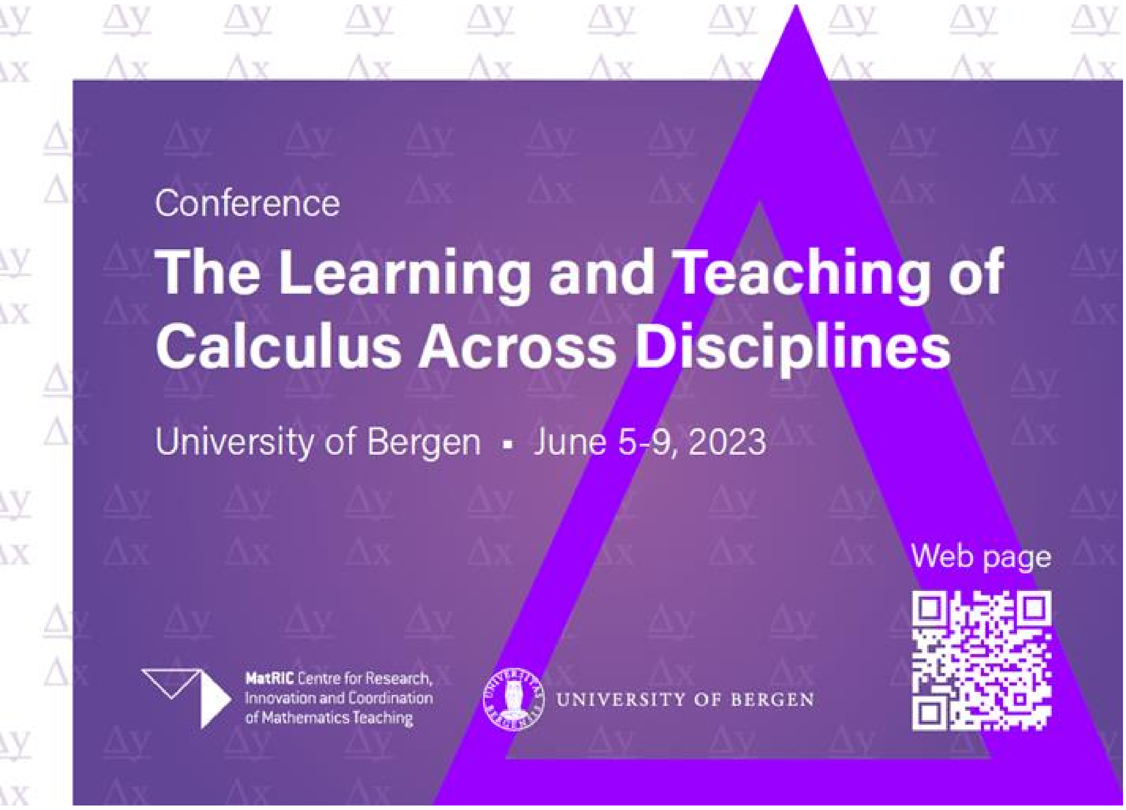

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The procedural-conceptual dichotomy is not invariant under transposition to applied fields

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Introduction

The dichotomy between procedural and conceptual knowledge has been studied extensively in mathematics education in general and in higher education (Rittle-Johnson & Schneider, 2015; Engelbrecht). Often it is conjectured that more conceptual understanding is needed while in actual teaching procedural techniques dominate (although already Rittle-Johnson and Schneider (2015) emphasized that they are intertwined). The importance of conceptual knowledge goes undoubtedly, although its components and the ways to measure it are less elaborate than for procedural knowledge. Among the research strands that investigate conceptual knowledge in calculus is the work on basic mental models (Greefrath et al., 2021). This identifies conceptual dimensions, but they are all rooted in the praxis of typical German high school teaching.

If strengthening conceptual knowledge is to be effective, it must be conceptualized in a way that respects the structure of the field that mathematics will finally be applied to. This paper investigates the techniques of multivariate integration by parts together in detail and some additional examples.

> Research hypothesis: When a body of mathematical knowledge gets transposed from pure mathematics to an application field, the classification as procedural or conceptual may change.

Methodology and Theory

The first underlying methodological framework of this research is the anthropological theory of the didactic (ATD; Chevallard, 2019). It models knowledge by praxeologies $\wp = [T / \tau / \theta / \Theta]$ where T denotes a task or a type of tasks, τ is a technique to solve it, θ is a technology that explains τ, and finally Θ is a theory that justifies θ. This is applied to integration by parts.

Integration by parts underlies the mathematical task (taken from Höllig & Hörner, 2021, p. 251) T_1="Calculate $\iint_{\mathbb{R}^2}(x+\cos(y))\cdot\partial_x \exp\left(-\sqrt{x^2+y^2}\right) \mathrm{d}x\mathrm{d}y$". The transposition of partial integration to physics is exemplified by T_2="Calculate the energy density in electrostatics". Both tasks use (among other things) the technique to rewrite an integral according to the multivariate rule of integration by parts. This part of θ is essentially the theorem that for scalar functions $u, v: \mathbb{R}^n \supset \Omega \to \mathbb{R}$ with adequate properties of integrability and differentiability one has $\int_\Omega u \cdot \partial_{x_j} v \, d\Omega = \int_{\partial\Omega} u \cdot v \cdot (e_j \cdot n) d\Gamma - \int_\Omega \partial_{x_j} u \cdot v \, d\Omega$, where e_j is the j-th unit vector, and n is a normal vector on the surface $\partial\Omega$ of the region Ω. This theorem and thus θ rests on the theory of multivariate integration Θ as foundational theory. We will investigate the praxeologies $\wp_i = [T_i / \tau_i / \theta / \Theta]$, $i \in \{1,2\}$.

The second underlying conceptual framework is the distinction between conceptual and procedural knowledge (Engelbrecht et al., 2012; Rittle-Johnson & Schneider, 2015). Rittle-Johnson and Schneider (2015) discuss several possible definitions but conclude simply "there is general consensus that conceptual knowledge should be defined as knowledge of concepts" and that "procedural knowledge is the ability to execute action sequences (i.e., procedures) to solve problems". Rittle-Johnson & Schneider (2015) have stated that "conceptual and procedural knowledge cannot always be separated". This is not by accident, similar to the fuzzy distinction line between syntactical and semantical test items (Oldenburg et al., 2013). Girard (1989, chapter 1) links this to the relevance of "sense" in the sense of Frege, and it is mainly the sense that is affected by transpositions.

The mathematical praxeology $\wp_1$

Solving task T_1 in mathematics will typically be taught and exercised in the following manner: One uses the above given theorem on partial integration for a circle of some large radius R and estimates that the surface integral (in this case a line integral) will approach 0 as $R \to \infty$ because the integrand decreases fast enough. Thus, one has in the limit of integrating over all of $\mathbb{R}^3$ that $\iint_{\mathbb{R}^2}(x + \cos(y)) \cdot \partial_x \exp\left(-\sqrt{x^2+y^2}\right) \mathrm{d}x\mathrm{d}y =$

$$0 - \iint_{\mathbb{R}^2} \partial_x(x+\cos(y)) \cdot \exp\left(-\sqrt{x^2+y^2}\right) \mathrm{d}x\mathrm{d}y = -\iint_{\mathbb{R}^2} \exp\left(-\sqrt{x^2+y^2}\right) \mathrm{d}x\mathrm{d}y = -2\pi$$

(where the last integral is worked out using radial coordinates). Thus, to solve T_1 one applies a technique τ_1 that relies on θ and consists of approximating the integral over the unbounded $\mathbb{R}^2$ by integrals over disks and taking the limit of the radius. I argue that the praxis part $[T_1 / \tau_1]$ is procedural because one is interested in the result (a number) and this result can be obtained by procedural, even algorithmic working style. This can be demonstrated by the fact that the whole calculation process can be carried out by computer algebra systems (although one may need to initiate the coordinate transformation by hand), it is guided by the structure of the expressions alone. No further conceptional considerations are necessary to arrive at the answer required by the task.

The physical praxeology $\wp_2$

Task T_2 must be described in a bit more detail. A distribution of charge in space described by a charge density function $\rho: \mathbb{R}^3 \to \mathbb{R}$ will store some energy due to the forces between charged particles (Jackson, 1975, p. 46). The charge in space defines a potential $\Phi: \mathbb{R}^3 \to \mathbb{R}$ that gives raise to the electric field strength $E = -\nabla\Phi$. Moreover, one knows that the Poisson equation $\nabla^2\Phi(x) = -4\pi\rho(x)$ holds. The potential and the charge combine to give the energy of the charge distribution according to $W = \frac{1}{2}\int \rho(x)\Phi(x)d^3x$. Plugging in the Poisson equation, one gets $W = \frac{-1}{8\pi}\int (\nabla^2\Phi(x))\Phi(x)d^3x$. The physical interpretation of this is that the total energy is summed up from the charges (i.e. $\nabla^2\Phi(x)$) weighted by their potential $\Phi(x)$. Now, perform partial integration to get $W = \frac{1}{8\pi}\int \nabla\Phi(x) \cdot \nabla\Phi(x)d^3x = \frac{1}{8\pi}\int |\mathrm{E}|^2 d^3x$ (the border integral is, just as in T_1, zero). Thus, the energy density is now expressed in terms of the electric field. This insight is not just the result of applying a procedure. Instead, it requires conceptual considerations and interpretations. This is true even if the technology τ_2 applied here contains much of the same calculations as τ_1, but it extends τ_1 by the physical interpretation of the transformations applied. Within τ_2, a transformation is not just a syntactical manipulation to get closer to the desired answer, but it is a transformation of the meaning of the expression. The difference is manifest as well in the sub-technology used to argue that the border integral does not contribute. In τ_1 this is a standard limit argument that relates the length of the curve and the maximum of the absolute value of the integrand to infer from the formula that for $R \to \infty$ the integral will vanish. The argument from τ_2 that allows to omit the border integral is different. No concrete function is given, and thus the vanishing cannot be inferred. Rather, it follows from general physical principles that fields tend quickly to infinity. Furthermore, the knowledge that integration by parts should be applied comes in τ_1 from the syntactical properties of the expression (the first factor is linear in x), while in τ_2 such a syntactical clue is completely missing. Thus, while in $\wp_1$ transformational procedures are guided by the structure of the expression which can itself be analyzed procedurally, in $\wp_2$ it is understanding of the concepts that guides the process and indicates what final form is sensible (i.e., has sense). Summarizing, the same integration by parts turned out to be a syntactical tool to solve T_1, but a conceptual tool to switch the way in which the physical reality is described in T_2. The passage from $\wp_1$ to $\wp_2$ is indeed an example of a transformation, not just an application of mathematics because the arguments and strategies differ. For example, the fact that the border integral vanishes is justified by different arguments.

Further examples

Such examples of conceptual use of partial integration (rather than just procedural use to calculate some results) are not rare and not limited to electrodynamics: proving that Coulomb's law and Maxwell's first equation are equivalent (Jackson, 1975, p. 33), derivation of the Klein-Gordon equation, derivation of Euler-Lagrange equations etc. The last example brings out the conceptual nature of partial integration obviously because it allows to trade in the change of a variation to the variation itself. Moreover, classifying integration by parts as procedural knowledge gives no adequate description of the way physicists calculate with the derivative of the Dirac delta distribution: To evaluate an integral like $\int_a^b f(x) \cdot \delta'(x) \cdot dx$ with $a < 0 < b$ physicists will apply integration by parts: $\int_a^b f(x) \cdot \delta'(x) \cdot dx = f(b)\delta(b) - f(a)\delta(a) - \int_a^b f'(x) \cdot \delta(x) \cdot dx = -f'(0)$. This is conceptual because it extends the notion of derivative to a new kind of object.

What has been established by now is that what may appear procedural in one praxeology is conceptual in another praxeology. But the situation may even be more drastic, as a look into computer science indicates. Knowledge of algorithms and how to carry them out seems to be procedural knowledge. On the other hand, proofs in mathematics are often considered carrying mathematics' conceptual knowledge (e.g., Hanna & Barbeau, 2008). However, the celebrated Curry-Howard correspondence (Girard, 1989; Thompson, 1991) states that proofs can be turned into strictly typed functional programs and vice versa. This blurs the distinction fully – at least at an abstract level. The lesson is obviously that being conceptual or being procedural is not a property of some piece of knowledge per se, but of the concrete praxeology where it is applied.

Conclusion and Outlook

The examples elaborated above have given support to the research hypothesis stated in the introduction. There are many further examples that go beyond the scope of this paper. A first consequence is that when teaching mathematical analysis and calculus for applied sciences, the lecturer should investigate the role of the subject played in the application domain. Especially, a teacher educated in pure mathematics may consider some learning objectives to be procedural and, based on this classification may decide to give them little weight, although in an application area these procedural techniques acquire a conceptual meaning. In the age of computer algebra systems, students do not need to know integration by parts and by substitution to find antiderivatives. Thus, these topics have been removed from German high school curricula. However, many students will need them as conceptual tools in STEM university courses. In research, one should be aware that the distinction between procedural and conceptual can only be made relative to a certain praxeology, but not absolutely. Hence, interpretation of results of such studies must investigate the concrete praxeology and should be careful when stating conclusions for distinct praxeologies.

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