# EXPANSION AND ATTRACTION OF RDS: LONG TIME BEHAVIOR OF THE SOLUTION TO SINGULAR SDE 

CHENGCHENG LING AND MICHAEL SCHEUTZOW


#### Abstract

We provide a framework for studying the expansion rate of the image of a bounded set under a flow in Euclidean space and apply it to stochastic differential equations (SDEs for short) with singular coefficients. If the singular drift of the SDE can be split into two terms, one of which is singular and the radial component of the other term has a radial component of sufficient strength in the direction of the origin, then the random dynamical system generated by the SDE admits a pullback attractor.


AMS 2020 Mathematics Subject Classification: 60H10, 60G17, 60J60, 60 H 50
Keywords: semi-flow, random dynamical system, pullback attractor, singular stochastic differential equation, Brownian motion, dispersion of random sets, chaining, Krylov estimate, regularization by noise, elliptic partial differential equation, Zvonkin transformation

## 1. Introduction

Regularization by noise, i.e. existence and uniqueness of solutions under the assumption of nondegenerate noise, has been established for a large class of singular stochastic differential equations (SDEs). It was shown recently that these equations also generate a random dynamical system (RDS), see [18], and like in the classical (non-singular) case it therefore seems natural to establish asymptotic properties of these RDS for large times, like expansion rates of bounded sets and the existence of attractors or even synchronization (meaning that the attractor is a single random point).

We consider an SDE on $\mathbb{R}^{d}$ with time homogeneous coefficients

$$
\begin{equation*}
d X_{t}=b\left(X_{t}\right) \mathrm{d} t+\sigma\left(X_{t}\right) \mathrm{d} W_{t}, \quad X_{s}=x \in \mathbb{R}^{d}, \quad t \geqslant s \geqslant 0, \tag{1.1}
\end{equation*}
$$

where $d \geqslant 1, b: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ and $\sigma=\left(\sigma_{i j}\right)_{1 \leqslant i, j \leqslant d}: \mathbb{R}^{d} \rightarrow L\left(\mathbb{R}^{d}\right)(:=d \times d$ real valued matrices $)$ are measurable, and $\left(W_{t}\right)_{t \geqslant 0}$ is a standard $d$-dimensional Brownian motion defined on some filtered probability space $\left(\Omega, \mathcal{F},\left(\mathcal{F}_{t}\right)_{t \geqslant 0}, \mathbb{P}\right)$. We assume that $b \in \tilde{L}_{p}\left(\mathbb{R}^{d}\right)$ (defined in Section 2.1), so $b$ does not have to be continuous nor bounded, and $\sigma \sigma^{*}\left(\sigma^{*}\right.$ denotes the matrix transpose of $\sigma$ ) is bounded and uniformly elliptic and $\nabla \sigma \in \tilde{L}_{p}\left(\mathbb{R}^{d}\right)$ with $p>d$ (time homogeneous Krylov-Röckner condition). These are sufficient conditions for the well-posedness of the equation (1.1), see [14] and [24]. They also imply the existence of a flow and random dynamical system (RDS) generated by the solution to (1.1) [18].

First we analyse the linear expansion rate of the flow generated by a singular SDE. In classical results, see e.g. [21],[7], Lipschitz continuity or one-sided Lipschitz continuity of the coefficients of the SDE is assumed to obtain bounds on the expansion rate. Obviously we lack these properties in our current setting. Instead, we assume the noise to be non-degenerate, so we can apply the Zvonkin transformation to get an SDE which has Lipschitz-like coefficients and this SDE is (in an appropriate sense) equivalent to the original one (1.1). The Zvonkin transformation was invented by A. K. Zvonkin in [32] for $d=1$ and then generalized by A. Yu. Veretennikov in [23] to $d \geqslant 1$. It has become a rather standard tool to study well-posedness of singular SDEs, see e.g. [27], [25] and [24]. This tool heavily relies on regularity estimates of the solution to Kolmogorov's equation corresponding to (1.1) which can be found for instance in [13] in the classical setting. In this paper we adapt the method to the study of the RDS induced by singular SDEs. We show that the flow expands linearly (see Theorem 5.4), a property which was established for non-singular SDEs with not necessarily non-degenerate noise in $[3,4,19,20,21]$. Our proof mainly depends on stability estimates (see Theorem 5.2). These kind of estimates were studied before, see for instance [12], [27] and [28], but the dependence of the constants on the coefficients was not specified. We give a formula in Theorem 5.2 which states this dependence explicitly. It also yields the expansion rate constant in Theorem 5.4.

Secondly, we aim at conditions which guarantee the existence of an attractor for the RDS generated by a singular SDE. Clearly, one can not expect that an attractor exists without further conditions (an example without attractor is the case in which the drift is zero and the diffusion is constant). Since [6], numerous papers appeared in which the existence of attractors for various finite and infinite dimensional RDS was shown, e.g. [2], [8], [9], [10], [11], [7], [15] and [31]. A common way to prove the existence of an attractor is to show the existence of a random compact absorbing set and then to apply the criterion from [6, Theorem 3.11]. Just like [7], we will use a different and more probabilistic criterion from [5] (Proposition 2.8). Roughly speaking, all one has to show is that the image of a very large ball will be contained inside a fixed large ball after a (deterministic) long time with high probability. In [7] this was shown under the assumption that the diffusion is bounded and Lipschitz and the drift $b(x)$ has a component of sufficient strength (compared to the diffusion) in the direction of the origin for large $|x|$. In our set-up, this condition is too restrictive. Instead, we assume that the drift can be written in the form $b=b_{1}+b_{2}$, in which $b_{1}$ is singular and $b_{2}$ has a component of sufficient strength (compared to the diffusion and the localized $L_{p}$-norm of $b_{1}$ ) in the direction of the origin for large $|x|$.

Structure of the paper. We introduce notation and the main results in Section 2. In Section 3 we study the expansion rate of the diameter of the image of a bounded set under a flow under rather general conditions. These results are minor modifications of results contained in [21] which are proved by chaining techniques. Section 4 contains estimates on functionals of the solution to the singular SDE, namely quantitative versions of Krylov's estimates and Khasminskii's lemma. The first part of the main results of this paper is presented in Section 5, i.e. the linear expansion rate of the diameter of the image of a bounded set under the flow generated by the solution to a singular SDE. In Section 6 we show the existence of an attractor of the RDS generated by the singular SDE. In Appendix A we study regularity estimates of elliptic partial differential equations with emphasis on the dependence on the coefficients. We believe that these estimates are of independent interest.

## 2. Notation and main results

2.1. Notation. We denote the Euclidean norm on $\mathbb{R}^{d}$ by $|$.$| and the induced norm on L\left(\mathbb{R}^{d}\right)$ or on $L\left(L\left(\mathbb{R}^{d}\right)\right)$ by $\|$.$\| . Recall that the trace of a:=\left(a_{i j}\right)_{1 \leqslant i, j \leqslant d}:=\sigma \sigma^{*}$ satisfies $\operatorname{tr}(a)=\sum_{i, j=1}^{d} \sigma_{i j}^{2}$, where $\sigma^{*}$ denotes the transpose of $\sigma \in L\left(\mathbb{R}^{d}\right)$. For $p \in[1, \infty)$, let $L_{p}\left(\mathbb{R}^{d}\right)$ denote the space of all real Borel measurable functions on $\mathbb{R}^{d}$ equipped with the norm

$$
\|f\|_{L_{p}}:=\left(\int_{\mathbb{R}^{d}}|f(x)|^{p} \mathrm{~d} x\right)^{1 / p}<+\infty
$$

and $L_{\infty}$ denotes the space of all bounded and measurable functions equipped with the norm

$$
\|f\|_{\infty}:=\|f\|_{L_{\infty}}:=\sup _{x \in \mathbb{R}^{d}}|f(x)|
$$

We introduce the notion of a localized $L_{p}$-space for $p \in[1, \infty]$ : for fixed $\delta>0$,

$$
\begin{equation*}
\tilde{L}_{p}\left(\mathbb{R}^{d}\right):=\left\{f:\|f\|_{\tilde{L}_{p}}:=\sup _{z}\left\|\xi_{\delta}^{z} f\right\|_{L_{p}}<\infty\right\}, \tag{2.1}
\end{equation*}
$$

where $\xi_{\delta}(x):=\xi\left(\frac{x}{\delta}\right)$ and $\xi_{\delta}^{z}(x):=\xi_{\delta}(x-z)$ for $x, z \in \mathbb{R}^{d}, \xi \in C_{c}^{\infty}\left(\mathbb{R}^{d} ;[0,1]\right)$ is a smooth function with $\xi(x)=1$ for $|x| \leqslant 1 / 2$, and $\xi(x)=0$ for $|x|>1$. For $(\alpha, p) \in \mathbb{R} \times[1, \infty)$, let $H^{\alpha, p}\left(\mathbb{R}^{d}\right)$ be the usual Bessel potential space with norm

$$
\|f\|_{H^{\alpha, p}}:=\left\|(\mathbb{I}-\Delta)^{\alpha / 2} f\right\|_{L_{p}},
$$

where $(\mathbb{I}-\Delta)^{\alpha / 2} f$ is defined via Fourier's transform

$$
(\mathbb{I}-\Delta)^{\alpha / 2} f:=\mathcal{F}^{-1}\left(\left(1+|\cdot|^{2}\right)^{\alpha / 2} \mathcal{F} f\right)
$$

The localized $H^{\alpha, p}$-space is defined as

$$
\tilde{H}^{\alpha, p}:=\left\{f:\|f\|_{\tilde{H}^{\alpha, p}}:=\sup _{z}\left\|\xi_{\delta}^{z} f\right\|_{H^{\alpha, p}}<\infty\right\} .
$$

From [24, Section 2] and [30, Proposition 4.1] we know that the space $\tilde{H}^{\alpha, p}$ does not depend on the choice of $\xi$ and $\delta$, but the norm does, of course. More precisely, by [30, Proposition 4.1], for the $\tilde{L}_{p}$-norms with different $\delta$, say $\delta_{1}$ and $\delta_{2}$ and $\delta_{1}<\delta_{2}$, if we use the notation $\left(\tilde{L}_{p}\right)_{\delta}$ to denote the $\tilde{L}_{p}$ space with support radius $\delta$ for localization, then

$$
\begin{equation*}
N_{1}\|\cdot\|_{\left(\tilde{L}_{p}\right) \delta_{1}} \leqslant\|\cdot\|_{\left(\tilde{L}_{p}\right)_{\delta_{2}}} \leqslant N_{2}\left(\frac{\delta_{2}}{\delta_{1}}\right)^{d}\|\cdot\|_{\left(\tilde{L}_{p}\right) \delta_{1}} \tag{2.2}
\end{equation*}
$$

where $N_{1}, N_{2}$ are constants independent of $\delta_{1}, \delta_{2}$. For convenience we take $\delta=1$ in the following. For further properties of these spaces we refer to [24]. In the following, all derivatives should be interpreted in the weak sense. Occasionally we will use Einstein's summation convention (omitting the summation sign for indices appearing twice). We will often use the notation $r_{+}=\max \{r, 0\}$ for the positive part of $r \in \mathbb{R}, a \vee b:=\max \{a, b\}$ and $a \wedge b:=\min \{a, b\}$.
2.2. Preliminaries. In the following, all random processes will be defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
Definition 2.1. A flow $\phi$ on a Polish (i.e. separable and completely metrizable) space $X$ equipped with its Borel- $\sigma$-algebra $\mathcal{X}=\mathcal{B}(X)$ is a measurable map

$$
\phi:\left\{(s, t, x, \omega) \in[0, \infty)^{2} \times X \times \Omega: s \leqslant t<\infty\right\} \rightarrow X
$$

such that, for each $\omega \in \Omega$,
(1) $\phi_{s, s}(x)=x$ for all $x \in X$ and $s \geqslant 0$,
(2) $(s, t, x) \mapsto \phi_{s, t}(x)$ is continuous,
(3) for each $s, t$, the map $x \mapsto \phi_{s, t}(x)$ is one-to-one,
(4) for all $0 \leqslant s \leqslant t<u$ and $x \in X$, the following identity holds

$$
\phi_{s, u}(x)=\phi_{t, u}\left(\phi_{s, t}(x)\right) .
$$

Next, we define the concepts of a metric dynamical system and a random dynamical system.
Definition 2.2. A metric dynamical system (MDS for short) $\theta=\left(\Omega, \mathcal{F}, \mathbb{P},\left\{\theta_{t}\right\}_{t \in \mathbb{R}}\right)$ is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a family of measure preserving transformations $\left\{\theta_{t}: \Omega \rightarrow \Omega, t \in \mathbb{R}\right\}$ such that
(1) $\theta_{0}=\mathrm{id}, \theta_{t} \circ \theta_{s}=\theta_{t+s}$ for all $t, s \in \mathbb{R}$;
(2) the map $(t, \omega) \mapsto \theta_{t} \omega$ is measurable and $\theta_{t} \mathbb{P}=\mathbb{P}$ for all $t \in \mathbb{R}$.

Definition 2.3 (RDS, [1]). A (global) random dynamical system (RDS) $(\theta, \varphi)$ on a Polish space $(X, d)$ over an MDS $\theta$ is a mapping

$$
\varphi:\{(s, x, \omega) \in[0, \infty) \times X \times \Omega\} \rightarrow X
$$

such that, for each $\omega \in \Omega$,
(1) measurability: $\varphi$ is $(\mathcal{B}([0, \infty)) \otimes \mathcal{X} \otimes \mathcal{F}, \mathcal{X})$-measurable,
(2) $(t, x) \mapsto \varphi_{t}(x)$ is continuous,
(3) $\varphi$ satisfies the following (perfect) cocycle property: for all $t, s \geqslant 0, x \in X$,

$$
\begin{equation*}
\varphi_{0}(., \omega)=\mathrm{id}, \quad \varphi_{t+s}(x, \omega)=\varphi_{t}\left(\varphi_{s}(x, \omega), \theta_{s} \omega\right) \tag{2.3}
\end{equation*}
$$

Clearly, an $\operatorname{RDS} \varphi$ induces a flow via $\phi_{s, t}(x):=\varphi_{t}\left(x, \theta_{s}\right.$. $)$. We say that an SDE generates a flow resp. an RDS if its solution map has a modification which is a flow resp. an RDS. The following study is based on the flow generated by the solution to the SDE with singular drift. Therefore we state the result from [18, Theorem 4.5, Corollary 4.10] on the existence of a global semi-flow and a global RDS for singular SDEs under the following condition.
Assumption 2.4. For $p, \rho \in(2 d, \infty)$ assume
(i) $b \in \tilde{L}_{p}\left(\mathbb{R}^{d}\right), \sigma: \mathbb{R}^{d} \rightarrow L\left(\mathbb{R}^{d}\right)$ is measurable, $\|\nabla \sigma\| \in \tilde{L}_{\rho}\left(\mathbb{R}^{d}\right)$.
(ii) There exist $K_{1}, K_{2}>0$ such that for $a:=\sigma \sigma^{*}$ we have

$$
K_{1}|\zeta|^{2} \leqslant\langle a(x) \zeta, \zeta\rangle \leqslant K_{2}|\zeta|^{2}, \quad \forall \zeta, x \in \mathbb{R}^{d} .
$$

Remark 2.5. Note that $\tilde{L}_{p} \subset \tilde{L}_{p^{\prime}}$ whenever $p>p^{\prime}$. Therefore, if Assumption 2.4 holds with different values of $p$ and $\rho$, then it also holds with the larger of the two numbers replaced by the smaller one. In particular, the following result which was formulated for $p=\rho$ can still be applied.

Theorem 2.6. [18, Theorem 4.5, Corollary 4.10] If Assumption 2.4 holds, then the SDE (1.1) admits a flow $\phi$ and a corresponding RDS $\varphi$.

We will often write $\psi_{t}(x)$ instead of $\phi_{0, t}(x)$. Abusing notation we will sometimes say "Let $\psi_{t}(x)$ (or just $\psi$ ) be a flow ..." instead of "Let $\phi_{s, t}(x), x \in \mathbb{R}^{d}, 0 \leqslant s \leqslant t<\infty$ be a flow and $\psi_{t}(x):=\phi_{0, t}(x)$, $t \geqslant 0, x \in \mathbb{R}^{d}$...".

Definition 2.7 (Attractor, [6]). Let $\varphi$ be an RDS over the $\operatorname{MDS} \theta=\left(\Omega, \mathcal{F}, \mathbb{P},\left\{\theta_{t}\right\}_{t \in \mathbb{R}}\right)$. The random set $A(\omega)$ is a (pullback) attractor if
(1) measurability: $A(\omega)$ is a random element in the metric space of nonempty compact subsets of $X$ equipped with the Hausdorff distance,
(2) invariance property: for $t>0$ there exists a set $\Omega_{t}$ with full measure such that

$$
\varphi(t, \omega)(A(\omega))=A\left(\theta_{t} \omega\right), \quad \forall \omega \in \Omega_{t}
$$

(3) pull-back limit: almost surely, for all bounded closed sets $B \subset X$,

$$
\lim _{t \rightarrow \infty} \sup _{x \in B} \operatorname{dist}\left(\varphi\left(t, \theta_{-t} \omega\right)(x), A(\omega)\right)=0
$$

One way to verify the existence of an attractor is the following criterion.
Proposition 2.8. ([5], [7, Proposition 2.3]) Let $\varphi$ be an $\operatorname{RDS}$ over the $\operatorname{MDS} \theta=\left(\Omega, \mathcal{F}, \mathbb{P},\left\{\theta_{t}\right\}_{t \in \mathbb{R}}\right)$. Then the following are equivalent:
(i) $\varphi$ has an attractor,
(ii) $\forall r>0, \lim _{R \rightarrow \infty} \mathbb{P}\left(\omega \in \Omega: B_{r} \subset \bigcup_{s=0}^{\infty} \bigcap_{t \geqslant s} \varphi^{-1}\left(t, B_{R}, \theta_{-t} \omega\right)\right)=1$.
2.3. Main results. Based on general estimates on the speed of dispersion of random sets in Section 3 (cf. Theorem 3.3) and on quantitative estimates of the solution to singular SDE in Section 4, we will show the following result in Section 5.

Theorem 2.9. If Assumption 2.4 holds, then there exists a constant $\kappa>0$ such that for the flow $\psi$ generated by the solution to (1.1) we have, for any compact $\mathcal{X} \subset \mathbb{R}^{d}$,

$$
\limsup _{T \rightarrow \infty}\left(\sup _{t \in[0, T]} \sup _{x \in \mathcal{X}} \frac{1}{T}\left|\psi_{t}(x)\right|\right) \leqslant \kappa \quad \text { a.s.. }
$$

The precise statement including a formula for $\kappa$ will be given in Theorem 5.4. There, we can see that $\kappa \rightarrow \infty$ as $K_{1} \rightarrow 0$ (when all other parameters remain unchanged). The following example explains this fact: as the noise becomes more and more degenerate, the linear bound on the dispersion of a bounded set under the flow approaches infinity, so our non-degeneracy assumption on the noise cannot be avoided.

Example 2.10. In $\mathbb{R}^{2}$, for $\epsilon>0$, we consider the system

$$
\left\{\begin{align*}
\mathrm{d} X_{t} & =B\left(Y_{t}\right) \mathrm{d} t+\epsilon \mathrm{d} W_{t}^{1}, \quad X_{0} \in \mathbb{R},  \tag{2.4}\\
\mathrm{~d} Y_{t} & =\left[\left(\left(-Y_{t}\right) \vee(-1)\right) \wedge 1\right] \mathrm{d} t+\epsilon \mathrm{d} W_{t}^{2}, \quad Y_{0} \in \mathbb{R},
\end{align*}\right.
$$

where

$$
B(y):=\left\{\begin{array}{cc}
|y|^{-q} & \text { if } y \neq 0, \\
0 & \text { else, }
\end{array} \quad q \in\left(0, \frac{1}{4}\right)\right.
$$

and $W^{1}, W^{2}$ are two independent 1-dimensional Brownian motions. Notice that for $b(x, y):=$ $(B(y),((-y) \vee(-1)) \wedge 1)^{*}$, we have $b \in \tilde{L}_{p}\left(\mathbb{R}^{2}\right)$ for $p \in\left(4, \frac{1}{q}\right)$. Clearly there exists a unique solution $(X, Y)$ to (2.4) and

$$
X_{t}=X_{0}+\int_{0}^{t} B\left(Y_{s}\right) \mathrm{d} s+W_{t}^{1}, \quad t \geqslant 0 .
$$

By the ergodic theorem, almost surely,

$$
\lim _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} B\left(Y_{s}\right) \mathrm{d} s=\int_{-\infty}^{\infty} B(y) \pi_{\epsilon}(\mathrm{d} y),
$$

where $\pi_{\epsilon}$ is the invariant probability measure of $Y$. Since $\pi_{\epsilon}$ converges to the point measure $\delta_{0}$ weakly as $\epsilon \downarrow 0$, we see that the linear expansion rate of $(X, Y)$ converges to $\infty$ when $\epsilon \downarrow 0$. In particular, we can not expect to have a linear expansion rate for the solution to a singular SDE with degenerate noise in general.

We will now assume that the singular drift $b$ in (1.1) is of the form $b=b_{1}+b_{2}$ with $b_{1} \in \tilde{L}_{p}\left(\mathbb{R}^{d}\right)$ and $b_{2}$ satisfies one of the following conditions.

Assumption 2.11. For a given $\beta \in \mathbb{R}, b_{2}(x): \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ satisfies
$\left(U^{\beta}\right) \quad \lim \sup _{|x| \rightarrow \infty} \frac{x}{|x|} \cdot b_{2}(x) \leqslant \beta$
or
$\left(U_{\beta}\right) \quad \quad \lim \inf _{|x| \rightarrow \infty} \frac{x}{|x|} \cdot b_{2}(x) \geqslant \beta$.
Theorem 2.12. Let Assumption 2.4 hold. If there exist vector fields $b_{1}$ and $b_{2}$ such that $b=b_{1}+b_{2}$ with $b_{1} \in \tilde{L}_{p}\left(\mathbb{R}^{d}\right)$. There exist positive constants $\beta_{1}$ (see Theorem 6.2) and $\beta_{2}$ (see Theorem 6.3) such that for the flow $\left(\psi_{t}(x)\right)_{t \geqslant 0}$ generated by the solution to (1.1)

1. if $b_{2}$ satisfies Assumption $2.11\left(U_{\beta}\right)$ for $\beta>\beta_{1}$, then for any $\gamma \in\left[0, \beta-\beta_{1}\right)$ we have

$$
\begin{equation*}
\lim _{r \rightarrow \infty} \mathbb{P}\left(B_{\gamma t} \subset \psi_{t}\left(B_{r}\right) \quad \forall \quad t \geqslant 0\right)=1 \tag{2.5}
\end{equation*}
$$

2. if $b_{2}$ satisfies Assumption $2.11\left(U^{\beta}\right)$ for $\beta<-\beta_{2}$, then for any $\gamma \in\left[0,-\beta-\beta_{2}\right)$ we have

$$
\begin{equation*}
\lim _{r \rightarrow \infty} \mathbb{P}\left(B_{\gamma t} \subset \psi_{-t, 0}^{-1}\left(B_{r}\right) \quad \forall \quad t \geqslant 0\right)=1 \tag{2.6}
\end{equation*}
$$

In particular, $\psi$ has a random attractor.
Correspondingly the detailed results are presented in Theorem 6.2 and Theorem 6.3.
In the end we give the following example on the special case that the drift is bounded (i.e. $p=\infty$ ) to conclude the results on the expansion rate and attractors.

Example 2.13 (A case study: bounded coefficients). We consider the flow $\left(\psi_{t}(x)\right)_{t \geqslant 0}$ generated by the solution to (1.1) when $b, \nabla \sigma$ are simply bounded, i.e., Assumption 2.4 holds with arbitrary $p=\rho \in(1, \infty)$.

1. Expansion rate of the flow: Theorem 5.4 shows that for each $\epsilon>0$ there exist constants $C_{1}$ (depending on $d$ and $\epsilon$ ) such that for each compact subset $\mathcal{X} \subset \mathbb{R}^{d}$

$$
\begin{align*}
& \limsup _{T \rightarrow \infty}\left(\sup _{t \in[0, T]} \sup _{x \in \mathcal{X}} \frac{1}{T}\left|\psi_{t}(x)\right|\right) \leqslant C_{1}\left(K_{2}+\|b\|_{\infty}^{2} \frac{K_{2}}{K_{1}^{2}}+\|\nabla \sigma\|_{\infty}^{2}\right) \\
& {\left[\left(\frac{K_{2}}{K_{1}}\right)^{16 d^{3}+\epsilon}+\left(\frac{\|\nabla \sigma\|_{\infty}^{2}}{K_{1}}\right)^{32 d^{3}+\epsilon}+\left(\frac{\|b\|_{\tilde{L}_{p}}}{K_{1}}\right)^{32 d^{2}+\epsilon}\right] . } \tag{2.7}
\end{align*}
$$

2. Existence of the attractor: if $b=b_{1}+b_{2}$ with $b_{1}$ bounded and $b_{2}$ satisfying $\left(U^{\beta}\right)$ in Assumption 2.11 and

$$
\beta<-C_{2} \frac{\left(\left\|b_{1}\right\|_{\infty}^{2}+K_{2}\left\|b_{1}\right\|_{\infty}\right)}{\sqrt{K_{1} K_{2}}}\left[\left(\frac{K_{2}}{K_{1}}\right)^{4 d^{2}+\epsilon}+\left(\frac{\|\nabla \sigma\|_{\infty}^{2}}{K_{1}}\right)^{4 d^{2}+\epsilon}+\left(\frac{\left\|b_{2}\right\|_{\infty}}{K_{1}}\right)^{4 d+\epsilon}\right]
$$

where $\epsilon>0$ and $C_{2}>0$ is an appropriate function depending on $d$ and $\epsilon$ only, then from Theorem 6.3 we know that $\psi$ has an attractor.

## 3. Expansion of sets under a flow

In this section, we assume that $\psi:[0, \infty) \times \mathbb{R}^{d} \times \Omega \rightarrow \mathbb{R}^{d}$ is measurable such that $t \mapsto \psi_{t}(x, \omega)$ is continuous for every $x \in \mathbb{R}^{d}$ and $\omega \in \Omega$ (we do not require that $\psi$ has any kind of flow property).

Lemma 3.1. Assume that there exists $\alpha>0$ and a constant $c_{1}>0$ such that for each $r>d$, there exists $c=c(r)>0$ such that for all $x, y \in \mathbb{R}^{d}$ and $T>0$, we have

$$
\begin{equation*}
\left(\mathbb{E} \sup _{0 \leqslant t \leqslant T}\left(\left|\psi_{t}(x)-\psi_{t}(y)\right|^{r}\right)\right)^{1 / r} \leqslant c|x-y| \mathrm{e}^{c_{1} r^{\alpha} T} . \tag{3.1}
\end{equation*}
$$

Then $\psi$ has a modification (which we denote by the same symbol) which is jointly continuous in $(t, x)$ and for each $\gamma>0$ and $u>0$,

$$
\begin{equation*}
\limsup _{T \rightarrow \infty} \frac{1}{T} \sup _{\chi T, \gamma} \log \mathbb{P}\left(\sup _{x, y \in \chi_{T, \gamma}} \sup _{0 \leqslant t \leqslant T}\left|\psi_{t}(x)-\psi_{t}(y)\right| \geqslant u\right) \leqslant-I(\gamma), \tag{3.2}
\end{equation*}
$$

where $\sup _{\chi_{T, \gamma}}$ means that we take the supremum over all cubes $\chi_{T, \gamma}$ in $\mathbb{R}^{d}$ with side length $\mathrm{e}^{-\gamma T}$, and $I:[0, \infty) \rightarrow \mathbb{R}$ is defined as

$$
I(\gamma):=\left\{\begin{array}{rlc}
\gamma^{1+1 / \alpha} \alpha(1+\alpha)^{-1-1 / \alpha} c_{1}^{-1 / \alpha} & \text { if } & \gamma \geqslant c_{1}(\alpha+1) d^{\alpha}  \tag{3.3}\\
d\left(\gamma-c_{1} d^{\alpha}\right) & \text { if } c_{1} d^{\alpha}<\gamma \leqslant c_{1}(\alpha+1) d^{\alpha} \\
0 & \text { if } & \gamma \leqslant c_{1} d^{\alpha} .
\end{array}\right.
$$

Proof. We follow the argument in [21, Proof of Theorem 3.1]. Without loss of generality we take $\chi:=\chi_{T, \gamma}=\left[0, \mathrm{e}^{-\gamma T}\right]^{d}$ and define $Z_{t}(x):=\phi_{t}\left(\mathrm{e}^{-\gamma^{T}} x\right), x \in \mathbb{R}^{d}$. From (3.1) we get

$$
\left(\mathbb{E} \sup _{0 \leqslant t \leqslant T}\left(\left|Z_{t}(x)-Z_{t}(y)\right|^{r}\right)\right)^{1 / r} \leqslant c e^{-\gamma T}|x-y| \mathrm{e}^{c_{1} r^{\alpha} T} .
$$

By Kolmogorov's Theorem (see, e.g. [21, Lemma 2.1]), $\phi$ admits a jointly continuous modification and for any $\rho \in\left(0, \frac{r-d}{r}\right)$ :

$$
\begin{equation*}
\mathbb{P}\left(\sup _{x, y \in \chi} \sup _{0, \gamma}\left|\psi_{t}(x)-\psi_{t}(y)\right| \geqslant u\right) \leqslant \tilde{c} \mathrm{e}^{\left(c_{1} r^{\alpha}-\gamma\right) r T} u^{-r} \tag{3.4}
\end{equation*}
$$

where $\tilde{c}$ depends on $r, d, \rho$ only. Taking logarithms, dividing by $T$, then letting $T \rightarrow \infty$ and optimizing over $r>d$ we get the desired result (3.2).
Remark 3.2. Since $I(\gamma)=\sup _{r>d}\left\{r\left(\gamma-c_{1} r^{\alpha}\right)\right\}$ is the supremum of affine functions, the map $\gamma \mapsto I(\gamma)$ is convex. Further, $I$ grows faster than linearly.

The following theorem is a reformulation of [21, Theorem 2.3].
Theorem 3.3. Let $\psi:[0, \infty) \times \mathbb{R}^{d} \times \Omega \rightarrow \mathbb{R}^{d}$ be jointly continuous and satisfy the assumptions of Lemma 3.1 and (3.1) hold with constants $c_{1}$ and $\alpha$. Assume further, that there exist $c_{2}$ and $c_{3} \geqslant 0$ such that, for each $k>0$ and each bounded set $S \subset \mathbb{R}^{d}$, the following holds

$$
\begin{equation*}
\limsup _{T \rightarrow \infty} \frac{1}{T} \log \sup _{x \in S} \mathbb{P}\left(\sup _{0 \leqslant t \leqslant T}\left|\psi_{t}(x)\right| \geqslant k T\right) \leqslant-c_{2} k^{2}+c_{3} . \tag{3.5}
\end{equation*}
$$

Let $\mathcal{X}$ be a compact subset of $\mathbb{R}^{d}$ with box (or upper entropy) dimension $\Delta>0$. Then

$$
\begin{equation*}
\limsup _{T \rightarrow \infty}\left(\sup _{t \in[0, T]} \sup _{x \in \mathcal{X}} \frac{1}{T}\left|\psi_{t}(x)\right|\right) \leqslant \kappa \quad \text { a.s., } \tag{3.6}
\end{equation*}
$$

where

$$
\kappa:=\left\{\begin{array}{lc}
\left(\frac{c_{3}+\gamma_{1} \Delta}{c_{2}}\right)^{\frac{1}{2}} & \text { if } \frac{d}{d-\Delta}<\alpha+1, \\
\left(\frac{c_{3}+\gamma_{2} \Delta}{c_{2}}\right)^{\frac{1}{2}} & \text { otherwise, }
\end{array} \quad \text { with } \quad \gamma_{1}=\frac{c_{1} d^{\alpha+1}}{d-\Delta}, \quad \gamma_{2}=c_{1}\left(\alpha^{-1} \Delta\right)^{\alpha}(1+\alpha)^{1+\alpha} .\right.
$$

Remark 3.4. In addition to the assumptions of the previous theorem, let us assume that $\psi_{t}(x)=\phi_{0, t}(x)$ where $\phi$ is a flow (later, we will only consider this case). Let $\mathcal{X} \subset \mathbb{R}^{d}$ be any compact set and let $B$ be a ball in $\mathbb{R}^{d}$ containing $\mathcal{X}$. Clearly, the boundary $\partial B$ of $B$ has box dimension $d-1$. The flow property of $\phi$ implies that for each $t \geqslant 0$, the boundary of $\phi_{0, t}(B)$ is contained in $\phi_{0, t}(\partial B)$ and therefore any almost sure upper bound $\kappa$ for the linear expansion rate of the set $\partial B$ is at the same time an upper bound for the linear expansion rate of the set $B$ and hence of $\mathcal{X}$. This means that in the case of a flow, the formula for $\kappa$ in the theorem always holds with $\Delta$ replaced by $d-1$ (or the minimum of $\Delta$ and $d-1$ ).

## 4. Quantitative version of Krylov estimates

We will show a quantitative version of Krylov estimates (4.1). One can find similar results in the literature with implicit constants, for instance [14], [27] and [24], which however do not fit our needs since some proofs in later sections rely on the explicit dependence of the constants on the coefficients of the SDE. In the following lemma, a constant $C_{\text {Kry }}$ appears which depends on $q, p, \rho, d$ only. While we will regard $p, \rho, d$ as fixed throughout, we will apply the formula with different values of $q$ and we will therefore write $C_{\mathrm{Kry}}(q)$ for clarity.

Lemma 4.1. If Assumption 2.4 holds and $\left(X_{t}\right)_{t \geqslant 0}$ solves (1.1), then, for $f \in \tilde{L}_{q}\left(\mathbb{R}^{d}\right)$ with $q \in(d, \infty]$, there exists a constant $C_{\mathrm{Kry}}(q)>0$ depending on $q, p, \rho, d$ only such that for $0 \leqslant s \leqslant t$,

$$
\begin{equation*}
\mathbb{E}\left[\int_{s}^{t}\left|f\left(X_{r}\right)\right| \mathrm{d} r \mid \mathcal{F}_{s}\right] \leqslant C_{\mathrm{Kry}}(q) \Gamma\left(K_{2}^{-\frac{1}{2}}(t-s)^{\frac{1}{2}}+(t-s)\right)\|f\|_{\tilde{L}_{q}} \tag{4.1}
\end{equation*}
$$

where $\Gamma:=\left(\frac{K_{2}}{K_{1}}\right)^{\frac{4 d{ }^{2}}{1-d / \rho}}+\left(\frac{\|\nabla \sigma\|_{\tilde{L}_{\rho}}^{2}}{K_{1}}\right)^{\frac{4 d^{2}}{1-d / \rho}}+\left(\frac{\|b\|_{\tilde{L}_{\rho}}}{K_{1}}\right)^{\frac{4 d}{1-d / p}}$.
Proof. It is sufficient to show the estimate for positive $f$. (4.1) clearly holds when $q=\infty$, so we assume $q \in(d, \infty)$. All positive constants $C_{i}, i=0, \cdots, 7$ appearing in the proof only depend on $p, \rho, q, d$. We will regard $p, \rho$ and $d$ as fixed but we will vary $q$ in the following proof and we will therefore highlight the dependence of constants on $q$ in some cases (for $C_{0}$ and $C_{1}$ ). First we show that $a:=\sigma \sigma^{*}$ is $1-\frac{d}{\rho}$-Hölder continuous using Sobolev's embedding theorem and the condition that $\sigma \in \tilde{H}^{1, \rho}$ with $\rho>d$. Indeed

$$
\begin{align*}
\omega_{1-d / \rho}(a) & :=\sup _{x, y \in \mathbb{R}^{d}, x \neq y,|x-y| \leqslant 1} \frac{\|a(x)-a(y)\|}{|x-y|^{1-d / \rho}} \\
& \leqslant \sup _{x, y \in \mathbb{R}^{d}, x \neq y,|x-y| \leqslant 1}\left(\frac{\left\|\left(\sigma \sigma^{*}\right)(x)-\sigma(x) \sigma^{*}(y)\right\|}{|x-y|^{1-d / \rho}}+\frac{\left\|\sigma(x) \sigma^{*}(y)-\left(\sigma \sigma^{*}\right)(y)\right\|}{|x-y|^{1-d / \rho}}\right) \\
& \leqslant \sup _{x, y \in \mathbb{R}^{d}, x \neq y,|x-y| \leqslant 1}\left(\frac{\left\|\sigma^{*}(x)-\sigma^{*}(y)\right\|\|\sigma\|_{\infty}}{|x-y|^{1-d / \rho}}+\frac{\|\sigma(x)-\sigma(y)\|\|\sigma\|_{\infty}}{|x-y|^{1-d / \rho}}\right) \\
& \leqslant C_{\rho, d} \sqrt{K_{2}}\|\nabla \sigma\|_{\tilde{L}_{\rho}} . \tag{4.2}
\end{align*}
$$

We follow the idea from [29, Theorem 3.4]. Applying Theorem A. 3 with $p^{\prime}=\infty$, we see that there is a unique solution $u \in \tilde{H}^{2, q}$ to

$$
\begin{equation*}
\lambda u-\frac{1}{2} a_{i j} \partial_{i j} u=f \tag{4.3}
\end{equation*}
$$

provided that $\lambda \geqslant C_{0}(q) \frac{K_{2}^{2}}{K_{1}}\left(\frac{K_{1}+\sqrt{K_{2}}\|\nabla \sigma\|_{\tilde{L}_{\rho}}}{K_{1}}\right)^{\frac{2}{1-d / \rho}}=: \lambda_{0}(q)$. Further, for $\lambda \geqslant \lambda_{0}(q)$, we have

$$
\begin{gather*}
\sup _{x \in \mathbb{R}^{d}}|u(x)| \leqslant C_{1}(q) \lambda^{-\frac{2-d / q}{2}} K_{1}^{-\frac{d}{2 q}}\left(\frac{K_{1}+\sqrt{K_{2}}\|\nabla \sigma\|_{\tilde{L}_{\rho}}}{K_{1}}\right)^{\frac{d}{1-d / \rho}}\|f\|_{\tilde{L}_{q}}=: U_{1, q}(\lambda)\|f\|_{\tilde{L}_{q}} \\
\sup _{x \in \mathbb{R}^{d}}|\nabla u(x)| \leqslant C_{1}(q) \lambda^{-\frac{1-d / q}{2}} K_{1}^{-\frac{1+d / q}{2}}\left(\frac{K_{1}+\sqrt{K_{2}}\|\nabla \sigma\|_{\tilde{L}_{\rho}}}{K_{1}}\right)^{\frac{d}{1-d / \rho}}\|f\|_{\tilde{L}_{q}}=: U_{2, q}(\lambda)\|f\|_{\tilde{L}_{q}} . \tag{4.4}
\end{gather*}
$$

Fix $t \geqslant s \geqslant 0$ and define the stopping time

$$
\tau_{R}:=\inf \left\{\bar{s}>s: \int_{s}^{\bar{s}}\left|b\left(X_{r}\right)\right| \mathrm{d} r \geqslant R\right\}, \quad 0<R<\infty .
$$

By the generalized Itô's formula (see e.g. [24, Lemma 4.1 (iii)])

$$
u\left(X_{t \wedge \tau_{R}}\right)-u\left(X_{s \wedge \tau_{R}}\right)
$$

$$
=\frac{1}{2} \int_{s \wedge \tau_{R}}^{t \wedge \tau_{R}} a_{i j}\left(X_{r}\right) \partial_{i j} u\left(X_{r}\right) \mathrm{d} r+\int_{s \wedge \tau_{R}}^{t \wedge \tau_{R}}\left(\nabla u\left(X_{r}\right)\right)^{*} \sigma\left(X_{r}\right) \mathrm{d} W_{r}+\int_{s \wedge \tau_{R}}^{t \wedge \tau_{R}} b\left(X_{r}\right) \cdot \nabla u\left(X_{r}\right) \mathrm{d} r .
$$

Using (4.3), the mean value theorem, (4.4) and BDG's inequality, we get that

$$
\begin{aligned}
& \mathbb{E}\left[\int_{s \wedge \tau_{R}}^{t \wedge \tau_{R}} f\left(X_{r}\right) \mathrm{d} r \mid \mathcal{F}_{s}\right] \\
& =\mathbb{E}\left[\left.\left(u\left(X_{s \wedge \tau_{R}}\right)-u\left(X_{t \wedge \tau_{R}}\right)\right)\right|_{\mathcal{F}_{s}}\right]+\mathbb{E}\left[\lambda \int_{s \wedge \tau_{R}}^{t \wedge \tau_{R}} u\left(X_{r}\right) \mathrm{d} r \mid \mathcal{F}_{s}\right]+\mathbb{E}\left[\int_{s \wedge \tau_{R}}^{t \wedge \tau_{R}} b\left(X_{r}\right) \cdot \nabla u\left(X_{r}\right) \mathrm{d} r \mid \mathcal{F}_{s}\right] \\
& \leqslant \\
& \leqslant \sup _{x \in \mathbb{R}^{d}}|\nabla u(x)| \mathbb{E}\left[\left|\int_{s \wedge \tau_{R}}^{t \wedge \tau_{R}} b\left(X_{r}\right) \mathrm{d} r+\int_{s \wedge \tau_{R}}^{t \wedge \tau_{R}} \sigma\left(X_{r}\right) \mathrm{d} W_{r}\right| \mathcal{F}_{s}\right]+\lambda(t-s) \sup _{x \in \mathbb{R}^{d}}|u(x)| \\
& \quad \quad+\sup _{x \in \mathbb{R}^{d}}|\nabla u(x)| \mathbb{E}\left[\int_{s \wedge \tau_{R}}^{t \wedge \tau_{R}}\left|b\left(X_{r}\right)\right| \mathrm{d} r \mid \mathcal{F}_{s}\right]
\end{aligned}
$$

$$
\leqslant \sup _{x \in \mathbb{R}^{d}}|\nabla u(x)| C_{2} \sqrt{K_{2}}(t-s)^{\frac{1}{2}}+\lambda(t-s) \sup _{x \in \mathbb{R}^{d}}|u(x)|+2 \sup _{x \in \mathbb{R}^{d}}|\nabla u(x)| \mathbb{E}\left[\int_{s \wedge \tau_{R}}^{t \wedge \tau_{R}}\left|b\left(X_{r}\right)\right| \mathrm{d} r \mid \mathcal{F}_{s}\right]
$$

$$
\leqslant C_{2} \sqrt{K_{2}}(t-s)^{\frac{1}{2}} U_{2, q}(\lambda)\|f\|_{\tilde{L}_{q}}+\lambda(t-s) U_{1, q}(\lambda)\|f\|_{\tilde{L}_{q}}
$$

$$
\begin{equation*}
+2 U_{2, q}(\lambda)\|f\|_{\tilde{L}_{q}} \mathbb{E}\left[\int_{s \wedge \tau_{R}}^{t \wedge \tau_{R}}\left|b\left(X_{r}\right)\right| \mathrm{d} r \mid \mathcal{F}_{s}\right] \tag{4.5}
\end{equation*}
$$

Here, the constant $C_{2}>0$ comes from BDG's inequality. We apply this inequality to $f=|b|$ with $q=p$. Then, for $\lambda \geqslant \lambda_{0}(p)$,

$$
\begin{aligned}
\mathbb{E}\left[\int_{s \wedge \tau_{R}}^{t \wedge \tau_{R}}\left|b\left(X_{r}\right)\right| \mathrm{d} r \mid \mathcal{F}_{s}\right] \leqslant & C_{2} \sqrt{K_{2}}(t-s)^{\frac{1}{2}} U_{2, p}(\lambda)\|b\|_{\tilde{L}_{p}}+\lambda(t-s) U_{1, p}(\lambda)\|b\|_{\tilde{L}_{p}} \\
& +2 U_{2, p}(\lambda)\|b\|_{\tilde{L}_{p}} \mathbb{E}\left[\int_{s \wedge \tau_{R}}^{t \wedge \tau_{R}}\left|b\left(X_{r}\right)\right| \mathrm{d} r \mid \mathcal{F}_{s}\right] .
\end{aligned}
$$

If $\lambda \geqslant \lambda_{0}(p)$ is so large that $U_{2, p}(\lambda)\|b\|_{\tilde{L}_{p}}=C_{1}(p) \lambda^{-\frac{1-d / p}{2}} K_{1}^{-\frac{1+d / p}{2}}\left(\frac{K_{1}+\sqrt{K_{2}}\|\nabla \sigma\|_{\tilde{L}_{\rho}}}{K_{1}}\right)^{\frac{d}{1-d / \rho}}\|b\|_{\tilde{L}_{p}} \leqslant \frac{1}{4}$, i.e.

$$
\begin{equation*}
\lambda \geqslant\left(4 C_{1}(p) K_{1} \frac{-1-d / p}{2}\left(\frac{K_{1}+\sqrt{K_{2}}\|\nabla \sigma\|_{\tilde{L}_{\rho}}}{K_{1}}\right)^{\frac{d}{1-d / \rho}}\|b\|_{\tilde{L}_{p}}\right)^{\frac{2}{1-d / p}} \tag{4.6}
\end{equation*}
$$

then we get

$$
\mathbb{E}\left[\int_{s \wedge \tau_{R}}^{t \wedge \tau_{R}}\left|b\left(X_{r}\right)\right| \mathrm{d} r \mid \mathcal{F}_{s}\right] \leqslant \frac{C_{2}}{2} \sqrt{K_{2}}(t-s)^{\frac{1}{2}}+2 \lambda(t-s) U_{1, p}(\lambda)\|b\|_{\tilde{L}_{p}}
$$

Plugging this into (4.5), observing that, by definition, $U_{1, p}(\lambda) U_{2, q}(\lambda)=U_{1, q}(\lambda) U_{2, p}(\lambda)$, and using (4.6) yields, for $\lambda \geqslant \lambda_{0}(p) \vee \lambda_{0}(q)$ satisfying (4.6),

$$
\mathbb{E}\left[\int_{s \wedge \tau_{R}}^{t \wedge \tau_{R}} f\left(X_{r}\right) \mathrm{d} r \mid \mathcal{F}_{s}\right]
$$

$$
\begin{aligned}
& \leqslant C_{3}\left(\sqrt{K_{2}}(t-s)^{\frac{1}{2}} U_{2, q}(\lambda)+\lambda(t-s)\left(U_{1, q}(\lambda)+U_{1, p}(\lambda) U_{2, q}(\lambda)\|b\|_{\tilde{L}_{p}}\right)\|f\|_{\tilde{L}_{q}}\right. \\
& \leqslant 2 C_{3}\left(\sqrt{K_{2}}(t-s)^{\frac{1}{2}} U_{2, q}(\lambda)+\lambda(t-s) U_{1, q}(\lambda)\right)\|f\|_{\tilde{L}_{q}} .
\end{aligned}
$$

Let $\lambda=C_{4}\left(\frac{K_{2}^{2}}{K_{1}}\left(\frac{K_{1}+\sqrt{K_{2}}\|\nabla \sigma\|_{\tilde{L}_{\rho}}}{K_{1}}\right)^{\frac{2}{1-d / \rho}}+\left(4 C_{1}(p) K_{1} \frac{-1-d / p}{2}\left(\frac{K_{1}+\sqrt{K_{2}}\|\nabla \sigma\|_{\tilde{L}_{\rho}}}{K_{1}}\right)^{\frac{d}{1-d / \rho}}\|b\|_{\tilde{L}_{p}}\right)^{\frac{2}{1-d / p}}\right)$ with $C_{4}>C_{0}(p) \vee$ $C_{0}(q) \vee 1$, which implies

$$
\begin{aligned}
\sqrt{K_{2}} U_{2, q}(\lambda) & =C_{1}(q) \sqrt{K_{2}}\left(\lambda K_{1}\right)^{-\frac{1}{2}}\left(\lambda K_{1}^{-1}\right)^{\frac{d}{2 q}}\left(\frac{K_{1}+\sqrt{K_{2}}\|\nabla \sigma\|_{\tilde{L}_{\rho}}}{K_{1}}\right)^{\frac{d}{1-d / \rho}} \\
& \leqslant C_{5} K_{2}^{-\frac{1}{2}}\left(\lambda K_{1}^{-1}\right)^{\frac{d}{2 q}}\left(\frac{K_{1}+\sqrt{K_{2}}\|\nabla \sigma\|_{\tilde{L}_{\rho}}}{K_{1}}\right)^{\frac{d}{1-d / \rho}} \\
& \leqslant C_{6} K_{2}^{-\frac{1}{2}}\left(\left(\frac{K_{2}}{K_{1}}\right)^{\frac{4 d^{2}}{1-d / \rho}}+\left(\frac{\|\nabla \sigma\|_{\tilde{L}_{\rho}}^{2}}{K_{1}}\right)^{\frac{4 d^{2}}{1-d / \rho}}+\left(\frac{\|b\|_{\tilde{L}_{p}}}{K_{1}}\right)^{\frac{4 d}{1-d / \rho}}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\lambda U_{1, q}(\lambda) & =C_{1}(q)\left(\lambda K_{1}^{-1}\right)^{\frac{d}{2 q}}\left(\frac{K_{1}+\sqrt{K_{2}}\|\nabla \sigma\|_{\tilde{L}_{\rho}}}{K_{1}}\right)^{\frac{d}{1-d / \rho}} \\
& \leqslant C_{7}\left(\left(\frac{K_{2}}{K_{1}}\right)^{\frac{4 d^{2}}{1-d / \rho}}+\left(\frac{\|\nabla \sigma\|_{\tilde{L}_{\rho}}^{2}}{K_{1}}\right)^{\frac{4 d^{2}}{1-d / \rho}}+\left(\frac{\|b\|_{\tilde{L}_{\rho}}}{K_{1}}\right)^{\frac{4 d}{1-d / p}}\right) .
\end{aligned}
$$

In the above estimates we used the fact that $p>2 d$ and $q>d$. Therefore,

$$
\begin{align*}
& \mathbb{E}\left[\int_{s \wedge \tau_{R}}^{t \wedge \tau_{R}} f\left(X_{r}\right) \mathrm{d} r \mid \mathcal{F}_{s}\right] \\
& \quad \leqslant C_{\mathrm{Kry}}(q)\left(\left(\frac{K_{2}}{K_{1}}\right)^{\frac{4 d^{2}}{1-1 d \rho}}+\left(\frac{\|\nabla \sigma\|_{\tilde{L}_{\rho}}^{2}}{K_{1}}\right)^{\frac{4 d^{2}}{1-d / \rho}}+\left(\frac{\|b\|_{\tilde{L}_{p}}}{K_{1}}\right)^{\frac{4 d}{1-d / p}}\right)\left[K_{2}^{-\frac{1}{2}}(t-s)^{\frac{1}{2}}+(t-s)\right]\|f\|_{\tilde{L}_{q}} . \tag{4.7}
\end{align*}
$$

Letting $R \rightarrow \infty$ we therefore get (4.1).
The following corollary is a quantitative version of Khasminskii's lemma. The constant $C_{\mathrm{Kry}}(q)$ appearing in there is the same as in the previous lemma.

Corollary 4.2. Let Assumption 2.4 hold, let $\Gamma:=\left(\left(\frac{K_{2}}{K_{1}}\right)^{\frac{4 d^{2}}{1-d / \rho}}+\left(\frac{\|\nabla \sigma\|_{\tilde{I}_{\rho}}^{2}}{K_{1}}\right)^{\frac{4 d^{2}}{1-d / \rho}}+\left(\frac{\|b\|_{\tilde{L}_{\rho}}}{K_{1}}\right)^{\frac{4 d}{1-d / p}}\right)$. Then, for any $f \in \tilde{L}_{q}\left(\mathbb{R}^{d}\right)$ with $q \in(d, \infty]$, any $0 \leqslant S \leqslant T$, and any $0<\lambda<\infty$, the solution $\left(X_{t}\right)_{t \geqslant 0}$ of (1.1) satisfies

$$
\begin{equation*}
\mathbb{E} \exp \left(\lambda \int_{S}^{T}\left|f\left(X_{r}\right)\right| \mathrm{d} r\right) \leqslant 2 \cdot 2^{(T-S)\left(\frac{\kappa}{2} K_{2}^{-1 / 2}+\sqrt{\frac{\kappa^{2}}{4} K_{2}^{-1}+\kappa}\right)^{2}} \leqslant 2 \cdot 2^{(T-S)\left(\frac{\kappa^{2}}{K_{2}}+2 \kappa\right)}, \tag{4.8}
\end{equation*}
$$

where $\kappa:=2 C_{\mathrm{Kry}}(q) \lambda \Gamma\|f\|_{\tilde{L}_{q}}$.

Proof. The second inequality is an application of the general inequality $(A+B)^{2} \leqslant 2 A^{2}+2 B^{2}$.
Lemma 4.1 shows that there exists some positive integer $n$ such that, for $j=0, \cdots, n-1$,

$$
\begin{equation*}
\lambda \mathbb{E}\left[\left.\int_{\frac{(T-S) j}{n}}^{\frac{(T-S)(j+1)}{n}}\left|f\left(X_{r}\right)\right| \mathrm{d} r \right\rvert\, \mathcal{F}_{\frac{(T-S) j}{n}}\right] \leqslant \frac{1}{2} \tag{4.9}
\end{equation*}
$$

and the proof of [26, Lemma 3.5] shows that for any such $n$ we have

$$
\mathbb{E} \exp \left(\lambda \int_{S}^{T}\left|f\left(X_{r}\right)\right| \mathrm{d} r\right) \leqslant 2^{n}
$$

(see also [17, Lemma 3.5]). By Lemma 4.1, any $n$ such that

$$
C_{\mathrm{Kry}}(q) \lambda \Gamma\|f\|_{\tilde{L}_{q}}\left[\left(\frac{T-S}{K_{2} n}\right)^{\frac{1}{2}}+\frac{T-S}{n}\right] \leqslant \frac{1}{2}
$$

satisfies (4.9). In particular, we can take

$$
n=\left\lfloor(T-S)\left(\frac{\kappa}{2} K_{2}^{-1 / 2}+\sqrt{\frac{\kappa^{2}}{4} K_{2}^{-1}+\kappa}\right)^{2}\right\rfloor+1
$$

Here, $\lfloor x\rfloor$ is the largest integer that is smaller than or equal to $x \in \mathbb{R}$. Therefore (4.8) holds.
Remark 4.3. Note that the right hand side of our version of Krylov's estimate contains the factor $(t-s)^{1 / 2}+(t-s)$ instead of $C(T)(t-s)^{1-\frac{d}{2 q}}$ in [29, Theorem 3.4 (3.8)]), where $C(T)$ depends on the final time $T$. Further, we require the condition $q>d$ instead of $q>d / 2$ in [29, Theorem 3.4 (3.8)]). The reason for our restriction to $q>d$ is that we use (4.4) which only holds for $q>d$. Since we will later apply Krylov's estimate to $f:=\left|b^{*} \cdot \sigma^{-1}\right|^{2}$ which is in $\tilde{L}_{p / 2}$ we will have to assume $p>2 d$.

Remark 4.4. More general versions of the quantitative Khasminskii's Lemma (but with less explicit constants) can be found in [16].

## 5. Upper bounds for the dispersion of sets induced by the flow generated by the solution to SDE

Depending on the regularity of the SDE's coefficients we show upper bounds for the dispersion of sets under the flow generated by the solution in the following two cases.
5.1. Stability estimates of the SDE with weakly differentiable coefficients. Consider the equation

$$
\begin{equation*}
\mathrm{d} Y_{t}^{i}=\tilde{b}\left(Y_{t}^{i}\right) \mathrm{d} t+\tilde{\sigma}\left(Y_{t}^{i}\right) \mathrm{d} W_{t}, \quad Y_{0}^{i}=y_{i} \in \mathbb{R}^{d}, \quad i=1,2 \tag{5.1}
\end{equation*}
$$

For $\tilde{b}$ and $\tilde{\sigma}$ we assume:
Assumption 5.1. For $p, \rho \in(2 d, \infty)$,

1. $\|\tilde{b}\|_{\tilde{H}^{1, p}}+\|\tilde{b}\|_{\infty}<\infty$;
2. $\|\nabla \tilde{\sigma}\|_{\tilde{L}_{\rho}}<\infty$;
3. for $\tilde{a}:=\tilde{\sigma} \tilde{\sigma}^{*}$, there exist some $\tilde{K}_{1}, \tilde{K}_{2}>0$ such that for all $x \in \mathbb{R}^{d}$,

$$
\tilde{K}_{1}|\zeta|^{2} \leqslant\langle\tilde{a}(x) \zeta, \zeta\rangle \leqslant \tilde{K}_{2}|\zeta|^{2}, \quad \forall \zeta \in \mathbb{R}^{d} .
$$

Theorem 5.2. Let Assumption 5.1 hold. There exist constants $\kappa_{0}, \kappa_{1}>0$ depending only on $p, d, \rho$, such that for any $r \geqslant 1, T \geqslant 0, y_{i} \in \mathbb{R}^{d}, i=1,2$, the solutions $Y^{i}:=Y^{i}\left(y_{i}\right)$ to equations (5.1) satisfy

$$
\begin{equation*}
\mathbb{E}\left[\sup _{t \in[0, T]}\left|Y_{t}^{1}\left(y_{1}\right)-Y_{t}^{2}\left(y_{2}\right)\right|^{r}\right] \leqslant \kappa_{0}\left|y_{1}-y_{2}\right|^{r} \exp \left(\kappa_{1} T \varrho\right), \tag{5.2}
\end{equation*}
$$

where

$$
\begin{aligned}
& \varrho:=r^{4}\left[\|\tilde{b}\|_{\infty}+\|\tilde{\sigma}\|_{\infty}^{2}+\left(\tilde{\Gamma}\|\nabla \tilde{b}\|_{\tilde{L}_{p}}\right)^{2} \tilde{K}_{2}^{-1}+\tilde{\Gamma}\|\nabla \tilde{b}\|_{\tilde{L}_{p}}+\tilde{\Gamma}^{2}\|\nabla \tilde{\sigma}\|_{\tilde{L}_{\rho}}^{4} \tilde{K}_{2}^{-1}+\tilde{\Gamma}\|\nabla \tilde{\sigma}\|_{\tilde{L}_{\rho}}^{2}\right] . \\
& \text { and } \tilde{\Gamma}:=\left(\left(\frac{\tilde{K}_{2}}{\tilde{K}_{1}}\right)^{\frac{4 d^{2}}{1-d / \rho}}+\left(\frac{\|\nabla \tilde{\sigma}\|_{\tilde{L}_{\rho}}^{2}}{\tilde{K}_{1}}\right)^{\frac{4 d^{2}}{1-d / \rho}}+\left(\frac{\|\tilde{b}\|_{\tilde{L}_{p}}}{\tilde{K}_{1}}\right)^{\frac{4 d}{1-d / p}}\right) .
\end{aligned}
$$

Proof. Again, all constants $C_{1}, \ldots$ depend on $p, \rho, d$ only. By Itô's formula we get for any $r \geqslant 1$,

$$
\begin{equation*}
\left|Y_{t}^{1}-Y_{t}^{2}\right|^{2 r}=\left|y_{1}-y_{2}\right|^{2 r}+\int_{0}^{t}\left|Y_{s}^{1}-Y_{s}^{2}\right|^{2 r} \mathrm{~d} A_{s}+M_{t} \leqslant\left|y_{1}-y_{2}\right|^{2 r}+\int_{0}^{t}\left|Y_{s}^{1}-Y_{s}^{2}\right|^{2 r} \mathrm{~d} \bar{A}_{s}+M_{t} \tag{5.4}
\end{equation*}
$$

where $M_{t}$ is an $\left(\mathcal{F}_{t}\right)$-local martingale defined as

$$
M_{t}:=\int_{0}^{t} 2 r\left|Y_{s}^{1}-Y_{s}^{2}\right|^{2 r-2}\left[\tilde{\sigma}\left(Y_{s}^{1}\right)-\tilde{\sigma}\left(Y_{s}^{2}\right)\right]^{*}\left(Y_{s}^{1}-Y_{s}^{2}\right) \mathrm{d} W_{s}
$$

and

$$
\begin{aligned}
A_{t}: & \int_{0}^{t} \frac{2 r\left\langle Y_{s}^{1}-Y_{s}^{2}, \tilde{b}\left(Y_{s}^{1}\right)-\tilde{b}\left(Y_{s}^{2}\right)\right\rangle+r\left\|\tilde{\sigma}\left(Y_{s}^{1}\right)-\tilde{\sigma}\left(Y_{s}^{2}\right)\right\|^{2}}{\left|Y_{s}^{1}-Y_{s}^{2}\right|^{2}} \mathrm{~d} s \\
& +\int_{0}^{t} \frac{2 r(r-1)\left|\left[\tilde{\sigma}\left(Y_{s}^{1}\right)-\tilde{\sigma}\left(Y_{s}^{2}\right)\right]^{*}\left(Y_{s}^{1}-Y_{s}^{2}\right)\right|^{2}}{\left|Y_{s}^{1}-Y_{s}^{2}\right|^{4}} \mathrm{~d} s
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{A}_{t}:= & \int_{0}^{t} \frac{2 r\left|\left\langle Y_{s}^{1}-Y_{s}^{2}, \tilde{b}\left(Y_{s}^{1}\right)-\tilde{b}\left(Y_{s}^{2}\right)\right\rangle\right|+r\left\|\tilde{\sigma}\left(Y_{s}^{1}\right)-\tilde{\sigma}\left(Y_{s}^{2}\right)\right\|^{2}}{\left|Y_{s}^{1}-Y_{s}^{2}\right|^{2}} \mathrm{~d} s \\
& +\int_{0}^{t} \frac{2 r(r-1)\left|\left[\tilde{\sigma}\left(Y_{s}^{1}\right)-\tilde{\sigma}\left(Y_{s}^{2}\right)\right]^{*}\left(Y_{s}^{1}-Y_{s}^{2}\right)\right|^{2}}{\left|Y_{s}^{1}-Y_{s}^{2}\right|^{4}} \mathrm{~d} s .
\end{aligned}
$$

There exists $C_{1}>0$ such that for each $x, y \in \mathbb{R}^{d}$

$$
\begin{aligned}
&|\tilde{\sigma}(x)-\tilde{\sigma}(y)| \leqslant C_{1}|x-y|\left(\mathcal{M}|\nabla \tilde{\sigma}|(x)+\mathcal{M}|\nabla \tilde{\sigma}|(y)+\|\tilde{\sigma}\|_{\infty}\right), \\
&|\tilde{b}(x)-\tilde{b}(y)| \leqslant C_{1}|x-y|\left(\mathcal{M}|\nabla \tilde{b}|(x)+\mathcal{M}|\nabla \tilde{b}|(y)+\|\tilde{b}\|_{\infty}\right),
\end{aligned}
$$

where $\mathcal{M} f$ is defined as $\mathcal{M} f(x):=\sup _{r \in(0,1)} \frac{1}{\left|B_{r}\right|} \int_{B_{r}} f(x+y) \mathrm{d} y$, which satisfies

$$
\begin{equation*}
\|\mathcal{M} f\|_{\tilde{L}_{\gamma}} \leqslant C(\gamma, d)\|f\|_{\tilde{L}_{\gamma}} \quad \text { for } \quad \gamma>1, \tag{5.5}
\end{equation*}
$$

see [24, Lemma 2.1].

Using these estimates and the Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
\bar{A}_{t} \leqslant & C_{2}\left(r\left(\int_{0}^{t} \mathcal{M}|\nabla \tilde{b}|\left(Y_{s}^{1}\right)+\mathcal{M}|\nabla \tilde{b}|\left(Y_{s}^{2}\right) \mathrm{d} s+t\|\tilde{b}\|_{\infty}\right)\right. \\
& +r\left(\int_{0}^{t} \mathcal{M}|\nabla \tilde{\sigma}|^{2}\left(Y_{s}^{1}\right)+\mathcal{M}|\nabla \tilde{\sigma}|^{2}\left(Y_{s}^{2}\right) \mathrm{d} s+t\|\tilde{\sigma}\|_{\infty}^{2}\right) \\
& \left.+2 r(r-1)\left(\int_{0}^{t} \mathcal{M}|\nabla \tilde{\sigma}|^{2}\left(Y_{s}^{1}\right)+\mathcal{M}|\nabla \tilde{\sigma}|^{2}\left(Y_{s}^{2}\right) \mathrm{d} s+t\|\tilde{\sigma}\|_{\infty}^{2}\right)\right) \\
= & t C_{2}\left(r\|\tilde{b}\|_{\infty}+\left(2 r^{2}-r\right)\|\tilde{\sigma}\|_{\infty}^{2}\right) \\
& +C_{2} \sum_{i=1}^{2} \int_{0}^{t} r \mathcal{M}|\nabla \tilde{b}|\left(Y_{s}^{i}\right)+\left(2 r^{2}-r\right) \mathcal{M}|\nabla \tilde{\sigma}|^{2}\left(Y_{s}^{i}\right) \mathrm{d} s .
\end{aligned}
$$

Applying Corollary 4.2 and (5.5) we get, for $\alpha>0$ and $t \geqslant 0$,

$$
\begin{equation*}
\mathbb{E}\left[\exp \left(\alpha \bar{A}_{t}\right)\right] \leqslant 16 \exp \left[C_{3} \varrho_{\alpha} t\right] \tag{5.6}
\end{equation*}
$$

where

$$
\begin{align*}
\varrho_{\alpha}= & \alpha\left(r\|\tilde{b}\|_{\infty}+r^{2}\|\tilde{\sigma}\|_{\infty}^{2}\right)+\left(r \alpha \tilde{\Gamma}\|\nabla \tilde{b}\|_{\tilde{L}_{p}}\right)^{2} \tilde{K}_{2}^{-1}+r \alpha \tilde{\Gamma}\|\nabla \tilde{b}\|_{\tilde{L}_{p}} \\
& +\left(\alpha r^{2} \tilde{\Gamma}\|\nabla \tilde{\sigma}\|_{\tilde{L}_{\rho}}^{2}\right)^{2} \tilde{K}_{2}^{-1}+\left(\alpha r^{2} \tilde{\Gamma}\|\nabla \tilde{\sigma}\|_{\tilde{L}_{\rho}}^{2}\right) . \tag{5.7}
\end{align*}
$$

Choosing $\alpha=1$ and applying stochastic Grönwall's inequality (see [22, Theorem 4] or [26, Lemma 3.7]) to (5.4) we get

$$
\mathbb{E}\left[\sup _{t \in[0, T]}\left|Y_{t}^{1}-Y_{t}^{2}\right|^{r}\right] \leqslant C_{4}\left|y_{1}-y_{2}\right|^{r}\left(\mathbb{E}\left[\exp \left(\bar{A}_{T}\right)\right]\right)^{1 / 2} \leqslant 4 C_{4}\left|y_{1}-y_{2}\right|^{r} \exp \left(\frac{1}{2} C_{3} \varrho_{1} T\right) .
$$

Observing that $\varrho_{1}$ is at most equal to $\varrho_{0}$ defined in (5.3) and defining $\kappa_{0}=4 C_{4}$ and $\kappa_{1}=\frac{1}{2} C_{3}$, (5.2) follows.

Remark 5.3. If $\tilde{\sigma}$ is even globally Lipschitz continuous with Lipschitz constant $L$, then there is no need to use Khasminskii's Lemma for the integral over $\tilde{\sigma}$ and we easily get (5.2) with

$$
\varrho=r^{2}\left[\|\tilde{b}\|_{\infty}+\left(\tilde{\Gamma}\|\nabla \tilde{b}\|_{\tilde{L}_{p}}\right)^{2} \tilde{K}_{2}^{-1}+\tilde{\Gamma}\|\nabla \tilde{b}\|_{\tilde{L}_{p}}+L^{2}\right]
$$

and

$$
\tilde{\Gamma}:=\left(\left(\frac{\tilde{K}_{2}}{\tilde{K}_{1}}\right)^{4 d^{2}}+\left(\frac{L}{\tilde{K}_{1}}\right)^{4 d^{2}}+\left(\frac{\|\tilde{b}\|_{\tilde{L}_{p}}}{\tilde{K}_{1}}\right)^{\frac{4 d}{1-d / p}}\right)
$$

### 5.2. Linear expansion rate of the SDE with singular coefficients.

Theorem 5.4. Let Assumption 2.4 hold. Let $\left(\psi_{t}\right)_{t \geqslant 0}$ denote the flow generated by the solution to (1.1). Let $\mathcal{X}$ be a compact subset of $\mathbb{R}^{d}$. Then there exists a positive constant $C_{p, \rho, d}$ depending on $p, d, \rho$ only such that

$$
\begin{equation*}
\limsup _{T \rightarrow \infty}\left(\sup _{t \in[0, T]} \sup _{x \in \mathcal{X}} \frac{1}{T}\left|\psi_{t}(x)\right|\right) \leqslant \kappa^{*} \quad \text { a.s., } \tag{5.8}
\end{equation*}
$$

where

$$
\begin{aligned}
\kappa^{*}=C_{p, \rho, d} & \left(K_{2}+\|b\|_{\tilde{L}_{p}}^{2} \frac{K_{2}}{K_{1}^{2}}+\|\nabla \sigma\|_{\tilde{L}_{\rho}}^{2}\right) \\
& {\left[\left(\frac{K_{2}}{K_{1}}\right)^{\frac{1-d /(\rho \wedge d)(1-d / \rho)}{(1-d)}}+\left(\frac{\|b\|_{\tilde{L}_{p}}}{K_{1}}\right)^{\frac{32 d^{2}}{1-d(p \wedge \rho)}}+\left(\frac{\|\nabla \sigma\|_{L_{\rho}}^{2}}{K_{1}}\right)^{\frac{32 d^{3}}{(1-d /(\rho \wedge \rho)(1-d \rho)}}\right] . }
\end{aligned}
$$

Proof. The idea is to apply Theorem 3.3. All constants $C_{1}^{*}, \ldots$ depend on $p, \rho, d$ only.
Step 1. We check the assumptions of Lemma 3.1.
Since, by (4.2), the map $x \mapsto a(x)=\sigma(x) \sigma^{*}(x)$ is $1-d / \rho$-Hölder continuous and $\omega_{1-d / \rho}(a) \leqslant$ $C_{\rho, d} \sqrt{K_{2}}\|\nabla \sigma\|_{\tilde{L}_{\rho}}$, Theorem A. 3 and Corollary A. 4 show that there exists a constant $C_{1}^{*}$ such that for

$$
\lambda:=C_{1}^{*} K_{1}\left(\frac{K_{2}^{2}}{K_{1}^{2}}\left(\frac{K_{1}+\sqrt{K_{2}}\|\nabla \sigma\|_{\tilde{L}_{\rho}}}{K_{1}}\right)^{\frac{2}{1-d / \rho}}+\left(\frac{K_{1}+\sqrt{K_{2}}\|\nabla \sigma\|_{\tilde{L}_{\rho}}}{K_{1}}\right)^{\frac{2 d}{(1-d / \rho)(1-d / p)}}\left(\frac{\|b\|_{\tilde{L}_{p}}}{K_{1}}\right)^{\frac{2}{1-d / p}}\right),
$$

the equation

$$
\frac{1}{2} a_{i j} \partial_{i j}^{2} u^{(l)}+b \cdot \nabla u^{(l)}-\lambda u^{(l)}=-b^{(l)}, \quad l=1, \cdots, d
$$

has a unique solution $U:=\left(u^{(l)}\right)_{1 \leqslant l \leqslant d}, u^{(l)} \in \tilde{H}^{2, p}$ and

$$
\begin{equation*}
\Phi(x):=x+U(x) \quad \text { for } x \in \mathbb{R}^{d} \tag{5.9}
\end{equation*}
$$

is a $C^{1}$-diffeomorphism on $\mathbb{R}^{d}$ (see also [29]). Let $\Psi:=(\Phi)^{-1}$. Then, by the generalized Itô's formula ([24]), $Y_{t}:=\Phi\left(\psi_{t}(x)\right)$ satisfies the following equation

$$
\begin{equation*}
\mathrm{d} Y_{t}=\tilde{b}\left(Y_{t}\right) \mathrm{d} t+\tilde{\sigma}\left(Y_{t}\right) \mathrm{d} W_{t}, \quad Y_{0}=y \in \mathbb{R}^{d} \tag{5.10}
\end{equation*}
$$

with

$$
\tilde{b}(x):=\lambda U(\Psi(x)), \quad \tilde{\sigma}(x):=[\nabla \Phi \cdot \sigma] \circ(\Psi(x)), \quad y=\Phi(x) .
$$

From [24, (4.5)] we know that

$$
\begin{equation*}
\|U\|_{\infty}<\frac{1}{2}, \quad\|\nabla U\|_{\infty}<\frac{1}{2} . \tag{5.11}
\end{equation*}
$$

Furthermore, by (A.17) and (A.4) we have

$$
\begin{array}{r}
\|\nabla U\|_{\tilde{L}_{p}} \leqslant \frac{1}{2}\left(\frac{K_{1}}{\lambda}\right)^{\frac{d}{2 p}} \leqslant \frac{1}{2}, \quad\|U\|_{\tilde{L}_{p}} \leqslant \frac{1}{2}\left(\frac{K_{1}}{\lambda}\right)^{\frac{1-d / p}{2}} \leqslant \frac{1}{2}, \\
\left\|\nabla^{2} U\right\|_{\tilde{L}_{p}} \leqslant C_{2}^{*} \frac{1}{K_{1}}\left(1+\frac{\sqrt{K_{2}}\|\nabla \sigma\|_{\tilde{L}_{\rho}}}{K_{1}}\right)^{\frac{d}{(1-d / \rho)}}\|b\|_{\tilde{L}_{p}} . \tag{5.12}
\end{array}
$$

Hence, by (5.11) (see also e.g. [24, p. 15]),

$$
\frac{1}{2} \leqslant|\nabla \Phi|=|\mathbb{I}+\nabla U| \leqslant \frac{3}{2}, \quad|\nabla \Psi| \leqslant 2
$$

which implies that for all $x \in \mathbb{R}^{d}$,

$$
\begin{equation*}
\frac{1}{4} K_{1}|\xi|^{2} \leqslant\left\langle\tilde{\sigma} \tilde{\sigma}^{*}(x) \xi, \xi\right\rangle \leqslant \frac{9}{4} K_{2}|\xi|^{2}, \quad \forall \xi \in \mathbb{R}^{d} \tag{5.13}
\end{equation*}
$$

and

$$
\begin{align*}
\|\tilde{b}\|_{\infty} & \leqslant \lambda\|U\|_{\infty} \leqslant \frac{1}{2} \lambda, \quad\|\tilde{b}\|_{\tilde{L}_{p}} \leqslant \lambda\|U\|_{\tilde{L}_{p}} \leqslant \frac{1}{2} \lambda, \\
\|\nabla \tilde{b}\|_{\tilde{L}_{p}} & \leqslant \lambda\|\operatorname{det}(\nabla \Phi)\|_{\infty}^{\frac{1}{p}}\|\nabla U\|_{\tilde{L}_{p}} \leqslant \lambda . \tag{5.14}
\end{align*}
$$

Moreover for $p^{\prime}=\min (p, \rho)$ we have by embedding

$$
\begin{align*}
\|\nabla \tilde{\sigma}\|_{\tilde{L}_{p^{\prime}}} & =\left\|\left(\left(\nabla^{2} \Phi \cdot \sigma+\nabla \Phi \nabla \sigma\right) \nabla \Psi\right) \circ \Psi\right\|_{\tilde{L}_{p^{\prime}}} \\
& \leqslant\left\|\left(\left(\nabla^{2} \Phi \cdot \sigma\right) \nabla \Psi\right) \circ \Psi\right\|_{\tilde{L}_{p}}+\|((\nabla \Phi \nabla \sigma) \nabla \Psi) \circ \Psi\|_{\tilde{L}_{\rho}} \\
& \leqslant 2\|\operatorname{det}(\nabla \Phi)\|_{\infty}^{\frac{1}{p \wedge \rho}}\left(\sqrt{K_{2}}\left\|\nabla^{2} \Phi\right\|_{\tilde{L}_{p}}+\|\nabla \Phi \cdot \nabla \Psi\|_{\infty}\|\nabla \sigma\|_{\tilde{L}_{\rho}}\right) \\
& \leqslant 9 C_{2}^{*} \frac{\sqrt{K_{2}}}{K_{1}}\left(1+\frac{\sqrt{K_{2}}\|\nabla \sigma\|_{\tilde{L}_{\rho}}}{K_{1}}\right) \frac{d}{(1-d / \rho)}\|b\|_{\tilde{L}_{p}}+9\|\nabla \sigma\|_{\tilde{L}_{\rho}} . \tag{5.15}
\end{align*}
$$

If $\left(\phi_{t}(x)\right)_{t \geqslant 0}$ is the flow generated by the solution to (5.10), then by definition of $\Phi\left(\psi_{t}(x)\right)$ from (5.9) and the fact that $U$ is uniformly bounded from (5.11), we get that

$$
\limsup _{T \rightarrow \infty}\left(\sup _{t \in[0, T]} \sup _{x \in \mathcal{X}} \frac{1}{T}\left|\psi_{t}(x)\right|\right)=\lim \sup _{T \rightarrow \infty}\left(\sup _{t \in[0, T]} \sup _{x \in \mathcal{X}} \frac{1}{T}\left|\phi_{t}(x)\right|\right) .
$$

Using the estimates (5.13), (5.14) and (5.15) we will establish (5.2) for $Y$. Indeed, let $\tilde{K}_{1}:=\frac{1}{4} K_{1}$ and $\tilde{K}_{2}=\frac{9}{4} K_{2}$ in Assumption 5.1. Then we define

$$
\begin{align*}
\tilde{\Gamma} & :=\left(\left(\frac{\tilde{K}_{2}}{\tilde{K}_{1}}\right)^{\frac{4 d^{2}}{1-d / p^{\prime}}}+\left(\frac{\|\tilde{\nabla} \sigma\|_{\tilde{L}_{p^{\prime}}}^{2}}{\tilde{K}_{1}}\right)^{\frac{4 d^{2}}{1-d / p^{\prime}}}+\left(\frac{\|\tilde{b}\|_{\tilde{L}_{p}}}{\tilde{K}_{1}}\right)^{\frac{4 d}{1-d / p}}\right) \\
& \leqslant C_{p, \rho, d}\left(\left(\frac{K_{2}}{K_{1}}\right)^{\frac{4 d^{2}}{1-d /(p \wedge \rho)}}+\left(\frac{K_{2}}{K_{1}} \frac{\|b\|_{\tilde{L}_{p}}^{2}}{K_{1}^{2}}\left(1+\frac{\sqrt{K_{2}}\|\nabla \sigma\|_{\tilde{L}_{\rho}}}{K_{1}}\right)^{\frac{2 d}{1-d / \rho}}\right)^{\frac{4 d^{2}}{1-d /(p \wedge \rho)}}+\left(\frac{\lambda}{K_{1}}\right)^{\frac{4 d}{p-d}}\right) \\
& \leqslant C_{p, \rho, d}\left(\left(\frac{K_{2}}{K_{1}}\right)^{\frac{8 d^{3}}{(1-d /(p \wedge \rho)(1-d / \rho)}}+\left(\frac{\|b\|_{\tilde{L}_{p}}}{K_{1}}\right)^{\frac{16 d^{2}}{1-d /(p \wedge \rho)}}+\left(\frac{\|\nabla \sigma\|^{2}}{K_{1}}\right)^{\frac{16 d^{3}}{(1-d /(p \wedge \rho)(1-d / \rho)}}\right) . \tag{5.16}
\end{align*}
$$

Using Theorem 5.2 and the fact that $|\nabla \Psi| \leqslant 2$ together with (5.13), (5.14) and (5.15), for the flows correspondingly $\psi_{t}^{1}\left(x_{1}\right), \psi_{t}^{2}\left(x_{2}\right)$ generated by the solutions $X_{t}^{1}\left(x_{1}\right), X_{t}^{1}\left(x_{2}\right)$ to (1.1) we get

$$
\begin{align*}
\mathbb{E}\left[\sup _{t \in[0, T]}\left|\psi_{t}^{1}\left(x_{1}\right)-\psi_{t}^{2}\left(x_{2}\right)\right|^{r}\right] & =\mathbb{E}\left[\sup _{t \in[0, T]}\left|\Psi\left(Y_{t}^{1}\left(y_{1}\right)\right)-\Psi\left(Y_{t}^{2}\left(y_{2}\right)\right)\right|^{r}\right] \\
& \leqslant 2^{r} \mathbb{E}\left[\sup _{t \in[0, T]}\left|Y_{t}^{1}\left(y_{1}\right)-Y_{t}^{2}\left(y_{2}\right)\right|^{r}\right] \leqslant 2^{r} C_{4}^{*}\left|y_{1}-y_{2}\right|^{r} \exp \left(C_{3}^{*} T \varrho\right) \tag{5.17}
\end{align*}
$$

with

$$
\begin{equation*}
\varrho:=r^{4}\left[\lambda+K_{2}+\tilde{\Gamma} \lambda+(\tilde{\Gamma} \lambda)^{2} K_{2}^{-1}+\tilde{\Gamma}^{2}\left(\frac{K_{2}}{K_{1}^{2}}\|b\|_{\tilde{L}_{p}}^{2}+\|\nabla \sigma\|_{\tilde{L}_{\rho}}^{2}\right)^{2} K_{2}^{-1}+\tilde{\Gamma}\left(\frac{K_{2}}{K_{1}^{2}}\|b\|_{\tilde{L}_{p}}^{2}+\|\nabla \sigma\|_{\tilde{L}_{\rho}}^{2}\right)\right] \tag{5.18}
\end{equation*}
$$

Step 2. Verification of estimate (3.5) in Theorem 3.3.
Let

$$
\rho_{t}:=\exp \left(\int_{0}^{t} b^{*}\left(\sigma^{-1}\right)^{*}\left(\varphi_{r}(x)\right) \mathrm{d} W_{r}-\frac{1}{2} \int_{0}^{t} b^{*}\left(\sigma \sigma^{*}\right)^{-1} b\left(\varphi_{r}(x)\right) \mathrm{d} r\right),
$$

where $\varphi_{t}(x)$ is the flow generated by the solution to

$$
\mathrm{d} \varphi_{t}=\sigma\left(\varphi_{t}\right) \mathrm{d} W_{t}, \quad \varphi_{0}(x)=x \in \mathbb{R}^{d}
$$

It follows from (4.8) that, for any $\beta>0$,

$$
\begin{equation*}
\mathbb{E} \exp \left(\beta \int_{0}^{T} b^{*}\left(\sigma \sigma^{*}\right)^{-1} b\left(\varphi_{r}(x)\right) \mathrm{d} r\right) \leqslant 2 \exp \left(T C_{5}^{*}\left(\left(K_{1}^{2} K_{2}\right)^{-1}\left(\Gamma^{\prime} \beta\right)^{2}\|b\|_{\tilde{L}_{p}}^{4}+\Gamma^{\prime} \beta K_{1}^{-1}\|b\|_{\tilde{L}_{p}}^{2}\right)\right) \tag{5.19}
\end{equation*}
$$

where

$$
\begin{equation*}
\Gamma^{\prime}=\left(\frac{K_{2}}{K_{1}}\right)^{\frac{4 d^{2}}{1-d / \rho}}+\left(\frac{\|\nabla \sigma\|_{\tilde{L}_{\rho}}^{2}}{K_{1}}\right)^{\frac{4 d^{2}}{1-d / \rho}} . \tag{5.20}
\end{equation*}
$$

Therefore $\left(\rho_{t}\right)_{t \geqslant 0}$ is a martingale. Let $\mathbb{P}^{\rho}:=\rho_{T} \mathbb{P}$. By Girsanov's theorem and Hölder's inequality,

$$
\begin{aligned}
\mathbb{P}\left(\sup _{0 \leqslant t \leqslant T}\left|\psi_{t}(x)-x\right| \geqslant k T\right) & =\mathbb{P}^{\rho}\left(\sup _{0 \leqslant t \leqslant T}\left|\varphi_{t}(x)-x\right| \geqslant k T\right) \\
& =\mathbb{E}\left[\rho_{T} \mathbb{I}_{\left\{\sup _{0 \leqslant t \leqslant T}\left|\varphi_{t}(x)-x\right| \geqslant k T\right\}}\right] \\
& \leqslant\left[\mathbb{E} \rho_{T}^{2}\right]^{\frac{1}{2}} \mathbb{P}\left[\sup _{0 \leqslant t \leqslant T}\left|\varphi_{t}(x)-x\right| \geqslant k T\right]^{\frac{1}{2}} .
\end{aligned}
$$

Applying Markov's inequality we obtain, for each $x \in \mathbb{R}^{d}$ and $\zeta \geqslant 0$,

$$
\begin{equation*}
\mathbb{P}\left(\sup _{0 \leqslant t \leqslant T}\left|\varphi_{t}(x)-x\right| \geqslant k T\right)^{1 / 2} \leqslant e^{-\frac{1}{2} \zeta k T}\left[\mathbb{E} \exp \left(\zeta \sup _{0 \leqslant t \leqslant T}\left|\int_{0}^{t} \sigma\left(\varphi_{r}(x)\right) \mathrm{d} W_{r}\right|\right)\right]^{\frac{1}{2}} \tag{5.21}
\end{equation*}
$$

(5.19) shows

$$
\begin{aligned}
{\left[\mathbb{E} \rho_{T}^{2}\right]^{1 / 2}=} & {\left[\mathbb { E } \operatorname { e x p } \left(2 \int_{0}^{T} b^{*}\left(\sigma^{-1}\right)^{*}\left(\varphi_{r}(x)\right) \mathrm{d} W_{r}-2 \int_{0}^{T} b^{*}\left(\sigma \sigma^{*}\right)^{-1} b\left(\varphi_{r}(x)\right) \mathrm{d} r\right.\right.} \\
& \left.\left.+\int_{0}^{T} b^{*}\left(\sigma \sigma^{*}\right)^{-1} b\left(\varphi_{r}(x)\right) \mathrm{d} r\right)\right]^{1 / 2} \\
\leqslant & \left(\mathbb{E}\left[\exp \left(2 \int_{0}^{T} b^{*}\left(\sigma^{-1}\right)^{*}\left(\varphi_{r}(x)\right) \mathrm{d} W_{r}-2 \int_{0}^{T} b^{*}\left(\sigma \sigma^{*}\right)^{-1} b\left(\varphi_{r}(x)\right) \mathrm{d} r\right]^{2}\right)^{1 / 4}\right. \\
& {\left[\mathbb{E} \exp \left(\int_{0}^{t} 2 b^{*}\left(\sigma \sigma^{*}\right)^{-1} b\left(\varphi_{r}(x)\right) \mathrm{d} r\right)\right]^{1 / 4} }
\end{aligned}
$$

$$
\begin{aligned}
& \leqslant\left[\mathbb{E} \exp \left(2 \int_{0}^{T} b^{*}\left(\sigma \sigma^{*}\right)^{-1} b\left(\varphi_{r}(x)\right) \mathrm{d} r\right)\right]^{1 / 4} \\
& \leqslant 2 \exp \left(C_{5}^{*} T\left(\left(K_{1}^{2} K_{2}\right)^{-1} \Gamma^{\prime 2}\|b\|_{\tilde{L}_{p}}^{4}+K_{1}^{-1} \Gamma^{\prime}\|b\|_{\tilde{L}_{p}}^{2}\right)\right)=: 2 \exp \left(T \kappa_{1}\right)
\end{aligned}
$$

and by time change $\int_{0}^{t} \sigma\left(\varphi_{r}(r)\right) \mathrm{d} W_{r}=W_{\int_{0}^{t}\left|\sigma\left(\varphi_{r}(x)\right)\right|^{2} \mathrm{~d} r}$, we also have

$$
\left[\mathbb{E} \exp \left(2 \zeta \sup _{0 \leqslant t \leqslant T}\left|\int_{0}^{t} \sigma\left(\varphi_{r}(x)\right) \mathrm{d} W_{r}\right|\right)\right]^{1 / 2} \leqslant \sqrt{2} \exp \left(C_{d} \zeta^{2}\|\sigma\|_{\infty}^{2} T\right)=: \sqrt{2} \exp \left(T \zeta^{2} \kappa_{2}\right)
$$

Inserting these estimate into (5.21) and optimizing over $\zeta \geqslant 0$ yields, for any $k>0$,

$$
\begin{align*}
\mathbb{P}\left(\sup _{0 \leqslant t \leqslant T}\left|\psi_{t}(x)-x\right| \geqslant k T\right) & \leqslant 2 \exp \left(C_{6}^{*} T\left(\kappa_{1}+\kappa_{2} \zeta^{2}-\zeta k\right)\right) \\
& \leqslant 2 \exp \left(C_{7}^{*} T\left(-\frac{1}{4 \kappa_{2}} k^{2}+\kappa_{1}\right)\right) . \tag{5.22}
\end{align*}
$$

With estimates (5.17) and (5.22) at hand we are ready to apply Theorem 3.3 by taking

$$
\begin{align*}
& c_{1}:=\lambda+K_{2}+\tilde{\Gamma} \lambda+(\tilde{\Gamma} \lambda)^{2} K_{2}^{-1}+\tilde{\Gamma}^{2}\left(\frac{K_{2}}{K_{1}^{2}}\|b\|_{\tilde{L}_{p}}^{2}+\|\nabla \sigma\|_{\tilde{L}_{\rho}}^{2}\right)^{2} K_{2}^{-1}+\tilde{\Gamma}\left(\frac{K_{2}}{K_{1}^{2}}\|b\|_{\tilde{L}_{p}}^{2}+\|\nabla \sigma\|_{\tilde{L}_{\rho}}^{2}\right) \\
& c_{2}:=\frac{1}{4\|\sigma\|_{\infty}^{2}}, \quad c_{3}:=C_{7}^{*}\left(K_{1}^{2} K_{2}\right)^{-1} \Gamma^{\prime 2}\|b\|_{\tilde{L}_{p}}^{4}+K_{1}^{-1} \Gamma^{\prime}\|b\|_{\tilde{L}_{p}}^{2}, \quad \alpha:=3, \tag{5.23}
\end{align*}
$$

with $\tilde{\Gamma}$ from (5.16) and $\Gamma^{\prime}$ from (5.20). Note that we can take $\Delta=d-1$ by Remark 3.4. The linear expansion rate $\kappa$ can now be estimated as follows (no matter which of the two cases in the definition of $\kappa$ in Theorem 3.3 applies):

$$
\begin{aligned}
& \kappa \leqslant C_{\alpha, d}\left(\frac{c_{1}+c_{3}}{c_{2}}\right)^{1 / 2} \\
& \leqslant C_{p, \rho, d}\|\sigma\|_{\infty}\left(\sqrt{\lambda}+\sqrt{K_{2}}+\sqrt{\tilde{\Gamma} \lambda}+\tilde{\Gamma} \lambda K_{2}^{-1 / 2}+\tilde{\Gamma}\left(K_{2} K_{1}^{-2}\|b\|_{\tilde{L}_{p}}^{2}+\|\nabla \sigma\|_{\tilde{L}_{\rho}}^{2}\right) K_{2}^{-1 / 2}\right. \\
& \left.+\sqrt{\tilde{\Gamma}}\left(\sqrt{K_{2}} K_{1}^{-1}\|b\|_{\tilde{L}_{p}}+\|\nabla \sigma\|_{\tilde{L}_{\rho}}\right)+\left(K_{1}^{2} K_{2}\right)^{-1 / 2} \Gamma^{\prime}\|b\|_{\tilde{L}_{p}}^{2}+\left(K_{1}^{-1} \Gamma^{\prime}\right)^{1 / 2}\|b\|_{\tilde{L}_{p}}\right) \\
& \leqslant C_{p, \rho, d} \sqrt{K_{2}}\left(\sqrt{K_{1}}+\sqrt{K_{2}}+\frac{K_{1}}{\sqrt{K_{2}}}+\|b\|_{\tilde{L}_{p}}^{2}\left(\frac{\sqrt{K_{2}}}{K_{1}^{2}}+\frac{1}{K_{1} \sqrt{K_{2}}}\right)+\frac{\sqrt{K_{2}}+\sqrt{K_{1}}}{K_{1}}\|b\|_{\tilde{L}_{p}}+\|\nabla \sigma\|_{\tilde{L}_{\rho}}\right. \\
& \left.+\frac{\|\nabla \sigma\|_{\tilde{L}_{\rho}}^{2}}{\sqrt{K_{2}}}\right)\left[\left(\frac{K_{2}}{K_{1}}\right)^{\frac{16 d^{3}}{(1-d /(p \wedge))(1-d / \rho)}}+\left(\frac{\|b\|_{\tilde{L}_{p}}}{K_{1}}\right)^{\frac{32 d^{2}}{1-d /(p \wedge \rho)}}+\left(\frac{\|b\|_{\tilde{L}_{p}}}{K_{1}}\right)^{\frac{8}{1-d / p}}\right. \\
& \left.+\left(\frac{\|\nabla \sigma\|_{\tilde{L}_{\rho}}^{2}}{K_{1}}\right)^{\frac{32 d^{3}}{(1-d /(\rho p) \rho)(1-d / \rho)}}+\left(\frac{\|\nabla \sigma\|_{\tilde{L}_{\rho}}^{2}}{K_{1}}\right)^{\frac{8 d}{(1-d / p)(1-d / \rho)}}+\left(\frac{\|\nabla \sigma\|_{\tilde{L}_{\rho}}^{2}}{K_{1}}\right)^{\frac{8}{1-d / \rho}}\right] \\
& \leqslant C_{p, \rho, d}\left(K_{2}+\|b\|_{\tilde{L}_{p}}^{2} \frac{K_{2}}{K_{1}^{2}}+\|\nabla \sigma\|_{\tilde{L}_{\rho}}^{2}\right)
\end{aligned}
$$

$$
\begin{equation*}
\left[\left(\frac{K_{2}}{K_{1}}\right)^{\frac{16 d^{3}}{(1-d /(\rho \wedge \rho)(1-d / \rho)}}+\left(\frac{\|b\|_{\tilde{L}_{p}}}{K_{1}}\right)^{\frac{32 d^{2}}{1-d /(\rho \wedge \rho)}}+\left(\frac{\|\nabla \sigma\|_{L_{\rho}}^{2}}{K_{1}}\right)^{\frac{32 d^{3}}{(1-d /(\rho \wedge \rho)(1-d / \rho)}}\right] . \tag{5.24}
\end{equation*}
$$

In the last inequality we used that $\max \left(\frac{32 d^{2}}{1-d /(p \wedge \rho)}, \frac{8}{1-d / p}\right) \leqslant \frac{32 d^{2}}{1-d /(p \wedge \rho)}$, and $\max \left(\frac{32 d^{3}}{(1-d /(p \wedge \rho))(1-d / \rho)}\right.$, $\left.\frac{8 d}{(1-d / p)(1-d / \rho)}, \frac{8}{1-d / \rho}\right) \leqslant \frac{32 d^{3}}{(1-d /(p \wedge \rho))(1-d / \rho)}$. In the end we get (5.8).

As a by-product from the proof of Theorem 5.4 we also have
Proposition 5.5. Let $\left(\psi_{t}(x)\right)_{t \geqslant 0}$ denote the flow generated by the solution to (1.1). Let $\chi_{T}$ be cubes of $\mathbb{R}^{d}$ with side length $\exp (-\gamma T), \gamma>0$. If Assumption 2.4 holds then for any $k>0$

$$
\limsup _{T \rightarrow \infty} \frac{1}{T} \sup _{\chi_{T}} \log \mathbb{P}\left(\sup _{x, y \in \chi_{T}} \sup _{0 \leqslant t \leqslant T}\left|\psi_{t}(x)-\psi_{t}(y)\right| \geqslant k\right) \leqslant-I(\gamma)
$$

where

$$
I(\gamma):=\left\{\begin{array}{rlc}
\gamma^{1+1 / \alpha} \alpha(1+\alpha)^{-1-1 / \alpha} c_{1}^{-1 / \alpha} & \text { if } & \gamma \geqslant c_{1}(\alpha+1) d^{\alpha}  \tag{5.25}\\
d\left(\gamma-c_{1} d^{\alpha}\right) & \text { if } & c_{1} d^{\alpha}<\gamma \leqslant c_{1}(\alpha+1) d^{\alpha} \\
0 & \text { if } & \gamma \leqslant c_{1} d^{\alpha} .
\end{array}\right.
$$

with $\alpha$ and $c_{1}$ as in (5.23).
Proof. This follows easily from (5.18) and Lemma 3.1.

## 6. Existence of random attractors to SDEs with singular drift

Inspired by the work [7], we are interested in the question whether there exists a random attractor of the RDS generated by the solution to the singular SDE. We start with estimates of the one-point motion (items 1-5 of the following lemma) and then move to estimates for the dispersion of sets (items 6 and 7).

Lemma 6.1. Let Assumption 2.4 hold. Further assume that there exist vector fields $b_{1}$ and $b_{2}$ such that $b=b_{1}+b_{2}$ with $b_{1} \in \tilde{L}_{p}\left(\mathbb{R}^{d}\right)$. Let $\left(\psi_{t}(x)\right)_{t \geqslant 0}$ be the flow generated by the solution to (1.1). Let $\Gamma:=C_{\mathrm{Kry}}\left(\frac{p}{2}\right)\left(\left(\frac{K_{2}}{K_{1}}\right)^{\frac{4 d^{2}}{1-d / \rho}}+\left(\frac{\|\nabla \sigma\|_{\tilde{I}_{\rho}}^{2}}{K_{1}}\right)^{\frac{4 d^{2}}{1-d / \rho}}+\left(\frac{\left\|b_{2}\right\|_{\tilde{L}_{p}}}{K_{1}}\right)^{\frac{4 d}{1-d / \rho}}\right)$ where $C_{\mathrm{Kry}}\left(\frac{p}{2}\right)$ is from (4.1) with $q=\frac{p}{2}$ depending on $p, \rho$ and $d$ only.

1. Let $1 \leqslant r$, and $r_{1}, r_{2}>r$. If $b_{2}$ satisfies Assumption $2.11\left(U^{\beta}\right)$ for some $\beta \in \mathbb{R}$, then, for each $|x|=r_{2}$,

$$
\begin{aligned}
& \mathbb{P}\left(\left|\psi_{T}(x)\right| \geqslant r_{1}, \inf _{0 \leqslant t \leqslant T}\left|\psi_{t}(x)\right| \geqslant r\right) \\
& \quad \leqslant 2 \exp \left(T \frac{\Gamma^{2}\left\|b_{1}\right\|_{\tilde{L}_{p}}^{4}+K_{2}^{2} \Gamma\left\|b_{1}\right\|_{\tilde{L}_{p}}^{2}}{K_{1} K_{2}^{2}}-\frac{1}{4}\left(-\frac{r_{2}-r_{1}}{\sqrt{K_{2} T}}-\frac{\sqrt{T} \beta^{*}(r)}{\sqrt{K_{2}}}\right)_{+}^{2}\right)
\end{aligned}
$$

with

$$
\begin{equation*}
\beta^{*}(r):=\sup _{|x| \geqslant r} \frac{x \cdot b_{2}(x)}{|x|}+(d-1) \frac{K_{2}}{2 r} . \tag{6.1}
\end{equation*}
$$

2. If $b_{2}$ satisfies Assumption $2.11\left(U^{\beta}\right)$ for some $\beta<0$ and $r_{0}>1$ is such that $\beta^{*}\left(r_{0}\right) \leqslant 0$ where $\beta^{*}\left(r_{0}\right)$ is from (6.1), then for every $R \geqslant r \geqslant r_{0}$ and every $x \in \mathbb{R}^{d}$, we have

$$
\mathbb{P}\left(\left|\psi_{T}(x)\right| \geqslant R, \inf _{0 \leqslant t \leqslant T}\left|\psi_{t}(x)\right| \leqslant r\right) \leqslant 4 \exp \left(T \frac{\Gamma^{2}\left\|b_{1}\right\|_{\tilde{L}_{p}}^{4}+K_{2}^{2} \Gamma\left\|b_{1}\right\|_{\tilde{L}_{p}}^{2}}{K_{1} K_{2}^{2}}-\frac{(R-r)^{2}}{16 K_{2} T}\right) .
$$

3. If $b_{2}$ satisfies Assumption $2.11\left(U^{\beta}\right)$ for some $\beta<0$ and $r_{0}>1$ such that $\beta^{*}\left(r_{0}\right) \leqslant 0$ where $\beta^{*}\left(r_{0}\right)$ is from (6.1) and if $R \geqslant r_{0}$, then for every $|x|=R, \delta, \delta_{1}>0$, we have

$$
\mathbb{P}\left(\sup _{0 \leqslant s \leqslant \delta_{1}}\left|\psi_{s}(x)\right| \geqslant R+\delta\right) \leqslant 6 \exp \left(T \frac{\Gamma^{2}\left\|b_{1}\right\|_{\tilde{L}_{p}}^{4}+K_{2}^{2} \Gamma\left\|b_{1}\right\|_{\tilde{L}_{p}}^{2}}{K_{1} K_{2}^{2}}-\frac{\delta^{2}}{16 K_{2} \delta_{1}}\right) .
$$

4. Let $1 \leqslant r$, and $r_{1}, r_{2}>r$. If $b_{2}$ satisfies Assumption $2.11\left(U_{\beta}\right)$ for some $\beta \in \mathbb{R}$, then for each $|x|=r_{1}$,

$$
\begin{aligned}
& \mathbb{P}\left(\left|\psi_{T}(x)\right| \leqslant r_{2}, \inf _{0 \leqslant t \leqslant T}\left|\psi_{t}(x)\right| \geqslant r\right) \\
& \leqslant 2 \exp \left(T \frac{\Gamma^{2}\left\|b_{1}\right\|_{\tilde{L}_{p}}^{4}+K_{2}^{2} \Gamma\left\|b_{1}\right\|_{\tilde{L}_{p}}^{2}}{K_{1} K_{2}^{2}}-\frac{1}{4}\left(\frac{\sqrt{T} \beta_{*}(r)}{\sqrt{K_{2}}}-\frac{r_{2}-r_{1}}{\sqrt{K_{2} T}}\right)_{+}^{2}\right)
\end{aligned}
$$

with
$\beta_{*}(r):=\inf _{|x| \geqslant r} \frac{x \cdot b_{2}(x)}{|x|}$.
5. If $b_{2}$ satisfies Assumption $2.11\left(U_{\beta}\right)$ for some $\beta \in \mathbb{R}$, then for each $|x|=r_{1}$, for $1 \leqslant r<r_{1}$
$\mathbb{P}\left(\inf _{t \geqslant 0}\left|\psi_{t}(x)\right| \leqslant r\right) \leqslant 2 \exp \left(T \frac{\Gamma^{2}\left\|b_{1}\right\|_{\tilde{L}_{p}}^{4}+K_{2}^{2} \Gamma\left\|b_{1}\right\|_{\tilde{L}_{p}}^{2}}{K_{1} K_{2}^{2}}-\left(r_{1}-r\right) \frac{\beta_{*}(r)}{K_{2}}\right)$
with $\beta_{*}\left(r_{1}\right)$ defined as (6.2).
6. Assume that $b_{2}$ satisfies Assumption $2.11\left(U_{\beta}\right)$ for

$$
\beta>\beta_{0}:=4 \frac{\left\|b_{1}\right\|_{\tilde{L}_{p}}^{2} \Gamma+K_{2}\left\|b_{1}\right\|_{\tilde{L}_{p}} \sqrt{\Gamma}}{\sqrt{K_{1} K_{2}}}
$$

Let $h:[1, \infty) \rightarrow[1, \infty)$ be strictly increasing such that $\lim _{x \rightarrow \infty} \frac{h(x)}{x}=0$ and $\lim _{x \rightarrow \infty} \frac{\log x}{h(x)}=0$. Let $\eta \in\left(0, \frac{1}{2}\right)$ and $\gamma>0$ with $\eta+\gamma<\beta-\beta_{0}$. For $R>2$, define $T:=h(R), r=(1-\eta) R$ and $r_{1}:=R+\gamma h(R)$. Then

$$
\limsup _{R \rightarrow \infty} \frac{1}{h(R)} \log \mathbb{P}_{R}:=\limsup _{R \rightarrow \infty} \frac{1}{h(R)} \log \mathbb{P}\left[\left(B_{r_{1}} \nsubseteq \psi_{T}\left(B_{R}\right)\right) \cup \cup_{t \in[0, T]}\left(B_{r} \nsubseteq \psi_{t}\left(B_{R}\right)\right)\right]<0
$$

7. Assume that $b_{2}$ satisfies Assumption $2.11\left(U^{\beta}\right)$ for

$$
\beta<-\beta_{0}:=-4 \frac{\left\|b_{1}\right\|_{\tilde{L}_{p}}^{2} \Gamma+K_{2}\left\|b_{1}\right\|_{\tilde{L}_{p}} \sqrt{\Gamma}}{20} \sqrt{K_{1} K_{2}}
$$

Let $h(R)=R^{\iota}$ for some $\iota \in\left(0, \frac{1}{3}\right)$. Let $\eta \in\left(0, \frac{1}{2}\right)$ and $\gamma>0$ with $\eta+\gamma<-\beta-\beta_{0}$. For $R>2$, define $T:=h(R), r=(1-\eta) R$ and $r_{1}:=R+\gamma h(R)$. Then

$$
\begin{aligned}
\limsup _{R \rightarrow \infty} & \frac{1}{h(R)} \log \mathbb{P}_{R} \\
& :=\limsup _{R \rightarrow \infty} \frac{1}{h(R)} \log \mathbb{P}\left[\bigcup_{|x|=r_{1}}\left(\left(\left|\psi_{T}(x)\right| \geqslant R\right) \cap\left(\inf _{t \in[0, T]}\left|\psi_{t}(x)\right| \geqslant r\right)\right)\right]<0 .
\end{aligned}
$$

Proof. Let us explain the idea of the proof of parts 1 to 5 : we express the probabilities on the left side by the corresponding ones for the flow $\psi^{2}$ generated by the SDE with drift $b$ replaced by $b_{2}$ by applying Girsanov's theorem. This is possible since $b_{1} \in \tilde{L}_{p}$. The required estimates for $\psi^{2}$ can then be obtained from results in [7]. Notice that strictly speaking the SDE generating $\psi^{2}$ cannot be applied since the assumptions in [7] require the coefficients to be one-sided Lipschitz continuous which is not necessarily true in our set-up. It is easy to check however that the estimates of the one-point motion in Propositions 4.2 to 4.6 in [7] hold without additional Lipschitz-type assumptions. Therefore, we divide the proof into two steps: a Girsanov argument and then estimates for the flow $\psi^{2}$.

Let

$$
\rho_{t}:=\exp \left\{\int_{0}^{t}\left(b_{1}\right)^{*}\left(\sigma^{-1}\right)^{*}\left(\psi_{r}^{2}(x)\right) \mathrm{d} W_{r}-\frac{1}{2} \int_{0}^{t}\left(b_{1}\right)^{*}\left(\sigma \sigma^{*}\right)^{-1} b_{1}\left(\psi_{r}^{2}(x)\right) \mathrm{d} r\right\}
$$

where $\psi_{t}^{2}(x)$ is the flow generated by the solution to

$$
\mathrm{d} \psi_{t}^{2}=b_{2}\left(\psi_{t}^{2}\right) \mathrm{d} t+\sigma\left(\psi_{t}^{2}\right) \mathrm{d} W_{t}, \quad \psi_{0}^{2}=x \in \mathbb{R}^{d} .
$$

From (4.8) and (5.19) we get for $T>1$ and any $\lambda>0$

$$
\begin{equation*}
\mathbb{E} \exp \left(\lambda \int_{0}^{T}\left(b_{1}\right)^{*}\left(\sigma \sigma^{*}\right)^{-1} b_{1}\left(\psi_{r}^{2}(x)\right) \mathrm{d} r\right) \leqslant 2 \exp \left(T\left(\left(K_{1}^{2} K_{2}\right)^{-1}(\Gamma \lambda)^{2}\left\|b_{1}\right\|_{\tilde{L}_{p}}^{4}+K_{1}^{-1} \lambda \Gamma\left\|b_{1}\right\|_{\tilde{L}_{p}}^{2}\right)\right) . \tag{6.3}
\end{equation*}
$$

Therefore, $\left(\rho_{t}\right)_{t \geqslant 0}$ is a martingale. Fix $T>0$ and let $\mathbb{P}^{\rho}:=\rho_{T} \mathbb{P}$. Girsanov's theorem and Hölder's inequality show for each measurable set $A \in C\left([0, T], \mathbb{R}^{d}\right)$

$$
\begin{aligned}
& \mathbb{P}\left(\left.\psi\right|_{[0, T]} \in A\right)=\mathbb{P}^{\rho}\left(\left.\psi^{2}\right|_{[0, T]} \in A\right) \\
&= \mathbb{E}\left[\rho_{T}:\left.\psi^{2}\right|_{[0, T]} \in A\right] \leqslant\left[\mathbb{E} \rho_{T}^{2}\right]^{1 / 2} \mathbb{P}\left(\left.\psi^{2}\right|_{[0, T]} \in A\right)^{1 / 2} \\
& \leqslant {\left[\mathbb { E } \operatorname { e x p } \left(2 \int_{0}^{T}\left(b_{1}\right)^{*}\left(\sigma^{-1}\right)^{*}\left(\psi_{r}^{2}(x)\right) \mathrm{d} W_{r}-2 \int_{0}^{T}\left(b_{1}\right)^{*}\left(\sigma \sigma^{*}\right)^{-1} b_{1}\left(\psi_{r}^{2}(x)\right) \mathrm{d} r\right.\right.} \\
&\left.\left.\quad+\int_{0}^{T}\left(b_{1}\right)^{*}\left(\sigma \sigma^{*}\right)^{-1} b^{1}\left(\psi_{r}^{2}(x)\right) \mathrm{d} r\right)\right]^{1 / 2}\left[\mathbb{P}\left(\left.\psi^{2}\right|_{[0, T]} \in A\right)\right]^{1 / 2} \\
& \leqslant\left(\mathbb{E}\left[\exp \left(2 \int_{0}^{T}\left(b_{1}\right)^{*}\left(\sigma^{-1}\right)^{*}\left(\psi_{r}^{2}(x)\right) \mathrm{d} W_{r}-2 \int_{0}^{T}\left(b_{1}\right)^{*}\left(\sigma \sigma^{*}\right)^{-1} b_{1}\left(\psi_{r}^{2}(x)\right) \mathrm{d} r\right]^{2}\right)^{1 / 4}\right. \\
& {\left[\mathbb{E} \exp \left(\int_{0}^{t} 2\left(b_{1}\right)^{*}\left(\sigma \sigma^{*}\right)^{-1} b_{1}\left(\psi_{r}^{2}(x)\right) \mathrm{d} r\right)\right]^{1 / 4}\left[\mathbb{P}\left(\left.\psi^{2}\right|_{[0, T]} \in A\right)\right]^{1 / 2} }
\end{aligned}
$$

$$
\begin{align*}
& \leqslant\left[\mathbb{E} \exp \left(2 \int_{0}^{T}\left(b_{1}\right)^{*}\left(\sigma \sigma^{*}\right)^{-1} b_{1}\left(\psi_{r}^{2}(x)\right) \mathrm{d} r\right)\right]^{1 / 4}\left[\mathbb{P}\left(\left.\psi^{2}\right|_{[0, T]} \in A\right)\right]^{1 / 2} \\
& \leqslant 2 \exp \left(T\left(\left(K_{1}^{2} K_{2}\right)^{-1} \Gamma^{2}\left\|b_{1}\right\|_{\tilde{L}_{p}}^{4}+K_{1}^{-1} \Gamma\left\|b_{1}\right\|_{\tilde{L}_{p}}^{2}\right)\right)\left[\mathbb{P}\left(\left.\psi^{2}\right|_{[0, T]} \in A\right)\right]^{1 / 2} \tag{6.4}
\end{align*}
$$

If $A_{i}$ denotes the set inside $\mathbb{P}$ on the left side of item i in the Lemma $(\mathrm{i}=1, \ldots, 5)$, then

$$
\mathbb{P}\left(\left.\psi\right|_{[0, T]} \in A_{i}\right) \leqslant 2 \exp \left(T \frac{\Gamma^{2}\left\|b_{1}\right\|_{\tilde{L}_{p}}^{4}+K_{2}^{2} \Gamma\left\|b_{1}\right\|_{\tilde{L}_{p}}^{2}}{K_{1} K_{2}^{2}}\right)\left[\mathbb{P}\left(\left.\psi^{2}\right|_{[0, T]} \in A_{i}\right)\right]^{1 / 2}
$$

finishing the first step in cases 1-5. It remains to estimate $\left[\mathbb{P}\left(\left.\psi^{2}\right|_{[0, T]} \in A_{i}\right)\right]^{1 / 2}$. Inserting the estimate in [7, Proposition 4.2 a)] under $\left(U^{\beta}\right)$, we obtain statement 1 . Inserting the estimate in [7, Proposition 4.5] under $\left(U^{\beta}\right)$, we obtain statement 2 . Inserting the estimate in [7, Proposition 4.6] under $\left(U^{\beta}\right)$, we obtain statement 3. Inserting the estimate in [7, Proposition 4.2 b$)$ ] under $\left(U_{\beta}\right)$, we obtain statement 4. and inserting the estimate in [7, Proposition 4.3] under $\left(U_{\beta}\right)$, we obtain statement 5.

Finally we show items 6 and 7. Without loss of generality we assume $\frac{1}{\eta}<R$. For a ball $B_{R}$ with radius $R$ we can cover its boundary $\partial B_{R}$ by $N=N_{\epsilon} \leqslant C_{d}\left(\frac{R}{\epsilon}\right)^{d-1}$ balls centered on $\partial B_{R}$ for any $\epsilon \in(0, R]$. Here we take $\epsilon=\exp (-\kappa h(R))$ for some $\kappa>0$ which will be chosen later and we label the balls by $L_{1}, \cdots, L_{N}$ with corresponding centers $x_{1}, \cdots, x_{N}$. Note that

$$
N \leqslant C_{d} R^{d-1} \exp ((d-1) \kappa h(R))
$$

Then

$$
\begin{aligned}
\mathbb{P}_{R} \leqslant & N \max _{1 \leqslant i \leqslant N}\left[\mathbb{P}\left(\left|\psi_{T}\left(x_{i}\right)\right| \leqslant r_{1}+1, \inf _{t \in[0, T]}\left|\psi_{t}\left(x_{i}\right)\right|>r+1\right)\right. \\
& \left.+\mathbb{P}\left(\inf _{t \in[0, T]}\left|\psi_{t}\left(x_{i}\right)\right| \leqslant r+1\right)+\mathbb{P}\left(\sup _{t \in[0, T]} \operatorname{diam} \psi_{t}\left(L_{i}\right) \geqslant 1\right)\right] \\
= & : N\left(P_{1}(R)+P_{2}(R)+P_{3}(R)\right) .
\end{aligned}
$$

Case 4 gives us the following upper bound (note $T=h(R)$ )

$$
\begin{aligned}
P_{1}(R) & \leqslant 2 \exp \left(T \frac{\Gamma^{2}\left\|b_{1}\right\|_{\tilde{L}_{p}}^{4}+K_{2}^{2} \Gamma\left\|b_{1}\right\|_{\tilde{L}_{p}}^{2}}{K_{1} K_{2}^{2}}-\frac{1}{4}\left(\frac{\sqrt{T} \beta_{*}(r+1)}{\sqrt{K_{2}}}-\frac{r_{1}+1-R}{\sqrt{K_{2} T}}\right)_{+}^{2}\right) \\
& =2 \exp \left(h(R) \frac{\Gamma^{2}\left\|b_{1}\right\|_{\tilde{L}_{p}}^{4}+K_{2}^{2} \Gamma\left\|b_{1}\right\|_{\tilde{L}_{p}}^{2}}{K_{1} K_{2}^{2}}-\frac{h(R)}{4 K_{2}}\left(\beta_{*}(r+1)-\gamma-\frac{1}{h(R)}\right)_{+}^{2}\right) .
\end{aligned}
$$

So

$$
\begin{equation*}
\limsup _{R \rightarrow \infty} \frac{1}{h(R)} \log \left(N \mathbb{P}_{1}(R)\right) \leqslant(d-1) \kappa+\frac{\Gamma^{2}\left\|b_{1}\right\|_{\tilde{L}_{p}}^{4}+K_{2}^{2} \Gamma\left\|b_{1}\right\|_{\tilde{L}_{p}}^{2}}{K_{1} K_{2}^{2}}-\frac{1}{4 K_{2}}(\beta-\gamma)^{2} \tag{6.5}
\end{equation*}
$$

Case 5 shows for $r=(1-\eta) R$

$$
\begin{aligned}
P_{2}(R) & \leqslant 2 \exp \left(T \frac{\Gamma^{2}\left\|b_{1}\right\|_{\tilde{L}_{p}}^{4}+K_{2}^{2} \Gamma\left\|b_{1}\right\|_{\tilde{L}_{p}}^{2}}{K_{1} K_{2}^{2}}-(R-r-1) \frac{\beta_{*}(r+1)}{K_{2}}\right) \\
& =2 \exp \left(h(R) \frac{\Gamma^{2}\left\|b_{1}\right\|_{\tilde{L}_{p}}^{4}+K_{2}^{2} \Gamma\left\|b_{1}\right\|_{\tilde{L}_{p}}^{2}}{K_{1} K_{2}^{2}}-(\eta R-1) \frac{\beta_{*}(r+1)}{K_{2}}\right) .
\end{aligned}
$$

Hence

$$
\begin{equation*}
\frac{1}{h(R)} \log \left(N \mathbb{P}_{2}(R)\right) \leqslant(d-1) \kappa+\frac{\Gamma^{2}\left\|b_{1}\right\|_{\tilde{L}_{p}}^{4}+K_{2}^{2} \Gamma\left\|b_{1}\right\|_{\tilde{L}_{p}}^{2}}{K_{1} K_{2}^{2}}-\frac{\eta R-1}{h(R)} \frac{\beta_{*}(r+1)}{K_{2}} . \tag{6.6}
\end{equation*}
$$

Furthermore, by Proposition 5.5,

$$
\begin{equation*}
\limsup _{R \rightarrow \infty} \frac{1}{h(R)} \log P_{3}(R) \leqslant-\kappa^{4 / 3} c_{1}^{-1 / 3} \text { with } \kappa>4 c_{1} d^{3} \tag{6.7}
\end{equation*}
$$

where $c_{1}$ is taken from (5.23) with $b$ replaced by $b_{2}$. Therefore, by (6.5), (6.6) and (6.7), it follows that, for $\kappa>4 c_{1} d^{3}$,

$$
\begin{align*}
& \limsup _{R \rightarrow \infty} \frac{1}{h(R)} \log \mathbb{P}_{R} \\
& \quad \leqslant \limsup _{R \rightarrow \infty} \frac{1}{h(R)} \log \left(N \mathbb{P}_{1}(R)+N \mathbb{P}_{2}(R)+N \mathbb{P}_{3}(R)\right) \\
& \quad \leqslant 2(d-1) \kappa+2 \frac{\Gamma^{2}\left\|b_{1}\right\|_{\tilde{L}_{p}}^{4}+K_{2}^{2} \Gamma\left\|b_{1}\right\|_{\tilde{L}_{p}}^{2}}{K_{1} K_{2}^{2}}-\frac{1}{4 K_{2}}(\beta-\gamma)^{2}+(d-1) \kappa-\kappa^{4 / 3} c_{1}^{-1 / 3} . \tag{6.8}
\end{align*}
$$

Notice that $\beta-\gamma>\beta_{0}+\eta \geqslant 4 \frac{\left\|b_{1}\right\|_{\tilde{L}_{p}}^{2} \Gamma+K_{2}\left\|b_{1}\right\|_{\tilde{L}_{p}} \sqrt{\Gamma}}{\sqrt{K_{1} K_{2}}}$. If we choose $\kappa \geqslant 3 c_{1}(d-1)^{3}$ initially, then get

$$
\limsup _{R \rightarrow \infty} \frac{1}{h(R)} \log \mathbb{P}_{R}<0
$$

Therefore case 6 holds.
We show case 7 in a similar way. We again cover $\partial B_{r_{1}}$ by $N \leqslant C_{d} r_{1}{ }^{d-1} e^{\kappa(d-1) T}$ balls centered on $\partial B_{r_{1}}$ for any with radius $e^{-\kappa T}$ for some $\kappa>0$ chosen later. Label the balls by $L_{1}, \cdots, L_{N}$ with corresponding centers $x_{1}, \cdots, x_{N}$. Then

$$
\begin{aligned}
\mathbb{P}_{R} \leqslant & N \max _{i}\left[\mathbb{P}\left(\left|\psi_{T}\left(x_{i}\right)\right| \geqslant R+1, \inf _{t \in[0, T]}\left|\psi_{t}\left(x_{i}\right)\right|>r+1\right)\right. \\
& \left.+\mathbb{P}\left(\left|\psi_{T}\left(x_{i}\right)\right| \geqslant R+1, \inf _{t \in[0, T]}\left|\psi_{t}\left(x_{i}\right)\right| \leqslant r+1\right)+\mathbb{P}\left(\sup _{t \in[0, T]} \operatorname{diam} \psi_{t}\left(L_{i}\right) \geqslant 1\right)\right] \\
= & : N\left(P_{1}(R)+P_{2}(R)+P_{3}(R)\right) .
\end{aligned}
$$

From case 1 we then get (note $T=h(R)$ )

$$
\begin{aligned}
P_{1}(R) & \leqslant 2 \exp \left(h(R) \frac{\Gamma^{2}\left\|b_{1}\right\|_{\tilde{L}_{p}}^{4}+K_{2}^{2} \Gamma\left\|b_{1}\right\|_{\tilde{L}_{p}}^{2}}{K_{1} K_{2}^{2}}-\frac{h(R)}{4 K_{2}}\left(\frac{R+1-r_{1}}{h(R)}-\beta^{*}(r+1)\right)_{+}^{2}\right) \\
& \leqslant 2 \exp \left(h(R) \frac{\Gamma^{2}\left\|b_{1}\right\|_{\tilde{L}_{p}}^{4}+K_{2}^{2} \Gamma\left\|b_{1}\right\|_{\tilde{L}_{p}}^{2}}{K_{1} K_{2}^{2}}-\frac{h(R)}{4 K_{2}}\left(\frac{1}{h(R)}-\gamma-\beta\right)^{2}\right) .
\end{aligned}
$$

Therefore,

$$
\begin{equation*}
\frac{1}{h(R)} \log P_{1}(R) \leqslant \frac{\Gamma^{2}\left\|b_{1}\right\|_{\tilde{L}_{p}}^{4}+K_{2}^{2} \Gamma\left\|b_{1}\right\|_{\tilde{L}_{p}}^{2}}{K_{1} K_{2}^{2}}-\frac{1}{4 K_{2}}\left(\frac{1}{h(R)}-\gamma-\beta\right)^{2} . \tag{6.9}
\end{equation*}
$$

Analogously, case 2 implies for $R$ such that $r=(1-\eta) R>r_{0}$ where $\beta^{*}\left(r_{0}\right)<0$,

$$
\begin{equation*}
\frac{1}{h(R)} \log P_{2}(R) \leqslant \frac{\Gamma^{2}\left\|b_{1}\right\|_{\tilde{L}_{p}}^{4}+K_{2}^{2} \Gamma\left\|b_{1}\right\|_{\tilde{L}_{p}}^{2}}{K_{1} K_{2}^{2}}-\frac{(R-r)^{2}}{16 K_{2} h(R)} . \tag{6.10}
\end{equation*}
$$

By (6.9), (6.10) and (6.7) we obtain

$$
\begin{align*}
& \frac{1}{h(R)} \log \mathbb{P}_{R} \\
& \leqslant 3(d-1) \kappa+2 \frac{\Gamma^{2}\left\|b_{1}\right\|_{\tilde{L}_{p}}^{4}+K_{2}^{2} \Gamma\left\|b_{1}\right\|_{\tilde{L}_{p}}^{2}}{K_{1} K_{2}^{2}}-\frac{1}{4 K_{2}}\left(\frac{1}{h(R)}-\beta-\gamma\right)^{2}-\frac{(R-r)^{2}}{16 K_{2} h(R)}-\kappa^{4 / 3} c_{1}^{-1 / 3} \tag{6.11}
\end{align*}
$$

and

$$
\limsup _{R \rightarrow \infty} \frac{1}{h(R)} \log \mathbb{P}_{R} \leqslant 3(d-1) \kappa+2 \frac{\Gamma^{2}\left\|b_{1}\right\|_{\tilde{L}_{p}}^{4}+K_{2}^{2} \Gamma\left\|b_{1}\right\|_{\tilde{L}_{p}}^{2}}{K_{1} K_{2}^{2}}-\frac{1}{4 K_{2}}(-\beta-\gamma)^{2}-\kappa^{4 / 3} c_{1}^{-1 / 3} .
$$

Under $\left(U^{\beta}\right)$,

$$
(-\gamma-\beta)^{2} \geqslant\left(-\beta_{0}-\eta\right)^{2} \geqslant 16 K_{2} \frac{\Gamma^{2}\left\|b_{1}\right\|_{\tilde{L}_{p}}^{4}+K_{2}^{2} \Gamma\left\|b_{1}\right\|_{\tilde{L}_{p}}^{2}}{K_{1} K_{2}^{2}} .
$$

Hence, choosing $\kappa \geqslant 3 c_{1}(d-1)^{3}$ above, we conclude that $\lim \sup _{R \rightarrow \infty} \frac{1}{h(R)} \log \mathbb{P}_{R}<0$.
Now we are ready to state the first main theorem of this section.
Theorem 6.2. Let Assumption 2.4 hold. Further assume that there exist vector fields $b_{1}$ and $b_{2}$ such that $b=b_{1}+b_{2}$ with $b_{1} \in \tilde{L}_{p}\left(\mathbb{R}^{d}\right)$. Let $\left(\psi_{t}(x)\right)_{t \geqslant 0}$ denote the flow generated by the solution to (1.1). Let $\Gamma:=C_{\mathrm{Kry}}\left(\frac{p}{2}\right)\left(\left(\frac{K_{2}}{K_{1}}\right)^{\frac{4 d^{2}}{1-d / \rho}}+\left(\frac{\|\nabla \sigma\|_{\tilde{L}_{\rho}}^{2}}{K_{1}}\right)^{\frac{4 d^{2}}{1-d / \rho}}+\left(\frac{\left\|b_{2}\right\|_{\tilde{L}_{p}}}{K_{1}}\right)^{\frac{4 d}{1-d / p}}\right)$ where $C_{\mathrm{Kry}}\left(\frac{p}{2}\right)$ is from (4.1) with $q=\frac{p}{2}$ depending on $p, \rho$ and $d$ only. If $b_{2}$ satisfies Assumption $2.11\left(U_{\beta}\right)$ for

$$
\beta>\beta_{0}:=4 \frac{\left\|b_{1}\right\|_{\tilde{L}_{p}}^{2} \Gamma+K_{2}\left\|b_{1}\right\|_{\tilde{L}_{p}} \sqrt{\Gamma}}{24} \sqrt{K_{1} K_{2}},
$$

then for any $\gamma \in\left[0, \beta-\beta_{0}\right)$ we have

$$
\begin{equation*}
\lim _{r \rightarrow \infty} \mathbb{P}\left(B_{\gamma t} \subset \psi_{t}\left(B_{r}\right) \quad \forall \quad t \geqslant 0\right)=1 \tag{6.12}
\end{equation*}
$$

Proof. For $\gamma \in\left[0, \beta-\beta_{0}\right)$, let $\eta \in\left(0, \frac{1}{2}\right)$ such that $\gamma+\eta<\beta-\beta_{0}$. Let $R_{0} \geqslant 2, R_{i+1}=R_{i}+\gamma h\left(R_{i}\right)$ by iteration, where $h:[1, \infty) \rightarrow[1, \infty)$ is strictly increasing and $\lim _{x \rightarrow \infty} \frac{h(x)}{x}=0$ and $\lim _{x \rightarrow \infty} \frac{\log x}{h(x)}=0$. For $i=0,1, \cdots$, take $r_{i}=(1-\eta) R_{i}, \bar{r}_{i}=R+\gamma h\left(R_{i}\right)$. Define

$$
\mathbb{P}_{R_{i}}:=\mathbb{P}\left[\left(B_{\bar{r}_{i}} \nsubseteq \psi_{T}\left(B_{R_{i}}\right)\right) \cup \cup_{t \in[0, T]}\left(B_{r_{i}} \nsubseteq \psi_{t}\left(B_{R_{i}}\right)\right)\right]
$$

Then Lemma 6.1 case 6 shows that

$$
\sum_{i=0}^{\infty} \mathbb{P}_{R_{i}}<\infty, \quad \text { if } \quad \sum_{i=0}^{\infty} \exp \left(-\kappa h\left(R_{i}\right)\right)<\infty, \quad \kappa>0
$$

If we take $h\left(R_{i}\right)=R_{i}^{\alpha}$ for some $\alpha \in(0,1)$, then Borel-Cantelli Lemma and time-homogeneity of flow $\psi$ yield the result (6.12).

Finally, we state the following theorem on the existence of random attractors.
Theorem 6.3. Let Assumption 2.4 hold. Further assume that there exist vector fields $b_{1}$ and $b_{2}$ such that $b=b_{1}+b_{2}$ with $b_{1} \in \tilde{L}_{p}\left(\mathbb{R}^{d}\right)$. Let $\left(\psi_{t}(x)\right)_{t \geqslant 0}$ denote the flow generated by the solution to (1.1). Let $\Gamma:=C_{\mathrm{Kry}}\left(\frac{p}{2}\right)\left(\left(\frac{K_{2}}{K_{1}}\right)^{\frac{4 d^{2}}{1-d / \rho}}+\left(\frac{\|\nabla \sigma\|_{\tilde{L}_{\rho}}^{2}}{K_{1}}\right)^{\frac{4 d^{2}}{1-d / \rho}}+\left(\frac{\left\|b_{2}\right\|_{\tilde{L}_{p}}}{K_{1}}\right)^{\frac{4 d}{1-d / p}}\right)$ where $C_{\mathrm{Kry}}\left(\frac{p}{2}\right)$ is from (4.1) with $q=\frac{p}{2}$ depending on $p, \rho$ and $d$ only. If $b_{2}$ satisfies Assumption $2.11\left(U^{\beta}\right)$ for

$$
\beta<-\beta_{0}:=-4 \frac{\left\|b_{1}\right\|_{\tilde{L}_{p}}^{2} \Gamma+K_{2}\left\|b_{1}\right\|_{\tilde{L}_{p}} \sqrt{\Gamma}}{\sqrt{K_{1} K_{2}}}
$$

then, for any $\gamma \in\left[0,-\beta-\beta_{0}\right)$, we have

$$
\begin{equation*}
\lim _{r \rightarrow \infty} \mathbb{P}\left(B_{\gamma t} \subset \psi_{-t, 0}^{-1}\left(B_{r}\right) \quad \forall \quad t \geqslant 0\right)=1 \tag{6.13}
\end{equation*}
$$

In particular, $\psi$ has a random attractor.
Proof. The existence of an attractor is an easy observation from Proposition 2.8 if we have (6.13). So we only need to show (6.13). The argument is essentially the same as [7, Proof of Theorem 3.1 a)]. We give the outline of the proof emphasising those arguments which are different.

For $\gamma \in\left[0,-\beta-\beta_{0}\right)$, let $\eta \in\left(0, \frac{1}{2}\right)$ such that $\gamma+\eta<-\beta-\beta_{0}$. Let $h(y)=y^{\alpha}$ for some $\alpha \in\left(0, \frac{1}{3}\right)$. Notice that such $h$ is strictly increasing and $\lim _{y \rightarrow \infty} \frac{h(y)}{y}=0$ and $\lim _{y \rightarrow \infty} \frac{\log y}{h(y)}=0$. For $T \in(1, \infty)$, take $R:=T^{1 / \alpha}, r_{1}=R+\gamma T$ and $r=(1-\eta) R$. Let $\left(\phi_{s, T}(x)\right)_{s \leqslant T}$ denote the flow starting from $x$ at initial time $s$. We define

$$
\mathbb{P}_{R}:=\mathbb{P}\left[\left(B_{r_{1}} \nsubseteq \psi_{T}^{-1}\left(B_{R}\right)\right) \underset{25}{\cup} \cup_{t \in[0, T]}\left(B_{r} \nsubseteq \phi_{t, T}^{-1}\left(B_{R}\right)\right)\right]
$$

Once we show that

$$
\begin{equation*}
\lim _{R \rightarrow \infty} \frac{1}{h(R)} \log \mathbb{P}_{R}<0 \tag{6.14}
\end{equation*}
$$

then, by the same argument as in the proof of Theorem 6.2, we can finish the proof by the Borel-Cantelli Lemma and time-homogeneity of the flow $\psi$.

To show (6.14), notice that

$$
\begin{aligned}
\mathbb{P}_{R} & \leqslant \mathbb{P}\left[\cup_{|x|=r_{1}}\left(\left(\left|\psi_{T}(x)\right| \geqslant R\right) \cap\left(\inf _{t \in[0, T]}\left|\psi_{t}(x)\right| \geqslant r\right)\right)\right]+\mathbb{P}\left(\sup _{|x|=r} \sup _{t \in[0, T]}\left|\psi_{t, T}(x)\right| \geqslant R\right) \\
& =: P_{1}(R)+P_{2}(R) .
\end{aligned}
$$

For $P_{1}(R)$, we get from Lemma 6.1, case 7 that (note $T=R^{\alpha}=h(R)$ )

$$
\lim _{R \rightarrow \infty} \frac{1}{h(R)} \log P_{1}(R)<0
$$

In the following we show

$$
\begin{equation*}
\lim _{R \rightarrow \infty} \frac{1}{h(R)} \log P_{2}(R)=-\infty \tag{6.15}
\end{equation*}
$$

which is sufficient to get (6.14).
Let $\xi_{s}:=\left(\sup _{|x|=r}\left|\psi_{s, T}(x)\right|-r\right)_{+}, \zeta_{s}:=\left(\sup _{|x|=r+R \eta / 2}\left|\psi_{s, T}(x)\right|-r\right)_{+}$. Then, as shown in [7, p.12051206], we have

$$
\begin{aligned}
& \limsup _{R \rightarrow \infty} \frac{1}{h(R)} \log P_{2}(R) \\
& \quad \leqslant \limsup _{R \rightarrow \infty} \frac{1}{h(R)} \log \max _{s \in[1, T]}\left[\mathbb{P}\left(\zeta_{s} \geqslant \eta R\right)+\mathbb{P}\left(\sup _{t \in[s-1, s]} \sup _{|x|=r}\left|\psi_{t, s}(x)\right| \geqslant r+\frac{\eta}{2} R\right)\right] \\
& \quad:=\limsup _{R \rightarrow \infty} \frac{1}{h(R)} \log \max _{s \in[1, T]}\left(P_{2,1}(s, R)+P_{2,2}(s, R)\right) .
\end{aligned}
$$

To estimate $P_{2,1}(s, R)$, for fixed $0 \leqslant s \leqslant T$, denote $r_{0}:=r+\frac{\eta}{2} R$, we cover $\partial B_{r_{0}}$ by $N \leqslant C_{d} r_{0}^{d-1} e^{\kappa(d-1) T}$ balls of radius $e^{-\kappa T}$ centered on $\partial B_{r_{0}}$ with $\kappa<\frac{c_{1} d^{2}}{3(d-1)}$ (the same choice as in the proof of Lemma 6.1 case 7. Label the balls by $L_{1}, \cdots, L_{N}$ and their centers correspondingly by $x_{1}, \cdots, x_{N}$. Then for a number $r_{2}$ such that $\beta^{*}\left(r_{2}\right)<0$ where $\beta^{*}\left(r_{2}\right)$ is from (6.1), we have

$$
\begin{aligned}
P_{2,1}(s, R) \leqslant N \max _{i} & {\left[\mathbb{P}\left(\left|\psi_{s, T}\left(x_{i}\right)\right| \geqslant r+\eta R-1\right)+\mathbb{P}\left(\operatorname{diam} \psi_{s, T}\left(L_{i}\right) \geqslant 1\right)\right] } \\
\leqslant N \max _{i}[ & \mathbb{P}\left(\left|\psi_{s, T}\left(x_{i}\right)\right| \geqslant r+\eta R-1, \inf _{s \leqslant t \leqslant T}\left|\psi_{s, t}\left(x_{i}\right)\right|>r_{2}\right) \\
& \left.+\mathbb{P}\left(\left|\psi_{s, T}\left(x_{i}\right)\right| \geqslant r+\eta R-1, \inf _{s \leqslant t \leqslant T}\left|\psi_{s, t}\left(x_{i}\right)\right| \leqslant r_{2}\right)+\mathbb{P}\left(\operatorname{diam} \psi_{s, T}\left(L_{i}\right) \geqslant 1\right)\right] .
\end{aligned}
$$

By the same argument from Lemma 6.1 case 7 (6.11) with $h(R)=R^{\alpha}=T$, and Lemma 6.1 case 2, and Proposition 5.5 we get

$$
\limsup _{R \rightarrow \infty} \frac{1}{h(R)} \log \max _{s \in[1, T]} P_{2,1}(s, R)=-\infty .
$$

Up to here, in order to get (6.15), we only need to show

$$
\begin{equation*}
\limsup _{T \rightarrow \infty} \frac{1}{T} \log \max _{s \in[1, T]} P_{2,2}\left(s, T^{1 / \alpha}\right)=-\infty . \tag{6.16}
\end{equation*}
$$

In [7, Proof of Theorem 3.1 a)], this is shown by using three statements: [7, (4.7)], [7, Proposition 4.5 ] and [7, Proposition 4.6]. In our setting, we already showed the second and the third statements: these are Lemma 6.1 case 2 and case 3 correspondingly. Therefore it is sufficient to show the estimate corresponding to [7, (4.7)] in our setting. In order to do so we first apply Girsanov Theorem as we did in Lemma 6.1. Let

$$
\rho_{t}:=\exp \left(\int_{0}^{t} b^{*}\left(\sigma^{-1}\right)^{*}\left(\phi_{r}(x)\right) \mathrm{d} W_{r}-\frac{1}{2} \int_{0}^{t} b^{*}\left(\sigma \sigma^{*}\right)^{-1} b\left(\phi_{r}(x)\right) \mathrm{d} r\right),
$$

where $\phi_{t}(x)$ is the flow generated by the solution to

$$
\mathrm{d} \phi_{t}=\sigma\left(\phi_{t}\right) \mathrm{d} W_{t}, \quad \phi_{0}(x)=x \in \mathbb{R}^{d} .
$$

Following from (4.8) we get for $T>1$ and any $\lambda>0$

$$
\mathbb{E} \exp \left(\lambda \int_{0}^{T} b^{*}\left(\sigma \sigma^{*}\right)^{-1} b\left(\phi_{r}(x)\right) d r\right) \leqslant \exp \left(T \frac{\|b\|_{\tilde{L}_{p}}^{4}\left(\lambda \Gamma^{\prime}\right)^{2}+K_{2}^{2}\|b\|_{\tilde{L}_{p}}^{2} \lambda \Gamma^{\prime}}{K_{1} K_{2}^{2}}\right)
$$

where $\Gamma^{\prime}=C_{\mathrm{Kry}}\left(\frac{p}{2}\right)\left(\left(\frac{K_{2}}{K_{1}}\right)^{\frac{4 d d^{2}}{1-d / \rho}}+\left(\frac{\|\nabla \sigma\|_{\tilde{L}_{\rho}}^{2}}{K_{1}}\right)^{\frac{4 d^{2}}{1-d / \rho}}\right)$ and $C_{\mathrm{Kry}}\left(\frac{p}{2}\right)$ is from (4.1) with $p=\frac{p}{2}$ and $b=0$. Therefore $\left(\rho_{t}\right)_{t \geqslant 0}$ is a martingale. Let $\mathbb{P}^{\rho}:=\rho_{1} \mathbb{P}$. As we already did in (6.4), by Girsanov theorem and Hölder's inequality, for $\epsilon>0$, for any $x, z \in \mathbb{R}^{d}$,

$$
\begin{align*}
& \mathbb{P}\left(\left|\psi_{t+\frac{1}{2^{n}}, 1}(x)-\psi_{t+\frac{1}{2^{n}, 1}}(z)\right| \geqslant \frac{\epsilon}{2}\right) \\
&=\mathbb{P}^{\rho}\left(\left|\phi_{t+\frac{1}{2^{n}, 1}}(x)-\phi_{t+\frac{1}{2^{n}}, 1}(z)\right| \geqslant \frac{\epsilon}{2}\right) \\
&=\mathbb{E}\left[\rho_{1} \mathbb{I}_{\left\{\left|\phi_{t+\frac{1}{2^{\prime}, 1}}(x)-\phi_{t+\frac{1}{2^{n}, 1}}(z)\right| \geqslant \frac{\epsilon}{2}\right\}}\right] \\
& \leqslant 2 \exp \left(\frac{\|b\|_{\tilde{L}_{p}}^{4} \Gamma^{\prime 2}+K_{2}^{2}\|b\|_{\tilde{L}_{p}}^{2} \Gamma^{\prime}}{K_{1} K_{2}^{2}}\right)\left[\mathbb{P}\left(\left|\phi_{t+\frac{1}{2^{n}, 1}}(x)-\phi_{t+\frac{1}{2^{n}}, 1}(z)\right| \geqslant \frac{\epsilon}{2}\right)\right]^{1 / 2} . \tag{6.17}
\end{align*}
$$

Let $B_{t}(x):=W_{\int_{t}^{1}|\sigma|^{2}\left(\phi_{r}(x)\right) d r}$, then by time change and the fact that for $\kappa_{1}, \kappa_{2} \in \mathbb{R}$

$$
\mathbb{P}\left(W_{t} \geqslant \kappa_{1}\right) \leqslant \frac{1}{2} e^{-\frac{\kappa_{1}^{2}}{2 t}}, \quad \mathbb{P}\left(\sup _{s \leqslant t} W_{s} \geqslant \kappa_{2}\right) \leqslant e^{-\frac{\kappa_{2}^{2}}{2 t}},
$$

we know for $x, z \in \mathbb{R}^{d}$ and $|x-z| \leqslant \delta$ with $\delta>0$

$$
\begin{aligned}
& {\left[\mathbb{P}\left(\left|\phi_{t+\frac{1}{2^{n}}, 1}(x)-\phi_{t+\frac{1}{2^{n}}, 1}(z)\right| \geqslant \frac{\epsilon}{2}\right)\right]^{1 / 2}} \\
& \leqslant\left[\mathbb{P}\left(\left|B_{t+\frac{1}{2^{n}}}(x)-B_{t+\frac{1}{2^{n}}}(z)\right| \geqslant \frac{\epsilon}{2}-\delta\right)\right]^{1 / 2} \\
& \leqslant\left[\exp \left(-\frac{(\epsilon-2 \delta)^{2}}{4} \frac{1}{2\left(\int_{t+\frac{1}{2^{n}}}^{1}|\sigma|^{2}\left(\phi_{r}(x)\right) \mathrm{d} r-\int_{t+\frac{1}{2^{n}}}^{1}|\sigma|^{2}\left(\phi_{r}(x)\right) \mathrm{d} r\right)}\right)\right]^{1 / 2} \\
& \leqslant \exp \left(-\frac{(\epsilon-2 \delta)^{2}}{16} \frac{1}{K_{2}-K_{1}}\right) .
\end{aligned}
$$

Accordingly by (6.17) for any $\epsilon, \delta>0$ and for any $x, z \in \mathbb{R}^{d}$ with $|x-z| \leqslant \delta$ we have

$$
\begin{aligned}
\mathbb{P}\left(\left|\psi_{t+\frac{1}{2^{n}}, 1}(x)-\psi_{t+\frac{1}{2^{n}}, 1}(z)\right| \geqslant \frac{\epsilon}{2}\right) & \leqslant 2 \exp \left(\frac{\|b\|_{\tilde{L}_{p}}^{4} \Gamma^{\prime 2}+K_{2}^{2}\|b\|_{\tilde{L}_{p}}^{2} \Gamma^{\prime}}{K_{1} K_{2}^{2}}-\frac{(\epsilon-2 \delta)^{2}}{16} \frac{1}{K_{2}-K_{1}}\right) \\
& \lesssim \exp \left(-\frac{(\epsilon-2 \delta)^{2}}{16} \frac{1}{K_{2}-K_{1}}\right)
\end{aligned}
$$

corresponding to [7, (4.7)]. Applying the argument from [7, Proof of Theorem 3.1 a )] we get that $P_{2,2}\left(s, T^{1 / \alpha}\right)$ decays super exponentially in $T$, therefore (6.16) holds. The proof is complete.

## Appendix A. Bounds for solutions of elliptic PDEs

Consider the following elliptic equation on $\mathbb{R}^{d}$ (recall the summation convention):

$$
\begin{equation*}
\lambda u-a_{i j} \partial_{i j} u+b \cdot \nabla u=f \tag{A.1}
\end{equation*}
$$

where $\lambda>0, a(\cdot): \mathbb{R}^{d} \rightarrow \mathbb{R}^{d} \otimes \mathbb{R}^{d}$ is a symmetric matrix-valued Borel measurable function, and $b(\cdot): \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ and $f: \mathbb{R}^{d} \rightarrow \mathbb{R}$ are Borel measurable functions such that $f \in \tilde{L}_{p}\left(\mathbb{R}^{d}\right)$ with $p \in(1, \infty)$. The definition of the solution to equation (A.1) is as follows:
Definition A.1. Let $\lambda>0$. We call $u \in \tilde{H}^{2, p}$ a strong solution to (A.1) if for a.e. $x \in \mathbb{R}^{d}$,

$$
\lambda u(x)-a_{i j}(x) \partial_{i j} u(x)+b(x) \cdot \nabla u(x)=f(x) .
$$

We assume
Assumption A.2. $\quad\left(H^{a}\right)$ there exist $0<K_{1} \leqslant K_{2}$ such that for all $x \in \mathbb{R}^{d}$,
$K_{1}|\zeta|^{2} \leqslant\langle a(x) \zeta, \zeta\rangle \leqslant K_{2}|\zeta|^{2}, \quad \forall \zeta \in \mathbb{R}^{d}$,
and $a(\cdot)$ is $\alpha$-Hölder continuous with
$\omega_{\alpha}(a):=\sup _{x, y \in \mathbb{R}^{d}, x \neq y,|x-y| \leqslant 1} \frac{\|a(x)-a(y)\|}{|x-y|^{\alpha}}<\infty$
for some $\alpha \in(0,1]$.
$\left(H^{b}\right) b \in \tilde{L}_{p_{1}}\left(\mathbb{R}^{d}\right)$ for some $p_{1} \in(d, \infty]$.

In this section we will show estimates of the solution of the elliptic PDE above. Such estimates were obtained in [29, Theorem 3.3] in the case where $a$ is uniformly elliptic and uniformly continuous and $b \in L_{p_{1}}$ for some $p_{1}>d$. These estimates were, however, not explicit in terms of the coefficients $a, b$ and $f$. We prove the following theorem which shows this dependence since we need it in the main text but it may also be of independent interest.

Theorem A.3. Suppose Assumption A. 2 holds. There exists a constant $C_{0}>0$ depending on $p, p_{1}, \alpha$ and d only, such that for $\lambda \geqslant C_{0} K_{1}\left(\frac{K_{2}^{2}}{K_{1}^{2}}\left(\frac{K_{1}+\omega_{\alpha}(a)}{K_{1}}\right)^{\frac{2}{\alpha}}+\left(\frac{K_{1}+\omega_{\alpha}(a)}{K_{1}}\right)^{\frac{d}{\alpha} \frac{2}{1-d / p_{1}}}\left(\frac{\|b\|_{\tilde{L}_{p_{1}}}}{K_{1}}\right)^{\frac{2}{1-d / p_{1}}}\right)$ and for any $f \in \tilde{L}_{p}\left(\mathbb{R}^{d}\right)$ with $p \in\left(d / 2 \vee 1, p_{1}\right]$, there is a unique solution $u \in \tilde{H}^{2, p}$ to (A.1). Further, for $p^{\prime} \in[1, \infty]$ there exists a constant $C$ depending on $\alpha, p, d, p^{\prime}$ and $p_{1}$ only, such that

$$
\left.\begin{array}{l}
\left\|\nabla^{2} u\right\|_{\tilde{L}_{p}} \leqslant C \frac{1}{K_{1}}\left(1+\frac{\omega_{\alpha}(a)}{K_{1}}\right)^{d / \alpha}\|f\|_{\tilde{L}_{p}} \\
\lambda^{\left(1+\frac{d}{p^{\prime}}-\frac{d}{p}\right) / 2}\|\nabla u\|_{\tilde{L}_{p^{\prime}}} \leqslant C K_{1}\left(\frac{d}{p^{\prime}}-\frac{d}{p}-1\right) / 2 \\
\left(1+\frac{\omega_{\alpha}(a)}{K_{1}}\right)^{d / \alpha}\|f\|_{\tilde{L}_{p}} \quad \text { if } \quad 1+\frac{d}{p^{\prime}}-\frac{d}{p}>0,  \tag{A.4}\\
\lambda^{\left(2+\frac{d}{p^{\prime}}-\frac{d}{p}\right) / 2}\|u\|_{\tilde{L}_{p^{\prime}}} \leqslant C K_{1}\left(\frac{d}{p^{\prime}}-\frac{d}{p}\right) / 2 \\
K_{1}
\end{array}\right)^{1+\frac{\omega_{\alpha}(a)}{d / \alpha}\|f\|_{\tilde{L}_{p}} \quad \text { if } \quad 2+\frac{d}{p^{\prime}}-\frac{d}{p}>0 .}
$$

Proof. Assume $u \in \tilde{H}^{2, p}$ is a solution to (A.1). We first show the $a$ priori estimates (A.4). Then the continuity method, as shown in [13], is a standard way to conclude the existence and uniqueness of the solution to (A.1) for those $\lambda$ for which (A.4) holds. We divide the proof into three steps. Note that all positive constants $C_{i}, i=1, \cdots$ appearing in the proof only depend on $d, p, p_{1}, p^{\prime}, \alpha$ (and not on $\lambda$, $f, b, a$, and $\left.\omega_{\alpha}(a)\right)$.

Step 1. Assume that $a$ is a constant (positive definite) matrix, $b=0$ and $f \in L_{p}$.
For $\lambda>0$, let $v \in H^{2, p}$ be the solution to the following equation

$$
\lambda v-\Delta v=\tilde{f}, \quad \tilde{f}(x):=f(\sigma x), \quad x \in \mathbb{R}^{d}
$$

where $\sigma$ is the unique positive definite matrix satisfying $\sigma \sigma^{*}=a$. Then $v=(\lambda-\Delta)^{-1} \tilde{f}$ is the unique solution in $H^{2, p}$. From [29, (3.3)] we know that, for each $p^{\prime} \in[1, \infty]$, there are constants $C_{1}, C_{2}, C_{3}$ such that

$$
\begin{align*}
& \left\|\nabla^{2} v\right\|_{L_{p}} \leqslant C_{1}\|\tilde{f}\|_{L_{p}}, \\
& \lambda^{\left(1+\frac{d}{p^{\prime}}-\frac{d}{p}\right) / 2}\|\nabla v\|_{L_{p^{\prime}}} \leqslant C_{2}\|\tilde{f}\|_{L_{p}}, \quad \text { if } \quad 1+\frac{d}{p^{\prime}}-\frac{d}{p}>0, \\
& \lambda^{\left(2+\frac{d}{p^{\prime}}-\frac{d}{p}\right) / 2}\|v\|_{L_{p^{\prime}}} \leqslant C_{3}\|\tilde{f}\|_{L_{p}} \quad \text { if } \quad 2+\frac{d}{p^{\prime}}-\frac{d}{p}>0 . \tag{A.5}
\end{align*}
$$

Let $u(x):=v\left(\sigma^{-1} x\right)$, i.e. $v(x)=u(\sigma x)$. Observe that

$$
\partial_{i} v(x)=\partial_{k} u(\sigma x) \sigma_{k i}, \quad \partial_{i j} v(x)=\partial_{k r} u(\sigma x) \sigma_{k i} \sigma_{r j} .
$$

Therefore

$$
(\lambda-\Delta) v(x)=\left(\lambda-a_{i j} \partial_{i j}\right) u(\sigma x)
$$

and hence $u$ solves (A.1). Uniqueness of a solution under the conditions of Step 1 holds since the map $v \mapsto u$ is a bijection between solutions of the corresponding PDEs. Considering

$$
\frac{1}{K_{1}^{p}}\left\|\nabla^{2} v\right\|_{L_{p}}^{p} \geqslant \operatorname{det} \sigma^{-1}\left\|\nabla^{2} u\right\|_{L_{p}}^{p}, \quad \frac{1}{K_{1}^{p^{\prime} / 2}}\|\nabla v\|_{L_{p^{\prime}}}^{p^{\prime}} \geqslant \operatorname{det} \sigma^{-1}\|\nabla u\|_{L_{p^{\prime}}}^{p^{\prime}}, \quad\|\tilde{f}\|_{L_{p}}^{p}=\operatorname{det} \sigma^{-1}\|f\|_{L_{p}}^{p},
$$

then (A.5) yields

$$
\begin{align*}
&\left\|\nabla^{2} u\right\|_{L_{p}} \leqslant C_{1} \frac{1}{K_{1}}\|f\|_{L_{p}}, \\
& \lambda^{\left(1+\frac{d}{p^{\prime}}-\frac{d}{p}\right) / 2}\|\nabla u\|_{L_{p^{\prime}}} \leqslant C_{2}\left(\operatorname{det} \sigma^{-1}\right)^{\frac{1}{p}-\frac{1}{p^{\prime}}} \frac{1}{\sqrt{K_{1}}}\|f\|_{L_{p}}, \quad \text { if } \quad 1+\frac{d}{p^{\prime}}-\frac{d}{p}>0, \\
& \lambda^{\left(2+\frac{d}{\left.p^{\prime}-\frac{d}{p}\right) / 2}\right.}\|u\|_{L_{p^{\prime}}} \leqslant C_{3}\left(\operatorname{det} \sigma^{-1}\right)^{\frac{1}{p}-\frac{1}{p^{\prime}}}\|f\|_{L_{p}} \quad \text { if } \quad 2+\frac{d}{p^{\prime}}-\frac{d}{p}>0 . \tag{A.6}
\end{align*}
$$

We know that $\operatorname{det} \sigma=\prod_{i=1}^{d} \sqrt{\lambda_{i}}$ where $\lambda_{i}>0, i=1, \cdots, d$, are the eigenvalues of $a$. From (A.2) we get $\lambda_{i} \in\left[K_{1}, K_{2}\right]$. Therefore

$$
\begin{equation*}
\operatorname{det} \sigma^{-1} \in\left[K_{2}^{-\frac{d}{2}}, K_{1}^{-\frac{d}{2}}\right] . \tag{A.7}
\end{equation*}
$$

Using (A.6) and (A.7), we finally get

$$
\begin{align*}
& \left\|\nabla^{2} u\right\|_{L_{p}} \leqslant C_{1} \frac{1}{K_{1}}\|f\|_{L_{p}}, \\
& \lambda^{\left(1+\frac{d}{p^{\prime}}-\frac{d}{p}\right) / 2}\|\nabla u\|_{L_{p^{\prime}}} \leqslant C_{2} K_{1}{ }^{\left(\frac{d}{p^{\prime}}-\frac{d}{p}-1\right) / 2}\|f\|_{L_{p}} \quad \text { if } \quad 1+\frac{d}{p^{\prime}}-\frac{d}{p}>0, \\
& \lambda^{\left(2+\frac{d}{p^{\prime}}-\frac{d}{p}\right) / 2}\|u\|_{L_{p^{\prime}}} \leqslant C_{3} K_{1}\left(\frac{d}{\left.p^{\prime}-\frac{d}{p}\right) / 2}\|f\|_{L_{p}} \quad \text { if } \quad 2+\frac{d}{p^{\prime}}-\frac{d}{p}>0 .\right. \tag{A.8}
\end{align*}
$$

Step 2. a satisfies Assumption A. $2\left(H^{a}\right), b=0$ and $f \in \tilde{L}_{p}$.
Here we apply the freezing coefficient argument. For $\delta>0$ which will be determined later, let $\xi^{\delta}(\cdot):=\xi(\dot{\bar{\delta}})$ where $\xi$ is the same function which we used to define the localized spaces. For $z \in \mathbb{R}^{d}$ denote

$$
\xi^{z, \delta}(x):=\xi^{\delta}(x-z), \quad a^{z}:=a(z), \quad u^{z}(x):=\xi^{z, \delta}(x) u(x), \quad f^{z}(x):=\xi^{z, \delta}(x) f(x)
$$

Observe that

$$
\lambda u^{z}-a_{i j}^{z} \partial_{i j} u^{z}=h_{z}
$$

where

$$
\begin{aligned}
h_{z} & :=f^{z}+\left(a_{i j} \partial_{i j} u\right) \xi^{z, \delta}-a_{i j}^{z} \partial_{i j} u^{z} \\
& =f^{z}+\left(a_{i j}-a_{i j}^{z}\right) \partial_{i j} u \cdot \xi^{z, \delta}-a_{i j}^{z}\left(\partial_{i} u \partial_{j} \xi^{z, \delta}+\partial_{j} u \partial_{i} \xi^{z, \delta}+u \partial_{i j} \xi^{z, \delta}\right) .
\end{aligned}
$$

From [13, p18, 2. Corollary], we know that there exists some $N_{0}>0$ such that for any $\bar{u} \in H^{2, p}$ and $\epsilon>0$ we have

$$
\|\nabla \bar{u}\|_{L_{p}} \leqslant \epsilon\left\|\nabla^{2} \bar{u}\right\|_{L_{p}}+N_{0} \epsilon^{-1}\|\bar{u}\|_{L_{p}} .
$$

Therefore

$$
\begin{align*}
&\left\|h_{z}\right\|_{L_{p}} \leqslant C_{4}\left(\left\|f^{z}\right\|_{L_{p}}\right.\left.+\omega_{\alpha}(a) \delta^{\alpha}\left\|\nabla^{2} u \cdot \xi^{z, \delta}\right\|_{L_{p}}+K_{2}\left\|\nabla u \cdot \nabla \xi^{z, \delta}\right\|_{L_{p}}+K_{2}\left\|u \cdot \nabla^{2} \xi^{z, \delta}\right\|_{L_{p}}\right) \\
& \leqslant C_{4}\left(\left\|f^{z}\right\|_{L_{p}}+2 \omega_{\alpha}(a) \delta^{\alpha}\left\|\nabla^{2}\left(u \cdot \xi^{z, \delta}\right)\right\|_{L_{p}}+\left(K_{2}+2 \omega_{\alpha}(a) \delta^{\alpha}\right)\left\|\nabla u \cdot \nabla \xi^{z, \delta}\right\|_{L_{p}}\right. \\
&\left.+\left(K_{2}+2 \omega_{\alpha}(a) \delta^{\alpha}\right)\left\|u \cdot \nabla^{2} \xi^{z, \delta}\right\|_{L_{p}}\right) \\
& \leqslant C_{5}\left(\left\|f^{z}\right\|_{L_{p}}+2 \omega_{\alpha}(a) \delta^{\alpha}\left\|\nabla^{2} u^{z}\right\|_{L_{p}}+\left(K_{2}+2 \omega_{\alpha}(a) \delta^{\alpha}\right) \delta^{-1}\left\|\nabla u \cdot \xi^{z, \delta}\right\|_{L_{p}}\right. \\
&\left.+\left(K_{2}+2 \omega_{\alpha}(a) \delta^{\alpha}\right) \delta^{-2}\left\|u \cdot \xi^{z, \delta}\right\|_{L_{p}}\right) \\
& \leqslant C_{5}\left(\left\|f^{z}\right\|_{L_{p}}+2 \omega_{\alpha}(a) \delta^{\alpha}\left\|\nabla^{2} u^{z}\right\|_{L_{p}}+\left(K_{2}+2 \omega_{\alpha}(a) \delta^{\alpha}\right) \delta^{-1}\left(\left\|\nabla u^{z}\right\|_{L_{p}}+\left\|u \cdot \nabla \xi^{z, \delta}\right\|_{L_{p}}\right)\right. \\
&\left.+\left(K_{2}+2 \omega_{\alpha}(a) \delta^{\alpha}\right) \delta^{-2}\left\|u \cdot \xi^{z, \delta}\right\|_{L_{p}}\right) \\
& \leqslant C_{6}\left(\left\|f^{z}\right\|_{L_{p}}+\left(2 \omega_{\alpha}(a) \delta^{\alpha}+\epsilon\left(K_{2}+2 \omega_{\alpha}(a) \delta^{\alpha}\right) \delta^{-1}\right)\left\|\nabla^{2} u^{z}\right\|_{L_{p}}\right. \\
&\left.+\left(K_{2}+2 \omega_{\alpha}(a) \delta^{\alpha}\right)\left(\epsilon^{-1} \delta^{-1}+\delta^{-2}\right)\left\|u \cdot \xi^{z, \delta}\right\|_{L_{p}}\right) \tag{A.9}
\end{align*}
$$

where $\omega_{\alpha}(a)$ is from (A.3). Assuming (without loss of generality) that $C_{6} \geqslant 1 / 6$, we define

$$
\begin{equation*}
\delta:=\left(\frac{K_{1}}{6 C_{6}\left(K_{1}+2 \omega_{\alpha}(a)\right)}\right)^{1 / \alpha}<1, \quad \epsilon:=\frac{K_{1} \delta}{6 C_{6}\left(K_{2}+2 \omega_{\alpha}(a) \delta^{\alpha}\right)} . \tag{A.10}
\end{equation*}
$$

It is easy to see that $C_{6} \frac{1}{K_{1}}\left(2 \omega_{\alpha}(a) \delta^{\alpha}+\epsilon\left(K_{2}+2 \omega_{\alpha}(a) \delta^{\alpha}\right) \delta^{-1}\right)<\frac{1}{2}$, and

$$
\left(K_{2}+2 \omega_{\alpha}(a) \delta^{\alpha}\right)\left(\epsilon^{-1} \delta^{-1}+\delta^{-2}\right) \leqslant C_{7} \frac{K_{2}^{2}}{K_{1}}\left(\frac{K_{1}+\omega_{\alpha}(a)}{K_{1}}\right)^{\frac{2}{\alpha}}
$$

So we get from (A.8) and (A.9) that

$$
\begin{equation*}
\left\|\nabla^{2} u^{z}\right\|_{L^{p}} \leqslant C_{8} \frac{1}{K_{1}}\left(\left\|f^{z}\right\|_{L_{p}}+\frac{K_{2}^{2}}{K_{1}}\left(\frac{K_{1}+\omega_{\alpha}(a)}{K_{1}}\right)^{\frac{2}{\alpha}}\left\|u^{z}\right\|_{L_{p}}\right) . \tag{A.11}
\end{equation*}
$$

Plugging this into (A.9) yields

$$
\left\|h_{z}\right\|_{L_{p}} \leqslant C_{6}\left(\left\|f^{z}\right\|_{L_{p}}+\frac{C_{8}}{2 C_{6}}\left(\left\|f^{z}\right\|_{L_{p}}+\frac{K_{2}^{2}}{K_{1}}\left(\frac{K_{1}+\omega_{\alpha}(a)}{K_{1}}\right)^{\frac{2}{\alpha}}\left\|u^{z}\right\|_{L_{p}}\right)+C_{7} \frac{K_{2}^{2}}{K_{1}}\left(\frac{K_{1}+\omega_{\alpha}(a)}{K_{1}}\right)^{\frac{2}{\alpha}}\left\|u^{z}\right\|_{L_{p}}\right)
$$

Using the second inequality in (A.8) we get for $1+\frac{d}{p^{\prime}}-\frac{d}{p}>0$

$$
\begin{equation*}
\lambda^{\left(1+\frac{d}{p^{\prime}}-\frac{d}{p}\right) / 2}\left\|\nabla u^{z}\right\|_{L^{p^{\prime}}} \leqslant C_{9} K_{1}\left(\frac{d}{p^{\prime}}-\frac{d}{p}-1\right) / 2\left(\left\|f^{z}\right\|_{L_{p}}+\frac{K_{2}^{2}}{K_{1}}\left(\frac{K_{1}+\omega_{\alpha}(a)}{K_{1}}\right)^{\frac{2}{\alpha}}\left\|u^{z}\right\|_{L_{p}}\right) . \tag{A.12}
\end{equation*}
$$

Similarly, for $2+\frac{d}{p^{\prime}}-\frac{d}{p}>0$

$$
\begin{equation*}
\lambda^{\left(2+\frac{d}{p^{\prime}}-\frac{d}{p}\right) / 2}\left\|u^{z}\right\|_{L_{p^{\prime}}} \leqslant C_{10} K_{1}\left(\frac{d}{p^{\prime}}-\frac{d}{p}\right) / 2\left(\left\|f^{z}\right\|_{L_{p}}+\frac{K_{2}^{2}}{K_{1}}\left(\frac{K_{1}+\omega_{\alpha}(a)}{K_{1}}\right)^{\frac{2}{\alpha}}\left\|u^{z}\right\|_{L_{p}}\right) . \tag{A.13}
\end{equation*}
$$

Let $p^{\prime}=p$. Then

$$
\begin{equation*}
\lambda\left\|u^{z}\right\|_{L_{p}} \leqslant C_{10}\left(\left\|f^{z}\right\|_{L_{p}}+\frac{K_{2}^{2}}{K_{1}}\left(\frac{K_{1}+\omega_{\alpha}(a)}{K_{1}}\right)^{\frac{2}{\alpha}}\left\|u^{z}\right\|_{L_{p}}\right) . \tag{A.14}
\end{equation*}
$$

Taking $\lambda \geqslant 2 C_{10} \frac{K_{2}^{2}}{K_{1}}\left(\frac{K_{1}+\omega_{\alpha}(a)}{K_{1}}\right)^{\frac{2}{\alpha}}=: C_{10} K$ we obtain

$$
\left\|u^{z}\right\|_{L_{p}} \leqslant \frac{C_{10}}{\lambda-C_{10} \frac{K_{2}^{2}}{K_{1}}\left(\frac{K_{1}+\omega_{\alpha}(a)}{K_{1}}\right)^{\frac{2}{\alpha}}}\left\|f^{z}\right\|_{L_{p}}, \quad \frac{K_{2}^{2}}{K_{1}}\left(\frac{K_{1}+\omega_{\alpha}(a)}{K_{1}}\right)^{\frac{2}{\alpha}}\left\|u^{z}\right\|_{L_{p}} \leqslant\left\|f^{z}\right\|_{L_{p}}
$$

Together with (A.11), (A.13), and (A.12), we have

$$
\begin{align*}
&\left\|\nabla^{2} u^{z}\right\|_{L^{p}} \leqslant C_{12} \frac{1}{K_{1}}\left\|f^{z}\right\|_{L_{p}}, \\
& \lambda^{\left(1+\frac{d}{p^{\prime}}-\frac{d}{p}\right) / 2}\left\|\nabla u^{z}\right\|_{L^{p^{\prime}}} \leqslant C_{13} K_{1}^{\left(\frac{d}{p^{\prime}}-\frac{d}{p}-1\right) / 2}\left\|f^{z}\right\|_{L_{p}}, \quad \text { if } \quad 1+\frac{d}{p^{\prime}}-\frac{d}{p}>0, \\
& \lambda^{\left(2+\frac{d}{p^{\prime}}-\frac{d}{p}\right) / 2}\left\|u^{z}\right\|_{L_{p^{\prime}}} \leqslant C_{14} K_{1}\left(\frac{d}{\left.p^{\prime}-\frac{d}{p}\right) / 2}\left\|f^{z}\right\|_{L_{p}}, \quad \text { if } \quad 2+\frac{d}{p^{\prime}}-\frac{d}{p}>0 .\right. \tag{A.15}
\end{align*}
$$

From definition (2.1) we know that, for each $z \in \mathbb{R}^{d},\left\|u^{z}\right\|_{L_{p}} \leqslant\|u\|_{\tilde{L}_{p}} \leqslant \delta^{-d} \sup _{\bar{z}}\left\|u^{\bar{z}}\right\|_{L_{p}}{ }^{1}$, so we get from (A.15) that for any $\lambda \geqslant C_{10} \kappa$ we have

$$
\begin{align*}
\lambda^{\left(2+\frac{d}{p^{\prime}}-\frac{d}{p}\right) / 2}\|u\|_{\tilde{L}_{p^{\prime}}} & \leqslant C_{15} K_{1}^{\left(\frac{d}{p^{\prime}}-\frac{d}{p}\right) / 2} \delta^{-d}\|f\|_{\tilde{L}_{p}} \quad \text { if } \quad 2+\frac{d}{p^{\prime}}-\frac{d}{p}>0 \\
\lambda^{\left(1+\frac{d}{p^{\prime}}-\frac{d}{p}\right) / 2}\|\nabla u\|_{\tilde{L}_{p^{\prime}}} & \leqslant \lambda^{\left(1+\frac{d}{p^{\prime}}-\frac{d}{p}\right) / 2} \sup _{z}\left(\left\|\nabla u^{z}\right\|_{L_{p^{\prime}}}+\left\|u \nabla \xi^{z, 1}\right\|_{L_{p^{\prime}}}\right) \\
& \leqslant C_{16}\left(K_{1}^{\left(\frac{d}{p^{\prime}}-\frac{d}{p}-1\right) / 2}+\lambda^{-1 / 2} K_{1}\left(\frac{d}{\left.p^{\prime}-\frac{d}{p}\right) / 2}\right) \delta^{-d}\|f\|_{\tilde{L}_{p}}\right. \\
& \leqslant C_{17} K_{1}^{\left(\frac{d}{p^{\prime}} \frac{d}{p}-1\right) / 2} \delta^{-d}\|f\|_{\tilde{L}_{p}} \quad \text { if } \quad 1+\frac{d}{p^{\prime}}-\frac{d}{p}>0, \\
\left\|\nabla^{2} u\right\|_{\tilde{L}^{p}} & \leqslant \sup _{z}\left(\left\|\nabla^{2} u^{z}\right\|_{L_{p}}+\left\|u \nabla^{2} \xi^{z, 1}\right\|_{L_{p}}+2\left\|\nabla u \nabla \xi^{z, 1}\right\|_{L_{p}}\right) \\
& \left.\leqslant C_{18}\left(\frac{1}{K_{1}}+\lambda^{-1}+\lambda^{-1 / 2} K_{1}^{-1 / 2}\right) \delta^{-d}\|f\|_{\tilde{L}_{p}}\right) \\
& \leqslant C_{19} \frac{1}{K_{1}} \delta^{-d}\|f\|_{\tilde{L}_{p}} . \tag{A.16}
\end{align*}
$$

Step 3. $a$ is Hölder continuous and Assumption A. $2\left(H^{a}\right)$ holds, $|b| \in \tilde{L}_{p_{1}}$ and $f \in \tilde{L}_{p}$.
By (A.16) and Hölder's inequality, we have for $\lambda \geqslant C_{10} \kappa$ and $1+\frac{d}{p^{\prime}}-\frac{d}{p}>0$

$$
\lambda^{\left(1+\frac{d}{p^{\prime}}-\frac{d}{p}\right) / 2}\|\nabla u\|_{\tilde{L}_{p^{\prime}}} \leqslant C_{17} K_{1}{ }^{\left(\frac{d}{p^{\prime}}-\frac{d}{p}-1\right) / 2} \delta^{-d}\|f+b \cdot \nabla u\|_{\tilde{L}_{p}}
$$

[^0]$$
\leqslant C_{17} K_{1}\left(\frac{d}{p^{\prime}}-\frac{d}{p}-1\right) / 2 \delta^{-d}\left(\|f\|_{\tilde{L}_{p}}+\|b\|_{\tilde{L}_{p_{1}}}\|\nabla u\|_{\tilde{L}_{p_{2}}}\right)
$$
where $p_{1}, p_{2} \in(p, \infty)$ and $\frac{1}{p_{1}}+\frac{1}{p_{2}}=\frac{1}{p}$. Let $p^{\prime}=p_{2}$. Then we get
$$
\lambda^{\left(1-\frac{d}{p_{1}}\right) / 2}\|\nabla u\|_{\tilde{L}_{p_{2}}} \leqslant C_{20} K_{1}{ }^{\left(-\frac{d}{p_{1}}-1\right) / 2} \delta^{-d}\left(\|f\|_{\tilde{L}_{p}}+\|b\|_{\tilde{L}_{p_{1}}}\|\nabla u\|_{\tilde{L}_{p_{2}}}\right) .
$$

Choosing $\lambda$ so large such that

$$
\lambda^{\left(1-\frac{d}{p_{1}}\right) / 2} \geqslant C_{20} K_{1} \frac{-d / p_{1}-1}{2} \delta^{-d}\|b\|_{\tilde{L}_{p_{1}}},
$$

we get

$$
\|\nabla u\|_{\tilde{L}_{p_{2}}} \leqslant \frac{C_{20} K_{1}{ }^{\left(-\frac{d}{p_{1}}-1\right) / 2}}{\lambda^{\left(1-\frac{d}{p_{1}}\right) / 2}-C_{20} K_{1} \frac{-d / p_{1}-1}{2}} \delta^{-d}\|b\|_{\tilde{L}_{p_{1}}} \delta^{-d}\|f\|_{\tilde{L}_{p}} .
$$

Moreover,

$$
\|b \cdot \nabla u\|_{\tilde{L}_{p}} \leqslant \frac{C_{20} K_{1}{ }^{\left(-\frac{d}{p_{1}}-1\right) / 2} \delta^{-d}\|b\|_{\tilde{L}_{p_{1}}}}{\lambda^{\left(1-\frac{d}{p_{1}}\right) / 2}-C_{20} K_{1}{ }^{\left(-\frac{d}{p_{1}}-1\right) / 2} \delta^{-d}\|b\|_{\tilde{L}_{p_{1}}}}\|f\|_{\tilde{L}_{p}}=: \gamma\|f\|_{\tilde{L}_{p}} .
$$

Using (A.16) we see that for any $\lambda$ such that $\lambda \geqslant C_{10} \kappa$ and $\lambda^{\left(1-\frac{d}{p_{1}}\right) / 2} \geqslant C_{20} K_{1} \frac{-1-d / p_{1}}{2} \delta^{-d}\|b\|_{\tilde{L}_{p_{1}}}$, we have

$$
\begin{gathered}
\left\|\nabla^{2} u\right\|_{\tilde{L}_{p}} \leqslant C_{21}(1+\gamma) \delta^{-d} \frac{1}{K_{1}}\|f\|_{\tilde{L}_{p}} \\
\lambda^{\left(1+\frac{d}{p^{\prime}}-\frac{d}{p}\right) / 2}\|\nabla u\|_{\tilde{L}_{p^{\prime}}} \leqslant C_{22}(1+\gamma) K_{1}\left(\frac{d}{\left.p^{\prime}-\frac{d}{p}-1\right) / 2} \delta^{-d}\|f\|_{\tilde{L}_{p}} \quad \text { if } \quad 1+\frac{d}{p^{\prime}}-\frac{d}{p}>0,\right. \\
\lambda^{\left(2+\frac{d}{p^{\prime}}-\frac{d}{p}\right) / 2}\|u\|_{\tilde{L}_{p^{\prime}}} \leqslant C_{23}(1+\gamma) K_{1}\left(\frac{d}{\left.p^{\prime}-\frac{d}{p}\right) / 2} \delta^{-d}\|f\|_{\tilde{L}_{p}} \quad\right. \text { if } \\
2+\frac{d}{p^{\prime}}-\frac{d}{p}>0 .
\end{gathered}
$$

Define $C_{24}:=\left(2 C_{10}\right) \vee C_{20}$. Then, for $\lambda \geqslant C_{24} \kappa$ and $C_{24} \lambda^{-\left(1-\frac{d}{p_{1}}\right) / 2} K_{1}{ }^{\left(-\frac{d}{p_{1}}-1\right) / 2} \delta^{-d}\|b\|_{\tilde{L}_{p_{1}}}<\frac{1}{2}$ (i.e. $\left.\lambda \geqslant C_{24} K_{1}\left(\delta^{-d} \frac{\|b\|_{\tilde{L}_{p}}}{K_{1}}\right)^{\frac{2}{1-d / p_{1}}}\right)$ by taking $\lambda \geqslant C_{24} K_{1}\left(\frac{K_{2}^{2}}{K_{1}^{2}}\left(\frac{K_{1}+\omega_{\alpha}(a)}{K_{1}}\right)^{\frac{2}{\alpha}}+\left(\frac{K_{1}+\omega_{\alpha}(a)}{K_{1}}\right)^{\frac{d}{\alpha} \frac{2}{1-d / p_{1}}}\left(\frac{\|b\|_{\tilde{L}_{p}}}{K_{1}}\right)^{\frac{2}{1-d / p_{1}}}\right)$ we get that there exists finite positive constant $C_{25}$ such that $1+\gamma \leqslant C_{25}$, which finally shows the desired result (A.4) after plugging in the value of $\delta$ from (A.10).

Corollary A.4. Let Assumption A. 2 hold and $f=b^{i}, i=1, \cdots, d$ in (A.1), let $p^{\prime} \in[1, \infty]$. There exists some $C_{0}>0$ depending on $\alpha, p_{1}$ and d only, such that if we choose $\lambda \geqslant C_{0} K_{1}\left(\frac{K_{2}^{2}}{K_{1}^{2}}\left(\frac{K_{1}+\omega_{\alpha}(a)}{K_{1}}\right)^{\frac{2}{\alpha}}+\right.$ $\left.\left(\frac{K_{1}+\omega_{\alpha}(a)}{K_{1}}\right)^{\frac{d}{\alpha} \frac{2}{1-d / p_{1}}}\left(\frac{\|b\|_{\tilde{L}_{p}}}{K_{1}}\right)^{\frac{2}{1-d / p_{1}}}\right)$ then for the solution $u^{i}$ to equation (A.1) we have

$$
\begin{gather*}
\left\|\nabla u^{i}\right\|_{\tilde{L}_{p^{\prime}}} \leqslant \frac{1}{2} \lambda^{-\frac{d}{2 p^{\prime}}} K_{1}^{\frac{d}{2 p^{\prime}}} \leqslant \frac{1}{2} \quad \text { if } \\
\|u\|_{\tilde{L}_{p^{\prime}}} \leqslant \frac{1}{2} \lambda^{-\frac{1+d / p^{\prime}}{2}} K_{1}^{\frac{1+d / p^{\prime}}{2}} \leqslant \frac{d}{p^{\prime}}-\frac{d}{p}>0,  \tag{A.17}\\
\text { if }
\end{gather*} 2+\frac{d}{p^{\prime}}-\frac{d}{p}>0 . .
$$

Proof. Notice that for such $\lambda$ we have $C_{0} \lambda^{-\left(1-\frac{d}{p_{1}}\right) / 2} K_{1}{ }^{\left(-\frac{d}{p_{1}}-1\right) / 2}\left(\frac{K_{1}+\omega_{\alpha}(a)}{K_{1}}\right)^{\frac{d}{\alpha}}\|b\|_{\tilde{L}_{p_{1}}}<\frac{1}{2}$, so by (A.4) for $f=b^{i}$,

$$
\begin{gathered}
\left\|\nabla u^{i}\right\|_{\tilde{L}_{p^{\prime}}} \leqslant C \lambda^{\frac{-1-d / p^{\prime}+d / p_{1}}{2}} K_{1} \frac{-1-d / p_{1}+d / p^{\prime}}{2}\left(\frac{K_{1}+\omega_{\alpha}(a)}{K_{1}}\right)^{\frac{d}{\alpha}}\left\|b^{i}\right\|_{\tilde{L}_{p_{1}}} \leqslant \frac{1}{2} \lambda^{-\frac{d}{2 p^{\prime}}} K_{1}^{\frac{d}{2 p^{\prime}}} \leqslant \frac{1}{2} . \\
\text { With the similar argument we get }\|u\|_{\tilde{L}_{p^{\prime}}} \leqslant C \lambda^{\frac{-2-d / p^{\prime}+d / p_{1}}{2}} K_{1} \frac{d / p^{\prime}-d / p_{1}}{2}\left(\frac{K_{1}+\omega_{\alpha}(a)}{K_{1}}\right)^{\frac{d}{\alpha}}\|b\|_{\tilde{L}_{p_{1}}} \leqslant \frac{1}{2} \lambda^{-\frac{1+d / p^{\prime}}{2}} K_{1}^{\frac{1+d / p^{\prime}}{2}} .
\end{gathered}
$$

## Acknowledgments

Inspiring suggestion from and fruitful discussions with Benjamin Gess (Bielefeld) are appreciated. Discussions with Xicheng Zhang (Beijing) and Zimo Hao (Bielefeld) are acknowledged. CL is supported by the DFG through the research unit (Forschergruppe) FOR 2402 and the Austrian Science Fund (FWF) via the project "Regularisation by noise in discrete and continuous systems" .

## References

[1] L. Arnold: Random Dynamical Systems. Springer, Berlin (1998).
[2] I. Chueshov and M. Scheutzow: On the structure of attractors and invariant measures for a class of monotone random systems. Dyn. Syst. 19 (2004) 127-144.
[3] M. Cranston, M. Scheutzow and D. Steinsaltz: Linear expansion of isotropic Brownian flows, Elect. Comm. in Probab. 4 (1999) 91-101.
[4] M. Cranston, M. Scheutzow and D. Steinsaltz: Linear bounds for stochastic dispersion, Ann. Probab. 28 (2000) 1852-1869.
[5] H. Crauel, G. Dimitroff and M. Scheutzow: Criteria for strong and weak random attractors. 7. Dynamics and Diff. Equations, 21 (2009) 233-247.
[6] H. Crauel and F. Flandoli: Attractors for random dynamical systems. Probab. Theory Relat. Fields $\mathbf{1 0 0}$ (1994) 365-393.
[7] G. Dimitroff and M. Scheutzow: Attractors and expansion for Brownian flows. Electronic F. Probab. 16 (2011) 1193-1213.
[8] F. Flandoli, B. Gess and M. Scheutzow: Synchronization by noise. Probab. Theory Relat. Fields 168 (2017) 511-556.
[9] F. Flandoli, B. Gess and M. Scheutzow: Synchronization by noise for order-preserving random dynamical systems. Ann. Probab. 45 (2017) 1325-1350.
[10] B. Gess: Random attractors for stochastic porous media equations perturbed by space-time linear multiplicative noise. Ann. Probab. 42 (2014) 818-864.
[11] B. Gess, W. Liu and A. Schenke: Random attractors for locally monotone stochastic partial differential equations. 7 . Differential Equations 269 (2020) 414-3455.
[12] L. Galeati and C. Ling: Stability estimates for singular SDEs and applications. Arxiv preprint https://arxiv.org/abs/2208.03670. (2022).
[13] N. V. Krylov: Lectures on Elliptic and Parabolic Equations in Sobolev spaces. American Mathematical Society (2008).
[14] N. V. Krylov and M. Röckner: Strong solutions of stochastic equations with singular time dependent drift. Probab. Theory Relat. Fields 131 (2005) 154-196.
[15] C. Kuehn, A. Neamtu and S. Sonner: Random attractors via pathwise mild solutions for parabolic stochastic evolution equations. Journal of Evolution Equations 21 (2021) 2631-2663.
[16] K. Lê: Quantitative John-Nirenberg inequality for stochastic processes of bounded mean oscillation. Arxiv preprint https://arxiv.org/pdf/2210.15736. (2022)
[17] K. Lê and C. Ling: Taming singular stochastic differential equations: A numerical method. Arxiv preprint https://arxiv.org/abs/2110.01343. (2021).
[18] C. Ling, M. Scheutzow and I. Vorkastner: The perfection of local semi-flows and local random dynamical systems with applications to SDEs. Stoch. Dyn. 22 (2022).
[19] H. Lisei and M. Scheutzow: Linear bounds and Gaussian tails in a stochastic dispersion model. Stoch. Dyn. 1 (2001) 389-403.
[20] H. Lisei and M. Scheutzow, M.: On the dispersion of sets under the action of an isotropic Brownian flow. In: Proceedings of the Swansea 2002 Workshop Probabilistic Methods in Fluids, World Scientific (2003) 224-238.
[21] M. Scheutzow: Chaining techniques and their application to stochastic flows. In: Trends in Stochastic Analysis, eds: Blath, ‥, Mörters, P., Scheutzow, M., Cambridge University Press, 35-63 (2009).
[22] M. Scheutzow: A stochastic Gronwall lemma. Infin. Dimens. Anal. Quantum Probab. Relat. Top. 16 (2013) 1350019, 4 p .
[23] A. Yu. Veretennikov: On the strong solutions of stochastic differential equations. Theory Probab. Appl. 24 (1979), 354-366.
[24] P. Xia, L. Xie, X. Zhang and G. Zhao: $L^{q}\left(L^{p}\right)$-theory of stochastic differential equations. Stochastic Process. Appl. 130 (2020) 5188-5211.
[25] L. Xie and X. Zhang: Sobolev differentiable flows of SDEs with local Sobolev and super-linear growth coefficients. Ann. Probab. 22 (2016) 3661-3687.
[26] L. Xie and X. Zhang: Ergodicity of stochastic differential equations with jumps and singular coefficients. Ann. Inst. H. Poincaré Probab. Statist. 56 (2020) 175-229.
[27] X. Zhang: Stochastic homeomorphism flows of SDE with singular drifts and Sobolev diffusion coefficients. Electron. 7. Probab. 16 (2011) 1096-1116.
[28] X. Zhang: Stochastic differential equations with Sobolev diffusion and singular drift and applications. Ann. Appl. Probab. 26 (2016) 2697-2732.
[29] X. Zhang and G. Zhao: Singular Brownian diffusion processes. Commun. Math. Stat. 6 (2018) 533-581.
[30] X. Zhang and G. Zhao: Stochastic Lagrangian path for Leray's solutions of 3D Navier-Stokes equations. Commun. Math. Phys 381 (2021) 491-525.
[31] R. Zhu and X. Zhu: Random attractor associated with the quasi-geostrophic equation. fournal of Dynamics and Differential Equations 29 (2017) 289-322.
[32] A. K. Zvonkin: A transformation of the phase space of a diffusion process that removes the drift. Math. Sbornik 135 (1974) 129-149.

Chengcheng Ling: Technische Universität Wien, Institute of Analysis and Scientific Computing, 1040 Wien, Austria, Email: chengcheng.ling@asc.tuwien.ac.at

Michael Scheutzow: Technische Universität Berlin, Fakultät II, Institut für Mathematik, 10623 Berlin, Germany, Email: ms@math.tu-berlin.de


[^0]:    ${ }^{1}$ Recall that in Section 2 we assumed that the localized spaces are defined using the function $\xi^{1}$

