)$ and define $b := \max\{\|f'\|_{C^0}, \|f''\|_{C^0}\}$.

For each $L > 0$, we get an induced map

$$D^n \times [0,1] \to \mathcal{R}(N \times S^{k-1}_{r_\infty} \times [0,L]); \ (x,\lambda) \mapsto g_{\lambda f(\frac{t}{L}),x} + \mathrm{d}t^2. \tag{3.28}$$

The first two derivatives of $t \mapsto \lambda f(\frac{t}{L})$ are

$$|\lambda f(\tfrac{t}{L})'| \leq \frac{\lambda}{L} b; \ |\lambda f(\tfrac{t}{L})''| \leq \frac{\lambda}{L^2} b.$$

- We pick L so large that $\frac{\lambda}{L} b, \frac{\lambda}{L^2} b \leq \Lambda$, where $\Lambda > 0$ is the constant provided by Lemma 2.5. With these choices, we obtain

$$\mathrm{scal}(g_{\lambda f(\frac{t}{L}),x} + \mathrm{d}t^2) \geq A + \eta. \tag{3.29}$$

- Finally, put $\ell := L + R$.

End of the proof of Proposition 3.3 We consider a diagram as in (3.25) and replace G_0 by $G'(0,\_)$, where $G'(s,x)$ is a family of Riemannian metrics with the properties stated in Lemma 3.26. Let ϵ_0 be as in Lemma 3.27. By Proposition 3.8, there exists a Gromov–Lawson curve Γ with parameters ϵ_0 and ℓ. Let $E_\Gamma : [0,1] \times M \to (M \times \{0\}) \cup (N \times \mathbb{R}^k \times \mathbb{R})$ be the isotopy of embeddings determined by Γ (as in (3.7)). Now we define $G(\lambda, x) \in \mathcal{R}^+(M)$ for $\lambda \in [0, \frac{1}{2}]$ by

$$G(\lambda, x) := E^*_{\Gamma, 2\lambda}(G'(0,x) + \mathrm{d}y^2)$$

and for $\lambda \in [\frac{1}{2}, 1]$ by

$$G(\lambda, x) := E^*_{\Gamma, 1}(G'((2\lambda - 1)f(\frac{y - y_5}{\ell}), x) + \mathrm{d}y^2).$$

By construction, $\mathrm{scal}(G(\lambda, x)) > 0$ for all x, λ, and if $x \in S^{n-1}$, then $\mathrm{scal}(G(\lambda, x)) > A$. These metrics are not normalized, but the curve $t \mapsto (p, tv)$ is a

variable speed geodesic. This can be rectified by a reparametrization (pull back by an isotopy of $N \times \mathbb{R}^k$ which is the identity outside a compact set and which is of the form $(p,v) \mapsto (p, h_\lambda(\|v\|)v)$ for a smooth odd function h_λ). After such a reparametrization, the metrics $G(\lambda,x)$ are normalized on the r_0-tube when $\lambda = 1$ and $x \in D^n$. If $x \in S^{n-1}$, they stay rotationally symmetric, and the geometric size of the region where they are does not decrease with λ. Hence, after reparametrization, $G(\lambda,x)$ is rotationally symmetric and normalized on the R-tube, for all $x \in S^{n-1}$ and all $\lambda \in [0,1]$. This completes the proof. $\square$

4 The space of rotationally symmetric metrics

In this section, we complete the proof of Theorem 3.1. Let us first recall some notation. Let $A := \inf \mathrm{scal}(g_N) \in \mathbb{R}$. We choose $\delta > 0$ so that $\frac{1}{\delta^2}(k-1)(k-2) + A > 0$ and pick a torpedo metric g_{tor}^k on $\mathbb{R}^k$ of radius δ, which is cylindrical outside the disc of radius R, for some $R > 0$. Recall that $\mathcal{R}_{\mathrm{rot}}^+(M) \subset \mathcal{R}^+(M)$ is the space of all psc metrics g on M such that

$$g|_{N \times B_R^k} = g_N + g_0$$

for some rotationally symmetric normalized psc metric g_0 on B_R^k. Furthermore, $\mathcal{R}^+(M, \varphi) \subset \mathcal{R}_{\mathrm{rot}}^+(M)$ is the subspace of those g such that $g_0 = g_{\mathrm{tor}}^k$. The goal is to prove the following result, which together with Proposition 3.3 completes the proof of Theorem 3.1.

Proposition 4.1 *The inclusion map*

$$\mathcal{R}^+(M, \varphi) \to \mathcal{R}_{\mathrm{rot}}^+(M)$$

is a weak homotopy equivalence.

4.1 Preliminary remarks

A rotationally symmetric normalized metric on B_R^k is of the form $g = \mathrm{d}t^2 + f(t)^2 \mathrm{d}\xi^2$, for some *warping function* $f : [0,R] \to \mathbb{R}$ with the properties stated in Lemma 2.6. We also recall the curvature formula

$$\sigma(f) := \mathrm{scal}(\mathrm{d}t^2 + f(t)^2 \mathrm{d}\xi^2) = (k-1)\left((k-2)\frac{1 - f'^2}{f^2} - 2\frac{f''}{f}\right). \tag{4.2}$$

The torpedo metric g_{tor}^k is given by the warping function h_δ as in (2.10). For the metric $g_N + \mathrm{d}t^2 + f(t)^2 \mathrm{d}\xi^2$ to have positive scalar curvature, we need to have $A + (k-1)\left((k-2)\frac{1-f'^2}{f^2} - 2\frac{f''}{f}\right) > 0$. To allow for more convenient notation when $A \leq 0$, we introduce

$$B := \max\{0, -A\} \geq 0.$$

Then

$$0 \leq B < \frac{1}{\delta^2}(k-1)(k-2),$$

and the condition on f becomes

$$(k-1)\left((k-2)\frac{1-f'^2}{f^2} - 2\frac{f''}{f}\right) > B.$$

The most delicate step in the proof of Proposition 4.1 is the following.

Proposition 4.3 *Let*

$$
\begin{array}{ccc}
S^{n-1} & \xrightarrow{G_0} & \mathcal{R}^+(M, \varphi) \\
\downarrow & & \downarrow \\
D^n & \xrightarrow{G_0} & \mathcal{R}^+_{\mathrm{rot}}(M)
\end{array}
$$

be a commutative diagram. Then there exists a homotopy $G : [0, 1] \times D^n \to \mathcal{R}^+_{\mathrm{rot}}(M)$ of maps of space pairs $(D^n, S^{n-1}) \to (\mathcal{R}^+_{\mathrm{rot}}(M), \mathcal{R}^+(M, \varphi))$ such that $G(0, \_) = G_0(\_)$ and such that the warping function $f_{t,x}$ of $G(t, x)$ satisfies

(1) $0 \leq f_{1,x} \leq \delta$ *and* $f''_{1,x} \leq 0$ *on* $[0, R]$,
(2) $f'_{1,x} \equiv 0$ *near R*.

We will prove Proposition 4.3 in Sect. 4.2, and in Sect. 4.3, we complete the proof of Proposition 4.1.

4.2 Introducing collars

To prove Proposition 4.3, we will change the warping function f by composition with another function h or a 1-parameter family thereof. The composition $f \circ h$ will have a different domain of definition. To obtain a well-defined family of Riemannian metrics on M, we introduce the following construction.

We fix, once and for all, diffeomorphisms $\varphi_{a,b}$ of $(0, \infty)$ for each $0 < a \leq b$ such that

- $\varphi_{a,b}(b) = a$,
- $\varphi_{a,a} = \mathrm{id}$,
- $\varphi_{a,b}|_{[0,\frac{a}{2}] \cup [2b,\infty)} = \mathrm{id}$,
- $\varphi'_{a,b} \equiv 1$ near b,
- $\varphi_{a,b}$ depends smoothly on a, b.

The formula $\phi_{a,b}(x,v) := (x, \frac{\varphi(\|v\|)}{\|v\|} v)$ defines diffeomorphisms of $N \times \mathbb{R}^k$. These are compactly supported and can be extended by the identity to M.

Lemma 4.4 *Let* $g \in \mathcal{R}_{\mathrm{rot}}^+(M)$ *given by* $g = g_N + \mathrm{d}t^2 + f(t)^2\mathrm{d}\xi^2$ *on* $N \times B_R^k$. *Let* $h : [0,\infty) \to [0,\infty)$ *be smooth with* $h(0) = 0$, $h' \equiv 1$ *near* 0 *and* $0 \leq h' \leq 1$. *Let* $S \in (0,\infty)$ *be such that* $h(S) = R$ *(this enforces* $S \geq R$*). Then the formula*

$$\Lambda(g,h,S) := \begin{cases} g_N + \mathrm{d}t^2 + f(h(t))^2\mathrm{d}\xi^2 & \text{on } N \times B_S^k, \\ \phi_{R,S}{}^* g & \text{else,} \end{cases}$$

defines a smooth Riemannian metric on M *in each of the following cases*:

(1) $f' \equiv 0$ *near* R, *or*
(2) $h' \equiv 1$ *near* S.

Proof We need to show that $g_N + \mathrm{d}t^2 + f(h(t))^2\mathrm{d}\xi^2$ and $\phi_{R,S}{}^* g$ coincide on $N \times B_S^k \setminus N \times B_{S-\epsilon}^k$ for some $\epsilon > 0$. But near $N \times S_S^{k-1}$, we have

$$\phi_{R,S}{}^* g = g_N + \varphi_{R,S}'(t)^2\mathrm{d}t^2 + f(\varphi_{R,S}(t))^2\mathrm{d}\xi^2 \overset{\text{near } S}{=} g_N + \mathrm{d}t^2 + f(t - S + R)^2\mathrm{d}\xi^2$$

because $\varphi_{R,S}' \equiv 1$ and $\varphi_{R,S}(t) = t - S + R$ near S. Now in either of the two cases we have $f(t - S + R) = f(h(t))$ for t near S: If $f' = 0$ near R, we have $f(t - S + R) = f(R)$ and $h(t)$ is close to R near S; and if $h' = 1$ near S, then $h(t) = t - S + R$ near S. $\square$

Let us record some further simple properties of this construction. We omit the easy proof.

Lemma 4.5

(1) *In the situation of Lemma 4.4, $\Lambda(g,h,S)$ is rotationally symmetric and normalized on $B_S^k \supset B_R^k$.*
(2) *Let X be a space and let $g : X \to \mathcal{R}_{\mathrm{rot}}(M)$, $h : X \to C^\infty([0,\infty),\mathbb{R})$ and $S : X \to (0,\infty)$ be continuous maps such that $h(x)$ and $S(x)$ satisfy the requirements of Lemma 4.4 and assume that for each $x \in X$, one of the two conditions from Lemma 4.4 is satisfied. Then $X \to \mathcal{R}_{\mathrm{rot}}(M), x \mapsto \Lambda(g(x),h(x),S(x))$ is continuous.*

From now on, we only change the warping function inside the R-disc. Note that the metric $\Lambda(g,h,S)$, restricted to the complement of $N \times B_S^k$, is isometric to the metric g. Hence we only need to control the scalar curvature of $\Lambda(g,h,S)$ inside B_S^k, where it is determined by (4.2). In particular, our consideration will only involve the metrics on B_S^k, not on N. Let us make a few observations: if $\mathrm{scal}(\mathrm{d}t^2 + f(t)^2\mathrm{d}\xi^2) \geq B' > 0$, then L'Hôpital's rule shows that

$$B' \leq \lim_{t \to 0}(k-1)\left((k-2)\frac{1-f'(t)^2}{f(t)^2} - 2\frac{f''(t)}{f(t)}\right)$$

$$= (k-1)\lim_{t \to 0}\left((k-2)\frac{-2f'(t)f''(t)}{2f(t)f'(t)} - 2\frac{f''(t)}{f(t)}\right)$$

$$= -(k-1)k\lim_{t \to 0}\frac{f''(t)}{f(t)} = -k(k-1)\lim_{t \to 0}\frac{f'''(t)}{f'(t)} = -k(k-1)f'''(0)$$

and hence

$$f'''(0) \leq \frac{-B'}{k(k-1)}. \tag{4.6}$$

If h is a function as in Lemma 4.4, then the scalar curvature of $dt^2 + (f \circ h)^2 d\xi^2$ is given by

$$
\begin{aligned}
(k-1)&\left((k-2)\frac{1-f'(h)^2h'^2}{f(h)^2} - 2\frac{f''(h)}{f(h)}h'^2 - 2\frac{f'(h)}{f(h)}h''\right) \\
&= h'^2(k-1)\left((k-2)\frac{1-f'(h)^2}{f(h)^2} - 2\frac{f''(h)}{f(h)}\right) + (1-h'^2)\frac{(k-1)(k-2)}{f(h)^2}
\end{aligned}
\tag{4.7}
$$

using the self-explanatory notation $f(h) := f \circ h$.

Lemma 4.8 *Let f be the warping function of a metric g satisfying $f' \in [0,1]$ and* $\mathrm{scal}(g) \geq B'' > B' > 0$ *and let h be as above. Assume also that for some $r > 0$*

(1) $B''f^2 \leq (k-1)(k-2)$ *on $[0, r]$ and*
(2) $h'' \leq \frac{1}{2}\frac{B''-B'}{k-1}f$ *whenever $h \leq r$, say $h([0,s]) \subset [0, r]$.*

Then $\mathrm{scal}(dt^2 + f(h(t))^2 d\xi^2) \geq B'$ *on $[0, s]$.*

Proof From (4.7), we get $\mathrm{scal}(dt^2 + f(h(t))^2 d\xi^2) \geq$

$$h'^2 B'' + (1-h'^2)B'' - f'(h)(B'' - B') \geq B'' - (B'' - B') = B'.$$

$\square$

Remark 4.9 If $f = h_\delta$ is the torpedo function of radius δ and if $B'' \leq \frac{1}{\delta^2}(k-1)(k-2)$, then hypothesis (1) of 4.8 is satisfied for each $r > 0$.

The following two elementary lemmas are slightly adapted versions of [3, Lemma 3.5 and Lemma 3.7].

Lemma 4.10 *(Existence of sloping functions) Let $b \in (0, R)$, $0 < a < \frac{8}{10}b$ and let $p > 0$. Then there exists $q \in (0, 1)$, only depending on p and b, a family $u_{r,s} : \mathbb{R} \to \mathbb{R}$ of functions and $c_{r,s} \in \mathbb{R}$, both depending continuously on $(r, s) \in [0, 1] \times [0, 1]$ such that*

(1) $u_{r,0} = id$ and $u_{r,s} = id$ on $(-\infty, \frac{8}{10}a]$ for all r, s,

(2) $u_{r,s}(c_{r,s}) = b$,

(3) $u_{r,s}'' \le pr$,

(4) $u_{r,s}''|_{[\frac{8}{10}a, a]} \le 0$, $u_{r,s}''|_{[\frac{8}{10}c_{r,s}, c_{r,s}]} \ge 0$ and $u_{r,s}'' = 0$ outside these intervals,

(5) $u_{r,s}' = 1 - sq$ on $[a, \frac{8}{10}c_{r,s}]$ and $u_{r,s}' = (1 - sq + rsq)$ on $[c_{r,s}, \infty)$,

(6) $0 \le u_{r,s}' \le 1$ for all r, s.

We call $u_{r,s}$ a sloping function with parameters a, b, p and q the resulting slope. The situation is depicted in the following figure.
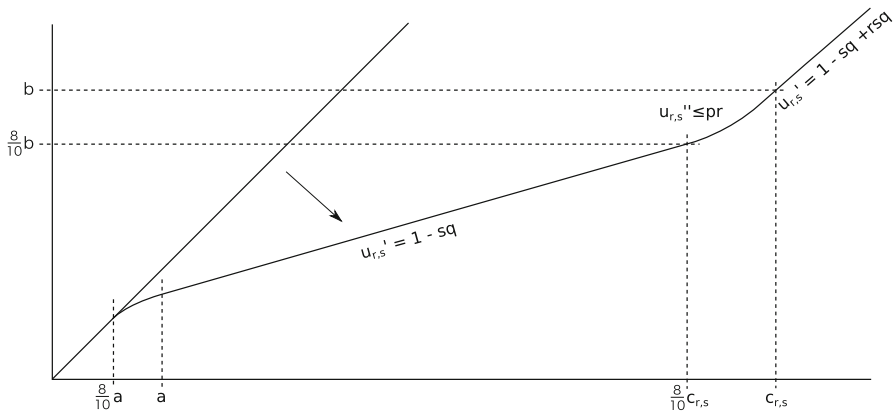
Proof We construct $u_{r,s}$ by constructing its second derivative, and do this by first constructing a piecewise continuous approximation to the second derivative of $u_{r,s}$. Choose $q \le \frac{b}{10}p$. Define a piecewise continuous function $w_{r,s} : \mathbb{R} \to \mathbb{R}$ by

$$w_{r,s} = -s\chi_{[\frac{17}{20}a, \frac{19}{20}a]}\frac{10q}{a} + rs\chi_{[\frac{17}{20}e_{r,s}, \frac{19}{20}e_{r,s}]}\frac{10q}{e_{r,s}},$$

where $e_{r,s}$ is to be determined. Let $v_{r,s}(x) := \int_0^x \int_0^t w_{r,s}(y)\mathrm{d}y$ and choose $e_{r,s} \ge b$ to be the unique point such that $v_{r,s}(\frac{8}{10}e_{r,s}) = \frac{8}{10}b$. Now let $\xi \ge 0$ be a smooth, even, nonnegative function with compact support and $\int_{\mathbb{R}} \xi(x)\mathrm{d}x = 1$ and let $\xi_\epsilon(x) := \frac{1}{\epsilon}\xi(\frac{x}{\epsilon})$. The function

$$u_{r,s}(x) := \xi_\epsilon * v_{r,s}$$

has, if ϵ is sufficiently small, all desired properties. Finally, we define $c_{r,s}$ as the unique point with $u_{r,s}(c_{r,s}) = b$.   □

Lemma 4.11 (*Existence of bending functions*) *Let $C > 0$ and $\beta > 0$. Then there exists $\alpha \in (0, \beta)$ and a family of functions $v_{r,s} : \mathbb{R} \to \mathbb{R}$ and $d_{r,s} \in \mathbb{R}$, depending continuously on $(r, s) \in (0, 1] \times [0, 1]$ such that*

(1) $v_{r,0} = id$ and $v_{r,s} = id$ on $(-\infty, \frac{\alpha}{2}]$,

(2) $v_{r,s}(d_{r,s}) = \beta$,

(3) $v'_{r,1} \equiv 0$ *near* α,

(4) $v''_{r,s} \leq C\frac{r}{t}$ *and* $v''_{r,s} \leq 0$ *on the complement of* $[2\alpha, d_{r,s}]$,

(5) $v'_{r,s} \equiv 1 - s + rs$ *on* $[d_{r,s}, \infty)$.

(6) $0 \leq v'_{r,s} \leq 1$ *for all* r, s.

We call $v_{r,s}$ *a family of bending functions with parameters* C, β. *The point* α *is called the attacking point. The following figure depicts the situation.*
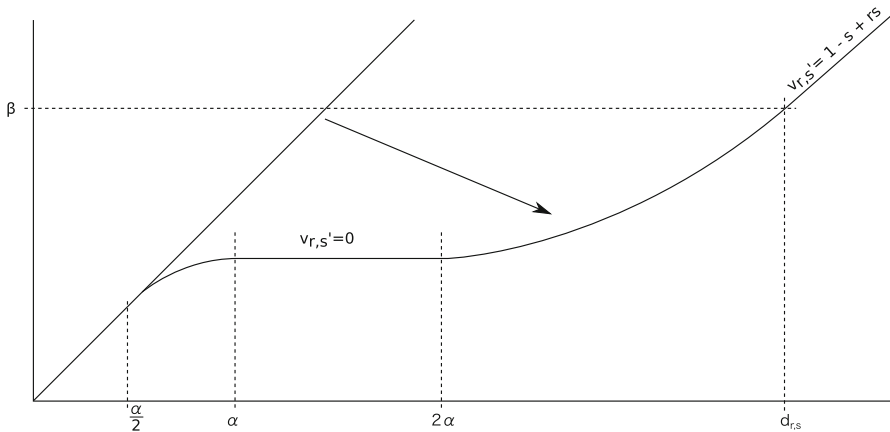
Proof This is by a similar method as the proof of Lemma 4.10. Let $\gamma := \frac{\beta}{2}$ and $\alpha := \frac{\beta}{4}e^{-\frac{1}{C}}$. We consider the piecewise continuous function

$$f_{r,s} = -s\chi_{[\frac{4}{6}\alpha, \frac{5}{6}\alpha]}\frac{6}{\alpha} + rs\chi_{[2\alpha,\gamma]}\frac{C}{t}$$

and set

$$w_{r,s}(t) := \int_0^t \left(1 + \int_0^x f_{r,s}(y)\mathrm{d}y\right)\mathrm{d}x.$$

A straightforward, but lengthy integral computation reveals that $w_{r,s}$ has all the desired properties, except that it is only piecewise C^2. For sufficiently small ϵ, consider the convoluted function

$$v_{r,s} := \xi_\epsilon * w_{r,s}.$$

The point $d_{r,s}$ is the unique point with $v_{r,s}(d_{r,s}) = \beta$. □

Proof of Proposition 4.3 We may assume that $G_0(x) = G_0(\frac{x}{\|x\|})$ for $\|x\| \geq \frac{1}{2}$, otherwise we perform an obvious homotopy beforehand to achieve this. We write the metric $G_0(x)$ as $g_N + g(x)$ for a rotationally symmetric metric $g(x)$ on B_R^k. Recall

that $B \in [0, \frac{1}{\delta^2}(k-1)(k-2))$ and $\mathrm{scal}(g(x)) > B$ for all $x \in D^n$. Let $f_x : [0, R] \to \mathbb{R}$ be the warping function of $G_0(x)$. The proof begins with making some choices:

(1) Pick

$$B < B' < B'' < \frac{1}{\delta^2}(k-1)(k-2)$$

so that $\mathrm{scal}(g(x)) \geq B''$ for all $x \in D^n$.

(2) By (4.6), there is $S \in (0, R)$ such that $f_x' \in [0, 1]$ and $f_x'' \leq 0$ on $[0, S]$ for all $x \in D^n$. We can furthermore pick S so that $B'' S^2 \leq (k-1)(k-2)$ and $S \leq \delta$.

(3) Let $F := \inf_{x \in D^n} f_x(\frac{8}{10} S) > 0$ and $p := \frac{1}{2(k-1)}(B'' - B')F \leq \frac{8}{10} \frac{1}{2(k-1)}(B'' - B')S$.

(4) Let q be the resulting slope (see Lemma 4.10) of a sloping function with parameters (α, S, p), for $\alpha < \frac{8}{10} S$ (recall the constant q only depends on p and S, not on α, which we have to pick later).

(5) Now we pick $T \in (0, \frac{8}{10} S]$ and $C > 0$ so that

$$B' T^2 + 2(k-1)C \leq (k-1)(k-2)(1 - (1 - q)^2).$$

(6) Pick a family of bending functions $v_{r,s}$ for the parameters (C, T) and let $\alpha > 0$ be the attacking point of this family of bending functions. The numbers $d_{r,s} \in \mathbb{R}$ are as in Lemma 4.11.

(7) Pick a family of sloping functions $u_{r,s}$ for the parameters (α, S, p). The numbers $c_{r,s}$ are as in Lemma 4.10.

Now we construct the homotopy $G : [0, 1] \times D^n \to \mathcal{R}_{\mathrm{rot}}^+(M)$. We first construct it on the part $[0, 1] \times D_{\frac{1}{2}}^n$, the disc of radius $\frac{1}{2}$.

(1) On $[0, \frac{1}{3}] \times D_{\frac{1}{2}}^n$, we define

$$G(\lambda, x) := \Lambda(g(x), u_{1,3\lambda}, c_{1,3\lambda})$$

and claim that $\mathrm{scal}(G(\lambda, x)) \geq B'$ for all such x and λ. In the region where $u_{1,3\lambda}'' \leq 0$, there is no problem: there $f_x(u_{1,3\lambda}) \leq S$, and we picked S small enough to satisfy the first hypothesis of Lemma 4.8. In the region where $u_{1,3\lambda}'' \geq 0$, we have by construction $u_{1,3\lambda} \geq \frac{8}{10} S$, and hence $f_x(u_{1,3\lambda}) \geq F$. Therefore

$$u_{1,3\lambda}'' \leq p = \frac{F(B'' - B')}{2(k-1)} \leq \frac{B'' - B'}{2(k-1)} f_x$$

and the claim follows from Lemma 4.8.

(2) The warping function $f = f_{x,\frac{1}{3}}$ of $G(x, \frac{1}{3})$ satisfies $f' \leq 1 - q$ on the interval $[\alpha, T]$. Furthermore, the scalar curvature of $G(x, \frac{1}{3})$ is bounded from below by B'. Now we define $G : [\frac{1}{3}, \frac{2}{3}] \times D_{\frac{1}{2}}^n$ by

$$G(\lambda, x) := \Lambda(G(\tfrac{1}{3}, x), v_{1,3\lambda-1}, d_{1,3\lambda-1}).$$

We claim that $\mathrm{scal}(g(x, \lambda)) \geq B'$ for all such λ and x. Again, there is no problem in the region where $v''_{1,3\lambda-1} \leq 0$. In the region where $v''_{1,3\lambda-1} \geq 0$, we have $f' \leq 1 - q$, $0 \leq v' \leq 1$, $f'' \leq 0$ and $f \leq T$. Therefore,

$$(k-1)\left((k-2)\frac{1 - f'^2 v'^2}{f^2} - 2\frac{f''}{f}v'^2 - 2\frac{f'}{f}v''\right) - B'$$

$$\geq (k-1)\left((k-2)\frac{1 - (1-q)^2}{f^2} - 2\frac{f'}{f}v''\right) - B'$$

$$= \frac{1}{f^2}\left((k-1)(k-2)(1 - (1-q)^2) - 2(k-1)ff'v'' - B'f^2\right).$$

But now $f' \leq 1$ in the relevant region, which implies $f(t) \leq t$. Since $0 \leq v'' \leq \frac{C}{t}$, the last term is bounded below by

$$\frac{1}{f^2}\left((k-1)(k-2)(1 - (1-q)^2) - 2(k-1)t\frac{C}{t} - B'T^2\right)$$

$$= \frac{1}{f^2}\left((k-1)(k-2)(1 - (1-q)^2) - 2(k-1)C - B'T^2\right),$$

using that $f \leq T$. But this is nonnegative, by our choice of T.

(3) Now we turn to the region $\|x\| \geq \frac{1}{2}$. Here we have $f_x = h_\delta$, the torpedo function of radius δ. In the region $\frac{1}{2} \leq \|x\| \leq \frac{2}{3}$ and $0 \leq \lambda \frac{2}{3}$, we merely change the point where the warping function obtained by composition with $v_{...}$ or $u_{...}$ is glued to the original metric, until this gluing is done in the region where the warping function $f_x = h_\delta$ is constant. There is no problem in doing this, as all the functions $u_{1,s}$ and $v_{1,s}$ are linear with slope 1 beyond $c_{1,s}$ and $d_{1,s}$. The concrete realization by formulas is

$$G(\lambda, x) := \begin{cases} \Lambda(g(x), u_{1,3\lambda}, c_{1,3\lambda} + 6(R - S)\|x\| + 3(S - R)) & \lambda \leq \frac{1}{3} \\ \Lambda(g(\tfrac{1}{3}, x), v_{1,3\lambda-1}, d_{1,3\lambda-1} - 3(c_{1,1} + R - S - T) + 6(c_{1,1} + R - S - T)\|x\|) & \frac{1}{3} \leq \lambda \leq \frac{2}{3}. \end{cases}$$

For $\|x\| = \frac{2}{3}$, we have

$$G(\lambda, x) := \begin{cases} \Lambda(g(x), u_{1,3\lambda}, c_{1,3\lambda} + R - S) & \lambda \leq \frac{1}{3} \\ \Lambda(g(\tfrac{1}{3}, x), v_{1,3\lambda-1}, d_{1,3\lambda} + c_{1,1} + R - S - T) & \frac{1}{3} \leq \lambda \leq \frac{2}{3}, \end{cases}$$

and since $u_{1,3\lambda}(c_{1,3\lambda} + R - S) = R$ and $v_{1,3\lambda-1}(d_{1,3\lambda-1} + c_{1,1} + R - S - T) = T + c_{1,1} + R - S - T = c_{1,1} + R - S$, the gluing now takes place in the region where $f_x = h_\delta$ is constant. Hence (compare Lemma 4.4), we are now free to change the functions $u_{1,3\lambda}$ and $v_{1,3\lambda-1}$ by functions whose derivative at the

relevant point is not equal to 1. We use this additional freedom to construct the homotopy in the region $\|x\| \geq \frac{2}{3}$.

(4) In the region $\frac{2}{3} \leq \|x\| \leq \frac{5}{6}$, we use the first parameter in the sloping and bending function and "dampen" those. To that end, let us pick $\eta \in (0,1)$ (we have to pick η small enough so that the next step goes through). We set, for $\frac{2}{3} \leq \|x\| \leq \frac{5}{6}$,

$$G(\lambda, x) := \begin{cases} \Lambda(g(x), u_{6(\eta-1)\|x\|+5-4\eta,3\lambda}, \tilde{c}_{\|x\|,3\lambda}) & \lambda \leq \dfrac{1}{3} \\[2ex] \Lambda(g(\frac{1}{3}, x), v_{6(\eta-1)\|x\|+5-4\eta,3\lambda-1}, \tilde{d}_{\|x\|,3\lambda} - 1) & \dfrac{1}{3} \leq \lambda \leq \dfrac{2}{3}, \end{cases}$$

where $\tilde{c}_{\|x\|,3\lambda}$ is the unique point with $u_{6(\eta-1)\|x\|+5-4\eta,3\lambda}(\tilde{c}_{\|x\|,3\lambda}) = R$ and $\tilde{d}_{\|x\|,3\lambda-1}$ is the unique point with $v_{6(\eta-1)\|x\|+5-4\eta,3\lambda-1}(\tilde{d}_{\|x\|,3\lambda} - 1) = \tilde{c}_{\|x\|,1}$. All the curvature estimates done in this proof so far apply as well when the sloping function $u_{1,3\lambda}$ is replaced by $u_{r,3\lambda}$ and the bending function $v_{1,3\lambda-1}$ is replaced by $v_{r,3\lambda-1}$, for $r > 0$. Hence, the above formula defines metrics of positive scalar curvature. For $\|x\| = \frac{5}{6}$, we can write

$$G(\lambda, x) = \Lambda(g(x), h_\lambda, a_\lambda),$$

where

$$h_\lambda := \begin{cases} u_{\eta,3\lambda} & \lambda \leq \dfrac{1}{3} \\[2ex] u_{\eta,1} \circ v_{\eta,3\lambda-1} & \lambda \geq \dfrac{1}{3}, \end{cases}$$

and $a_\lambda \in (0,\infty)$ is the point with $h_\lambda(a_\lambda) = R$. By Lemmas 4.10, 4.11 and the chain rule, we have
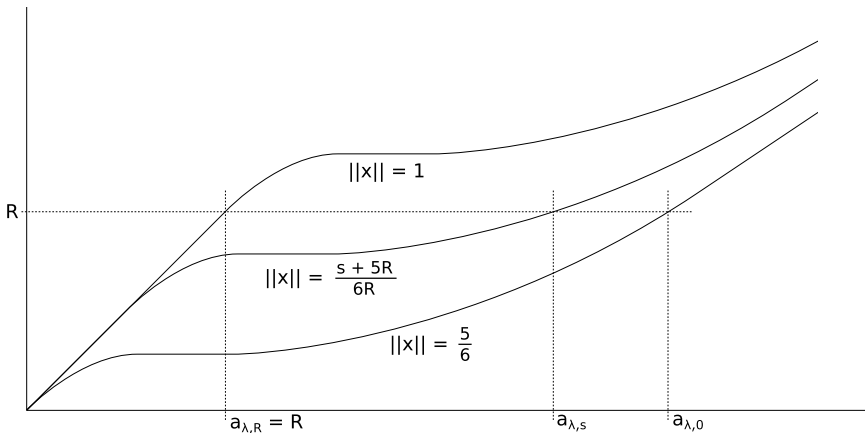

$\|x\| = 1$

R

$\|x\| = \dfrac{s + 5R}{6R}$

$\|x\| = \dfrac{5}{6}$

$a_{\lambda,R} = R$ $a_{\lambda,s}$ $a_{\lambda,0}$

Fig. 2 $h_{\lambda,6R\|x\|-5R}$ for various values of $\|x\|$

$$h''_\lambda \leq \max\left\{\eta p, \eta \frac{C}{2\alpha}\right\} = \eta \max\left\{p, \frac{C}{2\alpha}\right\}. \tag{4.12}$$

(5)   In the region $\frac{5}{6} \leq \|x\| \leq 1$, we change the function h_λ and "pull it away from zero to the region around R". More precisely, we set

$$h_{\lambda,s}(t) := h_\lambda(t - s) + s$$

and let $a_{\lambda,s}$ be the point with $h_{\lambda,s}(a_{\lambda,s}) = R$. Now we put, for $\lambda \in [0, \frac{2}{3}]$ and $\frac{5}{6} \leq \|x\| \leq 1$:

$$G(\lambda, x) = \Lambda(g(x), h_{\lambda, 6R\|x\|-5R}, a_{\lambda, 6R\|x\|-5R}). \tag{4.13}$$

If $\|x\| = 1$, then $h_{\lambda, 6R\|x\|-5R} \equiv \mathrm{id}$ on $[0, R]$ (Fig. 2). In other words, the metric $G(\lambda, x)$ coincides with the original metric on B_R^k (but it is changed outside this disc), so that we indeed get a relative homotopy. It remains to show that (4.13) defines a psc metric, and for this, we have to pick η sufficiently small. The functions $h_{\lambda,s}$ are just translated versions of h_λ, and so their second derivatives still satisfies (4.12). But recall Lemma 4.8 and Remark 4.9: together, they show that (4.13) defines a psc metric, as long as we pick η small enough so that

$$\eta \max\left\{p, \frac{C}{2\alpha}\right\} \leq \frac{B'' - B'}{2(k-1)}\delta.$$

(6)   Let us summarize what we have achieved so far: The map $G : [0, \frac{2}{3}] \times D^n \to \mathcal{R}^+_{\mathrm{rot}}(M)$ is continuous, $G([0, \frac{2}{3}] \times S^{n-1}) \subset \mathcal{R}^+(M, \varphi)$. For $\lambda = \frac{2}{3}$, the metric $G(x, \frac{2}{3})$ has a warping function $f_{\frac{2}{3},x}$, which we constructed in such a way that $f'_{\frac{2}{3},x} \equiv 0$ near

$$\tau(x) := \begin{cases} \alpha & \|x\| \leq \dfrac{5}{6} \\[2mm] \max R, \alpha + 6R\|x\| - 5R & \|x\| \geq \dfrac{5}{6}. \end{cases}$$

(7)   The last step is to stretch the collar around $\tau(x)$. This does not require us to be careful anymore. Let $\epsilon > 0$ so that $f'_{\frac{2}{3},x} \equiv 0$ on $[\tau(x) - \epsilon, \tau(x) + \epsilon]$. Pick a family $(h_{\lambda,x})_{(\lambda,x)\in[\frac{2}{3},1]\times D^n}$ of smooth functions and $a_{\lambda,x} \in \mathbb{R}$, depending continuously on λ and x so that

- $h_{\frac{2}{3},x} = \mathrm{id}$,
- $h_{\lambda,x}|_{[0,\tau(x)-\epsilon]} = \mathrm{id}$ for all λ,
- $h_{\lambda,x}|_{[\tau(x)+\epsilon,\infty)} \equiv \tau(x)$ for $\lambda \geq \frac{5}{6}$,
- $h''_{\lambda,x} \leq 0$, $0 \leq h'_{\lambda,x} \leq 1$ for all λ.

- $a_{\lambda,x} = \tau(x)$ for $\lambda \in [\frac{2}{3}, \frac{5}{6}]$,
- $a_{1,x} = R$ for all x,
- $a_{\lambda,x}$ is monotone increasing in λ.

The desired deformation of the metric on $[\frac{2}{3}, 1] \times D^n$ is

$$G(\lambda, x) := \Lambda(g(\frac{2}{3}, x), h_{\lambda,x}, a_{x,\lambda}).$$

$\square$

4.3 Deforming to a torpedo and adjusting widths

Now we are able to finish the proof of Proposition 4.1 and hence of Theorem 3.1. Consider a commutative diagram

$$
\begin{array}{ccc}
S^{n-1} & \xrightarrow{G_0} & \mathcal{R}^+(M, \varphi) \\
\downarrow & & \downarrow \\
D^n & \xrightarrow{G_0} & \mathcal{R}^+_{\mathrm{rot}}(M).
\end{array}
\tag{5.2}
$$

By Proposition 4.3, we may assume that $G_0(x)$ is given, on $N \times B_R^k$, by $g_N + \mathrm{d}t^2 + f_x(t)^2 \mathrm{d}\xi^2$, where the warping function f_x satisfies

- $f_x' \equiv 0$ near R,
- $f_x \leq \delta$, and we define $\delta_x := f_x(R)$,
- $0 \leq f_x' \leq 1$ and $f_x'' \leq 0$ on $[0, R]$,
- for $\|x\| \geq \frac{1}{2}$, f_x is the δ-torpedo function h_δ (in 4.3, this is only required for $\|x\| = 1$, but an obvious homotopy achieves this condition for $\|x\| \geq \frac{1}{2}$).

Furthermore, there are constants $\frac{(k-1)(k-2)}{\delta^2} > B'' > B' > B \geq 0$ so that

$$(k-1)\left((k-2)\frac{1 - f_x'^2}{f_x^2} - 2\frac{f_x''}{f_x}\right) \geq B''.$$

Let us make three observations.

Observation 4.15 *By a collar stretching homotopy as in the last step of the proof of Proposition 4.3, we may also assume that the metrics $G_0(x)$ are rotationally symmetric on a bigger disc $B_{R_\infty}^k$ and the warping function is constant on $[R, R_\infty]$, for R_∞ as large as we want. For a given $R_\infty > R$, pick a smooth function $a : \mathbb{R} \to [0, 1]$ such that $a|_{(-\infty, R]} \equiv 0$ and $a|_{[R_\infty, \infty)} \equiv 1$. For $\beta \in (0, \delta)$ and $p, q \in [\beta, \delta]$, let*

$$a_{p,q}(t) := (1 - a(t))p + a(t)q.$$

Then

$$\sigma(a_{p,q}) = (k-1)\left((k-2)\frac{1}{a_{p,q}^2} - (k-2)\frac{a_{p,q}'^2}{a_{p,q}^2} - 2\frac{a_{p,q}''}{a_{p,q}}\right)$$

$$\geq (k-1)\frac{k-2}{\delta^2} - \frac{(k-1)(k-2)(q-p)^2 a'^2}{\beta^2} + \frac{|p-q||a''|}{\beta}$$

$$\geq \underbrace{\frac{(k-1)(k-2)}{\delta^2}}_{\geq B''} - \delta^2\frac{(k-1)(k-2)a'^2}{\beta^2} + \delta\frac{|a''|}{\beta}.$$

So, if R_∞ is large enough, there exists a function a such that the latter term is at least B'.

Observation 4.16 *If f_0, f_1 are two warping functions such that $0 \leq f_i' \leq 1$ and $f_i'' \leq 0$, and such that $\sigma(f_i) > 0$ on $[0, R]$, then for any $\lambda \in [0, 1]$, $\sigma((1 - \lambda)f_0 + \lambda f_1) > 0$ on $[0, R]$. This follows from* (4.2) *and* (4.6). *Since by* (4.6), *we have $((1 - \lambda)f_0(0) + \lambda f_1(0))''' < 0$, it follows that $((1 - \lambda)f_0(t) + \lambda f_1(t))' < 1$ for $t > 0$ and $((1 - \lambda)f_0(t) + \lambda f_1(t))'' \leq 0$. Hence the scalar curvature is positive.*

Observation 4.17 *For $0 < \theta$ and a warping function f, let $f^\theta(t) := \theta f(\frac{t}{\theta})$. Then $f^\theta(t)' = f'(\frac{t}{\theta})$ and $f^\theta(t)'' = \frac{1}{\theta}f''(\frac{t}{\theta})$ and we get*

$$\sigma(f^\theta(t)) = \frac{1}{\theta^2}(k-1)\left((k-2)\frac{1 - f'(\frac{t}{\theta})^2}{f(\frac{t}{\theta})^2} - 2\frac{f''(\frac{t}{\theta})}{f(\frac{t}{\theta})}\right) = \frac{1}{\theta^2}\sigma(f(\frac{t}{\theta})).$$

Proof of Proposition 4.1 Consider a diagram as in (), with the properties provided by Proposition 4.3, and furthermore, by Observation 4.15, we may assume that the metric $G(0, x)$ is given by a warping function $f_x : [0, R_\infty] \to \mathbb{R}$, with R_∞ as in 4.15. The function f_x is constant on $[R, R_\infty]$, and it is convenient to extend it to all of $[0, \infty)$, by $f(t) := f(R_\infty)$ for $t > R_\infty$.

The goal is to deform f_x into a function which is equal to the torpedo function h_δ on $[0, R]$, while retaining this property if $\|x\| = 1$.

By 4.16, there exists $A > 0$, so that

$$\sigma((1 - \lambda)f_x + \lambda h_\delta) \geq A$$

for all $(x, \lambda) \in D^n \times [0, 1]$. It follows from 4.17 that there is a continuous function $\theta : D^n \to (0, 1]$ such that

$$\sigma((1 - \lambda)f_x^{\theta(x)} + \lambda h_\delta^{\theta(x)}) \geq B''$$

for all $\lambda \in [0, 1]$, $x \in D^n$ and $t \in [0, R]$. Since $f_x = h_\delta$ when $\|x\| \geq \frac{1}{2}$, we can, furthermore, assume that $\theta(x) = 1$ for $\|x\| = 1$.

The desired deformation of warping functions is given on the interval $[0, R]$ by the formula (note that $h_\delta^\theta = h_{\theta\delta}$)

$$f_{x,\lambda}(t) := \begin{cases} f_x^{3\lambda\theta(x)+1-3\lambda}(t) & \lambda \in [0, \frac{1}{3}] \\ (2-3\lambda)f_x^{\theta(x)}(t) + (3\lambda-1)h_{\delta_x}^{\theta(x)}(t) & \lambda \in [\frac{1}{3}, \frac{2}{3}] \\ h_{(3-3\lambda)\theta(x)\delta_x+(3\lambda-2)\delta}(t) & \lambda \in [\frac{2}{3}, 1]. \end{cases}$$

Now, let

$$\beta := \min_{(x,\lambda)\in D^n \times [0,1]} \{f_{x,\lambda}(R), f_x(R)\} \leq \delta,$$

and pick $R_\infty > R$ large enough so that there exists an a as in 4.15. Finally, on the interval $[R, R_\infty]$, we define

$$f_{x,\lambda}(t) = a_{f_{x,\lambda}(R), f_x(R)}(t).$$

$\square$

5 The fibration theorem and Theorem 1.6

5.1 Proof of the fibration theorem

We now present the proof of Theorem 1.1.

Lemma 5.1 *Let M be a closed manifold, P a compact space and $G : P \times [0,1] \to \mathcal{R}^+(M)$ be continuous. Then there is a continuous map*

$$C : P \times [0,1] \times [0,1] \to \mathcal{R}^+(M), (p, s, t) \mapsto C(p, s, t)$$

such that

(1) *C is smooth in t-direction,*
(2) *all derivatives in M- and t-direction are continuous,*
(3) *for all $(p, t) \in P \times [0,1]$, we have $C(p, 0, t) = G(p, 0)$,*
(4) *for all $(p, s) \in P \times [0,1]$, we have $C(p, s, 0) = G(p, 0)$, and $C(p, s, 1) = G(p, s)$,*
(5) *if $K \subset M$ is a codimension 0 submanifold and $G(p, t)|_K$ is independent of t, then $C(p, s, t)|_K$ is independent of s and t.*

Moreover, there is a continuous function $\Lambda : [0,1] \to (0, \infty]$ with $\Lambda(0) = \infty$, such that if $k : \mathbb{R} \to [0,1]$ is a smooth function with $|k'|, |k''| \leq \Lambda(s)$, then the metric

$$dt^2 + C(p, s, k(t))$$

on $\mathbb{R} \times M$ has positive scalar curvature for all $p \in P$.

Proof To construct C, let $U_{ni} = (\frac{i-1}{n}, \frac{i+1}{n}) \cap [0,1]$ and let $\mathcal{U}_n$ be the open cover $(U_{ni})_{i=0,\dots,n}$ of $[0,1]$. Let $(\lambda_{ni})_{i=0,\dots,n}$ be a subordinate smooth partition of unity. Define

$$C_n(p, s, t) := \sum_{i=0}^{n} G\left(p, s\frac{i}{n}\right) \lambda_{ni}(t) \in \mathcal{R}(M).$$

This has all the desired properties, except that the scalar curvature of $C_n(p, s, t)$ is not necessarily positive. Since $\mathcal{R}^+(M) \subset \mathcal{R}(M)$ is open, a routine compactness argument shows that that for sufficiently large n, the scalar curvature of $C_n(p, s, t)$ is positive for all p, s, t. Define $C := C_n$ for such an n.

   The existence of the function Λ with the asserted property follows from the properties of C, from the compactness of P and from Lemma 2.5.

Proof of Theorem 1.1   Let P be a disc and consider a lifting problem

$$
\begin{array}{ccc}
P \times \{0\} & \xrightarrow{\;\;F\;\;} & \mathcal{R}^+(W) \\
\downarrow & \overset{H}{\nearrow} & \downarrow{\scriptstyle \mathrm{res}} \\
P \times [0, 1] & \xrightarrow{\;\;G\;\;} & \mathcal{R}^+(M).
\end{array}
\tag{5.3}
$$

Since P is compact, we find $\delta > 0$ such that $F(P \times \{0\}) \subset \mathcal{R}^+(W)^{2\delta}$. This follows from a general property of colimits [24][Lemma 3.6] which can be applied here as the inclusion maps $\mathcal{R}^+(M)^b \to \mathcal{R}^+(M)^c$ are closed embeddings when $c < b$. We will construct a continuous map $K : P \times [0, 1] \to \mathcal{R}^+([0, \delta] \times M)$ with the properties that

(1) $K(p, s) = G(p, s)$ near $\{0\} \times M$,
(2) $K(p, s) = G(p, 0)$ near $\{\delta\} \times M$.

Then define $H(p, s) \in \mathcal{R}^+(W)$ to be equal to $K(p, s)$ on $[0, \delta] \times M$ and equal to $F(p)$ on $W \backslash ([0, \delta] \times M)$. This is a solution to the lifting problem (5.2).

   Let $C : P \times [0, 1] \times [0, 1] \to \mathcal{R}^+(M)$ and $\Lambda : [0, 1] \to (0, \infty]$ be as in Lemma 5.1. Now let

$$a(s) := \max\left(\sqrt{\frac{3}{\Lambda(s)}}, \frac{3}{\Lambda(s)} \right) + 1,$$

$$b := \sup_s a(s),$$

and fix a smooth function $f : \mathbb{R} \to [0, 1]$ such that

(1) $f = 0$ near $(-\infty, 0]$, $f = 1$ near $[1, \infty)$,
(2) $|f'|, |f''| \leq 3$.

With these choices, the Riemannian metric

$$L(p, s) := \mathrm{d}t^2 + C\left(p, s, f\left(\frac{t}{a(s)}\right)\right)
\tag{5.4}$$

on $\mathbb{R} \times M$ has positive scalar curvature, is cylindrical near $(-\infty, 0] \times M$ and near

$[b, \infty) \times M$ and lies in $\mathcal{R}^+([0,b] \times M)_{G(p,0),G(p,s)}$. Choose a diffeomorphism $h : [0, \delta] \to [0, b]$ such that $h' > 0$, $h' = 1$ near 0 and δ. Then $(p,s) \mapsto K(p,s) := (h \times \mathrm{id}_M)^* L(p,s)$ is the desired family of psc metrics on $[0, \delta] \times M$. $\square$

5.2 Proof of Theorem 1.6

The proof uses an auxiliary construction. For $r \geq 0$, we let $W_r := W \cup (M \times [0,r])$ be the result of gluing an external collar of length r to W and define N_r analogously. We extend the metric g_N on N to one denoted $g_{N,r}$ on N_r cylindrically. The embedding $\varphi : N \times \mathbb{R}^k \to W$ gets extended in the obvious way to an embedding $\varphi_r : N_r \times \mathbb{R}^k \to W_r$. We let $\mathcal{R}^+(W_r, \varphi_r) \subset \mathcal{R}^+(W_r)$ be the space of psc metrics which are of the form $g_{N,r} + g_{\mathrm{tor}}^k$ on $\varphi_r(N_r \times B_R^k)$. Extending psc metrics cylindrically over $M \times [0,r]$ defines maps $\mathcal{R}^+(W_r) \to \mathcal{R}^+(W_s)$ and $\mathcal{R}^+(W_r, \varphi_r) \to \mathcal{R}^+(W_s, \varphi_s)$ for $s > r$. Restriction to the boundary of W_r defines restriction maps $\mathrm{res}' : \mathcal{R}^+(W_r) \to \mathcal{R}^+(M)$ and $\mathcal{R}^+(W_r, \varphi_r) \to \mathcal{R}^+(M, \partial\varphi)$ which are compatible. In the colimit, we obtain a commutative diagram

$$
\begin{array}{ccc}
\mathrm{colim}_{r \to \infty} \mathcal{R}^+(W_r, \varphi_r) & \longrightarrow & \mathrm{colim}_{r \to \infty} \mathcal{R}^+(W_r) \\
\downarrow {\scriptstyle \mathrm{colim}_{r \to \infty} \, \mathrm{res}'} & & \downarrow {\scriptstyle \mathrm{colim}_{r \to \infty} \, \mathrm{res}'} \\
\mathcal{R}^+(M, \partial\varphi) & \longrightarrow & \mathcal{R}^+(M).
\end{array}
\tag{5.6}
$$

Lemma 5.5 *The left vertical map in* (5.4) *is a Serre fibration.*

Let us postpone the proof of Lemma 5.5 for the moment, and explain how to finish the proof of Theorem 1.6.

Proof of Theorem 1.6 The bottom horizontal map in (5.4) is a weak equivalence, by Theorem 1.2. The inclusion maps $\mathcal{R}^+(W_r) \to \mathcal{R}^+(W_s)$ are weak homotopy equivalences, and hence so is the inclusion map $\mathcal{R}^+(W) \to \mathrm{colim}_{r\to\infty}\mathcal{R}^+(W_r)$. This is proven in the same way as the elementary [2, Lemma 2.1 and Corollary 2.3].

Each individual map $\mathcal{R}^+(W_r, \varphi_r) \to \mathcal{R}^+(W_r)$ is a weak equivalence, by Theorem 1.2 and hence so is the top horizontal map in (5.4). It follows that the inclusion $\mathcal{R}^+(W, \varphi) \to \mathrm{colim}_{r\to\infty}\mathcal{R}^+(W_r, \varphi_r)$ is a weak equivalence.

The right vertical map in (5.4) is a Serre fibration. This follows easily from Theorem 1.1 and a colimit argument.

Together with Lemma 5.5, it follows that the induced map on fibres is a weak equivalence. Over $g \in \mathcal{R}^+(M, \partial\varphi)$, this is the bottom map of the diagram

$$
\begin{array}{ccc}
\mathcal{R}^+(W, \varphi)_g & \longrightarrow & \mathcal{R}^+(W)_g \\
\downarrow & & \downarrow \\
\mathrm{colim}_{r \to \infty} \mathcal{R}^+(W_r, \varphi_r)_g & \longrightarrow & \mathrm{colim}_{r \to \infty} \mathcal{R}^+(W_r)_g,
\end{array}
\tag{5.6}
$$

and we want to know that the top map is a weak equivalence. Thus to conclude the proof of Theorem 1.6, it enough to prove that the two vertical maps in (5.6) are weak equivalences. For the right hand vertical map, this follows immediately from

[2][Lemma 2.1], and for the left hand side map, we use [5][Corollary 2.5.4] (whose proof is elementary, but slightly lengthy). □

Proof of Lemma 5.5   This is similar to, but easier than the proof of Theorem 1.1. Let P be a disc and consider a lifting problem

$$
\begin{array}{ccc}
P \times \{0\} & \xrightarrow{\ F\ } & \operatorname{colim}_{r \to \infty} \mathcal{R}^+(W_r, \varphi_r) \\
\downarrow & \overset{H}{\nearrow} & \downarrow {\scriptstyle \operatorname{colim}_{r \to \infty} \operatorname{res}'} \\
P \times [0,1] & \xrightarrow{\ G\ } & \mathcal{R}^+(M, \partial\varphi).
\end{array}
$$

By compactness of P, there is $r \geq 0$ so that $F(P \times \{0\}) \subset \mathcal{R}^+(W_r, \varphi_r)$. Define $L(p, s)$ by the formula (5.3) and let b as in loc.cit. Define $H(p, s) \in \mathcal{R}^+(W_{r+b}, \varphi_{r+b})$ to be equal to $L(P, s)$ on $M \times [r, r + b]$ and equal to $F(p, s)$ on W_r. This has all the desired properties ($H(p, s)$ lies in $\mathcal{R}^+(W_{r+b}, \varphi_{r+b})$ because of property (5) of Lemma 5.1). □

6 Cobordism invariance of the space of psc metrics

This section is devoted to the proof of Theorem 1.5.

Definition 6.1   Let M_0 and M_1 be closed $(d-1)$-manifolds and let $W : M_0 \rightsquigarrow M_1$ be a cobordism. We say that W has *handle type* $[k, l]$ if there exists a handlebody decomposition of W relative to M_0 all whose handles have index in $\{k, k+1, \ldots, l-1, l\}$.

The first step in the proof of Theorem 1.5 is the following corollary of Theorem 1.2.

Corollary 6.2   (*to Theorem 1.2*) *Let M_0 and M_1 be two closed $(d-1)$-manifolds and assume that there is a cobordism $W : M_0 \rightsquigarrow M_1$ of handle type $[0, d-3]$. Then there exists a map $\mathcal{R}^+(M_0) \to \mathcal{R}^+(M_1)$, which can be chosen to be a weak homotopy equivalence if the handle type of W is $[3, d-3]$.*

Proof   Let M be a closed $(d-1)$-manifold and let $\phi : S^{m-1} \times \mathbb{R}^{d-m} \to M$ be an embedding. The result of a surgery on M is $M_\phi := M \backslash (\phi(S^{m-1} \times D^{d-m}) \cup D^m \times S^{d-m-1}$. The opposite embedding of ϕ is $\phi^{op} : \mathbb{R}^m \times S^{d-m-1} \to M_\phi$, and we obtain M back from M_ϕ by a surgery on ϕ^{op}. There is a zigzag of maps

$$
\mathcal{R}^+(M) \leftarrow \mathcal{R}^+(M, \phi) \cong \mathcal{R}^+(M_\phi; \phi^{op}) \to \mathcal{R}^+(M_\phi).
$$

By Theorem 1.2, the arrow pointing to the left is a weak equivalence if $m \leq d - 3$. By inverting the arrow up to homotopy, we obtain a map $\mathcal{R}^+(M) \to \mathcal{R}^+(M_\phi)$. If $m \geq 3$, then the arrow pointing to the right is a weak equivalence, and so is the map $\mathcal{R}^+(M) \to \mathcal{R}^+(M_\phi)$.

If W has handle type $[0, d-3]$, we can obtain M_1 from M_0 by a sequence of surgeries on embedded copies of $S^{m-1} \times \mathbb{R}^{d-m}$ with $d - m \geq 3$, which gives the

map, and if the handle type is $[3, d-3]$, this map is a weak homotopy equivalence.   □

Because of Corollary 6.2, Theorem 1.5 follows from the next two results.

Proposition 6.3 *Let* $\theta : B \to BO(d)$ *be a fibration and let* M_0, M_1 *be two closed* $(d-1)$*-dimensional* θ*-manifolds such that the structure maps* $M_i \to B$ *are 2-connected. Assume that* $d \geq 6$ *and that there is a* θ*-cobordism* $W : M_0 \rightsquigarrow M_1$*. Then there exists a* θ*-cobordism* $W' : M_0 \rightsquigarrow M_1$ *such that both inclusions* $M_i \to W'$ *are 2-connected.*

Proposition 6.4 *Let* $W : M_0 \rightsquigarrow M_1$ *be a d-dimensional cobordism between closed manifolds such that the inclusions* $M_i \to W$ *are 2-connected. Then*

(1) *if* $d \geq 7$*,* W *has handle type* $[3, d-3]$*,*
(2) *if* $d = 6$*, then there is* r *such that the connected sum* $W \natural^r(S^3 \times S^3)$ *with sufficiently many copies of* $S^3 \times S^3$ *has handle type* $[3, 3]$*.*

In both Propositions, the case $B = B\mathrm{Spin}$ follows quickly from the proof of the h-cobordism theorem for simply connected manifolds, which has found its way into textbooks [15, 17], and is described in [29]. The general case requires techniques from the proof of the s-cobordism theorem, which are not so well known. Therefore, we include the proof here, for sake of completeness.

Proof of Proposition 6.3   To simplify the notation, we assume that B is 0-connected. If we could assume that the space is of type (F_3) [25], we can perform θ-surgeries in the interior of W, giving a cobordism W' such that the structure map $W' \to B$ is 3-connected. It follows that the inclusion maps $M_i \to W$ are both 2-connected. However, the assumptions of the proposition only imply that B is of type (F_2), and an additional argument is required (which goes back to [20][Theorem 2.2] and is explained also in [10]). By θ-surgeries in the interior of W, we can replace W by a cobordism W' so that $W' \to B$ is 2-connected. Hence we can assume, without loss of generality, that $W \to B$ is 2-connected. This condition implies that both inclusions $M_i \to W$ induce isomorphisms on fundamental groups. Let π be the common fundamental group. The long exact homotopy sequence of the pair (W, M_0) gives the diagram

$$\begin{array}{ccccccc} \pi_2(M_0) & \longrightarrow & \pi_2(W) & \longrightarrow & \pi_2(W, M_0) & \longrightarrow & 0 \\ & \searrow & \downarrow {\scriptstyle (\ell_W)_*} & & & & \\ & & \pi_2(B) & & & & \end{array}$$

Now $\pi_2(W, M_0)$ is a finitely generated $\mathbb{Z}[\pi]$-module, by [25][§1]. Using that the structure map $M_0 \to B$ is 2-connected, a diagram chase shows that there are elements $x_1, \ldots, x_r \in \pi_2(W)$ whose image in $\pi_2(W, M_0)$ generate $\pi_2(W, M_0)$ as a $\mathbb{Z}[\pi]$-module and which lie in the kernel of $(\ell_W)_* : \pi_2(W) \to \pi_2(B)$.

Because $d \geq 5$, we can represent x_i by an embedded 2-sphere. Because $(\ell_W)_*(x_i) = 0$, it follows that the normal bundle of those embeddings is trivial,

and that x_i is represented by an embedding $S^2 \times \mathbb{R}^{d-2} \to W$. We may assume that those embeddings are disjoint, by general position. We can perform θ-surgeries on those spheres, and obtain a new θ-cobordism W' so that $M_0 \to W'$ is 2-connected. Rename $W := W'$.

To make the inclusion of M_1 2-connected as well, we use the same argument with M_0 replaced by M_1, but we have to be careful to not destroy the 2-connectivity of $M_0 \to W$. For an embedding $f : \coprod^r S^2 \times D^{d-2} \to W$, let $W^\circ := W \backslash \mathrm{Im}(f)^\circ$ and $W' = W^\circ \cup_{\partial \mathrm{Im}(f)} (\coprod^r D^3 \times S^{d-3})$. By general position, the inclusion $W^\circ \to W'$ is 2-connected, and $W^\circ \to W$ is $(d-3) \geq 3$-connected (here we use that $d \geq 6$). The diagram

$$W \xleftarrow{\text{3-co.}} W^\circ \xrightarrow{\text{2-co.}} W'$$

$$\underset{\text{2-co.}}{\nwarrow} \quad \uparrow \quad \nearrow$$

$$M_0$$

shows that $M_0 \to W^\circ$ is 2-connected, and so is $M_0 \to W'$ $\square$

Proof of Proposition 6.4 Part (1) follows quickly from handle trading [14, 26]. If $d = 6$, handle trading implies that W is of handle type [2, 3]. But handle trading is not enough: we can only barter the 2-handles for 4-handles. In fact, it is not correct that a 6-dimensional cobordism $W : M_0 \rightsquigarrow M_1$ with both inclusions 2-connected has handle type [3, 3]: look at a 6-dimensional h-cobordism with nontrivial Whitehead torsion to see why. To deal with part (2), we invoke the following result, whose proof is contained in the proof of [7, Lemma 6.21] (and which has its origins in [16]). Let $W : M_0 \rightsquigarrow M_1$ be a cobordism of dimension $2n \geq 6$ such that both inclusions $M_i \to W$ are $(n-1)$-connected. Then for sufficiently large r, the cobordism $W \natural^r (S^n \times S^n)$ has handle type $[n, n]$. $\square$

Appendix A: Curvature calculation

A.1. Proof of Lemma 2.4

Fix a local coordinate system $(x_1, \ldots, x_d)$ on M and define a coordinate system on $\mathbb{R} \times M$ by taking $x_0 = t$, the $\mathbb{R}$-variable. We now let g_{ij} be the components of g in these coordinates and h_{ij} those of h. Let g^{ij} and h^{ij} be the components of the inverses of the metric tensors. Note that

$$h_{ij} = \begin{cases} g_{ij} & i, j \geq 1, \\ 1 & i = j = 0, \\ 0 & \text{otherwise} \end{cases}$$

and

$$h^{ij} = \begin{cases} g^{ij} & i,j \geq 1, \\ 1 & i = j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

We write Γ_{ij}^k, R_{jkl}^i and S for the Christoffel symbols, the components of the curvature tensor and the scalar curvature of h and use the symbols γ_{ij}^k, r_{jkl}^i and s for those associated with g. Without any further comment, we use the Einstein summation convention. With these notations in place, we have, essentially by definition,

$$\Gamma_{ij}^k = \frac{1}{2} h^{kl}(h_{il,j} + h_{jl,i} - h_{ij,l})$$

$$R_{lij}^k = \Gamma_{jl,i}^k - \Gamma_{il,j}^k + \Gamma_{im}^k \Gamma_{jl}^m - \Gamma_{jm}^k \Gamma_{il}^m$$

$$S = h^{jl} R_{jkl}^k,$$

see [13, Corollary 3.3.1, (3.1.31), (3.3.6) and (3.3.19)]. The symmetry property

$$\Gamma_{ij}^k = \Gamma_{ji}^k$$

is obvious. The same formulas of course hold for g and its associated objects. Using these formulas, one computes

$$\Gamma_{ij}^k = \begin{cases} \gamma_{ij}^k & i,j,k \neq 0, \\ -\frac{1}{2} g_{ij,0} & k = 0,\ i,j \neq 0, \\ \frac{1}{2} g^{kl}(g_{jl,0}) & k,j \neq 0,\ i = 0, \\ \frac{1}{2} g^{kl}(g_{il,0}) & k,i \neq 0,\ j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The relevant components of the curvature tensor are

$$R_{0k0}^k = +\frac{1}{4} g^{ki} g^{jl} g_{ij,0} g_{kl,0} - \frac{1}{2} g^{kl} g_{kl,00},$$

$$R_{j0l}^0 = -\frac{1}{2} g_{jl,00} + \frac{1}{4} g^{mi} g_{lm,0} g_{ij,0}$$

and (for $j,k,l \neq 0$)

$$R_{jkl}^k = r_{jkl}^k + \frac{1}{4} g^{ki} g_{il,0} g_{kj,0} - \frac{1}{4} g^{ki} g_{ik,0} g_{lj,0}.$$

Altogether, we obtain (making an exception of the rule that we use the summation convention)

$$S = \sum_{k=0}^{d} R_{0k0}^{k} + \sum_{j,l \neq 0} g^{jl} R_{j0l}^{0} + \sum_{j,k,l \neq 0} g^{jl} R_{jkl}^{k} =$$

(switching back to the summation convention)

$$\frac{1}{4} g^{ki} g^{jl} g_{ij,0} g_{kl,0} - \frac{1}{2} g^{kl} g_{kl,00} - \frac{1}{2} g^{jl} g_{jl,00} + \frac{1}{4} g^{jl} g^{mi} g_{lm,0} g_{ij,0} + s$$

$$+ \frac{1}{4} g^{jl} g^{ki} g_{il,0} g_{kj,0} - \frac{1}{4} g^{jl} g^{ki} g_{ik,0} g_{lj,0} =$$

$$s + \frac{3}{4} g^{ki} g^{jl} g_{ij,0} g_{kl,0} - g^{kl} g_{kl,00} - \frac{1}{4} g^{jl} g^{ki} g_{ik,0} g_{lj,0}.$$

A.2: Proof of Lemma 3.11

We fix an orthonormal frame $(E_1, \ldots, E_d)$ at a point $p \in Q_\gamma \subset N \times \mathbb{R}^k \times \mathbb{R}$ such that $E_1 = \dot{\gamma}$, $E_2, \ldots, E_k$ are tangent to $S^{k-1}(r)$ and $E_{k+1}, \ldots, E_d$ are tangent to N. By the Gauß curvature equation, we have

$$\text{scal}(g_\gamma) = \underbrace{\text{scal}(g + \mathrm{d}t^2)}_{=\text{scal}(g)} + 2 \sum_{i<j} \Big(\langle \mathbb{I}(E_i, E_i), \mathbb{I}(E_j, E_j) \rangle - \langle \mathbb{I}(E_i, E_j), \mathbb{I}(E_i, E_j) \rangle \Big).$$

We will now estimate the entries of second fundamental form. First we note that for η the normal vector field of Q_γ in $N \times \mathbb{R}^k \times \mathbb{R}$ we have

$$\dot{\gamma} = \sin(\theta)\partial_t - \cos(\theta)\partial_r \qquad \eta = \cos(\theta)\partial_t + \sin(\theta)\partial_r.$$

We get for $i \geq 2$:

$$\begin{aligned}
\nabla_{\dot{\gamma}} \eta &= \nabla_{\dot{\gamma}}(\cos(\theta)\partial_t + \sin(\theta)\partial_r) \\
&= -\theta'(s)\sin(\theta)\partial_t + \cos(\theta)\underbrace{\nabla_{\dot{\gamma}}\partial_t}_{=0} + \theta'(s)\cos(\theta)\partial_r + \sin(\theta)\underbrace{\nabla_{\dot{\gamma}}\partial_r}_{=0} = -\theta'(s)\dot{\gamma} \\
\nabla_{E_i}\eta &= \nabla_{E_i}(\cos(\theta)\partial_t + \sin(\theta)\partial_r) = \\
&\quad - \theta'(s)\sin(\theta)\partial_t + \cos(\theta)\underbrace{\nabla_{E_i}\partial_t}_{=0} + \theta'(s)\cos(\theta)\partial_r + \sin(\theta)\nabla_{E_i}\partial_r \\
&= -\theta'(s)\dot{\gamma} + \sin(\theta)\nabla_{E_i}\partial_r
\end{aligned}$$

.

Since Q_γ is a hypersurface, the direction of $\mathbb{I}(E_i, E_j)$ is always η and we will omit this. We deduce for $2 \leq i < j$:

$$\mathbb{I}(E_1, E_1) = -\langle \nabla_{\dot{\gamma}} \eta, \ddot{\gamma} \rangle = \kappa$$
$$\mathbb{I}(E_1, E_i) = -\langle \nabla_{\dot{\gamma}} \eta, E_i \rangle = 0$$
$$\mathbb{I}(E_i, E_j) = -\sin(\theta) \langle \nabla_{E_i} \partial_r, E_j \rangle = \sin(\theta) \mathbb{I}^M(E_i, E_j),$$

where $\mathbb{I}^M$ denotes the second fundamental form of $N \times S^{k-1}(r)$ in $N \times \mathbb{R}^k$ with respect to the normal vector field ∂_r. We have the following estimate on $\mathbb{I}^M$ due to Hoelzel [12, Lemma 2.6][2]:

$$\mathbb{I}^M(E_i, E_j) = -\frac{1}{r} \cdot \langle \pi(E_i), E_j \rangle + O(1),$$

where $\pi : T(N \times S^{k-1}(r)) \to \mathscr{V}T(N \times S^{k-1}(r))$ denotes the projection onto the vertical tangent bundle with respect to the map $N \times S^{k-1}(r) \to N$. Hence, $\pi(E_i) = E_i$ if $2 \leq i \leq k$ and $\pi(E_i) = 0$ for $i \geq k+1$. This shows that $\mathbb{I}^M(E_i, E_j) = O(1)$ if $i \neq j$ or $i,j \geq k+1$. For $2 \leq i \leq k$ we have $\mathbb{I}^M(E_i, E_i) = -\frac{1}{r} + O(r)$ (see [28][Appendix]). This assembles to

$$\mathbb{I}(E_i, E_j) = \begin{cases} \kappa & \text{if } i = j = 1 \\ 0 & \text{if } 1 = i < j \\ \sin(\theta)(-\dfrac{1}{r} + O(r)) & \text{if } 2 \leq i = j \leq k \\ \sin(\theta)O(1) & \text{if } 2 \leq i < j \text{ or } j > k. \end{cases}$$

Therefore, we get[3]:

$$\sum_{i<j} \mathbb{I}(E_i, E_j)^2 = O(1) \cdot \sin(\theta)^2$$

---

[2] Note that there is a sign difference because Hoelzel uses a different sign convention for the second fundamental form: In his article, he defines $\mathbb{I}(V, W) = -g(\nabla_V W, \eta) \cdot \eta$ whereas we adapt to the convention $\mathbb{I}(V, W) = -g(\nabla_V \eta, W) \cdot \eta$. This difference is, however, not relevant to us since $\langle \pi(E_i), E_j \rangle = 0$ for $i \neq j$.

[3] This is where the error in [28] occurs: $\binom{k-1}{2} = \frac{(k-1)(k-2)}{2}$ is miscounted as $(k-1)(k-2)$.

$$\sum_{i<j} \mathbb{I}(E_i, E_i)\mathbb{I}(E_j, E_j) = \kappa \sum_{i=2}^{k} \mathbb{I}(E_i, E_i) + \kappa \sum_{i=k+1}^{d} \mathbb{I}(E_i, E_i)$$

$$+ \sum_{2 \le i < j \le k} \mathbb{I}(E_i, E_i)\mathbb{I}(E_j, E_j)$$

$$+ \sum_{2 \le i \le k < j \le d} \mathbb{I}(E_i, E_i)\mathbb{I}(E_j, E_j)$$

$$+ \sum_{k < i < j \le d} \mathbb{I}(E_i, E_i)\mathbb{I}(E_j, E_j)$$

$$= \sin(\theta) \cdot \kappa \cdot (k-1) \cdot \left(-\frac{1}{r} + O(r)\right) + \kappa \cdot O(1) \cdot \sin(\theta)$$

$$+ \underbrace{\frac{(k-1)(k-2)}{2}}_{=\binom{k-1}{2}} \cdot \underbrace{\left(-\frac{1}{r} + O(r)\right)^2}_{=\left(\left(\frac{1}{r}\right)^2 + O(1)\right)} \cdot \sin(\theta)^2$$

$$+ \left(-\frac{1}{r} + O(r)\right) \cdot O(1) \cdot \sin(\theta)^2 + O(1) \cdot \sin(\theta)^2.$$

Rearranging all terms finishes the proof.

Compliance with ethical standards

Conflict of interest The authors declare that they do not have conflict of interests.

Ethical standard This research complies with ethical standards

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