

PATH-BY-PATH UNIQUENESS FOR STOCHASTIC DIFFERENTIAL EQUATIONS UNDER KRYLOV–RÖCKNER CONDITION

LUKAS ANZELETTI, KHOA LÊ AND CHENGCHENG LING

ABSTRACT. We show that any stochastic differential equation (SDE) driven by Brownian motion with drift satisfying the Krylov–Röckner condition has exactly one solution in an ordinary sense for almost every trajectory of the Brownian motion. Additionally, we show that such SDE is strongly complete, i.e. for almost every trajectory of the Brownian motion, the family of solutions with different initial data forms a continuous semiflow for all nonnegative times.

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1. INTRODUCTION

For a measurable time-dependent vector field $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, $d \geq 1$, and a measurable driving signal $\gamma : [0, T] \rightarrow \mathbb{R}^d$, we consider the deterministic ordinary integral equation (ODE)

$$y_t = y_0 + \int_0^t b(r, y_r) dr + \gamma_t, \quad t \in [0, T]. \quad (1.1)$$

Here, $T > 0$ is a finite time horizon which is fixed but arbitrary. Because b is only measurable, a solution $(y_t)_{t \in [0, T]}$, if it exists, should be measurable and satisfy the integrability condition

$$\int_0^T |b(r, y_r)| dr < \infty, \quad (1.2)$$

which ensures that the integral in (1.1) is well-defined in Lebesgue sense. For such solution, it is immediate that $y - \gamma$ is continuous.

When γ is sampled from a stochastic process W on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P})$, equation (1.1) transforms into the stochastic differential equation (SDE)

$$Y_t = Y_0 + \int_0^t b(s, Y_s) ds + W_t, \quad t \in [0, T]. \quad (1.3)$$

A solution $Y : \Omega \times [0, T] \rightarrow \mathbb{R}^d$ to (1.3) is a stochastic process such that for a.s. ω , $(Y_t(\omega))_{t \in [0, T]}$ is a solution to (1.1) with $W(\omega)$ in place of γ .

It is now well-understood that the addition of fast oscillating driving signals γ , W can make both (1.1) and (1.3) well-posed for very irregular vector fields b , which is known as *regularization-by-noise* phenomenon. For SDEs driven by Brownian motion with bounded measurable vector

fields, [Zvo74, Ver79] showed that there exists a pathwise unique adapted solution to (1.3). In [KR05], Krylov and Röckner extended this result to drifts b fulfilling the condition

$$\left(\int_0^T \left(\int_{\mathbb{R}^d} |b(r, x)|^p dx \right)^{\frac{q}{p}} dr \right)^{\frac{1}{q}} < \infty, \quad (p, q) \in \mathcal{J} := \left\{ (p, q) \in (2, \infty)^2 : \frac{2}{q} + \frac{d}{p} < 1 \right\}. \quad (1.4)$$

This condition is also known as the *sub-critical Ladyzhenskaya–Prodi–Serrin* (LPS) condition in fluid dynamics, see more in [FF13b, BFGM19].

Pathwise uniqueness for SDEs with fractional noise was considered first by Nualart and Ouknine in [NO02, NO03] and more recently in [BNP19, Lê20, ART21, GG22a, BLM23b] among others.

Well-posedness for (1.1) is considered in more recent time. Starting from the seminal works of Davie [Dav07], Catellier–Gubinelli [CG16], there has been an increasing interest in the problem, [Sha16, HP21, GG22b, GG22a, DG22, CD22]. A typical result of these articles is that when γ is a sample path of an irregular stochastic process (such as fractional Brownian motion), then for almost every such sample paths, ODE (1.1) has a unique solution, a property known by the terminology *path-by-path uniqueness*. The necessary regularity for the vector field is inversely proportional to the roughness of the stochastic process from which γ is sampled. In a different direction, when the coefficients are sufficiently regular so that Lyons’ rough path theory can be applied, then uniqueness of the corresponding rough differential equation implies path-by-path uniqueness. This is because the stochastic basis is only used to construct a rough path lift of the driving signal and Lyons’ rough path analysis ([Lyo98]) is applicable for any fixed trajectory of the lift.

From a practical perspective, path-by-path uniqueness is most relevant in situations where only a handful samples of the signal can be observed. Theoretically, “path-by-path uniqueness” is stronger than “pathwise uniqueness” (which essentially allows to identify two adapted solutions to (1.3)). Having pathwise uniqueness for (1.3) does not provide uniqueness for the $W(\omega)$ -driven ODE

$$Y_t = Y_0 + \int_0^t b(r, Y_r) dr + W_t(\omega), \quad t \in [0, T], \quad (1.5)$$

for a given ω . In fact, examples of SDEs driven by Brownian motion which have pathwise uniqueness but not path-by-path uniqueness are given in [SW22, Anz22]. We point to Appendix B for further discussions on the related concepts.

The main goal of the current article is the following result, which will be stated more precisely later in Theorems 2.2, 2.4 and 2.5.

Theorem. *Assuming that W is a Brownian motion, then path-by-path uniqueness, path-by-path stability and strong completeness for (1.3) hold under the Krylov–Röckner condition (1.4).*

By path-by-path stability, we mean that for a sequence of functions (b^n) converging sufficiently fast to b in a suitable topology, there exists an event of full probability which depends only on (b^n) and T such that for every ω in such event, for every $x \in \mathbb{R}^d$, the solution Y^n to the

ODE

$$Y_t^n = x + \int_0^t b^n(r, Y_r^n) dr + W_t(\omega), \quad t \in [0, T]$$

converges uniformly to the unique solution to (1.5) with initial condition $Y_0 = x$. This is stronger than standard stability results because the event of full probability measure only depends on (b^n) . Usually, for each n , we have to choose Y^n to be the pathwise unique strong solution and the set of full measure on which convergence holds depends on the choice of Y^n (respectively the set of ω for which it fulfills the equation). Additionally, with this procedure the event of full measure would also depend on the choice of the initial condition.

Strong completeness refers to the property that for almost all trajectories of the Brownian motion, the equation can be solved globally regardless of the initial data. Strong completeness is a rather demanding property and may fail even for SDEs with bounded smooth (but with unbounded derivatives) coefficients, with examples given by Li and Scheutzow in [LS11]. Strong completeness for SDEs driven by Brownian motion with standard regularity conditions are considered in [Li94, SS17]. Alternatively, when the SDE under consideration can be lifted to a rough differential equation which has unique global solution, then it is strongly complete, see [RS17]. The SDE (1.3) under condition (1.4) falls outside of the standard Itô framework and Lyons’ rough path theory. Hence, our result provides the first example of strongly complete singular SDEs driven by Brownian motion.

To the best of our knowledge, path-by-path uniqueness under (1.4) has not been considered in the literature and path-by-path stability is introduced herein. [Dav07, Sha16] shows path-by-path uniqueness for SDEs with bounded drifts. For equations driven general fractional noise with distributional drifts, this kind of result is shown in [CG16, HP21, GG22a], by reformulating the original equation into the framework of nonlinear Young differential equations. In this framework, proofs are considerably simpler, however, the assumed regularity conditions on the drifts in the aforementioned works exclude the Krylov–Röckner condition (1.4). Although the solutions of nonlinear Young differential equations are apparently different from solutions of ordinary differential equations, it follows from our analysis that solutions of (1.5) can also be interpreted in nonlinear Young sense, see Remark 4.5.

Our approach is closer to that of [Dav07] taking into account the insights from [CG16, Sha16, GG22a] and the quantitative John–Nirenberg inequality recently obtained in [Lê22]. Davie in [Dav07] delivers all arguments from first principles, which unfortunately increases the level of technicality. His proof consists of two main parts, one is the regularizing estimates for averaging operators along Brownian motion ([Dav07, The basic estimate, Proposition 2.1]) and the other is the uniqueness argument. Catellier and Gubinelli in [CG16] reformulate and extend Davie’s approach in the framework of nonlinear Young integration, leading to path-by-path uniqueness for a large class of fractional SDEs. Although more general, results in [CG16] do not reproduce Davie’s result due to several technical reasons. Shaposhnikov in [Sha16] gives a simpler justification for Davie’s uniqueness arguments, calling out the importance of the flow generated by (1.3). Galeati and Gerencsér in [GG22a] adopt this argument in combination with estimates for nonlinear Young integrals. It is known that [Sha16] contains several technical

issues some of which were corrected in [Sha17]. Herein, we discover and justify another one which was not addressed in [Sha16, Sha17], see Remark 5.3.

As for our strategy, we apply John–Nirenberg inequality to show Davie estimates under (1.4) (see Proposition 3.1). The John–Nirenberg inequality effectively reduces estimating all positive moments to estimating only the second moment. This shortens existing arguments and improves known results from [Dav07, Rez14]. Building upon Davie estimates, we use nonlinear Young integration to obtain a key regularizing estimate for averaging operators along Brownian motion (Lemma 4.3). This allows us to show that equation (1.3) generates a random continuous semiflow and obtain a priori estimates for any solution to (1.5) (Propositions 5.1 and 5.2). Applying these priori estimates and the regularity of the semiflow generated by (1.3), we are able to identify any solution with the semiflow, see proof of Theorem 2.2 in Section 5. Although using similar tools as [CG16, Sha16, GG22a], our proofs are logically different from the aforementioned articles. Additionally, our arguments are generic and are not strictly tied to Brownian motion, as long as the random semiflow generated by the SDE (1.3) and the averaging operators along the driving signal W are sufficiently regular. This leads to a new uniqueness criterion for ODEs of the form (1.1), which does not require any regularity on the vector field, but instead relies on the regularizing effect of the driving signal γ . These results are summarized in Appendix A.

Structure of paper. Main results are stated in Section 2. In Section 3, we show Davie estimates under (1.4). In Section 4, we show some path-by-path regularizing estimates, which are consequences of Davie estimates and the sewing lemma. The proof of the main results are presented in Section 5. In Appendix A, we summarize a uniqueness criterion for (1.1) which is based on the regularization effect of the driving signal. Different notions of solutions and their relations are recalled in Appendix B.

Frequently used notation. Define for a random variable X , for any $m \geq 1$, $\|X\|_{L^m(\Omega)} := [\mathbb{E}(|X|^m)]^{1/m}$. For each $p \in [1, \infty]$, $L^p(\mathbb{R}^d)$ denotes the usual Lebesgue space on \mathbb{R}^d . For $0 \leq s \leq t$, $\mathbb{L}_p^q([s, t])$ contains all measurable functions $f : [s, t] \times \mathbb{R}^d \rightarrow \mathbb{R}$ such that

$$\|f\|_{\mathbb{L}_p^q([s, t])} := \left(\int_s^t \left(\int_{\mathbb{R}^d} |f(r, x)|^p dx \right)^{\frac{q}{p}} dr \right)^{\frac{1}{q}} < \infty$$

for $p, q \in [1, \infty)$ and with obvious modification when p or q is infinity. We abbreviate $\mathbb{L}_p^q = \mathbb{L}_p^q(\mathbb{R})$. For $0 \leq S \leq T$, let $[S, T]_{\leq}^2 := \{(s, t) \in [S, T]^2 : S \leq s \leq t \leq T\}$. We say that a continuous function $w : [S, T]_{\leq}^2 \rightarrow [0, \infty)$ is a control if $w(s, u) + w(u, t) \leq w(s, t)$ whenever $s \leq u \leq t$. For a function $(\psi_t)_{t \in [0, T]}$ of finite variation and each $(u, v) \in [0, T]_{\leq}^2$, we define

$$[\psi]_{\text{var}; [u, v]} := \sup_{\pi} \sum_{[s, t] \in \pi} |\psi_t - \psi_s|.$$

where the supremum is taken over all partitions π of $[u, v]$. It is known that $(u, v) \mapsto [\psi]_{\text{var}; [u, v]}$ is a control. We write $H \lesssim G$ if $H \leq CG$ for some universal finite positive constant C .

2. MAIN RESULTS

Let $\{\mathcal{F}_t\}_{t \geq 0}$ be a filtration and assume that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ satisfies the usual conditions. For each $s \in [0, T]$, \mathbb{E}_s denotes the conditional expectation with respect to \mathcal{F}_s , i.e. $\mathbb{E}_s[\cdot] = \mathbb{E}_s[\cdot | \mathcal{F}_s]$. We assume that W is a standard Brownian motion with respect to $\{\mathcal{F}_t\}_{t \geq 0}$.

A semiflow generated by the ODE (1.1) over the period $[0, T]$ is a measurable map $(s, t, x) \mapsto \phi_t^{s,x}$ defined on $[0, T]_{\leq}^2 \times \mathbb{R}^d$ such that for every $0 \leq s \leq u \leq t \leq T$ and $x \in \mathbb{R}^d$,

$$\int_s^T |b(r, \phi_r^{s,x})| dr < \infty, \quad \phi_t^{s,x} = x + \int_s^t b(r, \phi_r^{s,x}) dr + \gamma_t - \gamma_s \quad \text{and} \quad \phi_t^{s,x} = \phi_t^{u, \phi_u^{s,x}}.$$

We say that ϕ is *locally κ -Hölder continuous* if for every compact set $K \subset \mathbb{R}^d$, there exists a finite constant $C = C(\kappa, K, T)$ such that for every $x, y \in K$ and $(s, t) \in [0, T]_{\leq}^2$,

$$|\phi_t^{s,x} - \phi_t^{s,y}| \leq C|x - y|^\kappa.$$

Proposition 2.1 (Random Hölder semiflow). *Under (1.4), there exists an almost surely locally κ -Hölder continuous semiflow to (1.3) for every $\kappa \in (0, 1)$. In other words, there exists a jointly measurable map $(s, t, x, \omega) \mapsto X_t^{s,x}(\omega)$ defined on $[0, T]_{\leq}^2 \times \mathbb{R}^d \times \Omega$ such that for a.s. ω , $(s, t, x) \mapsto X_t^{s,x}(\omega)$ is a locally κ -Hölder continuous semiflow to (1.5) for every $\kappa \in (0, 1)$.*

The semiflow $(X_t^{s,x})$ is constructed by taking the continuous extension of the family of strong solutions over the dyadics. Hence, it inherits other properties and as such, it is adapted and is a flow of homeomorphisms. However, we will not use these properties. Proposition 2.1 thus improves upon [FF13a, Theorem 5.1] by showing joint continuity in all parameters and allowing the set of full measure on which the flow solves the equation to be uniform with respect to the initial data. Because b is not continuous, showing that the continuous extension X forms a family of solutions to (1.3) is actually a major obstacle. This is overcome thanks to the path-by-path regularizing estimates obtained by sewing techniques and Davie estimates, see Lemma 4.3 and Corollary 4.4. These arguments can be applied for other situations and it is expected that the conclusions of Proposition 2.1 hold for SDEs with bounded measurable drifts, more details are discussed in Remark 5.3. Furthermore, it follows from our arguments that a solution to (1.5) is also a nonlinear Young solution and a regularized solution, see Remark 4.5 for further details.

Theorem 2.2 (Path-by-path uniqueness). *There exists an event $\Omega_{b,T} \in \mathcal{F}$ which depends only on b, T and has full probability measure such that for every $\omega \in \Omega_{b,T}$, any solution Y to (1.5) is identical to $X^{0,Y_0}(\omega)$. As a consequence, path-by-path uniqueness for Equation (1.3) holds.*

Remark 2.3. The dependence on T can easily be removed by considering the event $\cup_{T=1}^\infty \Omega_{b,T}$. The dependence on b seems necessary. When an initial datum is specified so that $Y_0 = \xi$ for some $\xi \in \mathbb{R}^d$, the set $\Omega_{b,T}$ is independent from ξ . In addition, if ξ is a random variable which is not necessary \mathcal{F}_0 -measurable, Theorem 2.2 implies uniqueness among non-adapted solutions.

The event $\Omega_{b,T}$ essentially contains ω such that

- (i) the semiflow $X_t^{s,x}(\omega)$ is almost Lipschitz in x ,

(ii) the averaging operators

$$(t, x) \mapsto \int_0^t b(r, W_r(\omega) + x) dr, \int_0^t |b|(r, W_r(\omega) + x) dr$$

are jointly locally Hölder continuous with exponent $(\alpha, 1 - \varepsilon)$ for some $\alpha > 0$ and some $\varepsilon > 0$ sufficiently small.

These properties are rather generic and available not only when W is a Brownian motion but also for other stochastic processes. The case of fractional Brownian motion is currently investigated in [BLM23a].

To state the result on path-by-path stability, we introduce some additional notations. For $\nu \geq 0$, $W^{-\nu, p}(\mathbb{R}^d)$ is the Bessel potential space $W^{-\nu, p}(\mathbb{R}^d) := \{f : \|f\|_{W^{-\nu, p}(\mathbb{R}^d)} := \|(I - \Delta)^{-\frac{\nu}{2}} f\|_{L^p(\mathbb{R}^d)} < \infty\}$. When $\nu = 0$, we have $W^{-\nu, p}(\mathbb{R}^d) = L^p(\mathbb{R}^d)$. A sequence (z^n) in a metric space (\mathcal{Z}, ρ) is summable convergent to an element $z \in \mathcal{Z}$ if there is a constant $\eta > 0$ such that

$$\sum_{n=1}^{\infty} \rho(z^n, z)^\eta < \infty.$$

In such case, we write $\Sigma\text{-lim}_n z^n = z$ in \mathcal{Z} . It is evident that summable convergence implies convergence in the usual sense. Conversely, if $\lim_n z^n = z$, then for every $\eta > 0$, there exists a subsequence $(z^{\ell(n)})$ such that $\sum_{n=1}^{\infty} \rho(z^{\ell(n)}, z)^\eta < \infty$, in particular $\Sigma\text{-lim}_n z^{\ell(n)} = z$.

Theorem 2.4 (Path-by-path stability). *Let $(p, q) \in \mathcal{J}$ and (b^n) be a sequence of measurable functions in \mathbb{L}_p^q . We assume that $\Sigma\text{-lim}_n b^n = b$ in $L^q([0, T], W^{-\nu, p}(\mathbb{R}^d))$ where ν satisfies either*

$$\nu \in [0, 1 - \frac{2}{q}), \quad \frac{d}{p} + \frac{2}{q} + \nu < 2; \quad (2.1)$$

or

$$\nu \in [1 - \frac{2}{q}, 1], \quad \frac{d}{p} + \frac{4}{q} < 3 - 2\nu, \quad \frac{d}{p} < \nu. \quad (2.2)$$

Then there exists an event $\Omega'_{(b^n), b, T}$, which depends on (b^n) , b , T and has full probability measure, such that for every $\omega \in \Omega'_{(b^n), b, T}$, for every $(s, x) \in [0, T] \times \mathbb{R}^d$, whenever Y^n are solutions to ODEs

$$Y_t^n = x + \int_s^t b^n(r, Y_r^n) dr + W_t(\omega) - W_s(\omega), \quad t \in [s, T],$$

then $\lim_n Y^n = X^{s, x}(\omega)$ uniformly on $[s, T]$.

We say that the SDE (1.3) is *strongly complete* if for a.s. ω , the ODE (1.5) generates a continuous semiflow over $[0, \infty)$. This means that there exists a measurable map

$$[0, \infty)_{\leq}^2 \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^d \\ (s, t, x, \omega) \mapsto X_t^{s, x}(\omega)$$

such that for a.s. ω the following hold:

(i) $(s, t, x) \mapsto X_t^{s, x}(\omega)$ is continuous;

(ii) for every $0 \leq s \leq u \leq t$ and $x \in \mathbb{R}^d$

$$\int_s^t |b(r, X_r^{s,x}(\omega))| dr < \infty, \quad X_t^{s,x}(\omega) = x + \int_s^t b(r, X_r^{s,r}(\omega)) dr + W_t(\omega) - W_s(\omega)$$

and

$$X_t^{s,x}(\omega) = X_t^{u, X_u^{s,x}(\omega)}(\omega).$$

Our definition given above is different from the literatures' in that (ii) demands that the event on which the semiflow solves the equation is independent from initial data. Heuristically, strong completeness requires that the random semiflow is defined globally over all nonnegative times $[0, \infty)$ and continuously in (s, t, x) over $[0, \infty)_{\leq}^2 \times \mathbb{R}^d$. That the semiflow is generated by the equation means that for almost every ω and every initial data (s, x) , the map $t \mapsto X_t^{s,x}(\omega)$ is a global solution which does not blow up in finite time. The later property can be thought as "path-by-path non-explosion". SDEs which have global pathwise solutions without explosion may fail to have path-by-path non-explosion with examples given in [LS11].

Our next result provides a general principle which asserts that if over any finite period of time, an SDE generates a random continuous semiflow and has path-by-path uniqueness, then the SDE is strongly complete. Although this fact has been observed in [RS17] in the context of rough differential equations, singular SDEs is another distinct class of equations. It is therefore necessary to draw the connections.

Theorem 2.5 (Strong completeness). *Suppose that for every $T > 0$, (1.3) generates a random continuous semiflow over $[0, T]$ and that path-by-path uniqueness holds over $[0, T]$. Then (1.3) is strongly complete.*

Proof. Let $(X_t^{s,x})$ and $(\bar{X}_t^{s,x})$ be random continuous semiflows respectively over $[0, T]$ and $[0, \bar{T}]$ for some $T < \bar{T}$. Then by path-by-path uniqueness $\bar{X}|_{[0, T]_{\leq}^2 \times \mathbb{R}^d} = X$. This implies the existence (and uniqueness) of a random continuous semiflow over all nonnegative time. \square

The proofs of [Proposition 2.1](#) and [Theorems 2.2](#) and [2.4](#) are presented in [Section 5](#).

3. DAVIE ESTIMATES

We show a variant of *Davie estimates* in [Dav07, Propositions 2.1, 2.2].

Proposition 3.1. *Let f be a Schwartz function in \mathbb{L}_p^q for some $(p, q) \in \mathcal{J}$. Then there is a constant $C = C(d, p, q, T)$ such that for every $(s, t) \in [0, T]_{\leq}^2$ and $m \geq 1$,*

$$\left\| \int_s^t \nabla f(r, W_r) dr \right\|_{L^m(\Omega)} \leq C \Gamma \left(m \left(\frac{1}{2} + \frac{d}{2p} \right) + 1 \right) \|f\|_{\mathbb{L}_p^q([s, t])} (t - s)^{\frac{1}{2} - \frac{1}{q} - \frac{d}{2p}}, \quad (3.1)$$

where $\Gamma(r) = \int_0^\infty u^{r-1} e^{-u} du$ is the Gamma function. Consequently, for every bounded continuous function f in $\mathbb{L}_p^q([0, T])$, every $x, y \in \mathbb{R}^d$, every $(s, t) \in [0, T]_{\leq}^2$ and $m \geq 1$, we have

$$\left\| \int_s^t [f(r, W_r + x) - f(r, W_r + y)] dr \right\|_{L^m(\Omega)}$$

$$\leq C\Gamma\left(m\left(\frac{1}{2} + \frac{d}{2p}\right) + 1\right) \|f\|_{\mathbb{L}_p^q([s,t])} (t-s)^{\frac{1}{2} - \frac{1}{q} - \frac{d}{2p}} |x-y|. \quad (3.2)$$

The precise growth constant $\Gamma\left(m\left(\frac{1}{2} + \frac{d}{2p}\right) + 1\right)$ is not essential to our current purposes. Nevertheless, it improves upon previous known estimates [Rez14]. To prove this result, we follow a recent approach from [Lê22] based on the quantitative John–Nirenberg inequality. We prepare two auxiliary lemmas, putting

$$p_t(x) := \mathbf{1}_{(t>0)} (2\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{2t}}.$$

Lemma 3.2. *The operator $f \mapsto D(f)$ defined by*

$$D(f)(t, x) := \int_{\mathbb{R} \times \mathbb{R}^d} \nabla p_s(y) \cdot \nabla f(t-s, x-y) \, d(s, y),$$

is bounded on \mathbb{L}_p^q for every $(p, q) \in (1, \infty)^2$.

Proof. The operator D is associated to the Fourier multiplier

$$\Phi(x_0, x) = c \frac{x^2}{x^2 + ix_0}, \quad x^2 = x_1^2 + \dots + x_d^2,$$

for some absolute constant c , where i is the imaginary unit. A result from [Liz70, Corollary 1, pg. 234] asserts that D is a bounded operator on \mathbb{L}_p^q provided that Φ admits continuous (for $x_0, \dots, x_d \neq 0$) purely mixed derivatives of orders $k \leq d+1$ such that

$$\left| x_{j_1} \dots x_{j_k} \partial_{x_{j_1} \dots x_{j_k}}^k \Phi \right| \leq M$$

for all distinct j_1, \dots, j_k and for some finite constant M . Such condition can be verified directly. \square

Lemma 3.3 (Quantitative John–Nirenberg inequality). *Let V be an adapted continuous process, w be a deterministic control and $\alpha \in (0, 1)$. Assume that*

$$\|\mathbb{E}_s |V_t - V_s|\|_{L^\infty(\Omega)} \leq w(s, t)^\alpha \quad \forall (s, t) \in [0, T]_{\leq}^2.$$

Then, there exists a finite constant $C_\alpha > 0$ such that for every $(s, t) \in [0, T]_{\leq}^2$ and every $m \geq 1$,

$$\left\| \sup_{u \in [s, t]} |V_u - V_s| \right\|_{L^m(\Omega)} \leq C_\alpha \Gamma(m(1-\alpha) + 1) w(s, t)^\alpha.$$

Proof. Without the growth constant $\Gamma(m(1-\alpha) + 1)$, this result is a direct consequence of [SV06, Exercise A.3.2]. To obtain the precise growth constant, we follow [Lê22] closely. Continuity and the assumption imply that

$$\|\mathbb{E}_\sigma |V_\tau - V_\sigma|\|_{L^\infty(\Omega)} \leq w(s, t)^\alpha$$

for every $(s, t) \in [0, T]_{\leq}^2$ and all stopping times σ, τ satisfying $s \leq \sigma \leq \tau \leq t$. The class of all processes with such property is denoted by $\text{VMO}^{(1/\alpha)\text{-var}}$ in [Lê22, Section 3]. We then apply [Lê22, Corollary 3.5] to obtain the desired estimate. \square

Proof of Proposition 3.1. In view of Lemma 3.3, it suffices to estimate the conditional second moment. Define for each $(s, t) \in [0, T]_{\leq}^2$,

$$V_t := \int_0^t \nabla f(r, W_r) dr, \quad \text{and} \quad I_{s,t} := \frac{1}{2} \mathbb{E}_s |V_t - V_s|^2. \quad (3.3)$$

Using integration by parts, tower property of conditional expectation and Fubini's Theorem we have, almost surely,

$$\begin{aligned} I_{s,t} &= \mathbb{E}_s \int_s^t \int_{r_2}^t \mathbb{E}_{r_2} [\nabla f(r_1, W_{r_1})] \cdot \nabla f(r_2, W_{r_2}) dr_1 dr_2 \\ &= -\mathbb{E}_s \int_s^t \int_{r_2}^t \left[\int_{\mathbb{R}^d} f(r_1, y + W_{r_2}) \nabla p_{r_1-r_2}(y) dy \right] \cdot \nabla f(r_2, W_{r_2}) dr_1 dr_2 \\ &= -\int_s^t \int_{r_2}^t \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} f(r_1, y + z + W_s) \nabla p_{r_1-r_2}(y) dy \right] \cdot \nabla f(r_2, z + W_s) p_{r_2-s}(z) dz dr_1 dr_2 \\ &=: I_1 + I_2 \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_s^t \int_{r_2}^t \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} f(r_1, y + z + W_s) \nabla p_{r_1-r_2}(y) dy \right] \cdot f(r_2, z + W_s) \nabla p_{r_2-s}(z) dz dr_1 dr_2, \\ I_2 &= \int_s^t \int_{r_2}^t \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} \nabla f(r_1, y + z + W_s) \cdot \nabla p_{r_1-r_2}(y) dy \right] f(r_2, z + W_s) p_{r_2-s}(z) dz dr_1 dr_2. \end{aligned}$$

To estimate I_1 , we apply Hölder's inequality (below p' satisfies $\frac{1}{p} + \frac{1}{p'} = 1$) to see that for any $v \in \mathbb{R}^d$,

$$\begin{aligned} &\int_s^t \int_{r_2}^t \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} f(r_1, y + z + v) \nabla p_{r_1-r_2}(y) dy \right] f(r_2, z + v) \nabla p_{r_2-s}(z) dz dr_1 dr_2 \\ &\leq \int_s^t \int_{r_2}^t \|f(r_1, z + v + \cdot)\|_{L^p(\mathbb{R}^d)} \|\nabla p_{r_1-r_2}\|_{L^{p'}(\mathbb{R}^d)} \|f(r_2, v + \cdot)\|_{L^p(\mathbb{R}^d)} \|\nabla p_{r_2-s}\|_{L^{p'}(\mathbb{R}^d)} dr_1 dr_2 \\ &\lesssim \|f\|_{\mathbb{L}_p^q([s,t])} \int_s^t \left(\int_{r_2}^t (r_1 - r_2)^{\left(-\frac{1}{2} - \frac{d}{2p}\right) \frac{q}{q-1}} dr_1 \right)^{1-\frac{1}{q}} \|f(r_2)\|_{L^p(\mathbb{R}^d)} (r_2 - s)^{-\frac{1}{2} - \frac{d}{2p}} dr_2 \\ &\lesssim \|f\|_{\mathbb{L}_p^q([s,t])} \int_s^t (t - r_2)^{\frac{1}{2} - \frac{d}{2p} - \frac{1}{q}} \|f(r_2)\|_{L^p(\mathbb{R}^d)} (r_2 - s)^{-\frac{1}{2} - \frac{d}{2p}} dr_2 \\ &\lesssim \|f\|_{\mathbb{L}_p^q([s,t])} (t - s)^{\frac{1}{2} - \frac{d}{2p} - \frac{1}{q}} \int_s^t \|f(r_2)\|_{L^p(\mathbb{R}^d)} (r_2 - s)^{-\frac{1}{2} - \frac{d}{2p}} dr_2 \\ &\lesssim \|f\|_{\mathbb{L}_p^q([s,t])} (t - s)^{\frac{1}{2} - \frac{d}{2p} - \frac{1}{q}} \|f\|_{\mathbb{L}_p^q([s,t])} (t - s)^{\frac{1}{2} - \frac{d}{2p} - \frac{1}{q}} \lesssim \|f\|_{\mathbb{L}_p^q([s,t])}^2 (t - s)^{1 - \frac{2}{q} - \frac{d}{p}}. \end{aligned}$$

This yields that, almost surely,

$$|I_1| \lesssim \|f\|_{\mathbb{L}_p^q([s,t])}^2 (t - s)^{1 - \frac{2}{q} - \frac{d}{p}}.$$

To estimate I_2 , we apply Hölder inequality and [Lemma 3.2](#) to see that

$$|I_2| \lesssim \|f\|_{\mathbb{L}_p^q([s,t])} \|\mathbf{1}_{[s,t]} f^{W_s} p^s\|_{\mathbb{L}_{p'}^{q'}}$$

where

$$f^{W_s}(r, z) = f(r, z + W_s), \quad p_r^s(z) = p_{r-s}(z),$$

and p', q' denote the Hölder conjugates of p, q respectively. Applying Hölder inequality again, we have

$$\|\mathbf{1}_{[s,t]} f^{W_s} p^s\|_{\mathbb{L}_{p'}^{q'}} \leq \|\mathbf{1}_{[s,t]} p^s\|_{\mathbb{L}_{p/(p-2)}^{q/(q-2)}} \|f\|_{\mathbb{L}_p^q([s,t])}.$$

Using the fact that $(p, q) \in \mathcal{J}$ and some elementary calculations, we have

$$\|\mathbf{1}_{[s,t]} p^s\|_{\mathbb{L}_{p/(p-2)}^{q/(q-2)}} = c(t-s)^{1-\frac{d}{p}-\frac{2}{q}}$$

for some constant c depending on p, q . Combining with the previous estimates, we obtain that

$$|I_{s,t}| \lesssim \|f\|_{\mathbb{L}_p^q([s,t])}^2 (t-s)^{1-\frac{2}{q}-\frac{d}{p}} \quad a.s.$$

Consequently,

$$\|\mathbb{E}_s |V_t - V_s|\|_{L^\infty(\Omega)} \lesssim \|f\|_{\mathbb{L}_p^q([s,t])} (t-s)^{\frac{1}{2}-\frac{1}{q}-\frac{d}{2p}}.$$

It is evident that V is continuous. Applying [Lemma 3.3](#), we obtain [\(3.1\)](#).

Next we show [\(3.2\)](#). By approximations, we can assume that f has bounded continuous first derivatives. We observe that

$$\int_s^t f(r, W_r + x) - f(r, W_r + y) dr = (x - y) \cdot \int_0^1 \int_s^t \nabla f(r, W_r + \theta x + (1 - \theta)y) dr d\theta.$$

Then it follows from [\(3.1\)](#) that

$$\begin{aligned} & \left\| \int_s^t f(r, W_r + x) - f(r, W_r + y) dr \right\|_{L^m(\Omega)} \\ & \leq \sup_{\theta \in [0,1]} \left\| \int_s^t \nabla f(r, W_r + \theta x + (1 - \theta)y) dr \right\|_{L^m(\Omega)} |x - y| \\ & \leq |x - y| \sup_{z \in \mathbb{R}^d} \left\| \int_s^t \nabla f(r, W_r + z) dr \right\|_{L^m(\Omega)} \\ & \lesssim \Gamma \left(m \left(\frac{1}{2} + \frac{d}{2p} \right) + 1 \right) \|f\|_{\mathbb{L}_p^q([s,t])} (t-s)^{\frac{1}{2}-\frac{1}{q}-\frac{d}{2p}} |x - y|. \end{aligned}$$

The proof is completed. □

4. PATH-BY-PATH ESTIMATES

Let f be a measurable function on $\mathbb{R} \times \mathbb{R}^d$. If ω is such that $\int_0^T |f(r, W_r(\omega))| dr < \infty$ then the function $t \mapsto \int_0^t f(r, W_r(\omega)) dr$ is continuous. The following result extends this argument for the function $(t, x) \mapsto \int_0^t f(r, W_r(\omega) + x) dr$. The integrability condition $\int_0^T |f(r, W_r(\omega) + x)| dr < \infty$ is no longer sufficient and one has to replace it with the quantity $\Xi_{T,K}(f)(\omega)$ defined below.

Lemma 4.1. *Let $K \subset \mathbb{R}^d$ be a compact set containing 0, $\alpha \in (0, \frac{1}{2} - \frac{d}{2p} - \frac{1}{q})$ for some $(p, q) \in \mathcal{J}$ and $\varepsilon \in (0, 1)$. There exists a function $\Xi_{T,K,\alpha,\varepsilon} = \Xi_{T,K} : \mathbb{L}_p^q \rightarrow L^1(\Omega)$ such that*

- (i) $\Xi_{T,K}$ satisfies the triangle inequality, i.e. $\Xi_{T,K}(f + g) \leq \Xi_{T,K}(f) + \Xi_{T,K}(g)$;
- (ii) for every $f \in \mathbb{L}_p^q$,

$$\mathbb{E}[\Xi_{T,K}(f)] \lesssim \|f\|_{\mathbb{L}_p^q([0,T])}; \quad (4.1)$$

- (iii) for every bounded measurable function f in \mathbb{L}_p^q , there exists an event $\Omega_{f,T,K}$ of full measure such that for every $\omega \in \Omega_{f,T,K}$, every $(s, t) \in [0, T]_{\leq}^2$ and every $x, y \in K$,

$$\left| \int_s^t [f(r, W_r(\omega) + x) - f(r, W_r(\omega) + y)] dr \right| \leq \Xi_{T,K}(f)(\omega) |x - y|^{1-\varepsilon} (t - s)^\alpha, \quad (4.2)$$

$$\left| \int_s^t f(r, W_r(\omega) + x) dr \right| \leq \Xi_{T,K}(f)(\omega) (t - s)^\alpha. \quad (4.3)$$

Proof. For each function $f \in \mathbb{L}_p^q$, we define

$$\begin{aligned} \Xi_{T,K}(f) = C_{T,K} & \left(\iint_{[0,T]^2} \left| \frac{\int_s^t f(r, W_r) dr}{|t - s|^{\gamma_1}} \right|^m ds dt \right. \\ & \left. + \iint_{[0,T]^2 \times K^2} \left| \frac{\int_s^t [f(r, W_r + x) - f(r, W_r + y)] dr}{|t - s|^{\gamma_1} |x - y|^{\gamma_2}} \right|^m ds dt dx dy \right)^{\frac{1}{m}} \end{aligned}$$

for some absolute constants $C_{T,K} > 0$, $m \geq 2$ sufficiently large and $\gamma_1 \in (\frac{2}{m}, \frac{1}{m} + (\frac{1}{2} - \frac{d}{2p} - \frac{1}{q}))$, $\gamma_2 \in (\frac{2}{m}, 1 + \frac{1}{m})$ fixed. The constants $C_{T,K}$ and m will be determined at a later step. It is obvious that (i) holds. Using (3.2) and the following estimate from [LL21, Lemma 4.5]

$$\mathbb{E} \left(\int_s^t f(r, W_r) dr \right)^m \lesssim \|f\|_{\mathbb{L}_p^q([s,t])}^m (t - s)^{m(1 - \frac{d}{p} - \frac{2}{q})}, \quad (4.4)$$

we see that (ii) holds.

It remains to show that (iii) holds. Assume first that f is bounded continuous so that the map $(t, x) \mapsto \int_0^t f(r, W_r(\omega) + x) dr$ is continuous for a.s. ω . From (3.2) and (4.4), applying the multiparameter Garsia–Rodemich–Rumsey inequality ([HL13, Theorem 3.1]), there exist constants $C_{T,K}$ and m such that (4.2) and (4.3) hold.

Below, we remove the continuity restriction.

Step 1. We show (iii) for $f = \mathbf{1}_U$ where U is an open set of finite measure in $\mathbb{R} \times \mathbb{R}^d$. By Urysohn lemma, there exists a sequence of increasing continuous functions f^n converging pointwise to f . From (i) and (ii), we can choose a further subsequence, still denoted by (f^n) such that $\lim_n \Xi_{T,K}(f^n) = \Xi_{T,K}(f)$ a.s. We then apply (4.2) and (4.3) for f^n and take limit in n to obtain (iii).

Step 2. We show (iii) for a general bounded measurable f in \mathbb{L}_p^q . Let M be a constant such that $|f| \leq M$. By Lusin theorem, for any $n \in \mathbb{N}$, there exists a continuous function f^n and an open set U^n such that $|U^n| \leq 2^{-n}$ and $|f - f^n| \leq 2M\mathbf{1}_{U^n}$. This implies that $\lim_n f^n = f$ in \mathbb{L}_p^q . Using (ii), we can choose subsequences, still denoted by n such that a.s.

$$\lim_n \Xi_{T,K}(\mathbf{1}_{U^n}) = 0 \quad \text{and} \quad \lim_n \Xi_{T,K}(f^n) = \Xi_{T,K}(f). \quad (4.5)$$

Define

$$A[f](\omega) = \left| \int_s^t [f(r, W_r(\omega) + x) - f(r, W_r(\omega) + y)] dr \right|.$$

Then

$$A[f] \leq A[f^n] + A[f - f^n]. \quad (4.6)$$

Applying (4.2) for the continuous function f^n , we have a.s.

$$A[f^n] \leq \Xi_{T,K}(f^n) |x - y|^{1-\varepsilon} (t - s)^\alpha \quad \forall (s, t, x, y) \in [0, T]_{\leq}^2 \times K^2.$$

Applying (4.2) and (4.3) for $\mathbf{1}_{U^n}$, we have a.s.

$$\begin{aligned} A[f - f^n] &\leq 2M \int_s^t [\mathbf{1}_{U^n}(r, W_r + x) + \mathbf{1}_{U^n}(r, W_r + y)] dr \\ &\leq 4M \Xi_{T,K}(\mathbf{1}_{U^n}) (t - s)^\alpha \quad \forall (s, t, x, y) \in [0, T]_{\leq}^2 \times K^2. \end{aligned}$$

Hence, from (4.6), taking limit in n and using (4.5), we see that (4.2) holds. The estimate (4.3) is obtained in a similar way. \square

Remark 4.2. Via a truncation procedure as is done in the last step in the proof of Lemma 4.3 below, one can remove the assumption of boundedness in Lemma 4.1(iii). The current formulation is sufficient for our purpose.

The following result provides an alternative perspective to [Dav07, Lemmas 3.3 and 3.4] and [Sha16, Lemmas 3.3 and 3.4].

Lemma 4.3. *Let $f : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a function in \mathbb{L}_p^q for some $(p, q) \in \mathcal{J}$, $K \subset \mathbb{R}^d$ be a compact set containing 0, and $\alpha \in (0, \frac{1}{2} - \frac{d}{2p} - \frac{1}{q})$. Then there exists an event $\Omega_{f,T,K}$ with full probability such that for every $\omega \in \Omega_{f,T,K}$, every $\varepsilon \in (0, \alpha)$, there is a deterministic constant $C = C(\varepsilon, \alpha)$ such that for every $(s, t) \in [0, T]_{\leq}^2$, every function $\psi : [0, T] \rightarrow K$ of finite variation, we have*

$$\left| \int_s^t f(r, W_r(\omega) + \psi_r) dr - \int_s^t f(r, W_r(\omega) + \psi_s) dr \right| \leq C \Xi_{T,K}(f)(\omega) [\psi]_{\text{var};[s,t]}^{1-\varepsilon} (t - s)^\alpha \quad (4.7)$$

and

$$\left| \int_s^t f(r, W_r(\omega) + \psi_r) dr \right| \leq C \Xi_{T,K}(f)(\omega) (1 + [\psi]_{\text{var};[s,t]}^{1-\varepsilon}) (t-s)^\alpha. \quad (4.8)$$

Proof. Since (4.8) is a consequence of (4.7) and (4.3), we only focus on (4.7).

Step 1. We first show the result assuming that f is a bounded continuous function. For $(s, t) \in [0, T]_\leq^2$ define $A_{s,t} = \int_s^t f(r, W_r + \psi_s) dr$ so that, for $u \in [s, t]$,

$$\delta A_{s,u,t} := A_{s,t} - A_{s,u} - A_{u,t} = \int_u^t [f(r, W_r + \psi_s) - f(r, W_r + \psi_u)] dr.$$

Applying (4.2), we have

$$|\delta A_{s,u,t}| \leq \Xi_{T,K}(f) |\psi_u - \psi_s|^{1-\varepsilon} (t-s)^\alpha \leq \Xi_{T,K}(f) [\psi]_{\text{var};[s,t]}^{1-\varepsilon} (t-s)^\alpha.$$

That $\varepsilon < \alpha$ ensures that $1 - \varepsilon + \alpha > 1$. Since f is continuous, we have for every $t \in [0, T]$,

$$\int_0^t f(r, W_r + \psi_r) dr = \lim_{|\pi| \downarrow 0} \sum_{[a,b] \in \pi} A_{a,b}$$

where π is any partition of $[0, t]$. We apply the sewing lemma ([FH14]) to obtain (4.7).

Step 2. To remove the continuity assumption and show the result for a bounded measurable function f , we use similar arguments as in Lemma 4.1.

First, using Urysohn lemma, one can show that (4.7) holds for $f = \mathbf{1}_U$ for any open set U of finite measure. In particular, we have a.s.

$$\left| \int_s^t \mathbf{1}_U(r, W_r + \psi_r) dr \right| \lesssim \Xi_{T,K}(\mathbf{1}_U) (1 + [\psi]_{\text{var};[s,t]}^{1-\varepsilon}) (t-s)^\alpha \quad (4.9)$$

for every $(s, t) \in [0, T]_\leq^2$ and every ψ .

Next, applying Lusin theorem, we can find for each integer $n \geq 1$ a continuous bounded function f^n and an open set U^n such that $|U^n| \leq 2^{-n}$ and $|f - f^n| \leq 2M \mathbf{1}_{U^n}$, where M is a constant such that $|f| \leq M$. This implies that $\lim_n f^n = f$ in \mathbb{L}_p^q . For each $s \leq t$, we put

$$J[f] = \int_s^t f(r, W_r + \psi_r) dr - \int_s^t f(r, W_r + \psi_s) dr$$

so that $J[f] = J[f^n] + J[f - f^n]$. By the previous step,

$$|J[f^n]| \leq C \Xi_{T,K}(f^n) [\psi]_{\text{var};[s,t]}^{1-\varepsilon} (t-s)^\alpha.$$

Using (4.3) and (4.9),

$$\begin{aligned} |J[f - f^n]| &\lesssim M \left| \int_s^t \mathbf{1}_{U^n}(r, W_r + \psi_r) dr \right| + \sup_{x \in K} \left| \int_s^t \mathbf{1}_{U^n}(r, W_r + x) dr \right| \\ &\lesssim M T^\alpha (1 + [\psi]_{\text{var};[s,t]}^{1-\varepsilon}) \Xi_{T,K}(\mathbf{1}_{U^n}). \end{aligned}$$

By Lemma 4.1(i)(ii), we can further choose a subsequence, still denoted by n , such that

$$\lim_n \Xi_{T,K}(f^n) = \Xi_{T,K}(f) \quad \text{and} \quad \lim_n \Xi_{T,K}(\mathbf{1}_{U^n}) = 0 \quad \text{a.s.}$$

We emphasize that the null events only depend on f, T, K and are independent from s, t, ψ . Then, by passing through the limit in n , we obtain that

$$|J[f]| \leq C \Xi_{T,K}(f) [\psi]_{\text{var};[s,t]}^{1-\varepsilon} (t-s)^\alpha.$$

Step 3. Consider now the case $f \in \mathbb{L}_p^q$. For each integer $M \geq 1$, define $f^M = f \mathbf{1}_{(|f| \leq M)}$. Note that (f^M) (respectively $(|f^M|)$) is a sequence of bounded functions converging to f (respectively $|f|$) in \mathbb{L}_p^q . We can choose a sequence (M_n) such that $(\Xi_{T,K}(f^{M_n}), \Xi_{T,K}(|f^{M_n}|))$ converges to $(\Xi_{T,K}(f), \Xi_{T,K}(|f|))$ a.s. Using this, (4.8) (with $|f^{M_n}|$) and monotone convergence we then have a.s. ω

$$\begin{aligned} \int_s^t |f(r, W_r(\omega) + \psi_r)| dr &= \lim_{n \rightarrow \infty} \int_s^t |f^{M_n}(r, W_r(\omega) + \psi_r)| dr \\ &\leq C \Xi_{T,K}(|f|)(\omega) (1 + [\psi]_{\text{var};[s,t]}^{1-\varepsilon}) (t-s)^\alpha < \infty. \end{aligned}$$

From the above estimate and (4.3), applying the Lebesgue dominated convergence theorem, we have $\lim_n J[f - f^{M_n}] = 0$. As in the previous step, we have

$$\begin{aligned} |J[f]| &\leq |J[f^{M_n}]| + |J[f - f^{M_n}]| \\ &\leq \Xi_{T,K}(f^{M_n}) [\psi]_{\text{var};[s,t]}^{1-\varepsilon} (t-s)^\alpha + |J[f - f^{M_n}]|. \end{aligned}$$

We take the limit in n to obtain (4.7).

This concludes the proof. \square

Corollary 4.4. *Let f be a function in \mathbb{L}_p^q for some $(p, q) \in \mathcal{J}$. Let (\mathcal{Z}, ρ) be a metric space and $(t, z) \mapsto \psi_t^z$ be a bounded function from $[0, T] \times \mathcal{Z}$ to \mathbb{R} such that*

- (i) *for each $z \in \mathcal{Z}$, $t \mapsto \psi_t^z$ has finite variation and $\sup_{z \in \mathcal{Z}} [\psi^z]_{\text{var};[0,T]} < \infty$;*
- (ii) *the family $\{z \mapsto \psi_t^z\}_{t \in [0,T]}$ is uniformly equicontinuous, i.e.*

$$\lim_{h \downarrow 0} \sup_{t \in [0,T]} \sup_{\rho(z, \bar{z}) \leq h} |\psi_t^z - \psi_t^{\bar{z}}| = 0.$$

Then there exist an event $\Omega_{f,T}$ of full measure and a sequence of bounded continuous functions (f^n) which are independent from ψ and (\mathcal{Z}, ρ) , such that $\lim_n f^n = f$ in \mathbb{L}_p^q and for every $\omega \in \Omega_{f,T}$,

$$\lim_n \int_0^t f^n(r, W_r(\omega) + \psi_r^z) dr = \int_0^t f(r, W_r(\omega) + \psi_r^z) dr \text{ uniformly in } (t, z) \in [0, T] \times \mathcal{Z}.$$

Consequently, the map

$$(t, z) \mapsto \int_0^t f(r, W_r(\omega) + \psi_r^z) dr$$

is continuous on $[0, T] \times \mathcal{Z}$.

Proof. Let (K^i) be an increasing sequence of compacts such that $\bigcup_i K^i = \mathbb{R}$. Let Ω_{f,T,K^i} be as in Lemma 4.3. In view of (4.1), we can choose a sequence of bounded continuous functions (f^n) such that $\lim_n \Xi_{T,K^i}(f^n - f) = 0$ a.s. for all $i \in \mathbb{N}$. Because ψ is bounded, there exists

$j \in \mathbb{N}$ such that $\psi_t^z \in K^j$ for all $(t, z) \in [0, T] \times \mathcal{Z}$. Applying (4.8) and (i), we have for any $\omega \in \Omega_{f,T} := \bigcap_i \Omega_{f,T,K^i}$,

$$\lim_n \int_0^t f^n(r, W_r(\omega) + \psi_r^z) dr = \int_0^t f(r, W_r(\omega) + \psi_r^z) dr \text{ uniformly in } (t, z) \in [0, T] \times \mathcal{Z}.$$

Due to (ii), the function $(t, z) \mapsto \int_0^t f^n(r, W_r(\omega) + \psi_r^z) dr$ is continuous for each n . This shows the claim. \square

Remark 4.5. We record some consequential observations which may be useful for other purposes.

(i) From Lemma 4.3, it follows that for a.s. ω , a solution of (1.5) is also a solution in the framework of nonlinear Young integrals. More precisely, for a.s. ω , if Y is a solution to (1.5), then there is a control w and a number $\beta > 1$ such that for $\psi = Y - W$,

$$|\psi_t - \psi_s - \int_s^t b(r, W_r(\omega) + \psi_s) dr| \leq w(s, t)^\beta \quad \forall (s, t) \in [0, T]_{\leq}^2.$$

Via the sewing lemma, this means that ψ is a solution to the nonlinear Young integral equation ([HL17])

$$\psi_t = \psi_s + \int_s^t b^W(dr, \psi_r) \quad \forall (s, t) \in [0, T]_{\leq}^2,$$

where $\int_s^t b^W(dr, \psi_r)$ is the nonlinear Young integral defined as the limit of the Riemann sums

$$\sum_{[u,v]} \int_u^v b(r, W_r + \psi_u) dr.$$

While nonlinear Young integral equations have been a central theme in previous works [CG16, GG22b, HP21, GG22a, ART21], it was not known if (1.5) under (1.4) can be formulated in this framework.

(ii) From Corollary 4.4, it follows that for a.s. ω , a solution of (1.5) is also a *regularized solution* in the following sense: For a.s. ω , if Y is a solution to (1.5) then there exists a sequence of bounded continuous function (b^n) and a continuous function $V : [0, T] \rightarrow \mathbb{R}^d$ such that $V_t = \lim_n \int_0^t b^n(r, Y_r) dr$ uniformly on $[0, T]$ and $Y_t = Y_0 + V_t + W_t(\omega)$ for all $t \in [0, T]$. Regularized solutions of stochastic differential equations appear in [BC01, ABLM20, BLM23b, ART21].

5. PROOF OF MAIN RESULTS

For a given ω and a solution Y to (1.5), we obtain in Proposition 5.1 a priori estimates on the variations of $Y - W$ on arbitrary intervals. This allows us to show the existence of random Hölder continuous semiflow $(X_t^{s,x})_{s,t,x}$, i.e. Proposition 2.1. We then show in Proposition 5.2 another set of a priori estimates on $Y_t - X_t^{s,Y_s}(\omega)$ for any given $(s, t) \in [0, T]_{\leq}^2$. Having these properties at our disposal, we proceed to prove path-by-path uniqueness and stability, i.e. Theorems 2.2 and 2.4.

Proposition 5.1. Let $K \subset \mathbb{R}^d$ be a compact set containing 0, $(u, v) \in [0, T]_{\leq}^2$ and $\alpha \in (0, \frac{1}{2} - \frac{d}{2p} - \frac{1}{q})$. Let ω be such that (4.7) and (4.8) with $f = |b|$ hold. Let $Y : [u, v] \rightarrow \mathbb{R}^d$ be a solution to (1.5) on $[u, v]$, i.e. $\int_u^v |b(r, Y_r)| dr < \infty$ and

$$Y_t = Y_s + \int_s^t b(r, Y_r) dr + W_t(\omega) - W_s(\omega) \quad \forall (s, t) \in [u, v]_{\leq}^2.$$

Assume that $Y_t - W_t(\omega) \in K$ for all $t \in [u, v]$. Then there exists a finite constant $C = C(\varepsilon, \alpha)$ such that

$$\int_u^v |b|(r, Y_r) dr \leq \Xi_{T,K}(|b|)(\omega)(v-u)^\alpha + C(\Xi_{T,K}(|b|)(\omega)(v-u)^\alpha)^{\frac{1}{\varepsilon}}. \quad (5.1)$$

Proof. We omit the dependence on ω in this proof. Define $\psi = Y - W$ and note that

$$[\psi]_{\text{var}; [u, v]} \leq \int_u^v |b|(r, Y_r) dr.$$

Applying Lemma 4.3, we have

$$\int_u^v |b|(r, Y_r) dr \leq \Xi_{T,K}(|b|)(v-u)^\alpha + C\Xi_{T,K}(|b|) \left(\int_u^v |b|(r, Y_r) dr \right)^{1-\varepsilon} (v-u)^\alpha.$$

Applying Young's inequality, we have for every $\varepsilon' > 0$

$$\Xi_{T,K}(|b|) \left| \int_u^v |b|(r, Y_r) dr \right|^{1-\varepsilon'} (v-u)^\alpha \leq \varepsilon' \int_u^v |b|(r, Y_r) dr + C_{\varepsilon'} (\Xi_{T,K}(|b|)(v-u)^\alpha)^{\frac{1}{\varepsilon'}}.$$

We choose ε' sufficiently small to get (5.1). \square

Proof of Proposition 2.1. For each s, x , let $(\tilde{X}_t^{s,x})_{t \in [s, T]}$ be the unique strong solution to (1.3) started from x at time s . By definition of a solution, for every ω in a set of full measure that depends on s and x , we have $\int_s^T |b(r, \tilde{X}_r^{s,x}(\omega))| dr < \infty$ and

$$\tilde{X}_t^{s,x}(\omega) = x + \int_s^t b(r, \tilde{X}_r^{s,x}(\omega)) dr + W_t(\omega) - W_s(\omega) \quad \forall t \in [s, T]. \quad (5.2)$$

Let $m \geq 2$ be a fixed number. By [GL23, Theorem 1.2 (1.2)]

$$\|\tilde{X}_t^{s,x} - \tilde{X}_t^{s,y}\|_{L^m(\Omega)} \lesssim \|x - y\|_{L^m(\Omega)} \quad (5.3)$$

for any initial random points x, y which are \mathcal{F}_s -measurable. Applying [LL21, Lemma 4.5], we get

$$\|\tilde{X}_t^{s,x} - \tilde{X}_{t'}^{s,x}\|_{L^m(\Omega)} \lesssim \left\| \int_{t'}^t b(r, \tilde{X}_r^{s,x}) dr \right\|_{L^m(\Omega)} \lesssim |t - t'|^\alpha$$

for some $\alpha \in [\frac{1}{2}, 1)$. Using (5.3) and pathwise uniqueness, we have for $s' < s$,

$$\|\tilde{X}_t^{s,x} - \tilde{X}_t^{s',x}\|_{L^m(\Omega)} = \|\tilde{X}_t^{s,x} - \tilde{X}_t^{s, \tilde{X}_s^{s',x}}\|_{L^m(\Omega)} \lesssim \|x - \tilde{X}_s^{s',x}\|_{L^m(\Omega)} \lesssim |s - s'|^{\frac{1}{2}}.$$

It follows, that

$$\|\tilde{X}_t^{s,x} - \tilde{X}_{t'}^{s',y}\|_{L^m(\Omega)} \lesssim |x - y| + |t - t'|^\alpha + |s - s'|^{\frac{1}{2}}. \quad (5.4)$$

Since m can be arbitrarily large, applying Kolmogorov continuity criterion ([RY99, Theorem (2.1) Chapter I]), we see that for a.s. ω , the map $(s, t, x) \mapsto \tilde{X}_t^{s,x}(\omega)$ is locally Hölder continuous on

$$\{(s, t, x) \in [0, T]_{\leq}^2 \times \mathbb{R}^d : s, t, x \text{ are dyadic}\}$$

with exponents $(\alpha', \beta, \kappa) \in (0, \alpha) \times (0, 1/2) \times (0, 1)$. Because for each s, x , $(\tilde{X}_t^{s,x}(\omega))_{t \in [s, T]}$ is continuous, the map $(s, t, x) \mapsto \tilde{X}_t^{s,x}(\omega)$ is locally Hölder continuous on

$$G := \{(s, t, x) \in [0, T]_{\leq}^2 \times \mathbb{R}^d : s, x \text{ are dyadic}\}$$

with the same exponents.

Let Ω' be the event of full measure on which (5.2) holds whenever s, x are dyadic, and \tilde{X} is locally Hölder continuous on G . For each $\omega \in \Omega'$, let $X(\omega)$ be the (unique) continuous extension of $\tilde{X}(\omega)|_G$ to $[0, T]_{\leq}^2 \times \mathbb{R}^d$. We show that $(X_t^{s,x})_{s,t,x}$ is the desired semiflow. Hölder regularity is clear and hence, we focus on showing the semiflow properties. Since b is not continuous, it is non-trivial that for almost every ω and for each (s, x) , the map $t \mapsto X_t^{s,x}(\omega)$ satisfies the equation (1.5). The main difficulty is to show that the continuous extension of the map

$$(s, t, x) \mapsto \int_s^t b(r, \tilde{X}_r^{s,x}(\omega)) dr = \int_s^t b(r, X_r^{s,x}(\omega)) dr$$

which is defined on G , is identical to the map

$$(s, t, x) \mapsto \int_s^t b(r, X_r^{s,x}(\omega)) dr,$$

which is defined on $[0, T]_{\leq}^2 \times \mathbb{R}^d$.

Let $H \subset \mathbb{R}^d$ be a compact set and put $\mathcal{Z} = [0, T] \times H$. For each $\omega \in \Omega'$, we verify the conditions (i) and (ii) of Corollary 4.4. For each $(t, s, x) \in [0, T] \times \mathcal{Z}$, define $\psi_t^{s,x}(\omega) := X_{\min(s,t)}^{s,x}(\omega) - W_{\min(s,t)}(\omega)$, and if s, x are dyadic, define $\tilde{\psi}_t^{s,x}(\omega) := \tilde{X}_{\min(s,t)}^{s,x}(\omega) - W_{\min(s,t)}(\omega)$. Note that $\psi(\omega)$ is the continuous extension of $\tilde{\psi}(\omega)$. From (5.2), applying Proposition 5.1, we can find a constant $C_T(\omega)$ such that whenever s, x are dyadic,

$$[\psi^{s,x}(\omega)]_{\text{var}; [0, T]} = [\tilde{\psi}^{s,x}(\omega)]_{\text{var}; [0, T]} \leq \int_s^T |b|(r, \tilde{X}_r^{s,x}(\omega)) dr \leq C_T(\omega).$$

This means that for any partition π of $[0, T]$ and (s, x) dyadic in \mathcal{Z}

$$\sum_{[u,v] \in \pi} |\psi_v^{s,x}(\omega) - \psi_u^{s,x}(\omega)| \leq C_T(\omega).$$

By continuity, the above estimate also holds for every $(s, x) \in \mathcal{Z}$. This implies that

$$\sup_{(s,x) \in \mathcal{Z}} [\psi^{s,x}(\omega)]_{\text{var}; [0, T]} \leq C_T(\omega),$$

verifying (i). Uniform equicontinuity condition (ii) is satisfied because $X(\omega)$ is locally Hölder continuous on $[0, T]_{\leq}^2 \times \mathbb{R}^d$. Let $\Omega_{b,T}$ be the event in Corollary 4.4. Let $\omega \in \Omega' \cap \Omega_{b,T}$. Applying Corollary 4.4, we see that the map

$$(t, s, x) \mapsto \int_0^t b(r, W_r(\omega) + \psi_r^{s,x}(\omega)) dr = \int_0^t b(r, X_r^{s,x}(\omega)) dr$$

is continuous on $[0, T] \times \mathcal{Z}$.

It is now clear from (5.2) that for every $\omega \in \Omega' \cap \Omega_{b,T}$, for every $(s, t, x) \in [0, T]_{\leq}^2 \times H$, we have

$$X_t^{s,x}(\omega) = x + \int_s^t b(r, X_r^{s,x}(\omega)) dr + W_t(\omega) - W_s(\omega).$$

By exhausting \mathbb{R}^d with a sequence of increasing compact sets (H^i) , we see that the above equation holds for every $(s, t, x) \in [0, T]_{\leq}^2 \times \mathbb{R}^d$. By pathwise uniqueness, we have for every $s \leq u \leq t$ and every $x \in \mathbb{R}^d$

$$X_t^{s,x} = X_t^{u, X_u^{s,x}} \quad \text{a.s.} \quad (5.5)$$

Note that the exceptional null event (5.5) depends on s, u, t, x . However, because all processes in (5.5) are continuous, one can deduce that for a.s. ω , $X_t^{s,x} = X_t^{u, X_u^{s,x}}$ for any s, u, t, x , which means that $(X_t^{s,x}(\omega))_{s,t,x}$ is a semiflow. \square

The next result, together with Proposition 5.1, provide *a priori* estimates for any solution.

Proposition 5.2. *Let $K \subset \mathbb{R}^d$ be a compact set which contains 0. Let $\Omega_{b,T,K}$ be the event with full probability on which $(X_t^{s,x})$ is a κ -Hölder continuous semiflow for every $\kappa \in (0, 1)$ and (4.7) with $f \in \{b, |b|\}$ holds. Let $\omega \in \Omega_{b,T,K}$ and Y be a solution to (1.5). Suppose that $X_t^{s, Y_s}(\omega) - W_t(\omega)$ belongs to K for every $(s, t) \in [0, T]_{\leq}^2$ with $s \in Q_Y$. Then, there exist*

- (i) a control w which depends only on $Y, \Xi_{T,K}(b)(\omega), \Xi_{T,K}(|b|)(\omega), d, p, q$;
- (ii) a constant $\beta > 1$ which depends only on d, p, q

such that

$$|Y_t - X_t^{s, Y_s}(\omega)| \leq w(s, t)^\beta \quad \forall (s, t) \in [0, T]_{\leq}^2. \quad (5.6)$$

Proof. We omit the dependence of ω in the argument below. Define $\psi = Y - W$ and $\xi^{s,x} = X_{\min(s, \cdot)}^{s,x} - W_{\min(s, \cdot)}$. The proof consists of two steps.

Step 1. We show that for $\varepsilon \in (0, \alpha)$, there exist a constant $C = C(\Xi_{T,K}(|b|), \varepsilon)$ such that

$$\left| \int_s^t [|b|(r, X_r^{s, Y_s}) - |b|(r, Y_r)] dr \right| \leq C \left(\int_s^t |b|(r, Y_r) dr \right)^{1-\varepsilon} (t-s)^\alpha + C(t-s)^{\frac{\alpha}{\varepsilon}} \quad \forall (s, t) \in [0, T]_{\leq}^2. \quad (5.7)$$

As $\psi_s = \xi_s^{s, Y_s}$, we obtain that

$$\int_s^t [|b|(r, X_r^{s, Y_s}) - |b|(r, Y_r)] dr$$

$$= \int_s^t [|b|(r, X_r^{s, Y_s}) - |b|(r, W_r + \xi_s^{s, Y_s})] dr - \int_s^t [|b|(r, Y_r) - |b|(r, W_r + \psi_s)] dr.$$

We note that ξ^{s, Y_s} and ψ are contained in K and have finite variation such that

$$[\xi^{s, Y_s}]_{\text{var}; [s, t]} \leq \int_s^t |b|(r, X_r^{s, Y_s}) dr \quad \text{and} \quad [\psi]_{\text{var}; [s, t]} \leq \int_s^t |b|(r, Y_r) dr.$$

Applying [Lemma 4.3](#), we have for every $\varepsilon \in (0, \alpha)$ and every $(s, t) \in [0, T]_{\leq}^2$,

$$\begin{aligned} \left| \int_s^t [|b|(r, X_r^{s, Y_s}) - |b|(r, W_r + \xi_s^{s, Y_s})] dr \right| &\lesssim \Xi_{T, K}(|b|) \left(\int_s^t |b|(r, X_r^{s, Y_s}) dr \right)^{1-\varepsilon} (t-s)^\alpha, \\ \left| \int_s^t [|b|(r, Y_r) - |b|(r, W_r + \psi_s)] dr \right| &\lesssim \Xi_{T, K}(|b|) \left(\int_s^t |b|(r, Y_r) dr \right)^{1-\varepsilon} (t-s)^\alpha. \end{aligned}$$

We also have

$$\left| \int_s^t |b|(r, X_r^{s, Y_s}) dr \right|^{1-\varepsilon} \leq \left| \int_s^t [|b|(r, X_r^{s, Y_s}) - |b|(r, Y_r)] dr \right|^{1-\varepsilon} + \left| \int_s^t |b|(r, Y_r) dr \right|^{1-\varepsilon}. \quad (5.8)$$

It follows that

$$\begin{aligned} \left| \int_s^t [|b|(r, X_r^{s, Y_s}) - |b|(r, Y_r)] dr \right| &\lesssim \Xi_{T, K}(|b|) \left| \int_s^t [|b|(r, X_r^{s, Y_s}) - |b|(r, Y_r)] dr \right|^{1-\varepsilon} (t-s)^\alpha \\ &\quad + \Xi_{T, K}(|b|) \left(\int_s^t |b|(r, Y_r) dr \right)^{1-\varepsilon} (t-s)^\alpha. \end{aligned}$$

Applying Young's inequality, we have for every $\varepsilon' > 0$

$$\begin{aligned} \Xi_{T, K}(|b|) \left| \int_s^t [|b|(r, X_r^{s, Y_s}) - |b|(r, Y_r)] dr \right|^{1-\varepsilon} (t-s)^\alpha \\ \leq \varepsilon' \left| \int_s^t [|b|(r, X_r^{s, Y_s}) - |b|(r, Y_r)] dr \right| + C_{\varepsilon'} (\Xi_{T, K}(|b|) (t-s)^\alpha)^{\frac{1}{\varepsilon}}. \end{aligned} \quad (5.9)$$

Combining the previous two inequality and choosing ε' sufficiently small yields [\(5.7\)](#).

Step 2. We show [\(5.6\)](#). Let $(s, t) \in [0, T]_{\leq}^2$ be fixed. Using the equations and the identity $\psi_s = \xi_s^{s, Y_s}$, we obtain that

$$\begin{aligned} |X_t^{s, Y_s} - Y_t| &= \left| \int_s^t [b(r, X_r^{s, Y_s}) - b(r, Y_r)] dr \right| \\ &\leq \left| \int_s^t [b(r, X_r^{s, Y_s}) - b(r, W_r + \xi_s^{s, Y_s})] dr \right| + \left| \int_s^t [b(r, Y_r) - b(r, W_r + \psi_s)] dr \right|. \end{aligned}$$

Applying [Lemma 4.3](#), we have for every $\varepsilon \in (0, \alpha)$,

$$\left| \int_s^t [b(r, X_r^{s, Y_s}) - b(r, W_r + \xi_s^{s, Y_s})] dr \right| \lesssim \Xi_{T, K}(b) \left(\int_s^t |b|(r, X_r^{s, Y_s}) dr \right)^{1-\varepsilon} (t-s)^\alpha,$$

$$\left| \int_s^t [b(r, Y_r) - b(r, W_r + \psi_s)] dr \right| \lesssim \Xi_{T,K}(b) \left(\int_s^t |b|(r, Y_r) dr \right)^{1-\varepsilon} (t-s)^\alpha.$$

Combining with (5.8), we have

$$\begin{aligned} |X_t^{s, Y_s} - Y_t| &\lesssim \Xi_{T,K}(b) \left| \int_s^t [|b|(r, X_r^{s, Y_s}) - |b|(r, Y_r)] dr \right|^{1-\varepsilon} (t-s)^\alpha \\ &\quad + \Xi_{T,K}(b) \left(\int_s^t |b|(r, Y_r) dr \right)^{1-\varepsilon} (t-s)^\alpha. \end{aligned}$$

We apply (5.9) and (5.7) to get

$$|X_t^{s, Y_s} - Y_t| \leq C(\varepsilon, \Xi_{T,K}(|b|), \Xi_{T,K}(b)) \left(\left(\int_s^t |b|(r, Y_r) dr \right)^{1-\varepsilon} (t-s)^\alpha + (t-s)^{\alpha/\varepsilon} \right).$$

Choosing ε small enough such that $\alpha/\varepsilon > 1$ yields (5.6). \square

Proof of Theorem 2.2. Let $K_n = \{x \in \mathbb{R}^d : |x| \leq n\}$ ($n \in \mathbb{N}$) and $\Omega_{b,T} = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \Omega_{b,T,K_n}$ for Ω_{b,T,K_N} as in Proposition 5.2. Note that $\Omega_{b,T}$ has full probability. Let $\omega \in \bigcap_{n=k}^{\infty} \Omega_{b,T,K_n}$ for some k and let Y be a solution to (1.5). By continuity, we can choose $N \geq k$ such that K_N contains Y_t and $X_t^{s, Y_s}(\omega) - W_t(\omega)$ for every $(s, t) \in [0, T]_{\leq}^2$. Let w be the control and β be the constant found in Proposition 5.2. We note that w depends on N . Let $\tau \in (0, T]$ be a fixed but arbitrary number and define

$$F(t) = X_t^{t, Y_t}, \quad t \in [0, \tau].$$

For $(s, t) \in [0, \tau]_{\leq}^2$, we obtain by the semiflow property that

$$F(t) - F(s) = X_t^{t, Y_t} - X_t^{s, Y_s} = X_t^{t, Y_t} - X_t^{t, X_t^{s, Y_s}}.$$

Then, using Hölder continuity of the semiflow (Proposition 2.1), we have

$$|F(t) - F(s)| \lesssim |Y_t - X_t^{s, Y_s}|^\kappa,$$

for some $\kappa \in (0, 1)$ which can be chosen to be arbitrarily close to 1. Applying (5.6), we have

$$|F(t) - F(s)| \lesssim w(s, t)^{\kappa\beta}.$$

Choosing κ so that $\kappa\beta > 1$, for any sequence of partitions $\Pi_n = \{t_i\}_{i=0}^{N_n}$ of $[0, \tau]$ with mesh converging to 0, we get that

$$|F(\tau) - F(0)| \leq \sum_{i=1}^{N_n} |F(t_i) - F(t_{i-1})| \lesssim \left| \sup_{i \leq N_n} w(t_{i-1}, t_i) \right|^{\kappa\beta-1} w(0, \tau) \xrightarrow{n \rightarrow \infty} 0.$$

Hence $F(\tau) = F(0)$. Since τ was arbitrarily chosen in $(0, T]$, this means that Y and X^{0, Y_0} are identical. \square

Proof of Theorem 2.4. In the proof, we write $\|b^n - b\|$ for $\|b^n - b\|_{L^q([0,T], W^{-\nu,p}(\mathbb{R}^d))}$, where p, q and ν satisfy either (2.1) or (2.2) as in the statement. Denote by $(X_t^{s,x,n})_{s,t,x}$ the Hölder continuous semiflow of solutions to

$$X_t = X_0 + \int_0^t b^n(r, X_r) dr + W_t, \quad t \in [0, T]. \quad (5.10)$$

In view of Theorem 2.2, it suffices to show that there exists an event $\Omega'_{(b^n),b,T}$ such that for every $\omega \in \Omega'_{(b^n),b,T}$,

$$\lim_{n \rightarrow \infty} X^n(\omega) = X(\omega) \text{ uniformly over compact sets of } [0, T]_{\leq}^2 \times \mathbb{R}^d. \quad (5.11)$$

From (5.4) and [GL23, Theorem 1.2, Remark 2.13], putting $V_{s,t,x} = X_t^{s,x,n} - X_t^{s,x}$, we can find an $\alpha \in [\frac{1}{2}, 1)$ such that for every $m \geq 1, t, t' \in [0, T]$ and $x, x' \in \mathbb{R}^d$

$$\|V_{s,t,x} - V_{s',t',x'}\|_{L^m(\Omega)} \lesssim |s - s'|^{1/2} + |t - t'|^\alpha + |x - x'|,$$

and

$$\|V_{s,t,x}\|_{L^m(\Omega)} \lesssim \|b^n - b\|.$$

Hence,

$$\|V_{s,t,x} - V_{s',t',x'}\|_{L^m(\Omega)} \lesssim (|s - s'|^{1/2} + |t - t'|^\alpha + |x - x'|)^{1/2} \|b^n - b\|^{1/2}.$$

Applying Garsia–Rodemich–Rumsey inequality, for every compact set $K \subset \mathbb{R}^d$, every $m \geq 1$, we have

$$\left\| \sup_{(s,t,x) \in [0,T]_{\leq}^2 \times K} |V_{s,t,x}| \right\|_{L^m(\Omega)} \leq C_{m,T,K} \|b^n - b\|^{1/2} \quad (5.12)$$

for some finite constant $C_{m,T,K}$.

Let (K^i) be an increasing sequence of compacts such that $\bigcup_{i \in \mathbb{N}} K^i = \mathbb{R}^d$. By definition of summable convergence, we can choose $m \geq 2$ so that

$$\sum_n \|b^n - b\|_{\frac{m}{4}} < \infty.$$

Using (5.12) and Markov's inequality, we have for every i

$$\mathbb{P} \left(\sup_{(s,t,y) \in [0,T]_{\leq}^2 \times K^i} |X_t^{s,y,n} - X_t^{s,y}| > \|b^n - b\|_{\frac{m}{4}} \right) \leq C_{m,T,K^i}^m \|b^n - b\|_{\frac{m}{4}}^m.$$

Applying Borel–Cantelli lemma, we get an event $\Omega^i_{(b^n),b,T}$ of full measure on which $X^n(\omega)$ converges to $X(\omega)$ uniformly on $[0, T]_{\leq}^2 \times K^i$. Putting $\Omega'_{(b^n),b,T} = \bigcap_i \Omega^i_{(b^n),b,T}$, it follows that (5.11) holds for every $\omega \in \Omega'_{(b^n),b,T}$. \square

Remark 5.3. The fact that for almost every ω , for every s, x , $(X_t^{s,x}(\omega))_t$ is a solution to (1.5) is crucial for the proof of path-by-path uniqueness. When b is not continuous, this property becomes highly non-trivial. It was taken for granted without any justifications in [Sha16, Sha17]. We achieved this property in Proposition 2.1 by utilizing the regularizing estimates from

Lemma 4.3. This issue is irrelevant to [Dav07] because of its different arguments. We take this chance to note that the exceptional null events in [Dav07] depend on the initial condition. This dependence can be removed following our arguments herein. In fact, one just replaces Proposition 3.1 by Davie's basic estimate, then the rest of the arguments follows with minimal adjustments. In particular, Proposition 2.1 and Theorems 2.2 and 2.4 hold with $p = q = \infty$.

APPENDIX A. ODE UNIQUENESS

Most of the arguments in Section 5 are independent from the probability space, and hence, independent from the probability law of the driving noise. This forms a uniqueness criterion for ODE (1.1) which does not require any regularity on the vector field, but instead relies on the regularizing effect of the driving signal γ with Lemma 4.3 being a prototype. This regularizing effect is formalized by the following definition.

Definition A.1. Let $f : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a measurable function. We say that γ is $(1 - \varepsilon, \alpha)$ -regularizing for f if there exist a control η and constants $\Xi_{T,K}$ for each compact set $K \subset \mathbb{R}^d$ such that

$$\left| \int_s^t [f(r, \gamma_r + \psi_r) - f(r, \gamma_r + \psi_s)] dr \right| \leq \Xi_{T,K} [\psi]_{\text{var}; [s,t]}^{1-\varepsilon} \eta(s, t)^\alpha$$

for every $(s, t) \in [0, T]_{\leq}^2$ and every $\psi : [0, T] \rightarrow K$ of finite variation.

Theorem A.2. Let $K \subset \mathbb{R}^d$ be a compact set which contains 0. Suppose that γ is $(1 - \varepsilon, \alpha)$ -regularizing for $b, |b|$. Let y be a solution to (1.1). Suppose that $\phi_t^{s, y_s} - \gamma_t$ belongs to K for every $(s, t) \in [0, T]_{\leq}^2$. Then, there exists a control w which depends only on $y, \Xi_{T,K}, \eta$ such that

$$|y_t - \phi_t^{s, y_s}| \leq w(s, t)^{\min(1-\varepsilon+\alpha, \frac{\alpha}{\varepsilon})} \quad \forall (s, t) \in [0, T]_{\leq}^2. \quad (\text{A.1})$$

Suppose furthermore that ϕ is locally κ -Hölder continuous for some $\kappa \in (0, 1]$ such that

$$\kappa \cdot \min(1 - \varepsilon + \alpha, \frac{\alpha}{\varepsilon}) > 1.$$

Then $(y_t)_{t \in [0, T]}$ is identical to $(\phi_t^{0, y_0})_{t \in [0, T]}$.

The proof of this result follows analogous arguments used in proving Proposition 5.2 and Theorem 2.2 and hence is omitted.

APPENDIX B. ADAPTED SOLUTIONS AND PATH-BY-PATH SOLUTIONS

This section is devoted to summarize different concepts of solutions and uniqueness to (1.3) where $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is measurable. Even though the concept of adapted solutions is by now well covered in textbooks, we also recall these notions in order to make a clear comparison to path-by-path solutions and path-by-path uniqueness. In particular, we state and prove Theorem B.3 to point out the connection between path-by-path uniqueness and adaptedness of solutions.

Definition B.1 (Existence).

- (i) If there exists a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ equipped with a Brownian motion $(W_t)_{t \in [0, T]}$ and an (\mathcal{F}_t) -adapted process $(X_t)_{t \in [0, T]}$ such that $\int_0^T |b(s, X_s)| ds < \infty$ and (X, W) fulfills (1.3) almost surely, we say that (X, W) is a *weak solution* to (1.3). If the choice of W is clear from the context, we write that X is a weak solution.
- (ii) We call X a *strong solution* if X is a weak solution and $(X_t)_{t \in [0, T]}$ is adapted to the filtration $(\mathcal{F}_t^W)_{t \in [0, T]}$.
- (iii) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which a Brownian motion W is defined. We call a mapping $X: \Omega \rightarrow C([0, T])$ a *path-by-path solution* to (1.3) if there exists a set $\tilde{\Omega} \subset \Omega$ of full measure such that, for every $\omega \in \tilde{\Omega}$, $X(\omega)$ is a solution to (1.5).

Definition B.2 (Uniqueness).

- (i) We say that *pathwise uniqueness* for (1.3) holds if for any two weak solutions (X, W) , (\tilde{X}, W) defined on the same filtered probability space with the same Brownian motion W and the same initial condition, X and \tilde{X} are indistinguishable.
- (ii) We say that *path-by-path uniqueness* for (1.3) holds if for any probability space on which a Brownian motion W is defined, there exists a set of full measure $\tilde{\Omega}$, such that for any $x_0 \in \mathbb{R}^d$ and any $\omega \in \tilde{\Omega}$, there exists a unique solution on $[0, T]$ to

$$X_t(\omega) = x_0 + \int_0^t b(s, X_s(\omega)) ds + W_t(\omega).$$

Note that in the literature the definition of path-by-path uniqueness sometimes allows for the set of full measure to depend on the initial condition x_0 , see for example [Fla11].

The following implications follow directly from the definitions:

$$\boxed{\text{strong existence}} \implies \boxed{\text{weak existence}} \implies \boxed{\text{path-by-path existence}}$$

$$\boxed{\text{path-by-path uniqueness}} \implies \boxed{\text{pathwise uniqueness}}$$

The following theorem can be summarized in the following way: For drifts for which pathwise uniqueness is known to hold, losing path-by-path uniqueness implies existence of a non-adapted solution.

Theorem B.3. *Let $b: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be measurable. Assume that*

- (i) *pathwise uniqueness to (1.3) holds,*
- (ii) *all path-by-path solutions to (1.3) are weak solutions.*

Then path-by-path uniqueness holds.

Proof. Assume that path-by-path uniqueness does not hold. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which a Brownian motion W is defined. Then there exists a set $A \subset \Omega$ of positive measure such that for $\omega \in A$, there exist multiple solutions to

$$X_t(\omega) = X_0 + \int_0^t b(s, X_s(\omega)) ds + W_t(\omega),$$

with the same initial condition. Then we can define two path-by-path solutions X^1 and X^2 to (1.3) by letting them agree with the unique strong solution on A^c and letting $X^1(\omega) \neq X^2(\omega)$ on A , which is possible by the above and the axiom of choice. By assumption, both X^1 and X^2 are adapted w.r.t. filtrations (\mathcal{F}_t^1) and (\mathcal{F}_t^2) such that W is a Brownian motion w.r.t. these filtrations. Hence, W is also a Brownian motion with respect to $(\mathcal{F}_t^1 \cup \mathcal{F}_t^2)$. Therefore we constructed two solutions on the same filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t^1 \cup \mathcal{F}_t^2)_{t \in [0, T]}, \mathbb{P})$. By construction $X^1 \neq X^2$ on a set of positive measure. This contradicts pathwise uniqueness. \square

Remark B.4. There are examples of drifts b not fulfilling (1.4) such that there exist multiple (non-adapted) path-by-path solutions to (1.3), even though pathwise uniqueness holds (see [SW22] for $d > 1$ and [Anz22] for $d = 1$). These counterexamples heavily rely on the time-dependence of the drift. This is leading to non-uniqueness globally (i.e. considering the equation on the whole interval $[0, T]$). We are not aware of examples of drifts for which such a phenomenon occurs locally. Finally, note that Theorem 2.2 prevents such counterexamples for drifts fulfilling (1.4) since path-by-path solutions are identified on a set of full measure.

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REFERENCES

- [ABLM20] Siva Athreya, Oleg Butkovsky, Khoa Lê, and Leonid Mytnik. Well-posedness of stochastic heat equation with distributional drift and skew stochastic heat equation. *arXiv preprint arXiv:2011.13498*, 2020.
- [Anz22] Lukas Anzeletti. Comparison of classical and path-by-path solutions to SDEs. *arXiv preprint arXiv:2204.07866*, 2022.
- [ART21] Lukas Anzeletti, Alexandre Richard, and Etienne Tanré. Regularisation by fractional noise for one-dimensional differential equations with nonnegative distributional drift. *arXiv preprint arXiv:2112.05685*, 2021.
- [BC01] Richard F. Bass and Zhenqing Chen. Stochastic differential equations for Dirichlet processes. *Probab. Theory Related Fields*, 121(3):422–446, 2001.
- [BFGM19] Lisa Beck, Franco Flandoli, Massimiliano Gubinelli, and Mario Maurelli. Stochastic ODEs and stochastic linear PDEs with critical drift: regularity, duality and uniqueness. *Electron. J. Probab.*, 24:Paper No. 136, 72, 2019.
- [BLM23a] Oleg Butkovsky, Khoa Lê, and Toyomu Matsuda. Strong regularization of differential equations with integrable drifts by fractional noise. *in preparation*, 2023+.
- [BLM23b] Oleg Butkovsky, Khoa Lê, and Leonid Mytnik. Stochastic equations with singular drift driven by fractional Brownian motion. *arXiv preprint arXiv:2302.11937*, 2023.

- [BNP19] David Baños, Torstein Nilssen, and Frank Proske. Strong existence and higher order Fréchet differentiability of stochastic flows of fractional Brownian motion driven SDEs with singular drift. *J. Dyn. Diff. Equat.*, pages 1–48, 2019.
- [CD22] Rémi Catellier and Romain Duboscq. Regularization by noise for rough differential equations driven by Gaussian rough paths. *arXiv preprint arXiv:2207.04251*, 2022.
- [CG16] Rémi Catellier and Massimiliano Gubinelli. Averaging along irregular curves and regularisation of ODEs. *Stochastic Process. Appl.*, 126(8):2323–2366, 2016.
- [Dav07] Alexander M. Davie. Uniqueness of solutions of stochastic differential equations. *Int. Math. Res. Not. IMRN*, 2007(24):Art. ID rnm124, 26, 2007.
- [DG22] Konstantinos Dareiotis and Máté Gerencsér. Path-by-path regularisation through multiplicative noise in rough, Young, and ordinary differential equations. *arXiv preprint <https://arxiv.org/abs/2207.03476>*, 2022.
- [FF13a] Ennio Fedrizzi and Franco Flandoli. Hölder flow and differentiability for SDEs with nonregular drift. *Stoch. Anal. Appl.*, 31(4):708–736, 2013.
- [FF13b] Ennio Fedrizzi and Franco Flandoli. Noise prevents singularities in linear transport equations. *J. Funct. Anal.*, 264(6):1329–1354, 2013.
- [FH14] Peter K. Friz and Martin Hairer. *A course on rough paths*. Universitext. Springer, Cham, 2014. With an introduction to regularity structures.
- [Fla11] Franco Flandoli. *Random perturbation of PDEs and fluid dynamic models*, volume 2015 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2011. Lectures from the 40th Probability Summer School held in Saint-Flour, 2010.
- [GG22a] Lucio Galeati and Máté Gerencsér. Solution theory of fractional SDEs in complete subcritical regimes. *arXiv preprint arXiv:2207.03475*, 2022.
- [GG22b] Lucio Galeati and Massimiliano Gubinelli. Noiseless regularisation by noise. *Rev. Mat. Iberoam.*, 38(2):433–502, 2022.
- [GL23] Lucio Galeati and Chengcheng Ling. Stability estimates for singular SDEs and applications. *Electron. J. Probab.*, 28:1–31, 2023.
- [HL13] Yaozhong Hu and Khoa Lê. A multiparameter Garsia-Rodemich-Rumsey inequality and some applications. *Stochastic Process. Appl.*, 123(9):3359–3377, 2013.
- [HL17] Yaozhong Hu and Khoa Lê. Nonlinear Young integrals and differential systems in Hölder media. *Trans. Amer. Math. Soc.*, 369(3):1935–2002, 2017.
- [HP21] Fabian Andsem Harang and Nicolas Perkowski. C^∞ -regularization of ODEs perturbed by noise. *Stoch. Dyn.*, 21(8):Paper No. 2140010, 29, 2021.
- [KR05] Nikolay V. Krylov and Michael Röckner. Strong solutions of stochastic equations with singular time dependent drift. *Probab. Theory Related Fields*, 131(2):154–196, 2005.
- [Lê20] Khoa Lê. A stochastic sewing lemma and applications. *Electron. J. Probab.*, 25:1–55, 2020.
- [Lê22] Khoa Lê. Quantitative John–Nirenberg inequality for stochastic process of bounded mean oscillation. *arXiv preprint arXiv:2210.15736*, 2022.
- [Li94] Xuemei Li. Strong p -completeness of stochastic differential equations and the existence of smooth flows on noncompact manifolds. *Probab. Theory Related Fields*, 100(4):485–511, 1994.
- [Liz70] Pëtr I. Lizorkin. Multipliers of Fourier integrals and estimates of convolutions in spaces with mixed norm. Applications. *Izv. Akad. Nauk SSSR Ser. Mat.*, 34:218–247, 1970.
- [LL21] Khoa Lê and Chengcheng Ling. Taming singular stochastic differential equations: A numerical method. *arXiv preprint arXiv:2110.01343*, 2021.
- [LS11] Xuemei Li and Michael Scheutzow. Lack of strong completeness for stochastic flows. *Ann. Probab.*, 39(4):1407–1421, 2011.
- [Lyo98] Terry J. Lyons. Differential equations driven by rough signals. *Rev. Mat. Iberoamericana*, 14(2):215–310, 1998.

- [NO02] David Nualart and Youssef Ouknine. Regularization of differential equations by fractional noise. *Stochastic Process. Appl.*, 102(1):103–116, 2002.
- [NO03] David Nualart and Youssef Ouknine. Stochastic differential equations with additive fractional noise and locally unbounded drift. In *Stochastic inequalities and applications*, volume 56 of *Progr. Probab.*, pages 353–365. Birkhäuser, Basel, 2003.
- [Rez14] Fraydoun Rezakhanlou. Regular flows for diffusions with rough drifts. *arXiv preprint arXiv:1405.5856*, 2014.
- [RS17] Sebastian Riedel and Michael Scheutzow. Rough differential equations with unbounded drift term. *J. Differential Equations*, 262(1):283–312, 2017.
- [RY99] Daniel Revuz and Marc Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.
- [Sha16] Alexander Shaposhnikov. Some remarks on Davie’s uniqueness theorem. *Proc. Edinb. Math. Soc. (2)*, 59(4):1019–1035, 2016.
- [Sha17] Alexander Shaposhnikov. Correction to the paper "some remarks on davie’s uniqueness theorem". *arXiv preprint arXiv:1703.06598*, 2017.
- [SS17] Michael Scheutzow and Susanne Schulze. Strong completeness and semi-flows for stochastic differential equations with monotone drift. *J. Math. Anal. Appl.*, 446(2):1555–1570, 2017.
- [SV06] Daniel W. Stroock and S. R. Srinivasa Varadhan. *Multidimensional diffusion processes*. Classics in Mathematics. Springer-Verlag, Berlin, 2006. Reprint of the 1997 edition.
- [SW22] Alexander Shaposhnikov and Lukas Wresch. Pathwise vs. path-by-path uniqueness. *Ann. Inst. Henri Poincaré Probab. Stat.*, 58(3):1640–1649, 2022.
- [Ver79] Alexander Ju. Veretennikov. Strong solutions of stochastic differential equations. *Teor. Veroyatnost. i Primenen.*, 24(2):348–360, 1979.
- [Zvo74] Alexander K. Zvonkin. A transformation of the phase space of a diffusion process that will remove the drift. *Mat. Sb. (N.S.)*, 93(135):129–149, 1974.

LUKAS ANZELETTI: UNIVERSITÉ PARIS-SACLAY, CENTRALESUPÉLEC, MICS AND CNRS FR-3487, FRANCE,
EMAIL: LUKAS.ANZELETTI@CENTRALESUPELEC.FR

KHOA LÊ: UNIVERSITY OF LEEDS, SCHOOL OF MATHEMATICS, LEEDS LS29JT, U.K., EMAIL: K.LE@LEEDS.AC.UK

CHENGCHENG LING: TECHNISCHE UNIVERSITÄT WIEN, INSTITUTE OF ANALYSIS AND SCIENTIFIC COMPUTING, 1040 WIEN, AUSTRIA, EMAIL: CHENGCHENG.LING@ASC.TUWIEN.AC.AT