V-Shaped Action Functional with Delay

Urs Frauenfelder[1*]

[1] *Mathematisches Institut, Universität Augsburg, Universitätsstraße 14, 86159 Augsburg, Germany*

Abstract—In this note we introduce the V-shaped action functional with delay in a symplectization, which is an intermediate action functional between the Rabinowitz action functional and the V-shaped action functional. It lives on the same space as the V-shaped action functional, but its gradient flow equation is a delay equation as in the case of the Rabinowitz action functional. We show that there is a smooth interpolation between the V-shaped action functional and the V-shaped action functional with delay during which the critical points and its actions are fixed. Moreover, we prove that there is a bijection between gradient flow lines of the V-shaped action functional with delay and the ones of the Rabinowitz action functional.

1. INTRODUCTION

V-shaped symplectic homology was introduced in [2] by the author jointly with Cieliebak and Oancea, where it was shown that it is isomorphic to the Rabinowitz – Floer homology [1] and fits into a long exact sequence with symplectic homology and cohomology. This in particular allowed computations of the Rabinowitz – Floer homology of cotangent bundles. It turned out that these computations coincide with computations in Tate Hochschild cohomology by Rivera and Wang in [7], where they conjectured a connection with the algebraic structures discovered in V-shaped symplectic homology by Cieliebak and Oancea in [3]. That there is a close connection between the Tate homology and the Rabinowitz – Floer homology is, in fact, to be expected, given that the Rabinowitz action functional is antiinvariant under time reversal. On the other hand, the V-shaped action functional is not antiinvariant under time reversal and therefore the Poincaré duality that holds on the homology level gets lost on the chain level.

The isomorphism between the Rabinowitz – Floer homology and the V-shaped symplectic homology in [2] is based on a nonlinear deformation of the Lagrange multiplier in the Rabinowitz action functional and does not look very suitable to compare algebraic structures on the Rabinowitz side and on the V-shaped side. In view of this difficulty we introduce in this note an intermediate action functional, the V-shaped action functional with delay, which shares some features of the Rabinowitz action functional and some features of the V-shaped action functional. We hope that this intermediate action functional will be able to shed some light on the ongoing scientific debate about algebraic structures in the Rabinowitz – Floer homology and their connection with the Tate – Hochschild homology.

We are considering the symplectization $\mathbb{R} \times \Sigma$ of a contact manifold (Σ, λ). We are fixing the following V-shaped function

$$h\colon (0,\infty) \to [0,\infty), \quad \rho \mapsto \rho(\ln \rho - 1) + 1$$

*E-mail: `urs.frauenfelder@math.uni-augsburg.de`

whose derivative is the logarithm so that the function h attains all slopes in a monotone increasing way. We abbreviate

$$\mathcal{L} = C^\infty(S^1, \mathbb{R} \times \Sigma)$$

the free loop space of the symplectization where $S^1 = \mathbb{R}/\mathbb{Z}$ denotes the circle. The V-shaped action functional with respect to the V-shaped function h

$$\mathcal{A}_0 \colon \mathcal{L} \to \mathbb{R}$$

for a loop $v = (r, x) \in \mathcal{L}$ is defined as

$$\mathcal{A}_0(v) = -\int_{S^1} v^*\lambda + \int_0^1 h(e^r)dt.$$

For the V-shaped action functional with delay

$$\mathcal{A}_1 \colon \mathcal{L} \to \mathbb{R}$$

we simply interchange the order of integration and application of h in the second term

$$\mathcal{A}_1(v) = -\int_{S^1} v^*\lambda + h\bigg(\int_0^1 e^r dt\bigg).$$

The reason for the name is that, in contrast to the Floer equation which is a PDE, its gradient flow equation is a delay equation as we explain soon. We can interpolate between the two functionals as follows by defining for $\theta \in [0, 1]$ the functional

$$\mathcal{A}_\theta \colon \mathcal{L} \to \mathbb{R}$$

by

$$\mathcal{A}_\theta(v) = -\int_{S^1} v^*\lambda + \theta \cdot h\bigg(\int_0^1 e^r dt\bigg) + (1-\theta) \cdot \int_0^1 h(e^r)dt.$$

Our first lemma tells us that the critical points as well as their actions do not depend on the parameter θ and are in natural one-to-one correspondence with generalized periodic Reeb orbits on Σ, where the generalization corresponds to the fact that the period might as well be negative, meaning that the orbit is traversed backward or zero, meaning that the orbit is a constant point on Σ. We abbreviate by R the Reeb vector field.

Lemma 1. *Assume that $v = (r, x)$ is a critical point of $\mathcal{A}_\theta$. Then r is constant and x is a solution of the ODE*

$$\partial_t x(t) = rR(x(t)), \quad t \in S^1, \tag{1.1}$$

i.e., x is a periodic Reeb orbit of generalized period r. Moreover, the action of the critical point is given by

$$\mathcal{A}_\theta(v) = 1 - e^r. \tag{1.2}$$

The proof is postponed to the next section. Although the functionals have the same critical points their gradient flow lines are different. We fix a smooth family J_t for $t \in S^1$ of SFT-like almost complex structures and take the gradient with respect to the L^2-metric obtained from the family J_t. Gradient flow lines of the V-shaped action functional $\mathcal{A}_0$ are solutions $v = (r, x) \in C^\infty(\mathbb{R} \times S^1, \mathbb{R} \times \Sigma)$ of the PDE

$$\partial_s v(s,t) + J_t(v(s,t))\Big(\partial_t v(s,t) - r(s,t)R\big(v(s,t)\big)\Big) = 0. \tag{1.3}$$

On the other hand, gradient flow lines of the V-shaped action functional with delay are solutions of the problem

$$\partial_s v(s,t) + J_t\big(v(s,t)\big)\left(\partial_t v(s,t) - \ln\bigg(\int_0^1 e^{r(s,t')}dt'\bigg)R(v(s,t))\right) = 0. \tag{1.4}$$

In contrast to (1.3), this is not a PDE anymore since the integral in front of the Reeb vector field is not local anymore, but depends on the whole loop, i. e., this problem is a delay equation and that is the reason why we refer to $\mathcal{A}_1$ as the V-shaped action functional with delay. We rewrite the above equation equivalently as

$$\partial_s v(s,t) + J_t(v(s,t))\bigg(\partial_t v(s,t) - \tau(s) R\big(v(s,t)\big)\bigg) = 0,$$
$$\tau(s) = \ln\bigg(\int_0^1 e^{r(s,t')}dt'\bigg).$$

We have the following lemma, proved in the next section.

Lemma 2. *Suppose that v is a nonconstant solution of the gradient flow Eq.* (1.4) *converging asymptotically to critical points $v_\pm = \lim_{s\to\pm\infty} v(s)$. Then for every $s \in \mathbb{R}$ it holds that*

$$0 > \partial_s\tau(s) \geqslant -\frac{\mathcal{A}_1(v_-) - \mathcal{A}_1(v_+)}{1 - \mathcal{A}_1(v_+)} > -1.$$

In particular, τ is strictly monotone decreasing.

It follows from Lemma 2 by standard arguments (see, for instance, [2]) that, if $v = (r,x)$ is as in the lemma, then

$$\Delta r \geqslant -\frac{\mathcal{A}_1(v_-) - \mathcal{A}_1(v_+)}{1 - \mathcal{A}_1(v_+)} > -1, \tag{1.5}$$

i. e., the Laplacian of r is uniformly bounded from below. This guarantees an upper bound on r just in terms of asymptotic data of r.

We also recall the Rabinowitz action functional on a symplectization

$$\mathcal{A}_2 \colon \mathcal{L} \times \mathbb{R} \to \mathbb{R}$$

which for $(v,\tau) = (r,x,\tau) \in \mathcal{L} \times \mathbb{R}$ is given by

$$\mathcal{A}_2(v,\tau) = -\int_{S^1} v^*\lambda + \tau \int_0^1 e^r dt - \tau.$$

If one interprets τ as a Lagrange multiplier, then $\mathcal{A}_2$ is the Lagrange multiplier functional of the negative area functional with respect to the constraint given by the mean value of the function $e^r - 1$. By the theory of Lagrange multipliers, critical points of $\mathcal{A}_2$ correspond to critical points of the restriction of the negative area functional to

$$\overline{\mathcal{L}} = \bigg\{ v = (r,x) \in \mathcal{L} : \int_0^1 e^r dt = 1 \bigg\},$$

namely,

$$\mathcal{A}_3 \colon \overline{\mathcal{L}} \to \mathbb{R}, \quad v \mapsto -\int_{S^1} v^*\lambda.$$

This functional was discussed by the author in [6].

The gradient flow equations of the Rabinowitz action functionals $\mathcal{A}_2$ and $\mathcal{A}_3$ are delay equations as well. For $\mathcal{A}_2$ they are solutions $(v,\tau) = (r,x,\tau) \in C^\infty(\mathbb{R} \times S^1, \mathbb{R} \times \Sigma) \times C^\infty(\mathbb{R},\mathbb{R})$ of the problem

$$\partial_s v(s,t) + J_t(v(s,t))\Big(\partial_t v(s,t) - \tau(s) R(v(s,t))\Big) = 0, \tag{1.6}$$
$$\partial_s \tau(s) + \int_0^1 e^{r(s,t')}dt' = 1,$$

and for $\mathcal{A}_3$ solutions $v=(r,x)\in C^\infty(\mathbb{R}\times S^1,\mathbb{R}\times\Sigma)$ of the problem

$$\partial_s v(s,t)+J_t(v(s,t))\Big(\partial_t v(s,t)+\mathcal{A}_3(v_s)R(v(s,t))\Big)=0, \qquad \int_0^1 e^{r(s,t')}dt'=1, \tag{1.7}$$

where we abbreviate by $v_s\in\overline{\mathcal{L}}$ the loop $t\mapsto v(s,t)$, see [6].

All the Eqs. (1.3), (1.4), (1.6), and (1.7) have the form

$$\partial_s v+J(v)\big(\partial_t v-\tau R(v)\big)=0, \tag{1.8}$$

i. e., are perturbed Cauchy – Riemann equations where the perturbation is in the Reeb direction. In the last three equations, τ only depends on s and is independent of t, while in the first equation it depends on both variables. We see that the t-independence of τ is a feature the gradient flow equation for the V-shaped action functional with delay has in common with the gradient flow equations of the Rabinowitz action functionals. On the other hand, we have the following ascending chain of spaces:

$$\overline{\mathcal{L}}\subset\mathcal{L}\subset\mathcal{L}\times\mathbb{R},$$

where the codimension in each step is one. Both V-shaped action functionals live on the intermediate space, whereas the Rabinowitz action functionals live on the two extremal spaces.

We have a natural projection

$$\Pi\colon\mathcal{L}\to\overline{\mathcal{L}},\quad (r,x)\mapsto\left(r-\ln\int_0^1 e^r\,dt,x\right).$$

In the case of $\mathcal{A}_3$, i. e., the restriction of the negative area functional to the constraint $\overline{\mathcal{L}}$, the Lagrange multiplier $\tau(s)$ is uniquely determined by the loop $v_s\in\overline{\mathcal{L}}$. Therefore, the set of solutions of Eq. (1.7) can be uniquely characterized as the set of solutions of the perturbed Cauchy – Riemann equation (1.8) where τ only depends on s, but not on t and $v_s\in\overline{\mathcal{L}}$ for every $s\in\mathbb{R}$. We abbreviate by $\mathcal{M}_i$ for $0\leqslant i\leqslant 3$ the moduli spaces of finite energy solutions of the gradient flow lines of the four functionals $\mathcal{A}_i$. We have natural maps

$$\Pi_*\colon\mathcal{M}_1\to\mathcal{M}_3,\ \Pi_*\colon\mathcal{M}_2\to\mathcal{M}_3,\quad v\mapsto\Pi v.$$

This is not true for $\mathcal{M}_0$ as domain since in this case τ usually depends on t as well. In [6] it was shown that $\Pi_*\colon\mathcal{M}_2\to\mathcal{M}_3$ is a bijection, i. e., there is a natural bijection between finite energy gradient flow lines of the two Rabinowitz action functionals. The following theorem tells us that the same is true for the gradient flow lines of the V-shaped action functional with delay.

Theorem 1. *The map $\Pi_*\colon\mathcal{M}_1\to\mathcal{M}_3$ is a bijection.*

In particular, we see from the theorem combined with [6] that there is a natural bijection between the finite energy gradient flow lines of the V-shaped action functional with delay and the ones of the Rabinowitz action functional $\mathcal{A}_2$.

2. PROOF OF LEMMA 1 AND DERIVATION OF THE GRADIENT FLOW EQUATION

Before embarking on the proof of Lemma 1 we recall some notation. We assume that $\Sigma=\Sigma^{2n-1}$ is a closed odd-dimensional manifold of dimension $2n-1$ and $\lambda\in\Omega^1(\Sigma)$ is a contact form on Σ, i. e.,

$$\lambda\wedge(d\lambda)^{n-1}>0$$

is a volume form on Σ. We abbreviate by ξ the contact structure, i. e., the hyperplane distribution

$$\xi=\ker\lambda,$$

and by R the Reeb vector field on Σ implicitly defined by the conditions

$$\lambda(R)=1, \quad d\lambda(R,\cdot)=0.$$

By abuse of notation we use the same letter λ for the extension of the one-form to the symplectization $\mathbb{R}\times\Sigma$ of Σ, namely, we set

$$\lambda_{r,x}=e^r\lambda_x, \quad (r,x)\in\mathbb{R}\times\Sigma.$$

The exterior derivative of λ

$$\omega_{r,x}=d\lambda_{r,x}=e^r dr\wedge\lambda_x+e^r d\lambda_x$$

is then a symplectic form on $\mathbb{R}\times\Sigma$. We extend the bundle ξ and the Reeb vector field R trivially to the symplectization and use there by abuse of notation the same letters again.

Proof (*of Lemma* 1). Suppose that $v=(r,x)\in\mathcal{L}$ and $\widehat{v}$ is a vector field along v, i. e., a tangent vector

$$\widehat{v}=(\widehat{r},\widehat{x})\in T_v\mathcal{L}=\Gamma\big(v^*T(\mathbb{R}\times\Sigma)\big)=C^\infty(S^1,\mathbb{R})\times\Gamma(x^*T\Sigma).$$

In the following computation we denote by $L_{\widehat{v}}$ the Lie derivative in the direction of $\widehat{v}$ and use Cartan's formula $L_{\widehat{v}}=d\iota_{\widehat{v}}+\iota_{\widehat{v}}d$:

$$\begin{aligned}
d\mathcal{A}_\theta(v)\widehat{v}&=-\int_{S^1}v^*L_{\widehat{v}}\lambda+\theta\cdot h'\bigg(\int_0^1 e^{r(t)}dt\bigg)\int_0^1 e^{r(t)}\widehat{r}(t)dt\\
&\quad+(1-\theta)\cdot\int_0^1 h'\big(e^{r(t)}\big)e^{r(t)}\widehat{r}(t)dt\\
&=-\int_{S^1}v^*d\iota_{\widehat{v}}\lambda-\int_{S^1}v^*\iota_{\widehat{v}}d\lambda\\
&\quad+\theta\cdot\ln\bigg(\int_0^1 e^{r(t)}dt\bigg)\int_0^1 e^{r(t)}\lambda_{x(t)}\big(R(x(t))\big)\widehat{r}(t)dt\\
&\quad+(1-\theta)\cdot\int_0^1 \ln\big(e^{r(t)}\big)e^{r(t)}\lambda_{x(t)}\big(R(x(t))\big)\widehat{r}(t)dt\\
&=-\int_{S^1}dv^*\iota_{\widehat{v}}\lambda-\int_{S^1}v^*\iota_{\widehat{v}}\omega+\theta\cdot\ln\bigg(\int_0^1 e^r dt\bigg)\int_0^1\omega\big(\widehat{v},R(v)\big)dt\\
&\quad+(1-\theta)\cdot\int_0^1 r\omega\big(\widehat{v},R(v)\big)dt\\
&=-\int_0^1\omega(\widehat{v},\partial_t v)dt+\int_0^1\omega\Bigg(\widehat{v},\theta\cdot\ln\bigg(\int_0^1 e^r dt\bigg)R(v)\Bigg)dt\\
&\quad+\int_0^1\omega\Big(\widehat{v},(1-\theta)\cdot rR(v)\Big)dt\\
&=\int_0^1\omega\Bigg(\partial_t v-\theta\cdot\ln\bigg(\int_0^1 e^r dt\bigg)R(v)-(1-\theta)\cdot rR(v),\widehat{v}\Bigg)dt.
\end{aligned}$$

For later reference we summarize this to

$$d\mathcal{A}_\theta(v)\widehat{v}=\int_0^1\omega\Bigg(\partial_t v-\bigg(\theta\cdot\ln\bigg(\int_0^1 e^r dt\bigg)+(1-\theta)\cdot r\bigg)R(v),\widehat{v}\Bigg)dt. \tag{2.1}$$

In particular, we infer from (2.1) that $v\in\mathrm{crit}(\mathcal{A}_\theta)$ if and only if v is a solution of the problem

$$\partial_t v=\bigg(\theta\cdot\ln\bigg(\int_0^1 e^r dt\bigg)+(1-\theta)\cdot r\bigg)R(v). \tag{2.2}$$

Since $dr(R)=0$ we infer from (2.2) that r is constant. Therefore, the scaling factor in front of the Reeb vector field simplifies as follows:

$$\theta \cdot \ln\left(\int_0^1 e^r dt\right) + (1-\theta)\cdot r = \theta \cdot \ln e^r + (1-\theta)\cdot r = \theta\cdot r + (1-\theta)\cdot r = r.$$

This shows that problem (2.2) is equivalent to (1.1) and the first assertion of the lemma is proved.

It remains to prove the second assertion, namely, the formula (1.2) for the action of a critical point. Plugging (1.1) into the definition of $\mathcal{A}_\theta$, we compute using that r is independent of time

$$\begin{aligned}
\mathcal{A}_\theta(v) &= -\int_0^1 e^r \lambda_x(rR)dt + \theta h(e^r) + (1-\theta)h(e^r) \\
&= -re^r + h(e^r) \\
&= -re^r + e^r(r-1) + 1 \\
&= 1 - e^r.
\end{aligned}$$

This shows (1.2) and hence the lemma is proved. $\square$

We now derive the gradient flow equation for the functionals $\mathcal{A}_\theta$ simultaneously for every $\theta \in [0,1]$. The gradient flow equations (1.3) and (1.4) then follow by specializing to the cases $\theta=0$ and $\theta=1$, respectively. To write down the gradient flow equation we have to choose a metric on the free loop space. For that purpose we choose a smooth family of SFT-like almost complex structures J_t and take the L^2-metric with respect to this family. SFT-like almost complex structures are used in Symplectic Field Theory [4] and we recall their definition. The contact condition implies that the vector bundle $(\xi, d\lambda|_\xi) \to \Sigma$ is symplectic. Hence, we choose a family of $d\lambda|_\xi$ compatible almost complex structures J_t on ξ, i. e., $d\lambda|_\xi(\cdot, J_t\cdot)$ is a bundle metric on ξ. We extend this family canonically to a family of $\mathbb{R}$-invariant almost complex structures on the tangent space of the symplectization by requiring that J_t interchanges the Reeb vector field R and the Liouville vector field ∂_r, namely,

$$J_t R = -\partial_r, \quad J_t\partial_r = R.$$

In particular, J_t is compatible with the symplectic form ω in the sense that $\omega(\cdot, J_t\cdot)$ is a Riemannian metric on $\mathbb{R}\times\Sigma$. Following Floer [5], we use this family of ω-compatible almost complex structures to define an L^2-metric $g=g_J$ on the free loop space $\mathcal{L}$. If $v\in\mathcal{L}$ and $\widehat{v}_1, \widehat{v}_2 \in T_v\mathcal{L}$ are tangent vectors at v, i. e., vector fields along v, we define the L^2-inner product of $\widehat{v}_1$ and $\widehat{v}_2$ by

$$g(\widehat{v}_1, \widehat{v}_2) = \int_0^1 \omega\Big(\widehat{v}_1(t), J_t\big(v(t)\big)\widehat{v}_2(t)\Big)dt.$$

We define the L^2-gradient $\nabla\mathcal{A}_\theta(v)$ of $\mathcal{A}_\theta$ at $v\in\mathcal{L}$ implicitly by the requirement that

$$d\mathcal{A}_\theta(v)\widehat{v} = g\big(\nabla\mathcal{A}_\theta(v), \widehat{v}\big), \quad \forall\, \widehat{v}\in T_v\mathcal{L}.$$

We claim that

$$\nabla\mathcal{A}_\theta(v)(t) = J_t(v(t))\left(\partial_t v - \left(\theta\cdot\ln\left(\int_0^1 e^r dt\right) + (1-\theta)\cdot r\right)R(v)\right)(t), \quad t\in S^1.$$

To see this we rewrite (2.1) using $J^2 = -\mathrm{id}$,

$$\begin{aligned}
d\mathcal{A}_\theta(v)\widehat{v} &= \int_0^1 \omega\left(\widehat{v}, J(v)^2\left(\partial_t v - \left(\theta\cdot\ln\left(\int_0^1 e^r dt\right) + (1-\theta)\cdot r\right)R(v)\right)\right)dt \\
&= g\left(\widehat{v}, J(v)\left(\partial_t v - \left(\theta\cdot\ln\left(\int_0^1 e^r dt\right) + (1-\theta)\cdot r\right)R(v)\right)\right),
\end{aligned}$$

which implies the formula above for the gradient. A gradient flow line is formally a smooth map $v\in C^\infty(\mathbb{R},\mathcal{L})$ solving the "ODE" on $\mathcal{L}$

$$\partial_s v(s) + \nabla\mathcal{A}_\theta(v)(s) = 0, \quad s\in\mathbb{R},$$

which we interpret as a smooth map $v \in C^\infty(\mathbb{R} \times S^1, \mathbb{R} \times \Sigma)$ solving the problem

$$\partial_s v + J(v)\Bigg(\partial_t v - \bigg(\theta \cdot \ln\bigg(\int_0^1 e^r dt\bigg) + (1-\theta)\cdot r\bigg)R(v)\Bigg) = 0. \tag{2.3}$$

Let us spell out in detail the extremal cases $\theta = 0$ and $\theta = 1$. For $\theta = 0$ the problem (2.3) becomes

$$\partial_s v(s,t) + J_t(v(s,t))\Big(\partial_t v(s,t) - r(s,t)R(v(s,t))\Big) = 0, \quad (s,t) \in \mathbb{R} \times S^1. \tag{2.4}$$

This is a PDE on the cylinder. In contrast to (2.4) the problem (2.3) is not a PDE anymore for positive θ, since in this case the factor in front of the Reeb vector field not just depends on s and t, but on the whole loop

$$r_s \colon S^1 \to \mathbb{R}, \quad t' \mapsto r(s,t').$$

For $\theta = 1$ this becomes for $(s,t) \in \mathbb{R} \times S^1$

$$\partial_s v(s,t) + J_t(v(s,t))\Bigg(\partial_t v(s,t) - \ln\bigg(\int_0^1 e^{r(s,t')}dt'\bigg)R(v(s,t))\Bigg) = 0. \tag{2.5}$$

For every $\theta \in [0,1]$ the problem (2.3) is a perturbation of the Cauchy−Riemann equation in the direction of the Reeb vector field. In the case of (2.5) the scaling factor in front of the Reeb vector field only depends on s and not on t. We abbreviate this scaling factor for $(s,t) \in \mathbb{R} \times S^1$ by

$$\tau(s,t) = \theta \cdot \ln\bigg(\int_0^1 e^{r(s,t')}dt'\bigg) + (1-\theta)\cdot r(s,t).$$

With this notion we can write (2.3) more compactly as

$$\partial_s v + J(v)\big(\partial_t v - \tau R(v)\big) = 0.$$

3. PROOF OF LEMMA 2

Suppose that v is a gradient flow line of the V-shaped action functional with delay $\mathcal{A}_1$

$$\partial_s v + \nabla \mathcal{A}_1(v) = 0,$$

i. e., $v = (r,x) \in C^\infty(\mathbb{R} \times S^1, \mathbb{R} \times \Sigma)$ is a solution of (1.4). We abbreviate

$$T(s) := \int_0^1 e^{r(s,t)}dt$$

and

$$\tau(s) := \ln(T(s)) = \ln\bigg(\int_0^1 e^{r(s,t)}dt\bigg).$$

Since J_t is SFT-like we obtain from (1.4)

$$\partial_s r - \lambda_x(\partial_t x) + \tau = 0.$$

Abbreviating further v_s, the loops obtained by evaluating v at time s, we compute

$$\begin{aligned}
\partial_s T(s) &= \int_0^1 e^{r(s,t)}\partial_s r(s,t)dt \\
&= \int_0^1 e^{r(s,t)}\Big(\lambda_{x(s)}\big(\partial_t x(s,t)\big) - \tau(s)\Big)dt \\
&= \int_{S^1} v_s^*\lambda - \tau(s)T(s) \\
&= -\mathcal{A}_1(v_s) + h\bigg(\int_0^1 e^{r(s,t)}dt\bigg) - \ln(T(s))T(s)
\end{aligned}$$

$$
\begin{aligned}
&= -\mathcal{A}_1(v_s) + h(T(s)) - \ln(T(s))T(s) \\
&= -\mathcal{A}_1(v_s) + T(s)\ln(T(s)) - T(s) + 1 - \ln(T(s))T(s) \\
&= -\mathcal{A}_1(v_s) - T(s) + 1
\end{aligned}
$$

and therefore

$$
\partial_s \tau(s) = \frac{1 - \mathcal{A}_1(v_s)}{e^{\tau(s)}} - 1. \tag{3.1}
$$

It follows from (3.1) that at every local extremum we have

$$
e^{\tau(s)} = 1 - \mathcal{A}_1\big(v(s)\big).
$$

Since asymptotically the derivatives vanish, we further have for the asymptotics $\tau_\pm = \lim_{s\to\pm\infty} \tau(s)$

$$
e^{\tau_+} = 1 - \mathcal{A}_1(v_+), \qquad e^{\tau_-} = 1 - \mathcal{A}_1(v_-).
$$

Since τ is continuous, it either has to attain its supremum or converge to it asymptotically. Using that $\mathcal{A}_1$ decreases along v, we therefore conclude that

$$
e^{\tau(s)} \leqslant 1 - \mathcal{A}_1(v_+). \tag{3.2}
$$

Since

$$
1 - \mathcal{A}_1(v_s) \geqslant 1 - \mathcal{A}_1(v_-) > 0,
$$

we obtain from (3.1) and (3.2)

$$
\partial_s \tau(s) \geqslant \frac{1 - \mathcal{A}_1(v_s)}{1 - \mathcal{A}_1(v_+)} - 1 = -\frac{\mathcal{A}_1(v_s) - \mathcal{A}_1(v_+)}{1 - \mathcal{A}_1(v_+)} \geqslant -\frac{\mathcal{A}_1(v_-) - \mathcal{A}_1(v_+)}{1 - \mathcal{A}_1(v_+)} > -1.
$$

We next show that τ is strictly monotone decreasing when the gradient flow line is nonconstant. To see this we first show that τ does not have any local minima. In fact, assume that s_0 is a local extremum of τ. Then

$$
\partial_s \tau(s_0) = 0
$$

and we infer from (3.1) that

$$
\partial^2 \tau(s_0) = -\frac{\partial_s \mathcal{A}_1(v_{s_0})}{e^{\tau(s_0)}} > 0,
$$

where the inequality follows from the fact that $\mathcal{A}_1$ is strictly decreasing along its nonconstant gradient flow line. This excludes local minima of τ as well as inflection points. We next explain that τ cannot have local maxima either. Since τ attains its supremum asymptotically, a local maximum cannot be a global maximum, and therefore the existence of local maxima would imply the existence of local minima which do not exist, as we have just explained. Therefore, τ has to be strictly monotone. This finishes the proof of the lemma.

4. PROOF OF THEOREM 1

We construct a map $\mathcal{R}\colon \mathcal{M}_3 \to \mathcal{M}_1$ inverse to $\Pi_*\colon \mathcal{M}_1 \to \mathcal{M}_3$. Suppose that $v = (r, x) \in \mathcal{M}_3$, i. e., a finite energy solution of (1.7). We are looking for $\rho \in C^\infty(\mathbb{R}, \mathbb{R})$ such that

$$
w = \rho_* v = (r + \rho, x)
$$

is a solution of (1.4), i. e., a gradient flow line of $\mathcal{A}_1$. Since J_t is SFT-like and in particular R-invariant, w is a solution of the problem

$$
\partial_s w + J(w)\Big(\partial_t w + \big(\mathcal{A}_3(v) + \partial_s \rho\big) R(w)\Big) = 0.
$$

In order that this becomes a solution of (1.4), we need that for every $s \in \mathbb{R}$ it holds that

$$
\begin{aligned}
\mathcal{A}_3(v_s) + \partial_s \rho(s) &= \ln\left(\int_0^1 e^{r(s,t)+\rho(s)} dt\right) \\
&= \ln\left(e^{\rho(s)} \int_0^1 e^{r(s,t)} dt\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \rho(s) + \ln\left(\int_0^1 e^{r(s,t)}dt\right) \\
&= \rho(s),
\end{aligned}
$$

where in the last equality we have used that $v_s \in \overline{\mathcal{L}}$, since $v \in \mathcal{M}_3$. We see that ρ has to be a solution of the ODE

$$
\partial_s \rho(s) = \rho(s) - \mathcal{A}_3(v_s). \tag{4.1}
$$

Abbreviate

$$
\sigma(s) = \rho(s) - \mathcal{A}_3(v_s).
$$

From (4.1) we obtain

$$
\partial_s \sigma(s) = \partial_s \rho(s) - \partial_s \mathcal{A}_3(v_s) = \rho(s) - \mathcal{A}_3(v_s) - \partial_s \mathcal{A}_3(v_s) = \sigma(s) - \partial_s \mathcal{A}_3(v_s),
$$

so that σ is a solution of the ODE

$$
\partial_s \sigma(s) = \sigma(s) - \partial_s \mathcal{A}_3(v_s). \tag{4.2}
$$

Since v is a gradient flow line of $\mathcal{A}_3$ we have $\partial_s \mathcal{A}_3(v) \leqslant 0$, and since the gradient flow line has finite energy the integral of $\partial_s \mathcal{A}_3(v)$ is finite. In particular,

$$
\partial_s \mathcal{A}_3(v) \in L^1(\mathbb{R}).
$$

Since it is smooth as well we also have

$$
\partial_s \mathcal{A}_3(v) \in L^2(\mathbb{R}).
$$

The linear operator

$$
D\colon W^{1,2}(\mathbb{R}) \to L^2(\mathbb{R}), \quad \sigma \mapsto \partial_s \sigma - \sigma
$$

is an injective Fredholm operator of index zero, i.e., an isomorphism between the two Hilbert spaces. Therefore, there exists a unique $\sigma \in W^{1,2}(\mathbb{R})$ solving (4.2), namely,

$$
\sigma = D^{-1}\big(-\partial_s \mathcal{A}_3(v)\big).
$$

Now set

$$
\rho_v := \mathcal{A}_3(v) + D^{-1}\big(-\partial_s \mathcal{A}_3(v)\big),
$$

then $w := (\rho_v)_*(v)$ is a finite energy solution of (1.4), i.e., an element of the moduli space $\mathcal{M}_1$, and we have a well-defined map

$$
\mathcal{R}\colon \mathcal{M}_3 \to \mathcal{M}_1, \quad v \mapsto (\rho_v)_* v.
$$

It remains to check that $\mathcal{R}$ is inverse to Π_*. We carry this out in two steps.

Step 1: We have $\Pi_* \circ \mathcal{R} = \mathrm{id}\colon \mathcal{M}_3 \to \mathcal{M}_3$, i.e., $\mathcal{R}$ is right inverse to Π_*.

This is the same argument as the proof of Step 1 of Theorem 5.1 in [6]. Assume that $v = (r, x) \in \mathcal{M}_3$. Since Π_* and $\mathcal{R}$ are both t-independent translations in the $\mathbb{R}$-direction there exists $\rho \in C^\infty(\mathbb{R}, \mathbb{R})$ such that

$$
w := \Pi_* \circ \mathcal{R}(v) = \rho_* v = (r + \rho, x).
$$

Since both v and w belong to $\mathcal{M}_3$ we must have for every $s \in \mathbb{R}$

$$
\int_0^1 e^{r(s,t)}dt = 1, \quad \int_0^1 e^{r(s,t)+\rho(s)}dt = 1,
$$

implying that

$$
\rho = 0.
$$

Therefore, $w = v$ so that we have

$$
\Pi_* \circ \mathcal{R}(v) = v.
$$

Since $v \in \mathcal{M}_3$ was arbitrary, Step 1 follows.

Step 2: We have $\mathcal{R} \circ \Pi_* = \mathrm{id}\colon \mathcal{M}_1 \to \mathcal{M}_1$, i.e., $\mathcal{R}$ is left inverse to Π_*.

Assume that $v = (r, x) \in \mathcal{M}_1$. Again, since $\mathcal{R}$ and Π_* are both t-independent translations in the $\mathbb{R}$-direction, there exists $\rho \in C^\infty(\mathbb{R}, \mathbb{R})$ such that

$$w := \mathcal{R} \circ \Pi_*(v) = \rho_* v = (r + \rho, x).$$

Since v solves (1.4) we have

$$\begin{aligned} 0 &= \partial_s w + J(w)\left(\partial_t w + \left(\partial_s \rho - \ln \int_0^1 e^r dt'\right) R(w)\right) \\ &= \partial_s w + J(w)\left(\partial_t w + \left(\partial_s \rho + \rho - \ln \int_0^1 e^{r+\rho} dt'\right) R(w)\right). \end{aligned}$$

Since w is a solution of problem (1.4) as well, we must have

$$\partial_s \rho + \rho = 0$$

i. e.,

$$\rho(s) = \rho_0 e^{-s}.$$

Since w has finite energy, it follows that $\rho_0 = 0$ and therefore

$$\rho = 0.$$

This proves that $w = v$ and hence

$$\mathcal{R} \circ \Pi_*(v) = v.$$

Since $v \in \mathcal{M}_3$ was arbitrary, Step 2 follows and the theorem is proved.

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CONFLICT OF INTEREST

The author declares that he has no conflicts of interest.

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