A compactness theorem for frozen planets

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In this paper, we study the moduli space of frozen planet orbits in the Helium atom for an interpolation between instantaneous and mean interactions and show that this moduli space is compact.

1. Introduction

Since the beginning of quantum mechanics, the dynamics of the Helium atom is intriguing people. Different from the Hydrogen atom, the dynamics of the Helium atom is not completely integrable. In particular, its phase space is not foliated by invariant tori and the EKB method cannot be applied to it. On the other hand, the Hamiltonian of Helium is invariant under simultanuous rotation of the two electrons or particle interchange so that due to these symmetries periodic orbits are usually not isolated and Gutzwiller's trace formula [7] cannot be directly applied either.

An interesting periodic orbit for the Helium problem was discovered numerically by Wintgen, Richter and Tanner see [8, 9], which plays an important role in the semiclassical treatment of the Helium atom. For this orbit, both electrons lie on a ray emanating from the nucleus. The inner electron collides with the nucleus and bounces back, while the outer electron (the frozen planet) stays almost stationary simultaneously attracted by the nucleus and repelled by the inner electron. An interesting aspect of this periodic orbit is that to the authors' knowledge it does not fit into a family of periodic orbits starting from a completely integrable system. In fact, if one ignores the interaction between the two electrons, one obtains a completely integrable system. But by ignoring the interaction between the electrons, both electrons are just attracted by the nucleus and therefore have to fall into it and there is nothing like a frozen planet orbit.

In order to get a more tractable system, the author replaced in [5] the instantaneous interaction of the two electrons by a mean interaction. It turned out that for mean interaction between the electrons, the outer electron is really frozen, i.e. it is stationary. One obtains a delay equation for the inner electron and it is shown analytically in [5] that there exists a unique nondegenerate solution.

In this paper, we interpolate linearly between the instantaneous interaction and the mean interaction between the two electrons. It is interesting to note that there is a special type of frozen planet orbit. At time $t = 0$, both the inner and the outer electrons have vanishing velocity. The inner electron is then accelerated in direction of the nucleus in which it falls after some moment of time. Suppose now that at this moment the outer electron has vanishing velocity again. Then one can let the movie run backwards. The inner electron jumps out of the nucleus and goes back to its initial position. Meanwhile, the outer electron goes back as well to its initial condition so that at the end both electrons are back at their initial position again with zero velocity. One has a periodic orbit. We refer to such periodic orbits as *symmetric frozen planets*. Here is the description of this moduli problem in formulas. For a homotopy parameter $r \in [0, 1]$ one looks at solutions

$$q_1 \in C^\infty([0, 1], (0, \infty)), \quad q_2 \in C^0([0, 1], [0, \infty)) \cap C^\infty([0, 1), (0, \infty))$$

of the following moduli problem:

$$
\begin{cases}
\ddot{q}_1(t) = -\dfrac{2}{q_1(t)^2} + \dfrac{r}{(\overline{q}_1 - \overline{q}_2)^2} + \dfrac{1 - r}{(q_1(t) - q_2(t))^2}, & t \in [0, 1] \\[2mm]
\ddot{q}_2(t) = -\dfrac{2}{q_2(t)^2} - \dfrac{r}{(\overline{q}_1 - \overline{q}_2)^2} - \dfrac{1 - r}{(q_1(t) - q_2(t))^2}, & t \in [0, 1) \\[2mm]
\quad\quad\quad\quad q_2(t) < q_1(t), & t \in [0, 1] \\[2mm]
\dot{q}_1(0) = \dot{q}_1(1) = \dot{q}_2(0) = q_2(1) = 0.
\end{cases}
\tag{1.1}
$$

The variable $q_1(t)$ describes the outer electron on the ray $(0, \infty)$ at time t and the variable $q_2(t)$ the inner one. The nucleus lies at the origin with whom the inner electron collides at time $t = 1$. The positive numbers $\overline{q}_1$ and $\overline{q}_2$ denote the mean value of q_1, respectively, q_2, defined by

$$\overline{q}_1 = \int_0^1 q_1(t)dt, \quad \overline{q}_2 = \int_0^1 q_2(t)dt.$$

For $r = 0$, the first two equations become the second-order ODE

$$
\begin{cases}
\ddot{q}_1(t) = -\dfrac{2}{q_1(t)^2} + \dfrac{1}{(q_1 - q_2)^2}, & t \in [0, 1] \\[2mm]
\ddot{q}_2(t) = -\dfrac{2}{q_2(t)^2} - \dfrac{1}{(q_1 - q_2)^2}, & t \in [0, 1).
\end{cases}
$$

The first term on the right-hand side describes the attraction by the nucleus whose charge is two, since the nucleus of the Helium atom consists of two protons. The second term describes the repulsion of the two electrons. For $r = 1$, the first two equations become

$$
\begin{cases}
\ddot{q}_1(t) = -\dfrac{2}{q_1(t)^2} + \dfrac{1}{(\overline{q}_1 - \overline{q}_2)^2}, & t \in [0, 1] \\[2mm]
\ddot{q}_2(t) = -\dfrac{2}{q_2(t)^2} - \dfrac{1}{(\overline{q}_1 - \overline{q}_2)^2}, & t \in [0, 1).
\end{cases}
$$

In this case, the instantaneous interaction between the two electrons is replaced by the interaction of their mean positions. This is not an ODE anymore, but an equation involving delay and prolay.

The main result of this paper tells us that the moduli space of solutions of (1.1) is compact. There are two horror scenarii which have to be ruled out. The first horror scenario is ionization meaning that the outer electron escapes to infinity. The other horror scenario is that the two electrons come arbitrary close together. There are two ways how the later scenario could occur. Namely both electrons fall simultaneously into the nucleus, which leads to a triple collision, or the energy, which is not bounded *a priori*, explodes. The following is the main result of this paper which rules out these horror scenarii.

Theorem 1.1. *There exists a constant κ such that for every solution (q_1, q_2, r) of the problem (1.1) one has*

$$
\max_{t \in [0,1]} \left\{ q_1(t), \frac{1}{q_1(t) - q_2(t)} \right\} \le \kappa.
$$

In view of the theorem, given a sequence of solutions $(q_1^\nu, q_2^\nu, r^\nu)$ of problem (1.1) one can find a convergent subsequence which then by a usual bootstrapping argument is again a solution of problem (1.1). The author does not know if nonsymmetric frozen planet orbits actually exist. If they exist it is an open question if the compactness result can be extended to the nonsymmetric case.

In a joint work with Cieliebak and Volkov, the author is currently studying a variational approach to frozen planet orbits [4]. Theorem 1.1 together with the variational approach then leads to a well-defined Euler characteristic at least for symmetric frozen planet orbits. A more difficult question is whether a homology can be defined whose chain complex is generated by symmetric frozen planet orbits. To define such a homology, one needs to generalize the compactness result to gradient flow lines.

In joint work with Albers *et al.*, the author started to generalize Floer homology to Hamiltonian delay equations [1–3, 6]. The author believes that the frozen planet problem can trigger a lot of research in this direction and can become an important testing ground how far Floer homology can be further developed.

2. Proof of the Main Result

We assume that (q_1, q_2) is a solution of problem (1.1) for some $r \in [0, 1]$. Note that $\ddot{q}_2$ is always negative. Therefore, in view of its initial condition, we have $\dot{q}_2(t) < 0$ for every $t \in (0, 1)$, so that q_2 is strictly monoton decreasing. The following lemma tells us that in contrast to q_2 the variable q_1 is monotone increasing.

Lemma 2.1. *If $r < 1$, then for every $t \in (0, 1)$ we have $\dot{q}_1(t) > 0$. In particular, q_1 is strictly monotone increasing. If $r = 1$, then q_1 is constant.*

Proof. That q_1 is constant in the case $r = 1$ is proved in [5, Lemma 3.1]. It suffices therefore to consider the case $r < 1$. We prove it in two steps.

Step 1. For every $t_0 \in [0, 1)$ such that $\dot{q}_1(t_0) = 0$ we have $\ddot{q}_1(t_0) > 0$.
  We first note

$$
\ddot{q}_1 - \ddot{q}_2 = \frac{2}{q_2^2} - \frac{2}{q_1^2} + \frac{2r}{(\overline{q}_1 - \overline{q}_2)^2} + \frac{2(1-r)}{(q_1 - q_2)^2}
$$

$$
= \frac{2(q_1^2 - q_2^2)}{q_1^2 q_2^2} + \frac{2r}{(\overline{q}_1 - \overline{q}_2)^2} + \frac{2(1-r)}{(q_1 - q_2)^2}
$$

$$
> 0.
$$

In view of the initial condition, this implies

$$
\dot{q}_1(t) - \dot{q}_2(t) > 0, \quad t > 0. \tag{2.1}
$$

The jerk of q_1 is given by

$$
\dddot{q}_1 = \frac{4\dot{q}_1}{q_1^3} - \frac{2(1-r)(\dot{q}_1 - \dot{q}_2)}{(q_1 - q_2)^3}. \tag{2.2}
$$

In view of the initial conditions, we have

$$
\dddot{q}_1(0) = 0
$$

and in view of (2.1) and $r < 1$ it holds that

$$
\dddot{q}_1(t) < \frac{4\dot{q}_1(t)}{q_1^3}, \quad t > 0. \tag{2.3}
$$

We assume by contradiction that there exists

$$
t_0 \in (0, 1)
$$

satisfying

$$
\dot{q}_1(t_0) = 0, \quad \ddot{q}_1(t_0) \le 0. \tag{2.4}
$$

We define

$$
t_1 := \inf\{t \in (t_0, 1] : \dot{q}_1(t) \ge 0\}.
$$

Here we use the convention, that if there is no $t \in (t_0, 1]$ satisfying $\dot{q}_1(t) \geq 0$, then $t_1 = 1$. In view of (2.2), it follows that

$$\dddot{q}_1(t_0) = \frac{2(1-r)\dot{q}_2(t_0)}{(q_1(t_0) - q_2(t_0))^3} < 0.$$

Therefore,

$$t_1 > t_0.$$

In view of the definition of t_1 and (2.3) we have

$$\dddot{q}_1(t) < 0, \quad t \in (t_0, t_1).$$

In view of (2.4), this implies that

$$\ddot{q}_1(t) < 0, \quad t \in (t_0, t_1).$$

Combining this once more with (2.4) we conclude that

$$\dot{q}_1(t_1) < 0.$$

By definition of t_1 this implies that

$$t_1 = 1$$

and therefore

$$\dot{q}_1(1) < 0.$$

This contradicts the boundary condition in (1.1) and Step 1 is proved.

Step 2. We prove the lemma.

By Step 1 we conclude that q_1 cannot have a local maximum in $[0, 1)$ but all its critical points are strict local minima. By the boundary condition in (1.1), the function q_1 has a critical point at time 0 which therefore has to be a strict local minimum. Since there are no local maxima, there cannot be any additional critical points of q_1 in $(0, 1)$. In particular, the derivative $\dot{q}_1$ cannot change sign. Since q_1 has a strict local minimum at time 0 the sign of $\dot{q}_1(t)$ is positive for t close to 0 and consequently is positive always. This finishes the proof of the lemma. □

Lemma 2.2. *For the starting point of q_2, we have the following lower bound:*

$$q_2(0) \geq 1. \tag{2.5}$$

Proof. By (1.1) we have for every $t \in [0, 1)$ the estimate

$$\ddot{q}_2(t) \leq -\frac{2}{q_2(t)^2} \leq -\frac{2}{q_2(0)^2},$$

where the second inequality holds since q_2 is monotone decreasing. Using that $\dot{q}_2(0) = 0$ we obtain from this the estimate

$$q_2(t) \leq -\frac{t^2}{q_2(0)^2} + q_2(0).$$

Taking the limit $t \to 1$, we get

$$0 \leq -\frac{1}{q_2(0)^2} + q_2(0)$$

implying

$$1 \leq q_2(0)^3$$

and therefore

$$1 \leq q_2(0).$$

This finishes the proof of the lemma. □

We abbreviate

$$\Delta := q_1(0) - q_2(0)$$

the distance between q_1 and q_2 at time $t = 0$. Since the variable q_2 is decreasing and by Lemma 2.1 the variable q_1 is increasing we have for every $t \in [0, 1]$

$$q_1(t) - q_2(t) \geq \Delta. \tag{2.6}$$

The following lemma tells us that q_1 and q_2 cannot come too close to each other.

Lemma 2.3. *There exists a constant $c_1 > 0$ such that*

$$\Delta \geq (1 - r)c_1. \tag{2.7}$$

Proof. If $\Delta > \frac{1}{2}$, the constant c_1 clearly exists. We thus can assume that

$$\Delta \leq \frac{1}{2}.$$

We define $t_0 \in (0, 1)$ by the requirement

$$q_2(t_0) = q_2(0) - \Delta.$$

Note that

$$\ddot{q}_1 + \ddot{q}_2 = -\frac{2}{q_1^2} - \frac{2}{q_2^2} \geq -\frac{4}{q_2^2}$$

so that for $t \in [0, t_0]$ we have the estimate

$$\ddot{q}_1(t) + \ddot{q}_2(t) \geq -\frac{4}{(q_2(0) - \Delta)^2} \geq -\frac{4}{(1 - \Delta)^2} \geq -16.$$

Since the velocity of both variables q_1 and q_2 at time $t = 0$ vanishes, we get from that the estimate

$$\dot{q}_1(t_0) + \dot{q}_2(t_0) \geq -16t_0 \geq -16$$

which we can rearrange to

$$-\dot{q}_2(t_0) \leq 16 + \dot{q}_1(t_0).$$

Since $\dot{q}_2$ is negative but by Lemma 2.1 we have that $\dot{q}_1$ is positive we obtain from that the inequality

$$\dot{q}_2(t_0)^2 \leq 512 + 2\dot{q}_1(t_0)^2. \tag{2.8}$$

Since $\ddot{q}_1 + \ddot{q}_2 < 0$ we further have the estimate

$$q_1(t_0) - q_1(0) \leq q_2(0) - q_2(t_0) = \Delta$$

so that we obtain

$$q_1(t_0) - q_2(t_0) = (q_1(t_0) - q_1(0)) + (q_1(0) - q_2(0)) + (q_2(0) - q_2(t_0)) \leq 3\Delta \leq \frac{3}{2}. \tag{2.9}$$

We have the following preserved quantity:

$$E = \frac{1}{2}(\dot{q}_1(t)^2 + \dot{q}_2(t)^2) - \frac{2}{q_1(t)} - \frac{2}{q_2(t)} - \frac{r(q_1(t) - q_2(t))}{(\overline{q}_1 - \overline{q}_2)^2} + \frac{1 - r}{q_1(t) - q_2(t)}.$$

At time $t = 0$ this computes to be

$$E = -\frac{2}{q_2(0) + \Delta} - \frac{2}{q_2(0)} - \frac{r\Delta}{(\overline{q}_1 - \overline{q}_2)^2} + \frac{1 - r}{\Delta}. \tag{2.10}$$

Note that there exists a constant $0 < \varepsilon < 1$ such that

$$\overline{q}_2 \leq (1 - \varepsilon)q_2(0).$$

By the computations in Appendix A, the constant ε can be chosen around $\frac{1}{4}$, but we do not use this in the following. Hence, using Lemma 2.1 and (2.5), we estimate

$$\frac{1}{\overline{q}_1 - \overline{q}_2} \leq \frac{1}{q_1(0) - \overline{q}_2} \leq \frac{1}{q_2(0) - \overline{q}_2} \leq \frac{1}{\varepsilon q_2(0)} \leq \frac{1}{\varepsilon}. \tag{2.11}$$

Using (2.11) together with (2.5) we obtain from (2.10) the estimate

$$E \geq -4 - \frac{1}{2\varepsilon^2} + \frac{1 - r}{\Delta}. \tag{2.12}$$

Evaluating the preserved quantity E at time $t = t_0$ we obtain from that as well as (2.8)

$$E = \frac{1}{2}(\dot{q}_1(t_0)^2 + \dot{q}_2(t_0)^2) - \frac{2}{q_1(t_0)} - \frac{2}{q_2(t_0)}$$

$$- \frac{r(q_1(t_0) - q_2(t_0))}{(\overline{q}_1 - \overline{q}_2)^2} + \frac{1 - r}{q_1(t_0) - q_2(t_0)}$$

$$\leq \frac{3}{2}\dot{q}_1^2(t_0) + 256 + \frac{1 - r}{q_1(0) - q_2(0) + \Delta}$$

$$= \frac{3}{2}\dot{q}_1^2(t_0) + 256 + \frac{1 - r}{2\Delta}. \tag{2.13}$$

Combining (2.12) and (2.13), we get the inequality

$$\frac{1-r}{2\Delta} \leq \frac{3}{2}\dot{q}_1^2(t_0) + 260 + \frac{1}{2\varepsilon^2}.$$

We abbreviate

$$c_0 := 260 + \frac{1}{2\varepsilon^2}.$$

Since $\dot{q}_1(t_0)$ is positive by Lemma 2.1 we obtain the estimate

$$\dot{q}_1(t_0) \geq \frac{\sqrt{1-r}}{\sqrt{3\Delta}} - c_0. \tag{2.14}$$

Using Lemma 2.1 and (2.5) again, the acceleration of q_1 is estimated from below for every $t \in [0,1]$ by

$$\ddot{q}_1(t) \geq -\frac{2}{q_1(t)^2} \geq -\frac{2}{q_1(0)^2} \geq -\frac{2}{q_2(0)^2} \geq -2. \tag{2.15}$$

Since the velocity of q_1 at time $t = 1$ vanishes, we obtain from this combined with (2.14)

$$0 = \dot{q}_1(1)$$

$$= \dot{q}_1(t_0) + \int_{t_0}^1 \ddot{q}_1(t)dt$$

$$\geq \frac{\sqrt{1-r}}{\sqrt{3\Delta}} - c_0 - 2(1 - t_0)$$

$$\geq \frac{\sqrt{1-r}}{\sqrt{3\Delta}} - c_0 - 2$$

implying

$$\Delta \geq \frac{1}{3(c_0 + 2)^2}(1 - r)$$

so that (2.7) follows with:

$$c_1 := \frac{1}{3(c_0 + 2)^2}.$$

This finishes the proof of the lemma. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

We would like to replace the estimate for Δ in Lemma 2.3 by a uniform one not depending on r. For that purpose, we need the following result on the average positions of q_1 and q_2.

Lemma 2.4. *The mean values of q_1 and q_2 satisfy the following inequality:*

$$\overline{q}_2 \leq \left(1 - \sqrt{\frac{r}{1+r}}\right)\overline{q}_1. \tag{2.16}$$

Proof. From Lemma 2.1, it follows that

$$\overline{q}_1 \leq q_1(1)$$

and q_1 attains at time $t = 1$ a local maximum so that

$$\ddot{q}_1(1) \leq 0.$$

Combining these inequalities with (1.1), we estimate

$$0 \geq \ddot{q}_1(1)$$

$$= -\frac{2}{q_1(1)^2} + \frac{r}{(\overline{q}_1 - \overline{q}_2)^2} + \frac{1-r}{(q_1(1) - q_2(1))^2}$$

$$= -\frac{2}{q_1(1)^2} + \frac{r}{(\overline{q}_1 - \overline{q}_2)^2} + \frac{1-r}{q_1(1)^2}$$

$$= -\frac{1+r}{q_1(1)^2} + \frac{r}{(\overline{q}_1 - \overline{q}_2)^2}$$

$$\geq -\frac{1+r}{\overline{q}_1^2} + \frac{r}{(\overline{q}_1 - \overline{q}_2)^2}$$

which we rewrite as

$$\frac{\overline{q}_1^2}{1+r} \leq \frac{(\overline{q}_1 - \overline{q}_2)^2}{r}.$$

Taking square roots, we obtain

$$\overline{q}_1 \leq \sqrt{\frac{1+r}{r}} (\overline{q}_1 - \overline{q}_2)$$

so that

$$\sqrt{\frac{1+r}{r}} \overline{q}_2 \leq \left(\sqrt{\frac{1+r}{r}} - 1 \right) \overline{q}_1$$

from which (2.16) follows. □

Now we are in position to improve Lemma 2.6 by a uniform estimate from below for Δ not depending on r anymore.

Proposition 2.5. *There exists a constant $c_2 > 0$ such that $\Delta \geq c_2$.*

Proof. Using (2.15) and $\dot{q}_1(1) = 0$ we obtain the estimate

$$\dot{q}_1(1 - t) \leq 2t$$

and therefore using integration by parts

$$\bar{q}_1 = \int_0^1 q_1(t)dt$$

$$= \int_0^1 q_1(1-t)dt$$

$$= q_1(0) + \int_0^1 \dot{q}_1(1-t)t\,dt$$

$$\leq q_1(0) + \int_0^1 2t^2\,dt$$

$$= q_1(0) + \frac{2}{3}$$

so that

$$q_1(0) \geq \bar{q}_1 - \frac{2}{3}. \tag{2.17}$$

Since $\ddot{q}_2(t) \leq 0$ for every $t \in [0,1)$ and $\dot{q}_2(0) = 0$ we have

$$q_2(0) \leq 2\bar{q}_2 \tag{2.18}$$

so that combined with (2.5) we get the estimate

$$\bar{q}_2 \geq \frac{1}{2}. \tag{2.19}$$

From (2.16), we obtain for every $r \in [0,1]$ the inequality

$$\bar{q}_1 \geq \frac{1}{1 - \sqrt{\frac{r}{1+r}}}\,\bar{q}_2 = \frac{\sqrt{1+r}}{\sqrt{1+r} - \sqrt{r}}\,\bar{q}_2. \tag{2.20}$$

Using (2.17), (2.18) and (2.20), we estimate

$$\Delta = q_1(0) - q_2(0)$$

$$\geq \bar{q}_1 - \frac{2}{3} - 2\bar{q}_2$$

$$\geq \left(\frac{\sqrt{1+r}}{\sqrt{1+r} - \sqrt{r}} - 2 \right) \bar{q}_2 - \frac{2}{3}. \tag{2.21}$$

Note that the function

$$f \colon [0,1] \to (0, \infty), \quad r \mapsto \frac{\sqrt{1+r}}{\sqrt{1+r} - \sqrt{r}} = \frac{1}{1 - \sqrt{\frac{r}{1+r}}}$$

is monotone increasing, since $r \mapsto \frac{r}{1+r}$ is monotone increasing, and

$$f\left(\frac{1}{3}\right) = \frac{1}{1 - \sqrt{\frac{1}{4}}} = \frac{1}{1 - \frac{1}{2}} = 2$$

so that it follows from (2.19) and (2.21) that

$$\Delta \geq \left(\frac{\sqrt{1+r}}{\sqrt{1+r} - \sqrt{r}} - 2 \right) \frac{1}{2} - \frac{2}{3} = \frac{\sqrt{1+r}}{2(\sqrt{1+r} - \sqrt{r})} - \frac{5}{3}, \quad r \geq \frac{1}{3}. \quad (2.22)$$

Note that

$$f\left(\frac{144}{145} \right) = \frac{1}{1 - \sqrt{\frac{144}{289}}} = \frac{1}{1 - \frac{12}{17}} = \frac{17}{5}$$

so that since f is monotone increasing

$$f(r) \geq \frac{17}{5}, \quad r \geq \frac{144}{145}.$$

Therefore, it follows from (2.22) that

$$\Delta \geq \frac{17}{10} - \frac{5}{3} = \frac{51 - 50}{30} = \frac{1}{30}, \quad r \geq \frac{144}{145}.$$

Combining this estimate with Lemma 2.3 proves the proposition. □

Our next goal is to derive an upper bound for the outer electron. We start with an upper bound for the inner electron.

Lemma 2.6. *There exists a constant c_3 independent of r such that $q_2(0) \leq c_3$.*

Proof. We recall (2.6) which tells us that

$$q_1(t) - q_2(t) \geq \Delta, \quad t \in [0, 1],$$

which is an immediate consequence of Lemma 2.1. In particular, we have

$$\overline{q}_1 - \overline{q}_2 \geq \Delta.$$

We infer from (1.1) that

$$\ddot{q}_2(t) \geq -\frac{2}{q_2(t)^2} - \frac{1}{\Delta^2}, \quad t \in [0, 1].$$

With Proposition 2.5 it follows that

$$\ddot{q}_2(t) \geq -\frac{2}{q_2(t)^2} - \frac{1}{c_2^2}, \quad t \in [0, 1].$$

Let $t_0 \in [0, 1)$ be the time such that

$$q_2(t_0) = 1.$$

Because q_2 is motonote decreasing, we have

$$\ddot{q}_2(t) \geq -2 - \frac{1}{c_2^2}, \quad t \in [0, t_0].$$

Since $\dot{q}_2(0) = 0$ we deduce from that

$$q_2(0) \leq q_2(t_0) + \frac{t_0^2}{2} \left(2 + \frac{1}{c_2^2} \right) \leq 1 + \frac{1}{2} \left(2 + \frac{1}{c_2^2} \right) = 2 + \frac{1}{2c_2^2}.$$

Setting

$$c_3 := 2 + \frac{1}{2c_2^2}$$

the lemma follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now we are in position to obtain an upper bound for the outer electron.

Proposition 2.7. *There exists a constant c_4 independent of r such that $q_1(1) \leq c_4$.*

Proof. Since q_2 is monotone decreasing we have

$$q_2(t) \leq q_2(0), \quad t \in [0,1]$$

and in particular

$$\overline{q}_2 \leq q_2(0).$$

Recall the inequality (2.15) telling us that

$$\ddot{q}_1(t) \geq -2, \quad t \in [0,1].$$

Combining this with the fact that q_1 is monotone increasing by Lemma 2.1 we obtain the estimate

$$q_1(1) \geq q_1(t) \geq q_1(1) - 1, \quad t \in [0,1]$$

and in particular

$$\overline{q}_1 \geq q_1(1) - 1.$$

If $q_1(1) - 1 \leq q_2(0)$, then $q_1(1) \leq c_3 + 1$ by the previous proposition. Now assume that $q_1(1) - 1 > q_2(0)$. Then by the above estimates

$$q_2(t) \leq q_2(0) \leq q_1(1) - 1 \leq q_1(t) \leq q_1(1)$$

for all $t \in [0,1]$. Therefore,

$$|q_1(t) - q_2(t)| \leq |q_1(1) - 1 - q_2(0)| \quad \forall t \in [0,1].$$

In particular,

$$|\overline{q}_1 - \overline{q}_2| \leq |q_1(1) - 1 - q_2(0)|.$$

With this and (1.1) we can estimate

$$\ddot{q}_1(t) \leq -\frac{2}{q_1(1)^2} + \frac{r}{(q_1(1) - 1 - q_2(0))^2} + \frac{1-r}{(q_1(1) - 1 - q_2(0))^2}$$

$$= -\frac{2}{q_1(1)^2} + \frac{1}{(q_1(1) - 1 - q_2(0))^2}, \quad t \in [0,1].$$

Since $\dot{q}_1(0) = \dot{q}_1(1) = 0$ there exists t_0 satisfying $\ddot{q}_1(t_0) = 0$ so that we obtain the inequality

$$0 \leq -\frac{2}{q_1(1)^2} + \frac{1}{(q_1(1) - 1 - q_2(0))^2}$$

implying

$$q_1(1)^2 \geq 2(q_1(1) - 1 - q_1(0))^2.$$

Taking square roots on both sides, we obtain the inequality

$$q_1(1) \geq \sqrt{2}(q_1(1) - 1 - q_1(0)).$$

Consequently

$$\sqrt{2}(q_1(0) + 1) \geq (\sqrt{2} - 1)q_1(1)$$

so that

$$q_1(1) \leq \frac{\sqrt{2}(q_1(0) + 1)}{\sqrt{2} - 1} \leq \frac{\sqrt{2}(c_3 + 1)}{\sqrt{2} - 1},$$

where the last inequality follows from Lemma 2.6. Hence, setting

$$c_4 := \frac{\sqrt{2}(c_3 + 1)}{\sqrt{2} - 1}$$

the proposition follows. $\qquad\square$

Proof of Theorem 1.1. By Lemma 2.1, we know that $q_1(t)$ is monotone increasing so that combined with Proposition 2.7 we have the estimate

$$q_1(t) \leq c_4, \quad t \in [0, 1]. \tag{2.23}$$

From (2.6) and Proposition 2.5 we infer that

$$q_1(t) - q_2(t) \geq c_2, \quad t \in [0, 1]$$

so that

$$\frac{1}{q_1(t) - q_2(t)} \leq \frac{1}{c_2}. \tag{2.24}$$

Setting

$$\kappa := \max\left\{c_4, \frac{1}{c_2}\right\}$$

the theorem follows from inequalities (2.23) and (2.24). $\qquad\square$

Appendix A. The Average Mean Fall

The interior electron of the frozen planet problem is in the free fall. In this appendix, we discuss mean values of some free falls. This is not really needed for the proof of the main result. On the other hand, a careful analysis of the mean free fall could be used to determine some of the constants occurring in the proof more precisely and might be of use when trying to establish a homology theory which also involves compactness results for gradient flow lines. Moreover, the mean free fall has its own mathematical beauty.

If the acceleration is constant $g > 0$ then the free fall starting at height $q_0 > 0$ is the solution of the initial value problem

$$q(0) = q_0, \quad \dot{q}_0 = 0, \quad \ddot{q}(t) = -g$$

whose explicit solution is given by

$$q(t) = q_0 - \frac{g}{2}t^2.$$

If τ is the time of the free fall implicitly defined by

$$q(\tau) = 0$$

then from the above formula one obtains explicitly

$$\tau = \sqrt{\frac{2q_0}{g}}.$$

The average position is then given by

$$\bar{q} = \frac{1}{\tau}\int_0^\tau q(t)dt = \frac{1}{\tau}\int_0^\tau \left(q_0 - \frac{g}{2}t^2\right)dt = \frac{1}{\tau}\left(q_0\tau - \frac{g}{6}\tau^3\right) = q_0 - \frac{g}{6}\cdot\frac{2q_0}{g}$$

$$= \frac{2}{3}q_0,$$

so that the ratio of the average position and the initial position is given by

$$\kappa := \frac{\bar{q}}{q_0} = \frac{2}{3}.$$

More generally, if the acceleration is given by the derivative f' of a potential f, the free fall is a solution of the initial value problem

$$q(0) = q_0, \quad \dot{q}_0 = 0, \quad \ddot{q}(t) = f'(q(t)).$$

One has the preserved quantity

$$\frac{1}{2}\dot{q}^2(t) + f(q(t)) = f(q_0).$$

We assume that $f' < 0$ so that q is strictly decreasing. Hence, the velocity at time t is given by

$$\dot{q}(t) = -\sqrt{2(f(q_0) - f(q(t)))}.$$

Using this formula, the time of the free fall is

$$\tau = \int_0^\tau dt = \int_0^{q_0} \frac{1}{\sqrt{2(f(q_0) - f(q))}}dq,$$

and the average position is

$$\bar{q} = \frac{1}{\tau}\int_0^\tau q(t)dt = \frac{\int_0^{q_0} \frac{q}{\sqrt{2(f(q_0)-f(q))}}dq}{\int_0^{q_0} \frac{1}{\sqrt{2(f(q_0)-f(q))}}dq} = \frac{\int_0^{q_0} \frac{q}{\sqrt{f(q_0)-f(q)}}dq}{\int_0^{q_0} \frac{1}{\sqrt{f(q_0)-f(q)}}dq}.$$

Note that this expression is invariant under scaling the potential f to μf for $\mu > 0$. We now want to compute this for the homogeneous potentials

$$f_\alpha(q) = -\frac{1}{q^\alpha}$$

for $\alpha > 0$. We abbreviate by

$$\kappa(\alpha) := \frac{\bar{q}}{q_0}$$

the ratio between average position and initial position for the free fall with respect to the potential f_α. We have the following proposition.

Proposition 2.8. *The ratio $\kappa(\alpha)$ is given by*

$$\kappa(\alpha) := \frac{\Gamma\left(\frac{4+\alpha}{2\alpha}\right)\Gamma\left(\frac{1+\alpha}{\alpha}\right)}{\Gamma\left(\frac{2+\alpha}{\alpha}\right)\Gamma\left(\frac{2+\alpha}{2\alpha}\right)}.$$

Let us look at some special values. Using $\Gamma(x + 1) = x\Gamma(x)$ and $\Gamma(n + 1) = n!$ we obtain for the Newtonian potential $-\frac{1}{q}$, i.e. $\alpha = 1$,

$$\kappa(1) = \frac{\Gamma\left(\frac{5}{2}\right)\Gamma(2)}{\Gamma(3)\Gamma\left(\frac{3}{2}\right)} = \frac{\frac{3}{2}\Gamma\left(\frac{3}{2}\right)}{2\Gamma\left(\frac{3}{2}\right)} = \frac{3}{4}.$$

This is a bit bigger than $\frac{2}{3}$ one obtains for the free fall with constant acceleration. For the potential $-\frac{1}{q^2}$, i.e. $\alpha = 2$, one obtains using $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ the transzendental number

$$\kappa(2) = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(2)\Gamma(1)} = \frac{\Gamma\left(\frac{1}{2}\right)^2}{4} = \frac{\pi}{4}.$$

Proof of Proposition 2.8. Since the average position does not change if we scale the potential by a positive factor we work with the potential $-\frac{1}{2q^\alpha}$ to avoid annoying factors $\sqrt{2}$ in the computation. We then have

$$\dot{q} = -\sqrt{\frac{1}{q^\alpha} - \frac{1}{q_0^\alpha}} = -\frac{\sqrt{q_0^\alpha - q^\alpha}}{(q_0 q)^{\frac{\alpha}{2}}}$$

so that we obtain for the time of the free fall

$$\tau = q_0^{\frac{\alpha}{2}} \int_0^{q_0} \frac{q^{\frac{\alpha}{2}}}{\sqrt{q_0^\alpha - q^\alpha}} dq.$$

Changing variables

$$q = q_0(\cos\theta)^{\frac{2}{\alpha}}, \quad dq = -\frac{2q_0}{\alpha}(\cos\theta)^{\frac{2-\alpha}{\alpha}}\sin\theta$$

this becomes

$$\tau = \frac{2q_0^{\alpha+1}}{\alpha q_0^{\frac{\alpha}{2}}} \int_0^{\frac{\pi}{2}} \frac{(\cos\theta)^{\frac{2}{\alpha}}\sin\theta}{\sqrt{1-\cos^2\theta}}d\theta$$

$$= \frac{2q_0^{\frac{\alpha+2}{2}}}{\alpha} \int_0^{\frac{\pi}{2}} (\cos\theta)^{\frac{2}{\alpha}}d\theta$$

$$= \frac{q_0^{\frac{\alpha+2}{2}}}{\alpha} B\left(\frac{2+\alpha}{2\alpha}, \frac{1}{2}\right)$$

$$= \frac{q_0^{\frac{\alpha+2}{2}}}{\alpha} \frac{\Gamma\left(\frac{2+\alpha}{2\alpha}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1+\alpha}{\alpha}\right)},$$

where B is the Betafunction. Similarly, we have

$$\int_0^\tau q\,dt = q_0^{\frac{\alpha}{2}} \int_0^{q_0} \frac{q^{\frac{\alpha+2}{2}}}{\sqrt{q_0^\alpha - q^\alpha}}dq$$

$$= \frac{2q_0^{\frac{\alpha+4}{2}}}{\alpha} \int_0^{\frac{\pi}{2}} (\cos\theta)^{\frac{4}{\alpha}}d\theta$$

$$= \frac{q_0^{\frac{\alpha+4}{2}}}{\alpha} B\left(\frac{4+\alpha}{2\alpha}, \frac{1}{2}\right)$$

$$= \frac{q_0^{\frac{\alpha+4}{2}}}{\alpha} \frac{\Gamma\left(\frac{4+\alpha}{2\alpha}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{2+\alpha}{\alpha}\right)}$$

implying that

$$\overline{q} = \frac{1}{\tau} \int_0^\tau q\,dt$$

$$= \frac{\Gamma\left(\frac{4+\alpha}{2\alpha}\right)\Gamma\left(\frac{1+\alpha}{\alpha}\right)}{\Gamma\left(\frac{2+\alpha}{\alpha}\right)\Gamma\left(\frac{2+\alpha}{2\alpha}\right)}q_0$$

This proves the proposition. □

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