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Expansion and attraction of RDS: long time behavior of the solution to singular SDE*

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Abstract

We provide a framework for studying the expansion rate of the image of a bounded set under a semi-flow in Euclidean space and apply it to stochastic differential equations (SDEs for short) with singular coefficients. If the singular drift of the SDE can be split into two terms, one of which is singular and the radial component of the other term is negative then, under suitable conditions, the random dynamical system generated by the SDE admits a pullback attractor.

Keywords: semi-flow; random dynamical system; pullback attractor; singular stochastic differential equation; Brownian motion: dispersion of random sets; chaining; Krylov estimate; regularization by noise; elliptic partial differential equation; Zvonkin transformation.
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1 Introduction

Regularization by noise, i.e. existence and uniqueness of solutions under the assumption of non-degenerate noise, has been established for a large class of singular stochastic differential equations (SDEs). It was shown recently that these equations also generate a random dynamical system (RDS), see [20], and like in the classical (non-singular) case it therefore seems natural to establish asymptotic properties of these RDS for large times, like expansion rates of bounded sets and the existence of attractors or even *synchronization* (meaning that the attractor is a single random point).

We consider an SDE on $\mathbb{R}^d$ with time homogeneous coefficients

$$\mathrm{d}X_t = b(X_t)\ \mathrm{d}t + \sigma(X_t)\ \mathrm{d}W_t, \quad X_s = x \in \mathbb{R}^d, \quad t \geq s, \tag{1.1}$$

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where $d \geq 1$, $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma = (\sigma_{ij})_{1\leq i,j\leq d} : \mathbb{R}^d \to L(\mathbb{R}^d)$ $(:= d \times d$ real valued matrices) are measurable, and $(W_t)_{t\in\mathbb{R}}$ is a standard two-sided d-dimensional Brownian motion. We assume that $b \in \tilde{L}_p(\mathbb{R}^d)$ (defined in Section 2.1), so b does not have to be continuous nor bounded, and $\sigma\sigma^*$ (σ^* denotes the matrix transpose of σ) is bounded and uniformly elliptic and $\nabla\sigma \in \tilde{L}_p(\mathbb{R}^d)$ with $p > d$ (time homogeneous *Krylov-Röckner* condition). These are sufficient conditions for the well-posedness of the equation (1.1), see [16] and [27]. They also imply the existence of a flow and random dynamical system (RDS) generated by the solution to (1.1) [20].

First we analyse the linear expansion rate of the flow generated by a singular SDE. In simple stochastic models for the spread of passive tracers in a turbulent fluid, simulations supporting the conjecture that the expansion rate of a bounded set under a flow is linear in some non-singular models were first done in [2]. Later, this conjecture was proved in [4, 5, 23, 8] for various classes of stochastic flows. In classical results, see e.g. [23], [8], Lipschitz continuity or one-sided Lipschitz continuity of the coefficients of the SDE is assumed to obtain bounds on the expansion rate. Obviously we lack these properties in our current setting. Instead, we assume the noise to be non-degenerate, so we can apply the *Zvonkin transformation* to get an SDE which has Lipschitz-like coefficients and this SDE is (in an appropriate sense) equivalent to the original one (1.1). The *Zvonkin transformation* was invented by A. K. Zvonkin in [36] for $d = 1$ and then generalized by A. Yu. Veretennikov in [26] to $d \geq 1$. It has become a rather standard tool to study *well-posedness* of singular SDEs, see e.g. [30, 28] and [27]. This tool heavily relies on regularity estimates of the solution to Kolmogorov's equation corresponding to (1.1) which can be found for instance in [15] in the classical setting. In this paper we adapt the method to the study of the RDS induced by singular SDEs. We show that the flow expands linearly (see Theorem 5.4), a property which was established for non-singular SDEs with not necessarily non-degenerate noise in [4, 5, 21, 22, 23]. Our proof mainly depends on stability estimates (see Theorem 5.2). These kind of estimates were studied before, see for instance [13], [30] and [31], but the dependence of the constants on the coefficients was not specified. We give a formula in Theorem 5.2 which states this dependence explicitly. It also yields the expansion rate constant in Theorem 5.4.

Secondly, we aim at conditions which guarantee the existence of an attractor for the RDS generated by a singular SDE. Clearly, one can not expect that an attractor exists without further conditions (an example without attractor is the case in which the drift is zero and the diffusion is constant). Since [6], numerous papers appeared in which the existence of attractors for various finite and infinite dimensional RDS was shown, e.g. [3, 9, 10, 11, 12, 8, 17, 35]. A common way to prove the existence of an attractor is to show the existence of a random compact *absorbing* set and then to apply the criterion from [6, Theorem 3.11]. Just like [8], we will use a different and more probabilistic criterion from [7] (Proposition 2.8). Roughly speaking, all one has to show is that the image of a very large ball will be contained inside a fixed large ball after a (deterministic) long time with high probability. In [8] this was shown under the assumption that the diffusion is bounded and Lipschitz and the drift $b(x)$ has a component of sufficient strength (compared to the diffusion) in the direction of the origin for large $|x|$. In our set-up, this condition is too restrictive. Instead, we assume that the drift can be written in the form $b = b_1 + b_2$, in which b_1 is singular and b_2 is non-singular and has a component of sufficient strength (compared to the diffusion and the localized L_p-norm of b_1) in the direction of the origin for large $|x|$. Contrary to the non-singular case, adding a drift b_2 to a given function b_1 (which is bounded in the non-singular case and in $\tilde{L}_p$ in the singular case) will however not guarantee the existence of an attractor, no matter how strongly b_2 points towards the origin. We will explain this in Theorem 2.13 and Theorem 2.14. Roughly speaking, adding such a drift b_2 may cause solutions to spend more time in regions in which b_1 is “particularly singular” and pushes solutions more strongly away from the origin than b_2 pushes towards the origin.

The idea of splitting the drift b of a singular SDE into two parts b_1 and b_2 was also used in [32] where they assumed that b_1 is essentially as in our set-up and b_2 is Lipschitz. They showed well-posedness of the SDE (which is not covered by our assumptions). We decided not to work in this (essentially) more general set-up since this requires a new proof that the SDE generates a global RDS.

It seems to be an interesting and challenging question to study the attractor in more detail, e.g. to find sufficient conditions for *synchronization* (meaning that the attractor is a singleton). We will, however, not address these questions in this work.

Structure of the paper

We introduce notation and the main results in Section 2. In Section 3 we study the expansion rate of the diameter of the image of a bounded set under a flow under rather general conditions. These results are minor modifications of results contained in [23] which are proved by *chaining techniques*. Section 4 contains estimates on functionals of the solution to the singular SDE, namely quantitative versions of Krylov’s estimates and Khasminskii’s lemma. The first part of the main results of this paper is presented in Section 5, i.e. the linear expansion rate of the diameter of the image of a bounded set under the flow generated by the solution to a singular SDE. In Section 6 we show the existence of an attractor of the RDS generated by the singular SDE. In Appendix A we study regularity estimates of elliptic partial differential equations with emphasis on the dependence on the coefficients. We believe that these estimates are of independent interest.

2 Notation and main results

2.1 Notation

We denote the Euclidean norm on $\mathbb{R}^d$ by $|.|$ and the induced norm on $L(\mathbb{R}^d)$ or on $L(L(\mathbb{R}^d))$ by $\|.\|$. Recall that the trace of $a := (a_{ij})_{1\le i,j\le d} := \sigma\sigma^*$ satisfies $\mathrm{tr}(a) = \sum_{i,j=1}^d \sigma_{ij}^2$, where σ^* denotes the transpose of $\sigma \in L(\mathbb{R}^d)$. For $p \in [1,\infty)$, let $L_p(\mathbb{R}^d)$ denote the space of all real Borel measurable functions on $\mathbb{R}^d$ equipped with the norm

$$\|f\|_{L_p} := \Big(\int_{\mathbb{R}^d} |f(x)|^p \,\mathrm{d}x\Big)^{1/p} < +\infty$$

and L_∞ denotes the space of all bounded and measurable functions equipped with the norm

$$\|f\|_\infty := \|f\|_{L_\infty} := \sup_{x\in\mathbb{R}^d} |f(x)|.$$

We introduce the notion of a localized L_p-space for $p \in [1,\infty]$: for fixed $\delta > 0$,

$$\tilde{L}_p(\mathbb{R}^d) := \{f : \|f\|_{\tilde{L}_p} := \sup_z \|\xi_\delta^z f\|_{L_p} < \infty\}, \tag{2.1}$$

where $\xi_\delta(x) := \xi(\frac{x}{\delta})$ and $\xi_\delta^z(x) := \xi_\delta(x-z)$ for $x, z \in \mathbb{R}^d$, $\xi \in C_c^\infty(\mathbb{R}^d;[0,1])$ is a smooth function with $\xi(x) = 1$ for $|x| \le 1/2$, and $\xi(x) = 0$ for $|x| > 1$. For $(\alpha,p) \in \mathbb{R}\times[1,\infty)$, let $H^{\alpha,p}(\mathbb{R}^d)$ be the usual Bessel potential space with norm

$$\|f\|_{H^{\alpha,p}} := \|(\mathbb{I}-\Delta)^{\alpha/2} f\|_{L_p},$$

where $(\mathbb{I}-\Delta)^{\alpha/2} f$ is defined via Fourier’s transform

$$(\mathbb{I}-\Delta)^{\alpha/2} f := \mathcal{F}^{-1}((1+|\cdot|^2)^{\alpha/2}\mathcal{F}f).$$

The localized $H^{\alpha,p}$-space is defined as

$$\tilde{H}^{\alpha,p} := \{f : \|f\|_{\tilde{H}^{\alpha,p}} := \sup_z \|\xi_\delta^z f\|_{H^{\alpha,p}} < \infty\}.$$

From [27, Section 2] and [34, Proposition 4.1] we know that the space $\tilde{H}^{\alpha,p}$ does not depend on the choice of ξ and δ, but the norm does, of course. More precisely, by [34, Proposition 4.1], for the $\tilde{L}_p$-norms with different δ, say δ_1 and δ_2 and $\delta_1 < \delta_2$, if we use the notation $(\tilde{L}_p)_\delta$ to denote the $\tilde{L}_p$ space with support radius δ for localization, then

$$N_1 \|\cdot\|_{(\tilde{L}_p)_{\delta_1}} \le \|\cdot\|_{(\tilde{L}_p)_{\delta_2}} \le N_2 \Big(\frac{\delta_2}{\delta_1}\Big)^d \|\cdot\|_{(\tilde{L}_p)_{\delta_1}}, \tag{2.2}$$

where N_1, N_2 are constants independent of δ_1, δ_2. For convenience we take $\delta = 1$ in the following. For further properties of these spaces we refer to [27]. In the following, all derivatives should be interpreted in the weak sense. Occasionally we will use Einstein's summation convention (omitting the summation sign for indices appearing twice). We will often use the notation $r_+ = \max\{r, 0\}$ for the positive part of $r \in \mathbb{R}$, $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$.

2.2 Preliminaries

In the following, all random processes will be defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 2.1. *A semi-flow ϕ on a Polish (i.e. separable and completely metrizable) space X equipped with its Borel-σ-algebra $\mathcal{X} = \mathcal{B}(X)$ is a measurable map*

$$\phi : \big\{(s, t, x, \omega) \in \mathbb{R}^2 \times X \times \Omega : s \le t < \infty\big\} \to X$$

such that, for each $\omega \in \Omega$,

(1) $\phi_{s,s}(x) = x$ for all $x \in X$ and $s \in \mathbb{R}$,
(2) $(s, t, x) \mapsto \phi_{s,t}(x)$ is continuous,
(3) for all $s \le t \le u$ and $x \in X$, the following identity holds

$$\phi_{s,u}(x) = \phi_{t,u}(\phi_{s,t}(x)).$$

Next, we define the concepts of a metric dynamical system and a random dynamical system.

Definition 2.2. *A* metric dynamical system *(MDS for short) $\theta = (\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a family of measure preserving transformations $\{\theta_t : \Omega \to \Omega, t \in \mathbb{R}\}$ such that*

(1) $\theta_0 = \mathrm{id}, \theta_t \circ \theta_s = \theta_{t+s}$ for all $t, s \in \mathbb{R}$;
(2) the map $(t, \omega) \mapsto \theta_t\omega$ is measurable and $\theta_t\mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{R}$.

Definition 2.3 (RDS, [1])**.** *A (global)* random dynamical system *(RDS) (θ, φ) on a Polish space (X, d) over an MDS θ is a mapping*

$$\varphi : \{(s, x, \omega) \in [0, \infty) \times X \times \Omega\} \to X$$

such that, for each $\omega \in \Omega$,

(1) measurability: φ is $(\mathcal{B}([0, \infty)) \otimes \mathcal{X} \otimes \mathcal{F}, \mathcal{X})$-measurable,
(2) $(t, x) \mapsto \varphi_t(x)$ is continuous,

(3) *φ satisfies the following (perfect)* cocycle property*: for all $t, s \geq 0$, $x \in X$,*

$$\varphi_0(.,\omega) = \mathrm{id}, \quad \varphi_{t+s}(x,\omega) = \varphi_t(\varphi_s(x,\omega), \theta_s\omega) \tag{2.3}$$

In order to obtain a one-to-one correspondence between RDS φ and semi-flows ϕ via

$$\phi_{s,t}(x,\omega) := \varphi_{t-s}(x, \theta_s\omega),\ s \leq t; \qquad \varphi_t(x,\omega) = \phi_{0,t}(x,\omega),\ t \geq 0,$$

we will require from now on that, in addition to (1)-(3) in Theorem 2.1, a semi-flow satisfies $\phi_{s,t}(x,\omega) = \phi_{s+h,t+h}(x, \theta_h\omega)$ for all $s \leq t$ and all $h \in \mathbb{R}$ and, for an RDS φ, we will require that the map $(s,t,x) \mapsto \varphi_{t-s}(x, \theta_s\omega)$, $0 \leq s \leq t$ is continuous, see [25, p.609]. We say that an SDE generates a semi-flow resp. an RDS if its solution map has a modification which is a semi-flow resp. an RDS. The following study is based on the semi-flow generated by the solution to the SDE with singular drift. Therefore we state the result from [20, Theorem 4.3, Corollary 4.2] on the existence of a global semi-flow and a global RDS for singular SDEs under the following condition.

Assumption 2.4. *For $p, \rho \in (2d, \infty)$ assume*

- (i) $b \in \tilde{L}_p(\mathbb{R}^d)$*,* $\sigma : \mathbb{R}^d \to L(\mathbb{R}^d)$ *is measurable,* $\|\nabla\sigma\| \in \tilde{L}_\rho(\mathbb{R}^d)$*.*
- (ii) *There exist $K_1, K_2 > 0$ such that for $a := \sigma\sigma^*$ we have*

$$K_1|\zeta|^2 \leq \langle a(x)\zeta, \zeta\rangle \leq K_2|\zeta|^2, \quad \forall \zeta, x \in \mathbb{R}^d.$$

Remark 2.5. Note that $\tilde{L}_p \subset \tilde{L}_{p'}$ whenever $p > p'$. Therefore, if Theorem 2.4 holds with different values of p and ρ, then it also holds with the larger of the two numbers replaced by the smaller one. In particular, the following result which was formulated for $p = \rho$ can still be applied.

Theorem 2.6. *[20, Theorem 4.3, Corollary 4.2] If Theorem 2.4 holds, then the SDE* (1.1) *admits a semi-flow ϕ and a corresponding RDS φ.*

We will often write $\psi_t(x)$ instead of $\phi_{0,t}(x)$. Abusing notation we will sometimes say "Let $\psi_t(x)$ (or just ψ) be a flow..." instead of "Let $\phi_{s,t}(x), x \in \mathbb{R}^d, s \leq t < \infty$ be a semi-flow and $\psi_t(x) := \phi_{0,t}(x)$, $t \geq 0, x \in \mathbb{R}^d$...".

Definition 2.7 (Attractor, [6])**.** *Let φ be an RDS over the MDS $\theta = (\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$. The random set $A(\omega)$ is a* (pullback) attractor *if*

(1) *measurability: $A(\omega)$ is a random element in the metric space of nonempty compact subsets of X equipped with the Hausdorff distance,*

(2) *invariance property: for $t > 0$ there exists a set Ω_t with full measure such that*

$$\varphi(t,\omega)(A(\omega)) = A(\theta_t\omega), \quad \forall\, \omega \in \Omega_t,$$

(3) *pull-back limit: almost surely, for all bounded closed sets $B \subset X$,*

$$\lim_{t\to\infty} \sup_{x\in B} \mathit{dist}(\varphi(t, \theta_{-t}\omega)(x), A(\omega)) = 0.$$

One way to verify the existence of an attractor is the following criterion.

Proposition 2.8. *([7], [8, Proposition 2.3]) Let φ be an RDS over the MDS $\theta = (\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$. Then the following are equivalent:*

(i) *φ has an attractor,*

(ii) $\forall\, r > 0$, $\lim_{R\to\infty} \mathbb{P}\Big(\omega \in \Omega : B_r \subset \bigcup_{s=0}^\infty \bigcap_{t\geq s} \varphi^{-1}(t, B_R, \theta_{-t}\omega)\Big) = 1.$

2.3 Main results

Based on general estimates on the speed of dispersion of random sets in Section 3 (cf. Theorem 3.3) and on quantitative estimates of the solution to singular SDE in Section 4, we will show the following result in Section 5.

Theorem 2.9. *If Theorem 2.4 holds, then there exists a constant $\kappa > 0$ such that for the flow ψ generated by the solution to* (1.1) *we have, for any compact $\mathcal{X} \subset \mathbb{R}^d$,*

$$\limsup_{T\to\infty}\Big(\sup_{t\in[0,T]}\sup_{x\in\mathcal{X}}\frac{1}{T}|\psi_t(x)|\Big) \le \kappa \quad a.s..$$

The precise statement including a formula for κ will be given in Theorem 5.4. There, we can see that $\kappa \to \infty$ as $K_1 \to 0$ (when all other parameters remain unchanged). The following example explains this fact: as the noise becomes more and more degenerate, the linear bound on the dispersion of a bounded set under the flow approaches infinity, so our non-degeneracy assumption on the noise cannot be avoided.

Example 2.10. In $\mathbb{R}^2$, for $\varepsilon > 0$, we consider the system

$$\left\{\begin{array}{ll} \mathrm{d}X_t & = B(Y_t)\,\mathrm{d}t + \varepsilon\,\mathrm{d}W_t^1, \quad X_0 \in \mathbb{R}, \\ \mathrm{d}Y_t & = h(Y_t)\,\mathrm{d}t + \varepsilon\,\mathrm{d}W_t^2, \quad Y_0 \in \mathbb{R}, \end{array}\right. \tag{2.4}$$

where

$$B(y) := \left\{\begin{array}{cc} |y|^{-q} & \text{if } y \neq 0, \\ 0 & \text{else,} \end{array}\right. \qquad q \in (0, \frac{1}{4}), \qquad h(y) := \big((-y) \vee (-1)\big) \wedge 1.$$

and W^1, W^2 are two independent 1-dimensional Brownian motions. Notice that for $b(x,y) := (B(y), h(y))^*$, we have $b \in \tilde{L}_p(\mathbb{R}^2)$ for $p \in (4, \frac{1}{q})$. Clearly there exists a unique solution (X,Y) to (2.4) and

$$X_t = X_0 + \int_0^t B(Y_s)\,\mathrm{d}s + W_t^1, \quad t \ge 0.$$

By the ergodic theorem, almost surely,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t B(Y_s)\,\mathrm{d}s = \int_{-\infty}^{\infty} B(y)\pi_\epsilon(\,\mathrm{d}y),$$

where π_ε is the invariant probability measure of Y (an explicit formula for its density can be found in [14, p.353]). Since π_ε converges to the point measure δ_0 weakly as $\epsilon \downarrow 0$, we see that the linear expansion rate of (X,Y) converges to ∞ when $\varepsilon \downarrow 0$. In particular, we can not expect to have a linear expansion rate for the solution to a singular SDE with degenerate noise in general.

We will now assume that the singular drift b in (1.1) is of the form $b = b_1 + b_2$ with $b_1 \in \tilde{L}_p(\mathbb{R}^d)$ and b_2 satisfies one of the following conditions.

Assumption 2.11. *For a given $\beta \in \mathbb{R}$, $b_2(x) : \mathbb{R}^d \to \mathbb{R}^d$ satisfies*

(U^β) $\quad \limsup_{|x|\to\infty} \frac{x}{|x|} \cdot b_2(x) \le \beta$

or

(U_β) $\quad \liminf_{|x|\to\infty} \frac{x}{|x|} \cdot b_2(x) \ge \beta.$

Theorem 2.12. *Let Theorem 2.4 hold and let ϕ be the flow generated by the solution to* (1.1)*. If there exist vector fields b_1 and b_2 such that $b = b_1 + b_2$ with $b_1 \in \tilde{L}_p(\mathbb{R}^d)$, b_2 is bounded and b_2 and σ are Lipschitz continuous, then there exist positive constants β_1 (see Theorem 6.2) and β_2 (see Theorem 6.3) such that*

1. *if* b_2 *satisfies Theorem 2.11* (U_β) *for* $\beta > \beta_1$*, then for any* $\gamma \in [0, \beta - \beta_1)$ *we have*

$$\lim_{r\to\infty} \mathbb{P}\Big(B_{\gamma t} \subset \phi_{0,t}(B_r) \quad \forall \quad t \geq 0\Big) = 1. \tag{2.5}$$

2. *if* b_2 *satisfies Theorem 2.11* (U^β) *for* $\beta < -\beta_2$*, then for any* $\gamma \in [0, -\beta - \beta_2)$ *we have*

$$\lim_{r\to\infty} \mathbb{P}\Big(B_{\gamma t} \subset \phi_{-t,0}^{-1}(B_r) \quad \forall \quad t \geq 0\Big) = 1. \tag{2.6}$$

In particular, ϕ *(or the corresponding RDS* φ*) has a random attractor.*

Correspondingly, the detailed results are presented in Theorem 6.2 and Theorem 6.3.

Remark 2.13. In Theorem 6.3, the bound β_2 (or β_0 as it is called there) will not only depend on b_1 and σ but also (via Γ) on $\|b_2\|_{\tilde{L}_p}$ which may look strange and raises the question whether, for given b_1 and σ, there exists any bounded and Lipschitz continuous function b_2 which satisfies (U^β) for some $\beta < -\beta_2$. Observing that $\|b_2\|_{\tilde{L}_p} \geq \beta_2 c$ where $c = \|1\|_{\tilde{L}_p}$ one can easily see that there are cases in which no such b_2 exists and other cases in which it does. In particular, the explicit bound in Theorem 6.3 shows that for given σ and Lipschitz constant L_b of b_2 there exists some $\bar{\beta} > 0$ such that an attractor exists for any b_2 satisfying (U^β) for some $\beta > \bar{\beta}$ and for all b_1 with sufficiently small $\tilde{L}_p(\mathbb{R}^d)$ norm. The fact that, for given b_1, Theorem 6.3 does not guarantee the existence of an attractor even for a b_2 satisfying (U^β) for some very large β, is not just an artefact of our approach. The following example illustrates what is going on.

Example 2.14. Define B and h as in Theorem 2.10 and consider the SDE

$$\begin{aligned} \mathrm{d}X_t &= \delta B(Y_t)\ \mathrm{d}t + \gamma_1 h(X_t)\ \mathrm{d}t + \mathrm{d}W_t^1 \\ \mathrm{d}Y_t &= \gamma_2 h(Y_t)\ \mathrm{d}t + \mathrm{d}W_t^2, \end{aligned}$$

where $\gamma_1, \gamma_2 \geq 0$ and $\delta > 0$. The 1d diffusion Y has an invariant probability measure for every strictly positive value of γ_2 and it converges to a Dirac measure at 0 as $\gamma_2 \to \infty$ (see [14, p.353] for a formula for the density of the invariant probability measure). Arguing as in Theorem 2.10 we see that X converges to ∞ as long as γ_1 is not too large (compared to γ_2). If we split the drift into

$$b_1(x, y) = \begin{pmatrix} \delta B(x) \\ 0 \end{pmatrix}, \qquad b_2(x, y) = \begin{pmatrix} \gamma_1 h(x) \\ \gamma_2 h(y) \end{pmatrix},$$

then we see that for every negative value of β, there are (large) positive numbers γ_1, γ_2 for which b_2 satisfies (U^β) and X converges to ∞ so, in particular, the RDS does not have an attractor. Further, for any pair (γ_1, γ_2) for which the RDS has an attractor, we find a pair of larger values for which no attractor exists, so an attractor can even be destroyed by adding a strong drift towards the origin. Note that our sufficient condition for an attractor is, for example, satisfied provided that $\gamma_1 = \gamma_2$ is sufficiently large and $\delta > 0$ is sufficiently small.

We now state what the upper bounds look like in the special case when the drift b is bounded (i.e. $p = \infty$).

Example 2.15 (A case study: bounded coefficients)**.** We consider the flow ϕ generated by the solution to (1.1) when b, $\nabla\sigma$ are simply bounded, i.e., Theorem 2.4 holds with arbitrary $p = \rho \in (1, \infty)$.

1. Expansion rate of the flow: Theorem 5.4 shows that for each $\epsilon > 0$ there exist

constants C_1 (depending on d and ϵ) such that for each compact subset $\mathcal{X} \subset \mathbb{R}^d$

$$\limsup_{T\to\infty}\Big(\sup_{t\in[0,T]}\sup_{x\in\mathcal{X}}\frac{1}{T}|\phi_{0,t}(x)|\Big) \le C_1\Big(K_2 + \|b\|^2_{\tilde{L}_p}\frac{K_2}{K_1^2} + \|\nabla\sigma\|^2_\infty\Big)$$
$$\Big[\Big(\frac{K_2}{K_1}\Big)^{16d^3+\epsilon} + \Big(\frac{\|\nabla\sigma\|^2_\infty}{K_1}\Big)^{32d^3+\epsilon} + \Big(\frac{\|b\|_{\tilde{L}_p}}{K_1}\Big)^{32d^2+\epsilon}\Big]. \tag{2.7}$$

2. Existence of the attractor: if $b = b_1 + b_2$, and b_2 and σ are Lipschitz continuous with Lipschitz constants L_b and L_σ respectively. Further assume that b_2 satisfies (U^β) in Theorem 2.11 with

$$\beta < -C_2\Big(\frac{(\|b_1\|^2_{\tilde{L}_p} + K_2\|b_1\|_{\tilde{L}_p})}{K_1}\Big)\Big[\Big(\frac{K_2}{K_1}\Big)^{16d^3+\epsilon} + \Big(\frac{\|\nabla\sigma\|^2_\infty}{K_1}\Big)^{32d^3+\epsilon} + \Big(\frac{\|b_2\|_{\tilde{L}_p}}{K_1}\Big)^{32d^2+\epsilon}\Big]$$
$$-C_3(\sqrt{(d-1)K_2(L_b+L_\sigma))} + K_2(d-1)), \tag{2.8}$$

where $\epsilon > 0$ and $C_2, C_3 > 0$ are appropriate functions depending on d and ϵ only, then from Theorem 6.3 we know that ϕ (or the associated RDS φ) has an attractor.

3 Expansion of sets under a flow

In this section, we assume that $\psi : [0,\infty) \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ is measurable such that $t \mapsto \psi_t(x,\omega)$ is continuous for every $x \in \mathbb{R}^d$ and $\omega \in \Omega$ (we do not require that ψ has any kind of flow property).

Lemma 3.1. *Assume that there exist $\alpha > 0$ and a constant $c_1 > 0$ such that for each $r > d$, there exists $c = c(r) > 0$ such that for all $x, y \in \mathbb{R}^d$ and $T > 0$, we have*

$$\Big(\mathbb{E}\sup_{0\le t\le T}(|\psi_t(x) - \psi_t(y)|^r)\Big)^{1/r} \le c|x-y|e^{c_1 r^\alpha T}. \tag{3.1}$$

Then ψ has a modification (which we denote by the same symbol) which is jointly continuous in (t,x) and for each $\gamma > 0$ and $u > 0$,

$$\limsup_{T\to\infty}\frac{1}{T}\sup_{\chi_{T,\gamma}}\log\mathbb{P}\Big(\sup_{x,y\in\chi_{T,\gamma}}\sup_{0\le t\le T}|\psi_t(x)-\psi_t(y)| \ge u\Big) \le -I(\gamma), \tag{3.2}$$

where $\sup_{\chi_{T,\gamma}}$ means that we take the supremum over all cubes $\chi_{T,\gamma}$ in $\mathbb{R}^d$ with side length $e^{-\gamma T}$, and $I : [0,\infty) \to \mathbb{R}$ is defined as

$$I(\gamma) := \begin{cases} \gamma^{1+1/\alpha}\alpha(1+\alpha)^{-1-1/\alpha}c_1^{-1/\alpha} & \text{if} \quad \gamma \ge c_1(\alpha+1)d^\alpha \\ d(\gamma - c_1 d^\alpha) & \text{if} \quad c_1 d^\alpha < \gamma \le c_1(\alpha+1)d^\alpha \\ 0 & \text{if} \quad \gamma \le c_1 d^\alpha. \end{cases} \tag{3.3}$$

Proof. We follow the argument in [23, Proof of Theorem 3.1]. Without loss of generality we take $\chi := \chi_{T,\gamma} = [0, e^{-\gamma T}]^d$ and define $Z_t(x) := \phi_t(e^{-\gamma T}x)$, $x \in \mathbb{R}^d$. From (3.1) we get

$$\Big(\mathbb{E}\sup_{0\le t\le T}(|Z_t(x) - Z_t(y)|^r)\Big)^{1/r} \le ce^{-\gamma T}|x-y|e^{c_1 r^\alpha T}.$$

By Kolmogorov's Theorem (see, e.g. [23, Lemma 2.1]), ϕ admits a jointly continuous modification and for any $\rho \in (0, \frac{r-d}{r})$:

$$\mathbb{P}\Big(\sup_{x,y\in\chi_{T,\gamma}}\sup_{0\le t\le T}|\psi_t(x)-\psi_t(y)| \ge u\Big) \le \tilde{c}e^{(c_1 r^\alpha - \gamma)rT}u^{-r}, \tag{3.4}$$

where $\tilde{c}$ depends on r, d, ρ only. Taking logarithms, dividing by T, then letting $T \to \infty$ and optimizing over $r > d$ we get the desired result (3.2). ☐

Remark 3.2. Since $I(\gamma) = \sup_{r>d} \{r(\gamma - c_1 r^\alpha)\}$ is the supremum of affine functions, the map $\gamma \mapsto I(\gamma)$ is convex. Further, I grows faster than linearly.

The following theorem is a reformulation of [23, Theorem 2.3].

Theorem 3.3. *Let $\psi : [0,\infty) \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ be jointly continuous and satisfy the assumptions of Theorem 3.1 and* (3.1) *hold with constants c_1 and α. Assume further, that there exist c_2 and $c_3 \geq 0$ such that, for each $k > 0$ and each bounded set $S \subset \mathbb{R}^d$, the following holds*

$$\limsup_{T\to\infty} \frac{1}{T} \log \sup_{x\in S} \mathbb{P}\Big(\sup_{0\leq t\leq T} |\psi_t(x)| \geq kT\Big) \leq -c_2 k^2 + c_3. \tag{3.5}$$

Let $\mathcal{X}$ be a compact subset of $\mathbb{R}^d$ with box (or upper entropy) dimension $\Delta > 0$. Then

$$\limsup_{T\to\infty} \Big(\sup_{t\in[0,T]} \sup_{x\in\mathcal{X}} \frac{1}{T}|\psi_t(x)|\Big) \leq \kappa \quad a.s., \tag{3.6}$$

where

$$\kappa := \begin{cases} \left(\frac{c_3+\gamma_1\Delta}{c_2}\right)^{\frac{1}{2}} & \text{if } \frac{d}{d-\Delta} < \alpha+1, \\ \left(\frac{c_3+\gamma_2\Delta}{c_2}\right)^{\frac{1}{2}} & \text{otherwise}, \end{cases} \qquad \text{with} \quad \gamma_1 = \frac{c_1 d^{\alpha+1}}{d-\Delta}, \gamma_2 = c_1(\alpha^{-1}\Delta)^\alpha (1+\alpha)^{1+\alpha}.$$

Remark 3.4. In addition to the assumptions of the previous theorem, let us assume that $\psi_t(x) = \phi_{0,t}(x)$ where ϕ is a semi-flow (later, we will only consider this case). Let $\mathcal{X} \subset \mathbb{R}^d$ be any compact set and let B be a ball in $\mathbb{R}^d$ containing $\mathcal{X}$. Clearly, the boundary ∂B of B has box dimension $d-1$. The flow property of ϕ implies that for each $t \geq 0$, the boundary of $\phi_{0,t}(B)$ is contained in $\phi_{0,t}(\partial B)$ and therefore any almost sure upper bound κ for the linear expansion rate of the set ∂B is at the same time an upper bound for the linear expansion rate of the set B and hence of $\mathcal{X}$. This means that in the case of a flow, the formula for κ in the theorem always holds with Δ replaced by $d-1$ (or the minimum of Δ and $d-1$).

4 Quantitative version of Krylov estimates

We will show a quantitative version of Krylov estimates (4.1). One can find similar results in the literature with implicit constants, for instance [16], [30] and [27], which however do not fit our needs since some proofs in later sections rely on the explicit dependence of the constants on the coefficients of the SDE. In the following lemma, a constant C_{Kry} appears which depends on q, p, ρ, d only. While we will regard p, ρ, d as fixed throughout, we will apply the formula with different values of q and we will therefore write $C_{\mathrm{Kry}}(q)$ for clarity. We denote the filtration generated by W_t, $t \geq 0$ by $\mathcal{F}_t$, $t \geq 0$.

Lemma 4.1. *If Theorem 2.4 holds and $(X_t)_{t\geq 0}$ solves* (1.1)*, then, for $f \in \tilde{L}_q(\mathbb{R}^d)$ with $q \in (d,\infty]$, there exists a constant $C_{\mathrm{Kry}}(q) > 0$ depending on q, p, ρ, d only such that for $0 \leq s \leq t$,*

$$\mathbb{E}\Big[\int_s^t |f(X_r)|\,\mathrm{d}r\Big|\mathcal{F}_s\Big] \leq C_{\mathrm{Kry}}(q)\Gamma\big(K_2^{-\frac{1}{2}}(t-s)^{\frac{1}{2}} + (t-s)\big)\|f\|_{\tilde{L}_q}, \tag{4.1}$$

where $\Gamma := \big(\frac{K_2}{K_1}\big)^{\frac{4d^2}{1-d/\rho}} + \big(\frac{\|\nabla\sigma\|^2_{\tilde{L}_\rho}}{K_1}\big)^{\frac{4d^2}{1-d/\rho}} + \big(\frac{\|b\|_{\tilde{L}_p}}{K_1}\big)^{\frac{4d}{1-d/p}}$.

Proof. It is sufficient to show the estimate for positive f. (4.1) clearly holds when $q = \infty$, so we assume $q \in (d,\infty)$. All positive constants C_i, $i = 0, \cdots, 7$ appearing in the proof only depend on p, ρ, q, d. We will regard p, ρ and d as fixed but we will vary q in the following proof and we will therefore highlight the dependence of constants on q in some cases (for C_0 and C_1). First we show that $a := \sigma\sigma^*$ is $1 - \frac{d}{\rho}$-Hölder continuous using Sobolev's embedding theorem and the condition that $\sigma \in \tilde{H}^{1,\rho}$ with $\rho > d$. Indeed

$$\begin{aligned}\omega_{1-d/\rho}(a) :=& \sup_{x,y\in\mathbb{R}^d, x\neq y, |x-y|\leq 1} \frac{\|a(x)-a(y)\|}{|x-y|^{1-d/\rho}} \\ \leq& \sup_{x,y\in\mathbb{R}^d, x\neq y, |x-y|\leq 1} \Big(\frac{\|(\sigma\sigma^*)(x)-\sigma(x)\sigma^*(y)\|}{|x-y|^{1-d/\rho}} + \frac{\|\sigma(x)\sigma^*(y)-(\sigma\sigma^*)(y)\|}{|x-y|^{1-d/\rho}}\Big) \\ \leq& \sup_{x,y\in\mathbb{R}^d, x\neq y, |x-y|\leq 1} \Big(\frac{\|\sigma^*(x)-\sigma^*(y)\|\|\sigma\|_\infty}{|x-y|^{1-d/\rho}} + \frac{\|\sigma(x)-\sigma(y)\|\|\sigma\|_\infty}{|x-y|^{1-d/\rho}}\Big) \\ \leq& C_{\rho,d}\sqrt{K_2}\|\nabla\sigma\|_{\tilde{L}_\rho}. \end{aligned} \tag{4.2}$$

We follow the idea from [33, Theorem 3.4]. Applying Theorem A.3 with $p' = \infty$, we see that there is a unique solution $u \in \tilde{H}^{2,q}$ to

$$\lambda u - \frac{1}{2}a_{ij}\partial_{ij}u = f \tag{4.3}$$

provided that $\lambda \geq C_0(q)\frac{K_2^2}{K_1}\big(\frac{K_1+\sqrt{K_2}\|\nabla\sigma\|_{\tilde{L}_\rho}}{K_1}\big)^{\frac{2}{1-d/\rho}} =: \lambda_0(q)$. Further, for $\lambda \geq \lambda_0(q)$, we have

$$\begin{aligned} \sup_{x\in\mathbb{R}^d}|u(x)| &\leq C_1(q)\lambda^{-\frac{2-d/q}{2}}K_1^{-\frac{d}{2q}}\Big(\frac{K_1+\sqrt{K_2}\|\nabla\sigma\|_{\tilde{L}_\rho}}{K_1}\Big)^{\frac{d}{1-d/\rho}}\|f\|_{\tilde{L}_q} =: U_{1,q}(\lambda)\|f\|_{\tilde{L}_q}, \\ \sup_{x\in\mathbb{R}^d}|\nabla u(x)| &\leq C_1(q)\lambda^{-\frac{1-d/q}{2}}K_1^{-\frac{1+d/q}{2}}\Big(\frac{K_1+\sqrt{K_2}\|\nabla\sigma\|_{\tilde{L}_\rho}}{K_1}\Big)^{\frac{d}{1-d/\rho}}\|f\|_{\tilde{L}_q} =: U_{2,q}(\lambda)\|f\|_{\tilde{L}_q}. \end{aligned} \tag{4.4}$$

Fix $t \geq s \geq 0$ and define the stopping time

$$\tau_R := \inf\Big\{\bar{s} > s : \int_s^{\bar{s}} \big|b(X_r)\big|\,\mathrm{d}r \geq R\Big\}, \quad 0 < R < \infty.$$

By the generalized Itô's formula (see e.g. [27, Lemma 4.1 (iii)])

$$\begin{aligned} &u(X_{t\wedge\tau_R}) - u(X_{s\wedge\tau_R}) \\ &\quad= \frac{1}{2}\int_{s\wedge\tau_R}^{t\wedge\tau_R} a_{ij}(X_r)\partial_{ij}u(X_r)\,\mathrm{d}r + \int_{s\wedge\tau_R}^{t\wedge\tau_R}\big(\nabla u(X_r)\big)^*\sigma(X_r)\,\mathrm{d}W_r \\ &\qquad+ \int_{s\wedge\tau_R}^{t\wedge\tau_R} b(X_r)\cdot\nabla u(X_r)\,\mathrm{d}r. \end{aligned}$$

Using (4.3), the mean value theorem, (4.4) and BDG's inequality, we get that

$$\begin{aligned} &\mathbb{E}\Big[\int_{s\wedge\tau_R}^{t\wedge\tau_R} f(X_r)\,\mathrm{d}r\Big|\mathcal{F}_s\Big] \\ =&\mathbb{E}\big[(u(X_{s\wedge\tau_R}) - u(X_{t\wedge\tau_R}))\big|\mathcal{F}_s\big] + \mathbb{E}\Big[\lambda\int_{s\wedge\tau_R}^{t\wedge\tau_R} u(X_r)\,\mathrm{d}r\Big|\mathcal{F}_s\Big] \\ &\quad+ \mathbb{E}\Big[\int_{s\wedge\tau_R}^{t\wedge\tau_R} b(X_r)\cdot\nabla u(X_r)\,\mathrm{d}r\Big|\mathcal{F}_s\Big] \\ \leq& \sup_{x\in\mathbb{R}^d}|\nabla u(x)|\mathbb{E}\Big[\Big|\int_{s\wedge\tau_R}^{t\wedge\tau_R} b(X_r)\,\mathrm{d}r + \int_{s\wedge\tau_R}^{t\wedge\tau_R}\sigma(X_r)\,\mathrm{d}W_r\Big|\Big|\mathcal{F}_s\Big] + \lambda(t-s)\sup_{x\in\mathbb{R}^d}|u(x)| \end{aligned}$$

$$
\begin{aligned}
&\quad + \sup_{x\in\mathbb{R}^d}|\nabla u(x)|\mathbb{E}\big[\int_{s\wedge\tau_R}^{t\wedge\tau_R}\big|b(X_r)\big|\,\mathrm{d}r\Big|\mathcal{F}_s\big]\\
&\le \sup_{x\in\mathbb{R}^d}|\nabla u(x)|C_2\sqrt{K_2}(t-s)^{\frac12}+\lambda(t-s)\sup_{x\in\mathbb{R}^d}|u(x)|\\
&\quad + 2\sup_{x\in\mathbb{R}^d}|\nabla u(x)|\mathbb{E}\big[\int_{s\wedge\tau_R}^{t\wedge\tau_R}\big|b(X_r)\big|\,\mathrm{d}r\Big|\mathcal{F}_s\big]\\
&\le C_2\sqrt{K_2}(t-s)^{\frac12}U_{2,q}(\lambda)\|f\|_{\tilde L_q}+\lambda(t-s)U_{1,q}(\lambda)\|f\|_{\tilde L_q}\\
&\quad + 2U_{2,q}(\lambda)\|f\|_{\tilde L_q}\mathbb{E}\big[\int_{s\wedge\tau_R}^{t\wedge\tau_R}\big|b(X_r)\big|\,\mathrm{d}r\Big|\mathcal{F}_s\big].
\end{aligned}\tag{4.5}
$$

Here, the constant $C_2>0$ comes from BDG's inequality. We apply this inequality to $f=|b|$ with $q=p$. Then, for $\lambda\ge\lambda_0(p)$,

$$
\begin{aligned}
\mathbb{E}\big[\int_{s\wedge\tau_R}^{t\wedge\tau_R}|b(X_r)|\,\mathrm{d}r\Big|\mathcal{F}_s\big] \le & C_2\sqrt{K_2}(t-s)^{\frac12}U_{2,p}(\lambda)\|b\|_{\tilde L_p}+\lambda(t-s)U_{1,p}(\lambda)\|b\|_{\tilde L_p}\\
&+2U_{2,p}(\lambda)\|b\|_{\tilde L_p}\mathbb{E}\big[\int_{s\wedge\tau_R}^{t\wedge\tau_R}|b(X_r)|\,\mathrm{d}r\Big|\mathcal{F}_s\big].
\end{aligned}
$$

If $\lambda\ge\lambda_0(p)$ is so large that $U_{2,p}(\lambda)\|b\|_{\tilde L_p}=C_1(p)\lambda^{-\frac{1-d/p}{2}}K_1^{-\frac{1+d/p}{2}}\big(\frac{K_1+\sqrt{K_2}\|\nabla\sigma\|_{\tilde L_\rho}}{K_1}\big)^{\frac{d}{1-d/\rho}}\|b\|_{\tilde L_p}\le\frac14$, i.e.

$$
\lambda\ge\Big(4C_1(p)K_1^{\frac{-1-d/p}{2}}\big(\frac{K_1+\sqrt{K_2}\|\nabla\sigma\|_{\tilde L_\rho}}{K_1}\big)^{\frac{d}{1-d/\rho}}\|b\|_{\tilde L_p}\Big)^{\frac{2}{1-d/p}},\tag{4.6}
$$

then we get

$$
\mathbb{E}\big[\int_{s\wedge\tau_R}^{t\wedge\tau_R}|b(X_r)|\,\mathrm{d}r\Big|\mathcal{F}_s\big]\le\frac{C_2}{2}\sqrt{K_2}(t-s)^{\frac12}+2\lambda(t-s)U_{1,p}(\lambda)\|b\|_{\tilde L_p}.
$$

Plugging this into (4.5), observing that, by definition, $U_{1,p}(\lambda)U_{2,q}(\lambda)=U_{1,q}(\lambda)U_{2,p}(\lambda)$, and using (4.6) yields, for $\lambda\ge\lambda_0(p)\vee\lambda_0(q)$ satisfying (4.6),

$$
\begin{aligned}
&\mathbb{E}\big[\int_{s\wedge\tau_R}^{t\wedge\tau_R}f(X_r)\,\mathrm{d}r\Big|\mathcal{F}_s\big]\\
\le& C_3\big(\sqrt{K_2}(t-s)^{\frac12}U_{2,q}(\lambda)+\lambda(t-s)(U_{1,q}(\lambda)+U_{1,p}(\lambda)U_{2,q}(\lambda)\|b\|_{\tilde L_p}\big)\|f\|_{\tilde L_q}\\
\le& 2C_3\big(\sqrt{K_2}(t-s)^{\frac12}U_{2,q}(\lambda)+\lambda(t-s)U_{1,q}(\lambda)\big)\|f\|_{\tilde L_q}.
\end{aligned}
$$

Let $\lambda=C_4\big(\frac{K_2^2}{K_1}(\frac{K_1+\sqrt{K_2}\|\nabla\sigma\|_{\tilde L_\rho}}{K_1})^{\frac{2}{1-d/\rho}}+(4C_1(p)K_1^{\frac{-1-d/p}{2}}(\frac{K_1+\sqrt{K_2}\|\nabla\sigma\|_{\tilde L_\rho}}{K_1})^{\frac{d}{1-d/\rho}}\|b\|_{\tilde L_p})^{\frac{2}{1-d/p}}\big)$ with $C_4>C_0(p)\vee C_0(q)\vee 1$, which implies

$$
\begin{aligned}
\sqrt{K_2}U_{2,q}(\lambda)&=C_1(q)\sqrt{K_2}(\lambda K_1)^{-\frac12}(\lambda K_1^{-1})^{\frac{d}{2q}}\big(\frac{K_1+\sqrt{K_2}\|\nabla\sigma\|_{\tilde L_\rho}}{K_1}\big)^{\frac{d}{1-d/\rho}}\\
&\le C_5K_2^{-\frac12}(\lambda K_1^{-1})^{\frac{d}{2q}}\big(\frac{K_1+\sqrt{K_2}\|\nabla\sigma\|_{\tilde L_\rho}}{K_1}\big)^{\frac{d}{1-d/\rho}}\\
&\le C_6K_2^{-\frac12}\Big(\big(\frac{K_2}{K_1}\big)^{\frac{4d^2}{1-d/\rho}}+\big(\frac{\|\nabla\sigma\|_{\tilde L_\rho}^2}{K_1}\big)^{\frac{4d^2}{1-d/\rho}}+\big(\frac{\|b\|_{\tilde L_p}}{K_1}\big)^{\frac{4d}{1-d/p}}\Big)
\end{aligned}
$$

and

$$
\lambda U_{1,q}(\lambda)=C_1(q)(\lambda K_1^{-1})^{\frac{d}{2q}}\big(\frac{K_1+\sqrt{K_2}\|\nabla\sigma\|_{\tilde L_\rho}}{K_1}\big)^{\frac{d}{1-d/\rho}}
$$

$$\leq C_7\Big(\big(\frac{K_2}{K_1}\big)^{\frac{4d^2}{1-d/\rho}}+\big(\frac{\|\nabla\sigma\|^2_{\tilde{L}_\rho}}{K_1}\big)^{\frac{4d^2}{1-d/\rho}}+\big(\frac{\|b\|_{\tilde{L}_p}}{K_1}\big)^{\frac{4d}{1-d/p}}\Big).$$

In the above estimates we used the fact that $p>2d$ and $q>d$. Therefore,

$$\begin{aligned}&\mathbb{E}\Big[\int_{s\wedge\tau_R}^{t\wedge\tau_R} f(X_r)\,\mathrm{d}r\Big|\mathcal{F}_s\Big]\\&\leq C_{\mathrm{Kry}}(q)\Big(\big(\frac{K_2}{K_1}\big)^{\frac{4d^2}{1-d/\rho}}+\big(\frac{\|\nabla\sigma\|^2_{\tilde{L}_\rho}}{K_1}\big)^{\frac{4d^2}{1-d/\rho}}+\big(\frac{\|b\|_{\tilde{L}_p}}{K_1}\big)^{\frac{4d}{1-d/p}}\Big)[K_2^{-\frac12}(t-s)^{\frac12}+(t-s)]\|f\|_{\tilde{L}_q}.\end{aligned}\tag{4.7}$$

Letting $R\to\infty$ we therefore get (4.1). □

The following corollary is a quantitative version of Khasminskii's lemma. The constant $C_{\mathrm{Kry}}(q)$ appearing in there is the same as in the previous lemma.

Corollary 4.2. *Let Theorem 2.4 hold, let* $\Gamma:=\Big(\big(\frac{K_2}{K_1}\big)^{\frac{4d^2}{1-d/\rho}}+\big(\frac{\|\nabla\sigma\|^2_{\tilde{L}_\rho}}{K_1}\big)^{\frac{4d^2}{1-d/\rho}}+\big(\frac{\|b\|_{\tilde{L}_p}}{K_1}\big)^{\frac{4d}{1-d/p}}\Big)$. *Then, for any* $f\in\tilde{L}_q(\mathbb{R}^d)$ *with* $q\in(d,\infty]$, *any* $0\leq S\leq T$, *and any* $0<\lambda<\infty$, *the solution* $(X_t)_{t\geq0}$ *of* (1.1) *satisfies*

$$\mathbb{E}\exp\Big(\lambda\int_S^T|f(X_r)|\,\mathrm{d}r\Big)\leq 2\cdot 2^{(T-S)\left(\frac{\kappa}{2}K_2^{-1/2}+\sqrt{\frac{\kappa^2}{4}K_2^{-1}+\kappa}\right)^2}\leq 2\cdot 2^{(T-S)\left(\frac{\kappa^2}{K_2}+2\kappa\right)},\tag{4.8}$$

where $\kappa:=2C_{\mathrm{Kry}}(q)\lambda\Gamma\|f\|_{\tilde{L}_q}$.

Proof. The second inequality is an application of the general inequality $(A+B)^2\leq 2A^2+2B^2$.

Theorem 4.1 shows that there exists some positive integer n such that, for $j=0,\cdots,n-1$,

$$\lambda\mathbb{E}\Big[\int_{\frac{(T-S)j}{n}}^{\frac{(T-S)(j+1)}{n}}\Big|f(X_r)\Big|\,\mathrm{d}r\Big|\mathcal{F}_{\frac{(T-S)j}{n}}\Big]\leq\frac12\tag{4.9}$$

and the proof of [29, Lemma 3.5] shows that for any such n we have

$$\mathbb{E}\exp\Big(\lambda\int_S^T|f(X_r)|\,\mathrm{d}r\Big)\leq 2^n$$

(see also [19, Lemma 3.5]). By Theorem 4.1, any n such that

$$C_{\mathrm{Kry}}(q)\lambda\Gamma\|f\|_{\tilde{L}_q}\Big[\Big(\frac{T-S}{K_2 n}\Big)^{\frac12}+\frac{T-S}{n}\Big]\leq\frac12$$

satisfies (4.9). In particular, we can take

$$n=\Big\lfloor(T-S)\Big(\frac{\kappa}{2}K_2^{-1/2}+\sqrt{\frac{\kappa^2}{4}K_2^{-1}+\kappa}\Big)^2\Big\rfloor+1$$

Here, $\lfloor x\rfloor$ is the largest integer that is smaller than or equal to $x\in\mathbb{R}$. Therefore (4.8) holds. □

Remark 4.3. Note that the right hand side of our version of Krylov's estimate contains the factor $(t-s)^{1/2}+(t-s)$ instead of $C(T)(t-s)^{1-\frac{d}{2q}}$ in [33, Theorem 3.4 (3.8)]), where $C(T)$ depends on the final time T. Further, we require the condition $q>d$ instead of $q>d/2$ in [33, Theorem 3.4 (3.8)]). The reason for our restriction to $q>d$ is that we use (4.4) which only holds for $q>d$. Since we will later apply Krylov's estimate to $f:=|b^*\cdot\sigma^{-1}|^2$ which is in $\tilde{L}_{p/2}$ we will have to assume $p>2d$.

Remark 4.4. More general versions of the quantitative Khasminskii's Lemma (but with less explicit constants) can be found in [18].

5 Upper bounds for the dispersion of sets induced by the flow generated by the solution to SDE

This section is devoted to providing upper bounds for the dispersion of bounded sets induced by the flow generated by the solution to a singular SDE. It essentially follows from Theorem 3.3 once we verify the stability estimate (3.1). To establish (3.1), we treat the special case of weakly differentiable coefficients in Section 5.1 and then use *Zvonkin's transformation* in Section 5.2 to prove the general case.

5.1 Stability estimates of the SDE with weakly differentiable coefficients

Consider the equation

$$\mathrm{d}Y_t^i = \tilde{b}(Y_t^i)\ \mathrm{d}t + \tilde{\sigma}(Y_t^i)\ \mathrm{d}W_t, \quad Y_0^i = y_i \in \mathbb{R}^d, \quad i = 1, 2. \tag{5.1}$$

For $\tilde{b}$ and $\tilde{\sigma}$ we assume:

Assumption 5.1. *For $p, \rho \in (2d, \infty)$,*

1. $\|\tilde{b}\|_{\tilde{H}^{1,p}} < \infty$;[1]
2. $\|\nabla\tilde{\sigma}\|_{\tilde{L}_\rho} < \infty$;
3. *for $\tilde{a} := \tilde{\sigma}\tilde{\sigma}^*$, there exist some $\tilde{K}_1, \tilde{K}_2 > 0$ such that for all $x \in \mathbb{R}^d$,*

$$\tilde{K}_1|\zeta|^2 \le \langle \tilde{a}(x)\zeta, \zeta\rangle \le \tilde{K}_2|\zeta|^2, \quad \forall \zeta \in \mathbb{R}^d.$$

Theorem 5.2. *Let Theorem 5.1 hold. There exist constants $\kappa_0, \kappa_1 > 0$ depending only on p, d, ρ, such that for any $r \ge 1$, $T \ge 0$, $y_i \in \mathbb{R}^d$, $i = 1, 2$, the solutions $Y^i := Y^i(y_i)$ to equations* (5.1) *satisfy*

$$\mathbb{E}[\sup_{t\in[0,T]} |Y_t^1(y_1) - Y_t^2(y_2)|^r] \le \kappa_0 |y_1 - y_2|^r \exp(\kappa_1 T\varrho), \tag{5.2}$$

where

$$\varrho := r^4\Big[\|\tilde{b}\|_\infty + \|\tilde{\sigma}\|_\infty^2 + (\tilde{\Gamma}\|\nabla\tilde{b}\|_{\tilde{L}_p})^2\tilde{K}_2^{-1} + \tilde{\Gamma}\|\nabla\tilde{b}\|_{\tilde{L}_p} + \tilde{\Gamma}^2\|\nabla\tilde{\sigma}\|_{\tilde{L}_\rho}^4\tilde{K}_2^{-1} + \tilde{\Gamma}\|\nabla\tilde{\sigma}\|_{\tilde{L}_\rho}^2\Big], \tag{5.3}$$

and $\tilde{\Gamma} := \Big(\big(\frac{\tilde{K}_2}{\tilde{K}_1}\big)^{\frac{4d^2}{1-d/\rho}} + \big(\frac{\|\nabla\tilde{\sigma}\|_{\tilde{L}_\rho}^2}{\tilde{K}_1}\big)^{\frac{4d^2}{1-d/\rho}} + \big(\frac{\|\tilde{b}\|_{\tilde{L}_p}}{\tilde{K}_1}\big)^{\frac{4d}{1-d/p}}\Big)$.

Proof. Again, all constants $C_1, \ldots$ depend on p, ρ, d only. By Itô's formula we get for any $r \ge 1$,

$$\begin{aligned}|Y_t^1 - Y_t^2|^{2r} =& |y_1 - y_2|^{2r} + \int_0^t |Y_s^1 - Y_s^2|^{2r}\,\mathrm{d}A_s + M_t \le |y_1 - y_2|^{2r} \\ &+ \int_0^t |Y_s^1 - Y_s^2|^{2r}\,\mathrm{d}\bar{A}_s + M_t,\end{aligned} \tag{5.4}$$

where M_t is an $(\mathcal{F}_t)$-local martingale defined as

$$M_t := \int_0^t 2r|Y_s^1 - Y_s^2|^{2r-2}[\tilde{\sigma}(Y_s^1) - \tilde{\sigma}(Y_s^2)]^*(Y_s^1 - Y_s^2)\ \mathrm{d}W_s$$

[1] Since $\tilde{b} \in H^{1,p}$, we can find a continuous and bounded version $\bar{b} \in H^{1,p}$. Due to the uniformly elliptic noise, the solutions Y^i, $i = 1, 2$ with drift $\bar{b}$ are indistinguishable from those with drift $\tilde{b}$. Therefore will always assume that $\tilde{b}$ is bounded and continuous.

and

$$A_t := \int_0^t \frac{2r\langle Y_s^1 - Y_s^2, \tilde{b}(Y_s^1) - \tilde{b}(Y_s^2)\rangle + r\|\tilde{\sigma}(Y_s^1) - \tilde{\sigma}(Y_s^2)\|^2}{|Y_s^1 - Y_s^2|^2}\,\mathrm{d}s + \int_0^t \frac{2r(r-1)|[\tilde{\sigma}(Y_s^1) - \tilde{\sigma}(Y_s^2)]^*(Y_s^1 - Y_s^2)|^2}{|Y_s^1 - Y_s^2|^4}\,\mathrm{d}s$$

and

$$\bar{A}_t := \int_0^t \frac{2r|\langle Y_s^1 - Y_s^2, \tilde{b}(Y_s^1) - \tilde{b}(Y_s^2)\rangle| + r\|\tilde{\sigma}(Y_s^1) - \tilde{\sigma}(Y_s^2)\|^2}{|Y_s^1 - Y_s^2|^2}\,\mathrm{d}s + \int_0^t \frac{2r(r-1)|[\tilde{\sigma}(Y_s^1) - \tilde{\sigma}(Y_s^2)]^*(Y_s^1 - Y_s^2)|^2}{|Y_s^1 - Y_s^2|^4}\,\mathrm{d}s.$$

There exists $C_1 > 0$ such that for each $x, y \in \mathbb{R}^d$

$$|\tilde{\sigma}(x) - \tilde{\sigma}(y)| \leq C_1|x-y|(\mathcal{M}|\nabla\tilde{\sigma}|(x) + \mathcal{M}|\nabla\tilde{\sigma}|(y) + \|\tilde{\sigma}\|_\infty),$$
$$|\tilde{b}(x) - \tilde{b}(y)| \leq C_1|x-y|(\mathcal{M}|\nabla\tilde{b}|(x) + \mathcal{M}|\nabla\tilde{b}|(y) + \|\tilde{b}\|_\infty),$$

where $\mathcal{M}f$ is defined as $\mathcal{M}f(x) := \sup_{r\in(0,1)} \frac{1}{|B_r|}\int_{B_r} f(x+y)\,\mathrm{d}y$, which satisfies

$$\|\mathcal{M}f\|_{\tilde{L}_\gamma} \leq C(\gamma, d)\|f\|_{\tilde{L}_\gamma} \quad \text{for} \quad \gamma > 1, \tag{5.5}$$

see [27, Lemma 2.1].

Using these estimates and the Cauchy–Schwarz inequality, we get

$$\begin{aligned}\bar{A}_t \leq& C_2\Big(r\Big(\int_0^t \mathcal{M}|\nabla\tilde{b}|(Y_s^1) + \mathcal{M}|\nabla\tilde{b}|(Y_s^2)\,\mathrm{d}s + t\|\tilde{b}\|_\infty\Big) \\ &+ r\Big(\int_0^t \mathcal{M}|\nabla\tilde{\sigma}|^2(Y_s^1) + \mathcal{M}|\nabla\tilde{\sigma}|^2(Y_s^2)\,\mathrm{d}s + t\|\tilde{\sigma}\|_\infty^2\Big) \\ &+ 2r(r-1)\Big(\int_0^t \mathcal{M}|\nabla\tilde{\sigma}|^2(Y_s^1) + \mathcal{M}|\nabla\tilde{\sigma}|^2(Y_s^2)\,\mathrm{d}s + t\|\tilde{\sigma}\|_\infty^2\Big)\Big) \\ =& tC_2\big(r\|\tilde{b}\|_\infty + (2r^2 - r)\|\tilde{\sigma}\|_\infty^2\big) \\ &+ C_2\sum_{i=1}^2 \int_0^t r\mathcal{M}|\nabla\tilde{b}|(Y_s^i) + (2r^2 - r)\mathcal{M}|\nabla\tilde{\sigma}|^2(Y_s^i)\ \mathrm{d}s.\end{aligned}$$

Applying Theorem 4.2 and (5.5) we get, for $\alpha > 0$ and $t \geq 0$,

$$\mathbb{E}[\exp(\alpha\bar{A}_t)] \leq 16\exp\big[C_3\varrho_\alpha t\big], \tag{5.6}$$

where

$$\begin{aligned}\varrho_\alpha =& \alpha\big(r\|\tilde{b}\|_\infty + r^2\|\tilde{\sigma}\|_\infty^2\big) + (r\alpha\tilde{\Gamma}\|\nabla\tilde{b}\|_{\tilde{L}_p})^2\tilde{K}_2^{-1} + r\alpha\tilde{\Gamma}\|\nabla\tilde{b}\|_{\tilde{L}_p} \\ &+ (\alpha r^2\tilde{\Gamma}\|\nabla\tilde{\sigma}\|_{\tilde{L}_\rho}^2)^2\tilde{K}_2^{-1} + (\alpha r^2\tilde{\Gamma}\|\nabla\tilde{\sigma}\|_{\tilde{L}_\rho}^2).\end{aligned} \tag{5.7}$$

Choosing $\alpha = 1$ and applying stochastic Grönwall's inequality (see [24, Theorem 4] or [29, Lemma 3.7]) to (5.4) we get

$$\mathbb{E}[\sup_{t\in[0,T]} |Y_t^1 - Y_t^2|^r] \leq C_4|y_1 - y_2|^r\Big(\mathbb{E}\big[\exp\big(\bar{A}_T\big)\big]\Big)^{1/2} \leq 4C_4|y_1 - y_2|^r\exp\Big(\frac{1}{2}C_3\varrho_1 T\Big).$$

Observing that ϱ_1 is at most equal to ϱ_0 defined in (5.3) and defining $\kappa_0 = 4C_4$ and $\kappa_1 = \frac{1}{2}C_3$, (5.2) follows. □

Remark 5.3. If $\tilde{\sigma}$ is even globally Lipschitz continuous with Lipschitz constant L, then there is no need to use Khasminskii's Lemma for the integral over $\tilde{\sigma}$ and we easily get (5.2) with

$$\varrho = r^2\Big[\|\tilde{b}\|_\infty + (\tilde{\Gamma}\|\nabla\tilde{b}\|_{\tilde{L}_p})^2\tilde{K}_2^{-1} + \tilde{\Gamma}\|\nabla\tilde{b}\|_{\tilde{L}_p} + L^2\Big]$$

and

$$\tilde{\Gamma} := \Big(\big(\frac{\tilde{K}_2}{\tilde{K}_1}\big)^{4d^2} + \big(\frac{L}{\tilde{K}_1}\big)^{4d^2} + \big(\frac{\|\tilde{b}\|_{\tilde{L}_p}}{\tilde{K}_1}\big)^{\frac{4d}{1-d/p}}\Big).$$

5.2 Linear expansion rate of the SDE with singular coefficients

Theorem 5.4. *Let Theorem 2.4 hold. Let ψ denote the flow generated by the solution to* (1.1)*. Let $\mathcal{X}$ be a compact subset of $\mathbb{R}^d$. Then there exists a positive constant $C_{p,\rho,d}$ depending on p, d, ρ only such that*

$$\limsup_{T\to\infty}\Big(\sup_{t\in[0,T]}\sup_{x\in\mathcal{X}}\frac{1}{T}|\psi_t(x)|\Big) \leq \kappa^* \quad a.s., \tag{5.8}$$

where

$$\begin{aligned}\kappa^* =& C_{p,\rho,d}\Big(K_2 + \|b\|^2_{\tilde{L}_p}\frac{K_2}{K_1^2} + \|\nabla\sigma\|^2_{\tilde{L}_\rho}\Big)\\ &\Big[\Big(\frac{K_2}{K_1}\Big)^{\frac{16d^3}{(1-d/(p\wedge\rho))(1-d/\rho)}} + \Big(\frac{\|b\|_{\tilde{L}_p}}{K_1}\Big)^{\frac{32d^2}{1-d/(p\wedge\rho)}} + \Big(\frac{\|\nabla\sigma\|^2_{\tilde{L}_\rho}}{K_1}\Big)^{\frac{32d^3}{(1-d/(p\wedge\rho))(1-d/\rho)}}\Big].\end{aligned}$$

Proof. The idea is to apply Theorem 3.3. All following constants $C_1^*, ..., C_7^*$ will depend on p, ρ, d only.

Step 1. We check the assumptions of Theorem 3.1.

To verify (3.1) we apply *Zvonkin's transformation* and Theorem 5.2. Since, by (4.2), the map $x \mapsto a(x) = \sigma(x)\sigma^*(x)$ is $1-d/\rho$-Hölder continuous and $\omega_{1-d/\rho}(a) \leq C_{\rho,d}\sqrt{K_2}\|\nabla\sigma\|_{\tilde{L}_\rho}$, Theorem A.3 and Theorem A.4 show that there exists a constant C_1^* such that for

$$\lambda := C_1^* K_1\Big(\frac{K_2^2}{K_1^2}\big(\frac{K_1 + \sqrt{K_2}\|\nabla\sigma\|_{\tilde{L}_\rho}}{K_1}\big)^{\frac{2}{1-d/\rho}} + \big(\frac{K_1 + \sqrt{K_2}\|\nabla\sigma\|_{\tilde{L}_\rho}}{K_1}\big)^{\frac{2d}{(1-d/\rho)(1-d/p)}}\big(\frac{\|b\|_{\tilde{L}_p}}{K_1}\big)^{\frac{2}{1-d/p}}\Big),$$

the equation

$$\frac{1}{2}a_{ij}\partial^2_{ij}u^{(l)} + b\cdot\nabla u^{(l)} - \lambda u^{(l)} = -b^{(l)}, \quad l = 1, \cdots, d,$$

has a unique solution $U := (u^{(l)})_{1\leq l\leq d}$, $u^{(l)} \in \tilde{H}^{2,p}$ and

$$\Phi(x) := x + U(x) \quad \text{ for } x \in \mathbb{R}^d \tag{5.9}$$

is a C^1-diffeomorphism on $\mathbb{R}^d$ (see also [33]), which is also known as *Zvonkin's transformation* map. Let $\Psi := (\Phi)^{-1}$. Then, by the generalized Itô's formula ([27]), $Y_t := \Phi(\psi_t(x))$ satisfies the following equation

$$\mathrm{d}Y_t = \tilde{b}(Y_t)\ \mathrm{d}t + \tilde{\sigma}(Y_t)\ \mathrm{d}W_t, \quad Y_0 = y \in \mathbb{R}^d \tag{5.10}$$

with

$$\tilde{b}(x) := \lambda U(\Psi(x)), \quad \tilde{\sigma}(x) := [\nabla\Phi\cdot\sigma]\circ(\Psi(x)), \quad y = \Phi(x).$$

From [27, (4.5)] we know that

$$\|U\|_\infty < \frac{1}{2}, \quad \|\nabla U\|_\infty < \frac{1}{2}. \tag{5.11}$$

Furthermore, by (A.17) and (A.4) we have

$$\|\nabla U\|_{\tilde{L}_p} \le \frac{1}{2}\big(\frac{K_1}{\lambda}\big)^{\frac{d}{2p}} \le \frac{1}{2}, \quad \|U\|_{\tilde{L}_p} \le \frac{1}{2}\big(\frac{K_1}{\lambda}\big)^{\frac{1-d/p}{2}} \le \frac{1}{2},$$
$$\|\nabla^2 U\|_{\tilde{L}_p} \le C_2^* \frac{1}{K_1}\big(1+\frac{\sqrt{K_2}\|\nabla\sigma\|_{\tilde{L}_\rho}}{K_1}\big)^{\frac{d}{(1-d/\rho)}}\|b\|_{\tilde{L}_p}. \tag{5.12}$$

Hence, by (5.11) (see also e.g. [27, p. 15]),

$$\frac{1}{2} \le |\nabla\Phi| = |\mathbb{I} + \nabla U| \le \frac{3}{2}, \quad |\nabla\Psi| \le 2$$

which implies that for all $x \in \mathbb{R}^d$,

$$\frac{1}{4}K_1|\xi|^2 \le \langle\tilde{\sigma}\tilde{\sigma}^*(x)\xi,\xi\rangle \le \frac{9}{4}K_2|\xi|^2, \quad \forall \xi \in \mathbb{R}^d, \tag{5.13}$$

and

$$\|\tilde{b}\|_\infty \le \lambda\|U\|_\infty \le \frac{1}{2}\lambda, \quad \|\tilde{b}\|_{\tilde{L}_p} \le \lambda\|U\|_{\tilde{L}_p} \le \frac{1}{2}\lambda,$$
$$\|\nabla\tilde{b}\|_{\tilde{L}_p} \le \lambda\|\mathrm{det}(\nabla\Phi)\|_\infty^{\frac{1}{p}}\|\nabla U\|_{\tilde{L}_p} \le \lambda. \tag{5.14}$$

Moreover for $p' = \min(p,\rho)$ we have by embedding

$$\begin{aligned}
\|\nabla\tilde{\sigma}\|_{\tilde{L}_{p'}} &= \|\Big((\nabla^2\Phi\cdot\sigma + \nabla\Phi\nabla\sigma)\nabla\Psi\Big)\circ\Psi\|_{\tilde{L}_{p'}} \\
&\le \|\Big((\nabla^2\Phi\cdot\sigma)\nabla\Psi\Big)\circ\Psi\|_{\tilde{L}_p} + \|\Big((\nabla\Phi\nabla\sigma)\nabla\Psi\Big)\circ\Psi\|_{\tilde{L}_\rho} \\
&\le 2\|\mathrm{det}(\nabla\Phi)\|_\infty^{\frac{1}{p\wedge\rho}}(\sqrt{K_2}\|\nabla^2\Phi\|_{\tilde{L}_p} + \|\nabla\Phi\cdot\nabla\Psi\|_\infty\|\nabla\sigma\|_{\tilde{L}_\rho}) \\
&\le 9C_2^*\frac{\sqrt{K_2}}{K_1}\big(1+\frac{\sqrt{K_2}\|\nabla\sigma\|_{\tilde{L}_\rho}}{K_1}\big)^{\frac{d}{(1-d/\rho)}}\|b\|_{\tilde{L}_p} + 9\|\nabla\sigma\|_{\tilde{L}_\rho}.
\end{aligned} \tag{5.15}$$

If $(\phi_t(x))_{t\ge0}$ is the flow generated by the solution to (5.10), then by definition of $\Phi(\psi_t(x))$ from (5.9) and the fact that U is uniformly bounded from (5.11), we get that

$$\limsup_{T\to\infty}\Big(\sup_{t\in[0,T]}\sup_{x\in\mathcal{X}}\frac{1}{T}|\psi_t(x)|\Big) = \limsup_{T\to\infty}\Big(\sup_{t\in[0,T]}\sup_{x\in\mathcal{X}}\frac{1}{T}|\phi_t(x)|\Big).$$

Using the estimates (5.13), (5.14) and (5.15) we will establish (5.2) for Y. Indeed, let $\tilde{K}_1 := \frac{1}{4}K_1$ and $\tilde{K}_2 = \frac{9}{4}K_2$ in Theorem 5.1. Then we define

$$\begin{aligned}
\tilde{\Gamma} :=& \Big(\big(\frac{\tilde{K}_2}{\tilde{K}_1}\big)^{\frac{4d^2}{1-d/p'}} + \big(\frac{\|\tilde{\nabla}\sigma\|^2_{\tilde{L}_{p'}}}{\tilde{K}_1}\big)^{\frac{4d^2}{1-d/p'}} + \big(\frac{\|\tilde{b}\|_{\tilde{L}_p}}{\tilde{K}_1}\big)^{\frac{4d}{1-d/p}}\Big) \\
\le& C_{p,\rho,d}\Big(\big(\frac{K_2}{K_1}\big)^{\frac{4d^2}{1-d/(p\wedge\rho)}} + \big(\frac{K_2}{K_1}\frac{\|b\|^2_{\tilde{L}_p}}{K_1^2}(1+\frac{\sqrt{K_2}\|\nabla\sigma\|_{\tilde{L}_\rho}}{K_1})^{\frac{2d}{1-d/\rho}}\big)^{\frac{4d^2}{1-d/(p\wedge\rho)}} + \big(\frac{\lambda}{K_1}\big)^{\frac{4d}{p-d}}\Big) \\
\le& C_{p,\rho,d}\Big(\big(\frac{K_2}{K_1}\big)^{\frac{8d^3}{(1-d/(p\wedge\rho))(1-d/\rho)}} + \big(\frac{\|b\|_{\tilde{L}_p}}{K_1}\big)^{\frac{16d^2}{1-d/(p\wedge\rho)}} + \big(\frac{\|\nabla\sigma\|^2}{K_1}\big)^{\frac{16d^3}{(1-d/(p\wedge\rho))(1-d/\rho)}}\Big).
\end{aligned} \tag{5.16}$$

Using Theorem 5.2 and the fact that $|\nabla\Psi| \le 2$ together with (5.13), (5.14) and (5.15), for the flows correspondingly $\psi_t^1(x_1), \psi_t^2(x_2)$ generated by the solutions $X_t^1(x_1)$, $X_t^1(x_2)$ to (1.1) we get

$$\begin{aligned}
\mathbb{E}[\sup_{t\in[0,T]}|\psi_t^1(x_1) - \psi_t^2(x_2)|^r] &= \mathbb{E}[\sup_{t\in[0,T]}|\Psi(Y_t^1(y_1)) - \Psi(Y_t^2(y_2))|^r] \\
&\le 2^r\mathbb{E}[\sup_{t\in[0,T]}|Y_t^1(y_1) - Y_t^2(y_2)|^r] \le 2^rC_4^*|y_1-y_2|^r\exp(C_3^*T\varrho)
\end{aligned} \tag{5.17}$$

with

$$\varrho :=r^4\Big[\lambda+K_2+\tilde{\Gamma}\lambda+(\tilde{\Gamma}\lambda)^2K_2^{-1}+\tilde{\Gamma}^2\big(\frac{K_2}{K_1^2}\|b\|^2_{\tilde{L}_p}+\|\nabla\sigma\|^2_{\tilde{L}_\rho}\big)^2K_2^{-1}+\tilde{\Gamma}\big(\frac{K_2}{K_1^2}\|b\|^2_{\tilde{L}_p}+\|\nabla\sigma\|^2_{\tilde{L}_\rho}\big)\Big]. \tag{5.18}$$

Step 2. Verification of estimate (3.5) in Theorem 3.3.

Let

$$\rho_t := \exp\Big(\int_0^t b^*(\sigma^{-1})^*(\varphi_r(x))\,\mathrm{d}W_r-\frac{1}{2}\int_0^t b^*(\sigma\sigma^*)^{-1}b(\varphi_r(x))\,\mathrm{d}r\Big),$$

where $\varphi_t(x)$ is the flow generated by the solution to

$$\mathrm{d}\varphi_t=\sigma(\varphi_t)\,\mathrm{d}W_t,\quad \varphi_0(x)=x\in\mathbb{R}^d.$$

It follows from (4.8) that, for any $\beta>0$,

$$\mathbb{E}\exp\Big(\beta\int_0^T b^*(\sigma\sigma^*)^{-1}b(\varphi_r(x))\,\mathrm{d}r\Big)\le 2\exp\Big(TC_5^*\big((K_1^2K_2)^{-1}(\Gamma'\beta)^2\|b\|^4_{\tilde{L}_p}+\Gamma'\beta K_1^{-1}\|b\|^2_{\tilde{L}_p}\big)\Big) \tag{5.19}$$

where

$$\Gamma'=\big(\frac{K_2}{K_1}\big)^{\frac{4d^2}{1-d/\rho}}+\big(\frac{\|\nabla\sigma\|^2_{\tilde{L}_\rho}}{K_1}\big)^{\frac{4d^2}{1-d/\rho}}. \tag{5.20}$$

Therefore $(\rho_t)_{t\ge 0}$ is a martingale. Let $\mathbb{P}^\rho:=\rho_T\mathbb{P}$. By Girsanov's theorem and Hölder's inequality,

$$\begin{aligned}\mathbb{P}\Big(\sup_{0\le t\le T}|\psi_t(x)-x|\ge kT\Big) &=\mathbb{P}^\rho\Big(\sup_{0\le t\le T}|\varphi_t(x)-x|\ge kT\Big)\\ &=\mathbb{E}[\rho_T\mathbb{I}_{\{\sup_{0\le t\le T}|\varphi_t(x)-x|\ge kT\}}]\\ &\le[\mathbb{E}\rho_T^2]^{\frac12}\mathbb{P}[\sup_{0\le t\le T}|\varphi_t(x)-x|\ge kT]^{\frac12}.\end{aligned}$$

Applying Markov's inequality we obtain, for each $x\in\mathbb{R}^d$ and $\zeta\ge 0$,

$$\mathbb{P}\Big(\sup_{0\le t\le T}|\varphi_t(x)-x|\ge kT\Big)^{1/2}\le e^{-\frac12\zeta kT}\Big[\mathbb{E}\exp\Big(\zeta\sup_{0\le t\le T}\Big|\int_0^t\sigma(\varphi_r(x))\,\mathrm{d}W_r\Big|\Big)\Big]^{\frac12}. \tag{5.21}$$

(5.19) shows

$$\begin{aligned}\Big[\mathbb{E}\rho_T^2\Big]^{1/2} &= \Big[\mathbb{E}\exp\Big(2\int_0^T b^*(\sigma^{-1})^*(\varphi_r(x))\,\mathrm{d}W_r-2\int_0^T b^*(\sigma\sigma^*)^{-1}b(\varphi_r(x))\,\mathrm{d}r\\ &\qquad+\int_0^T b^*(\sigma\sigma^*)^{-1}b(\varphi_r(x))\,\mathrm{d}r\Big)\Big]^{1/2}\\ &\le\Big(\mathbb{E}\Big[\exp\Big(2\int_0^T b^*(\sigma^{-1})^*(\varphi_r(x))\,\mathrm{d}W_r-2\int_0^T b^*(\sigma\sigma^*)^{-1}b(\varphi_r(x))\,\mathrm{d}r\Big]^2\Big)^{1/4}\\ &\quad\Big[\mathbb{E}\exp\Big(\int_0^t 2b^*(\sigma\sigma^*)^{-1}b(\varphi_r(x))\,\mathrm{d}r\Big)\Big]^{1/4}\\ &\le\Big[\mathbb{E}\exp\Big(2\int_0^T b^*(\sigma\sigma^*)^{-1}b(\varphi_r(x))\,\mathrm{d}r\Big)\Big]^{1/4}\\ &\le 2\exp\Big(C_5^*T\big((K_1^2K_2)^{-1}\Gamma'^2\|b\|^4_{\tilde{L}_p}+K_1^{-1}\Gamma'\|b\|^2_{\tilde{L}_p}\big)\Big)=:2\exp(T\kappa_1)\end{aligned}$$

and by time change $\int_0^t \sigma(\varphi_r(r))\,\mathrm{d}W_r = W_{\int_0^t |\sigma(\varphi_r(x))|^2\,\mathrm{d}r}$, we also have

$$\Big[\mathbb{E}\exp\Big(2\zeta\sup_{0\le t\le T}\Big|\int_0^t \sigma(\varphi_r(x))\,\mathrm{d}W_r\Big|\Big)\Big]^{1/2} \le \sqrt{2}\exp(C_d\zeta^2\|\sigma\|_\infty^2 T) =: \sqrt{2}\exp(T\zeta^2\kappa_2).$$

Inserting these estimate into (5.21) and optimizing over $\zeta \ge 0$ yields, for any $k > 0$,

$$\begin{aligned}\mathbb{P}\Big(\sup_{0\le t\le T}|\psi_t(x) - x| \ge kT\Big) &\le 2\exp\Big(C_6^* T\big(\kappa_1 + \kappa_2\zeta^2 - \zeta k\big)\Big)\\ &\le 2\exp\Big(C_7^* T\big(-\frac{1}{4\kappa_2}k^2 + \kappa_1\big)\Big). \end{aligned}\tag{5.22}$$

With estimates (5.17) and (5.22) at hand we are ready to apply Theorem 3.3 by taking

$$\begin{aligned} c_1 &:= \lambda + K_2 + \tilde\Gamma\lambda + (\tilde\Gamma\lambda)^2 K_2^{-1} + \tilde\Gamma^2\big(\frac{K_2}{K_1^2}\|b\|_{\tilde L_p}^2 + \|\nabla\sigma\|_{\tilde L_\rho}^2\big)^2 K_2^{-1} + \tilde\Gamma\big(\frac{K_2}{K_1^2}\|b\|_{\tilde L_p}^2 + \|\nabla\sigma\|_{\tilde L_\rho}^2\big),\\ c_2 &:= \frac{1}{4\|\sigma\|_\infty^2}, \qquad c_3 := C_7^*(K_1^2K_2)^{-1}\Gamma'^2\|b\|_{\tilde L_p}^4 + K_1^{-1}\Gamma'\|b\|_{\tilde L_p}^2, \quad \alpha := 3, \end{aligned}\tag{5.23}$$

with $\tilde\Gamma$ from (5.16) and Γ' from (5.20). Note that we can take $\Delta = d-1$ by Remark 3.4. The linear expansion rate κ can now be estimated as follows (no matter which of the two cases in the definition of κ in Theorem 3.3 applies):

$$\begin{aligned} \kappa &\le C_{\alpha,d}\Big(\frac{c_1+c_3}{c_2}\Big)^{1/2}\\ &\le C_{p,\rho,d}\|\sigma\|_\infty\Big(\sqrt{\lambda} + \sqrt{K_2} + \sqrt{\tilde\Gamma\lambda} + \tilde\Gamma\lambda K_2^{-1/2} + \tilde\Gamma\big(K_2K_1^{-2}\|b\|_{\tilde L_p}^2 + \|\nabla\sigma\|_{\tilde L_\rho}^2\big)K_2^{-1/2}\\ &\qquad + \sqrt{\tilde\Gamma}\big(\sqrt{K_2}K_1^{-1}\|b\|_{\tilde L_p} + \|\nabla\sigma\|_{\tilde L_\rho}\big) + (K_1^2K_2)^{-1/2}\Gamma'\|b\|_{\tilde L_p}^2\\ &\qquad + (K_1^{-1}\Gamma')^{1/2}\|b\|_{\tilde L_p}\Big)\\ &\le C_{p,\rho,d}\sqrt{K_2}\Big(\sqrt{K_1} + \sqrt{K_2} + \frac{K_1}{\sqrt{K_2}} + \|b\|_{\tilde L_p}^2\big(\frac{\sqrt{K_2}}{K_1^2} + \frac{1}{K_1\sqrt{K_2}}\big) + \frac{\sqrt{K_2}+\sqrt{K_1}}{K_1}\|b\|_{\tilde L_p}\\ &\qquad + \|\nabla\sigma\|_{\tilde L_\rho}\\ &\qquad + \frac{\|\nabla\sigma\|_{\tilde L_\rho}^2}{\sqrt{K_2}}\Big)\Big[\Big(\frac{K_2}{K_1}\Big)^{\frac{16d^3}{(1-d/(p\wedge\rho))(1-d/\rho)}} + \big(\frac{\|b\|_{\tilde L_p}}{K_1}\big)^{\frac{32d^2}{1-d/(p\wedge\rho)}} + \big(\frac{\|b\|_{\tilde L_p}}{K_1}\big)^{\frac{8}{1-d/p}}\\ &\qquad + \big(\frac{\|\nabla\sigma\|_{\tilde L_\rho}^2}{K_1}\big)^{\frac{32d^3}{(1-d/(p\wedge\rho))(1-d/\rho)}} + \big(\frac{\|\nabla\sigma\|_{\tilde L_\rho}^2}{K_1}\big)^{\frac{8d}{(1-d/p)(1-d/\rho)}} + \big(\frac{\|\nabla\sigma\|_{\tilde L_\rho}^2}{K_1}\big)^{\frac{8}{1-d/\rho}}\Big]\\ &\le C_{p,\rho,d}\Big(K_2 + \|b\|_{\tilde L_p}^2\frac{K_2}{K_1^2} + \|\nabla\sigma\|_{\tilde L_\rho}^2\Big)\\ &\qquad \Big[\Big(\frac{K_2}{K_1}\Big)^{\frac{16d^3}{(1-d/(p\wedge\rho))(1-d/\rho)}} + \Big(\frac{\|b\|_{\tilde L_p}}{K_1}\Big)^{\frac{32d^2}{1-d/(p\wedge\rho)}} + \Big(\frac{\|\nabla\sigma\|_{\tilde L_\rho}^2}{K_1}\Big)^{\frac{32d^3}{(1-d/(p\wedge\rho))(1-d/\rho)}}\Big]. \end{aligned}\tag{5.24}$$

In the last inequality we used that $\max(\frac{32d^2}{1-d/(p\wedge\rho)}, \frac{8}{1-d/p}) \le \frac{32d^2}{1-d/(p\wedge\rho)}$, and $\max(\frac{32d^3}{(1-d/(p\wedge\rho))(1-d/\rho)}, \frac{8d}{(1-d/p)(1-d/\rho)}, \frac{8}{1-d/\rho}) \le \frac{32d^3}{(1-d/(p\wedge\rho))(1-d/\rho)}$. In the end we get (5.8). □

As a by-product from the proof of Theorem 5.4 we also have

Proposition 5.5. *Let ψ denote the flow generated by the solution to* (1.1)*. Let χ_T be cubes of $\mathbb{R}^d$ with side length $\exp(-\gamma T)$, $\gamma > 0$. If Theorem 2.4 holds then for any $k > 0$*

$$\limsup_{T\to\infty}\frac{1}{T}\sup_{\chi_T}\log\mathbb{P}\Big(\sup_{x,y\in\chi_T}\sup_{0\le t\le T}|\psi_t(x) - \psi_t(y)| \ge k\Big) \le -I(\gamma)$$

where

$$I(\gamma) := \begin{cases} \gamma^{1+1/\alpha}\alpha(1+\alpha)^{-1-1/\alpha}c_1^{-1/\alpha} & \text{if} \quad \gamma \geq c_1(\alpha+1)d^\alpha \\ d(\gamma - c_1 d^\alpha) & \text{if} \quad c_1 d^\alpha < \gamma \leq c_1(\alpha+1)d^\alpha \\ 0 & \text{if} \quad \gamma \leq c_1 d^\alpha. \end{cases} \tag{5.25}$$

with α and c_1 as in (5.23).

Proof. This follows easily from (5.18) and Theorem 3.1. □

6 Existence of random attractors to SDEs with singular drift

Inspired by the work [8], we are interested in the question whether there exists a random attractor of the RDS generated by the solution to the singular SDE. To formulate our results, we assume that the drift b can be decomposed into $b = b_1 + b_2$ where $|b_1| \in \tilde{L}_p(\mathbb{R}^d)$ and b_2 is non-singular and points towards the origin. In order to be able to use results from [8] we assume, in addition to the previous assumptions, that b_2 and σ are both globally Lipschitz continuous and that b_2 is bounded. We will obtain the required bounds for the one-point motion of the flow ψ by using the corresponding bounds for the flow generated by the SDE without drift b_1 and apply Girsanov's theorem. To obtain Theorem 6.2 and Theorem 6.3 we use the *chaining technique* to control the two-point motion and hence the growth of sets under the action of the solution flow. We start with estimates of the one-point motion (items 1-5 of the following lemma) and then move to estimates for the dispersion of sets (items 6 and 7).

Lemma 6.1. *Let Theorem 2.4 hold. Further assume that there exist vector fields b_1 and b_2 such that $b = b_1 + b_2$ with $b_1 \in \tilde{L}_p(\mathbb{R}^d)$ and b_2 and σ Lipschitz with constants L_b and L_σ respectively. We also assume b_2 to be bounded (which implies $b_2 \in \tilde{L}_p(\mathbb{R}^d)$). Let ψ be the flow generated by the solution to* (1.1). *Let $\Gamma := C_{\mathrm{Kry}}(\frac{p}{2})\Big(\big(\frac{K_2}{K_1}\big)^{\frac{4d^2}{1-d/\rho}} + \big(\frac{\|\nabla\sigma\|^2_{\tilde{L}_\rho}}{K_1}\big)^{\frac{4d^2}{1-d/\rho}} + \big(\frac{\|b_2\|_{\tilde{L}_p}}{K_1}\big)^{\frac{4d}{1-d/p}}\Big)$ where $C_{\mathrm{Kry}}(\frac{p}{2})$ is from* (4.1) *with $q = \frac{p}{2}$ depending on p, ρ and d only.*

1. *Let $1 \leq r$, and $r_1, r_2 > r$. If b_2 satisfies Theorem 2.11 (U^β) for some $\beta \in \mathbb{R}$, then, for each $|x| = r_2$,*

$$\begin{aligned} &\mathbb{P}\Big(|\psi_T(x)| \geq r_1, \inf_{0\leq t\leq T} |\psi_t(x)| \geq r\Big) \\ &\qquad \leq 2\exp\Big(T\frac{\Gamma^2\|b_1\|^4_{\tilde{L}_p} + K_2^2\Gamma\|b_1\|^2_{\tilde{L}_p}}{K_1^2 K_2} - \frac{1}{4}\Big(-\frac{r_2-r_1}{\sqrt{K_2 T}} - \frac{\sqrt{T}\beta^*(r)}{\sqrt{K_2}}\Big)_+^2\Big) \end{aligned}$$

   with

$$\beta^*(r) := \sup_{|x|\geq r} \frac{x\cdot b_2(x)}{|x|} + (d-1)\frac{K_2}{2r}. \tag{6.1}$$

2. *If b_2 satisfies Theorem 2.11 (U^β) for some $\beta < 0$ and $r_0 > 1$ is such that $\beta^*(r_0) \leq 0$ where $\beta^*(r_0)$ is from* (6.1)*, then for every $R \geq r \geq r_0$ and every $x \in \mathbb{R}^d$, we have*

$$\mathbb{P}\Big(|\psi_T(x)| \geq R, \inf_{0\leq t\leq T}|\psi_t(x)| \leq r\Big) \leq 4\exp\Big(T\frac{\Gamma^2\|b_1\|^4_{\tilde{L}_p} + K_2^2\Gamma\|b_1\|^2_{\tilde{L}_p}}{K_1^2K_2} - \frac{(R-r)^2}{16K_2T}\Big).$$

3. *If b_2 satisfies Theorem 2.11 (U^β) for some $\beta < 0$ and $r_0 > 1$ such that $\beta^*(r_0) \leq 0$ where $\beta^*(r_0)$ is from* (6.1) *and if $R \geq r_0$, then for every $|x| = R$, $\delta, \delta_1 > 0$, we have*

$$\mathbb{P}\Big(\sup_{0\leq s\leq\delta_1}|\psi_s(x)| \geq R+\delta\Big) \leq 6\exp\Big(T\frac{\Gamma^2\|b_1\|^4_{\tilde{L}_p} + K_2^2\Gamma\|b_1\|^2_{\tilde{L}_p}}{K_1^2K_2} - \frac{\delta^2}{16K_2\delta_1}\Big).$$

4. *Let* $1 \le r$*, and* $r_1, r_2 > r$*. If* b_2 *satisfies Theorem 2.11* (U_β) *for some* $\beta \in \mathbb{R}$*, then for each* $|x| = r_1$*,*

$$\mathbb{P}\Big(|\psi_T(x)| \le r_2, \inf_{0\le t\le T} |\psi_t(x)| \ge r\Big) \\ \le 2\exp\Big(T\frac{\Gamma^2\|b_1\|^4_{\tilde{L}_p} + K_2^2\Gamma\|b_1\|^2_{\tilde{L}_p}}{K_1^2K_2} - \frac{1}{4}\Big(\frac{\sqrt{T}\beta_*(r)}{\sqrt{K_2}} - \frac{r_2-r_1}{\sqrt{K_2T}}\Big)^2_+\Big)$$

with

$$\beta_*(r) := \inf_{|x|\ge r} \frac{x\cdot b_2(x)}{|x|}. \tag{6.2}$$

5. *If* b_2 *satisfies Theorem 2.11* (U_β) *for some* $\beta \in \mathbb{R}$*, then for each* $|x| = r_1$*, for* $1 \le r < r_1$

$$\mathbb{P}\Big(\inf_{t\ge 0} |\psi_t(x)| \le r\Big) \le 2\exp\Big(T\frac{\Gamma^2\|b_1\|^4_{\tilde{L}_p} + K_2^2\Gamma\|b_1\|^2_{\tilde{L}_p}}{K_1^2K_2} - (r_1 - r)\frac{\beta_*(r)}{K_2}\Big)$$

with $\beta_*(r_1)$ *defined as* (6.2)*.*

6. *Assume that* b_2 *satisfies Theorem 2.11* (U_β) *for*

$$\beta > \beta_0 := 4\frac{\|b_1\|^2_{\tilde{L}_p}\Gamma + K_2\|b_1\|_{\tilde{L}_p}\sqrt{\Gamma}}{K_1} + (2\sqrt{3(d-1)K_2(L_b+L_\sigma)} + 6K_2(d-1)).$$

Let $h : [1,\infty) \to [1,\infty)$ *be strictly increasing such that* $\lim_{x\to\infty} \frac{h(x)}{x} = 0$ *and* $\lim_{x\to\infty} \frac{\log x}{h(x)} = 0$*. Let* $\eta \in (0, \frac{1}{2})$ *and* $\gamma > 0$ *with* $\eta + \gamma < \beta - \beta_0$*. For* $R > 2$*, define* $T := h(R)$*,* $r = (1-\eta)R$ *and* $r_1 := R + \gamma h(R)$*. Then*

$$\limsup_{R\to\infty} \frac{1}{h(R)} \log \mathbb{P}_R \\ := \limsup_{R\to\infty} \frac{1}{h(R)} \log \mathbb{P}\Big[\Big(B_{r_1} \not\subseteq \psi_T(B_R)\Big) \cup \cup_{t\in[0,T]}\Big(B_r \not\subseteq \psi_t(B_R)\Big)\Big] < 0.$$

7. *Assume that* b_2 *satisfies Theorem 2.11* (U^β) *for*

$$\beta < -\beta_0 := -4\frac{\|b_1\|^2_{\tilde{L}_p}\Gamma + K_2\|b_1\|_{\tilde{L}_p}\sqrt{\Gamma}}{K_1} - (2\sqrt{3(d-1)K_2(L_b+L_\sigma)} + 6K_2(d-1)).$$

Let $h(R) = R^\iota$ *for some* $\iota \in (0, \frac{1}{3})$*. Let* $\eta \in (0, \frac{1}{2})$ *and* $\gamma > 0$ *with* $\eta + \gamma < -\beta - \beta_0$*. For* $R > 2$*, define* $T := h(R)$*,* $r = (1-\eta)R$ *and* $r_1 := R + \gamma h(R)$*. Then*

$$\limsup_{R\to\infty} \frac{1}{h(R)} \log \mathbb{P}_R \\ := \limsup_{R\to\infty} \frac{1}{h(R)} \log \mathbb{P}\Big[\bigcup_{|x|=r_1}\Big((|\psi_T(x)| \ge R) \cap (\inf_{t\in[0,T]} |\psi_t(x)| \ge r)\Big)\Big] < 0.$$

Proof. Let

$$\rho_t := \exp\Big\{\int_0^t (b_1)^*(\sigma^{-1})^*(\psi_r^2(x))\,\mathrm{d}W_r - \frac{1}{2}\int_0^t (b_1)^*(\sigma\sigma^*)^{-1}b_1(\psi_r^2(x))\,\mathrm{d}r\Big\},$$

where $\psi_t^2(x)$ is the flow generated by the solution to

$$\mathrm{d}\psi_t^2 = b_2(\psi_t^2)\,\mathrm{d}t + \sigma(\psi_t^2)\,\mathrm{d}W_t, \quad \psi_0^2 = x \in \mathbb{R}^d.$$

From (4.8) we get for $T>1$ and any $\lambda>0$

$$\begin{aligned}&\mathbb{E}\exp\Big(\lambda\int_0^T(b_1)^*(\sigma\sigma^*)^{-1}b_1(\psi_r^2(x))\,\mathrm{d}r\Big)\\&\qquad\le 2\exp\Big(T\big((K_1^2K_2)^{-1}(\Gamma\lambda)^2\|b_1\|_{\tilde L_p}^4+K_1^{-1}\lambda\Gamma\|b_1\|_{\tilde L_p}^2\big)\Big).\end{aligned}\tag{6.3}$$

(since $\log 2<1$). Therefore, $(\rho_t)_{t\ge0}$ is a martingale. Fix $T>0$ and let $\mathbb{P}^\rho:=\rho_T\mathbb{P}$. Girsanov's theorem and Hölder's inequality show for each measurable set $A\subseteq C([0,T],\mathbb{R}^d)$

$$\begin{aligned}&\mathbb{P}(\psi|_{[0,T]}\in A)=\mathbb{P}^\rho\big(\psi^2|_{[0,T]}\in A\big)\\&\quad=\mathbb{E}\big[\rho_T:\,\psi^2|_{[0,T]}\in A\big]\le\big[\mathbb{E}\rho_T^2\big]^{1/2}\mathbb{P}\big(\psi^2|_{[0,T]}\in A\big)^{1/2}\\&\quad\le\Big[\mathbb{E}\exp\Big(2\int_0^T(b_1)^*(\sigma^{-1})^*(\psi_r^2(x))\,\mathrm{d}W_r-2\int_0^T(b_1)^*(\sigma\sigma^*)^{-1}b_1(\psi_r^2(x))\,\mathrm{d}r\\&\qquad\qquad+\int_0^T(b_1)^*(\sigma\sigma^*)^{-1}b^1(\psi_r^2(x))\,\mathrm{d}r\Big)\Big]^{1/2}\Big[\mathbb{P}\big(\psi^2|_{[0,T]}\in A\big)\Big]^{1/2}\\&\quad\le\Big(\mathbb{E}\Big[\exp\Big(2\int_0^T(b_1)^*(\sigma^{-1})^*(\psi_r^2(x))\,\mathrm{d}W_r-2\int_0^T(b_1)^*(\sigma\sigma^*)^{-1}b_1(\psi_r^2(x))\,\mathrm{d}r\Big]^2\Big)^{1/4}\\&\qquad\Big[\mathbb{E}\exp\Big(\int_0^T2(b_1)^*(\sigma\sigma^*)^{-1}b_1(\psi_r^2(x))\,\mathrm{d}r\Big)\Big]^{1/4}\Big[\mathbb{P}\big(\psi^2|_{[0,T]}\in A\big)\Big]^{1/2}\\&\quad\le\Big[\mathbb{E}\exp\Big(2\int_0^T(b_1)^*(\sigma\sigma^*)^{-1}b_1(\psi_r^2(x))\,\mathrm{d}r\Big)\Big]^{1/4}\Big[\mathbb{P}\big(\psi^2|_{[0,T]}\in A\big)\Big]^{1/2}\\&\quad\le 2\exp\Big(T\big((K_1^2K_2)^{-1}\Gamma^2\|b_1\|_{\tilde L_p}^4+K_1^{-1}\Gamma\|b_1\|_{\tilde L_p}^2\big)\Big)\Big[\mathbb{P}\big(\psi^2|_{[0,T]}\in A\big)\Big]^{1/2}.\end{aligned}\tag{6.4}$$

If A_i denotes the set inside $\mathbb{P}$ on the left side of item i in the Lemma (i $=1,...,5$), then

$$\mathbb{P}(\psi|_{[0,T]}\in A_i)\le 2\exp\Big(T\frac{\Gamma^2\|b_1\|_{\tilde L_p}^4+K_2^2\Gamma\|b_1\|_{\tilde L_p}^2}{K_1^2K_2}\Big)\Big[\mathbb{P}\big(\psi^2|_{[0,T]}\in A_i\big)\Big]^{1/2}$$

finishing the first step in cases 1-5. It remains to estimate $\Big[\mathbb{P}\big(\psi^2|_{[0,T]}\in A_i\big)\Big]^{1/2}$. Inserting the estimate in [8, Proposition 4.2 a)] under (U^β), we obtain statement 1. Inserting the estimate in [8, Proposition 4.5] under (U^β), we obtain statement 2. Inserting the estimate in [8, Proposition 4.6] under (U^β), we obtain statement 3. Inserting the estimate in [8, Proposition 4.2 b)] under (U_β), we obtain statement 4. and inserting the estimate in [8, Proposition 4.3] under (U_β), we obtain statement 5.

Finally we show items 6 and 7. Without loss of generality we assume $\frac{1}{\eta}<R$. For a ball B_R with radius R we can cover its boundary ∂B_R by $N=N_\epsilon\le C_d(\frac{R}{\epsilon})^{d-1}$ balls with radius ϵ centered on ∂B_R for any $\epsilon\in(0,1]$. Here we take $\epsilon=\exp(-\kappa h(R))$ for some $\kappa>0$ which will be chosen later and we label the balls by $L_1,\cdots,L_N$ with corresponding centers $x_1,\cdots,x_N$. Note that

$$N\le C_dR^{d-1}\exp\Big((d-1)\kappa h(R)\Big).$$

Then

$$\begin{aligned}\mathbb{P}_R\le&N\max_{1\le i\le N}\Big[\mathbb{P}\Big(|\psi_T(x_i)|\le r_1+1,\inf_{t\in[0,T]}|\psi_t(x_i)|>r+1\Big)\\&\qquad+\mathbb{P}\Big(\inf_{t\in[0,T]}|\psi_t(x_i)|\le r+1\Big)+\mathbb{P}\Big(\sup_{t\in[0,T]}\operatorname{diam}\psi_t(L_i)\ge1\Big)\Big]\\=:&N(P_1(R)+P_2(R)+P_3(R)).\end{aligned}$$

Item 5. shows for $r=(1-\eta)R$

$$\begin{aligned}P_2(R)&\le 2\exp\Big(T\frac{\Gamma^2\|b_1\|^4_{\tilde L_p}+K_2^2\Gamma\|b_1\|^2_{\tilde L_p}}{K_1^2K_2}-(R-r-1)\frac{\beta_*(r+1)}{K_2}\Big)\\&=2\exp\Big(h(R)\frac{\Gamma^2\|b_1\|^4_{\tilde L_p}+K_2^2\Gamma\|b_1\|^2_{\tilde L_p}}{K_1^2K_2}-(\eta R-1)\frac{\beta_*(r+1)}{K_2}\Big).\end{aligned}$$

Hence

$$\frac{1}{h(R)}\log(N\mathbb{P}_2(R))\le(d-1)\kappa+\frac{\Gamma^2\|b_1\|^4_{\tilde L_p}+K_2^2\Gamma\|b_1\|^2_{\tilde L_p}}{K_1^2K_2}-\frac{\eta R-1}{h(R)}\frac{\beta_*(r+1)}{K_2}\tag{6.5}$$

which is tending to $-\infty$ since $\frac{\beta_*(r+1)}{K_2}>0$ and $\lim_{R\to\infty}\frac{\eta R-1}{h(R)}=\infty$.

By [8, (4.2)], choosing $\kappa\ge(L_b+L_\sigma)+3K_2(d-1)$,

$$\limsup_{R\to\infty}\frac{1}{h(R)}\log NP_3(R)\le(d-1)\kappa-\frac{1}{2K_2}(\kappa-(L_b+L_\sigma))^2<0.\tag{6.6}$$

Further, item 4. gives us the following upper bound (note $T=h(R)$)

$$\begin{aligned}P_1(R)&\le 2\exp\Big(T\frac{\Gamma^2\|b_1\|^4_{\tilde L_p}+K_2^2\Gamma\|b_1\|^2_{\tilde L_p}}{K_1^2K_2}-\frac14\Big(\frac{\sqrt{T}\beta_*(r+1)}{\sqrt{K_2}}-\frac{r_1+1-R}{\sqrt{K_2T}}\Big)_+^2\Big)\\&=2\exp\Big(h(R)\frac{\Gamma^2\|b_1\|^4_{\tilde L_p}+K_2^2\Gamma\|b_1\|^2_{\tilde L_p}}{K_1^2K_2}-\frac{h(R)}{4K_2}\Big(\beta_*(r+1)-\gamma-\frac{1}{h(R)}\Big)_+^2\Big).\end{aligned}$$

So

$$\limsup_{R\to\infty}\frac{1}{h(R)}\log(N\mathbb{P}_1(R))\le(d-1)\kappa+\frac{\Gamma^2\|b_1\|^4_{\tilde L_p}+K_2^2\Gamma\|b_1\|^2_{\tilde L_p}}{K_1^2K_2}-\frac{1}{4K_2}(\beta-\gamma)^2.$$

For $\kappa\ge(L_b+L_\sigma)+3K_2(d-1)$, notice

$$\beta-\gamma>\beta_0+\eta\ge 4\frac{\|b_1\|^2_{\tilde L_p}\Gamma+K_2\|b_1\|_{\tilde L_p}\sqrt{\Gamma}}{K_1}+(2\sqrt{3(d-1)K_2(L_b+L_\sigma)}+6K_2(d-1)).$$

Then we have

$$\limsup_{R\to\infty}\frac{1}{h(R)}\log(N\mathbb{P}_1(R))\le(d-1)\kappa+\frac{\Gamma^2\|b_1\|^4_{\tilde L_p}+K_2^2\Gamma\|b_1\|^2_{\tilde L_p}}{K_1^2K_2}-\frac{1}{4K_2}(\beta-\gamma)^2<0.\tag{6.7}$$

Therefore, by (6.7), (6.5) and (6.6), it follows that, for $\kappa\ge(L_b+L_\sigma)+3K_2(d-1)$,

$$\limsup_{R\to\infty}\frac{1}{h(R)}\log\mathbb{P}_R\le\limsup_{R\to\infty}\frac{1}{h(R)}\max\Big(\log N\mathbb{P}_1(R),\log N\mathbb{P}_2(R),\log N\mathbb{P}_3(R)\Big)<0.\tag{6.8}$$

Therefore part 6. holds.

We show part 7. in a similar way. We again cover ∂B_{r_1} by $N\le C_d{r_1}^{d-1}e^{\kappa(d-1)T}$ balls centered on ∂B_{r_1} with radius $e^{-\kappa T}$ for some $\kappa>0$ chosen later. Label the balls by $L_1,\cdots,L_N$ with corresponding centers $x_1,\cdots,x_N$. Then

$$\mathbb{P}_R\le N\max_i\Big[\mathbb{P}\Big(|\psi_T(x_i)|\ge R+1,\inf_{t\in[0,T]}|\psi_t(x_i)|>r+1\Big)$$

$$
\begin{aligned}
&+\mathbb{P}\Big(|\psi_T(x_i)|\ge R+1,\inf_{t\in[0,T]}|\psi_t(x_i)|\le r+1\Big)+\mathbb{P}\Big(\sup_{t\in[0,T]}\text{diam }\psi_t(L_i)\ge 1\Big)\Big]\\
=:&N(P_1(R)+P_2(R)+P_3(R)).
\end{aligned}
$$

From case 1 we then get (note $T=h(R)$)

$$
\begin{aligned}
P_1(R)\le&2\exp\Big(h(R)\frac{\Gamma^2\|b_1\|^4_{\tilde{L}_p}+K_2^2\Gamma\|b_1\|^2_{\tilde{L}_p}}{K_1^2K_2}-\frac{h(R)}{4K_2}\Big(\frac{R+1-r_1}{h(R)}-\beta^*(r+1)\Big)_+^2\Big)\\
\le&2\exp\Big(h(R)\frac{\Gamma^2\|b_1\|^4_{\tilde{L}_p}+K_2^2\Gamma\|b_1\|^2_{\tilde{L}_p}}{K_1^2K_2}-\frac{h(R)}{4K_2}(\frac{1}{h(R)}-\gamma-\beta)^2\Big).
\end{aligned}
$$

Therefore,

$$
\frac{1}{h(R)}\log P_1(R)\le 2\big(\frac{\Gamma^2\|b_1\|^4_{\tilde{L}_p}+K_2^2\Gamma\|b_1\|^2_{\tilde{L}_p}}{K_1^2K_2}-\frac{1}{4K_2}(\frac{1}{h(R)}-\gamma-\beta)^2\big).
$$

Under (U^β), taking $\kappa\ge(L_b+L_\sigma)+3K_2(d-1)$,

$$
(-\gamma-\beta)^2\ge(-\beta_0-\eta)^2\ge 16\frac{\Gamma^2\|b_1\|^4_{\tilde{L}_p}+K_2^2\Gamma\|b_1\|^2_{\tilde{L}_p}}{K_1^2}+12(d-1)\kappa K_2-2(\kappa-(L_b+L_\sigma))^2,
$$

which implies that

$$
\frac{1}{h(R)}\log NP_1(R)\le 2\big(\frac{\Gamma^2\|b_1\|^4_{\tilde{L}_p}+K_2^2\Gamma\|b_1\|^2_{\tilde{L}_p}}{K_1^2K_2}-\frac{1}{4K_2}(\frac{1}{h(R)}-\gamma-\beta)^2\big)+(d-1)\kappa<0. \tag{6.9}
$$

Analogously, case 2 implies for R such that $r=(1-\eta)R>r_0$ where $\beta^*(r_0)<0$,

$$
\frac{1}{h(R)}\log NP_2(R)\le\frac{\Gamma^2\|b_1\|^4_{\tilde{L}_p}+K_2^2\Gamma\|b_1\|^2_{\tilde{L}_p}}{K_1^2K_2}-\frac{(R-r)^2}{16K_2h(R)}+(d-1)\kappa\to-\infty\quad\text{as }R\to\infty. \tag{6.10}
$$

By (6.9), (6.10) and (6.6) we conclude that $\limsup_{R\to\infty}\frac{1}{h(R)}\log\mathbb{P}_R<0$. □

Now we are ready to state the first main theorem of this section.

Theorem 6.2. *Let Theorem 2.4 hold. Further assume that there exist vector fields b_1 and b_2 such that $b=b_1+b_2$ with $b_1\in\tilde{L}_p(\mathbb{R}^d)$, b_2 is bounded and b_2 and σ are Lipschitz continuous with Lipschitz constants L_b and L_σ respectively. Let ψ denote the flow generated by the solution to* (1.1)*. Let $\Gamma:=C_{\text{Kry}}(\frac{p}{2})\Big(\big(\frac{K_2}{K_1}\big)^{\frac{4d^2}{1-d/\rho}}+\big(\frac{\|\nabla\sigma\|^2_{\tilde{L}_\rho}}{K_1}\big)^{\frac{4d^2}{1-d/\rho}}+\big(\frac{\|b_2\|_{\tilde{L}_p}}{K_1}\big)^{\frac{4d}{1-d/p}}\Big)$ where $C_{\text{Kry}}(\frac{p}{2})$ is from* (4.1) *with $q=\frac{p}{2}$ depending on p,ρ and d only. c_1 is from* (5.23)*. If b_2 satisfies Theorem 2.11 (U_β) for*

$$
\beta>\beta_0:=4\frac{\|b_1\|^2_{\tilde{L}_p}\Gamma+K_2\|b_1\|_{\tilde{L}_p}\sqrt{\Gamma}}{K_1}+(2\sqrt{3(d-1)K_2(L_b+L_\sigma)}+6K_2(d-1)),
$$

then for any $\gamma\in[0,\beta-\beta_0)$ we have

$$
\lim_{r\to\infty}\mathbb{P}\Big(B_{\gamma t}\subset\psi_t(B_r)\quad\forall\quad t\ge0\Big)=1. \tag{6.11}
$$

Proof. For $\gamma \in [0, \beta - \beta_0)$, let $\eta \in (0, \frac{1}{2})$ such that $\gamma + \eta < \beta - \beta_0$. Let $R_0 \geq 2$, $R_{i+1} = R_i + \gamma h(R_i)$ by iteration, where $h : [1, \infty) \to [1, \infty)$ is strictly increasing and $\lim_{x\to\infty} \frac{h(x)}{x} = 0$ and $\lim_{x\to\infty} \frac{\log x}{h(x)} = 0$. For $i = 0, 1, \cdots$, take $r_i = (1 - \eta)R_i$, $\bar{r}_i = R + \gamma h(R_i)$. Define

$$\mathbb{P}_{R_i} := \mathbb{P}\Big[\Big(B_{\bar{r}_i} \nsubseteq \psi_T(B_{R_i})\Big) \cup \cup_{t\in[0,T]} \Big(B_{r_i} \nsubseteq \psi_t(B_{R_i})\Big)\Big].$$

Then Theorem 6.1 case 6 shows that

$$\sum_{i=0}^{\infty} \mathbb{P}_{R_i} < \infty, \quad \text{if} \quad \sum_{i=0}^{\infty} \exp(-\kappa h(R_i)) < \infty, \quad \kappa > 0.$$

If we take $h(R_i) = R_i^\alpha$ for some $\alpha \in (0, 1)$, then Borel-Cantelli Lemma and time-homogeneity of flow ψ yield the result (6.11). $\square$

Finally, we state the following theorem on the existence of random attractors.

Theorem 6.3. *Let Theorem 2.4 hold. Further assume that there exist vector fields b_1 and b_2 such that $b = b_1 + b_2$ with $b_1 \in \tilde{L}_p(\mathbb{R}^d)$ and b_2 and σ are Lipschitz continuous with Lipschitz constants L_b and L_σ respectively. Let ϕ denote the flow generated by the solution to* (1.1)*. Let $\Gamma := C_{\mathrm{Kry}}(\frac{p}{2})\Big(\big(\frac{K_2}{K_1}\big)^{\frac{4d^2}{1-d/\rho}} + \big(\frac{\|\nabla\sigma\|^2_{\tilde{L}_\rho}}{K_1}\big)^{\frac{4d^2}{1-d/\rho}} + \big(\frac{\|b_2\|_{\tilde{L}_p}}{K_1}\big)^{\frac{4d}{1-d/p}}\Big)$ where $C_{\mathrm{Kry}}(\frac{p}{2})$ is from* (4.1) *with $q = \frac{p}{2}$ depending on p, ρ and d only. If b_2 satisfies Theorem 2.11 (U^β) for*

$$\beta < -\beta_0 := -4\frac{\|b_1\|^2_{\tilde{L}_p}\Gamma + K_2\|b_1\|_{\tilde{L}_p}\sqrt{\Gamma}}{K_1} - (2\sqrt{3(d-1)K_2(L_b + L_\sigma)} + 6K_2(d-1)),$$

then, for any $\gamma \in [0, -\beta - \beta_0)$, we have

$$\lim_{r\to\infty} \mathbb{P}\Big(B_{\gamma t} \subset \phi^{-1}_{-t,0}(B_r) \quad \forall \quad t \geq 0\Big) = 1. \tag{6.12}$$

In particular, ψ has a random attractor.

Proof. The existence of an attractor is an easy observation from Theorem 2.8 if we have (6.12). So we only need to show (6.12). The argument is essentially the same as [8, Proof of Theorem 3.1 a)]. We give the outline of the proof emphasising those arguments which are different.

For $\gamma \in [0, -\beta - \beta_0)$, let $\eta \in (0, \frac{1}{2})$ such that $\gamma + \eta < -\beta - \beta_0$. Let $h(y) = y^\alpha$ for some $\alpha \in (0, \frac{1}{3})$. Notice that such h is strictly increasing and $\lim_{y\to\infty} \frac{h(y)}{y} = 0$ and $\lim_{y\to\infty} \frac{\log y}{h(y)} = 0$. For $T \in (1, \infty)$, take $R := T^{1/\alpha}$, $r_1 = R + \gamma T$ and $r = (1 - \eta)R$. Define

$$\mathbb{P}_R := \mathbb{P}\Big[\Big(B_{r_1} \nsubseteq \phi^{-1}_{0,T}(B_R)\Big) \cup \cup_{t\in[0,T]} \Big(B_r \nsubseteq \phi^{-1}_{t,T}(B_R)\Big)\Big].$$

Once we show that

$$\lim_{R\to\infty} \frac{1}{h(R)} \log \mathbb{P}_R < 0, \tag{6.13}$$

then, by the same argument as in the proof of Theorem 6.2, we can finish the proof by the Borel-Cantelli Lemma and time-homogeneity of the flow ψ.

To show (6.13), notice that

$$\begin{aligned}\mathbb{P}_R &\leq \mathbb{P}\Big[\cup_{|x|=r_1} \Big((|\phi_{0,T}(x)| \geq R) \cap (\inf_{t\in[0,T]} |\phi_{0,t}(x)| \geq r)\Big)\Big] + \mathbb{P}\Big(\sup_{|x|=r} \sup_{t\in[0,T]} |\phi_{t,T}(x)| \geq R\Big) \\ &=: P_1(R) + P_2(R).\end{aligned}$$

For $P_1(R)$, we get from Theorem 6.1, case 7 that (note $T = R^\alpha = h(R)$)

$$\lim_{R\to\infty} \frac{1}{h(R)} \log P_1(R) < 0.$$

In the following we show

$$\lim_{R\to\infty} \frac{1}{h(R)} \log P_2(R) = -\infty, \tag{6.14}$$

which is sufficient to get (6.13).

Let $\xi_s := (\sup_{|x|=r} |\phi_{s,T}(x)| - r)_+$, $\zeta_s := (\sup_{|x|=r+R\eta/2} |\phi_{s,T}(x)| - r)_+$. Then, as shown in [8, p.1205-1206], we have

$$\begin{aligned}
&\limsup_{R\to\infty} \frac{1}{h(R)} \log P_2(R) \\
&\quad \le \limsup_{R\to\infty} \frac{1}{h(R)} \log \max_{s\in[1,T]} \Big[\mathbb{P}\Big(\zeta_s \ge \eta R\Big) + \mathbb{P}\Big(\sup_{t\in[s-1,s]} \sup_{|x|=r} |\phi_{t,s}(x)| \ge r + \frac{\eta}{2}R \Big) \Big] \\
&\qquad =: \limsup_{R\to\infty} \frac{1}{h(R)} \log \max_{s\in[1,T]} (P_{2,1}(s,R) + P_{2,2}(s,R)).
\end{aligned}$$

To estimate $P_{2,1}(s,R)$, for fixed $0 \le s \le T$, denote $r_0 := r + \frac{\eta}{2}R$, we cover ∂B_{r_0} by $N \le C_d r_0^{d-1} e^{\kappa(d-1)T}$ balls of radius $e^{-\kappa T}$ centered on ∂B_{r_0} withfor $\kappa \ge (L_b + L_\sigma) + 3K_2(d-1)$, (the same choice as in the proof of Theorem 6.1 case 7. Label the balls by $L_1, \cdots, L_N$ and their centers correspondingly by $x_1, \cdots, x_N$. Then for a number r_2 such that $\beta^*(r_2) < 0$ where $\beta^*(r_2)$ is from (6.1), we have

$$\begin{aligned}
P_{2,1}(s,R) &\le N \max_i \Big[\mathbb{P}\Big(|\phi_{s,T}(x_i)| \ge r + \eta R - 1 \Big) + \mathbb{P}\Big(\text{diam } \phi_{s,T}(L_i) \ge 1 \Big) \Big] \\
&\le N \max_i \Big[\mathbb{P}\Big(|\phi_{s,T}(x_i)| \ge r + \eta R - 1, \inf_{s\le t\le T} |\phi_{s,t}(x_i)| > r_2 \Big) \\
&\qquad\qquad + \mathbb{P}\Big(|\phi_{s,T}(x_i)| \ge r + \eta R - 1, \inf_{s\le t\le T} |\phi_{s,t}(x_i)| \le r_2 \Big) \\
&\qquad\qquad + \mathbb{P}\Big(\text{diam } \phi_{s,T}(L_i) \ge 1 \Big) \Big].
\end{aligned}$$

By the same argument from Theorem 6.1 case 7 with $h(R) = R^\alpha = T$, and Theorem 6.1 case 2, and Theorem 5.5 we get

$$\limsup_{R\to\infty} \frac{1}{h(R)} \log \max_{s\in[1,T]} P_{2,1}(s,R) = -\infty.$$

Up to here, in order to get (6.14), we only need to show

$$\limsup_{T\to\infty} \frac{1}{T} \log \max_{s\in[1,T]} P_{2,2}(s, T^{1/\alpha}) = -\infty. \tag{6.15}$$

In [8, Proof of Theorem 3.1 a)], this is shown by using three statements: [8, (4.7)], [8, Proposition 4.5] and [8, Proposition 4.6]. In our setting, we already showed the second and the third statements: these are Theorem 6.1 case 2 and case 3 correspondingly. Therefore it is sufficient to show the estimate corresponding to [8, (4.7)] in our setting. In order to do so we first apply Girsanov Theorem as we did in Theorem 6.1. Let

$$\rho_t := \exp\Big(\int_0^t b^*(\sigma^{-1})^*(\bar\phi_r(x)) \, \mathrm{d}W_r - \frac{1}{2} \int_0^t b^*(\sigma\sigma^*)^{-1} b(\bar\phi_r(x)) \, \mathrm{d}r \Big),$$

where $\bar\phi$ is the flow generated by the solution to

$$\mathrm{d}\bar\phi_t = \sigma(\bar\phi_t) \, \mathrm{d}W_t, \quad \bar\phi_0(x) = x \in \mathbb{R}^d.$$

Following from (4.8) we get for $T > 1$ and any $\lambda > 0$

$$\mathbb{E} \exp\Big(\lambda \int_0^T b^*(\sigma\sigma^*)^{-1} b(\bar\phi_r(x)) dr\Big) \le \exp\Big(T \frac{\|b\|_{\tilde L_p}^4 (\lambda\Gamma')^2 + K_2^2 \|b\|_{\tilde L_p}^2 \lambda\Gamma'}{K_1^2 K_2}\Big)$$

where $\Gamma' = C_{\rm Kry}(\frac{p}{2})\Big(\big(\frac{K_2}{K_1}\big)^{\frac{4d^2}{1-d/\rho}} + \big(\frac{\|\nabla\sigma\|_{\tilde L_\rho}^2}{K_1}\big)^{\frac{4d^2}{1-d/\rho}}\Big)$ and $C_{\rm Kry}(\frac{p}{2})$ is from (4.1) with $p = \frac{p}{2}$ and $b = 0$. Therefore $(\rho_t)_{t\ge 0}$ is a martingale. Let $\mathbb{P}^\rho := \rho_1 \mathbb{P}$. As we already did in (6.4), by Girsanov theorem and Hölder's inequality, for $\epsilon > 0$, for any $x, z \in \mathbb{R}^d$,

$$\begin{aligned}
\mathbb{P}\Big(&|\phi_{t+\frac{1}{2^n},1}(x) - \phi_{t+\frac{1}{2^n},1}(z)| \ge \frac{\epsilon}{2}\Big) \\
&= \mathbb{P}^\rho\Big(|\bar\phi_{t+\frac{1}{2^n},1}(x) - \bar\phi_{t+\frac{1}{2^n},1}(z)| \ge \frac{\epsilon}{2}\Big) \\
&= \mathbb{E}[\rho_1 \mathbb{I}_{\big\{|\bar\phi_{t+\frac{1}{2^n},1}(x) - \bar\phi_{t+\frac{1}{2^n},1}(z)| \ge \frac{\epsilon}{2}\big\}}] \\
&\le 2\exp\Big(\frac{\|b\|_{\tilde L_p}^4 \Gamma'^2 + K_2^2 \|b\|_{\tilde L_p}^2 \Gamma'}{K_1^2 K_2}\Big)\Big[\mathbb{P}\Big(|\bar\phi_{t+\frac{1}{2^n},1}(x) - \bar\phi_{t+\frac{1}{2^n},1}(z)| \ge \frac{\epsilon}{2}\Big)\Big]^{1/2}.
\end{aligned} \tag{6.16}$$

Let $B_t(x) := W_{\int_t^1 |\sigma|^2(\phi_r(x))dr}$, then by time change and the fact that for $\kappa_1, \kappa_2 \in \mathbb{R}$

$$\mathbb{P}\Big(W_t \ge \kappa_1\Big) \le \frac{1}{2} e^{-\frac{\kappa_1^2}{2t}}, \quad \mathbb{P}(\sup_{s\le t} W_s \ge \kappa_2) \le e^{-\frac{\kappa_2^2}{2t}},$$

we know for $x, z \in \mathbb{R}^d$ and $|x - z| \le \delta$ with $\delta > 0$

$$\begin{aligned}
&\Big[\mathbb{P}\Big(|\bar\phi_{t+\frac{1}{2^n},1}(x) - \bar\phi_{t+\frac{1}{2^n},1}(z)| \ge \frac{\epsilon}{2}\Big)\Big]^{1/2} \\
&\le \Big[\mathbb{P}\Big(|B_{t+\frac{1}{2^n}}(x) - B_{t+\frac{1}{2^n}}(z)| \ge \frac{\epsilon}{2} - \delta\Big)\Big]^{1/2} \\
&\le \Big[\exp\Big(-\frac{(\epsilon-2\delta)^2}{4} \frac{1}{2(\int_{t+\frac{1}{2^n}}^1 |\sigma|^2(\bar\phi_r(x))\,{\rm d}r - \int_{t+\frac{1}{2^n}}^1 |\sigma|^2(\bar\phi_r(x))\,{\rm d}r)}\Big)\Big]^{1/2} \\
&\le \exp\Big(-\frac{(\epsilon-2\delta)^2}{16} \frac{1}{K_2 - K_1}\Big).
\end{aligned}$$

Accordingly by (6.16) for any $\epsilon, \delta > 0$ and for any $x, z \in \mathbb{R}^d$ with $|x - z| \le \delta$ we have

$$\begin{aligned}
\mathbb{P}\Big(|\phi_{t+\frac{1}{2^n},1}(x) - \phi_{t+\frac{1}{2^n},1}(z)| \ge \frac{\epsilon}{2}\Big) &\le 2\exp\Big(\frac{\|b\|_{\tilde L_p}^4 \Gamma'^2 + K_2^2 \|b\|_{\tilde L_p}^2 \Gamma'}{K_1^2 K_2} - \frac{(\epsilon-2\delta)^2}{16} \frac{1}{K_2 - K_1}\Big) \\
&\lesssim \exp\Big(-\frac{(\epsilon-2\delta)^2}{16} \frac{1}{K_2 - K_1}\Big)
\end{aligned}$$

corresponding to [8, (4.7)]. Applying the argument from [8, Proof of Theorem 3.1 a)] we get that $P_{2,2}(s, T^{1/\alpha})$ decays super exponentially in T, therefore (6.15) holds. The proof is complete. □

A Bounds for solutions of elliptic PDEs

Consider the following elliptic equation on $\mathbb{R}^d$ (recall the summation convention):

$$\lambda u - a_{ij}\partial_{ij} u + b \cdot \nabla u = f, \tag{A.1}$$

where $\lambda > 0$, $a(\cdot) : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ is a symmetric matrix-valued Borel measurable function, and $b(\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ and $f : \mathbb{R}^d \to \mathbb{R}$ are Borel measurable functions such that $f \in \tilde L_p(\mathbb{R}^d)$ with $p \in (1, \infty)$. The definition of the solution to equation (A.1) is as follows:

Definition A.1. *Let $\lambda > 0$. We call $u \in \tilde{H}^{2,p}$ a strong solution to* (A.1) *if for a.e. $x \in \mathbb{R}^d$,*

$$\lambda u(x) - a_{ij}(x)\partial_{ij}u(x) + b(x) \cdot \nabla u(x) = f(x).$$

We assume

Assumption A.2. *(H^a) there exist $0 < K_1 \le K_2$ such that for all $x \in \mathbb{R}^d$,*

$$K_1|\zeta|^2 \le \langle a(x)\zeta, \zeta\rangle \le K_2|\zeta|^2, \quad \forall \zeta \in \mathbb{R}^d, \tag{A.2}$$

and $a(\cdot)$ is α-Hölder continuous with

$$\omega_\alpha(a) := \sup_{x,y\in\mathbb{R}^d, x\neq y, |x-y|\le 1} \frac{\|a(x) - a(y)\|}{|x-y|^\alpha} < \infty \tag{A.3}$$

for some $\alpha \in (0, 1]$.

(H^b) $b \in \tilde{L}_{p_1}(\mathbb{R}^d)$ for some $p_1 \in (d, \infty]$.

In this section we will show estimates of the solution of the elliptic PDE above. Such estimates were obtained in [33, Theorem 3.3] in the case where a is uniformly elliptic and uniformly continuous and $b \in L_{p_1}$ for some $p_1 > d$. These estimates were, however, not explicit in terms of the coefficients a, b and f. We prove the following theorem which shows this dependence since we need it (and also Corollary A.4) in the main text for the proofs of Theorem 4.1 and Theorem 5.4, but it may also be of independent interest.

Theorem A.3. *Suppose Theorem A.2 holds. There exists a constant $C_0 > 0$ depending on p, p_1, α and d only, such that for $\lambda \ge C_0 K_1\Big(\frac{K_2^2}{K_1^2}(\frac{K_1+\omega_\alpha(a)}{K_1})^{\frac{2}{\alpha}} + (\frac{K_1+\omega_\alpha(a)}{K_1})^{\frac{d}{\alpha}\frac{2}{1-d/p_1}}$ $(\frac{\|b\|_{\tilde{L}_{p_1}}}{K_1})^{\frac{2}{1-d/p_1}}\Big)$ and for any $f \in \tilde{L}_p(\mathbb{R}^d)$ with $p \in (d/2 \vee 1, p_1]$, there is a unique solution $u \in \tilde{H}^{2,p}$ to* (A.1)*. Further, for $p' \in [1, \infty]$ there exists a constant C depending on α, p, d, p' and p_1 only, such that*

$$\begin{aligned}
&\|\nabla^2 u\|_{\tilde{L}_p} \le C\frac{1}{K_1}\Big(1 + \frac{\omega_\alpha(a)}{K_1}\Big)^{d/\alpha}\|f\|_{\tilde{L}_p}, \\
&\lambda^{(1+\frac{d}{p'}-\frac{d}{p})/2}\|\nabla u\|_{\tilde{L}_{p'}} \le C K_1{}^{(\frac{d}{p'}-\frac{d}{p}-1)/2}\Big(1 + \frac{\omega_\alpha(a)}{K_1}\Big)^{d/\alpha}\|f\|_{\tilde{L}_p} \quad \text{if} \quad 1 + \frac{d}{p'} - \frac{d}{p} > 0, \\
&\lambda^{(2+\frac{d}{p'}-\frac{d}{p})/2}\|u\|_{\tilde{L}_{p'}} \le C K_1{}^{(\frac{d}{p'}-\frac{d}{p})/2}\Big(1 + \frac{\omega_\alpha(a)}{K_1}\Big)^{d/\alpha}\|f\|_{\tilde{L}_p} \quad \text{if} \quad 2 + \frac{d}{p'} - \frac{d}{p} > 0.
\end{aligned} \tag{A.4}$$

Proof. Assume $u \in \tilde{H}^{2,p}$ is a solution to (A.1). We first show the *a priori* estimates (A.4). Then the *continuity method*, as shown in [15], is a standard way to conclude the existence and uniqueness of the solution to (A.1) for those λ for which (A.4) holds. We divide the proof into three steps. Note that all positive constants $C_i, i = 1, \cdots$ appearing in the proof only depend on d, p, p_1, p', α (and not on λ, f, b, a, and $\omega_\alpha(a)$).

Step 1. Assume that a is a constant (positive definite) matrix, $b = 0$ and $f \in L_p$.

For $\lambda > 0$, let $v \in H^{2,p}$ be the solution to the following equation

$$\lambda v - \Delta v = \tilde{f}, \quad \tilde{f}(x) := f(\sigma x), \quad x \in \mathbb{R}^d,$$

where σ is the unique positive definite matrix satisfying $\sigma\sigma^* = a$. Then $v = (\lambda - \Delta)^{-1}\tilde{f}$ is the unique solution in $H^{2,p}$. From [33, (3.3)] we know that, for each $p' \in [1, \infty]$, there are constants C_1, C_2, C_3 such that

$$\|\nabla^2 v\|_{L_p} \le C_1\|\tilde{f}\|_{L_p},$$

$$\lambda^{(1+\frac{d}{p'}-\frac{d}{p})/2}\|\nabla v\|_{L_{p'}} \le C_2\|\tilde f\|_{L_p}, \quad \text{if} \quad 1+\frac{d}{p'}-\frac{d}{p}>0,$$
$$\lambda^{(2+\frac{d}{p'}-\frac{d}{p})/2}\|v\|_{L_{p'}} \le C_3\|\tilde f\|_{L_p} \quad \text{if} \quad 2+\frac{d}{p'}-\frac{d}{p}>0. \tag{A.5}$$

Let $u(x):=v(\sigma^{-1}x)$, i.e. $v(x)=u(\sigma x)$. Observe that

$$\partial_i v(x)=\partial_k u(\sigma x)\sigma_{ki}, \quad \partial_{ij}v(x)=\partial_{kr}u(\sigma x)\sigma_{ki}\sigma_{rj}.$$

Therefore

$$(\lambda-\Delta)v(x)=(\lambda-a_{ij}\partial_{ij})u(\sigma x)$$

and hence u solves (A.1). Uniqueness of a solution under the conditions of Step 1 holds since the map $v\mapsto u$ is a bijection between solutions of the corresponding PDEs. Considering

$$\frac{1}{K_1^p}\|\nabla^2 v\|_{L_p}^p \ge \det\sigma^{-1}\|\nabla^2 u\|_{L_p}^p, \quad \frac{1}{K_1^{p'/2}}\|\nabla v\|_{L_{p'}}^{p'} \ge \det\sigma^{-1}\|\nabla u\|_{L_{p'}}^{p'},$$
$$\|\tilde f\|_{L_p}^p=\det\sigma^{-1}\|f\|_{L_p}^p,$$

then (A.5) yields

$$\|\nabla^2 u\|_{L_p} \le C_1\frac{1}{K_1}\|f\|_{L_p},$$
$$\lambda^{(1+\frac{d}{p'}-\frac{d}{p})/2}\|\nabla u\|_{L_{p'}} \le C_2(\det\sigma^{-1})^{\frac1p-\frac1{p'}}\frac{1}{\sqrt{K_1}}\|f\|_{L_p}, \quad \text{if} \quad 1+\frac{d}{p'}-\frac{d}{p}>0,$$
$$\lambda^{(2+\frac{d}{p'}-\frac{d}{p})/2}\|u\|_{L_{p'}} \le C_3(\det\sigma^{-1})^{\frac1p-\frac1{p'}}\|f\|_{L_p} \quad \text{if} \quad 2+\frac{d}{p'}-\frac{d}{p}>0. \tag{A.6}$$

We know that $\det\sigma=\prod_{i=1}^d\sqrt{\lambda_i}$ where $\lambda_i>0, i=1,\cdots,d$, are the eigenvalues of a. From (A.2) we get $\lambda_i\in[K_1,K_2]$. Therefore

$$\det\sigma^{-1}\in[K_2^{-\frac{d}{2}},K_1^{-\frac{d}{2}}]. \tag{A.7}$$

Using (A.6) and (A.7), we finally get

$$\|\nabla^2 u\|_{L_p} \le C_1\frac{1}{K_1}\|f\|_{L_p},$$
$$\lambda^{(1+\frac{d}{p'}-\frac{d}{p})/2}\|\nabla u\|_{L_{p'}} \le C_2K_1{}^{(\frac{d}{p'}-\frac{d}{p}-1)/2}\|f\|_{L_p} \quad \text{if} \quad 1+\frac{d}{p'}-\frac{d}{p}>0,$$
$$\lambda^{(2+\frac{d}{p'}-\frac{d}{p})/2}\|u\|_{L_{p'}} \le C_3K_1{}^{(\frac{d}{p'}-\frac{d}{p})/2}\|f\|_{L_p} \quad \text{if} \quad 2+\frac{d}{p'}-\frac{d}{p}>0. \tag{A.8}$$

Step 2. a satisfies Theorem A.2 (H^a), $b=0$ and $f\in\tilde L_p$.

Here we apply the freezing coefficient argument. For $\delta>0$ which will be determined later, let $\xi^\delta(\cdot):=\xi(\frac{\cdot}{\delta})$ where ξ is the same function which we used to define the localized spaces. For $z\in\mathbb{R}^d$ denote

$$\xi^{z,\delta}(x):=\xi^\delta(x-z), \quad a^z:=a(z), \quad u^z(x):=\xi^{z,\delta}(x)u(x), \quad f^z(x):=\xi^{z,\delta}(x)f(x).$$

Observe that

$$\lambda u^z-a_{ij}^z\partial_{ij}u^z=h_z$$

where

$$\begin{aligned}h_z :=& f^z + (a_{ij}\partial_{ij}u)\xi^{z,\delta} - a^z_{ij}\partial_{ij}u^z\\ =& f^z + (a_{ij} - a^z_{ij})\partial_{ij}u \cdot \xi^{z,\delta} - a^z_{ij}(\partial_i u\partial_j\xi^{z,\delta} + \partial_j u\partial_i\xi^{z,\delta} + u\partial_{ij}\xi^{z,\delta}).\end{aligned}$$

From [15, p18, 2. Corollary], we know that there exists some $N_0 > 0$ such that for any $\bar{u} \in H^{2,p}$ and $\epsilon > 0$ we have

$$\|\nabla\bar{u}\|_{L_p} \le \epsilon\|\nabla^2\bar{u}\|_{L_p} + N_0\epsilon^{-1}\|\bar{u}\|_{L_p}.$$

Therefore

$$\begin{aligned}\|h_z\|_{L_p} \le& C_4\big(\|f^z\|_{L_p} + \omega_\alpha(a)\delta^\alpha\|\nabla^2 u\cdot\xi^{z,\delta}\|_{L_p} + K_2\|\nabla u\cdot\nabla\xi^{z,\delta}\|_{L_p} + K_2\|u\cdot\nabla^2\xi^{z,\delta}\|_{L_p}\big)\\ \le& C_4\big(\|f^z\|_{L_p} + 2\omega_\alpha(a)\delta^\alpha\|\nabla^2(u\cdot\xi^{z,\delta})\|_{L_p} + (K_2 + 2\omega_\alpha(a)\delta^\alpha)\|\nabla u\cdot\nabla\xi^{z,\delta}\|_{L_p}\\ &\quad + (K_2 + 2\omega_\alpha(a)\delta^\alpha)\|u\cdot\nabla^2\xi^{z,\delta}\|_{L_p}\big)\\ \le& C_5\big(\|f^z\|_{L_p} + 2\omega_\alpha(a)\delta^\alpha\|\nabla^2 u^z\|_{L_p} + (K_2 + 2\omega_\alpha(a)\delta^\alpha)\delta^{-1}\|\nabla u\cdot\xi^{z,\delta}\|_{L_p}\\ &\quad + (K_2 + 2\omega_\alpha(a)\delta^\alpha)\delta^{-2}\|u\cdot\xi^{z,\delta}\|_{L_p}\big)\\ \le& C_5\big(\|f^z\|_{L_p} + 2\omega_\alpha(a)\delta^\alpha\|\nabla^2 u^z\|_{L_p} + (K_2 + 2\omega_\alpha(a)\delta^\alpha)\delta^{-1}(\|\nabla u^z\|_{L_p} + \|u\cdot\nabla\xi^{z,\delta}\|_{L_p})\\ &\quad + (K_2 + 2\omega_\alpha(a)\delta^\alpha)\delta^{-2}\|u\cdot\xi^{z,\delta}\|_{L_p}\big)\\ \le& C_6\big(\|f^z\|_{L_p} + (2\omega_\alpha(a)\delta^\alpha + \epsilon(K_2 + 2\omega_\alpha(a)\delta^\alpha)\delta^{-1})\|\nabla^2 u^z\|_{L_p}\\ &\quad + (K_2 + 2\omega_\alpha(a)\delta^\alpha)(\epsilon^{-1}\delta^{-1} + \delta^{-2})\|u\cdot\xi^{z,\delta}\|_{L_p}),\end{aligned} \tag{A.9}$$

where $\omega_\alpha(a)$ is from (A.3). Assuming (without loss of generality) that $C_6 \ge 1/6$, we define

$$\delta := \Big(\frac{K_1}{6C_6(K_1 + 2\omega_\alpha(a))}\Big)^{1/\alpha} < 1, \quad \epsilon := \frac{K_1\delta}{6C_6(K_2 + 2\omega_\alpha(a)\delta^\alpha)}. \tag{A.10}$$

It is easy to see that $C_6\frac{1}{K_1}(2\omega_\alpha(a)\delta^\alpha + \epsilon(K_2 + 2\omega_\alpha(a)\delta^\alpha)\delta^{-1}) < \frac{1}{2}$, and

$$(K_2 + 2\omega_\alpha(a)\delta^\alpha)(\epsilon^{-1}\delta^{-1} + \delta^{-2}) \le C_7\frac{K_2^2}{K_1}(\frac{K_1 + \omega_\alpha(a)}{K_1})^{\frac{2}{\alpha}}.$$

So we get from (A.8) and (A.9) that

$$\|\nabla^2 u^z\|_{L^p} \le C_8\frac{1}{K_1}(\|f^z\|_{L_p} + \frac{K_2^2}{K_1}(\frac{K_1 + \omega_\alpha(a)}{K_1})^{\frac{2}{\alpha}}\|u^z\|_{L_p}). \tag{A.11}$$

Plugging this into (A.9) yields

$$\begin{aligned}\|h_z\|_{L_p} \le& C_6\Big(\|f^z\|_{L_p} + \frac{C_8}{2C_6}\Big(\|f^z\|_{L_p} + \frac{K_2^2}{K_1}(\frac{K_1 + \omega_\alpha(a)}{K_1})^{\frac{2}{\alpha}}\|u^z\|_{L_p}\Big)\\ &+ C_7\frac{K_2^2}{K_1}(\frac{K_1 + \omega_\alpha(a)}{K_1})^{\frac{2}{\alpha}}\|u^z\|_{L_p}\Big).\end{aligned}$$

Using the second inequality in (A.8) we get for $1 + \frac{d}{p'} - \frac{d}{p} > 0$

$$\lambda^{(1+\frac{d}{p'}-\frac{d}{p})/2}\|\nabla u^z\|_{L^{p'}} \le C_9 K_1{}^{(\frac{d}{p'}-\frac{d}{p}-1)/2}\Big(\|f^z\|_{L_p} + \frac{K_2^2}{K_1}(\frac{K_1 + \omega_\alpha(a)}{K_1})^{\frac{2}{\alpha}}\|u^z\|_{L_p}\Big). \tag{A.12}$$

Similarly, for $2 + \frac{d}{p'} - \frac{d}{p} > 0$

$$\lambda^{(2+\frac{d}{p'}-\frac{d}{p})/2}\|u^z\|_{L_{p'}} \le C_{10} K_1{}^{(\frac{d}{p'}-\frac{d}{p})/2}\Big(\|f^z\|_{L_p} + \frac{K_2^2}{K_1}(\frac{K_1 + \omega_\alpha(a)}{K_1})^{\frac{2}{\alpha}}\|u^z\|_{L_p}\Big). \tag{A.13}$$

Let $p' = p$. Then

$$\lambda\|u^z\|_{L_p} \le C_{10}\Big(\|f^z\|_{L_p} + \frac{K_2^2}{K_1}(\frac{K_1+\omega_\alpha(a)}{K_1})^{\frac{2}{\alpha}}\|u^z\|_{L_p}\Big). \tag{A.14}$$

Taking $\lambda \ge 2C_{10}\frac{K_2^2}{K_1}(\frac{K_1+\omega_\alpha(a)}{K_1})^{\frac{2}{\alpha}} =: C_{10}\kappa$ we obtain

$$\|u^z\|_{L_p} \le \frac{C_{10}}{\lambda - C_{10}\frac{K_2^2}{K_1}(\frac{K_1+\omega_\alpha(a)}{K_1})^{\frac{2}{\alpha}}}\|f^z\|_{L_p}, \quad \frac{K_2^2}{K_1}(\frac{K_1+\omega_\alpha(a)}{K_1})^{\frac{2}{\alpha}}\|u^z\|_{L_p} \le \|f^z\|_{L_p}.$$

Together with (A.11), (A.13), and (A.12), we have

$$\begin{aligned}
\|\nabla^2 u^z\|_{L^p} \le& C_{12}\frac{1}{K_1}\|f^z\|_{L_p}, \\
\lambda^{(1+\frac{d}{p'}-\frac{d}{p})/2}\|\nabla u^z\|_{L^{p'}} \le& C_{13}{K_1}^{(\frac{d}{p'}-\frac{d}{p}-1)/2}\|f^z\|_{L_p}, \quad \text{if} \quad 1+\frac{d}{p'}-\frac{d}{p} > 0, \\
\lambda^{(2+\frac{d}{p'}-\frac{d}{p})/2}\|u^z\|_{L_{p'}} \le& C_{14}{K_1}^{(\frac{d}{p'}-\frac{d}{p})/2}\|f^z\|_{L_p}, \quad \text{if} \quad 2+\frac{d}{p'}-\frac{d}{p} > 0.
\end{aligned} \tag{A.15}$$

From definition (2.1) we know that, for each $z \in \mathbb{R}^d$, $\|u^z\|_{L_p} \le \|u\|_{\tilde{L}_p} \lesssim \delta^{-d}\sup_{\tilde{z}}\|u^{\tilde{z}}\|_{L_p}$[2], so we get from (A.15) that for any $\lambda \ge C_{10}\kappa$ we have

$$\begin{aligned}
\lambda^{(2+\frac{d}{p'}-\frac{d}{p})/2}\|u\|_{\tilde{L}_{p'}} &\le C_{15}{K_1}^{(\frac{d}{p'}-\frac{d}{p})/2}\delta^{-d}\|f\|_{\tilde{L}_p} \quad \text{if} \quad 2+\frac{d}{p'}-\frac{d}{p} > 0, \\
\lambda^{(1+\frac{d}{p'}-\frac{d}{p})/2}\|\nabla u\|_{\tilde{L}_{p'}} &\le \lambda^{(1+\frac{d}{p'}-\frac{d}{p})/2}\sup_z(\|\nabla u^z\|_{L_{p'}} + \|u\nabla\xi^{z,1}\|_{L_{p'}}) \\
&\le C_{16}({K_1}^{(\frac{d}{p'}-\frac{d}{p}-1)/2} + \lambda^{-1/2}{K_1}^{(\frac{d}{p'}-\frac{d}{p})/2})\delta^{-d}\|f\|_{\tilde{L}_p} \\
&\le C_{17}{K_1}^{(\frac{d}{p'}-\frac{d}{p}-1)/2}\delta^{-d}\|f\|_{\tilde{L}_p} \quad \text{if} \quad 1+\frac{d}{p'}-\frac{d}{p} > 0, \\
\|\nabla^2 u\|_{\tilde{L}^p} &\le \sup_z(\|\nabla^2 u^z\|_{L_p} + \|u\nabla^2\xi^{z,1}\|_{L_p} + 2\|\nabla u\nabla\xi^{z,1}\|_{L_p}) \\
&\le C_{18}\Big(\frac{1}{K_1} + \lambda^{-1} + \lambda^{-1/2}{K_1}^{-1/2}\Big)\delta^{-d}\|f\|_{\tilde{L}_p}) \\
&\le C_{19}\frac{1}{K_1}\delta^{-d}\|f\|_{\tilde{L}_p}.
\end{aligned} \tag{A.16}$$

Step 3. a is Hölder continuous and Theorem A.2 (H^a) holds, $|b| \in \tilde{L}_{p_1}$ and $f \in \tilde{L}_p$.

By (A.16) and Hölder's inequality, we have for $\lambda \ge C_{10}\kappa$ and $1+\frac{d}{p'}-\frac{d}{p} > 0$

$$\begin{aligned}
\lambda^{(1+\frac{d}{p'}-\frac{d}{p})/2}\|\nabla u\|_{\tilde{L}_{p'}} \le& C_{17}{K_1}^{(\frac{d}{p'}-\frac{d}{p}-1)/2}\delta^{-d}\|f + b\cdot\nabla u\|_{\tilde{L}_p} \\
\le& C_{17}{K_1}^{(\frac{d}{p'}-\frac{d}{p}-1)/2}\delta^{-d}(\|f\|_{\tilde{L}_p} + \|b\|_{\tilde{L}_{p_1}}\|\nabla u\|_{\tilde{L}_{p_2}})
\end{aligned}$$

where $p_1, p_2 \in (p, \infty)$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. Let $p' = p_2$. Then we get

$$\lambda^{(1-\frac{d}{p_1})/2}\|\nabla u\|_{\tilde{L}_{p_2}} \le C_{20}{K_1}^{(-\frac{d}{p_1}-1)/2}\delta^{-d}(\|f\|_{\tilde{L}_p} + \|b\|_{\tilde{L}_{p_1}}\|\nabla u\|_{\tilde{L}_{p_2}}).$$

Choosing λ so large such that

$$\lambda^{(1-\frac{d}{p_1})/2} \ge C_{20}{K_1}^{\frac{-d/p_1-1}{2}}\delta^{-d}\|b\|_{\tilde{L}_{p_1}},$$

[2] Recall that in Section 2 we assumed that the localized spaces are defined using the function ξ^1.

we get

$$\|\nabla u\|_{\tilde{L}_{p_2}} \le \frac{C_{20} K_1{}^{(-\frac{d}{p_1}-1)/2}}{\lambda^{(1-\frac{d}{p_1})/2} - C_{20} K_1{}^{\frac{-d/p_1-1}{2}} \delta^{-d} \|b\|_{\tilde{L}_{p_1}}} \delta^{-d} \|f\|_{\tilde{L}_p}.$$

Moreover,

$$\|b \cdot \nabla u\|_{\tilde{L}_p} \le \frac{C_{20} K_1{}^{(-\frac{d}{p_1}-1)/2} \delta^{-d} \|b\|_{\tilde{L}_{p_1}}}{\lambda^{(1-\frac{d}{p_1})/2} - C_{20} K_1{}^{(-\frac{d}{p_1}-1)/2} \delta^{-d} \|b\|_{\tilde{L}_{p_1}}} \|f\|_{\tilde{L}_p} =: \gamma \|f\|_{\tilde{L}_p}.$$

Using (A.16) we see that for any λ such that $\lambda \ge C_{10}\kappa$ and $\lambda^{(1-\frac{d}{p_1})/2} \ge C_{20} K_1{}^{\frac{-1-d/p_1}{2}} \delta^{-d} \|b\|_{\tilde{L}_{p_1}}$, we have

$$\begin{aligned}
\|\nabla^2 u\|_{\tilde{L}_p} &\le C_{21}(1+\gamma)\delta^{-d} \frac{1}{K_1} \|f\|_{\tilde{L}_p}, \\
\lambda^{(1+\frac{d}{p'}-\frac{d}{p})/2} \|\nabla u\|_{\tilde{L}_{p'}} &\le C_{22}(1+\gamma) K_1{}^{(\frac{d}{p'}-\frac{d}{p}-1)/2} \delta^{-d} \|f\|_{\tilde{L}_p} \quad \text{if} \quad 1 + \frac{d}{p'} - \frac{d}{p} > 0, \\
\lambda^{(2+\frac{d}{p'}-\frac{d}{p})/2} \|u\|_{\tilde{L}_{p'}} &\le C_{23}(1+\gamma) K_1{}^{(\frac{d}{p'}-\frac{d}{p})/2} \delta^{-d} \|f\|_{\tilde{L}_p} \quad \text{if} \quad 2 + \frac{d}{p'} - \frac{d}{p} > 0.
\end{aligned}$$

Define $C_{24} := \big(2C_{10}\big) \vee C_{20}$. Then, for $\lambda \ge C_{24}\kappa$ and $C_{24} \lambda^{-(1-\frac{d}{p_1})/2} K_1{}^{(-\frac{d}{p_1}-1)/2} \delta^{-d} \|b\|_{\tilde{L}_{p_1}} < \frac{1}{2}$ (i.e. $\lambda \ge C_{24} K_1 (\delta^{-d} \frac{\|b\|_{\tilde{L}_p}}{K_1})^{\frac{2}{1-d/p_1}}$) by taking

$$\lambda \ge C_{24} K_1 \Big(\frac{K_2^2}{K_1^2} (\frac{K_1 + \omega_\alpha(a)}{K_1})^{\frac{2}{\alpha}} + (\frac{K_1 + \omega_\alpha(a)}{K_1})^{\frac{d}{\alpha}\frac{2}{1-d/p_1}} (\frac{\|b\|_{\tilde{L}_p}}{K_1})^{\frac{2}{1-d/p_1}} \Big)$$

we get that there exists finite positive constant C_{25} such that $1 + \gamma \le C_{25}$, which finally shows the desired result (A.4) after plugging in the value of δ from (A.10). □

Corollary A.4. *Let Theorem A.2 hold and $f = b^i, i = 1, \cdots, d$ in* (A.1)*, let $p' \in [1, \infty]$. There exists some $C_0 > 0$ depending on α, p_1 and d only, such that if we choose $\lambda \ge C_0 K_1 \Big(\frac{K_2^2}{K_1^2} (\frac{K_1+\omega_\alpha(a)}{K_1})^{\frac{2}{\alpha}} + (\frac{K_1+\omega_\alpha(a)}{K_1})^{\frac{d}{\alpha}\frac{2}{1-d/p_1}} (\frac{\|b\|_{\tilde{L}_p}}{K_1})^{\frac{2}{1-d/p_1}} \Big)$ then for the solution u^i to equation* (A.1) *we have*

$$\begin{aligned}
\|\nabla u^i\|_{\tilde{L}_{p'}} &\le \frac{1}{2} \lambda^{-\frac{d}{2p'}} K_1^{\frac{d}{2p'}} \le \frac{1}{2} \quad \textit{if} \quad 1 + \frac{d}{p'} - \frac{d}{p} > 0, \\
\|u\|_{\tilde{L}_{p'}} &\le \frac{1}{2} \lambda^{-\frac{1+d/p'}{2}} K_1^{\frac{1+d/p'}{2}} \le \frac{1}{2} \quad \textit{if} \quad 2 + \frac{d}{p'} - \frac{d}{p} > 0.
\end{aligned} \tag{A.17}$$

Proof. Notice that for such λ we have $C_0 \lambda^{-(1-\frac{d}{p_1})/2} K_1{}^{(-\frac{d}{p_1}-1)/2} (\frac{K_1+\omega_\alpha(a)}{K_1})^{\frac{d}{\alpha}} \|b\|_{\tilde{L}_{p_1}} < \frac{1}{2}$, so by (A.4) for $f = b^i$,

$$\|\nabla u^i\|_{\tilde{L}_{p'}} \le C \lambda^{\frac{-1-d/p'+d/p_1}{2}} K_1^{\frac{-1-d/p_1+d/p'}{2}} (\frac{K_1 + \omega_\alpha(a)}{K_1})^{\frac{d}{\alpha}} \|b^i\|_{\tilde{L}_{p_1}} \le \frac{1}{2} \lambda^{-\frac{d}{2p'}} K_1^{\frac{d}{2p'}} \le \frac{1}{2}.$$

With the similar argument we get $\|u\|_{\tilde{L}_{p'}} \le C \lambda^{\frac{-2-d/p'+d/p_1}{2}} K_1^{\frac{d/p'-d/p_1}{2}} (\frac{K_1+\omega_\alpha(a)}{K_1})^{\frac{d}{\alpha}} \|b\|_{\tilde{L}_{p_1}} \le \frac{1}{2} \lambda^{-\frac{1+d/p'}{2}} K_1^{\frac{1+d/p'}{2}}$. □

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