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Nonnegative scalar curvature on manifolds with at least two ends

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Abstract

Let M be an orientable connected n-dimensional manifold with $n \in \{6,7\}$ and let $Y \subset M$ be a two-sided closed connected incompressible hypersurface that does not admit a metric of positive scalar curvature (abbreviated by psc). Moreover, suppose that the universal covers of M and Y are either both spin or both nonspin. Using Gromov's μ-bubbles, we show that M does not admit a complete metric of psc. We provide an example showing that the spin/nonspin hypothesis cannot be dropped from the statement of this result. This answers, up to dimension 7, a question by Gromov for a large class of cases. Furthermore, we prove a related result for submanifolds of codimension 2. We deduce as special cases that, if Y does not admit a metric of psc and $\dim(Y) \neq 4$, then $M := Y \times \mathbb{R}$ does not carry a complete metric of psc and $N := Y \times \mathbb{R}^2$ does not carry a complete metric of uniformly psc, provided that $\dim(M) \leqslant 7$ and $\dim(N) \leqslant 7$, respectively. This solves, up to dimension 7, a conjecture due to Rosenberg and Stolz in the case of orientable manifolds.

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1 | INTRODUCTION

In the 1994 survey article [28, Section 7], Rosenberg and Stolz proposed the following conjectures concerning the (non-)existence of *positive scalar curvature* (abbreviated by *psc*) metrics on certain noncompact manifolds.

Conjecture 1.1. *Let Y be a closed manifold of dimension $(n-1) \neq 4$ which does not admit a metric of positive scalar curvature. Then $Y \times \mathbb{R}$ does not admit a complete metric of positive scalar curvature.*

Conjecture 1.2. *Let Y be a closed manifold of dimension $(n-2) \neq 4$ which does not admit a metric of positive scalar curvature. Then $Y \times \mathbb{R}^2$ does not admit a complete metric with uniformly positive scalar curvature.*

Recently, Gromov phrased a related conjecture [14, 11.12, Conjecture C], which is a cornerstone of his program regarding the study of metric inequalities with scalar curvature.

Conjecture 1.3. *Let Y be a closed manifold of dimension $(n-1) \neq 4$ which does not admit a metric of positive scalar curvature and $X = Y \times [-1,1]$. If g is a Riemannian metric on X with $\operatorname{scal}(X,g) \geqslant n(n-1)$, then*

$$\operatorname{width}(X,g) := \operatorname{dist}_g(Y \times \{-1\}, Y \times \{1\}) \leqslant \frac{2\pi}{n}.$$

Remark 1.4. The condition $\dim(Y) \neq 4$ (which we have added in our formulation of1.1, 1.2) is necessary since in this case, there are well-known counterexamples using Seiberg–Witten obstructions to positive scalar curvature. It is possible to show that there exists a closed simply connected 4-manifold Y which does not admit a metric of positive scalar curvature, while $Y \times S^1$ does (see [29, Counterexample 4.16]). But then $Y \times \mathbb{R}$ (and consequently $Y \times \mathbb{R}^2$) admits a complete metric with uniformly positive scalar curvature that would violate1.1, 1.2, 1.3.

As is common in the study of scalar curvature, there are two broad families of methods to approach these conjectures: One is based on the spinor Dirac operator on spin manifolds originating in the work of Lichnerowicz [21]. Indeed, using different variants of index theory on noncompact manifolds, it is now well established that1.1, 1.2 can be proved whenever Y admits an index-theoretic obstruction to positive scalar curvature such as the Rosenberg index [27], see [1, 17, 35]. The other family of methods is based on geometric measure theory and originates in the minimal hypersurface method of Schoen and Yau [31]. The first established cases of Conjecture 1.3 by Gromov [14, Section 2] used the classical minimal hypersurface method. Subsequently, also a Dirac operator approach to Conjecture 1.3 was developed by Cecchini and Zeidler [2, 3, 36, 37].

In [15, Sections 3.6, 5], Gromov proposes a different approach toward Conjecture 1.3 using a modified version of the minimal hypersurface method involving so-called μ-bubbles. Following this idea, Räde [25] proved Conjecture 1.3 and generalizations thereof in case that Y is orientable and $n \leqslant 7$.1.1, 1.2 have not been directly approached via minimal hypersurface techniques so far, in particular, due to the noncompactness inherent to the problem. However, in recent work of various authors (e.g., [6, 19, 20, 39]), μ-bubbles have turned out to be a useful tool to deal with noncompact situations. Even though there is no direct formal implication between Conjecture 1.1 and Conjecture 1.3, it was observed in [36] that the Dirac operator methods used in [2, 37] to attack

Conjecture 1.3 can be refined to prove a more general statement (compare [36, Conjecture 7.1]), which imply both1.1, 1.3 for closed spin manifolds with nonvanishing Rosenberg index.

The main objective of this article is to combine the ideas from [36] with μ-bubble methods, in particular from [25], to prove a generalization of Conjecture 1.1 in case Y is orientable and of dimension $\leqslant 6$. By a reduction argument to a codimension 1 situation (compare [13, Theorem 7.5][17]), we also establish Conjecture 1.2 in case Y is orientable and of dimension $\leqslant 5$. We note that ideas related to Conjecture 1.1 have also appeared recently in the work of Chen, Liu, Shi, and Zhu [5] in connection with the positive mass theorem.

More generally than Conjecture 1.1, one may ask under which circumstances the existence of a hypersurface $Y \subset M$ which does not admit psc already is an obstruction to the existence of a complete psc metric on the ambient manifold M. This question has been discussed by Gromov in [14, §11.6], where, in particular, he asked if it would be enough to assume that Y is a two-sided *incompressible hypersurface*, that is, the map $\pi_1 Y \to \pi_1 M$ induced by the inclusion is injective. In the case that the ambient manifold is spin of dimension $n \in \{6, 7\}$ and under further geometric conditions, a proof confirming this was sketched in [14, pp. 708 sqq.] and it was asked if the spin hypothesis can be dropped. We answer this question in our first theorem below together with the following Example 1.6. Here we say that a connected manifold M is *almost spin*, if its universal covering $\tilde{M}$ is spin, and we say that it is *totally nonspin* if $\tilde{M}$ is nonspin. Since spin structures lift to coverings, being almost spin is equivalent to the existence of some covering which is spin. Unless explicitly stated otherwise, we consider manifolds without boundary.

Theorem 1.5. *Let M be an orientable connected n-dimensional manifold with $n \in \{6, 7\}$ and let $Y \subset M$ be a two-sided closed connected incompressible hypersurface that does not admit a metric of positive scalar curvature. Suppose that one of the following two conditions holds.*

(a) *M is almost spin.*
(b) *Y is totally nonspin.*

Then M does not admit a complete metric of positive scalar curvature. More precisely, if g is a complete metric of nonnegative scalar curvature on M, then (M, g) admits a connected Riemannian covering isometric to $(N \times \mathbb{R}, g_N + \mathrm{d}x^2)$, where (N, g_N) is a closed Ricci flat manifold.

Note that if M is almost spin, then any two-sided hypersurface $Y \subset M$ is almost spin itself. Conversely, if a two-sided hypersurface is totally nonspin, then so is the ambient manifold. Thus, the alternative hypotheses (a) and (b) of Theorem 1.5 simply say that either M and Y are both almost spin or neither is. The following example shows that this restriction cannot be dropped.

Example 1.6. Fix $n \geqslant 6$. Let L be the K3 surface, that is, a four-dimensional simply-connected spin manifold such that $\widehat{\mathrm{A}}(L) \neq 0$. Consider the closed manifold $M := (L \times \mathrm{T}^{n-4}) \#(\mathbb{C}\mathrm{P}^2 \times \mathrm{S}^{n-4})$. Then, since L is oriented cobordant to a psc manifold by [12, §3], the cobordism class represented by M in $\Omega_n^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}^{n-4})$ has a psc representative. Thus, the totally nonspin manifold M itself admits a psc metric, for example, by [26, Theorem 2.13] or [8, Theorem 1.5].† It contains $Y = L \times \mathrm{T}^{n-5}$ as an incompressible hypersurface which does not admit psc by [13, Corollary 5.22]. By passing to the covering $\hat{M} \to M$ with $\pi_1 \hat{M} = \mathbb{Z}^{n-5}$ corresponding to $\pi_1 Y$, we even find an example of a complete manifold that admits a complete uniform psc metric although it contains a closed incompressible *separating* hypersurface which does not.

† Alternatively, the existence of a psc metric on M can also be deduced from [23, Theorem 5.6].

This shows that the almost spin condition is fundamentally relevant for this kind of problem even though the proof Theorem 1.5 does not use the Dirac operator at all. Moreover, even in the spin case, Theorem 1.5 is stronger (in the dimension range where it applies) than what can be proved using index-theoretic techniques because there are examples of closed spin manifolds that do not admit psc even though all known index invariants vanish [30].

The upper dimension bound $n \leqslant 7$ comes from the usual problem pertaining to singularities of minimal hypersurfaces and μ-bubbles. Thus, if this issue was resolved, the upper-dimensional bound in Theorem 1.5 (and in all other results of this paper) should conjecturally be removable. The case $n = 8$ likely is more easily accessible via an adaptation of the work of N. Smale [32].

Remark 1.7. In [14, §11.6], Gromov conjectured that, in general, given a complete Riemannian manifold M, the existence of a two-sided incompressible closed embedded hypersurface which does not carry a psc metric obstructs the existence of a function h on M such that

$$\frac{n}{n-1}h^2 - 2|\mathrm{d}h| + \mathrm{scal}_g \geqslant 0.$$

This condition is motivated by the potential function used to construct μ-bubbles. From this point of view, Theorem 1.5 gives a complete description of the case when $h = 0$, which includes many geometrically interesting cases. We point out that our method can be adapted to treat cases with nontrivial h. In the direction of Gromov's motivating example [14, pp. 708 sqq.], compare Corollary 3.7. We also point out that, compared to the method sketched in [14, §11.6], our technique does not require any extra assumption on the geometry of the manifold.

For $n = 5$, Theorem 1.5 fails as mentioned in Remark 1.4. On the other hand, for $2 \leqslant n \leqslant 4$, it holds even independently of the hypotheses (a) and (b), but for a different reason than in high dimensions. To formulate this, we consider the following notion.

Definition 1.8 (Compare [25, Definition 2.20]“$\mathcal{C}_{\deg}$”CPSZ21). A closed connected oriented manifold Y is called NPSC$^+$ if it satisfies the following property: No closed oriented manifold Z that admits a continuous map of nonzero degree $Z \to Y$ admits a metric of positive scalar curvature.

For instance, it has recently been shown [6, 7, 16] that closed oriented aspherical manifolds of dimension $\leqslant 5$ are NPSC$^+$. Moreover, a closed oriented manifold of dimension $\leqslant 3$ which does not admit psc necessarily admits a nonzero degree map to an aspherical NPSC$^+$ manifold.[†] Thus, the low-dimensional counterpart to Theorem 1.5 is contained in the following theorem that already appeared recently in [5].

Theorem 1.9 (Compare [5, Theorem 1.1]). *Let M be an orientable connected n-dimensional manifold with $n \leqslant 7$ and let $\iota\colon Y \hookrightarrow M$ be a two-sided closed hypersurface that admits a map of nonzero degree $\phi\colon Y \to Y_0$ to an aspherical* NPSC$^+$ *manifold Y_0 and such that* $\ker(\pi_1 Y \xrightarrow{\iota_*} \pi_1 M) \subseteq \ker(\pi_1 Y \xrightarrow{\phi_*} \pi_1 Y_0)$. *Then M does not admit a complete metric of positive scalar curvature. More precisely, if g is a complete metric of nonnegative scalar curvature on M, then (M, g) admits a connected Riemannian covering isometric to $(N \times \mathbb{R}, g_N + \mathrm{d}x^2)$, where (N, g_N) is a closed Ricci flat manifold.*

[†] In dimension 2, this is a consequence of the classification of surfaces and the Gauß–Bonnet theorem, whereas in dimension 3, it is a consequence of the classification of 3-manifolds which admit psc following from Perelman's work, see, for example, the discussion in [22].

In particular, applying1.5, 1.9 to $M = Y \times \mathbb{R}$, we deduce the following.

Corollary 1.10. *Conjecture 1.1 holds for orientable manifolds in dimensions* $5 \neq n \leqslant 7$.

However, we note that the low-dimensional cases $n \leqslant 4$ of Corollary 1.10 were already known because an oriented manifold of dimension $\leqslant 3$ which does not admit positive scalar curvature is necessarily spin and has rationally nonvanishing Rosenberg index, and so, the situation is within the scope of index-theoretic results such as [1, Theorem A].

We now turn to our results corresponding to Conjecture 1.2.

Theorem 1.11. *Let M be an orientable connected seven-dimensional manifold and let $Y \subset M$ be a closed connected five-dimensional submanifold with trivial normal bundle such that the inclusion induces an injection $\pi_1 Y \hookrightarrow \pi_1 M$ and a surjection $\pi_2 Y \twoheadrightarrow \pi_2 M$. Suppose that Y does not admit a metric of positive scalar curvature. Then M does not admit a complete metric of uniformly positive scalar curvature.*

Note that, unlike Theorem 1.5, we do not need to impose any conditions involving spin structures. The intuitive reason for this is that the hypotheses already imply that the induced map $\tilde{Y} \to \tilde{M}$ between universal covers is 2-connected, and so, M is almost spin if and only if Y is. On the other hand, the surjectivity condition $\pi_2 Y \twoheadrightarrow \pi_2 M$ cannot be omitted as the example $M = Y \times S^2$ shows.

We also have a codimension 2 counterpart to Theorem 1.9 exploiting the NPSC^+ property.

Theorem 1.12. *Let M be an orientable connected n-dimensional manifold with $n \leqslant 7$ and let $Y \subset M$ be a closed connected $(n-2)$-dimensional submanifold with trivial normal bundle such that the inclusion induces an injection $\pi_1 Y \hookrightarrow \pi_1 M$ and a surjection $\pi_2 Y \twoheadrightarrow \pi_2 M$. Suppose that Y admits a map of nonzero degree $Y \to Y_0$ to an aspherical* NPSC^+ *manifold Y_0. Then M does not admit a complete metric of uniformly positive scalar curvature.*

Similarly as before, this includes a version of Theorem 1.11 in dimensions $n \leqslant 5$.

Corollary 1.13. *Let* $6 \neq n \leqslant 7$. *Let M be an orientable connected n-dimensional manifold and let $Y \subset M$ be a closed connected $(n-2)$-dimensional submanifold with trivial normal bundle such that the inclusion induces an injection $\pi_1 Y \hookrightarrow \pi_1 M$ and a surjection $\pi_2 Y \twoheadrightarrow \pi_2 M$. Suppose that Y does not admit a metric of positive scalar curvature. Then M does not admit a complete metric of uniformly positive scalar curvature.*

Proof. For $n = 7$, this is a restatement of Theorem 1.11. If $n \leqslant 5$, then $\dim(Y) \leqslant 3$, and so, it admits a nonzero degree map $Y \to Y_0$, where Y_0 is aspherical and NPSC^+. Thus, Theorem 1.12 is applicable in this case. □

Specializing to $M = Y \times \mathbb{R}^2$, we finally obtain the following.

Corollary 1.14. *Conjecture 1.2 holds for orientable manifolds in dimensions* $6 \neq n \leqslant 7$.

Again, we like to point out that the low-dimensional cases $n \leqslant 5$ of Corollary 1.14 were already known due to index-theoretic results [17, 37, Theorem 1.10].

This article is organized as follows: In Section 2, we prepare an abstract setup for the study of manifolds with at least two ends which underpins our work. In Section 3, we state quantitative comparison results in the spirit of Conjecture 1.3 which are then used together with topological arguments in Section 4 to prove our codimension 1 results. The codimension 2 results are deduced in Section 5. Finally, in Section 6, we provide the analytic proofs of the comparison statements from Section 3.

2 | BANDS

A natural class of manifolds generalizing the situation of Conjecture 1.1 are connected noncompact manifolds, where we partition the set of ends into two parts and we consider hypersurfaces separating these parts from each other. To make this precise, we will make use of the Freudenthal end compactification [9] of a connected manifold M, denoted by $\mathcal{F}M = M \cup \mathcal{E}M$, where $\mathcal{E}M$ is the space of ends. By definition, the space of ends is the inverse limit $\mathcal{E}M = \varprojlim \pi_0(M \setminus K)$, where K runs over compact subsets of M and each set of components $\pi_0(M \setminus K)$ is endowed with the discrete topology.

Definition 2.1.

(i) An *open band* is a connected noncompact manifold M without boundary and a decomposition

$$\mathcal{E}M = \mathcal{E}_{-}M \sqcup \mathcal{E}_{+}M,$$

where $\mathcal{E}_{\pm}M$ are nonempty closed[†] subsets $\mathcal{E}_{\pm}M \subset \mathcal{E}M$; in particular, M has at least two ends.

(ii) Given an open band M, a *separating hypersurface* $\Sigma \subset M$ is a compact hypersurface which separates each end in $\mathcal{E}_{-}M$ from every end in $\mathcal{E}_{+}M$. Moreover, we say that a separating hypersurface $\Sigma \subset M$ is *properly separating* if every component of Σ can be connected to both $\mathcal{E}_{+}M$ and $\mathcal{E}_{-}M$ inside $M \setminus \Sigma$.

(iii) Let $\Sigma_{-}, \Sigma_{+} \subset M$ be two properly separating hypersurfaces in an open band M. Then we write $\Sigma_{-} \prec \Sigma_{+}$ if the hypersurface Σ_{-} is contained in the union of those components of $M \setminus \Sigma_{+}$ that contain the ends in $\mathcal{E}_{-}M$ (or equivalently, Σ_{+} is contained in the union of those components of $M \setminus \Sigma_{-}$ containing $\mathcal{E}_{+}M$).

The condition of being *properly* separating simply means that there are no superfluous components as the observation recorded in the following lemma illustrates.

Lemma 2.2. *Let $\Sigma \subset M$ be a separating hypersurface in an open band M. Then there exists a union of components of Σ which is a properly separating hypersurface in M.*

Proof. Suppose that Σ is a separating hypersurface that contains a component not connected to both $\mathcal{E}_{-}M$ and $\mathcal{E}_{+}M$ inside $M \setminus \Sigma$. Then the hypersurface Σ' obtained from Σ by deleting this component is still a separating hypersurface. This shows that a minimal collection of components of Σ such that its union is still separating yields the desired properly separating hypersurface. $\square$

[†] While the space of ends is totally disconnected, it is in general not necessarily discrete. In this case, it is important to assume $\mathcal{E}_{\pm}M$ to be closed (and thus clopen) subsets of $\mathcal{E}M$.

The next elementary lemma shows that (properly) separating hypersurfaces always exist and we can find them arbitrarily far out.

Lemma 2.3. *Let M be an open band and $K \subset M$ be an arbitrary compact subset. Then there exists a properly separating hypersurface $\Sigma \subset M$ which also separates K from $\mathcal{E}_+M$ (or $\mathcal{E}_-M$, respectively).*

Proof. Note that the end compactification $\mathcal{F}M$ is a compact Hausdorff space that in our case of a connected manifold is also second countable. Thus, since $\mathcal{E}_\pm M$ are two disjoint closed subsets of $\mathcal{F}M$, Urysohn's lemma implies the existence of a continuous function $f : \mathcal{F}M \to [-1, 1]$ such that $\mathcal{E}_\pm M = f^{-1}(\pm 1)$. Since $K \subseteq M$ is compact, there exists $0 < r < 1$ such that $K \subseteq f^{-1}([-r, r])$. Choose $s \in (r, 1)$. Then $f^{-1}(s) \subseteq M$ is a compact subset which separates $\mathcal{E}_-M$ from $\mathcal{E}_+M$. Now choose a connected compact n-dimensional submanifold $V \subset M$ with boundary, where $n = \dim(M)$, such that $f^{-1}(s) \subseteq \mathring{V} \subseteq f^{-1}([s - \varepsilon, s + \varepsilon])$ for some $\varepsilon > 0$ with $r < s - \varepsilon$. Then ∂V is a separating hypersurface and it contains a properly separating hypersurface $\Sigma \subseteq \partial V$ by Lemma 2.2. Since by construction $f(x) \leqslant r < s - \varepsilon \leqslant f(y) \leqslant s + \varepsilon < 1$ for each $x \in K$ and $y \in \Sigma$, it follows that Σ must separate K from $\mathcal{E}_+M$. A completely analogous argument also provides a properly separating hypersurface that separates K from $\mathcal{E}_-M$. □

In the spirit of Conjecture 1.1, we will be interested in open bands with:

Property A. No separating hypersurface admits a metric of positive scalar curvature.

We will also work with compact bands, which may be viewed as a special case of open bands in the following sense.

Definition 2.5. A *compact band*, to which we will often simply refer to as a "*band*," is a connected compact manifold X together with the structure of an open band on its interior $\mathring{X}$. In other words, this amounts to a decomposition $\partial X = \partial_- X \sqcup \partial_+ X$, where $\partial_\pm X$ are (nonempty) unions of boundary components.

The notions of (properly) separating hypersurfaces and Property A make sense for compact bands by considering them for the interior.

We also note that if M is an open band and $\Sigma_\pm$ are two properly separating hypersurfaces such that $\Sigma_- \prec \Sigma_+$, then the hypersurfaces $\Sigma_\pm$ bound a compact band $X \subset M$. In this case, if M has Property A, then so has X.

3 | THE PARTITIONED COMPARISON PRINCIPLE

In this section, we establish the main analytic results on which our main theorems rely. The central concept we study here is the width(X, g) of a compact band X endowed with a Riemannian metric g, that is, the distance between $\partial_- X$ and $\partial_+ X$ with respect to g. As is explained in [15, Section 3.6], the width of an n-dimensional Riemannian band (X, g) with scal $\geqslant n(n-1)$ is bounded from above by $\frac{2\pi}{n}$ if X has Property A and $n \leqslant 7$. We will work in the setting where (X, g) has Property A and is partitioned into multiple segments with possibly different lower scalar curvature bounds. It turns out that in this case, positivity of the scalar curvature in a single segment can often have global effects on the geometry of (X, g).

Definition 3.1. Let X be a compact band and let Σ_i, for $i \in \{1, \dots, k\}$, be properly separating hypersurfaces such that $\Sigma_{i-1} \prec \Sigma_i$ for $1 < i \leqslant k$. We call the triple (X, Σ_i, k) a *partitioned band* and denote by V_j, for $j \in \{1, \dots, k+1\}$, the segment of X bounded by Σ_{j-1} and Σ_j, where $\Sigma_0 = \partial_- X$ and $\Sigma_{k+1} = \partial_+ X$.

Definition 3.2. A smooth function $\varphi : [a, b] \to \mathbb{R}_+$ is called log-*concave* if

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \log(\varphi(t)) = \left(\frac{\varphi'(t)}{\varphi(t)}\right)' \leqslant 0$$

for all $t \in [a, b]$. If the inequality is strict, then φ is called *strictly* log-*concave*.

Definition 3.3. Let (N, g_N) be a closed scalar flat Riemannian manifold. A warped product

$$(M, g_\varphi) = (N \times [a, b], \varphi^2(t) g_N + \mathrm{d}t^2)$$

is called a *model space* if $\mathrm{scal}(M, g_\varphi)$ is constant and φ is strictly log-concave.

Remark 3.4. This notion of a model space arises in the scalar and mean curvature comparison theory surrounding Conjecture 1.3. In [3, 25], a compact Riemannian band (X, g) is compared, in scalar curvature, mean curvature and width, to a warped product (M, g_φ) over an arbitrary scalar flat manifold (N, g_N) with strictly log-concave warping function. It turns out that, if X has Property A, $\mathrm{scal}(X, g) \geqslant \mathrm{scal}(M, g_\varphi)$ and $\mathrm{H}(\partial X, g) \geqslant \mathrm{H}(\partial M, g_\varphi)$, then $\mathrm{width}(X, g) \leqslant \mathrm{width}(M, g_\varphi)$. Furthermore, Cecchini and Zeidler [3, Theorem 8.3, Theorem 9.1] showed that, under further topological assumptions on X, equality of widths can only be achieved if (X, g) itself is isometric to such a warped product. In this section, we adapt these ideas and compare a compact partitioned Riemannian band with Property A to a sequence of model spaces, whose scalar curvatures may differ, but whose mean curvatures fit together pairwise.

Let (N, h) be a compact Riemannian manifold with boundary and let $\partial_j N$ be a connected component of the boundary of N. Then we denote by $\mathrm{H}(\partial_j N, h)$ the mean curvature of $\partial_j N$ with respect to the interior unit normal field. According to our convention, the boundary of the unit disk in $\mathbb{R}^3$ has mean curvature 2. In the following theorem, we give estimates on the widths of segments inside a partitioned band based on scalar and mean curvature.

Theorem 3.5. *Let $n \leqslant 7$ and let (X, Σ_i, k) be an orientable partitioned n-dimensional band with Property A. For $j \in \{1, \dots, k+1\}$, let (M_j, g_{φ_j}) be model spaces over a fixed closed scalar flat Riemannian manifold (N, g). If*

- ▷ $\mathrm{scal}(V_j, g) \geqslant \mathrm{scal}(M_j, g_{\varphi_j})$ *for all* $j \in \{1, \dots, k+1\}$,
- ▷ $\mathrm{H}(\partial_- X, g) \geqslant \mathrm{H}(\partial_- M_1, g_{\varphi_1})$ *and* $\mathrm{H}(\partial_+ X, g) \geqslant \mathrm{H}(\partial_+ M_{k+1}, g_{\varphi_{k+1}})$,
- ▷ $\mathrm{H}(\partial_+ M_j, g_{\varphi_j}) = -\mathrm{H}(\partial_- M_{j+1}, g_{\varphi_{j+1}})$ *for all* $j \in \{1, \dots, k\}$,

then $\mathrm{width}(V_j, g) \leqslant \mathrm{width}(M_j, g_{\varphi_j})$ *for at least one* $j \in \{1, \dots, k+1\}$.

The following result is more or less a direct application of Theorem 3.5.

Corollary 3.6. *Let $n \leqslant 7$ and let $(X, \Sigma_i, 2)$ be an orientable partitioned n-dimensional band with Property A. Let g be a metric on X and let $\varkappa$ be a positive constant. If*

▷ $\mathrm{scal}(V_2, g) \geqslant \varkappa n(n-1)$,
▷ $\mathrm{scal}(X, g) \geqslant 0$,

then $\min\{\mathrm{width}(V_1, g), \mathrm{width}(V_3, g)\} < \ell = \frac{2}{\sqrt{\varkappa n}} \cot(\frac{\sqrt{\varkappa n}d}{4})$, *where* $d := \mathrm{width}(V_2, g) < \frac{2\pi}{\sqrt{\varkappa n}}$.

If, instead of $\mathrm{scal}(X, g) \geqslant 0$, one assumes that the scalar curvature of the partitioned band is bounded from below by a negative constant, Theorem 3.5 provides the following estimate, which is very much in the same spirit of Corollary 3.6 and should be compared with Zeidler's result [36, Theorem 1.4] in the spin setting.

Corollary 3.7. *Let $n \leqslant 7$ and let $(X, \Sigma_i, 2)$ be an orientable partitioned n-dimensional band with Property A. Let g be a metric on X and let $\varkappa$ be a positive constant. If*

▷ $\mathrm{scal}(V_2, g) \geqslant \varkappa n(n-1)$,
▷ $\mathrm{scal}(X, g) \geqslant -\sigma > -\varkappa n(n-1)\tan(\frac{\sqrt{\varkappa n}d}{4})^2$, *where* $d := \mathrm{width}(V_2, g) < \frac{2\pi}{\sqrt{\varkappa n}}$,

then $\min\{\mathrm{width}(V_1, g), \mathrm{width}(V_3, g)\} < \ell$, *where ℓ is such that*

$$\sqrt{\varkappa}(n-1)\tan\left(\frac{\sqrt{\varkappa n}d}{4}\right) = \sqrt{\frac{\sigma(n-1)}{n}}\coth\left(\frac{\sqrt{\sigma n}\ell}{2\sqrt{n-1}}\right).$$

We postpone the proofs of Theorem 3.5 and Corollaries 3.6 and 3.7 to Section 6.

4 | OBSTRUCTIONS ON OPEN BANDS

We want to use Corollary 3.6 or Corollary 3.7 to attack Conjecture 1.1. If $X = Y \times \mathbb{R}$ and g is a complete metric on X, we consider the compact segment $Y \times [-C, C]$ for any $C > 1$, which is partitioned into the bands $Y \times [-C, -1]$, $Y \times [-1, 1]$ and $Y \times [1, C]$. If the scalar curvature of (X, g) is assumed to be positive and $Y \times [-C, C]$ has Property A, the minimum of the widths of $(Y \times [-C, -1], g)$ and $(Y \times [1, C], g)$ is bounded from above in terms of $\mathrm{width}(Y \times [-1, 1], g)$ and the infimum of $\mathrm{scal}(Y \times [-1, 1], g)$. For $C > 1$ large enough, this produces a contradiction.

In this section, we formulate the most general result that one can prove in this manner based on our notion of an open band from Definition 2.1. Indeed, Corollary 3.6 or Corollary 3.7 will yield obstructions to the existence of a complete metric of positive scalar curvature on an open band with Property A. Moreover, since any open band M has at least two ends by definition, we can also establish a rigidity result using the Cheeger–Gromoll splitting theorem.

Theorem 4.1. *Let $n \leqslant 7$ and let M be an open n-dimensional band with Property A. If g is a complete metric on M with nonnegative scalar curvature, then (M, g) is isometric to $(Y \times \mathbb{R}, g_Y + dt^2)$, where (Y, g_Y) is a closed Ricci flat manifold.*

Proof. If (M, g) is not Ricci flat, then M admits a complete metric $\hat{g}$ of positive scalar curvature by [18, Theorem B]. Let $\Sigma \subset M$ be a properly separating hypersurface which exists by Lemma 2.3.

There is a $\kappa > 0$ such that $\mathrm{scal}(M, \hat{g}) \geqslant \kappa n(n-1)$ in a neighborhood of Σ with width $d < \frac{2\pi}{\sqrt{\kappa n}}$, which is bounded by two properly separating hypersurfaces Σ_1 and Σ_2 such that $\Sigma_1 \prec \Sigma \prec \Sigma_2$. For every $C > 0$, we can use Lemma 2.3 to find further properly separating hypersurfaces $\Sigma^C_\pm$ such that $\Sigma^C_- \prec \Sigma_1 \prec \Sigma_2 \prec \Sigma^C_+$ and $\mathrm{dist}_{\hat{g}}(\Sigma^C_-, \Sigma_1) \geqslant C \leqslant \mathrm{dist}_{\hat{g}}(\Sigma_2, \Sigma^C_+)$. Let X^C be the compact band bounded by $\Sigma^C_\pm$. Then for C large enough, Corollary 3.6 or Corollary 3.7 applied to X^C yields a contradiction. Hence, (X, g) is Ricci flat.

Since M is an open band, it is disconnected at infinity and so admits a geodesic line (see, e.g., [24, Lemma 7.3.1]). Thus, by the Cheeger–Gromoll splitting theorem (see, e.g., [24, Theorem 7.3.5]), (M, g) is isometric to $(Y \times \mathbb{R}, g_Y + dt^2)$, where (Y, g_Y) is a Ricci flat manifold. Furthermore, M has more than one end by the definition of an open band, and so, Y must be compact. □

Using a similar (and somewhat simpler argument), we also obtain the following statement:

Proposition 4.2. *Let $n \leqslant 7$ and let M be an open n-dimensional band with Property A. Then M does not admit a complete metric which has uniformly positive scalar curvature outside a compact subset.*

Proof. Assume, by contradiction, that M admits a complete metric g which has scalar curvature $\mathrm{scal}_g \geqslant \kappa n(n-1) > 0$ on $M \setminus K$ for some compact subset $K \subset M$ and some $\kappa > 0$. Then, applying Lemma 2.3 twice, we can find a compact band $X \subset M \setminus K$ which is bounded by two properly separating hypersurfaces $\partial_- X \prec \partial_+ X \subset M$ such that $\mathrm{width}(X, g) > \frac{2\pi}{\sqrt{\kappa n}}$. By construction, we also have $\mathrm{scal}_g \geqslant \kappa n(n-1) > 0$ on X. But this contradicts Corollary 3.7 or even the usual band width estimate on compact bands [15, Section 3.6][25] of dimension $\leqslant 7$. □

In the rest of this section, we investigate topological conditions which imply the existence of open bands with Property A and use them together with Theorem 4.1 to prove the main1.5, 1.9. Let us start with the notion of an incompressible hypersurface.

Definition 4.3. Let X be a connected manifold and let $\Sigma \subset X$ be a connected hypersurface. We say that Σ is incompressible if the induced map $\iota_* : \pi_1(\Sigma) \to \pi_1(X)$ is injective.

We now provide topological conditions such that if an incompressible hypersurface does not admit psc, then the manifold is covered by an open band with Property A: see Proposition 4.6. The arguments will be based on positive scalar curvature bordism and surgery techniques that usually require a case distinction depending on the presence of a spin structure. To treat this in a unified way, we use the language of tangential structures, for details, see, for example, [10, §5]. We briefly recall that an n-dimensional tangential structure is given by a fibration $\theta : B \to \mathrm{BO}(n)$ and a θ-structure on an n-manifold M is a lift $M \to B$ of the classifying map of the tangent bundle $M \to \mathrm{BO}(n)$ along $\theta : B \to \mathrm{BO}(n)$. Then, given an n-dimensional tangential structure $\theta : B \to \mathrm{BO}(n)$, one may define a notion of θ-cobordism for $(n-1)$-dimensional θ-manifolds† and form the corresponding cobordism group which we denote by Ω^θ_{n-1}. Now the positive scalar curvature surgery principle can be phrased as follows: Let $\theta : B \to \mathrm{BO}(n)$ be an n-dimensional tangential structure and fix an $(n-1)$-dimensional θ-manifold N such that the map $N \to B$ is 2-connected.

† To be precise, here one implicitly considers the $(n-1)$-dimensional tangential structure obtained from θ via the pullback along $\mathrm{BO}(n-1) \to \mathrm{BO}(n)$.

If N is θ-cobordant to a θ-manifold that admits a metric of psc, then N itself already admits psc, see [8, Theorem 1.5].

We now connect this back to the study of open bands and first observe that an open band endowed with a θ-structure induces a well-defined θ-cobordism class of $(n-1)$-manifolds.

Lemma 4.4. *Let $\theta : B \to \mathrm{BO}(n)$ be a tangential structure and let M be an open band endowed with a θ-structure. Then any two properly separating hypersurfaces $\Sigma_1, \Sigma_2 \subset M$ are θ-cobordant.*

Proof. First consider the case that $\Sigma_1 \prec \Sigma_2$. Then the band in M bounded in between Σ_1 and Σ_2 is a cobordism witnessing that Σ_1 and Σ_2 are θ-cobordant. In general, Lemma 2.3 implies the existence of a properly separating hypersurface $\Sigma' \subset M$ such that $\Sigma_1 \prec \Sigma'$ and $\Sigma_2 \prec \Sigma'$. Thus, by the first case, both Σ_1 and Σ_2 are θ-cobordant to Σ' which proves the desired statement. □

Next we use the psc surgery principle as stated above to derive a sufficient condition for the existence of open bands with Property A.

Lemma 4.5. *Let $n \geqslant 6$ and let $\theta : B \to \mathrm{BO}(n)$ be a tangential structure. Let M be a connected θ-manifold without boundary and let $Y \subset M$ be a closed connected two-sided hypersurface. Assume that Y does not support a psc metric and that the composition $Y \subset M \to B$ induces an injection $\pi_1 Y \hookrightarrow \pi_1 B$ and a surjection $\pi_2 Y \twoheadrightarrow \pi_2 B$. Then the connected covering $\hat{M}$ of M with $\pi_1 \hat{M} = \pi_1 Y$ is an open band with Property A.*

Proof. First, let $\hat{B} \to B$ be the covering with $\pi_1 \hat{B} = \pi_1 Y$. Then $\hat{M} \to M$ is the pullback of the covering $\hat{B} \to B$ along $\theta : M \to B$. The embedding $Y \subset M$ lifts to an embedding $Y \subset \hat{M}$ as a hypersurface that separates $\hat{M}$ into two components (otherwise there existed a loop in $\hat{M}$ intersecting Y transversally in precisely one point which would contradict $\pi_1 Y \cong \pi_1 \hat{M}$). Moreover, we obtain a $\hat{\theta}$-structure on $\hat{M}$, where $\hat{\theta} : \hat{B} \to B \xrightarrow{\theta} \mathrm{BO}(n)$.

$$\begin{tikzcd}
 & \hat{M} \arrow[r] \arrow[d] & \hat{B} \arrow[d] \arrow[dr, "\hat{\theta}"] & \\
Y \arrow[ur, hook] \arrow[r, hook] & M \arrow[r] & B \arrow[r, "\theta"] & \mathrm{BO}(n)
\end{tikzcd}$$

By assumption, the composition $Y \hookrightarrow \hat{M} \to \hat{B}$ is 2-connected. Let $U_\pm$ be the two components of $\hat{M} \setminus Y$. We observe that $U_\pm$ are both noncompact because otherwise Y would be $\hat{\theta}$-nullbordant and thus support a psc metric by [8, Theorem 1.5]. We can now define the ends of U_- to be $\mathcal{E}_- \hat{M}$ and the ends of U_+ to be $\mathcal{E}_+ \hat{M}$ in order to turn $\hat{M}$ into an open band. Then, by construction, the open band $\hat{M}$ contains Y as a properly separating hypersurface.

Finally, we need to verify Property A. To this end, assume by contradiction that $\Sigma \subset \hat{M}$ is a separating hypersurface that admits a psc metric. By Lemma 2.2, we may assume without loss of generality that Σ is properly separating. Then it follows from Lemma 4.4 that Σ and Y are $\hat{\theta}$-cobordant. But since the map $Y \to \hat{B}$ is 2-connected and Σ admits a psc metric, [8, Theorem 1.5] implies that Y also admits a psc metric, a contradiction. □

To describe the possible concrete applications of Lemma 4.5, it turns out to be enough to go through the possible tangential 2-types of M. Recall that for an arbitrary n-manifold, its tangential

2-type is the (unique up to homotopy) tangential structure θ_M which factors the tangent bundle as $M \to B_M \xrightarrow{\theta_M} \mathrm{BO}(n)$, where $M \to B_M$ is 2-connected and $\theta_M : B_M \to \mathrm{BO}(n)$ is 2-co-connected. In other words, B_M is the second stage of the Moore–Postnikov tower for the map $M \to \mathrm{BO}(n)$. Now a simple diagram chase shows that if the hypotheses of Lemma 4.5 are satisfied for some tangential structure θ on M, then they are already satisfied for $\theta = \theta_M$. Moreover, the tangential 2-type can be described algebraically in terms of the fundamental group $\pi = \pi_1 M$, the first Stiefel–Whitney class $w : \pi \to \mathbb{Z}/2$, and an extension $\hat{\pi} \twoheadrightarrow \pi$ determined by the second Stiefel–Whitney class whose kernel has order at most 2, see [33, §2] for details. However, for our purposes, we do not need such a full description, and we only need to distinguish two cases depending on whether $\pi_2 B_M \cong \mathbb{Z}/2$ or $\pi_2 B_M = 0$ as the following proposition demonstrates.

Proposition 4.6. *Let $n \geqslant 6$. Let M be a connected n-dimensional manifold without boundary and $Y \subset M$ a closed two-sided incompressible hypersurface that does not admit a psc metric. Suppose that one of the following conditions holds:*

(a) *M is almost spin.*
(b) *Y is totally nonspin.*

Then there exists a covering $\hat{M} \to M$ which is an open band with Property A.

Proof. In light of the discussion in the previous paragraph, we need to check that in either case we can apply Lemma 4.5 for the tangential structure $\theta = \theta_M$ given by its 2-type.

(a) If M is almost spin, that is, the universal covering $\tilde{M}$ of M is spin, then the map $M \to \mathrm{BO}(n)$ classifying the tangent bundle induces the zero map $\pi_2 M \to \pi_2 \mathrm{BO}(n) = \mathbb{Z}/2$ because this map can be identified with the second Stiefel–Whitney class of $\tilde{M}$ via the Hurewicz isomorphism $\mathrm{H}_2(\tilde{M}) \cong \pi_2(\tilde{M}) \cong \pi_2(M)$. Thus, it follows that $\pi_2(B_M) = 0$ and so Lemma 4.5 applies to every incompressible hypersurface $Y \subset M$ which does not admit psc.
(b) On the other hand, if M is totally nonspin, that is, $\tilde{M}$ is nonspin, then by an analogous consideration involving the second Stiefel–Whitney class of $\tilde{M}$, we necessarily have $\pi_2 B_M \cong \mathbb{Z}/2$ and $\pi_2 M \to \pi_2 B_M = \mathbb{Z}/2$ is surjective. Now in this situation, the condition on the hypersurface $Y \subset M$ in Lemma 4.5 is that it is incompressible, itself totally nonspin, and does not admit psc. □

We are now ready to deduce Theorem 1.5.

Proof of Theorem 1.5. By Proposition 4.6, there exists a covering $\hat{M}$ which is an open band with Property A. Now suppose that M admits a complete metric of nonnegative scalar curvature g. Let $\hat{g}$ denote its lift to $\hat{M}$. Then Theorem 4.1 implies that $(\hat{M}, \hat{g})$ must be isometric to $(N \times \mathbb{R}, g_N + \mathrm{d}t^2)$ for a closed Ricci flat manifold (N, g_N). □

We now turn to Theorem 1.9. Note that a proof of Theorem 1.9 has already appeared recently in [5, Theorem 1.1], but we give a separate argument here because it also fits directly into our topological setup. In fact, it is simpler than Theorem 1.5 as it does not need surgery arguments and instead only relies on Lemma 4.4 together with some homological considerations.

Lemma 4.7. *Let M be an oriented open band and $Y \subseteq M$ a properly separating hypersurface together with a map $\phi : M \to Y_0$ to an NPSC$^+$ manifold Y_0 such that the restriction $\phi|_Y : Y \to Y_0$ has nonzero degree. Then M has Property A.*

Proof. We define a tangential structure $M \xrightarrow{l} B \xrightarrow{\theta} \mathrm{BO}(n)$, where we set $B = Y_0 \times \mathrm{BSO}(n)$ and l is induced by the map $M \to Y_0$ together with the orientation of M. Note that a θ-structure on an $(n-1)$-manifold N is the same as an orientation on N together with a map $N \to Y_0$, and thus, $\Omega^{\theta}_{n-1} = \Omega^{\mathrm{SO}}_{n-1}(Y_0)$. In this picture, the degree of the map $N \to Y_0$ can be read off from the transformation $\Omega^{\mathrm{SO}}_{n-1}(Y_0) \to \mathrm{H}_{n-1}(Y_0; \mathbb{Z}) \cong \mathbb{Z}$. Suppose that $\Sigma \subset M$ is a separating hypersurface, and assume without loss of generality that it is properly separating. Then, by Lemma 4.4, $\Sigma \to Y_0$ and $Y \to Y_0$ represent the same class of $\Omega^{\mathrm{SO}}_{n-1}(Y_0)$. In particular, $\deg(\Sigma \to Y_0) = \deg(Y \to Y_0) \neq 0$. Since Y_0 is NPSC$^+$, this proves that Σ does not admit a psc metric and so M must have Property A. □

Proposition 4.8. *Let M be an oriented connected n-dimensional manifold and let $\iota\colon Y \hookrightarrow M$ be a two-sided closed connected hypersurface that admits a map of nonzero degree $\phi\colon Y \to Y_0$ to an aspherical* NPSC$^+$ *manifold Y_0 and such that* $\ker(\pi_1 Y \xrightarrow{\iota_*} \pi_1 M) \subseteq \ker(\pi_1 Y \xrightarrow{\phi_*} \pi_1 Y_0)$. *Then there exists a connected covering $\hat{M} \to M$ that is an open band with Property A.*

Proof. Let $\Lambda := \iota_*(\pi_1 Y) \subseteq \pi_1 M$ and let $\hat{M} \to M$ be the covering with $\pi_1 \hat{M} = \Lambda$. Then $Y \hookrightarrow M$ lifts to an embedding $Y \hookrightarrow \hat{M}$ which induces a surjection $\pi_1 Y \twoheadrightarrow \Lambda$. By the assumption on the kernels of the induced maps on fundamental groups, it follows that the homomorphism $\phi_*\colon \pi_1 Y \to \pi_1 Y_0$ factors as a composition $\pi_1 Y \twoheadrightarrow \Lambda \to \pi_1 Y_0$. Since Y_0 is aspherical and $\pi_1 \hat{M} = \Lambda$, this implies that the map $\phi\colon Y \to Y_0$ extends to a map $\hat{M} \to Y_0$. As in the proof of Lemma 4.5, let $U_\pm$ be the two components of $\hat{M} \setminus Y$. Then $U_\pm$ must be noncompact because otherwise $[Y \to Y_0] = 0 \in \Omega^{\mathrm{SO}}_{n-1}(Y_0)$ which would contradict the hypothesis $\deg(Y \to Y_0) \neq 0$ (compare the proof of Lemma 4.7 above). Thus, $\hat{M}$ can be turned into an open band such that Y is a properly separating hypersurface. Thus, the proposition follows from Lemma 4.7. □

Proof of Theorem 1.9. Combine Proposition 4.8 and Theorem 4.1 analogously as in the proof of Theorem 1.5 above. □

5 | THE CODIMENSION 2 OBSTRUCTION

In this section, we prove our codimension 2 obstruction results. These are based on a reduction to a codimension 1 problem essentially following original ideas of Gromov and Lawson [13, Theorem 7.5] and their adaptation by Hanke, Pape, and Schick [17].

Lemma 5.1 (cf. [17, Theorem 4.3][34, Lemma 4.1.4]). *Let X be a connected manifold without boundary. Let $Y \subset X$ be a submanifold without boundary of codimension 2 whose normal bundle is trivial and suppose that the pair (X, Y) is 2-connected. Consider the manifold $W := X \setminus \mathcal{U}$ obtained by deleting a small open tubular neighborhood $Y \times \mathrm{B}^2 \cong \mathcal{U} \subseteq X$. Then the map $\partial W \cong Y \times \mathrm{S}^1 \xrightarrow{\mathrm{pr}_2} \mathrm{S}^1$ given by projection onto the second factor extends to a continuous map $W \to \mathrm{S}^1$. In particular, the map $\pi_1 Y \times \mathbb{Z} = \pi_1(\partial W) \to \pi_1 W$ induced by the inclusion $\partial W \hookrightarrow W$ is split-injective.*

Proof. It suffices to show that the induced homomorphism $(\mathrm{pr}_2)_*\colon \pi_1(\partial W) \to \pi_1(\mathrm{S}^1) = \mathbb{Z}$ extends to a homomorphism $\pi_1(W) \to \mathbb{Z}$. The hypotheses imply that the pair $(X, \mathcal{U})$ is also 2-connected, and so, excision and the Hurewicz theorem show that $\mathrm{H}_k(W, \partial W) \cong \mathrm{H}_k(X, \mathcal{U}) = 0$ for $0 \leqslant k \leqslant 2$. In particular, the map $\mathrm{H}_1(\partial W) \xrightarrow{\cong} \mathrm{H}_1(W)$ induced by the inclusion $\partial W \hookrightarrow W$ is an isomorphism. The existence of the desired extension now follows from the following diagram because the map $(\mathrm{pr}_2)_* : \pi_1(\partial W) \to \pi_1(\mathrm{S}^1) = \mathbb{Z}$ factors through the Hurewicz homomorphism.

$$\begin{tikzcd}
\pi_1(\partial W) \arrow[r, "\mathrm{hur}"] \arrow[d] & \mathrm{H}_1(\partial W) \arrow[r, "(\mathrm{pr}_2)_*"] \arrow[d, "\cong"] & \mathrm{H}_1(\mathrm{S}^1) \arrow[r, equal, "\mathrm{hur}"] & \pi_1(\mathrm{S}^1) = \mathbb{Z} \\
\pi_1(W) \arrow[r, "\mathrm{hur}"] & \mathrm{H}_1(W) \arrow[ur, dashed] & &
\end{tikzcd}$$

This also implies that the map $\pi_1 Y \times \mathbb{Z} = \pi_1(\partial W) \to \pi_1 W$ is split injective because a retraction can be constructed using the map $\pi_1 W \to \mathbb{Z}$ from the previous paragraph together with the map $\pi_1 W \to \pi_1 X \cong \pi_1 Y$ induced by the inclusion $W \hookrightarrow X$. □

Proposition 5.2. *Let $n \geqslant 6$. Let X be a connected n-dimensional manifold and let $Y \subset X$ be a closed connected submanifold of codimension 2 with trivial normal bundle such that the pair (X, Y) is 2-connected. Consider the manifold $W := X \setminus \mathcal{U}$ obtained by deleting a small open tubular neighborhood $Y \times \mathrm{B}^2 \cong \mathcal{U} \subseteq X$. If $Y \times \mathrm{S}^1$ does not admit a metric of positive scalar curvature, then the double $\mathrm{d}W$ of W is an open band with Property A.*

Proof. We start with considering the tangential 2-type $X \to B_X \to \mathrm{BO}(n)$ of X. By restriction to $W \subset X$, this induces a tangential structure $l' : W \to B_X$. Let $p : W \to \mathrm{S}^1$ be a map extending the projection $\partial W \cong Y \times \mathrm{S}^1 \to \mathrm{S}^1$ which exists by Lemma 5.1. We obtain a new tangential structure $W \xrightarrow{l} B := (B_X \times \mathrm{S}^1) \xrightarrow{\theta} \mathrm{BO}(n)$, where $l = (l', p)$, and θ is defined as projection to B_X followed by $B_X \to \mathrm{BO}(n)$. Furthermore, this tangential structure l extends to the double $\mathrm{d}W$ by reflection. Its restriction to the hypersurface $\mathrm{d}W \supset \partial W \cong Y \times \mathrm{S}^1$ is a 2-connected map $Y \times \mathrm{S}^1 \to B = B_X \times \mathrm{S}^1$ because by construction, it is homotopic to $l_Y \times \mathrm{id}_{\mathrm{S}^1}$, where l_Y denotes the restriction of $X \to B_X$ to $Y \subset X$ and l_Y is 2-connected. Thus, Lemma 4.5 (applied to the trivial covering) proves the desired conclusion. □

Proof of Theorem 1.11. Note that since Y does not admit positive scalar curvature and $\dim(Y) = 5$, Theorem 1.5 implies that $Y \times \mathrm{S}^1$ does not admit positive scalar curvature either. Then we let X be the connected covering of M with $\pi_1 X = \pi_1 Y$. It follows $Y \hookrightarrow X$ and the pair (X, Y) satisfies the hypotheses of Proposition 5.2. Thus, if we let W be as in the statement of Proposition 5.2, then its double $\mathrm{d}W$ is an open band with Property A. Now assume by contradiction that M admits a complete metric of uniformly positive scalar curvature. Then by first lifting it to X and restricting it to W, we obtain a complete metric of uniformly positive scalar curvature on W. After changing this metric in a compact neighborhood of ∂W, we obtain a smooth complete metric on $\mathrm{d}W$ that has uniformly positive scalar curvature outside a compact subset, a contradiction to Proposition 4.2. □

Proof of Theorem 1.12. We first verify that $Y_0 \times \mathrm{S}^1$ is also NPSC^+. To this end, let N be an oriented $(n-1)$-manifold and $\phi : N \to Y_0 \times \mathrm{S}^1$ a map of nonzero degree, where we assume N to be connected without loss of generality. Then let $Z = \phi^{-1}(Y_0 \times \{*\})$ be a transversal preimage. It follows that $\iota : Z \hookrightarrow N$ and $\phi : Z \to Y_0 \times \{*\} = Y_0$ satisfies the hypotheses of Theorem 1.9, and so, N does not admit a psc metric. This proves that $Y_0 \times \mathrm{S}^1$ is NPSC^+.

To prove the theorem, we again consider the connected covering of $X \to M$ with $\pi_1 X = \pi_1 Y$, and we let W be as in the statement of Proposition 5.2. By Lemma 5.1, the map $\pi_1 Y \times \mathbb{Z} =$

$\pi_1(\partial W) \to \pi_1 W$ induced by the inclusion $Y \times S^1 = \partial W \hookrightarrow W$ is injective and admits a retraction $r : \pi_1 W \twoheadrightarrow \pi_1 Y \times \mathbb{Z}$. Since $Y_0 \times S^1$ is aspherical, this implies that the map $Y \times S^1 \to Y_0 \times S^1$ extends to a map $W \to Y_0 \times S^1$ and subsequently to a map $dW \to Y_0 \times S^1$ on the double. In summary, dW is an open band that contains $Y \times S^1$ as a properly separating hypersurface and it satisfies the hypotheses of Lemma 4.7 because $Y_0 \times S^1$ is NPSC$^+$. Thus, dW has Property A and the theorem now follows again from Proposition 4.2 as in the proof of Theorem 1.11 above. □

6 | PROOF OF THE PARTITIONED COMPARISON PRINCIPLE

We will prove Theorem 3.5 by contradiction. We stress that all bands considered in this section are compact. Under the assumption that $\mathrm{width}(V_j, g) > \mathrm{width}(M_j, g_{\varphi_j})$ for all $j \in \{1, \dots, k+1\}$, we will produce a closed embedded hypersurface $\Sigma \subset X$ which separates $\partial_- X$ and $\partial_+ X$ and admits a metric of positive scalar curvature.

This hypersurface Σ will appear as the boundary of a μ-bubble. The key ingredient for the corresponding functional is the potential function $h : X \to \mathbb{R}$. We use the ideas from [25, Section 3] for each band (V_j, g) and model space (M_j, g_{φ_j}) separately to produce $h_j : V_j \to \mathbb{R}$ as the concatenation of a strictly 1-Lipschitz band map $(V_j, g) \to (M_j, g_{\varphi_j})$ and the function $h_{\varphi_j} : M_j \to \mathbb{R}$. Subsequently, we use a gluing construction to paste all of the h_j together to obtain a smooth function $h : X \to \mathbb{R}$ which is suitable for our purposes.

The idea to combine potential functions in this way was already used in [4, 36]. The gluing construction is based on the following result.

Lemma 6.1. *Let $h : [a, b] \to \mathbb{R}$ be a strictly monotonously decreasing smooth function such that*

$$-\frac{n}{n-1} h^2 - 2h' = \sigma,$$

for some constant $\sigma \in \mathbb{R}$. Then, for every sufficiently small $\varepsilon > 0$, there exists a function $\hat{h} : [a - \varepsilon, b + \varepsilon] \to \mathbb{R}$ such that:

- ▷ *$\hat{h}(t) = h(t)$ for $t \in [a + \varepsilon, b - \varepsilon]$,*
- ▷ *$\hat{h}(t) = h(a)$ in a neighborhood of $a - \varepsilon$ and $\hat{h}(t) = h(b)$ in a neighborhood of $b + \varepsilon$,*
- ▷ *$\hat{h}' \leqslant 0$,*
- ▷ *$-\frac{n}{n-1}\hat{h}^2 - 2\hat{h}' \leqslant \sigma$ and $-\frac{n}{n-1}\hat{h}^2(t) - 2\hat{h}'(t) < \sigma$ if $\hat{h}'(t) = 0$.*

Proof. Let $\rho : \mathbb{R} \to [a, b]$ be a smooth function with:

- ▷ $\rho(t) = a$ for $t \in (-\infty, a - \frac{\varepsilon}{2}]$, $\rho(t) = t$ for $t \in [a + \frac{\varepsilon}{2}, b - \frac{\varepsilon}{2}]$ and $\rho(t) = b$ for $t \in [b + \frac{\varepsilon}{2}, \infty)$.
- ▷ $0 < \rho'(t) < 1$ for $t \in (a - \frac{\varepsilon}{2}, a + \frac{\varepsilon}{2})$ and $t \in (b - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2})$.

Then the function $\hat{h} : [a - \varepsilon, b + \varepsilon] \to \mathbb{R}$ defined by $\hat{h} = h \circ \rho$ has all of the desired properties. The first two are immediate from the definition. The third one holds since $\hat{h}'(t) = h'(\rho(t))\rho'(t)$ and $h' < 0$ while $\rho' \geqslant 0$. To check the last property, we point out that

$$-\frac{n}{n-1}\hat{h}^2(t) - 2\hat{h}'(t) = -\frac{n}{n-1} h(\rho(t)) - 2h'(\rho(t))\rho'(t) = \sigma + 2h'(\rho(t))(1 - \rho'(t)).$$

Since $h'(\rho(t)) < 0$ and $0 \leqslant \rho' \leqslant 1$, the above is always $\leqslant \sigma$ and it is $< \sigma$ if $\rho'(t) < 1$. This holds true in particular when $\hat{h}'(t) = 0$, that is, $\rho'(t) = 0$. □

In order to construct our functions $h_j : V_j \to \mathbb{R}$, we make use of the following basic existence result of band maps; for the proof, we refer to [38, Lemma 4.1][3, Lemma 7.2].

Lemma 6.2. *Let (V, g) be a Riemannian band and let $a < b$ be two real numbers. If* $\operatorname{width}(V, g) > b - a$*, then there exists a smooth function $\beta : V \to [a, b]$ with $\beta(\partial_- V) = a$, $\beta(\partial_+ V) = b$ and* $\operatorname{Lip}(\beta) < 1$.

Next, for a Riemannian band (X, g), we give conditions on the scalar curvature, the mean curvature of ∂X, and on the potential function h so that the μ-bubble associated to h produces a separating closed hypersurface admitting a metric of positive scalar curvature. For more details on μ-bubbles, we refer the reader to [15, 38, Section 5].

Proposition 6.3. *Let $n \leqslant 7$ and let (X, g) be an n-dimensional oriented Riemannian band. Let $h : X \to \mathbb{R}$ be a smooth function with the property that*

$$\operatorname{scal}(X, g) + \frac{n}{n-1} h^2 - 2|\nabla h| > 0. \tag{6.1}$$

Furthermore, suppose that the mean curvature satisfies

$$\mathrm{H}(\partial_\pm X, g) > \pm h|_{\partial_\pm X}. \tag{6.2}$$

Then there exist a closed embedded hypersurface Σ that separates $\partial_- X$ and $\partial_+ X$ and a constant $b > 0$ such that

$$\int_\Sigma \left(|\nabla_\Sigma \psi|^2 + \frac{1}{2} \operatorname{scal}(\Sigma, g) \psi^2 \right) d\mathrm{vol}_\Sigma \geqslant b \int_\Sigma \psi^2 \, d\mathrm{vol}_\Sigma, \qquad \forall \psi \in C^\infty(\Sigma). \tag{6.3}$$

Proof. Denote by $C(X)$ the set of all Caccioppoli sets in X which contain an open neighborhood of $\partial_- X$ and are disjoint from $\partial_+ X$. For $\hat{\Omega} \in C(X)$, consider the functional

$$\mathcal{A}_h(\hat{\Omega}) = \mathcal{H}^{n-1}(\partial^* \hat{\Omega} \cap \mathring{X}) - \int_{\hat{\Omega}} h \, d\mathcal{H}^n,$$

where $\partial^* \hat{\Omega}$ is the reduced boundary [11, Chapters 3,4] of $\hat{\Omega}$. By Condition (6.2) and [25, Lemma 4.2], there exists a smooth *μ-bubble* $\Omega \in C(X)$, that is, a smooth Caccioppoli set with

$$\mathcal{A}_h(\Omega) = \mathcal{I} := \inf\{\mathcal{A}_h(\hat{\Omega}) | \hat{\Omega} \in C(X)\}.$$

Then $\Sigma := \partial\Omega \cap \mathring{X}$ is a closed embedded hypersurface that separates $\partial_- X$ and $\partial_+ X$. Let ν be the outward pointing unit normal vector field to Σ. By the first variation formula for $\mathcal{A}_h$ (see [25, Lemma 4.3]), the mean curvature of Σ (computed with respect to $-\nu$) is equal to $h|_\Sigma$. By stability, from the second variation formula (see [25, Lemma 4.4]), we deduce

$$\begin{aligned}\int_\Sigma (2|\nabla_\Sigma \psi|^2 + \operatorname{scal}(\Sigma, g)\psi^2) \, d\mathrm{vol}_\Sigma &\geqslant \int_\Sigma \left(\operatorname{scal}(X, g) + \frac{n}{n-1} h^2 + 2g(\nabla_X h, \nu) \right) \psi^2 \, d\mathrm{vol}_\Sigma \\ &\geqslant \min_\Sigma \left\{ \operatorname{scal}(X, g) + \frac{n}{n-1} h^2 - 2|\nabla h| \right\} \int_\Sigma \psi^2 \, d\mathrm{vol}_\Sigma\end{aligned}$$

for all $\psi \in C^\infty(\Sigma)$. By Condition (6.1), the previous inequality yields a constant $b > 0$ such that Inequality (6.3) holds. □

We have gathered all the ingredients we need to prove Theorem 3.5.

Proof of Theorem 3.5. Assume, by contradiction, that $\mathrm{width}(V_j, g) > \mathrm{width}(M_j, g_{\varphi_j})$ for all $j \in \{1, \dots, k+1\}$. Consider the functions

$$h_{\varphi_j}(t) = (n-1)\frac{\varphi_j'(t)}{\varphi_j(t)} : [a_j, b_j] \to \mathbb{R}.$$

Since the φ_j are strictly log-concave, the functions h_{φ_j} are strictly monotonously decreasing and the scalar curvature of (M_j, g_{φ_j}) is given by

$$\sigma_j = \mathrm{scal}(M_j, g_{\varphi_j}) = -\frac{n}{n-1}h_{\varphi_j}^2 - 2h_{\varphi_j}'. \tag{6.4}$$

For $j \in \{2, \dots, k\}$, we apply Lemma 6.1 to h_{φ_j} and obtain smooth functions

$$\hat{h}_{\varphi_j} : [a_j - \varepsilon, b_j + \varepsilon] \to \mathbb{R}$$

with the aforementioned properties.

For $j = 1$, we extend the domain of h_{φ_j} to $[a_1 - \varepsilon, b_1]$ and apply the interpolation procedure of Lemma 6.1 only on the right-hand side of the interval to produce $\hat{h}_{\varphi_1} : [a_1 - \varepsilon, b_1 + \varepsilon] \to \mathbb{R}$. For $j = k+1$, we extend the domain of $h_{\varphi_{k+1}}$ to $[a_{k+1}, b_{k+1} + \varepsilon]$ and apply the interpolation procedure of Lemma 6.1 only on the left-hand side of the interval to produce $\hat{h}_{\varphi_{k+1}} : [a_{k+1} - \varepsilon, b_{k+1} + \varepsilon] \to \mathbb{R}$.

By Lemma 6.2, there are smooth maps

$$\beta_j : V_j \to [a_j - \varepsilon, b_j + \varepsilon]$$

such that $\beta_j(\partial_- V_j) = a_j - \varepsilon$, $\beta_j(\partial_+ V_j) = b_j + \varepsilon$ and $\mathrm{Lip}(\beta_j) < 1$. Define $h : X \to \mathbb{R}$ by $h(x) = \hat{h}_{\varphi_j} \circ \beta_j(x)$ if $x \in V_j$. The function h is continuous since

$$\hat{h}_{\varphi_j}(b_j + \varepsilon) = \mathrm{H}(\partial_+ M_j, g_{\varphi_j}) = -\mathrm{H}(\partial_- M_{j+1}, g_{\varphi_{j+1}}) = \hat{h}_{\varphi_{j+1}}(a_{j+1} - \varepsilon)$$

for all $j \in \{1, \dots, k\}$. It is smooth since the $\hat{h}_{\varphi_j} \circ \phi_j$ are constant in a neighborhood of the separating hypersurfaces Σ_i, which partition the band.

Note that h satisfies Condition (6.1) by the chain rule, the fourth property of the $\hat{h}_{\varphi_j}$ from Lemma 6.1, and since $\mathrm{Lip}(\beta_j) < 1$. For the mean curvature of the boundary, the following holds true:

$$\mathrm{H}(\partial_- X, g) \geqslant \mathrm{H}(\partial_- M_1, g_{\varphi_1}) = -\hat{h}_{\varphi_1}(a_1) > -\hat{h}_{\varphi_1}(a_1 - \varepsilon) = -h|_{\partial_- X}$$

and

$$\mathrm{H}(\partial_+ X, g) \geqslant \mathrm{H}(\partial_+ M_{k+1}, g_{\varphi_{k+1}}) = \hat{h}_{\varphi_{k+1}}(b_{k+1}) > \hat{h}_{\varphi_{k+1}}(b_{k+1} + \varepsilon) = h|_{\partial_+ X}.$$

Hence, Condition (6.2) is satisfied as well. By Proposition 6.3, there exist a closed embedded hypersurface Σ which separates $\partial_- X$ and $\partial_+ X$ and a constant $b > 0$ such that Inequality (6.3) holds.

If $n = 2$, this yields an immediate contradiction by choosing $\psi = 1$ in (6.3). If $n = 3$, we again choose $\psi = 1$ and use Gauß-Bonnet to see that Σ admits a psc metric. If $n \geqslant 4$, then Σ admits a metric of positive scalar curvature by the conformal change argument of Schoen and Yau [31]. This contradicts the fact that X has Property A. □

Proof of Corollary 3.6. Consider the function

$$\varphi_2 : \left(-\frac{\pi}{\sqrt{\kappa n}}, \frac{\pi}{\sqrt{\kappa n}}\right) \to \mathbb{R}_+ \qquad t \mapsto \cos\left(\frac{\sqrt{\kappa n} t}{2}\right)^{\frac{2}{n}},$$

which is strictly log-concave and has

$$h_{\varphi_2}(t) = -\sqrt{\kappa}(n-1)\tan\left(\frac{\sqrt{\kappa n} t}{2}\right).$$

Consider the function

$$\varphi_1 : \mathbb{R}_+ \to \mathbb{R}_+ \qquad t \mapsto t^{\frac{2}{n}},$$

which is strictly log-concave and has

$$h_{\varphi_1}(t) : = \frac{2(n-1)}{nt}.$$

Since $h_{\varphi_1}(t) \to \infty$ as $t \to 0$, there is a value $t_- > 0$ such that $H(\partial_- X, g) \geqslant -h_{\varphi_1}(t_-)$. By continuity, there are $\delta_1, \delta_2 > 0$ small enough such that

$$h_{\varphi_2}\left(\frac{-d+\delta_1}{2}\right) = h_{\varphi_1}(\ell + \delta_2),$$

while $\delta_2 < t_-$ and hence $\ell + \delta_2 - t_- < \ell$. Let (N, g_N) be a closed scalar flat Riemannian manifold. We fix the model space

$$(M_1, g_{\varphi_1}) = (N \times [t_-, \ell + \delta_2], \varphi_1^2(t) g_N + dt^2)$$

with scalar curvature equal to zero and width $< \ell$.

Let $\varphi_3 : \mathbb{R}_- \to \mathbb{R}_+$ be defined by $\varphi_3(t) = \varphi_1(-t)$. This function is strictly log-concave and $h_{\varphi_3}(t) = -h_{\varphi_1}(-t)$. Since $h_{\varphi_3}(t) \to -\infty$ as $t \to 0$, there is a value $t_+ < 0$ such that $H(\partial_+ X, g) \geqslant h_{\varphi_3}(t_+)$. Similarly as before, we find $\delta_3, \delta_4 > 0$ such that

$$h_{\varphi_2}\left(\frac{d-\delta_3}{2}\right) = h_{\varphi_3}(-\ell - \delta_4),$$

while $\delta_4 < -t_+$ and hence $\ell + \delta_4 + t_+ < \ell$. We fix the model space

$$(M_3, g_{\varphi_3}) = (N \times [t_-, \ell + \delta_4], \varphi_3^2(t) g_N + dt^2)$$

with scalar curvature equal to zero and width $< \ell$.

Finally, we fix the model space

$$(M_2, g_{\varphi_2}) = \left(N \times \left[\frac{-d+\delta_1}{2}, \frac{d-\delta_3}{2}\right], \varphi_2^2(t) g_N + dt^2\right)$$

with scalar curvature equal to $\kappa n(n-1)$ and width $< d$.

It follows from Theorem 3.5 that $\mathrm{width}(V_j, g) \leqslant \mathrm{width}(M_j, g_{\varphi_j})$ for at least one $i \in \{1,2,3\}$. Since $d = \mathrm{width}(V_2, g) > \mathrm{width}(M_2, g_{\varphi_2})$, we conclude that

$$\min\{\mathrm{width}(V_1, g), \mathrm{width}(V_3, g)\} < \ell,$$

which is what we wanted to prove. □

Proof of Corollary 3.7. Consider the function

$$\varphi_2 : \left(-\frac{\pi}{\sqrt{\kappa n}}, \frac{\pi}{\sqrt{\kappa n}}\right) \to \mathbb{R}_+ \qquad t \mapsto \cos\left(\frac{\sqrt{\kappa n} t}{2}\right)^{\frac{2}{n}},$$

which is strictly log-concave and has

$$h_{\varphi_2}(t) = -\sqrt{\kappa}(n-1)\tan\left(\frac{\sqrt{\kappa n} t}{2}\right).$$

Consider the function

$$\varphi_1 : \mathbb{R}_+ \to \mathbb{R}_+ \qquad t \mapsto \sinh\left(\frac{\sqrt{\sigma n} t}{2\sqrt{n-1}}\right)^{\frac{2}{n}},$$

which is strictly log-concave and has

$$h_{\varphi_1}(t) := \sqrt{\frac{\sigma(n-1)}{n}}\coth\left(\frac{\sqrt{\sigma n} t}{2\sqrt{n-1}}\right).$$

Since $h_{\varphi_1}(t) \to \infty$ as $t \to 0$, there is a value $t_- > 0$ such that the mean curvature $H(\partial\_X, g)$ is greater or equal to $-h_{\varphi_1}(t_-)$. By continuity, there are $\delta_1, \delta_2 > 0$ small enough such that $-\sigma > -\kappa n(n-1)\tan(\frac{\sqrt{\kappa n}(d-2\delta_1)}{4})^2$ and

$$h_{\varphi_2}\left(\frac{-d+\delta_1}{2}\right) = h_{\varphi_1}(\ell + \delta_2),$$

while $\delta_2 < t_-$ and hence $\ell + \delta_2 - t_- < \ell$. Let (N, g_N) be a closed scalar flat Riemannian manifold. We fix the model space

$$(M_1, g_{\varphi_1}) = (N \times [t_-, \ell + \delta_2], \varphi_1^2(t) g_N + dt^2)$$

with scalar curvature equal to $-\sigma$ and width $< \ell$.

Let $\varphi_3 : \mathbb{R}_- \to \mathbb{R}_+$ be defined by $\varphi_3(t) = \varphi_1(-t)$. This function is strictly log-concave and $h_{\varphi_3}(t) = -h_{\varphi_1}(-t)$. Since $h_{\varphi_3}(t) \to -\infty$ as $t \to 0$, there is a value $t_+ < 0$ such that $H(\partial_X, g) \geqslant h_{\varphi_3}(t_+)$. Similarly as before, we find $\delta_3, \delta_4 > 0$ such that $-\sigma > -\kappa n(n-1)\tan(\frac{\sqrt{\kappa}n(d-2\delta_3)}{4})^2$ and

$$h_{\varphi_2}\left(\frac{d-\delta_3}{2}\right) = h_{\varphi_3}(-\ell - \delta_4),$$

while $\delta_4 < -t_+$ and hence $\ell + \delta_4 + t_+ < \ell$. We fix the model space

$$(M_3, g_{\varphi_3}) = (N \times [-\ell - \delta_4, t_+], \varphi_3^2(t)g_N + dt^2)$$

with scalar curvature equal to $-\sigma$ and width $< \ell$.

Finally, we fix the model space

$$(M_2, g_{\varphi_2}) = \left(N \times \left[\frac{-d+\delta_1}{2}, \frac{d-\delta_3}{2}\right], \varphi_2^2(t)g_N + dt^2\right)$$

with scalar curvature equal to $\kappa n(n-1)$ and width $< d$.

It follows from Theorem 3.5 that $\mathrm{width}(V_j, g) \leqslant \mathrm{width}(M_j, g_{\varphi_j})$ for at least one $i \in \{1,2,3\}$. Since $d = \mathrm{width}(V_2, g) > \mathrm{width}(M_2, g_{\varphi_2})$, we conclude that

$$\min\{\mathrm{width}(V_1, g), \mathrm{width}(V_3, g)\} < \ell,$$

which is what we wanted to prove. □

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