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ON THE CONNECTION BETWEEN BASIC MENTAL MODELS AND THE UNDERSTANDING OF EQUATIONS

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Basic mental models (BMMs) of equations have been proposed as structures describing conceptual understanding of equations. Two of these BMMs are those of equations as relations and equations as objects. We are interested in the relation between these BMMs and special errors associated with working with equations. In this study we concentrate on very basic equations in the form of $a \cdot x = b$ and $a + x = b$. We are interested in obstacles, errors and misunderstandings concerning these prototypes of equations. An empirical investigation shows that two types of errors, the reversal error and the attribute error, are statistically related to the BMMs students have established.

INTRODUCTION

Equations are basic elements in all fields of mathematics and mathematics education. However, many studies have revealed that students have problems with the understanding of equations, especially with the equals sign, and the solving of equations. The equals sign is seen as an instruction “to work it out now” (Kieran, 1981) or “to do a calculation” (Arcavi et al. 2017, p. 55). While this perspective is important and correct in primary school, it is still present and becomes problematic in lower secondary school (see Borromeo Ferri & Blum 2011).

“A limited conception for what the equals sign means is one of the major stumbling blocks in learning algebra. Virtually all manipulations on equations require understanding that the equals sign represents a relation.” (Carpenter et al. 2003, p. 22)

Without this “relational view” students will have problems in interpreting expressions correctly and setting up equations properly. This paper concentrates on two particular types of errors, the *reversal error* and the *attribute error*, and investigates how they are linked with each other on the one side and with the two basic mental models of equations as objects and as relations on the other.

The paper first describes the theory of basic mental models (BMMs) and the two error types in detail. Then an empirical investigation is presented which looks for relations between BMMs and these two types of errors.

THEORY

The equals sign

The equals sign has different meanings or perspectives in mathematics and mathematics lessons. It can be seen as an operation sign, e.g., in $3 + 7 = 10$ with the meaning of “results in”, it is a relational sign, e.g., in $29 + 36 = 30 + 35$ or $3 \cdot x + 5 =$

$x - 1$, it can express an identity or an equivalence, e.g., in $a \cdot (b + c) = a \cdot b + a \cdot c$ or it can give a functional relationship, e.g., $V(r) = \frac{4}{3}r^3\pi$ or $f(x) = x^2$. These different meanings result in different conceptions and perceptions connected with equations.

Basic mental models of equation (BMMs)

The concept of BMMs has a long tradition in German didactics where they are called “Grundvorstellungen” (vom Hofe & Blum, 2016). They describe, from a normative point of view, the conceptual mental models students should develop in order to grasp the meaning of concepts and apply them in an adequate and sensible way. Weigand et al. (2022) describe four BMMs of equations, based on mathematical aspects of equations. These are:

- **Operational BMM:** An equation is understood as a calculation or transformation. The equals sign is seen as an operational sign, which indicates a reading direction of the equation in the sense of a “resulting-in” sign.
- **Relational BMM:** An equation is understood as a task to determine numbers or quantities for the expressions on both sides of the equation to get the same value or quantity on both sides. The equals sign is seen as a relational sign. The variable here is understood as an unknown which has to be determined.³
- **Functional BMM:** An equation $T_1(x) = T_2(x)$ is a comparison of two expressions which are understood as functions with $y = T_1(x)$ and $y = T_2(x)$. Here, too, the equals sign is understood relationally, but the the variable is seen as varying over its domain.
- **Object-BMM:** An equation is regarded as a mathematical object that has characteristic properties, such as the number of possible solutions, the definition range or special solution algorithms.

Meanwhile, there are some empirical investigations concerning the structure and the independence of different BMMs of a concept, e.g., of the concepts of function, derivatives and integral (see Greefrath et al., 2021). However, there is a lack of research concerning the relevance of BMMs for solving problems in special fields of mathematics. The BMMs of equations are still a theoretical concept and it is not much known about their effect on solving problems. The present paper is a step in this direction. We especially emphasize the *Relational BMM* and the *Object-BMM* in analysing errors while formulating and interpreting equations.

³ We refer to the three central BMMs for variables without explaining their background (see e.g., MacGregor and Stacey (1997) for details): The variable as a general number, the variable as an unknown number and the variable as changing number or quantity.

Basic difficulties while working with equations

Students have problems while working with equations (see e.g., Arcavi et al., 2017, p. 95 ff.). In this study we concentrate on the very basic equations in the form of $a \cdot x = b$ and $a + x = b$. We are interested in obstacles, errors and misunderstandings concerning these prototypes of equations. Without the competence of interpreting and operating with these kinds of equations, the understanding of more complex equations is not possible. Moreover, we concentrate on two errors in relation to these equations, the *reversal error* and the *attribute error*. In particular, these errors show misunderstandings when dealing with the equals sign as a relational sign.

The prototype of the *reversal error* is provoked by the professor-and-students task (Clement, Lochhead, & Monk, 1981), that reads in its original version:

Write an equation for the following statement: There are six times as many students as professors at this university. Use S for the number of students and P for the number of professors.

While the correct solution is $S = 6 \cdot P$, many students write the reversed relation: $P = 6 \cdot S$. A lot of explanations for this error have been proposed. Already Clement et al. (1981) investigated the possibility of a syntactical transformation of the sentence into an equation. However, MacGregor & Stacey (1993) found that even relations presented in pictures can lead to the error. The error also occurs with additive relations.

We came up with another idea of an explanation of the reversal error within a test on this error. Students should write down an equation that expresses that the river Rhine (length r km) is 200 km longer than the river Elbe (which is e km long). The reversal error, $r + 200 = e$, can be explained if the expression $r + 200$ is not interpreted as a summation, but as “ r is 200 more than another quantity”. Moreover, this view explains a variant of the error that we observed in our studies: There was also the (wrong) answer “ $e - 200 = r + 200$ ”. While this error version resists explanations by syntactical translation or other approaches in the literature, it can be seen as an *attribute error*: The Elbe has the attribute (property) of being 200 km shorter ($-$) while ($=$) the Rhine has the property of being 200 km longer ($+$). This supports the thesis of MacGregor & Stacey (1993), according to which the equals sign is not necessarily understood as a numerical equality, but as a sign for a comparison of a different kind.

Attributes are quite common in mathematics. E.g., arrows are attributes to declare that \vec{v} is a vector, or the plus sign in \mathbb{R}^+ denotes positive numbers. Confusion with operations is likely because some operations look very similar to attributes. E.g., the complex conjugate \bar{z} of a complex number z is an operation that maps one number to another, yet it looks similar to the vector attribute. The absolute value $|x|$ is an operation that maps \mathbb{R} to \mathbb{R}_0^+ , but it may be misunderstood as giving x the attribute of being non-negative. Similarly, in “ $-x$ ” the minus sign should be understood as an operation (namely the opposite of x) but may be misinterpreted as a negative number.

We name this type of error *attribute error*. Students may look at e.g., $x + 5$ as a declaration that x is 5 more than some reference quantity. This attitude is supported by textbooks that contain tasks like this: “Write in symbols: x is increased by 5“. The students are then expected to write $x + 5$ which might be understood as changing the value of x by 5 or as statement that x is larger by 5 than some reference.

This discussion leads to the hypothesis that at least some reversal errors might result from an underlying attribute error and hence there should be a correlation between their occurrences. The attribute error was to the best of our knowledge first discussed in Oldenburg & Henz (2015). This present paper investigates the hypothesis that BMMs of equations, the reversal and the attribute error are correlated. We try to answer the following research questions:

- Is there a relation between the *attribute error* and the *reversal error*?
- Is there a relation between the *Relational BMM* and the *Object-BMM*?
- Are there relations between these error types and the BMMs?

THE TEST

To answer the research question, we use data for a subset of the items of an algebra test. The whole test takes a broader view on algebraic competence and includes e.g., items on substitution and on simplifying expressions. In this subtest we analyse measures of four scales, two on basic mental models and two on the error types described above:

- RevErr: 4 items about the reversal error
- AtrErr: 5 items about the attribute error
- RelBMM: 5 items about the Relational BMM
- ObjBMM: 7 items about the Object-BMM

The *Relational BMM* was measured e.g., by the following items (translated versions):

- It is known that $r = s + t$ and $r + t + s = 30 + 2x$. Determine r .
- In Phantasia you don't measure the temperature in Celsius. Our temperature 0°C corresponds to 10° and 100°C corresponds to 50° . Give a formula for the conversion from Celsius temperature T to fantasy temperature P .

The items for the *Object-BMM* require to look at equations as a whole, e.g.,

- A solution of $(x + 1)^3 + x = 349$ is given by $x = 6$. Use this knowledge to find a solution of $(5x + 1)^3 + 5x = 349$. (from Küchemann, 1979)
- Solve the equation $x^2 + 2x + 1 = 0$.

Most items of these two scales haven been graded on partial credit scale with 0 points for a wrong answer, 1 point for a partially correct answer and 2 points for a fully correct answer. Some easier items have been graded only by 0 (wrong) or 1 point (correct).

To measure the *reversal error* four items have been used. For each item an equation had to be set up. Three of the equations are of additive type (such as the Rhine-Elbe-example above), the last one is multiplicative:

- At a school, there are 20 times as many students as teachers. Let s stand for the number of teachers and S for the number of students. Write this as an equation.

For each of the four tasks a score was given to measure competence in avoiding the error: -1 for explicitly writing the erroneous version of the equation, 0 für writing nothing interpretable, 1 for writing almost the correct equation and 2 für the correct equation. The scale made of these 4 items is called RevErr. Note that the scale is oriented such that high values indicate a high competence in avoiding the error.

The items to measure the competence to avoid the attribute error were all of the following form: An expression was given and students should judge whether a given verbal statement expresses the same information. Some examples:

- x may be any real number. Is it true that $-x$ is negative?
- If x is any real number. Is $|x - 1|$ the same as $+x + 1$?
- Does $|x - 1|$ mean that $x - 1$ is not negative?

As with RevErr the scale AtrErr is oriented so that high values indicate high performance, i.e., few errors of that type.

The test has been completed by 123 teacher students from two second year courses. Participation was not mandatory and no further information (such as age, sex) has been recorded to avoid privacy issues. The students had not had lessons on algebra education before, but they had studied some mathematics at university level. This explains that the solution rate for many items is quite good. For example, only 51 reversal errors were committed (each of 123 students had 4 tasks, i.e., rate 10,3%). However, 42% marked falsely as correct that $|x - 1|$ means that $x - 1$ is nonnegative.

RESULTS AND FIRST INTERPRETATIONS

The written test results were coded and analysed with the R statistical program.

The internal consistency of the scales was assessed by means of the Cronbach alpha coefficient. Results show good values for all scales: RevErr: 0.74, AtrErr: 0.83, RelBMM: 0.79, ObjBMM: 0.86.

The four scales all correlate positively, as shown in table 1. All coefficients are significant (correlation test with Kendall's correlation).

Table 1: Correlation matrix of the four scales.

Correlation	RevErr	AtrErr	RelBMM	ObjBMM
RevErr	1	0.317**	0.213*	0.264**
AtrErr		1	0.525***	0.583***
RelBMM			1	0.686***
ObjBMM				1

First, there is a strong correlation between the two BMM scales. Either of these BMMs correlates strongly and highly significantly with a higher attribute error avoidance competence. On the other hand, the reversal error avoidance competence only correlates moderately, although significantly with the BMMs.

The strong correlations of the AtrErr scale with the other scales can also be confirmed by a linear regression model $AtrErr \sim RelBMM + ObjBMM + RevErr$. The standardized beta coefficients are $AtrErr \sim 0.135RelBMM + 0.153ObjBMM + 0.106RevErr$ and all three are significant, the coefficient of ObjBMM is even highly significant. Of interest is also a model that predicts AtrErr simply from the basic mental models, i.e., $AtrErr \sim RelBMM + ObjBMM$. Here the standardized coefficients are $AtrErr \sim 0.141RelBMM + 0.168ObjBMM$, and both are significant.

However, in the other regression $RevErr \sim RelBMM + ObjBMM + AtrErr$ only the last coefficient is significant: $RevErr \sim 0.002RelBMM + 0.075ObjBMM + 0.386AtrErr$. Similarly, if one just wants to predict RevErr by means of the BMMs: The regression equation $RevErr \sim RelBMM + ObjBMM$ is fitted to $RevErr \sim 0.056RelBMM + 0.140ObjBMM$ and this is not significant ($p \approx 0.06$).

The fact that the two types of errors are connected can also be demonstrated by comparing the group consisting of those students that made no reversal error, and the remaining ones. The mean of AtrErr for the first group is 3.34, while for the second group is only 2.50, which is a significant difference by the Wilcoxon test with $p=0.017$ and an effect size of Cohen $d=0.33$. In a complementary decomposition two groups were defined by scoring in AtrErr below resp. above average. The RevErr score shows a highly significant group difference, with $p=0.001$ and an effect size of $d=0.40$.

The relevance of the acquisition of BMMs for not committing errors can also be seen when looking at the sum scales $BMM := RelBMM + ObjBMM$ and $Err := AtrErr + RevErr$. They correlate with 0.48.

To shed further light on these relations a statistical implicative analysis (Gras et al., 2008) gave the following implications sorted by implicative intensity φ :

- 1) ObjBMM \rightarrow AtrErr 0.997
- 2) RelBMM \rightarrow AtrErr 0.996
- 3) RelBMM \rightarrow ObjBMM 0.992
- 4) ObjBMM \rightarrow RelBMM 0.98
- 5) AtrErr \rightarrow ObjBMM 0.948
- 6) ObjBMM \rightarrow RevErr 0.943

- | | |
|---------------------------------------|---------------------------------------|
| 7) AtrErr \rightarrow RevErr 0.941 | 8) RelBMM \rightarrow RevErr 0.909 |
| 9) AtrErr \rightarrow RelBMM 0.902 | 10) RevErr \rightarrow AtrErr 0.744 |
| 11) RevErr \rightarrow ObjBMM 0.651 | 12) RevErr \rightarrow RelBMM 0.609 |

First, look at 7) and 10). The difference of the implication weight shows that 7) is more important, so mastering the attribute obstacle predicts a good performance on reversal errors tasks but not vice versa. This gives support to the hypothesis that attribute error misconceptions may underly many occurrences of the reversal error. Implications 3) and 4) simply reflect the high correlation between ObjBMM and RelBMM and show that there is no particular direction on their mutual relation.

Implications 1) and 2) show that high BMMs predict good performance in attribute error tasks. It is instructive to interpret this from the logical contraposition: The implication $A \Rightarrow B$ is logically equivalent to $\neg B \Rightarrow \neg A$. Hence, one may read 1) and 2) as expressing that mastering attribute error tasks may be a necessary (in the statistical sense) requisite for high BMMs. However, the opposite implications 5) and 9) have high implicative intensities as well so that the directional effect is not very strong.

INTERPRETATIONS

Both basic mental models considered in this paper correlate highly but still can be clearly separated. Concerning the two error types, regressions, correlations, group comparisons and implications indicate that they are related so that the first research question can be answered affirmatively. Moreover, results show that having especially a distinct *Object-BMM* indicates a strong resistance against these errors. Overall, the test results indicate that putting more emphasis in developing BMMs may be beneficial for avoiding the reversal and the attribute errors.

The statistical implicative analysis given above sheds some further light on directional effects between these scales. This may give hints (but not proofs) on possible causal connections. When interpreting the above numbers on the relevance of *Object-BMM* one should have in mind that this BMM is usually considered to be the most advanced form of understanding equations and this may explain its importance.

CONCLUSIONS

The study presented here underpins that BMMs of equations are an important part of understanding equations, which means here the ability of formulating and interpreting equations. They are positively correlated with the avoidance of the reversal error and the attribute error. This especially means that developing the BMMs of equations is a good strategy not only for understanding equations but also for avoiding errors like the reversal and attribute error. The fact that even teacher students at university show considerable difficulties with these tasks further supports the suggestion to address the semantics of equations more deeply by building up these BMMs. A first step is the early development of the relational view already in primary school, e.g., with examples like $39 + 17 = 40 + 16$. Learning steps for this development can be found in Stacey

(2011). In the following years in lower secondary school the *Relational BMM* has to be developed on manifold representations especially on the enactive and iconic level, also integrating environmental situations. Moreover, the meaning of the *Object-BMM* has been underestimated in relation to understanding and interpreting equations already in lower algebra, starting with very basic types of equations. However, a learning strategy for the development of this BMM has still to be constructed.

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