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## CONTROL SETS FOR AFFINE SYSTEMS, SPECTRAL PROPERTIES, AND PROJECTIVE SPACES\*

FRITZ COLONIUS<sup>†</sup>, ALEXANDRE J. SANTANA<sup>‡</sup>, AND JULIANA SETTI<sup>‡</sup>

**Abstract.** For affine control systems with bounded control range, the control sets, i.e., the maximal subsets of complete approximate controllability, are studied using spectral properties. For hyperbolic systems there is a unique control set with nonvoid interior, and it is bounded. For nonhyperbolic systems, these control sets are unbounded. In an appropriate compactification of the state space there is a unique chain control set, and the relations to the homogeneous part of the control system are worked out.

**Key words.** affine control systems, control sets, boundary at infinity

**MSC codes.** 93B05, 34H05

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**1. Introduction.** We study controllability properties for affine control systems of the form

$$(1.1) \quad \dot{x}(t) = Ax(t) + \sum_{i=1}^m u_i(t)(B_i x(t) + c_i) + d, \quad u(t) \in \Omega,$$

where  $A, B_1, \dots, B_m \in \mathbb{R}^{n \times n}$  and  $c_1, \dots, c_m, d \in \mathbb{R}^n$ . The controls  $u = (u_1, \dots, u_m)$  have values in a bounded set  $\Omega \subset \mathbb{R}^m$  with  $0 \in \Omega$ . The set of admissible controls is  $\mathcal{U} = \{u \in L^\infty(\mathbb{R}, \mathbb{R}^m) | u(t) \in \Omega \text{ for almost all } t\}$  or the set  $\mathcal{U}_{pc}$  of all piecewise constant functions defined on  $\mathbb{R}$  with values in  $\Omega$ . We also write (1.1) as

$$\dot{x}(t) = A(u(t))x(t) + Cu(t) + d, \quad u(t) \in \Omega,$$

with  $A(u) := A + \sum_{i=1}^m u_i B_i$  for  $u \in \Omega$ , and  $C := (c_1, \dots, c_m)$ . With the vector fields  $f_0(x) = Ax + d$  and  $f_i(x) = B_i x + c_i$  on  $\mathbb{R}^n$ , we assume throughout that the following accessibility (or Lie algebra) rank condition holds:

$$(1.2) \quad \dim \mathcal{LA}(f_0, f_1, \dots, f_m)(x) = n \text{ for all } x \in \mathbb{R}^n,$$

where  $\mathcal{LA}(f_0, f_1, \dots, f_m)$  is the set of vector fields in the Lie algebra generated by  $f_0, f_1, \dots, f_m$ .

Controllability properties of bilinear and affine systems have been studied for over than 50 years. Early contributions are due to Rink and Mohler [21] who took the set of equilibria as a starting point for establishing results on complete controllability. On the other hand, Lie-algebraic methods have yielded important insights. A classical result due to Jurdjevic and Sallet [15, Theorem 2] (cf. also Do Rocio, Santana, and

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Verdi [11]) shows that affine system (1.1) is controllable on  $\mathbb{R}^n$  if it has no fixed points and its homogeneous part, the bilinear system

$$(1.3) \quad \dot{x}(t) = A(u(t))x(t), \quad u(t) \in \Omega,$$

is controllable on  $\mathbb{R}^n \setminus \{0\}$  (only the first condition is necessary). Initially, bilinear control systems were considered as “nearly linear” (cf. Bruni, Di Pillo, and Koch [4], which also contains many early references). However it has turned out that characterizing controllability of such systems (even with unrestricted controls) is a very difficult problem. As Jurdjevic [14, p. 182] emphasizes, the controllability properties of affine systems are substantially richer and may require “entirely different geometrical considerations” (based on Lie-algebraic methods). There is a substantial literature on controllability properties of bilinear and affine control systems. Here we only refer to the monographs of Mohler [18], Elliott [12], and Jurdjevic [14]. Further references are also included in [9], where we analyze controllability properties near equilibria.

The present paper concentrates on control sets, i.e., maximal subsets of approximate controllability; cf. Definition 2.1 (another relaxation of complete controllability is the notion of “near controllability” studied by Tie [19] for discrete-time bilinear control systems). Key to our analysis are properties of the interior of the system semigroup and spectral properties of the homogeneous part (1.3). In the hyperbolic case (cf. Definition 4.1) Theorem 4.4 shows that an affine control system has a unique control set  $D$  with nonvoid interior. In the uniformly hyperbolic case (cf. Definition 4.5) Theorem 4.8 shows that  $D$  is bounded. Hence these systems enjoy similar controllability properties as linear control systems of the form  $\dot{x} = Ax + Bu$  with hyperbolic matrix  $A$ ; cf. Colonius and Kliemann [7, Example 3.2.16]. We remark that a generalization in another direction is given for control sets of linear control systems on Lie groups by Ayala and Da Silva [2].

Nonhyperbolic affine systems may possess several control sets with nonvoid interior. By Theorem 6.1 each of them is unbounded. The proof takes up ideas from Rink and Mohler [21], replacing the set of equilibria by the set of periodic solutions. Then we compactify the state space using an embedding into projective space  $\mathbb{P}^n$ . This is in line with the study of the behavior at infinity for ordinary differential equations based on the Poincaré sphere; cf. Perko [20, section 3.10]. The special case of linear control systems discussed in the beginning of section 6 shows that here chain transitivity (a classical notion in the theory of dynamical systems; cf. Robinson [22]) plays an important role. Proposition 6.3 shows that for a control set with nonvoid interior the “boundary at infinity” (cf. Definition 5.3) intersects a chain control set of the projectivized homogeneous part (here small jumps in the trajectories are allowed; cf. Definition 2.5). The main result on the nonhyperbolic case is Theorem 6.9, showing that there is a single chain control set in  $\mathbb{P}^n$  containing the images of all control sets  $D$  with nonvoid interior in  $\mathbb{R}^n$ . The boundary at infinity of this chain control set contains all chain control sets of the projectivized homogeneous part having nonvoid intersection with the boundary at infinity of one of the control sets  $D$ . These results cast new light on the relations between affine systems and their homogeneous parts and are intuitively appealing, since one may expect that for unbounded  $x$ -values the inhomogeneous part  $Cu(t) + d$  will become less relevant.

The contents of the present paper are as follows. In section 2 we first recall some notation and general properties of nonlinear control systems and cite results on spectral properties and controllability for homogeneous bilinear control systems. A result on periodic solutions of periodic inhomogeneous differential equations is stated. Its proof is given in the appendix. Section 3 analyzes the system semigroup of affine

control systems and their homogeneous parts. Section 4 shows that for hyperbolic systems a unique control set exists and is bounded. Section 5 discusses nonhyperbolic systems by embedding affine control systems into homogeneous bilinear systems and associated systems in projective spaces. Section 6 describes control sets and their boundaries at infinity for nonhyperbolic systems.

**Notation.** The set of eigenvalues of a matrix  $A \in \mathbb{R}^{n \times n}$  is  $\text{spec}(A)$ , and the real eigenspace for an eigenvalue  $\mu \in \mathbb{C}$  is  $\mathbf{E}(A; \mu)$ . In a metric space  $X$  with distance  $d$ , the distance of  $x \in X$  to a nonvoid subset  $A \subset X$  is  $d(x, A) = \inf\{d(x, a) | a \in A\}$ .

**2. Preliminaries.** In subsection 2.1 notation and some basic properties of control systems are recalled. Subsection 2.2 cites results on control sets for homogeneous bilinear control systems, and subsection 2.3 characterizes periodic solutions of inhomogeneous periodic linear differential equations.

**2.1. Basic properties of nonlinear control systems.** In this subsection we introduce some terminology and notation for control-affine systems, including control sets and chain control sets.

We will consider control-affine systems on a smooth (real analytic) manifold  $M$  of the form

$$(2.1) \quad \dot{x}(t) = f_0(x(t)) + \sum_{i=1}^m u_i(t) f_i(x(t)), \quad u \in \mathcal{U} \text{ or } u \in \mathcal{U}_{pc},$$

where  $f_0, f_1, \dots, f_m$  are smooth vector fields on  $M$ , and the control range  $\Omega \subset \mathbb{R}^m$  is bounded with  $0 \in \Omega$ . We assume that for every initial state  $x \in M$  and every control function  $u \in \mathcal{U}$  there exists a unique solution  $\varphi(t, x, u), t \in \mathbb{R}$ , with  $\varphi(0, x, u) = x$  of (2.1) depending continuously on  $x$ . For the general theory of nonlinear control systems we refer the reader to Sontag [23] and Jurdjevic [14].

The sets of points reachable from  $x \in M$  and controllable to  $x \in M$  up to time  $T > 0$  are defined by

$$\mathcal{O}_{\leq T}^+(x) := \{y \in M \mid \text{there are } 0 \leq t \leq T \text{ and } u \in \mathcal{U} \text{ with } y = \varphi(t, x, u)\},$$

$$\mathcal{O}_{\leq T}^-(x) := \{y \in M \mid \text{there are } 0 \leq t \leq T \text{ and } u \in \mathcal{U} \text{ with } x = \varphi(t, y, u)\},$$

respectively. Furthermore, the reachable set (or “positive orbit”) from  $x$  and the set controllable to  $x$  (or “negative orbit” of  $x$ ) are

$$\mathcal{O}^+(x) = \bigcup_{T>0} \mathcal{O}_{\leq T}^+(x), \quad \mathcal{O}^-(x) = \bigcup_{T>0} \mathcal{O}_{\leq T}^-(x),$$

respectively. The system is called locally accessible in  $x$  if  $\mathcal{O}_{\leq T}^+(x)$  and  $\mathcal{O}_{\leq T}^-(x)$  have nonvoid interior for all  $T > 0$ , and the system is called locally accessible if this holds in every point  $x \in M$ . This is equivalent to the accessibility rank condition

$$(2.2) \quad \dim \mathcal{LA}\{f_0, f_1, \dots, f_m\}(x) = \dim M \text{ for all } x \in M;$$

here  $\mathcal{LA}\{f_0, f_1, \dots, f_m\}(x)$  is the subspace of the tangent space  $T_x M$  corresponding to the vector fields, evaluated in  $x$ , in the Lie algebra generated by  $f_0, f_1, \dots, f_m$ . The sets  $\mathcal{O}^-(x)$  are the reachable sets of the time reversed system given by  $\dot{x}(t) = -f_0(x(t)) - \sum_{i=1}^m u_i(t) f_i(x(t))$ . The trajectories for controls in  $\mathcal{U}$  can be uniformly approximated on bounded intervals by trajectories for controls in  $\mathcal{U}_{pc}$ .

The following definition introduces sets of complete approximate controllability.

DEFINITION 2.1. A nonvoid set  $D \subset M$  is called a control set of system (2.1) if it has the following properties: (i) for all  $x \in D$  there is a control function  $u \in \mathcal{U}$  such that  $\varphi(t, x, u) \in D$  for all  $t \geq 0$ ; (ii) for all  $x \in D$  one has  $D \subset \overline{\mathcal{O}^+(x)}$ ; and (iii)  $D$  is maximal with these properties; that is, if  $D' \supset D$  satisfies conditions (i) and (ii), then  $D' = D$ . A control set  $D \subset M$  is called an invariant control set if  $\overline{D} = \overline{\mathcal{O}^+(x)}$  for all  $x \in D$ . All other control sets are called variant.

We recall some properties of control sets; cf. Colonius and Kliemann [7, Chap. 3].

Remark 2.2. If the intersection of two control sets is nonvoid, the maximality property (iii) implies that they coincide. If the system is locally accessible in all  $x \in \overline{D}$ , then by [7, Lemma 3.2.13(i)],  $\overline{D} = \text{int}(\overline{D})$  and  $D = \mathcal{O}^-(x) \cap \overline{\mathcal{O}^+(x)}$  for all  $x \in \text{int}(D)$ , and  $\text{int}(D) \subset \mathcal{O}^+(x)$  for all  $x \in D$  (here it suffices to consider controls in  $\mathcal{U}_{pc}$ ). If local accessibility holds on  $M$ , the control sets with nonvoid interior for controls in  $\mathcal{U}$  coincide with those defined analogously for controls in  $\mathcal{U}_{pc}$ . This is proved using the approximation by trajectories for controls in  $\mathcal{U}_{pc}$ .

LEMMA 2.3. Suppose that the system is locally accessible and that there is  $x \in \text{int}(\mathcal{O}^-(x)) \cap \text{int}(\mathcal{O}^+(x))$ . Then  $D := \mathcal{O}^-(x) \cap \overline{\mathcal{O}^+(x)}$  is a control set, and  $x \in \text{int}(D)$ .

Proof. The set  $\text{int}(\mathcal{O}^-(x)) \cap \text{int}(\mathcal{O}^+(x))$  satisfies properties (i) and (ii) of control sets, and hence it is contained in a control set  $D'$  and  $x \in \text{int}(D')$ . Then  $D' = \mathcal{O}^-(x) \cap \overline{\mathcal{O}^+(x)}$  by Remark 2.2.  $\square$

Next we introduce a notion of controllability allowing for (small) jumps between pieces of trajectories. Here we fix a metric  $d$  on  $M$ .

DEFINITION 2.4. Fix  $x, y \in M$ , and let  $\varepsilon, T > 0$ . A controlled  $(\varepsilon, T)$ -chain  $\zeta$  from  $x$  to  $y$  is given by  $n \in \mathbb{N}$ ,  $x_0 = x, \dots, x_{n-1}, x_n = y \in M$ ,  $u_0, \dots, u_{n-1} \in \mathcal{U}$ , and  $t_0, \dots, t_{n-1} \geq T$ , with

$$d(\varphi(t_j, x_j, u_j), x_{j+1}) \leq \varepsilon \text{ for all } j = 0, \dots, n-1.$$

If for every  $\varepsilon, T > 0$  there is a controlled  $(\varepsilon, T)$ -chain from  $x$  to  $y$ , then the point  $x$  is chain controllable to  $y$ .

In analogy to control sets, chain control sets are defined as maximal regions of chain controllability; cf. [7, Chap. 4].

DEFINITION 2.5. A nonvoid set  $E \subset M$  is called a chain control set of system (2.1) if (i) for all  $x \in E$  there is  $u \in \mathcal{U}$  such that  $\varphi(t, x, u) \in E$  for all  $t \in \mathbb{R}$ ; (ii) for all  $x, y \in E$  and  $\varepsilon, T > 0$  there is a controlled  $(\varepsilon, T)$ -chain from  $x$  to  $y$ ; and (iii)  $E$  is maximal (with respect to set inclusion) with these properties.

Obviously, every equilibrium and every periodic trajectory are contained in a control set and a chain control set. Since the concatenation of two controlled  $(\varepsilon, T)$ -chains again yields a controlled  $(\varepsilon, T)$ -chain, two chain control sets coincide if their intersection is nonvoid.

For a continuous dynamical system  $\psi : \mathbb{R} \times X \rightarrow X$  on a metric space  $X$ , a subset  $Y \subset X$  is called chain transitive if for all  $x, y \in Y$  and all  $\varepsilon, T > 0$  there is an  $(\varepsilon, T)$ -chain from  $x$  to  $y$  given by  $n \in \mathbb{N}$ ,  $x_0 = x, x_1, \dots, x_n = y \in X$ , and  $t_0, \dots, t_{n-1} \geq T$  with  $d(\psi(t_j, x_j), x_{j+1}) \leq \varepsilon$  for all  $j = 0, \dots, n-1$ .

For compact and convex control range  $\Omega$ , a control system of the form (2.1) defines a continuous dynamical system, the control flow, given by  $\Psi : \mathbb{R} \times \mathcal{U} \times M \rightarrow \mathcal{U} \times M$ ,  $(t, u, x) \rightarrow (u(t + \cdot), \varphi(t, x, u))$ , where  $u(t + \cdot)(s) := u(t + s)$ ,  $s \in \mathbb{R}$ , and

$\mathcal{U} \subset L^\infty(\mathbb{R}, \mathbb{R}^m)$  considered in a metric for the weak\* topology is compact; cf. Kawan [16, Proposition 1.17]. The following assertions are shown in [16, Proposition 1.24]: chain control sets are closed, and for locally accessible systems every control set with nonvoid interior is contained in a chain control set. The chain control sets  $E$  uniquely correspond to the maximal invariant chain transitive sets  $\mathcal{E}$  of the control flow  $\Psi$  via

$$(2.3) \quad \mathcal{E} := \{(u, x) \in \mathcal{U} \times E \mid \varphi(t, x, u) \in E \text{ for all } t \in \mathbb{R}\}.$$

**2.2. Control sets for homogeneous bilinear systems.** In this subsection we cite several results on control sets and chain control sets for homogeneous bilinear control systems of the form

$$(2.4) \quad \dot{x}(t) = A(u(t))x(t), \quad u(t) \in \Omega, \text{ with } A(u) := A + \sum_{i=1}^m u_i B_i, u \in \Omega,$$

where  $A, B_1, \dots, B_m \in \mathbb{R}^{n \times n}$ , and the controls  $u = (u_1, \dots, u_m)$  have values in a compact convex neighborhood  $\Omega$  of the origin in  $\mathbb{R}^m$ . The solutions are denoted by  $\varphi_{\text{hom}}(t, x, u), t \in \mathbb{R}$ .

By homogeneity, a system of the form (2.4) induces control systems on the unit sphere  $\mathbb{S}^{n-1}$  and on projective space  $\mathbb{P}^{n-1}$ . The projections of  $\mathbb{R}^n \setminus \{0\}$  to the unit sphere  $\mathbb{S}^{n-1}$  and projective space  $\mathbb{P}^{n-1}$  are denoted by  $\pi_{\mathbb{S}}$  and  $\pi_{\mathbb{P}}$ , respectively.

**DEFINITION 2.6.** Let  $\lambda(u, x) = \limsup_{t \rightarrow \infty} \frac{1}{t} \log \|\varphi_{\text{hom}}(t, x, u)\|$  be the Lyapunov exponent for  $(u, x) \in \mathcal{U} \times (\mathbb{R}^n \setminus \{0\})$ .

(i) The Floquet spectrum of a control set  $_{\mathbb{S}}D$  on the unit sphere  $\mathbb{S}^{n-1}$  is

$$\Sigma_{Fl}(_{\mathbb{S}}D) = \{\lambda(u, x) \mid \pi_{\mathbb{S}}x \in \text{int}(_{\mathbb{S}}D), u \in \mathcal{U}_{pc} \text{ } \tau\text{-periodic with } \pi_{\mathbb{S}}\varphi_{\text{hom}}(\tau, x, u) = \pi_{\mathbb{S}}x\}.$$

(ii) The Floquet spectrum of a control set  $_{\mathbb{P}}D$  on projective space  $\mathbb{P}^{n-1}$  is

$$\Sigma_{Fl}(_{\mathbb{P}}D) = \{\lambda(u, x) \mid \pi_{\mathbb{P}}x \in \text{int}(_{\mathbb{P}}D), u \in \mathcal{U}_{pc} \text{ } \tau\text{-periodic with } \pi_{\mathbb{P}}\varphi_{\text{hom}}(\tau, x, u) = \pi_{\mathbb{P}}x\}.$$

(iii) The Lyapunov spectrum of a control set  $_{\mathbb{P}}D$  on projective space  $\mathbb{P}^{n-1}$  is

$$\Sigma_{Ly}(_{\mathbb{P}}D) = \{\lambda(u, x) \mid u \in \mathcal{U} \text{ and } \pi_{\mathbb{P}}\varphi(t, x, u) \in \overline{{}_{\mathbb{P}}D} \text{ for all } t \geq 0\}.$$

**Remark 2.7.** The Floquet spectrum can be characterized using the system semigroups  $_{\mathbb{R}}\mathcal{S}^{\text{hom}}$  and  $_{\mathbb{P}}\mathcal{S}^{\text{hom}}$  of the systems on  $\mathbb{R}^n \setminus \{0\}$  and on  $\mathbb{P}^{n-1}$ , respectively (cf. section 3 for the definition of system semigroups). Suppose that the accessibility rank condition in  $\mathbb{P}^{n-1}$  holds. Corollary 7.3.18 in Colonius and Kliemann [7] implies that the Floquet spectrum of a control set  $_{\mathbb{P}}D$  consists of the numbers  $\frac{1}{\tau} \log |\rho|$ , where  $\rho$  is a real eigenvalue of an element  $\Phi_u(\tau, 0) \in {}_{\mathbb{R}}\mathcal{S}_\tau^{\text{hom}}$  with eigenspace  $\pi_{\mathbb{P}}(\mathbf{E}(\Phi_u(\tau, 0); \rho)) \subset \text{int}(_{\mathbb{P}}D)$  and such that  $\Phi_u(\tau, 0)$  induces an element of the system semigroup in  $\text{int}(_{\mathbb{P}}\mathcal{S}_{\leq \tau+1}^{\text{hom}})$ .

The following theorem analyzes the control sets of the homogeneous system (2.4).

**THEOREM 2.8.** Consider the systems on the unit sphere  $\mathbb{S}^{n-1}$  and on projective space  $\mathbb{P}^{n-1}$  obtained by projection of the homogeneous bilinear control system (2.4). Assume that the accessibility rank condition on  $\mathbb{P}^{n-1}$  is satisfied.

(i) There are  $1 \leq k_0 \leq n$  control sets  $_{\mathbb{P}}D_j$  with nonvoid interior in  $\mathbb{P}^{n-1}$ , and exactly one of these control sets is an invariant control set.

(ii) There are  $1 \leq k_1 \leq 2k_0$  control sets  ${}_{\mathbb{S}}D_i$  on  $\mathbb{S}^{n-1}$ . In each case, one or two of the control sets  ${}_{\mathbb{S}}D_i$  on  $\mathbb{S}^{n-1}$  project to a single control set  ${}_{\mathbb{P}}D_j$  on  $\mathbb{P}^{n-1}$ , and then  $\Sigma_{Fl}({}_{\mathbb{S}}D_i) = \Sigma_{Fl}({}_{\mathbb{P}}D_j)$  and  $\Sigma_{Ly}({}_{\mathbb{S}}D_i) = \Sigma_{Ly}({}_{\mathbb{P}}D_j)$ .

(iii) If 0 is in the interior of the Floquet spectrum  $\Sigma_{Fl}({}_{\mathbb{S}}D_i)$ , the cone

$${}_{\mathbb{R}}D_i := \{\alpha x \in \mathbb{R}^n \mid \alpha > 0 \text{ and } x \in {}_{\mathbb{S}}D_i\}$$

generated by  ${}_{\mathbb{S}}D_i$  is a control set with nonvoid interior in  $\mathbb{R}^n \setminus \{0\}$ . Conversely, if  ${}_{\mathbb{R}}D$  is a control set with nonvoid interior in  $\mathbb{R}^n \setminus \{0\}$ , then  $0 \in \overline{\Sigma_{Ly}({}_{\mathbb{P}}D)}$ , where  ${}_{\mathbb{P}}D \supset \pi_{\mathbb{P}}({}_{\mathbb{R}}D)$ .

*Proof.* For (i) see [7, Theorem 7.1.1]. Assertion (ii) follows from Colonius, Santana, and Setti [9, Theorem 3.15] and the observation that the equality of the Lyapunov spectra is obvious. Statement (iii) follows from [9, Proposition 3.18] and (ii).  $\square$

We turn to the chain control sets in projective space. By [7, Theorem 7.1.2], every chain control set  ${}_{\mathbb{P}}E_j$  contains a control set  ${}_{\mathbb{P}}D_i$  with nonvoid interior, and hence the number  $l$  of chain control sets satisfies  $1 \leq l \leq k_0$ . Furthermore, [7, Theorem 7.3.16] shows that for every chain control set  ${}_{\mathbb{P}}E_j$  in  $\mathbb{P}^{n-1}$  and every  $u \in \mathcal{U}$ ,

$$\{x \in \mathbb{R}^n \mid x = 0 \text{ implies } \pi_{\mathbb{P}}\varphi_{\text{hom}}(t, x, u) \in {}_{\mathbb{P}}E_j \text{ for all } t \in \mathbb{R}\}$$

is a linear subspace, and its dimension is independent of  $u \in \mathcal{U}$ . By (2.3), the chain control sets  ${}_{\mathbb{P}}E_j$  uniquely correspond to the maximal chain transitive subsets  ${}_{\mathbb{P}}\mathcal{E}_j$  of the control flow on  $\mathcal{U} \times \mathbb{P}^{n-1}$  via

$$(2.5) \quad {}_{\mathbb{P}}\mathcal{E}_j := \{(u, \pi_{\mathbb{P}}x) \in \mathcal{U} \times \mathbb{P}^{n-1} \mid \pi_{\mathbb{P}}\varphi_{\text{hom}}(t, x, u) \in {}_{\mathbb{P}}E_j \text{ for all } t \in \mathbb{R}\}.$$

**2.3. Periodic solutions.** We state some facts on periodic solutions of inhomogeneous periodic differential equations of the form

$$(2.6) \quad \dot{x}(t) = P(t)x(t) + z(t),$$

where  $P(\cdot) \in L^\infty(\mathbb{R}, \mathbb{R}^{n \times n})$  and  $z(\cdot) \in L^\infty(\mathbb{R}, \mathbb{R}^n)$  are  $\tau$ -periodic, i.e.,  $P(t + \tau) = P(t)$  and  $z(t + \tau) = z(t)$  for almost all  $t \in \mathbb{R}$ . The principal fundamental solution  $\Phi(t, s) \in \mathbb{R}^{n \times n}$ ,  $t, s \in \mathbb{R}$ , is given by

$$\frac{d}{dt}\Phi(t, s) = P(t)\Phi(t, s) \text{ with } \Phi(s, s) = I.$$

The homogeneous equation with  $z(t) \equiv 0$  has nontrivial (nonunique)  $\tau$ -periodic solutions if and only if 1 is an eigenvalue of  $\Phi(\tau, 0)$ . Here and in the following, uniqueness of a periodic solution means that it is unique up to time shifts.

The Floquet multipliers  $\rho_j \in \mathbb{C}$ ,  $j = 1, \dots, n$ , are defined as the eigenvalues of  $\Phi(\tau, 0)$ , and the Floquet exponents are  $\lambda_j := \frac{1}{\tau} \log |\rho_j|$ . They coincide with the Lyapunov exponents; cf. Chicone [5, Proposition 2.61] and Colonius and Kliemann [8, Theorem 7.2.9]. In particular, 0 is a Floquet exponent if 1 is a Floquet multiplier. We also refer the reader to Chicone [5, section 2.4] and Teschl [24, section 3.6] for background on Floquet theory (note that the Floquet exponents as defined above are the real parts of the Floquet exponents defined in [5] and [24]).

We need the following results on periodic solutions.

**PROPOSITION 2.9.** *Consider the  $\tau$ -periodic differential equation (2.6).*

(i) *There is a unique  $\tau$ -periodic solution if and only if  $1 \in \text{spec}(\Phi(\tau, 0))$ . Its initial value (at time 0) is  $x^0 = (I - \Phi(\tau, 0))^{-1} \int_0^\tau \Phi(\tau, s)z(s)ds$ .*

(ii) If for  $z(t) \equiv 0$  there does not exist a  $\tau$ -periodic solution, then the principal fundamental solution satisfies  $1 \in \text{spec}(\Phi(\tau, 0))$  and  $\int_0^\tau \Phi(\tau, s)z(s)ds \in \text{Im}(I - \Phi(\tau, 0))$ .

(iii) For  $k = 0, 1, \dots$ , let  $P^k(\cdot)$  and  $z^k(\cdot)$  be  $\tau_k$ -periodic, and suppose that for a  $c > 0$  the norms in  $L^\infty([0, \tau_0 + 1]; \mathbb{R}^n)$  satisfy  $\|P^k(\cdot)\|_\infty, \|z^k(\cdot)\|_\infty \leq c$  for all  $k$ . Assume that  $\tau_k \rightarrow \tau_0$ ,  $P^k(\cdot) \rightarrow P^0(\cdot)$  in  $L^1([0, \tau_0 + 1], \mathbb{R}^{n \times n})$ , and  $z^k(\cdot) \rightarrow z^0(\cdot)$  in  $L^1([0, \tau_0 + 1]; \mathbb{R}^n)$  for  $k \rightarrow \infty$ . Then for  $k \rightarrow \infty$ , the corresponding principal fundamental matrices  $\Phi^k(t, s)$  converge to  $\Phi^0(t, s)$  uniformly in  $t, s \in [0, \tau_0 + 1]$ .

(iv) In the situation of (iii), assume additionally that for  $k = 0, 1, \dots$ , the corresponding principal fundamental solutions  $\Phi^k(t, s)$  satisfy  $1 \in \text{spec}(\Phi^k(\tau_k, 0))$ . Then the initial values  $x^k$  of the corresponding unique  $\tau_k$ -periodic solutions converge for  $k \rightarrow \infty$  to the initial value  $x^0$  of the unique  $\tau_0$ -periodic solution for  $P^0(\cdot)$  and  $z^0(\cdot)$ .

*Proof.* See Appendix A.  $\square$

**3. System semigroups of affine systems in  $\mathbb{R}^n$ .** In this section we analyze system semigroups for affine systems of the form (1.1); cf. Jurdjevic and Sallet [15].

We start with the following general remarks on the relevant Lie group which is the semidirect product  $G = \mathbb{R}^n \ltimes GL(\mathbb{R}^n)$  with product given by

$$(v, g) \cdot (w, h) = (v + gw, gh).$$

Its Lie algebra is given by the semidirect product  $\mathfrak{g} = \mathbb{R}^n \ltimes \mathfrak{gl}(\mathbb{R}^n)$ , where the Lie bracket is

$$[(a, A), (b, B)] = (Ab - Ba, AB - BA).$$

If we consider  $X = (a, A), Y = (b, B)$  as vector fields on  $\mathbb{R}^n$  through the relation  $X(x) = Ax + a, Y(x) = Bx + b$ , the Lie bracket is

$$[X, Y](x) = -(AB - BA)x - (Ab - Ba).$$

If we denote by  $e = (0, I) \in G$  the identity element, then the tangent space is  $T_e G = \mathfrak{g}$ , and an element  $X \in \mathfrak{g}$  can be identified with a right-invariant vector field, a smooth vector field on  $G$ , given by  $X^R(g) := (dR_g)_e X$ , where  $R_g$  stands for the right-translation and  $(dR_g)_e$  is its differential at the identity element. By standard results, the vector fields  $X^R$  are complete, and their flows satisfy

$$(3.1) \quad \phi_t^{X^R}(g) = R_g(\phi_t^{X^R}(e)) \text{ for all } g \in G.$$

Note also that

$$(\exp tX)(x) = e^{tA}x + \int_0^t e^{(t-s)A}a ds \text{ for } x \in \mathbb{R}^n \text{ and } X = (a, A)$$

gives us exactly the expression for the exponential map for  $G = \mathbb{R}^n \ltimes GL(\mathbb{R}^n)$  and  $\mathfrak{g} = \mathbb{R}^n \ltimes \mathfrak{gl}(\mathbb{R}^n)$ , meaning that the Lie group exponential is given by

$$\exp tX = \left( \int_0^t e^{(t-s)A}a ds, e^{tA} \right) \text{ for } X = (a, A).$$

If  $\mathcal{F} \subset \mathfrak{g}$  is a nonempty subset, consider the subgroups of  $G$ ,

$$\begin{aligned} \mathcal{G}(\mathcal{F}) &:= \{ \exp t_1 X_1 \cdot \exp t_2 X_2 \cdots \exp t_k X_k \mid t_i \in \mathbb{R} \text{ and } X_i \in \mathcal{F} \}, \\ \mathcal{G}^R(\mathcal{F}) &:= \{ (\phi_{t_1}^{X_1^R} \circ \phi_{t_2}^{X_2^R} \circ \cdots \circ \phi_{t_k}^{X_k^R})(e) \mid t_i \in \mathbb{R} \text{ and } X_i \in \mathcal{F} \}, \end{aligned}$$



and the semigroups  $\mathcal{S}(\mathcal{F})$ ,  $\mathcal{S}^R(\mathcal{F})$ , where only  $t_i > 0$  are allowed. Thus  $\mathcal{S}^R(\mathcal{F})$  is the set of points on  $G$  that can be attained from  $e \in G$  by concatenations of the flows of  $\mathcal{F}^R = \{X^R | X \in \mathcal{F}\}$ .

**PROPOSITION 3.1.** *If  $\mathcal{F} \subset \mathfrak{g}$  is a nonempty subset, the groups  $\mathcal{G}(\mathcal{F})$  and  $\mathcal{G}^R(\mathcal{F})$ , as well as the semigroups  $\mathcal{S}(\mathcal{F})$  and  $\mathcal{S}^R(\mathcal{F})$ , respectively, coincide.*

*Proof.* Since  $X^R(e) = X$ , the exponential map  $\exp: \mathfrak{g} \rightarrow G$  is defined by  $\exp X = \phi_1^{X^R}(g)$ . Consequently, we get for all  $t_i \in \mathbb{R}$  and  $X_i \in \mathcal{F} \subset \mathfrak{g}$ , using (3.1),

$$\begin{aligned} & \exp t_1 X_1 \cdot \exp t_2 X_2 \cdots \exp t_k X_k \\ &= R_{e^{t_1 X_1} e^{t_2 X_2} \cdots e^{t_{k-1} X_{k-1}}} (e^{t_k X_k}) = R_{e^{t_1 X_1} e^{t_2 X_2} \cdots e^{t_{k-1} X_{k-1}}} (\phi_{t_k}^{X_k^R}(e)) \\ &= \phi_{t_k}^{X_k^R}(e^{t_1 X_1} e^{t_2 X_2} \cdots e^{t_{k-1} X_{k-1}}) = \phi_{t_k}^{X_k^R}(R_{e^{t_1 X_1} \cdots e^{t_{k-2} X_{k-2}}}(e^{t_{k-1} X_{k-1}})) \\ &= \phi_{t_k}^{X_k^R}(R_{e^{t_1 X_1} e^{t_2 X_2} \cdots e^{t_{k-2} X_{k-2}}}(\phi_{t_{k-1}}^{X_{k-1}^R}(e))) = \phi_{t_k}^{X_k^R} \circ \phi_{t_{k-1}}^{X_{k-1}^R}(e^{t_1 X_1} \cdots e^{t_{k-2} X_{k-2}}) \\ &= \cdots = \phi_{t_k}^{X_k^R} \circ \phi_{t_{k-1}}^{X_{k-1}^R} \circ \cdots \circ \phi_{t_1}^{X_1^R}(e). \end{aligned}$$

This implies the assertion.  $\square$

The family of affine vector fields on  $\mathbb{R}^n$  associated with (1.1) is given by

$$(3.2) \quad \mathcal{F} = \{X^u(x) = A(u)x + Cu + d | u \in \Omega\}.$$

Then  $\mathcal{LA}(\mathcal{F}) = \mathcal{LA}(f_0, f_1, \dots, f_m)$ ; cf. (1.2). The system group  $\mathcal{G} = \mathcal{G}(\mathcal{F})$  is a subgroup of the semidirect product  $\mathbb{R}^n \ltimes GL(\mathbb{R}^n)$ .

Since we assume the accessibility rank condition (1.2), Jurdjevic [14, Theorem 3, p. 44] implies that the set  $\mathcal{F}$  of vector fields is transitive on  $\mathbb{R}^n$ ; i.e., for all  $x, y \in \mathbb{R}^n$  there is  $g \in \mathcal{G}$  with  $y = gx$ . Denote by  $\mathcal{S}_\tau = \mathcal{S}_\tau(\mathcal{F})$  the set of those elements of  $\mathcal{S}(\mathcal{F})$  with  $t_1 + \cdots + t_k = \tau$ , and let  $\mathcal{S}_{\leq \tau} = \bigcup_{\tau \in [0, T]} \mathcal{S}_\tau$ ; do the same for  $\mathcal{S}_\tau(\mathcal{F}^R)$ . The trajectories for  $u \in \mathcal{U}_{pc}$  of control system (1.1) are given by the action of the semigroup  $\mathcal{S}$  on  $\mathbb{R}^n$ : For  $g \in \mathcal{S}_\tau$  and  $\tau = t_k + \cdots + t_1, t_i > 0$ ,

$$gx = g(u)x = \exp(t_k X^{u^k}) \cdots \exp(t_1 X^{u^1})x = \varphi(\tau, x, u),$$

where  $u^j \in \Omega, X^{u^j} \in \mathcal{F}$ , and  $\varphi(t, x, u), t \in [0, \tau]$ , is the solution of (1.1) with piecewise constant control  $u$  defined, with  $t_0 = 0$ , by

$$(3.3) \quad u(t) := u^{j+1} \text{ for } t \in \left[ \sum_{i=0}^j t_i, \sum_{i=0}^{j+1} t_i \right), \quad j = 0, \dots, k-1.$$

Note that  $u$  is not uniquely determined by  $g$ . We will always consider the interior of  $\mathcal{S}$  in the system group  $\mathcal{G}$ .

**THEOREM 3.2.** (i) *The system semigroup  $\mathcal{S} = \mathcal{S}(\mathcal{F})$  of (1.1) satisfies  $\mathcal{S}_{\leq \tau} \subset \overline{\text{int}(\mathcal{S}_{\leq \tau})}$  in  $\mathcal{G}$  for every  $\tau > 0$ .*

(ii) *If  $g \in \text{int}(\mathcal{S}_{\leq \tau})$  for a  $\tau > 0$ , then  $gx \in \text{int}(\mathcal{O}_{\leq \tau}^+(x))$  for every  $x \in \mathbb{R}^n$ .*

*Proof.* (i) The right-invariant vector fields in  $\mathcal{F}^R$  on  $\mathcal{G}$  are real analytic. Since this implies that they are Lie-determined, we can apply Jurdjevic [14, Corollary, p. 67] which shows that for every open set  $U$  in an orbit of  $\mathcal{F}_r$ , any  $y \in U$ , and any  $\tau > 0$ , the reachable set  $\mathcal{S}_{\leq \tau}(\mathcal{F}^R)(y) \cap U$  contains an open set in the orbit topology.

In particular, this applies to the reachable set up to time  $\tau$  of the identity which by Proposition 3.1 coincides with  $\mathcal{S}_{\leq \tau}$ . Furthermore, [14, Corollary 1, p. 68] implies  $\mathcal{S}_{\leq \tau} \subset \overline{\text{int}(\mathcal{S}_{\leq \tau})}$ .

(ii) The maps  $\mathcal{G} \rightarrow \mathbb{R}^n : g \rightarrow gx$  are open, and hence  $g \in \text{int}(\mathcal{S}_{\leq \tau})$  implies  $gx \in \text{int}(\mathcal{O}_{\leq \tau}^+(x))$ .  $\square$

Next, we relate the system semigroup and fixed points in control sets.

**PROPOSITION 3.3.** (i) *Let  $D \subset \mathbb{R}^n$  be a control set with nonvoid interior for (1.1). Then for every  $x \in \text{int}(D)$ , there are  $\tau > 0$  and  $g \in \mathcal{S}_\tau \cap \text{int}(\mathcal{S}_{\leq \tau+1})$  such that  $gx = x$ .*

(ii) *Conversely, let  $g \in \text{int}(\mathcal{S})$  with  $gx = x$  for some point  $x \in \mathbb{R}^n$ . Then  $x \in \text{int}(D)$  for some control set  $D \subset \mathbb{R}^n$ .*

*Proof.* (i) Let  $x \in \text{int}(D)$ . By continuity of the action, the set  $H = \{h \in \mathcal{G} | hx \in \text{int}(D)\}$  is open in  $\mathcal{G}$ , and hence for all  $t > 0$  small enough there exists an element  $h \in \mathcal{S}_t \cap H$ . By Proposition 3.1(i), it follows that for  $\sigma_k \rightarrow 0^+$  there are elements  $g_k \in \text{int}(\mathcal{S}_{\leq \sigma_k})$  converging to the identity in  $\mathcal{G}$ , and hence  $h_k := g_k h \in \text{int}(\mathcal{S}_{\leq t+\sigma_k}) \rightarrow h$ . Since  $H$  is open, there is  $k \in \mathbb{N}$  large enough such that  $h_k \in H \cap \mathcal{S}_{t+\sigma} \cap \text{int}(\mathcal{S}_{\leq t+\sigma_k})$  for some  $\sigma \in [0, \sigma_k]$ , and hence  $h_k x \in \text{int}(D)$ . By Remark 2.2, exact controllability in  $\text{int}(D)$  holds. Thus we find  $h_0 \in \mathcal{S}_s, s > 0$ , such that  $h_0 h_k x = x$ . It follows that  $g := h_0 h_k \in \mathcal{S}_\tau \cap \text{int}(\mathcal{S}_{\leq \tau+1})$  with  $\tau := t + \sigma + s$  and  $gx = x$ .

(ii) Every  $g \in \text{int}(\mathcal{S})$  satisfies  $gx \in \text{int}(\mathcal{O}^+(x))$  and  $x \in \text{int}(\mathcal{O}^-(gx))$  for all  $x$ . Now  $gx = x$  implies that  $x \in \text{int}(\mathcal{O}^+(x)) \cap \text{int}(\mathcal{O}^-(x))$ , and hence Lemma 2.3 shows that  $D = \mathcal{O}^-(x) \cap \mathcal{O}^+(x)$  is a control set with  $x \in \text{int}(D)$ .  $\square$

Note that  $g \in \text{int}(\mathcal{S})$  implies that  $g \in \text{int}(\mathcal{S}_{\leq \tau})$  for some  $\tau > 0$ ; cf. Colonius and Kliemann [7, Lemma 4.5.2]. The control  $u$  in (3.3) also determines the element  $\Phi_u(\tau, 0)$  of the system semigroup of the homogeneous bilinear control system (2.4). We denote the corresponding semigroup by  ${}_{\mathbb{R}}\mathcal{S}^{\text{hom}} \subset GL(n, \mathbb{R})$ . Theorem 3.2 and Proposition 3.3 are also valid for system (2.4) and  ${}_{\mathbb{R}}\mathcal{S}^{\text{hom}}$  provided that the corresponding accessibility rank condition in  $\mathbb{R}^n \setminus \{0\}$  holds.

Next, we describe the relation between the action of the system semigroup and periodic control functions. We may extend the control defined by (3.3) to a  $\tau$ -periodic control function in  $\mathcal{U}_{pc}$ . The next lemma follows immediately from Proposition 2.9(i).

**LEMMA 3.4.** *Let  $g(u) \in \mathcal{S}_\tau$ . Then  $g(u)x = x$  for some  $x \in \mathbb{R}^n$  if and only if the corresponding  $\tau$ -periodic differential equation in (1.1) has a  $\tau$ -periodic solution with initial value  $x(0) = x$ . This solution is unique if and only if  $1 \in \text{spec}(\Phi_u(\tau, 0))$ .*

Note the following results on continuous dependence.

**LEMMA 3.5.** *Let  $u, v \in \mathcal{U}_{pc}$  be  $\sigma$ -periodic and  $\tau$ -periodic, respectively, for some  $\sigma, \tau > 0$ . Define, for  $\alpha \in [\sigma, \sigma + \tau]$ ,*

$$u^\alpha(t) := u(t) \text{ for } t \in [0, \sigma], u^\alpha(t) := v(t - \sigma) \text{ for } t \in [\sigma, \alpha],$$

*and extend  $u^\alpha$  to an  $\alpha$ -periodic control  $u^\alpha \in \mathcal{U}_{pc}$ . Then the controls  $u_{[0, \sigma+\tau]}^\alpha$  depend continuously on  $\alpha$  as elements of  $L^1([0, \sigma + \tau], \mathbb{R}^m)$ . Furthermore, the principal fundamental solutions  $\Phi_{u^\alpha}(\alpha, 0) \in \mathbb{R}^{n \times n}$  and the elements  $g(u^\alpha) \in \mathbb{R}^n \ltimes GL(\mathbb{R}^n)$  depend, in a continuous and piecewise analytic way, on  $\alpha$ .*

*Proof.* We may write the control  $v$  as  $v(t) = v^j$  for  $t \in [t_0 + \dots + t_{j-1}, t_0 + \dots + t_j]$ , where  $v^j \in \Omega, t_0 := 0$ , and  $t_j > 0$  for  $j = 1, \dots, \ell$  with  $\tau = t_1 + \dots + t_\ell$ . The assertions follow from the explicit expressions for  $\alpha \in [\sigma + \sum_{i=0}^{j-1} t_i, \sigma + \sum_{i=0}^j t_i]$  and  $j = 1, \dots, \ell$ ,

$$\Phi_{u^\alpha}(\alpha, 0) = e^{(\alpha - \sigma - \sum_{i=0}^{j-1} t_i)A(v^j)} e^{t_{i-1}A(v^{j-1})} \dots e^{t_1 A(v^1)} \Phi_u(\sigma, 0),$$

$$g(u^\alpha) = \exp\left(\left(\alpha - \sigma - \sum_{i=0}^{j-1} t_i\right) X^{v^j}\right) \exp\left(t_{j-1} X^{v^{j-1}}\right) \dots \exp\left(t_1 X^{v^1}\right) g(u),$$

where  $X^{v^j}$  is the affine vector field  $X^{v^j}(x) = A(v^j)x + Cv^j + d$ .  $\square$

This result can be used to analyze the system semigroup  $\mathcal{S}$  of the affine system and the system semigroup  $\mathbb{R}\mathcal{S}^{\text{hom}}$  of the homogeneous part.

LEMMA 3.6. *Let  $g(u) \in \mathcal{S}_\sigma \cap \text{int}(\mathcal{S})$  and  $g(v) \in \mathcal{S}_\tau \cap \text{int}(\mathcal{S})$  for some  $\sigma, \tau > 0$ . Then there exist for  $\alpha \in [0, 1]$ ,  $\tau_\alpha$ -periodic controls  $u^\alpha$  with  $u^0 = u, \tau_0 = \sigma$  and  $u^1 = v, \tau_1 = \tau$  such that the maps  $p: [0, 1] \rightarrow \text{int}(\mathcal{S})$  and  $p^{\text{hom}}: [0, 1] \rightarrow \mathbb{R}\mathcal{S}^{\text{hom}}$ ,*

$$p(\alpha) := g(u^\alpha) \in \text{int}(\mathcal{S}) \text{ and } p^{\text{hom}}(\alpha) := \Phi_{u^\alpha}(\tau_\alpha, 0) \in \mathbb{R}\mathcal{S}^{\text{hom}}, \alpha \in [0, 1],$$

*are continuous paths. Furthermore, the continuity and smoothness properties from Lemma 3.5 hold not only for  $u^\alpha, \Phi_{u^\alpha}(\tau_\alpha, 0) \in \mathbb{R}\mathcal{S}^{\text{hom}}$  and  $g(u^\alpha) \in \mathcal{S}$  but also for  $\tau_\alpha$ .*

*Proof.* By Lemma 3.5, one finds a path in  $\mathcal{S}$  from  $g(u)$  to  $g(v)g(u)$ , and an analogous construction yields a path in  $\mathcal{S}$  from  $g(v)$  to  $g(v)g(u)$ . Since for any  $g' \in \mathcal{S}$  and  $g'' \in \text{int}(\mathcal{S})$  it follows that  $g'g'', g''g' \in \text{int}(\mathcal{S})$ , the elements on the paths are in  $\text{int}(\mathcal{S})$ . Combining these paths, one obtains a path from  $g(u)$  to  $g(v)$ . This can be reparametrized to obtain a path with  $\alpha \in [0, 1]$  and periods  $\tau_\alpha$  in  $[0, \sigma + \tau]$ . Analogously, one obtains the path  $p^{\text{hom}}$ . The smoothness properties remain valid.  $\square$

Next, we use these lemmas to prove spectral properties of elements in the interior of the system semigroup and corresponding periodic solutions.

PROPOSITION 3.7. *Let  $g(u) \in \mathcal{S}_\sigma \cap \text{int}(\mathcal{S})$  for some  $\sigma > 0$ , with  $1 \in \text{spec}(\Phi_u(\sigma, 0))$ , and  $g(v) \in \mathcal{S}_\tau \cap \text{int}(\mathcal{S})$  for some  $\tau > 0$ . Consider the paths  $p$  in  $\text{int}(\mathcal{S})$  and  $p^{\text{hom}}$  in  $\mathbb{R}\mathcal{S}^{\text{hom}}$  constructed in Lemma 3.6.*

(i) *For every  $\varepsilon > 0$ , there are  $\tau_\alpha$ -periodic controls  $w^\alpha \in \mathcal{U}_{pc}$  and continuous paths  $p_1: [0, 1] \rightarrow \text{int}(\mathcal{S})$  with  $p_1(\alpha) = g(w^\alpha)$  for  $\alpha \in [0, 1]$ , and  $p_1^{\text{hom}}: [0, 1] \rightarrow \mathbb{R}\mathcal{S}^{\text{hom}}$  with  $p_1^{\text{hom}}(\alpha) = \Phi_{w^\alpha}(\tau_\alpha, 0)$ , with*

$$\begin{aligned} \|p_1(1) - g(v)\| &= \|g(w^1) - g(v)\| < \varepsilon, \\ \|p_1^{\text{hom}}(1) - \Phi_v(\tau, 0)\| &= \|\Phi_{w^1}(\tau_1, 0) - \Phi_v(\tau, 0)\| < \varepsilon, \end{aligned}$$

*such that  $1 \in \text{spec}(\Phi_{w^\alpha}(\tau_\alpha, 0))$  for all but at most finitely many  $\alpha \in [0, 1]$ , and the continuity and smoothness properties from Lemma 3.5 hold for  $u^\alpha, \tau_\alpha, \Phi_{w^\alpha}(\alpha, 0) \in \mathbb{R}\mathcal{S}^{\text{hom}}$ , and  $g(w^\alpha) \in \mathcal{S}$ .*

(ii) *If  $1 \in \text{spec}(\Phi_{u^{\alpha_0}}(\tau_{\alpha_0}, 0))$  (or  $1 \in \text{spec}(\Phi_{w^{\alpha_0}}(\tau_{\alpha_0}, 0))$ ) for some  $\alpha_0 \in [0, 1]$ , then for all  $\alpha$  in a neighborhood of  $\alpha_0$  there are unique  $\tau^\alpha$ -periodic solutions with initial values  $x^\alpha$  depending continuously on  $\alpha$ .*

*Proof.* (i) First, we prove these properties for the  $\alpha$ -periodic controls  $u^\alpha$  constructed in Lemma 3.5. The proof will proceed inductively for  $j = 0, \dots, \ell - 1$  and  $t \in [t_j, t_{j+1}]$ . Since  $1 \in \text{spec}(\Phi_u(\sigma, 0))$ , it follows that the analytic function  $\alpha \rightarrow \det(I - \Phi_{u^\alpha}(\tau^\alpha, 0))$  is not identically 0. Hence, there are at most finitely many  $\alpha_i \in [0, t_1]$  with  $1 \in \text{spec}(\Phi_{u^{\alpha_i}}(\tau^{\alpha_i}, 0))$ . If  $\det(I - \Phi_{u^{t_1}}(\tau_{t_1}, 0)) = 0$ , choose  $s_1 = t_1$ ; otherwise, choose  $s_1 < t_1$  arbitrarily close to  $t_1$  with  $\det(I - \Phi_{u^{s_1}}(\tau_{s_1}, 0)) = 0$ , and define, for  $\alpha \in [\sigma, \sigma + s_1]$ ,

$$w^\alpha(t) = u^\alpha(t), t \in [0, \alpha].$$

Then the  $\alpha$ -periodic extension of  $w^\alpha$  satisfies  $p_1(\alpha) := g(w^\alpha) \in \text{int}(\mathcal{S})$  and  $p_1^{\text{hom}}(\alpha) := \Phi_{w^\alpha}(\alpha, 0) \in {}_{\mathbb{R}}\mathcal{S}^{\text{hom}}$  with  $1 \in \text{spec}(\Phi_{w^\alpha}(\alpha, 0))$  for all but at most finitely many  $\alpha \in [\sigma, \sigma + s_1]$ . Consider

$$I - e^{(\alpha - \sigma - s_1)A(v^2)} e^{s_1 A(v^1)} \Phi_u(\sigma, 0) \text{ for } \alpha \in [\sigma + s_1, \sigma + t_2].$$

For  $\alpha = \sigma + s_1$ , the determinant of this matrix is unequal zero, and hence it has at most finitely many zeros in  $[s_1, t_2]$ .

Proceeding in this way up to  $j = \ell - 1$ , one constructs  $\alpha$ -periodic controls  $w^\alpha$  such that  $p_1(\alpha) := g(w^\alpha) \in \text{int}(\mathcal{S})$  and  $p_1^{\text{hom}}(\alpha) := \Phi_{w^\alpha}(\alpha, 0) \in {}_{\mathbb{R}}\mathcal{S}^{\text{hom}}$  with  $1 \in \text{spec}(\Phi_{w^\alpha}(\alpha, 0))$  for all but at most finitely many  $\alpha \in [\sigma, \sigma + \tau]$ . Since  $t_i - s_i > 0$  is arbitrarily small, it also follows that

$$p_1^{\text{hom}}(\sigma + \tau) - p^{\text{hom}}(\sigma + \tau) \|\| < \varepsilon, \|p_1(\sigma + \tau) - p(\sigma + \tau)\| < \varepsilon.$$

The same constructions as in the proof of Lemma 3.6 can also be applied here to yield assertion (i).

(ii) Continuous dependence on  $\alpha$  of  $\Phi_{w^\alpha}(\tau_\alpha, 0)$  implies that  $1 \in \text{spec}(\Phi_{w^\alpha}(\tau_\alpha, 0))$  for all  $\alpha$  in a neighborhood of  $\alpha_0$ , and hence by Proposition 2.9(i) there are unique  $\tau_\alpha$ -periodic solutions. Since the controls  $w^\alpha$  depend continuously on  $\alpha$  as elements of  $L^1([0, 1], \mathbb{R}^m)$ , we see that

$$A(w^\alpha(\cdot)) = A + \sum_{i=1}^m w_i^\alpha(\cdot) B_i \in L^1([0, \sigma + \tau]; \mathbb{R}^{n \times n}), Cw^\alpha(\cdot) + d \in L^1([0, \sigma + \tau]; \mathbb{R}^n)$$

also depend continuously on  $\alpha$ . Thus Proposition 2.9(iv) shows that their initial values  $x^\alpha$  depend continuously on  $\alpha$ .  $\square$

The next two lemmas discuss the periodic solutions when 1 is in the spectrum.

LEMMA 3.8. *Consider, for  $k = 0, 1, \dots$ , the differential equations*

$$\dot{x}(t) = A(u^k(t))x(t) + Cu^k(t) + d,$$

where  $u^k$  is  $\tau_k$ -periodic with both  $\tau_k \rightarrow \tau_0 > 0$  and  $u^k \rightarrow u^0$  in  $L^1([0, \tau_0 + 1]; \mathbb{R}^m)$  with  $\|u_k\|_\infty \leq c, k \in \mathbb{N}$ , for some  $c > 0$ , and

(i) *the principal fundamental solutions  $\Phi_{u^k}(t, s)$  of  $\dot{x}(t) = A(u^k(t))x(t)$  satisfy  $1 \in \text{spec}(\Phi_{u^0}(\tau_0, 0))$  and  $1 \in \text{spec}(\Phi_{u^k}(\tau_k, 0))$  for  $k = 1, 2, \dots$ ; and*

(ii)  $\int_0^{\tau_0} \Phi_{u^0}(\tau_0, s)(Cu^0(s) + d)ds \in \text{Im}(I - \Phi_{u^0}(\tau_0, 0))$ .

*Then there are unique  $\tau_k$ -periodic solutions for  $u^k$ , and their initial values  $x^k$  satisfy*

$$(3.4) \quad \|x^k\| \rightarrow \infty \text{ and } \frac{x^k}{\|x^k\|} \rightarrow \ker(I - \Phi_{u^0}(\tau_0, 0)) = \mathbf{E}(\Phi_{u^0}(\tau_0, 0); 1) \text{ for } k \rightarrow \infty.$$

*Proof.* By Proposition 2.9(i) it follows that for every  $k = 1, 2, \dots$ , there is a unique  $\tau_k$ -periodic solution with initial value  $x^k$  satisfying

$$(3.5) \quad (I - \Phi_{u^k}(\tau_k, 0))x^k = \int_0^{\tau_k} \Phi_{u^k}(\tau_k, s)(Cu^k(s) + d)ds.$$

If  $x^k$  remains bounded, we may suppose that there is  $x^0 \in \mathbb{R}^n$  with  $x^k \rightarrow x^0$ . Since  $u^k \rightarrow u^0$  in  $L^1([0, \tau_0 + 1]; \mathbb{R}^m)$ , it follows that also

$$A(u^k(\cdot)) \rightarrow A(u^0(\cdot)) \text{ and } Cu^k(\cdot) + d \rightarrow Cu^0(\cdot) + d$$

in  $L^1([0, \tau_0 + 1]; \mathbb{R}^{n \times n})$  and in  $L^1([0, \tau_0 + 1]; \mathbb{R}^n)$ , respectively. Thus, by Proposition 2.9(iii), the right-hand sides of (3.5) converge to  $\int_0^{\tau_0} \Phi_{u^0}(\tau_0, s)(Cu^0(s) + d)ds$ , and one obtains a contradiction to assumption (ii). This shows that  $\|x^k\| \rightarrow \infty$ . Similarly, the second assertion in (3.4) also follows when we divide (3.5) by  $\|x^k\|$ .  $\square$

The next lemma describes the case where assumption (ii) above is violated.

LEMMA 3.9. *Consider, for a  $\tau_0$ -periodic control  $u^0$ ,*

$$(3.6) \quad \dot{x}(t) = A(u^0(t))x(t) + Cu^0(t) + d,$$

*and suppose that the principal fundamental solution of  $\dot{x}(t) = A(u^0(t))x(t)$  satisfies  $1 \in \text{spec}(\Phi_{u^0}(\tau_0, 0))$  and  $\int_0^{\tau_0} \Phi_{u^0}(\tau_0, s)(Cu^0(s) + d)ds = (I - \Phi_{u^0}(\tau_0, 0))y^0$  for some  $y^0 \in \mathbb{R}^n$ . Then the nontrivial affine subspace  $Y := y^0 + \mathbf{E}(\Phi_{u^0}(\tau_0, 0); 1)$  has the property that there is a  $\tau_0$ -periodic solution of (3.6) starting in  $y$  if and only if  $y \in Y$ , and there are  $x^k \in Y, k \in \mathbb{N}$ , satisfying the conditions in (3.4).*

*Proof.* The first assertion is clear by the definitions. The second assertion follows by choosing  $x^k := y^0 + kz, k \in \mathbb{N}$ , where  $0 = z \in \mathbf{E}(\Phi_{u^0}(\tau_0, 0); 1)$ .  $\square$

**4. Control sets for hyperbolic systems.** In this section we present definitions of hyperbolicity for affine control systems and show that hyperbolic systems have a unique control set with nonvoid interior and that it is bounded.

Since for any  $\tau$ -periodic control, the homogeneous part (1.3) of affine system (1.1) is a homogeneous periodic differential equation, we can define corresponding Floquet multipliers which are the eigenvalues of the principal fundamental solution  $\Phi_u(\tau, 0)$ ; cf. subsection 2.3.

DEFINITION 4.1. *An affine system of the form (1.1) is hyperbolic if*

$$(4.1) \quad 1 \in \text{spec}(\Phi_u(\tau, 0)) \text{ for all } \tau\text{-periodic } u \in \mathcal{U}_{pc} \text{ with } \tau > 0, g(u) \in \mathcal{S}_\tau \cap \text{int}(\mathcal{S}).$$

*Otherwise, it is called nonhyperbolic.*

REMARK 4.2. If there is a  $\tau$ -periodic control  $u$  with  $g(u) \in \mathcal{S}_\tau \cap \text{int}(\mathcal{S})$ , and  $\rho \in \text{spec}(\Phi_u(\tau, 0))$  with  $\rho^k = 1$  for some  $k \in \mathbb{N}$ , then the system is nonhyperbolic. In fact, we may consider  $u$  as a  $k\tau$ -periodic control and find that  $1 \in \text{spec}(\Phi_u(k\tau, 0))$  with  $\hat{g}(u) \in \mathcal{S}_{k\tau} \cap \text{int}(\mathcal{S})$ .

REMARK 4.3. If our sufficient condition for the existence of a control set  $\mathbb{R}D^{\text{hom}}$  of the homogeneous part of (1.1) holds (cf. Theorem 2.8(iii)), there is a Floquet exponent  $0 = \frac{1}{\tau} \log |\rho|$  for a Floquet multiplier  $\rho \in \text{spec}(\Phi_u(\tau, 0))$ , and hence  $|\rho| = 1$ . According to the preceding remark, the system can only be hyperbolic if  $\rho$  is not a root of unity.

Next, we show that for hyperbolic affine systems, there is a unique control set with nonvoid interior.

THEOREM 4.4. *Suppose that the affine system (1.1) is hyperbolic. Then there is a unique control set  $D$  with nonvoid interior, and for every  $g \in \text{int}(\mathcal{S})$  there is a unique  $x \in \mathbb{R}^n$  with  $x = gx$  and*

$$\text{int}(D) = \{x \in \mathbb{R}^n \mid \text{there is } g \in \text{int}(\mathcal{S}) \text{ with } x = gx\}.$$

*Proof.* Let  $g = g(u) \in \mathcal{S}_\tau \cap \text{int}(\mathcal{S})$ . By hyperbolicity, 1 is not an eigenvalue of the principal fundamental solution  $\Phi_u(\tau, 0)$ . Proposition 2.9(i) implies that there is a unique  $\tau$ -periodic solution starting in some  $x \in \mathbb{R}^n$ , and hence  $gx = x$  by Lemma 3.4.

By Proposition 3.3(ii) it follows that  $x \in \text{int}(D)$  for some control set  $D$ . In order to show that  $D$  does not depend on  $g$ , consider  $g, h \in \text{int}(\mathcal{S})$ . By Lemma 3.6, one finds a continuous path  $p$  in  $\text{int}(\mathcal{S})$  from  $g$  to  $h$ . For all  $\alpha \in [0, 1]$  hyperbolicity implies that  $1 \in \text{spec}(\Phi_{u^\alpha}(\tau_\alpha, 0))$ , and hence there are unique fixed points  $x^\alpha$  of  $g(u^\alpha)$  and a control set  $D^\alpha$  with  $x^\alpha \in \text{int}(D^\alpha)$ . As in Proposition 3.7(ii) it follows that  $x^\alpha$  also depends continuously on  $\alpha$ . Hence, for small  $\alpha > 0$  all points  $x^\alpha$  are contained in a single control set  $D$ , showing

$$\alpha^* := \sup \left\{ \alpha \mid x^{\alpha'} \in D \text{ for all } \alpha' \in [0, \alpha] \right\} > 0.$$

Since  $x^{\alpha^*} \in \text{int}(D^{\alpha^*})$ , it follows from Remark 2.2 that  $D^{\alpha^*} = D$  which shows that  $x^{\alpha^*} \in \text{int}(D)$ . If  $\alpha^* < 1$ , this implies that  $D^\alpha = D$  for all  $\alpha \in [\alpha^*, \alpha^* + \varepsilon]$  for some  $\varepsilon > 0$ , contradicting the definition of  $\alpha^*$ . It follows that  $\alpha^* = 1$ , and hence there is a single control set  $D$  containing all  $x = gx$  for  $g \in \text{int}(\mathcal{S})$ . The corresponding periodic controls generate periodic solutions which are also contained in  $D$ .

It remains to show that  $D$  is the unique control set with nonvoid interior. By Proposition 3.3(i), for a point  $x$  in the interior of any control set, there are  $\tau > 0$  and  $g \in \mathcal{S}_\tau \cap \text{int}(\mathcal{S}_{\leq \tau+1})$  with  $gx = x$ . Hence it follows that  $x \in D$ .  $\square$

The question arises of whether the control set  $D$  is bounded. We will give a positive answer provided that the following uniform hyperbolicity condition holds, assuming that the control range  $\Omega$  is a compact and convex neighborhood of the origin in  $\mathbb{R}^m$ , and hence, for system (2.4), the control flow  $\Psi$  on  $\mathcal{U} \times \mathbb{R}^n$  is well defined (cf. subsection 2.1).

**DEFINITION 4.5.** *The homogeneous bilinear system (2.4) is uniformly hyperbolic if the vector bundle  $\mathcal{U} \times \mathbb{R}^n$  can be decomposed into the Whitney sum of two invariant subbundles  $\mathcal{V}^1$  and  $\mathcal{V}^2$  such that the restrictions  $\Psi^1$  and  $\Psi^2$  of the control flow  $\Psi$  to  $\mathcal{V}^1$  and  $\mathcal{V}^2$ , respectively, satisfy, for constants  $\alpha > 0$  and  $K \geq 1$  and for all  $(u, x_i) \in \mathcal{V}^i$ ,*

$$\begin{aligned} \|\varphi(t, x_1, u)\| &= \|\Psi_t^1(u, x_1)\| \leq Ke^{-\alpha t} \|x_1\| \text{ for } t \geq 0, \\ \|\varphi(t, x_2, u)\| &= \|\Psi_t^2(u, x_2)\| \leq Ke^{\alpha t} \|x_2\| \text{ for } t \leq 0. \end{aligned}$$

*Remark 4.6.* The uniform hyperbolicity condition is also used in Kawan [17] and Da Silva and Kawan [10]. It is equivalent to the condition that 0 is not in the Sacker–Sell spectrum of the linear flow  $\Psi$ ; cf. Colonius and Kliemann [7, section 5.5].

Then, for  $i = 1, 2$ , one obtains that  $\mathcal{V}^i(u) := \{x \in \mathbb{R}^n \mid (u, x) \in \mathcal{V}^i\}$  is a subspace of  $\mathbb{R}^n$ , and its dimension is independent of  $u \in \mathcal{U}$ . For all  $u \in \mathcal{U}$ ,

$$\mathbb{R}^n = \mathcal{V}^1(u) \oplus \mathcal{V}^2(u) \text{ and } \varphi(t, x_i, u) \in \mathcal{V}^i(u(t + \cdot)) \text{ for all } t \in \mathbb{R},$$

and hence, for  $x = x_1 \oplus x_2$  with  $x_i \in \mathcal{V}^i(u)$  and  $\Phi_u^i(t, s) := \Phi_u(t, s)|_{\mathcal{V}^i(u(s+\cdot))}$ . for  $t, s \in \mathbb{R}$ ,

$$\varphi(t, x, u) = \varphi(t, x_1, u) \oplus \varphi(t, x_2, u) \text{ and } \Phi_u(t, s) = \Phi_u^1(t, s) + \Phi_u^2(t, s).$$

The uniform hyperbolicity condition above implies that system (1.1) is hyperbolic in the sense of Definition 4.1 since

$$\text{spec}(\Phi_u(\tau, 0)) = \text{spec}(\Phi_u^1(\tau, 0)) \cup \text{spec}(\Phi_u^2(\tau, 0)),$$

and  $\rho \in \text{spec}(\Phi_u^1(\tau, 0))$  implies  $|\rho| \leq e^{-\alpha\tau}$ ,  $\rho \in \text{spec}(\Phi_u^2(\tau, 0))$  implies  $|\rho| \geq e^{\alpha\tau}$ .

LEMMA 4.7. *Suppose that the uniform hyperbolicity assumption holds. Then there exists  $c > 0$  such that for  $(u, x_1) \in \mathcal{V}^1$  and  $(u, x_2) \in \mathcal{V}^2$ ,*

$$\|\varphi(t, x_1, u)\| \leq K \|x_1\| + \frac{Kc}{\alpha} \text{ for } t \geq 0, \quad \|\varphi(t, x_2, u)\| \leq K \|x_2\| + \frac{Kc}{\alpha} \text{ for } t \leq 0.$$

*Proof.* Denote the projections of  $\mathbb{R}^n$  to  $\mathcal{V}^1(u)$  along  $\mathcal{V}^2(u)$  by  $P_u$ , and choose  $c > 0$  such that  $\|P_u\| \|Cv + d\| \leq c$  for all  $u \in \mathcal{U}, v \in \Omega$ . By invariance of  $\mathcal{V}^1$ ,  $P_{u(t+\cdot)}\Phi_u(t, s) = \Phi_u(t, s)P_{u(s+\cdot)}$ , and hence,

$$\begin{aligned} \varphi(t, x_1, u) &= P_{u(t+\cdot)}\varphi(t, x_1, u) = P_{u(t+\cdot)}\Phi_u(t, 0)x_1 + \int_0^t P_{u(t+\cdot)}\Phi_u(t, s)[Cu(s) + d]ds \\ &= \Phi_u^1(t, 0)x_1 + \int_0^t \Phi_u^1(t, s)P_{u(s+\cdot)}[Cu(s) + d]ds. \end{aligned}$$

Then it follows for all  $u \in \mathcal{U}$  and  $t \geq 0$  that

$$\begin{aligned} \|\varphi(t, x_1, u)\| &\leq \|\Phi_u^1(t, 0)x_1\| + \int_0^t \|\Phi_u^1(t, s)P_{u(s+\cdot)}[Cu(s) + d]\| ds \\ &\leq Ke^{-\alpha t} \|x_1\| + Kc \int_0^t e^{-\alpha(t-s)} ds \leq K \|x_1\| + \frac{Kc}{\alpha}. \end{aligned}$$

The second assertion holds analogously.  $\square$

THEOREM 4.8. *Let  $\Omega$  be a compact and convex neighborhood of the origin in  $\mathbb{R}^m$ . Suppose that the homogeneous part (1.3) of affine system (1.1) satisfies the uniform hyperbolicity condition in Definition 4.5. Then the unique control set  $D$  with nonvoid interior is bounded.*

*Proof.* We show that  $\text{int}(D)$  is bounded. This will yield the assertion, since by Remark 2.2 the accessibility rank condition implies that  $D \subset \overline{\text{int}(D)}$ . Fix  $x \in \text{int}(D)$ , and consider an arbitrary point  $y \in \text{int}(D) = \mathcal{O}^+(x) \cap \mathcal{O}^-(x)$ . Thus there are controls  $u^1, u^2 \in \mathcal{U}$  and times  $t_1, t_2 > 0$  with  $y = \varphi(t_1, x, u^1) = \varphi(-t_2, x, u^2)$ . Define

$$u(t) = \begin{cases} u^1(t) & \text{for } t \in [0, t_1], \\ u^2(t) & \text{for } t \in (-t_2, 0), \end{cases}$$

and extend  $u$  to a  $t_1 + t_2$ -periodic function on  $\mathbb{R}$ . Then  $u(t_1 + \cdot) = u(-t_2 + \cdot)$  in  $\mathcal{U}$ , implying

$$(4.2) \quad \mathcal{V}^i(u(t_1 + \cdot)) = \mathcal{V}^i(u(-t_2 + \cdot)) \text{ for } i = 1, 2.$$

We decompose

$$x = x_1 \oplus x_2 \text{ with } x_i \in \mathcal{V}^i(u) \text{ and } y = y_1 \oplus y_2 \text{ with } y_i \in \mathcal{V}^i(u(t_1 + \cdot)) \text{ for } i = 1, 2.$$

The invariance of the complementary subbundles  $\mathcal{V}^i$ , together with (4.2), shows that

$$y_1 = \varphi(t_1, x_1, u) = \varphi(-t_2, x_1, u), \quad y_2 = \varphi(t_1, x_2, u) = \varphi(-t_2, x_2, u),$$

and  $y = \varphi(t_1, x_1, u) \oplus \varphi(-t_2, x_2, u)$ . By Lemma 4.7  $\|\varphi(t_1, x_1, u)\|$  and  $\|\varphi(-t_2, x_2, u)\|$  and hence all  $y \in \text{int}(D)$  satisfy bounds which are independent of  $t_1$  and  $t_2$ .  $\square$

Next, we present a simple example of a uniformly hyperbolic affine system.

*Example 4.9.* Consider the following system with control range  $\Omega = [-1, 1]$ :

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + u(t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \end{pmatrix} u(t) + \begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$

For the homogeneous part  $A(u) = A + uB$  the exponential growth rates of the  $x$ - and  $y$ -components are for  $u \in [-1, 1]$  given by  $\lambda_1(u) = 2 + u \geq 1$  and  $\lambda_2(u) = -2 + u \leq -1$ , respectively. Thus the system is uniformly hyperbolic with  $\mathcal{V}^1 = \mathcal{U} \times (\{0\} \times \mathbb{R})$  and  $\mathcal{V}^2 = \mathcal{U} \times (\mathbb{R} \times \{0\})$ . We claim that the unique control set with nonvoid interior is  $D = (-2, 0) \times [-1, 3]$ . For the proof, first observe that the equilibria are given by

$$0 = (2 + u)x_u + 3u + 3, \quad 0 = (-2 + u)y_u + 3u; \quad \text{hence } x_u = -\frac{3u + 3}{2 + u} \text{ and } y_u = \frac{3u}{2 - u}.$$

The maps  $u \rightarrow x_u$  and  $u \rightarrow y_u$  are monotonically decreasing and increasing, respectively, since  $\frac{d}{du}x_u < 0$  and  $\frac{d}{du}y_u > 0$ . This implies that the set of equilibria is contained in

$$[x_1, x_{-1}] \times [y_{-1}, y_1] = [-2, 0] \times [-1, 3].$$

Inspection of the phase portraits for constant  $u$  shows that any control set is contained in this set, and the phase portraits for  $u = -1$  and  $u = 1$  show that one can approximately reach (with a combination of these controls) from any point  $(x, y)^\top$  in  $(-2, 0) \times [-1, 3]$  to any other point in this set, while this is not possible from points  $(-2, y)^\top$  and  $(0, y)^\top, y \in [-1, 3]$ . This proves the claim.

**5. Affine control systems and projective spaces.** In this section we construct, for affine control systems and their homogeneous parts, induced systems on projective spaces. In order to distinguish explicitly between control sets and chain control sets referring to the affine system and its homogeneous part, we will mark the latter by the superscript “hom” in the rest of this paper.

Systems (1.1) and (1.3) can be embedded into a homogeneous bilinear control system in  $\mathbb{R}^{n+1}$  of the form (cf. Elliott [12, subsection 3.8.1])

$$(5.1) \quad \begin{pmatrix} \dot{x}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{pmatrix} A & d \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ z(t) \end{pmatrix} + \sum_{i=1}^m u_i(t) \begin{pmatrix} B_i & c_i \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ z(t) \end{pmatrix}.$$

Denote the solutions of (5.1) with initial condition  $(x(0), z(0)) = (x^0, z^0) \in \mathbb{R}^n \times \mathbb{R}$  by  $\psi(t, (x^0, z^0), u), t \in \mathbb{R}$ . For initial values of the form  $(x^0, 1) \in \mathbb{R}^{n+1}$  one finds

$$(5.2) \quad \psi(t, (x^0, 1), u) = (\varphi(t, x^0, u), 1) \text{ in } \mathbb{R}^{n+1},$$

and for initial values of the form  $(x^0, 0) \in \mathbb{R}^{n+1}$  one finds

$$(5.3) \quad \psi(t, (x^0, 0), u) = (\varphi_{\text{hom}}(t, x^0, u), 0) \text{ in } \mathbb{R}^{n+1}.$$

Thus the trajectories (5.2) and (5.3) are copies of the trajectories of (1.1) and its homogeneous part (1.3), respectively, obtained by adding a trivial  $(n+1)$ st component. An immediate consequence is the following proposition.

**PROPOSITION 5.1.** (i) *A subset  $D \subset \mathbb{R}^n$  is a control set of (1.1) if and only if the set  $D^1 := \{(x, 1) | x \in D\}$  is a control set of (5.1) in  $\mathbb{R}^{n+1} \setminus \{0\}$ .*

(ii) *A subset  ${}_{\mathbb{R}}D^{\text{hom}} \subset \mathbb{R}^n \setminus \{0\}$  is a control set of (1.3) if and only if the set  $D^0 := \{(x, 0) | x \in {}_{\mathbb{R}}D^{\text{hom}}\}$  is a control set of (5.1) in  $\mathbb{R}^{n+1} \setminus \{0\}$ .*



Next, we discuss associated systems in projective spaces. Recall that  $\mathbb{P}^{n-1} = (\mathbb{R}^n \setminus \{0\}) / \sim$ , where  $\sim$  is the equivalence relation  $x \sim y$  if  $y = \lambda x$  with some  $\lambda \neq 0$ . An atlas of  $\mathbb{P}^{n-1}$  is given by  $n$  charts  $(U_i, \psi_i)$ , where  $U_i$  is the set of equivalence classes  $[x_1 : \cdots : x_n]$  with  $x_i \neq 0$  (using homogeneous coordinates), and  $\psi_i : U_i \rightarrow \mathbb{R}^{n-1}$  is defined by

$$\psi_i([x_1 : \cdots : x_n]) = \left( \frac{x_1}{x_i}, \dots, \widehat{\frac{x_i}{x_i}}, \dots, \frac{x_n}{x_i} \right);$$

here the hat means that the  $i$ th entry is missing. Denote by  $\pi_{\mathbb{P}}$  both projections  $\mathbb{R}^n \rightarrow \mathbb{P}^{n-1}$  and  $\mathbb{R}^{n+1} \rightarrow \mathbb{P}^n$ . A metric on  $\mathbb{P}^n$  is given by defining, for elements  $p_1 = \pi_{\mathbb{P}}x, p_2 = \pi_{\mathbb{P}}y$ ,

$$(5.4) \quad d(p_1, p_2) = \min \left\{ \frac{\|x\|}{\|y\|} - \frac{\|y\|}{\|x\|}, \frac{\|x\|}{\|y\|} + \frac{\|y\|}{\|x\|} \right\}.$$

Projecting the homogeneous bilinear control system (5.1) in  $\mathbb{R}^{n+1}$  to  $\mathbb{P}^n$  one obtains the following system given in homogeneous coordinates by

$$(5.5) \quad \begin{bmatrix} \dot{x}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{pmatrix} A & d \\ 0 & 0 \end{pmatrix} + \sum_{i=1}^m u_i(t) \begin{pmatrix} B_i & c_i \\ 0 & 0 \end{pmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}.$$

Projective space  $\mathbb{P}^n$  can be written as the disjoint union  $\mathbb{P}^n = \mathbb{P}^{n,1} \dot{\cup} \mathbb{P}^{n,0}$ , where, in homogeneous coordinates, the levels  $\mathbb{P}^{n,i}$  are given by

$$\mathbb{P}^{n,i} := \{[x_1 : \cdots : x_n : i] \mid (x_1, \dots, x_n) \in \mathbb{R}^n\} \text{ for } i = 0, 1.$$

Observe that, by homogeneity,  $\mathbb{P}^{n,0} = \{[x_1 : \cdots : x_n : 0] \mid \|(x_1, \dots, x_n)\| = 1\}$ . Any trajectory of system (5.5) is obtained as the projection of a trajectory of (5.1) with initial condition satisfying  $z^0 = 0$  or 1, since any initial value  $[x_1^0 : \cdots : x_n^0 : z^0]$  with  $z^0 = 0$  coincides with  $[\frac{x_1^0}{z^0} : \cdots : \frac{x_n^0}{z^0} : 1]$ .

Loosely speaking,  $\mathbb{P}^{n,0}$  is projective space  $\mathbb{P}^{n-1}$  (embedded into  $\mathbb{P}^n$ ), and  $\mathbb{P}^{n,1}$  is  $\mathbb{P}^n$  without  $\mathbb{P}^{n-1}$ . In fact, as noted above, an atlas of  $\mathbb{P}^n$  is given by  $n+1$  charts  $(U_i, \psi_i)$ . A trivial atlas for  $\mathbb{P}^{n,1}$  is given by  $\{(U_{n+1}, \psi_{n+1})\}$ , proving that  $\mathbb{P}^{n,1}$  is a manifold which is diffeomorphic to  $\mathbb{R}^n$ . The space  $\mathbb{P}^{n,0}$  is closed in  $\mathbb{P}^n$ , and the spaces  $\mathbb{P}^{n-1}$  and  $\mathbb{P}^{n,0}$  are diffeomorphic under the map

$$(5.6) \quad e : \mathbb{P}^{n-1} \rightarrow \mathbb{P}^{n,0} : [x_1 : \cdots : x_n] \rightarrow [x_1 : \cdots : x_n : 0].$$

For any trajectory  $\psi(t, (x^0, 1), u) = (\varphi_1(t, x^0, u), \dots, \varphi_n(t, x^0, u), 1)^\top$  of system (5.1) in  $\mathbb{R}^{n+1} \setminus \{0\}$ , the projection to  $\mathbb{P}^{n,1} \subset \mathbb{P}^n$  is  $[\varphi_1(t, x^0, u) : \cdots : \varphi_n(t, x^0, u) : 1]$ .

The proof of the following proposition is straightforward and thus omitted.

**PROPOSITION 5.2.** *Consider in  $\mathbb{R}^n$  the affine control system (1.1) and its homogeneous part (1.3), and in  $\mathbb{R}^{n+1}$  the homogeneous bilinear control system (5.1), and consider as well the system in  $\mathbb{P}^{n-1}$  induced by (1.3) and the system (5.5) in  $\mathbb{P}^n$  induced by (5.1).*

(i) *Every control set  $D \subset \mathbb{R}^n$  of the affine system (1.1) yields a control set  $\pi_{\mathbb{P}}D^1$  of the system (5.5) in  $\mathbb{P}^n$  via the map*

$$(x_1, \dots, x_n) \longrightarrow [x_1 : \cdots : x_n : 1] : \mathbb{R}^n \rightarrow \mathbb{P}^{n,1} \subset \mathbb{P}^n.$$

*Furthermore,  $D$  is an invariant control set if and only if  $\pi_{\mathbb{P}}D^1$  is an invariant control set. The control set  $D$  is unbounded if and only if  $\partial(\pi_{\mathbb{P}}D^1) \cap \mathbb{P}^{n,0} = \emptyset$ . More precisely,*

if  $x^k \in D$  with  $\|x^k\| \rightarrow \infty$ , then every cluster point  $y$  of  $\frac{x^k}{\|x^k\|}$  satisfies, in homogeneous coordinates,

$$[x_1^{k_i} : \cdots : x_n^{k_i} : 1] \rightarrow [y_1 : \cdots : y_n : 0] \in \mathbb{P}^{n,0} \text{ for a subsequence } k_i \rightarrow \infty.$$

(ii) Every control set  $\mathbb{P}D^{\text{hom}}$  and every chain control set  $\mathbb{P}E^{\text{hom}}$  of the system in  $\mathbb{P}^{n-1}$  induced by (1.3) corresponds to a unique control set  $e(\mathbb{P}D^{\text{hom}})$  and chain control set  $e(\mathbb{P}E^{\text{hom}})$ , respectively, of the system (5.5) restricted to  $\mathbb{P}^{n,0}$  and, conversely, via the map  $e$ . The invariant control sets in  $\mathbb{P}^{n-1}$  correspond to the invariant control sets in  $\mathbb{P}^{n,0}$ .

We remark that the assertions in Proposition 5.2(ii) also hold if the accessibility rank condition in  $\mathbb{P}^{n-1}$  is violated (this is the case in Example 6.5).

The intersection  $\partial(\pi_{\mathbb{P}}D^1) \cap \mathbb{P}^{n,0}$  will be of relevance below. Hence we give it a suggestive name.

**DEFINITION 5.3.** For a control set  $D \subset \mathbb{R}^n$  with associated control set  $\pi_{\mathbb{P}}D^1$  in  $\mathbb{P}^{n,1}$ , the set  $\partial_{\infty}(D) := \partial(\pi_{\mathbb{P}}D^1) \cap \mathbb{P}^{n,0}$  is the boundary at infinity of  $D$ .

Proposition 5.2(i) shows, in particular, that the boundary at infinity  $\partial_{\infty}(D)$  is nonvoid if and only if  $D$  is unbounded.

*Remark 5.4.* The construction of the boundary at infinity of a control set bears some similarity to the ideal boundary used by Firer and do Rocio [13] in the analysis of invariant control sets for sub-semigroups of a semisimple Lie group.

Next, we clarify the relations between the accessibility rank conditions on the relevant spaces.

**THEOREM 5.5.** (i) If the accessibility rank condition holds for affine system (1.1) on  $\mathbb{R}^n$ , then it also holds for the system on the submanifold  $\mathbb{P}^{n,1} \subset \mathbb{P}^n$  induced by the bilinear system (5.1) on  $\mathbb{R}^{n+1}$ .

(ii) If the accessibility rank condition holds for the system on  $\mathbb{P}^{n-1}$  induced by the homogeneous part (1.3) of system (1.1), then it holds for the system on the invariant submanifold  $\mathbb{P}^{n,0} \subset \mathbb{P}^n$  induced by the bilinear system (5.1) on  $\mathbb{R}^{n+1}$ .

*Proof.* The proof is based on the local coordinate description of vector fields in projective space obtained by projection of linear vector fields; cf. Bacciotti and Vivalda [3, section 4]. We omit the details.  $\square$

**6. Control sets for nonhyperbolic systems.** This section shows that all control sets with nonvoid interior are unbounded if the hyperbolicity condition specified in Definition 4.1 is violated. Using the compactification of the state space constructed in the previous section, we show that there is a single chain control set in  $\mathbb{P}^n$  containing the images of all control sets  $D$  with nonvoid interior in  $\mathbb{R}^n$ , and the boundary at infinity of this chain control set contains all chain control sets of the homogeneous part having nonvoid intersection with the boundary at infinity of one of the control sets  $D$ .

We begin with the following motivation. Consider a linear control system

$$(6.1) \quad \dot{x}(t) = Ax(t) + Bu(t), \quad u(t) \in \Omega,$$

where the control range  $\Omega \subset \mathbb{R}^m$  is a compact convex neighborhood of the origin. This is a special case of system (1.1) for  $B_1 = \cdots = B_m = 0$  and  $d = 0$ . We assume that the system without control restriction is controllable. By Colonius and Kliemann

[7, Example 3.2.16] there is a unique control set  $D$  with nonvoid interior, and  $0 \in \text{int}(D)$ . Let  $\mathbf{GE}(A; \mu)$  denote the real generalized eigenspace for an eigenvalue  $\mu$  of  $A$ . Then

$$(6.2) \quad \mathbf{E}^0 \subset D \subset \overline{K} + \mathbf{E}^0 + F,$$

where  $\mathbf{E}^0 := \bigoplus_{\text{Re } \mu = 0} \mathbf{GE}(A; \mu)$  is the central spectral subspace, and the sets  $K \subset \mathbf{E}^+ := \bigoplus_{\text{Re } \mu < 0} \mathbf{GE}(A; \mu)$  and  $F \subset \mathbf{E}^- := \bigoplus_{\text{Re } \mu > 0} \mathbf{GE}(A; \mu)$  are bounded. This follows from Sontag [23, Corollary 3.6.7], showing that  $\mathcal{O}^+(0) = K + \mathbf{E}_0$ . Then time reversal yields  $\mathcal{O}^-(0) = \mathbf{E}_0 + F$ , and hence by Remark 2.2,

$$D = \overline{\mathcal{O}^+(0)} \cap \mathcal{O}^-(0) = (\overline{K} + \mathbf{E}^0) \cap (\mathbf{E}^0 + F).$$

Due to the decomposition  $\mathbb{R}^n = \mathbf{E}^+ \oplus \mathbf{E}^0 \oplus \mathbf{E}^-$ , this implies (6.2). In particular,  $D$  is bounded if and only if  $\mathbf{E}^0 = \{0\}$ , i.e., if  $A$  is a hyperbolic matrix. If  $A$  is nonhyperbolic, we embed system (6.1) into a homogeneous bilinear control system in  $\mathbb{R}^{n+1}$  as explained in section 5 and find that the boundary at infinity satisfies

$$(6.3) \quad \partial_\infty(D) = \partial(\pi_{\mathbb{P}} D^1) \cap \mathbb{P}^{n,0} = \{[x_1 : \cdots : x_n : 0] \mid [x_1 : \cdots : x_n] \in \pi_{\mathbb{P}} \mathbf{E}^0\}.$$

This follows from (6.2) noting that for  $0 = x \in \mathbf{E}^0$  and every  $j \in \mathbb{N}$ , one obtains an element of  $D$  given by  $k_j + jx + f_j$  with  $k_j \in \overline{K}$  and  $f_j \in F$ . Considering the homogeneous coordinates and dividing by  $j$ , one finds for  $j \rightarrow \infty$  that (6.3) holds. The set  $\pi_{\mathbb{P}} \mathbf{E}^0$  is a maximal invariant chain transitive set for the flow induced by the homogeneous part  $\dot{x} = Ax$  on  $\mathbb{P}^{n-1}$  (cf. Colonius and Kliemann [8, Theorem 4.1.3]). Thus the boundary at infinity  $\partial_\infty(D)$  is a maximal invariant chain transitive set for the induced flow on  $\mathbb{P}^{n,0}$ .

For general affine control systems of the form (1.1) it stands to reason to replace the maximal chain transitive set  $\pi_{\mathbb{P}} \mathbf{E}^0$  by maximal chain transitive sets of the control flow in  $\mathcal{U} \times \mathbb{P}^{n-1}$  associated with the homogeneous part or, equivalently, by chain control sets in  $\mathbb{P}^{n-1}$  (cf. (2.5)) and to replace the spectral property of  $\mathbf{E}^0$  by appropriate generalized spectral properties. However, the situation for general affine control systems will turn out to be more intricate than for linear control systems.

Now we start our discussion of the nonhyperbolic case for (1.1). Here several control sets with nonvoid interior may coexist as illustrated by Example 6.10 and [9, Examples 5.16 and 5.17]. The following theorem shows that in the nonhyperbolic case all control sets with nonvoid interior are unbounded.

**THEOREM 6.1.** *Assume that the affine control system (1.1) on  $\mathbb{R}^n$  is nonhyperbolic.*

(i) *If there is a  $\sigma$ -periodic control  $u \in \mathcal{U}_{pc}$  with  $g(u) \in \mathcal{S}_\sigma \cap \text{int}(\mathcal{S})$  and  $1 \in \text{spec}(\Phi_u(\sigma, 0))$ , then there exists a control set  $D$  with nonvoid interior.*

(ii) *Every control set  $D$  with nonvoid interior is unbounded. More precisely, there are  $x^k \in \text{int}(D)$ ,  $k \in \mathbb{N}$ , and  $g(v) \in \mathcal{S}_\tau \cap \text{int}(\mathcal{S})$ ,  $\tau > 0$ , with  $1 \in \text{spec}(\Phi_v(\tau, 0))$  and*

$$(6.4) \quad \|x^k\| \rightarrow \infty \text{ and } d \left( \frac{x^k}{\|x^k\|}, \mathbf{E}(\Phi_v(\tau, 0); 1) \right) \rightarrow 0 \text{ for } k \rightarrow \infty.$$

*Proof.* (i) By Lemma 3.4 there is a unique  $\sigma$ -periodic trajectory of (1.1) for  $u$ . Proposition 3.3(ii) implies that it is contained in the interior of a control set  $D$ .

(ii) Let  $x$  be in the interior of a control set  $D$ . By Proposition 3.3(i) there are  $\sigma > 0$  and  $g(u) \in \mathcal{S}_\sigma \cap \text{int}(\mathcal{S})$  such that  $g(u)x = x$ . Then the  $\sigma$ -periodic control  $u$  yields the  $\sigma$ -periodic trajectory  $\varphi(\cdot, x, u) \subset \text{int}(D)$  and hence,

$$\int_0^\sigma \Phi_u(\sigma, s) (Cu(s) + d) ds = (I - \Phi_u(\sigma, 0))x.$$

*Case 1:* If  $1 \in \text{spec}(\Phi_u(\sigma, 0))$ , the affine subspace  $Y = x + \mathbf{E}(\Phi_u(\sigma, 0); 1)$  is contained in  $\text{int}(D)$ . For the proof, an application of Lemma 3.9 shows that there is a  $\sigma$ -periodic solution of (3.6) starting in  $y$  if and only if  $y \in Y = x + \mathbf{E}(\Phi_u(\sigma, 0); 1)$ . Thus  $g(u)y = y$  for all  $y \in Y$ . Proposition 3.3(ii) implies that every  $y$  is in the interior of some control set, and hence  $Y \subset \text{int}(D)$ . Furthermore, Lemma 3.9 also yields points  $x^k \in Y$  such that assertion (6.4) holds with  $v := u$  and  $\tau := \sigma$ .

*Case 2:* Suppose that  $1 \in \text{spec}(\Phi_u(\sigma, 0))$ . Since the system is nonhyperbolic, there is a  $\tau^*$ -periodic control  $v^*$  with  $1 \in \text{spec}(\Phi_{v^*}(\tau^*, 0))$  and  $g(v^*) \in \mathcal{S}_{\tau^*} \cap \text{int}(\mathcal{S})$ .

Consider the continuous paths  $p$  and  $p^{\text{hom}}$  from  $g(u)$  to  $g(v^*)$  and  $\Phi_u(\sigma, 0)$  to  $\Phi_{v^*}(\tau^*, 0)$ , respectively, given by Lemma 3.6. Let

$$\alpha_0 := \sup\{\alpha \in [0, 1] \mid \text{for all } \alpha' \in [0, \alpha] : 1 \in \text{spec}(\Phi_{u^{\alpha'}}(\tau_{\alpha'}, 0))\}.$$

Hence, for  $\alpha \in [0, \alpha_0)$ , Proposition 2.9(i) shows that there are unique  $\tau_\alpha$ -periodic trajectories for  $u^\alpha$  which by Proposition 3.3(ii) are in the interior of a control set. By Proposition 3.7(ii)  $x^\alpha$  depends continuously on  $\alpha \in [0, \alpha_0)$ , and then the arguments in the proof of Theorem 4.4 show that the initial values satisfy  $x^\alpha \in \text{int}(D)$  for all  $\alpha \in [0, \alpha_0)$ .

Now consider a sequence  $\alpha_k \rightarrow \alpha_0$  with  $\alpha_k < \alpha_0$ . Suppose first that

$$(6.5) \quad \int_0^{\tau_{\alpha_0}} \Phi_{u^{\alpha_0}}(\tau_{\alpha_0}, s) (Cu^{\alpha_0}(s) + d) ds \in \text{Im}(I - \Phi_{u^{\alpha_0}}(\tau_{\alpha_0}, 0)).$$

For  $k = 0, 1, 2, \dots$  let

$$b_k := \int_0^{\tau_{\alpha_k}} \Phi_{u^{\alpha_k}}(\tau_{\alpha_k}, s) (Cu^{\alpha_k}(s) + d) ds, \quad A_k := I - \Phi_{u^{\alpha_k}}(\tau_{\alpha_k}, 0).$$

Then  $A_k x^{\alpha_k} = b_k$  and  $A_k \rightarrow A_0, b_k \rightarrow b_0$  for  $k \rightarrow \infty$ , and  $\ker A_0 = \mathbf{E}(\Phi_{u^{\alpha_0}}(\tau_{\alpha_0}, 0); 1)$ . If  $x^{\alpha_k}$  remains bounded, we may assume that  $x^{\alpha_k} \rightarrow x^0$  for some  $x^0 \in \mathbb{R}^n$  and hence it follows that  $A_0 x^0 = b_0$ . Since  $1 \in \text{spec}(\Phi_{u^{\alpha_0}}(0))$ , Lemma 3.9 implies assertion (6.4) for  $v := u^{\alpha_0}$ , as in Case 1. If  $x^{\alpha_k}$  becomes unbounded then

$$A_0 \frac{x^{\alpha_k}}{\|x^{\alpha_k}\|} \leq \|A_0 - A_k\| + A_k \frac{x^{\alpha_k}}{\|x^{\alpha_k}\|} \leq \|A_0 - A_k\| + \frac{b_k}{\|x^{\alpha_k}\|} \rightarrow 0 \text{ for } k \rightarrow \infty,$$

and again (6.4) follows.

If (6.5) does not hold, Lemma 3.8 implies that, for  $k = 1, 2, \dots$ , the initial values  $x^k$  of the  $\tau_{\alpha_k}$ -periodic solutions satisfy (6.4) with  $v := u^{\alpha_0}, \tau = \tau_{\alpha_0}$ .  $\square$

Next we discuss the relation of the boundary at infinity to control sets of the homogeneous part of the affine control system, motivated by the case of linear control systems introduced in the beginning of this section. First, we obtain the following result for invariant control sets.

**THEOREM 6.2.** *Assume that the affine system (1.1) is nonhyperbolic, and suppose that  $D$  is an invariant control set.*

(i) *Then the interior of  $D$  is nonvoid, the set  $D$  is unbounded in  $\mathbb{R}^n$ , and the boundary at infinity  $\partial_\infty(D)$  contains an invariant control set  $e(\mathbb{P}D^{\text{hom}})$  of the system restricted to  $\mathbb{P}^{n,0}$ .*

(ii) *If the control range  $\Omega$  is a compact convex neighborhood of the origin and the system on  $\mathbb{P}^{n-1}$  satisfies the accessibility rank condition, then the boundary at infinity  $\partial_\infty(D)$  contains the unique invariant control set  $e(\mathbb{P}D^{\text{hom}})$  where  $\mathbb{P}D^{\text{hom}}$  is the unique invariant control set on  $\mathbb{P}^{n-1}$ .*

*Proof.* (i) By local accessibility, the interior of the invariant control set  $D$  is nonvoid, and hence Theorem 6.1 shows that  $D$  is unbounded. It follows that there is a point  $\pi_{\mathbb{P}}(x, 0) \in \overline{\pi_{\mathbb{P}}D^1} \cap \mathbb{P}^{n,0}$ , and by Proposition 5.2(i)  $\pi_{\mathbb{P}}D^1$  is an invariant control set contained in  $\mathbb{P}^{n,1}$ . Since  $\mathbb{P}^{n,0}$  is invariant and closed in  $\mathbb{P}^n$ , it follows that  $\overline{\mathcal{O}^+(\pi_{\mathbb{P}}(x, 0))} \subset \mathbb{P}^{n,0}$ , and hence every point in this set has the form  $\pi_{\mathbb{P}}(y, 0)$ . For fixed  $\varepsilon > 0$  there are  $T > 0$  and  $u \in \mathcal{U}$  with  $d(\pi_{\mathbb{P}}\psi(T, (x, 0), u), \pi_{\mathbb{P}}(y, 0)) < \varepsilon$  (recall (5.3) and (5.2)). Furthermore, there are  $\pi_{\mathbb{P}}(x^k, 1) \in \pi_{\mathbb{P}}D^1$  with  $\pi_{\mathbb{P}}(x^k, 1) \rightarrow \pi_{\mathbb{P}}(x, 0)$ . Since  $\pi_{\mathbb{P}}D^1$  is an invariant control set, it follows that  $\pi_{\mathbb{P}}\psi(T, (x^k, 1), u) \in \overline{\pi_{\mathbb{P}}D^1}$ , and continuous dependence on the initial values implies that  $d(\pi_{\mathbb{P}}\psi(T, (x^k, 1), u), \pi_{\mathbb{P}}(y, 0)) < \varepsilon$  for  $k$  large enough. Since  $\varepsilon > 0$  is arbitrary, we have shown that  $\pi_{\mathbb{P}}(y, 0) \in \overline{\pi_{\mathbb{P}}D^1}$  and hence that  $\overline{\mathcal{O}^+(\pi_{\mathbb{P}}(x, 0))} \subset \partial(\pi_{\mathbb{P}}D^1) \cap \mathbb{P}^{n,0} = \partial_{\infty}(D)$ . By Colonius and Kliemann [7, Theorem 3.2.8], for every point  $\pi_{\mathbb{P}}(x, 0)$  in the compact space  $\mathbb{P}^{n,0}$  there is an invariant control set contained in the closure of the reachable set  $\overline{\mathcal{O}^+(\pi_{\mathbb{P}}(x, 0))}$ . By Proposition 5.2(ii) this invariant control set has the form  $e(\pi_{\mathbb{P}}D^{\text{hom}})$ , implying (i).

(ii) This follows from Proposition 5.2(ii) since the accessibility rank condition implies, by Theorem 2.8(i), that the invariant control set of the system on  $\mathbb{P}^{n-1}$  is unique.  $\square$

We proceed to prove the following result on the relation between the boundary at infinity of a control set in  $\mathbb{R}^n$  and the chain control sets of the homogeneous part in  $\mathbb{P}^{n-1}$ .

**PROPOSITION 6.3.** *Assume that the affine control system (1.1) on  $\mathbb{R}^n$  is nonhyperbolic. Then for every control set  $D \subset \mathbb{R}^n$  with nonvoid interior of (1.1) there is a chain control set  ${}_{\mathbb{P}}E^{\text{hom}} \subset \mathbb{P}^{n-1}$  such that  $\partial_{\infty}(D) \cap e({}_{\mathbb{P}}E^{\text{hom}}) = \emptyset$ .*

*Proof.* Theorem 6.1 shows that  $D$  is unbounded and that there are  $g(v) \in \mathcal{S}_{\tau} \cap \text{int}(\mathcal{S})$  with  $1 \in \text{spec}(\Phi_v(\tau, 0))$  and  $x^k \in \text{int}(D)$  satisfying  $\|x^k\| \rightarrow \infty$  and  $d(\frac{x^k}{\|x^k\|}, \mathbf{E}(\Phi_v(\tau, 0); 1)) \rightarrow 0$  for  $k \rightarrow \infty$ . Since  $x = \Phi_v(\tau, 0)x$  for all  $x \in \mathbf{E}(\Phi_v(\tau, 0); 1)$ , the path connected set  $\pi_{\mathbb{P}}\mathbf{E}(\Phi_v(\tau, 0); 1)$  consists of points on  $\tau$ -periodic solutions for the  $\tau$ -periodic control  $v$  and hence is contained in a chain control set  ${}_{\mathbb{P}}E^{\text{hom}}$ . Hence one obtains  $\partial_{\infty}(D) \cap e({}_{\mathbb{P}}E^{\text{hom}}) = \emptyset$  in  $\mathbb{P}^{n,0}$ .  $\square$

The following theorem presents a partial converse of Theorem 6.2(i).

**THEOREM 6.4.** *Assume that the homogeneous part (1.3) of the affine system (1.1) satisfies the accessibility rank condition on  $\mathbb{R}^n \setminus \{0\}$ , that there is  $g(u) \in \mathcal{S}_{\sigma} \cap \text{int}(\mathcal{S})$  with  $1 \in \text{spec}(\Phi_u(\sigma, 0))$  for some  $\sigma > 0$ , and that there are at most finitely many control sets with nonvoid interior of system (1.1). Then for every control set  ${}_{\mathbb{R}}D_i^{\text{hom}}$  with nonvoid interior there exists such a control set  $D$  of (1.1) with boundary at infinity satisfying*

$$(6.6) \quad \partial_{\infty}(D) \cap e({}_{\mathbb{P}}D_i^{\text{hom}}) = \emptyset \text{ for } {}_{\mathbb{P}}D_i^{\text{hom}} \supset \pi_{\mathbb{P}}({}_{\mathbb{R}}D_i^{\text{hom}}).$$

*Proof.* Fix a point  $x \in \text{int}({}_{\mathbb{R}}D_i^{\text{hom}})$ . Since by Theorem 3.2,  $\text{int}(\mathcal{S}_{\leq \tau}) = \emptyset$  for all  $\tau > 0$  there are  $\tau_0 > 0$  small enough and  $u^0 \in \mathcal{U}_{pc}$  with  $g(u^0) \in \mathcal{S}_{\tau_0} \cap \text{int}(\mathcal{S})$  and  $x^0 := \Phi_{u^0}(\tau_0, 0)x \in \text{int}({}_{\mathbb{R}}D_i^{\text{hom}})$ . Since also  $\text{int}({}_{\mathbb{R}}\mathcal{S}_{\leq \tau}^{\text{hom}}) = \emptyset$  for all  $\tau > 0$ , there are  $\tau_1 > 0$  small enough and  $u^1 \in \mathcal{U}_{pc}$  such that the corresponding element  $\Phi_{u^1}(\tau_1, 0) \in {}_{\mathbb{R}}\mathcal{S}_{\tau_1}^{\text{hom}} \cap \text{int}({}_{\mathbb{R}}\mathcal{S}^{\text{hom}})$  satisfies

$$x^1 := \Phi_{u^1}(\tau_1, 0)x^0 = \Phi_{u^1}(\tau_1, 0)\Phi_{u^0}(\tau_0, 0)x \in \text{int}({}_{\mathbb{R}}D_i^{\text{hom}}).$$

By Remark 2.2 controllability in the interior of  ${}_{\mathbb{R}}D_i^{\text{hom}}$  holds, and hence there are  $\tau_2 > 0$  and  $u^2 \in \mathcal{U}_{pc}$  satisfying  $\Phi_{u^2}(\tau_2, 0)x^1 = x$ . Define  $\tau := \tau_0 + \tau_1 + \tau_2$  and a control  $u \in \mathcal{U}_{pc}$  by  $\tau$ -periodic extension of

$$u(t) := \begin{cases} u^0(t) & \text{for } t \in [0, \tau_0), \\ u^1(t - \tau_0) & \text{for } t \in [\tau_0, \tau_0 + \tau_1), \\ u^2(t - \tau_0 - \tau_1) & \text{for } t \in [\tau_0 + \tau_1, \tau_0 + \tau_1 + \tau_2). \end{cases}$$

Then  $\Phi_u(\tau, 0)x = x$ , hence  $1 \in \text{spec}(\Phi_u(\tau, 0))$ , and

$$g(u) = g(u^2)g(u^1)g(u^0) \in \mathcal{S}_\tau \cap \text{int}(\mathcal{S}),$$

$$\Phi_u(\tau, 0) = \Phi_{u^2}(\tau_2, 0)\Phi_{u^1}(\tau_1, 0)\Phi_{u^0}(\tau_0, 0) \in \text{int}({}_{\mathbb{R}}\mathcal{S}^{\text{hom}}).$$

By Proposition 3.3(ii) (for  ${}_{\mathbb{R}}\mathcal{S}^{\text{hom}}$ ) this implies that the eigenspace  $\mathbf{E}(\Phi_u(\tau, 0); 1)$  of  $\Phi_u(\tau, 0)$  for the eigenvalue 1 is contained in the interior of some control set in  $\mathbb{R}^n \setminus \{0\}$ . Since  $x \in \mathbf{E}(\Phi_u(\tau, 0); 1) \cap {}_{\mathbb{R}}D_i^{\text{hom}}$ , it follows that

$$(6.7) \quad \mathbf{E}(\Phi_u(\tau, 0); 1) \subset \text{int}({}_{\mathbb{R}}D_i^{\text{hom}}) \text{ and hence } \pi_{\mathbb{P}}\mathbf{E}(\Phi_u(\tau, 0); 1) \subset \text{int}({}_{\mathbb{P}}D_i^{\text{hom}}).$$

By Proposition 3.7(i) there are  $g(u^k) \in \mathcal{S}_{\tau_k} \cap \text{int}(\mathcal{S})$  with  $1 \in \text{spec}(\Phi_{u^k}(\tau_k, 0))$  and  $\Phi_{u^k}(\tau_k, 0) \rightarrow \Phi_u(\tau, 0)$  for  $k \rightarrow \infty$ . Proposition 2.9(i) implies that there are unique  $\tau_k$ -periodic solutions denoted by  $\varphi(\cdot, x^k, u^k)$  of the affine equation for the  $\tau_k$ -periodic extension of  $u^k$ . By Proposition 3.3(ii) each of them is in the interior of a control set for the affine system (1.1). Since, by assumption, there are only finitely many of them, infinitely many  $x^k$  are contained in the interior of a single control set  $D$ . We may assume that all  $x^k$  are in  $\text{int}(D)$ .

Suppose that assumption (ii) in Lemma 3.8 is satisfied. Then it follows that

$$(6.8) \quad \|x^k\| \rightarrow \infty \text{ and } \frac{x^k}{\|x^k\|} \rightarrow \mathbf{E}(\Phi_u(\tau, 0); 1) \text{ for } k \rightarrow \infty.$$

Hence, for  $k \rightarrow \infty$ , the points  $\pi_{\mathbb{P}}(x^k, 1) \in \pi_{\mathbb{P}}D^1$  converge to  $e(\pi_{\mathbb{P}}\mathbf{E}(\Phi_u(\tau, 0); 1))$ , showing that

$$\overline{\pi_{\mathbb{P}}D^1} \cap e(\pi_{\mathbb{P}}\mathbf{E}(\Phi_u(\tau, 0); 1)) = \emptyset.$$

Together with (6.7) this implies that the boundary at infinity of  $D$  satisfies (6.6).

If assumption (ii) in Lemma 3.8 does not hold, then Lemma 3.9 shows that there are  $x^k \in \mathbf{E}(\Phi_u(\tau, 0); 1)$  with (6.8) for  $k \rightarrow \infty$ . Then the assertion follows also in this case.  $\square$

The following examples (cf. Mohler [18, Example 2, p. 32] and Colonius, Santana, and Setti [9, Example 5.16]) show that, in general, the boundary at infinity of a control set  $D$  may intersect more than one control set of the projectivized homogeneous part.

*Example 6.5.* Consider the affine control system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2u & 1 \\ 1 & 2u \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \quad u(t) \in \Omega = [-1, 1].$$

The eigenvalues of  $A(u) = A + uB$  are given by  $\lambda_1(u) = 2u + 1 > \lambda_2(u) = 2u - 1$  and  $\lambda_1(-1/2) = \lambda_2(1/2) = 0$ . For every  $u \in \mathbb{R}$ , the eigenspaces for  $\lambda_1(u)$  and  $\lambda_2(u)$  are  $\mathbf{E}(A + uB; \lambda_1(u)) = \{(z, z)^\top | z \in \mathbb{R}\}$  and  $\mathbf{E}(A + uB; \lambda_2(u)) = \{(z, -z)^\top | z \in \mathbb{R}\}$ , respectively. In the northern part of the unit circle (hence in  $\mathbb{P}^1$ ) this yields the two one-point control sets given by the equilibria for any  $u \in [-1, 1]$ ,

$${}_{\mathbb{P}}D_1^{\text{hom}} = \left\{ \left( 1/\sqrt{2}, 1/\sqrt{2} \right)^\top \right\} \text{ and } {}_{\mathbb{P}}D_2^{\text{hom}} = \left\{ \left( -1/\sqrt{2}, 1/\sqrt{2} \right)^\top \right\}.$$

Here the trajectories not starting in one of these equilibria approach  $\mathbb{P}D_1^{\text{hom}}$  and  $\mathbb{P}D_2^{\text{hom}}$  for  $t \rightarrow +\infty$  and  $t \rightarrow -\infty$ , respectively. As shown in [9, Examples 4.4 and 5.16] there is a connected branch of equilibria of the affine system

$$\mathcal{B}_1 = \{(x_u, y_u)^\top \mid u \in (-1/2, 1/2)\} \text{ with } (x_0, y_0)^\top = (0, 0)^\top \in \mathcal{B}_1.$$

They become unbounded for  $|u| \rightarrow \frac{1}{2}$ , and there is a single control set  $D$  containing the equilibria in  $\mathcal{B}_1$  in the interior. The equilibria in  $\mathcal{B}_1$  satisfy  $\frac{(x_u, y_u)}{\|(x_u, y_u)\|} \rightarrow \left(\mp \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$  for  $u \rightarrow \pm \frac{1}{2}$ . Consequently, one obtains, for the control sets of the homogeneous part,

$$e(\mathbb{P}D_1^{\text{hom}}) \cup e(\mathbb{P}D_2^{\text{hom}}) \subset \partial(\pi_{\mathbb{P}}D^1) \cap \mathbb{P}^{2,0} = \partial_\infty(D).$$

The homogeneous part of Example 6.5 violates the accessibility rank condition in  $\mathbb{P}^1$ , and the control sets  $\mathbb{P}D_1^{\text{hom}}$  and  $\mathbb{P}D_2^{\text{hom}}$  in  $\mathbb{P}^1$  have void interiors. We modify this example in order to get control sets in  $\mathbb{P}^1$  with nonvoid interiors. Note that here an arbitrarily small perturbation suffices to change the system behavior drastically.

*Example 6.6.* Consider, for small  $\varepsilon > 0$  and  $\Omega = [-1, 1]$ ,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2u & 1 \\ 1 & (2+\varepsilon)u \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u = (A + uB(\varepsilon)) \begin{pmatrix} x \\ y \end{pmatrix} + Cu.$$

We will show that there is a control set  $D$  in  $\mathbb{R}^2$  such that the boundary at infinity  $\partial_\infty(D)$  intersects two control sets with nonvoid interiors for the homogeneous part.

**Step 1:** The eigenvalues of  $A + uB(\varepsilon)$  are given by

$$\lambda_{1,2}(u, \varepsilon) = \frac{u}{2}(4 + \varepsilon) \pm \frac{1}{2}\sqrt{4 + u^2[(4 + \varepsilon)^2 - 4(4 + 2\varepsilon)]}.$$

Note that  $\lambda_1(u, \varepsilon) > \lambda_2(u, \varepsilon)$  for all  $u \in [-1, 1]$ . For  $\varepsilon = 0$ , it is clear that the functions  $u \rightarrow \lambda_{1,2}(u, 0) = 2u \pm 1$  are strictly increasing, and hence this also holds for small  $\varepsilon > 0$ . Thus there are unique values  $u^1(\varepsilon), u^2(\varepsilon) \in (-1, 1)$  with  $\lambda_1(u^1(\varepsilon), \varepsilon) = 0$  and  $\lambda_2(u^2(\varepsilon), \varepsilon) = 0$ , and  $u^1(\varepsilon) \rightarrow -\frac{1}{2}$  and  $u^2(\varepsilon) \rightarrow \frac{1}{2}$  for  $\varepsilon \rightarrow 0$ . The eigenvectors  $(x, y)^\top$  satisfy  $y = (\lambda_{1,2}(u, \varepsilon) - 2u)x$ . For  $\varepsilon \rightarrow 0$  and all  $u \in [-1, 1]$  the eigenspace  $\mathbf{E}(A + uB(\varepsilon); \lambda_i(u, \varepsilon))$  converges to the eigenspace  $\mathbf{E}(A + uB(0); \lambda_i(u, 0))$ . In the northern part of the unit circle (hence in  $\mathbb{P}^1$ ) this yields two equilibria  $e_1(u, \varepsilon)$  and  $e_2(u, \varepsilon)$ , and the other trajectories in  $\mathbb{P}^1$  converge for  $t \rightarrow \infty$  to  $e_1(u, \varepsilon)$  and for  $t \rightarrow -\infty$  to  $e_2(u, \varepsilon)$ . Hence there are control sets  $\mathbb{P}D_1^{\text{hom}}$  and  $\mathbb{P}D_2^{\text{hom}}$  (depending on  $\varepsilon$ ) with nonvoid interiors consisting of the equilibria  $e_1(u, \varepsilon)$  and  $e_2(u, \varepsilon), u \in [-1, 1]$ , respectively. The control set  $\mathbb{P}D_1^{\text{hom}}$  is invariant. One easily verifies the accessibility rank condition in  $\mathbb{R}^2 \setminus \{0\}$ . Since  $0 \in \text{int}(\Sigma_{Fl}(\mathbb{P}D_i^{\text{hom}}))$ , it follows that  $\mathbb{P}D_i^{\text{hom}}$  is the projection to  $\mathbb{P}^1$  of a control set  $\mathbb{R}D_i^{\text{hom}}$  in  $\mathbb{R}^2 \setminus \{0\}$ ,  $i = 1, 2$ .

**Step 2:** The equilibria  $(x_u(\varepsilon), y_u(\varepsilon))^\top$  approach  $\mathbf{E}(A + u^i(\varepsilon)B(\varepsilon); 0)$  for  $u \rightarrow u^i(\varepsilon), i = 1, 2$ . In both cases, the equilibria become unbounded. In particular, there is a connected unbounded branch of equilibria

$$\mathcal{B}_1(\varepsilon) = \{(x_u(\varepsilon), y_u(\varepsilon))^\top \mid u \in (u^1(\varepsilon), u^2(\varepsilon))\}$$

and a single control set  $D$  (again depending on  $\varepsilon$ ) containing the equilibria in  $\mathcal{B}_1(\varepsilon)$ .

**Step 3:** Embedding the control system into a homogeneous bilinear system in  $\mathbb{R}^3$  and projecting it to  $\mathbb{P}^2$ , one obtains from the control set  $D$  a control set in  $\mathbb{P}^{2,1}$  given by  $\pi_{\mathbb{P}}D^1 = \{[x : y : 1] \mid (x, y)^\top \in D\}$ . As the equilibria  $(x_u(\varepsilon), y_u(\varepsilon))^\top \in \mathcal{B}_1$

become unbounded for  $u \rightarrow u^i(\varepsilon)$ , they approach the eigenspace  $\mathbf{E}(A + u^i(\varepsilon)B(\varepsilon); 0)$ , and hence

$$e(\mathbb{P}D_1^{\text{hom}}) \cap \partial_\infty(D) = \emptyset \text{ and } e(\mathbb{P}D_2^{\text{hom}}) \cap \partial_\infty(D) = \emptyset.$$

In the following we require that  $\Omega$  be a convex and compact neighborhood of  $0 \in \mathbb{R}^m$ , and we consider chain control sets of the affine system in  $\mathbb{P}^n$ .

**DEFINITION 6.7.** *The boundary at infinity of a chain control set  $\mathbb{P}E$  for the affine system (5.5) in  $\mathbb{P}^n$  is  $\partial_\infty(\mathbb{P}E) := \partial(\mathbb{P}E) \cap \mathbb{P}^{n,0}$ .*

This definition is similar to the boundary at infinity for control sets, but it refers to chain control sets in  $\mathbb{P}^n$  not requiring that they are obtained from chain control sets in  $\mathbb{R}^n$ .

**LEMMA 6.8.** *Let  $\mathbb{P}E$  be a chain control set in  $\mathbb{P}^n$ .*

(i) *If  $\partial_\infty(\mathbb{P}E) \cap e(\mathbb{P}E_j^{\text{hom}}) = \emptyset$  for a chain control set  $\mathbb{P}E_j^{\text{hom}}$  in  $\mathbb{P}^{n-1}$  of the homogeneous part, then  $e(\mathbb{P}E_j^{\text{hom}}) \subset \partial_\infty(\mathbb{P}E)$ .*

(ii) *If  $\partial_\infty(\mathbb{P}E)$  is nonvoid, it contains a chain control set  $e(\mathbb{P}E_j^{\text{hom}})$  for a chain control set  $\mathbb{P}E_j^{\text{hom}}$  of the homogeneous part.*

*Proof.* (i) Recall from Proposition 5.2(ii) that  $e(\mathbb{P}E_j^{\text{hom}})$  is a chain control set of the system restricted to  $\mathbb{P}^{n,0}$ . We will show that the set  $\mathbb{P}E' := \mathbb{P}E \cup e(\mathbb{P}E_j^{\text{hom}})$  satisfies the properties (i) and (ii) of a chain control set; cf. Definition 2.5. Then the maximality property (iii) of the chain control set  $\mathbb{P}E$  implies that  $\mathbb{P}E' = \mathbb{P}E$ , showing that  $e(\mathbb{P}E_j^{\text{hom}}) \subset \partial_\infty(\mathbb{P}E)$ .

It is clear that  $\mathbb{P}E'$  satisfies (i), since this holds for  $\mathbb{P}E$  and  $e(\mathbb{P}E_j^{\text{hom}})$ . For property (ii), consider  $x \in \mathbb{P}E$  and  $y \in e(\mathbb{P}E_j^{\text{hom}})$  and  $\varepsilon, T > 0$ . Fix  $z \in \partial_\infty(\mathbb{P}E) \cap e(\mathbb{P}E_j^{\text{hom}}) = \mathbb{P}E \cap e(\mathbb{P}E_j^{\text{hom}})$ . There are controlled  $(\varepsilon, T)$ -chains  $\zeta_1$  and  $\zeta_2$  from  $x$  to  $z$  and from  $z$  to  $y$ , respectively. For the system restricted to  $\mathbb{P}^{n,0}$ , there exist controlled  $(\varepsilon, T)$ -chains  $\zeta_3$  and  $\zeta_4$  from  $z$  to  $y$  and from  $y$  to  $z$ , respectively. Then the concatenations  $\zeta_3 \circ \zeta_1$  and  $\zeta_3 \circ \zeta_4$  are controlled  $(\varepsilon, T)$ -chains from  $x$  to  $y$  and from  $y$  to  $x$ , respectively. This concludes the proof of assertion (i) since  $\varepsilon, T > 0$  are arbitrary.

(ii) Let  $x \in \partial_\infty(\mathbb{P}E)$ . Then there exists a control  $u \in \mathcal{U}$  with  $\pi_{\mathbb{P}}\varphi(t, x, u) \in \partial_\infty(\mathbb{P}E) = \mathbb{P}E \cap \mathbb{P}^{n,0}$  for all  $t \geq 0$  by property (i) of chain control sets and invariance of  $\mathbb{P}^{n,0}$ . Since  $\mathbb{P}E \cap \mathbb{P}^{n,0} = \partial(\mathbb{P}E) \cap \mathbb{P}^{n,0}$  is compact, it follows that the  $\omega$ -limit set

$$\omega_{\mathbb{P}}(u, x) := \{y = \lim_{k \rightarrow \infty} \pi_{\mathbb{P}}\varphi(t_k, x, u) \mid t_k \rightarrow \infty\} \subset \mathbb{P}E \cap \mathbb{P}^{n,0}.$$

is nonvoid. Hence, Colonius and Kliemann [7, Corollary 4.3.12] imply that there exists a chain control set of the system restricted to  $\mathbb{P}^{n,0}$  containing  $\omega_{\mathbb{P}}(u, x)$ . Thus there is a chain control set  $\mathbb{P}E_j^{\text{hom}}$  in  $\mathbb{P}^{n-1}$  of the homogeneous part with  $\partial_\infty(\mathbb{P}E) \cap e(\mathbb{P}E_j^{\text{hom}}) = \emptyset$ . Now the assertion follows from (i).  $\square$

The next theorem is a main result on the control sets  $D$  with nonvoid interiors in  $\mathbb{R}^n$  in the nonhyperbolic case.

**THEOREM 6.9.** *Assume that the affine control system (1.1) is nonhyperbolic. Furthermore, let the control range  $\Omega$  be a compact convex neighborhood of the origin, and assume that there is  $g(u) \in S_\sigma \cap \text{int}(\mathcal{S})$  with  $1 \in \text{spec}(\Phi_u(\sigma, 0))$  for some  $\sigma > 0$ .*

*Then there exists a single chain control set  $\mathbb{P}E$  in  $\mathbb{P}^n$  containing the control sets  $\pi_{\mathbb{P}}D^1$  for all control sets  $D$  with nonvoid interior in  $\mathbb{R}^n$ . Furthermore, the boundary at infinity  $\partial_\infty(\mathbb{P}E)$  contains all  $\partial_\infty(D)$  and the chain control sets  $e(\mathbb{P}E_j^{\text{hom}})$ , where*



$\mathbb{P}E_j^{\text{hom}}$  are the chain control sets in  $\mathbb{P}^{n-1}$  for the homogeneous part (1.3) with  $\partial_\infty(D) \cap e(\mathbb{P}E_j^{\text{hom}}) = \emptyset$  for some  $D$ .

*Proof.* By Proposition 2.9(i) there is a unique  $\sigma$ -periodic solution of the affine system with  $g(u)x^0 = x^0$ . Proposition 3.3 implies that  $x^0$  is in the interior of a control set  $D_0$ . Now let  $D_1$  be any control set in  $\mathbb{R}^n$  with nonvoid interior. It suffices to show that there is a chain control set  $\mathbb{P}E$  in  $\mathbb{P}^n$  containing  $\pi_{\mathbb{P}}D_0^1$  and  $\pi_{\mathbb{P}}D_1^1$  and that its boundary at infinity  $\partial_\infty(\mathbb{P}E)$  contains all chain control sets  $e(\mathbb{P}E_j^{\text{hom}})$  with  $\partial_\infty(D_1) \cap e(\mathbb{P}E_j^{\text{hom}}) = \emptyset$ .

Pick  $x^1 \in \text{int}(D_1)$ . Then Proposition 3.3(i) implies that there are  $\tau_1 > 0$  and  $g_1 = g(u^1) \in \mathcal{S}_{\tau_1} \cap \text{int}(\mathcal{S})$  with  $x^1 = g(u^1)x^1$ . Proposition 3.7(i) yields  $\tau_\alpha$ -periodic controls  $w^\alpha \in \mathcal{U}_{pc}$  and continuous paths  $p_1 : [0, 1] \rightarrow \text{int}(\mathcal{S})$  with  $p_1(0) = g(u)$ ,  $p_1(\alpha) = g(w^\alpha)$  for  $\alpha \in [0, 1]$ , and  $p_1^{\text{hom}} : [0, 1] \rightarrow \mathbb{R}\mathcal{S}^{\text{hom}}$  with  $p_1^{\text{hom}}(0) = \Phi_u(\sigma, 0)$ ,  $p_1^{\text{hom}}(\alpha) = \Phi_{w^\alpha}(\tau_\alpha, 0)$  such that

$$p_1^{\text{hom}}(1) - \Phi_{u^1}(\tau_1, 0) \parallel < \varepsilon, \parallel p_1(1) - g(u^1) \parallel < \varepsilon,$$

and  $1 \in \text{spec}(\Phi_{w^\alpha}(\tau_\alpha, 0))$  for all but at most finitely many  $\alpha \in [0, 1]$ . Denote the  $\alpha$ -values with  $1 \in \text{spec}(\Phi_{w^\alpha}(\tau_\alpha, 0))$  by  $0 < \alpha_1 < \dots < \alpha_r < 1$ , where the last inequality holds without loss of generality. The continuity and smoothness properties from Proposition 3.7(i) hold for  $w^\alpha, \tau_\alpha, \Phi_{w^\alpha}(\tau_\alpha, 0) \in \mathbb{R}\mathcal{S}^{\text{hom}}$  and  $g(w^\alpha) \in \mathcal{S}$ . For  $\alpha$  with  $1 \in \text{spec}(\Phi_{w^\alpha}(\tau_\alpha, 0))$  Proposition 3.7(ii) shows that there are unique  $\tau_\alpha$ -periodic solutions with initial values  $x^\alpha$  depending continuously on  $\alpha$ . Define

$$A := \{\alpha \in [0, 1] \mid 1 \in \text{spec}(\Phi_{w^\alpha}(\tau_\alpha, 0))\}.$$

The set  $A$  consists of  $r + 1$  intervals. For  $\alpha \in A$ , Proposition 3.3(ii) implies that  $x^\alpha \in \text{int}(D^\alpha)$  for some control set  $D^\alpha$  since  $g(w^\alpha) \in \text{int}(\mathcal{S})$ . For each  $\alpha$  in an interval contained in  $A$ , the  $x^\alpha$  depend continuously on  $\alpha$ , and hence they are contained in the interior of a single control set. By construction,  $x^0 \in \text{int}(D_0)$  and, for  $\varepsilon > 0$  small enough,  $x^1 \in \text{int}(D_1)$ . Denote the other control sets containing the  $x^\alpha$  by  $D_i, i \geq 2$ . By Theorem 6.1 the control sets  $D_i$  are unbounded, and hence their boundaries at infinity  $\partial_\infty(D_i)$  are nonvoid.

The control sets  $\pi_{\mathbb{P}}D_i^1$  in  $\mathbb{P}^{n,1}$  are contained in chain control sets  $\mathbb{P}E_i \subset \mathbb{P}^n$ , and it follows that  $\partial_\infty(\mathbb{P}E_i) \supset \partial_\infty(D_i)$  is nonvoid. Thus Lemma 6.8(ii) implies that  $\partial_\infty(\mathbb{P}E_i)$  contains a chain control set  $e(\mathbb{P}E_j^{\text{hom}})$  where  $\mathbb{P}E_j^{\text{hom}}$  is a chain control set of the homogeneous part in  $\mathbb{P}^{n-1}$ . By Lemma 6.8(i),  $\partial_\infty(\mathbb{P}E_i)$  contains every chain control set of the homogeneous part that it intersects. The theorem follows from the next claim.

**Claim.** All chain control sets  $\mathbb{P}E_i, i \geq 2$ , in  $\mathbb{P}^n$  coincide.

For every point  $\alpha_i \in (0, 1) \setminus A$  there are control sets which we denote by  $D_i$  and  $D_{i+1}$  such that all  $\alpha$  in a neighborhood of  $\alpha_i$  satisfy  $x^\alpha \in \text{int}(D_i)$  for  $\alpha < \alpha_i$  and  $x^\alpha \in \text{int}(D_{i+1})$  for  $\alpha > \alpha_i$ .

We have to show that the chain control sets  $\mathbb{P}E_i$  for  $\alpha < \alpha_i$  and  $\mathbb{P}E_{i+1}$  for  $\alpha > \alpha_i$  coincide. The projected eigenspace  $\pi_{\mathbb{P}}\mathbf{E}(\Phi_{u^{\alpha_i}}(\tau_{\alpha_i}, 0); 1)$  consists of  $\tau_{\alpha_i}$ -periodic solutions for the  $\tau_{\alpha_i}$ -periodic control  $u^{\alpha_i}$  and hence is contained in a chain control set  $\mathbb{P}E_j^{\text{hom}}$  in  $\mathbb{P}^{n-1}$ . We will show that

$$e(\mathbb{P}E_j^{\text{hom}}) \subset \partial_\infty(\mathbb{P}E_i) \cap \partial_\infty(\mathbb{P}E_{i+1}),$$

which implies that  $\mathbb{P}E_i$  and  $\mathbb{P}E_{i+1}$  have a nonvoid intersection and hence coincide.

First, consider parameters  $\beta_k \rightarrow \alpha_i, \beta_k < \alpha_i$ . As stated above, the points  $x^{\beta_k}$  with  $x^{\beta_k} = g(u^{\beta_k})x^{\beta_k}$  satisfy  $x^{\beta_k} \in \text{int}(D_i)$ .

*Case 1:*  $\int_0^{\tau_{\alpha_i}} \Phi_{u^{\alpha_i}}(\tau_{\alpha_i}, s)[Cu^{\alpha_i}(s) + d]ds \in \text{Im}(I - \Phi_{u^{\alpha_i}}(\tau_{\alpha_i}, 0))$ .

Lemma 3.8 implies that the  $x^{\beta_k} \in D_i$  satisfy

$$(6.9) \quad \|x^{\beta_k}\| \rightarrow \infty \text{ and } \frac{x^{\beta_k}}{\|x^{\beta_k}\|} \rightarrow \mathbf{E}(\Phi^{\alpha_i}(\tau_{\alpha_i}, 0); 1),$$

and by Lemma 6.8(i),  $e(\mathbb{P}E_j^{\text{hom}}) \subset \partial_\infty(\mathbb{P}E_i)$  since

$$(6.10) \quad \emptyset = \partial_\infty(D_i) \cap e(\pi_{\mathbb{P}}\mathbf{E}(\Phi^{\alpha_i}(\tau_{\alpha_i}, 0); 1)) \subset \partial_\infty(\mathbb{P}E_i) \cap e(\mathbb{P}E_j^{\text{hom}}).$$

*Case 2:*  $\int_0^{\tau_{\alpha_i}} \Phi_{u^{\alpha_i}}(\tau_{\alpha_i}, s)[Cu^{\alpha_i}(s) + d]ds \in \text{Im}(I - \Phi_{u^{\alpha_i}}(\tau_{\alpha_i}, 0))$ .

For  $k = 1, 2, \dots$ , let

$$A_k := I - \Phi_{u^{\beta_k}}(\tau_{\beta_k}, 0), \quad b_k := \int_0^{\tau_{\beta_k}} \Phi_{u^{\beta_k}}(\tau_{\beta_k}, s)(Cu^{\beta_k}(s) + d)ds.$$

Then  $x^{\beta_k} = g(u^{\beta_k})x^{\beta_k}$  implies  $A_k x^{\beta_k} = b_k$ , and by Proposition 2.9(iii), (iv),

$$A_k \rightarrow A_0 := I - \Phi_{u^{\alpha_i}}(\tau_{\alpha_i}, 0), \quad b_k \rightarrow b_0 := \int_0^{\tau_{\alpha_i}} \Phi_{u^{\alpha_i}}(\tau_{\alpha_i}, s)(Cu^{\alpha_i}(s) + d)ds.$$

If  $x^{\beta_k}$  remains bounded, we may assume that  $x^{\beta_k} \rightarrow y^0$  for some  $y^0 \in \overline{D_i} \subset \mathbb{R}^n$  and hence that  $A_0 y^0 = b_0$ . By Lemma 3.9 there are  $x^k \in y^0 + \mathbf{E}(\Phi_{u^{\alpha_i}}(\tau_{\alpha_i}, 0); 1)$  with  $\|x^k\| \rightarrow \infty$  and  $\frac{x^k}{\|x^k\|} \rightarrow \mathbf{E}(\Phi^{\alpha_i}(\tau_{\alpha_i}, 0); 1)$ , and again (6.9) follows, implying  $e(\mathbb{P}E_j^{\text{hom}}) \subset \partial_\infty(\mathbb{P}E_i)$ .

If  $x^{\beta_k}$  becomes unbounded, we obtain

$$A_0 \frac{x^{\beta_k}}{\|x^{\beta_k}\|} \leq \|A_0 - A_k\| + A_k \frac{x^{\beta_k}}{\|x^{\beta_k}\|} = \|A_0 - A_k\| + \frac{\|b_k\|}{\|x^{\beta_k}\|} \rightarrow 0,$$

and (6.9) follows, implying  $e(\mathbb{P}E_j^{\text{hom}}) \subset \partial_\infty(\mathbb{P}E_i)$ .

We have shown this inclusion using  $\beta_k \rightarrow \alpha_i, \beta_k < \alpha_i$ . The same arguments can be applied to parameters  $\beta_k \rightarrow \alpha_i, \beta_k > \alpha_i$ , showing that also for the chain control set  $\mathbb{P}E_{i+1}$  the boundary at infinity  $\partial_\infty(\mathbb{P}E_{i+1})$  contains the chain control set  $e(\mathbb{P}E_j^{\text{hom}})$ . This proves the claim and concludes the proof of the theorem.  $\square$

The following controlled linear oscillator illustrates Theorem 6.9.

*Example 6.10.* Consider the affine control system (cf. [9, Example 5.17])

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1-u & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + u(t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ d \end{pmatrix}, \quad u(t) \in [-\rho, \rho],$$

where  $\rho \in (1, \frac{5}{4})$  and  $d < 1$ . The equilibria are, for  $u \in [-\rho, -1]$  and  $u \in (-1, \rho]$ ,

$$\mathcal{C}_1 = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} : x \in \left[ \frac{d-\rho}{1-\rho}, \infty \right) \right\}, \quad \mathcal{C}_2 = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} : x \in \left( -\infty, \frac{d+\rho}{1+\rho} \right] \right\},$$

respectively. The equilibria in  $\mathcal{C}_1$  are hyperbolic, since here the eigenvalues of  $A(u)$  are  $\lambda_1(u) < 0 < \lambda_2(u)$ . The equilibria in  $\mathcal{C}_2$  are stable nodes since here  $\lambda_1(u) < \lambda_2(u) < 0$ . For  $u^0 = -1$  the matrix  $A(-1) = \begin{pmatrix} 0 & 1 \\ 0 & -3 \end{pmatrix}$  has the eigenvalue  $\lambda_2(-1) = 0$  with eigenspace  $\mathbb{R} \times \{0\}$ , and hence  $e^{A(-1)t}$  has the eigenvalue 1. There are control sets  $D_1 = D_2$

containing the equilibria in  $\mathcal{C}_1$  and  $\mathcal{C}_2$ , respectively, in the interior. For  $u^k \nearrow u^0 = -1$ , the equilibria in  $D_1$  satisfy  $(x_{u^k}, 0) \rightarrow (\infty, 0)$ , and for  $u^k \searrow u^0 = -1$ , the equilibria in  $D_2$  satisfy  $(x_{u^k}, 0) \rightarrow (-\infty, 0)$  for  $k \rightarrow \infty$ . There is a single chain control set  ${}_{\mathbb{P}}E$  in  $\mathbb{P}^2$  containing the images of  $D_1$  and  $D_2$  since the eigenspace  $\mathbf{E}(e^{A(-1)\tau}; 1) = \mathbb{R} \times \{0\}$  satisfies

$$e(\pi_{\mathbb{P}}\mathbf{E}(e^{A(-1)\tau}; 1)) \subset \partial_{\infty}(D_1) \cap \partial_{\infty}(D_2) \quad \text{for any } \tau > 0.$$

Concerning the homogeneous part in  $\mathbb{P}^1$ , the projectivized eigenspace  $\pi_{\mathbb{P}}\mathbf{E}(e^{A(-1)\tau}; 1)$  is contained in the invariant control set  ${}_{\mathbb{P}}D_2^{\text{hom}} = \pi_{\mathbb{P}}\{(x, \lambda_2(u)x)^{\top} | x = 0, u \in [-\rho, \rho]\}$ , and  ${}_{\mathbb{P}}D_2^{\text{hom}}$  is the projection of a control set  ${}_{\mathbb{R}}D_2^{\text{hom}}$  in  $\mathbb{R}^2$  since  $0 \in \text{int}(\Sigma_{Fl}({}_{\mathbb{P}}D_2^{\text{hom}}))$ .

**Appendix A.** This appendix presents the proof of Proposition 2.9. Assertion

(i) follows from the variation-of-parameters formula, and (ii) is a consequence of (i).

(iii) The principal fundamental solutions  $\Phi^k(t, s)$  satisfy, for  $t, s \in [0, \tau_0 + 1]$ ,

$$\|\Phi^k(t, s)\| \leq 1 + \int_s^t \|P^k(\sigma)\| \|\Phi^k(\sigma, s)\| d\sigma.$$

By the generalized Gronwall inequality (cf. Amann [1, Lemma 6.1]) it follows that  $c_1 = \sup_{k \in \mathbb{N}, t, s \in [0, \tau_0 + 1]} \|\Phi^k(t, s)\| < \infty$ . For all  $t, s \in [0, \tau_0 + 1]$ ,

$$\begin{aligned} \|\Phi^k(t, s) - \Phi^0(t, s)\| &= \left\| \int_s^t [P^k(\sigma)\Phi^k(\sigma, s) - P^0(\sigma)\Phi^0(\sigma, s)] d\sigma \right\| \\ &\leq \int_s^t \|P^k(\sigma) - P^0(\sigma)\| \|\Phi^k(\sigma, s)\| d\sigma + \int_s^t \|P^0(\sigma)\| \|\Phi^k(\sigma, s) - \Phi^0(\sigma, s)\| d\sigma. \end{aligned}$$

The first term is bounded by  $c_k := c_1 \int_0^{\tau_0+1} \|P^k(\sigma) - P^0(\sigma)\| d\sigma \rightarrow 0$  for  $k \rightarrow \infty$ . Again, by [1, Lemma 6.1] it follows that for  $t, s \in [0, \tau_0 + 1]$ ,

$$\|\Phi^k(t, s) - \Phi^0(t, s)\| \leq c_k + \int_s^t c_k \|P^0(r)\| \exp \int_r^t \|P^0(\sigma)\| d\sigma dr.$$

The right-hand side converges to 0 uniformly in  $t, s$  for  $k \rightarrow \infty$  since  $c_k \rightarrow 0$ .

(iv) The assumption implies that there are unique  $\tau_k$ -periodic solutions given by

$$x^k := (I - \Phi^k(\tau_k, 0))^{-1} \int_0^{\tau_k} \Phi^k(\tau_k, s) z^k(s) ds.$$

Then  $\|\int_0^{\tau_k} \Phi^k(\tau_k, s) z^k(s) ds\| \rightarrow 0$  since  $\tau_k \rightarrow \tau_0$ , and the integrands are uniformly bounded, and

$$\begin{aligned} &\int_0^{\tau_0} [\Phi^k(\tau_k, s) z^k(s) - \Phi^0(\tau_0, s) z^0(s)] ds \\ &\leq \int_0^{\tau_0} [\Phi^k(\tau_k, s) - \Phi^0(\tau_0, s)] z^k(s) ds + \int_0^{\tau_0} \Phi^0(\tau_0, s) [z^k(s) - z^0(s)] ds \\ &\leq \left[ \sup_{s \in [0, \tau_0]} \|\Phi^k(\tau_k, s) - \Phi^0(\tau_0, s)\| + \sup_{s \in [0, \tau_0]} \|\Phi^0(\tau_k, s) - \Phi^0(\tau_0, s)\| \right] \int_0^{\tau_0} \|z^k(s)\| ds \\ &\quad + \sup_{s \in [0, \tau_0]} \|\Phi^0(\tau_0, s)\| \int_0^{\tau_0} \|z^k(s) - z^0(s)\| ds. \end{aligned}$$

This converges to 0, and it follows that  $x^k \rightarrow x^0$ .

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