

Conformal transformations and integrable mechanical billiards

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1. Introduction

1.1. General setting of mechanical billiard systems

The dynamics of billiards in the plane in which a particle moves freely along straight lines in a “billiard table” and reflects elastically at a reflection wall is a widely-studied subject. In this paper we study a type of variants of such systems, namely planar mechanical billiards, in which the particle is assumed to move under the additional influence of a conservative force field derived from a potential.

Our general setting is the following: We consider a mechanical system on two-dimensional Riemannian manifold (M, g) with a force function $U : M \rightarrow \mathbb{R}$. The negative of the force function gives the potential $V = -U$ of the system. The dynamics is given by the corresponding second-order Newton’s equation

$$\nabla_{\dot{q}} \dot{q} = \nabla_g U(q), \quad q \in M,$$

in which ∇ is the Levi-Civita connection of g . The use of the force function over the potential avoids the negative sign in the right hand side of the above equation. Moreover, we assume that the motion is elastically reflected against a C^1 -smooth curve $\mathcal{B} \subset M$. This then defines a billiard system when we specify (when necessary) a component of $M \setminus \mathcal{B}$ as a billiard table where the motions of interest take place. We shall not need this specification for the purpose of this article. We thus define the corresponding mechanical billiard system as the quadruplet $(M, g, U, \mathcal{B})$.

Note that as compared to the case of free billiards, it is not always necessary to assume that the billiard table is bounded in order for the billiard mapping to be well-defined, for example when the force forces the trajectories to meet the reflection wall $\mathcal{B}$ again. Moreover, in such cases we may as well remove part of $\mathcal{B}$ which may possibly lead to a still well-defined, albeit discontinuous billiard mapping.

A first integral of the system $(M, g, U, \mathcal{B})$ is a first integral of (M, g, U) which is invariant under the reflections at $\mathcal{B}$. The energy $E = T - U$ is always a first integral of

the system. As we are in dimension 2, such a system is called integrable if there exists another first-integral of this system functionally independent of E .

Due to the conservation of energy E , we can moreover restrict the mechanical billiard system to an energy hypersurface $\{E = e\}$. We denote the corresponding billiard system by $(M, g, U, \mathcal{B}, e)$. Accordingly, this restricted system is called integrable if there exists an additional non-trivial first integral of the system defined on $\{E = e\}$. In this article, we primarily use this definition of integrability since it is natural to fix its energy when we consider a billiard system.

The free motion case ($U = 0$) corresponds to the classical free billiards. In this case, any of its positive energy hypersurfaces carry the same dynamics. In contrast to this, general mechanical systems can have essentially different behaviors on different energy surfaces and analogously also the mechanical billiard systems. Therefore it is often necessary to specify the energy value e or the subset of possible energy values $\mathcal{E}$ under consideration. We write $(M, g, U, \mathcal{B}, \mathcal{E})$ to emphasize also the region of energy under consideration. Such a system is integrable if the system is integrable for all $e \in \mathcal{E}$. On the other hand, a “reflection wall” $\mathcal{B}$ such that $(M, g, U, \mathcal{B}, \mathcal{E})$ is integrable, is called an integrable reflection wall for the mechanical system $(M, g, U, \mathcal{B}, \mathcal{E})$. Note that for the discussion of integrability, we do not require that the billiard mapping be always well-defined.

Already in the free billiard case with no additional force, billiard systems may carry rich dynamics and offer the class of examples illustrating many dynamical phenomena [32]. The book [21] also discusses several aspects of mechanical billiards.

1.2. Known examples of integrable mechanical billiard systems

For free motion in 2-dimensional plane $\mathbb{R}^2$, there are two types of integrable billiard systems. The simplest one is the one with a circular reflection wall. In this case, one can easily see that the angle of reflection is preserved, hence it is an additional first integral. The second example is provided with an elliptic reflection wall. The integrability of such a system has been shown by Birkhoff [3]. This integrability can be generalized in the case of free motions in 2-dimensional sphere $\mathbb{S}^2$ and the hyperbolic space $\mathbb{H}^2$, in which circular and elliptic reflection walls are also integrable [34][30][31]. Additionally, a conjecture attributed to Birkhoff and Poritsky states that any closed convex reflection wall of an integrable billiard system is either a circle or an ellipse [27]. This conjecture has not been fully proven yet, but there are important progresses recently made [16][17][1]. Also, an algebraic version of the conjecture for billiards on the plane and constant curvature surfaces has recently been proved by Glutsyuk [12][13].

Many examples of integrable mechanical billiard systems with the presence of a non-constant potential function have been identified as well. We start our list with a class of relatively easy examples: In a central force problem in $\mathbb{R}^2$, in which U is a function of $|q|$ only, then circles with center at O and lines passing through the center O are integrable reflection walls: In both cases, it is direct to check that the

norm of the angular momentum is preserved under reflections at these reflection walls. The very same argument works also on the sphere $\mathbb{S}^2$, and on the hyperbolic plane $\mathbb{H}^2$.

A number of integrable mechanical billiards are defined for the Kepler problem and the Hooke problem, with respectively force functions of the forms $U = \frac{s}{r}$ and $U = fr^2$, where r is the distance of the particle from a fixed center $O \in \mathbb{R}^2$ and the factors $f, s \in \mathbb{R}$ can take both signs, allowing both attractive and repulsive forces.

In the Hooke problem, it is direct to see that any line is integrable. Centered conic sections are also integrable, for which the case of a centered ellipse follows from the classical work of Jacobi on the integrability of a quadratic radial potential of the form r^2 restricted to a triaxial ellipsoid in $\mathbb{R}^3$, by letting one of the axes of the ellipsoid tends to zero [15][7]. Additionally, the integrability of two centered confocal elliptic reflection walls is shown by Pustovoitov in [28]. Later, by the same author, the integrability of reflection walls consisting of centered confocal ellipses and centered confocal hyperbola is also established [29]. In addition, the centered elliptic reflection walls are integrable for certain potentials given by certain polynomials of even degrees in $\mathbb{R}^2$ [21][36].

The case of integrable mechanical billiards for the Kepler problem in $\mathbb{R}^2$ has been first noticed by Panov [26] based on the use of the planer Kepler-Hooke correspondence ultimately related to this paper.

The Kepler problem in $\mathbb{R}^2$ with a line not passing through the attractive center is contained in a class of mechanical billiard systems proposed by Boltzmann in [4], who expected such systems to be ergodic and in particular non-integrable. Based on a close examination of Boltzmann's argument and some numerical investigations, Gallavotti has conjectured that the contrary is actually true, namely this mechanical billiard system should actually be integrable. This has been confirmed by Gallavotti and Jauslin in [11], with alternative proofs in [8] and [37]. Moreover, such systems can be generalized to $\mathbb{S}^2$ and $\mathbb{H}^2$ [37].

It has also been known that a parabolic reflection wall whose focus is at the origin is integrable for the Stark problem in which the force function is a linear combination of a Kepler and a uniform gravitational force function $U = gy$ with constant $g \in \mathbb{R}$ [19]. This result has significance in optics, and such a parabolic mirror has been constructed in experiments [9]. In Section 3, we shall give a short alternative proof of the theorem of [19] as well as bring certain extensions.

More recently, for the planar system with the force function $U := \frac{k}{2}(x^2 + y^2) + \frac{\alpha^2}{2x^2} + \frac{\beta^2}{2y^2}$, Kobtsev showed that any centered ellipse with semi-axis a, b forms an integrable reflection wall [18]. This is a particular case of the result by Kozlov [20], who actually showed that the case with two additional Kepler centers at the foci added to the force function does not change the integrability.

The integrable dynamics of some of these integrable mechanical billiards have been extensively investigated as well. For this, we refer to [10] and the references therein.

1.3. Purpose of this article

These examples of integrable mechanical billiards have been found independently under different contexts. In this article, our main goal is to illustrate how conformal transformations transform integrable mechanical billiard systems.

As an application, we shall start by showing that via conformal transformations one gets from integrable free billiards in the plane some classes of planar immersed curves which are integrable reflection walls for certain central force problems in the plane on its zero-energy level. The complexity of these curves makes us wonder whether this simple corollary admits different but as simple solutions, if we first fix the force function and ask to identify these integrable reflection walls.

We shall then apply the well-known complex square mapping, [24][14][23][22] which induces a duality between the Hooke and the Kepler problems, to obtain a duality between integrable Hooke and Kepler billiards as indicated in [26]. The same transformation also leads to many new classes of integrable mechanical billiards in systems similar to the Stark problem. We shall also apply a closely-related conformal mapping due to Birkhoff to Euler's two-center problem and identify its integrable reflection walls.

In this way many known examples of integrable mechanical billiards are related. Besides, we have also identified some classes of integrable billiards which we think are new, namely

- hyperbola focused at the center for the Kepler billiards;
- well-oriented parabola focused at the center for Stark-type billiards;
- confocal conic sections for Euler's two-center problem.
- Moreover, some of these integrable conic section reflection walls in the Kepler and in the two-center problem are allowed to be combined when they are confocal.

We organize this article as follows:

In Section 2, we introduce conformal transformations between mechanical billiard systems. In particular, we explain that conformal transformations preserve integrability of mechanical billiards. As a first application, we show that with conformal transformations we get infinitely many families of planar mechanical billiards which are integrable at one particular energy level.

In Section 3, we explain the duality between the Hooke billiard and the Kepler billiard and establish our results concerning them.

In Section 4, we study the integrability of Stark-type mechanical billiards. In particular, we provide a short alternative proof to the theorem of Korsch-Lang [19].

In Section 5, we apply Birkhoff's conformal transformation to the classical Euler's two-center problem and establish our results concerning this system.

2. Conformal transformations and mechanical billiards

2.1. Duality between integrable mechanical billiards

We start our discussion by the following definition of integrable mechanical system.

Definition 1. Let (M, g) be a 2-dimensional Riemannian manifold, U a smooth function on M , $\mathcal{B} \subset M$ a C^1 -curve, and $\mathcal{E} \subset \mathbb{R}$ such that $(M, g, U, \mathcal{B}, \mathcal{E})$ is a 2-dimensional mechanical billiard, meaning that (M, g, U) is a natural mechanical system and the motions are assumed to carry energies from $\mathcal{E}$ and are reflected elastically at $\mathcal{B}$. We call the system $(M, g, U, \mathcal{B}, \mathcal{E})$ *integrable* when there exists an additional C^∞ function

$$G : T^*M \rightarrow \mathbb{R}$$

functionally independent of its energy E , which is preserved by the motions and by reflections at $\mathcal{B}$.

Definition 2. Let M and M' be two smooth manifolds and $\phi : M \rightarrow M'$ be a k -to-1 regular mapping. Then its cotangent lift $\Phi : T^*M \rightarrow T^*M'$ is defined as

$$\Phi(x, \xi) = (x', \xi'), \quad x \in M, \quad \xi \in T_x^*M, \quad x' \in M', \quad \xi' \in T_{x'}^*M',$$

with

$$x' = \phi(x), \quad \xi' = (d\phi_x^*)^{-1}\xi,$$

where $(d\phi_x^*)^{-1}$ is the inverse mapping of the isomorphism $d\phi_x^* : T_{\phi(x)}^*M' \rightarrow T_x^*M$ that is an adjoint of the derivative $d\phi_x : T_xM \rightarrow T_{\phi(x)}M$ at x .

Moreover, Φ preserves the canonical symplectic forms on the cotangent bundles. More precisely we shall show that the cotangent lift Φ pulls the tautological one-form α' on T^*M' back to the tautological one-form α on T^*M , i.e. $\Phi^*\alpha' = \alpha$. This means pointwise

$$(d\Phi)_p^*(\alpha')_{p'} = (\alpha)_p,$$

where $(d\Phi)_p^*$ is the adjoint of the derivative $d\Phi$ at p and $p' = \phi(p)$. Let $\pi : T^*M \rightarrow M$ and $\pi' : T^*M' \rightarrow M'$ be footprint projections such that

$$\pi(x, \xi) = x, \quad \pi'(x', \xi') = x', \quad x \in M, \xi \in T_x^*M, x' \in M', \xi' \in T_{x'}^*M'.$$

The tautological one-forms α, α' are defined pointwise as

$$(\alpha)_p = (d\pi)_p^*\xi, \quad (\alpha')_{p'} = (d\pi')_{p'}^*\xi',$$

where $p = (x, \xi)$, $p' = (x', \xi')$ and $(d\pi)_p^*$, $(d\pi')_{p'}^*$ are adjoints of the derivatives of π and π' at p and p' respectively. We now have

$$\begin{aligned} (d\Phi)_p^*(\alpha')_{p'} &= (d\Phi)_p^*(d\pi')_{p'}^*\xi' = (d(\pi' \circ \Phi))_p^*\xi' = (d(\phi \circ \pi))_p^*\xi' \\ &= (d\pi)_p^*(d\phi)_p^*\xi' = (d\pi)_p^*\xi = (\alpha)_p. \end{aligned}$$

Now we are ready to state our first theorem.

Theorem 1. *Let $(M, g, U, \mathcal{B}, \mathcal{E})$ and $(M', g', U', \mathcal{B}', \mathcal{E}')$ be two 2-dimensional natural mechanical systems, where $\mathcal{E}$ and $\mathcal{E}'$ consist of regular values of the energies. Let $\phi : M \rightarrow M'$ be a conformal k -to-1 smooth regular mapping for some $k \in \mathbb{N}_+$ and assume that $\phi(\mathcal{B}) \subset \mathcal{B}'$. Suppose also that its cotangent lift $\Phi : T^*M \rightarrow T^*M'$ sends each energy hypersurface with energy $e \in \mathcal{E}$ to an energy hypersurface with energy $e' \in \mathcal{E}'$.*

*Under these assumptions, if $(M', g', U', \mathcal{B}', \mathcal{E}')$ is integrable, then $(M, g, U, \mathcal{B}, \mathcal{E})$ is also integrable. Additionally, if $(M, g, U, \mathcal{B}, \mathcal{E})$ is integrable and $\psi(\mathcal{B}') \subset \mathcal{B}$ for a smooth inverse branch $\psi : M' \rightarrow M$ of $\phi : M \rightarrow M'$, and its cotangent lift $\Psi : T^*M' \rightarrow T^*M$ sends each energy hypersurface with energy $e' \in \mathcal{E}'$ to an energy hypersurface with energy $e \in \mathcal{E}$, then $(M', g', U', \mathcal{B}', \mathcal{E}')$ is also integrable.*

Proof. We first suppose that $(M', g', U', \mathcal{B}', \mathcal{E}')$ is integrable. Since the energies from $\mathcal{E}'$ and $\mathcal{E}$ are mapped to each other, the vector fields X_H and $X_{\Phi^*H'}$ leave the common energy hypersurface

$$\{H = e\} = \{\Phi^*H' = e'\} \quad e \in \mathcal{E}, e' \in \mathcal{E}'$$

invariant, on which both vector fields are non-vanishing by the assumption that e is a regular value of H . Thus, there exists a smooth function $\rho : T^*M \rightarrow \mathbb{R} \setminus \{0\}$ such that $X_H = \rho X_{\Phi^*H'}$. This means X_H and $X_{\Phi^*H'}$ agree up to time parametrization.

From integrability of $(M', g', U', \mathcal{B}', \mathcal{E}')$, there exists a first integral G' that is functionally independent of energy H' . Thus

$$\mathcal{L}_{X_{H'}}G'|_{H'=e'} = \{H', G'\}|_{H'=e'} = 0,$$

where $\mathcal{L}_{X_{H'}}$ is the Lie derivative along the vector field $X_{H'}$. By setting $G := \Phi^*G'$, we obtain

$$\mathcal{L}_{X_H}G|_{H=e} = \rho \mathcal{L}_{X_{\Phi^*H'}}G|_{H=e} = \rho \{\Phi^*H', G\}|_{H=e} = 0.$$

So, G is conserved along the flow of X_H on $\{H = e\}$.

Now, we check the conservation of G before and after the reflection at $\mathcal{B}$. Take a point $b \in \mathcal{B}$, then $b' = \phi(b)$ lies in $\mathcal{B}'$ from the assumption $\phi(\mathcal{B}) \subset \mathcal{B}'$. Let (v'_-, v'_+) be a pair of incoming and outgoing vectors at $b' \in \mathcal{B}'$ so that v'_- and v'_+ have the same length with respect to the g' -metric and the angles they made with the normal agree up to sign.

There exist (v_-, v_+) such that $(d\phi_b(v_-), d\phi_b(v_+)) = (v'_-, v'_+)$. From the conformality of ϕ , the vectors v_- and v_+ have the same g -metric and the angle with the normal agree at $b \in \mathcal{B}$ up to sign. Therefore (v_-, v_+) are vectors before and after an elastic reflection at b . Since G' is invariant under the reflection at $\mathcal{B}'$, $G := \Phi^*G'$ is then invariant under the reflection at $\mathcal{B}$.

We now suppose that $(M, g, U, \mathcal{B}, \mathcal{E})$ is integrable. Since $\phi : M \rightarrow M'$ is a regular k -to-1 covering map, there exists k smooth regular inverse branches of ϕ . Let $\psi : M' \rightarrow M$ be such an inverse branch. The above argument now works the same for ψ in place of ϕ . $\square$

Remark 1. When the billiard mappings are well-defined, then the above theorem actually shows that they are (semi-)conjugate which implies their equivalence (up to covering) in the sense of dynamical systems.

Remark 2. The theorem can be directly generalized to certain multi-dimensional cases as well. Nevertheless, in view of Liouville's theorem, conformal mappings on a domain of $\mathbb{R}^d$, $d \geq 3$ are rather limited. Thus we may in general expect more non-trivial applications in the two-dimensional case.

Pulling back a mechanical billiard system by the cotangent lift of a conformal mapping without restricting its energy gives another mechanical billiard system without the necessity to change time, where the kinetic energy is transformed into a quadratic form of velocity depending on the base point in the configuration space. In our applications, we shall rather fix its energy and make a proper time change in order to have an iso-energetic correspondence between mechanical billiards in the plane with standard kinetic energies.

Before applying this theorem to concrete problems, we first state a lemma concerning the time-reparametrization of a Hamiltonian system on a fixed energy hypersurface.

Lemma 1. *Let $H(p, q)$ be a Hamiltonian function defined on $T^*\mathbb{R}^2$ equipped with its canonical symplectic form. Set $\hat{H} := g(q) \cdot H$ where $g(q) > 0$ is a C^∞ -smooth function of q . Then the two systems defined by H and $\hat{H}$ are equivalent up to a time-reparametrization given by $d\hat{t} = dt/g(q)$ on their zero energy-hypersurfaces.*

Proof. The statement immediately follows from the equation of motion:

$$\begin{aligned}\dot{q} &= \frac{\partial \hat{H}}{\partial p} = g(q) \cdot \frac{\partial H}{\partial p}, \\ \dot{p} &= \frac{\partial \hat{H}}{\partial p} = g'(q) \cdot H + g(q) \cdot \frac{\partial H}{\partial q} = g(q) \cdot \frac{\partial H}{\partial q},\end{aligned}$$

when being restricted to their common zero energy-hypersurfaces. $\square$

We now apply Theorem 1 to some central force problems.

Theorem 2. *Let $f, s \in \mathbb{R}$ be two real parameters. For any $k \in \mathbb{N}, k \geq 2$ the cotangent lift of the conformal mapping*

$$\mathbb{C} \setminus O \mapsto \mathbb{C} \setminus O, z \mapsto q = z^k$$

gives the transformation between two hamiltonians

$$\frac{|w|^2}{2} + f|z|^{2k-2} + s$$

and

$$\frac{|p|^2}{2} + \frac{s}{|q|^{2-2/k}} + f$$

on their zero-energy surface, up to time parametrization.

In particular, it gives

- *for $(f > 0, s < 0)$, $(f < 0, s > 0)$, or $(f < 0, s < 0)$, an iso-energetic transformation between two central force systems. In particular, when $k = 2$ and $f > 0, s < 0$, between the Hooke system of isotropic harmonic oscillators and the Kepler system in the plane.*
- *for $(f = 0, s < 0)$ or $(s = 0, f < 0)$, an iso-energetic transformation between the free motion in the plane with positive energy and some homogeneous central force systems at their energy zero.*
- *for $f = s = 0$, a trivial iso-energetic transformation between zero-energy free motions in the plane.*

Proof. The cotangent lift for $\phi : z \mapsto z^k$ is given by

$$\Phi : (z, w) \mapsto \left(q = z^k, p = \frac{w}{kz^{k-1}} \right),$$

and is a symplectic map. This follows from our discussions above but it is also direct to have a verification with complex notations. Indeed, the canonical symplectic form $\omega_0 = \sum dq_i \wedge dp_i$ is given by $\omega_0 = d\alpha_0$, where $\alpha_0 = \sum p_i dq_i$ is the tautological one-form. When we identify $\mathbb{R}^2$ and $\mathbb{C}$ and describe $p = p_1 + ip_2$ and $q = q_1 + iq_2$, we can rewrite the tautological one-form into $\alpha_0 = \text{Re}(\bar{p}dq)$. By substituting $q = z^k$ and $p = \frac{w}{kz^{k-1}}$, we obtain $\text{Re}(\bar{p}dq) = \text{Re}(\bar{\omega}dz)$, thus $\omega_0(p, q) = \omega_0(z, w)$.

For normalization purpose, we would prefer the conformal symplectic transformation

$$\Phi : (z, w) \mapsto \left(q = z^k, p = \frac{w}{z^{k-1}} \right),$$

which is equivalent to making an additional inessential constant change of time which then pulls the system

$$\frac{|p|^2}{2} + \frac{s}{|q|^{2-2/k}} + f = 0$$

back to

$$\frac{|w|^2}{2|z|^{2k-2}} + \frac{s}{|z|^{2k-2}} + f = 0.$$

On this energy level we may now apply Lemma 1 and multiply the Hamiltonian by the factor $|z|^{2k-2}$ which just reparametrizes the flow on this energy hypersurface. With this, we get

$$\frac{|w|^2}{2} + f|z|^{2k-2} + s = 0,$$

which is the system with Hamiltonian $\frac{|w|^2}{2} + f|z|^{2k-2} + s$ on its zero-energy level. $\square$

We remark that this has been used by McGehee for regularization purpose [25].

2.2. Mechanical billiards from free billiards

We now draw our first consequences in the case $f = 0$. In this case one of the two systems is the system of free motions in the plane. It is classically known that a free billiard with a conic section as a reflection wall is integrable [3][32][21]. Namely, it allows elliptic, hyperbolic, parabolic, and line boundaries as integrable reflection walls. We shall deduce this also from our discussions on integrable Hooke/Kepler billiards in Section 3

A conic section in the plane is described with six parameters as

$$Az_1^2 + Bz_1z_2 + Cz_2^2 + Dz_1 + Ez_2 + F = 0, \tag{1}$$

where all coefficients are real numbers and A, B , and C are not all zero. Since multiplication by a common factor to all the coefficients does not change the curve that it describes, only five out of the six parameters are free. In addition, when we identify conic sections which differ from each other just by scalings and rotations, then only three of the parameters are free.

From this fact and Theorem 1, we directly get the following proposition as an easy corollary.

Proposition 1. *For any $k \in \mathbb{N}, k \geq 2$, the system $(\mathbb{C}, g_{flat}, 1/|q|^{2-2/k}, 0)$ on zero-energy surface admits 5-parameter family of smooth integrable reflection walls without ruling out the scalings and the rotations, and 3-parameter family of smooth integrable reflection walls while ruling out the scalings and the rotations.*

We may as well consider the case $s = 0$ which also gives rise to free motion. With the same argument we get the following proposition:

Proposition 2. *For any $k \in \mathbb{N}, k \geq 2$, the system $(\mathbb{C}, g_{flat}, |z|^{2k-2}, 0)$ on the zero-energy surface admits 5-parameter family of smooth integrable reflection walls without ruling out the scalings and the rotations, and 3-parameter family of smooth integrable reflection walls while ruling out the scalings and the rotations.*

We illustrate these propositions in the case $k = 2$. The complex square mapping $z \mapsto z^2 = q$ is lifted to the cotangent bundle explicitly given as

$$\begin{aligned} q_1 &= z_1^2 - z_2^2 \\ q_2 &= 2z_1 z_2 \\ p_1 &= \frac{z_1 w_1 - z_2 w_2}{z_1^2 + z_2^2} \\ p_2 &= \frac{z_1 w_2 + z_2 w_1}{z_1^2 + z_2^2}. \end{aligned}$$

Fig. 1 shows the reflection walls that are transformed from the ellipses/hyperbolae

$$\frac{(z_1 - c_1)^2}{a^2} \pm \frac{(z_2 - c_2)^2}{b^2} = 1$$

by the mapping above $z \mapsto z^2$, in the case of $f = 0$. Notice that for a non-centrally symmetric curve, its centrally symmetric reflection is another branch of the pre-image of its image under the complex square mapping. Since centrally symmetric points are mapped to the same point under the complex square mapping which is locally a diffeomorphism, the image may thus have self-intersection points. For a non-centered ellipse, its image contains self-intersection points when the center of the ellipse is not too far away from the origin. The situation is exactly the contrary for a non-centered hyperbola: its image contains self-intersections points when its center is sufficiently far from the origin.

The law of reflections should also correspond to each other through the conformal complex square mapping. This gives the following law of reflection at the target space of the mapping: A particle is supposed to be reflected against the curve in the target space when the corresponding motions in the source space do so. Otherwise the particle just crosses the curve. Fig. 2 illustrates this rule, in which the left picture shows what happens in the source space, and the right picture in the target space. The dashed curve in the left picture is the centrally symmetric image of the reflection wall. A pair of centrally symmetric points lying in the reflection wall colored in green in the source space are mapped to one point which is a self-intersection point in the target space. A reflection point colored in red in the source space is mapped again to a reflection point in the target space, and vice versa. The blue points indicate where the particle just crosses without reflections in the source and the target spaces.

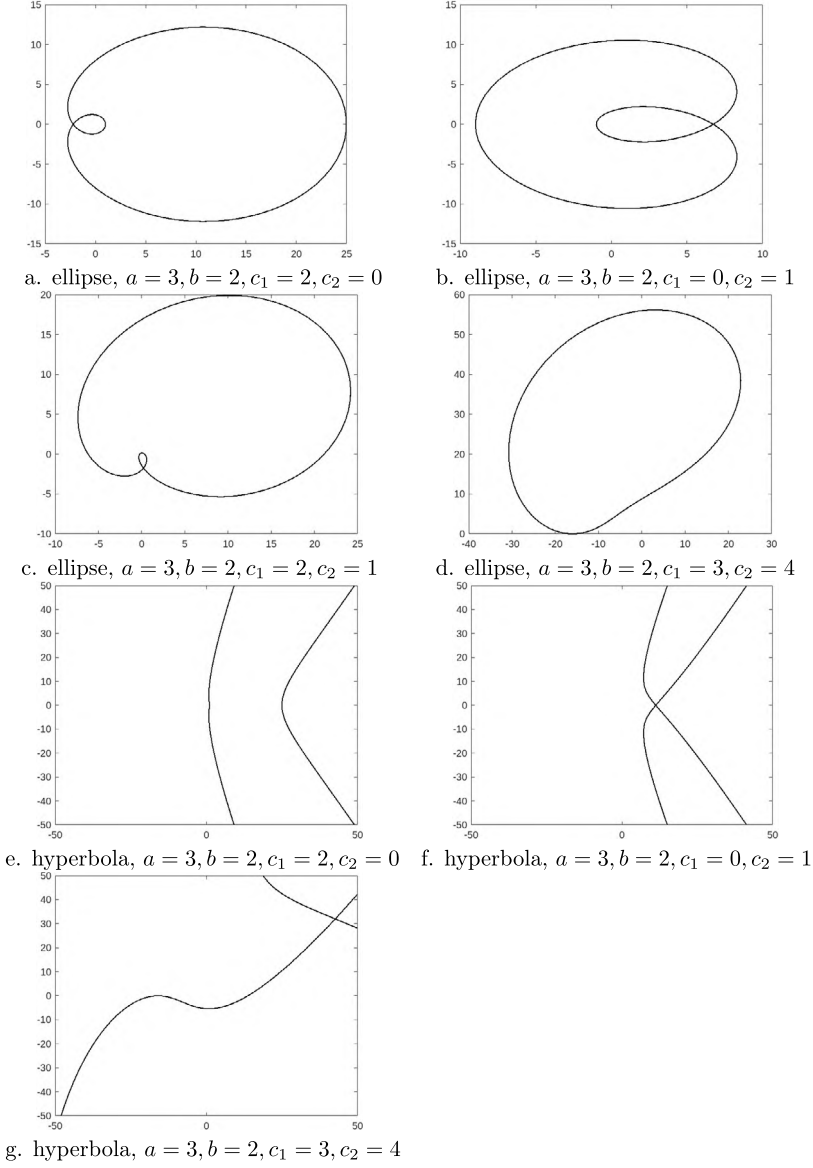


Fig. 1. Transformed ellipses/hyperbolae by $z \mapsto z^2$.

In the case of the complex square mapping $z \mapsto q = z^2$, the two inverse branches are given respectively as

$$z_1 = \frac{q_2}{\sqrt{-2q_1 + 2\sqrt{q_1^2 + q_2^2}}}, \quad z_2 = \frac{\sqrt{-2q_1 + 2\sqrt{q_1^2 + q_2^2}}}{2}$$

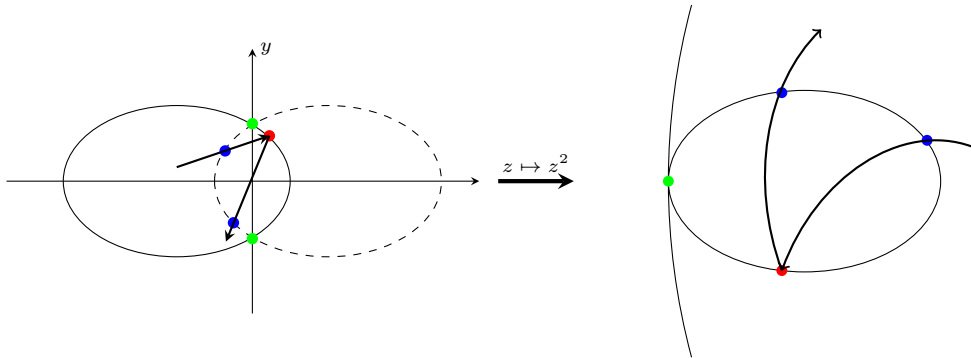


Fig. 2. The laws of reflections before and after the transformation induced by the complex square mapping. (For interpretation of the colors in the figure, the reader is referred to the web version of this article.)

and

$$z_1 = -\frac{q_2}{\sqrt{-2q_1 + 2\sqrt{q_1^2 + q_2^2}}}, \quad z_2 = -\frac{\sqrt{-2q_1 + 2\sqrt{q_1^2 + q_2^2}}}{2}.$$

After eliminating the square roots in the equations, the quadratic curves given by Equation (1) are mapped into some fourth-order curves of q_1 and q_2 , so the transformed curve is more complicated and can be hard to identify via a more direct method.

Next we consider the case $k = 3$. Fig. 3 shows the image of ellipses/hyperbolae

$$\frac{(z_1 - c_1)^2}{a^2} \pm \frac{(z_2 - c_2)^2}{b^2} = 1$$

by the conformal mapping $z \mapsto z^3 = q$. One can see that the transformed curves from centered ellipses give four self-intersection points. As the center of an ellipse moves away from the origin, the number of self-intersection points is reduced to two, and further to zero when the center is sufficiently far from the origin. For hyperbolae, there always exists at least one self-intersection in the transformed curves.

In the case of $fs \neq 0$, much less integrable reflection walls are known. In the next section we shall analyze the situation in the Hooke and Kepler problems.

3. Hooke, Kepler billiards and their dualities

Recall that by Hooke problem we refer to the mechanical system defined in the plane with force function fr^2 , in which r is the distance of the particle to the center and $f \neq 0$ is a real parameter. The force field is attractive/repulsive when f is negative/positive. We accept both cases. Similarly by Kepler problem we refer to the mechanical system defined in the plane with force function sr^{-1} in which $s \neq 0$ is a real parameter. Again we accept both signs of s .

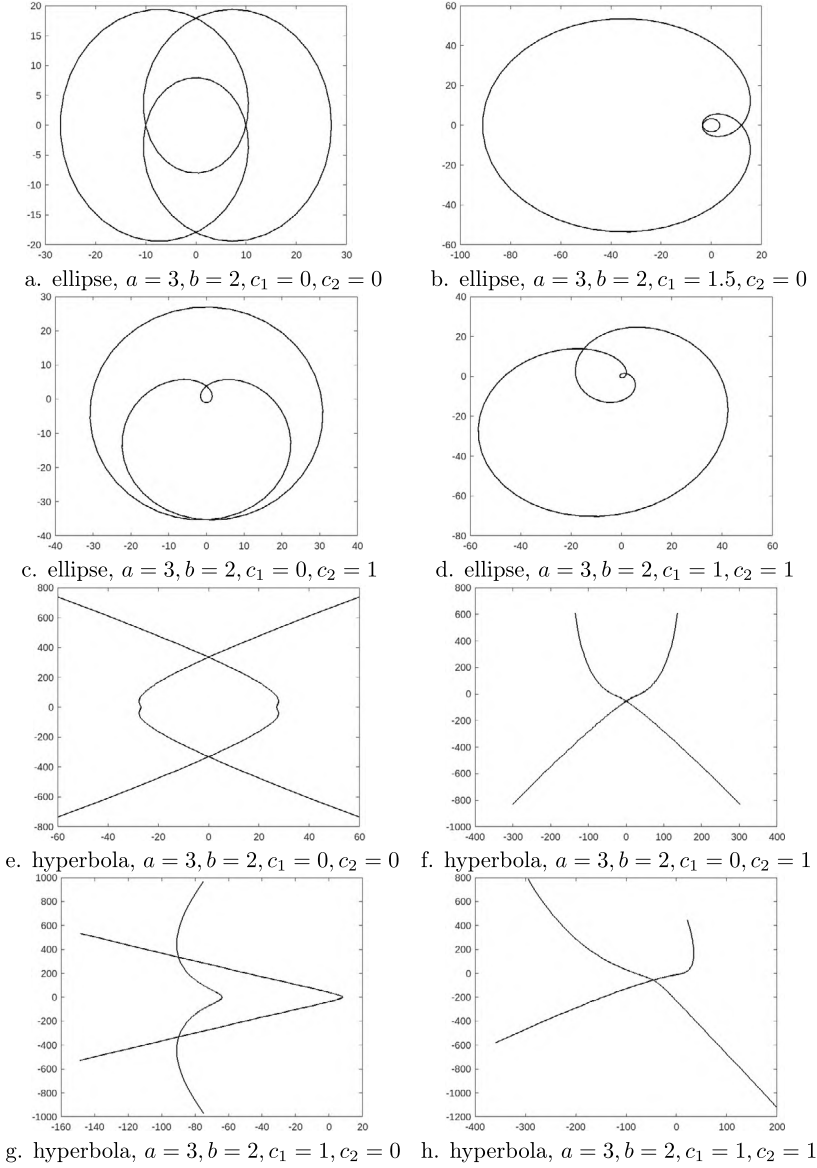


Fig. 3. Transformed ellipses/hyperbolae by $z \mapsto z^3$.

3.1. Integrable Hooke billiards

Billiard systems defined with the Hooke problem are relatively well-studied and the following integrable reflection walls are known: Centered ellipses [15][7], lines, and combinations of confocal centered conic sections [28][29]. Note that we call a conic section centered, when its center is at the origin. In the following theorem we provide a di-

rect verification for the integrability of centered conic section reflection walls in Hooke billiards.

Theorem 3. *The attractive/repulsive Hooke billiard*

$$(\mathbb{C}, g_{flat}, -f|z|^2, \mathcal{B})$$

with $\mathcal{B}$ being a centered conic section reflection wall is integrable.

Proof. By using the rotational symmetry, we can write centered ellipses as

$$F := \frac{z_1^2}{a^2} + \frac{z_2^2}{b^2} - 1 = 0.$$

Without loss of generality we can assume that $a \geq b$. Let $w := (w_1, w_2)$ and $w' = (w'_1, w'_2)$ denote the linear momenta, respectively, before and after the reflection against this centered elliptic reflection wall. Set $F_1 := \partial F / \partial z_1$ and $F_2 := \partial F / \partial z_2$.

The normal vector at a point on $F = 0$ is given by

$$n := (F_1, F_2).$$

The normal component of w is thus

$$w_n := \frac{w \cdot n}{|n|^2} n = \left(\frac{(w_1 F_1 + w_2 F_2) F_1}{F_1^2 + F_2^2}, \frac{(w_1 F_1 + w_2 F_2) F_2}{F_1^2 + F_2^2} \right)$$

From the law of elastic reflection, the momenta after the reflection w' is

$$w' = w - 2w_n,$$

that is

$$\begin{aligned} w'_1 &= w_1 - \frac{2(w_1 F_1 + w_2 F_2) F_1}{F_1^2 + F_2^2} \\ &= \frac{a^4 w_1 z_2^2 - 2a^2 b^2 w_2 z_1 z_2 - b^4 w_1 z_1^2}{a^4 z_2^2 + b^4 z_1^2}, \\ w'_2 &= w_2 - \frac{2(w_1 F_1 + w_2 F_2) F_2}{F_1^2 + F_2^2} \\ &= \frac{-a^4 w_2 z_2^2 - 2a^2 b^2 w_1 z_1 z_2 + b^4 w_2 z_1^2}{a^4 z_2^2 + b^4 z_1^2}. \end{aligned}$$

We now search for possible first integrals of this Hooke billiard system. The Hamiltonian of the system is $|w|^2/2 + f|z|^2$, which admits three independent first integrals from its separability and rotational symmetry:

$$2fz_1^2 + w_1^2, \quad 2fz_2^2 + w_2^2, \quad z_1w_2 - z_2w_1.$$

So any combination of (functions of) these first integrals is again a first integral. We set

$$\tilde{G}(z_1, z_2, w_1, w_2) := k_1(2fz_1^2 + w_1^2) + k_2(2fz_2^2 + w_2^2) + (z_1w_2 - z_2w_1)^2,$$

in which we square the angular momentum in order to have a quadratic function on (w_1, w_2) and $k_1, k_2 \in \mathbb{R}$ are coefficients to be determined. After the reflection at the reflection wall, (z_1, z_2, w_1, w_2) is mapped to (z_1, z_2, w'_1, w'_2) . The difference between the values of $\tilde{G}$ before and after the reflection becomes

$$\begin{aligned} \tilde{G}(z_1, z_2, w_1, w_2) - \tilde{G}(z_1, z_2, w'_1, w'_2) = \\ \frac{4z_1z_2(a^4w_1w_2z_2^2 + a^2b^2w_1^2z_1z_2 - a^2b^2w_2^2z_1z_2 - b^4w_1w_2z_1^2)}{a^4z_2^2 + b^4z_1^2} \times \\ \frac{a^4z_2^2 + a^2b^2z_1^2 - a^2b^2z_2^2 - b^4z_1^2 - a^2b^2k_1 + a^2b^2k_2}{a^4z_2^2 + b^4z_1^2}. \end{aligned}$$

Set

$$Z(z_1, z_2) := a^4z_2^2 + a^2b^2z_1^2 - a^2b^2z_2^2 - b^4z_1^2 - a^2b^2k_1 + a^2b^2k_2.$$

This is a factor in the numerator which does not depend on the momenta, and therefore its nullity implies preservation of G under reflections.

We deduce from $F = 0$ that

$$Z = a^2b^2(a^2 - b^2 - k_1 + k_2).$$

Thus, Z equals to 0 if and only if

$$k_1 - k_2 = a^2 - b^2 = a^2e^2 \tag{2}$$

in which e is the eccentricity. After normalizing the coefficients, we thus get the following expression of the additional first integral as

$$G(z_1, z_2, w_1, w_2) := \frac{a^2e^2}{1 + a^2e^2}(2fz_1^2 + w_1^2) + \frac{1}{1 + a^2e^2}(z_1w_2 - z_2w_1)^2. \tag{3}$$

The case of a centered hyperbola given by

$$\frac{z_1^2}{a^2} - \frac{z_2^2}{b^2} - 1 = 0$$

can be treated similarly and we get the condition on k_1 and k_2 as

$$k_1 - k_2 = a^2 + b^2 = a^2 e^2. \quad (4)$$

Thus, the first integral is given by the same formula (3). $\square$

From the conditions (2) and (4) on the coefficients k_1 and k_2 , we can immediately deduce the integrability of reflection walls consisting of confocal ellipses and hyperbolae.

Corollary 1. *The attractive/repulsive Hooke billiard*

$$(\mathbb{C}, g_{flat}, -f|z|^2, \mathcal{B})$$

with $\mathcal{B}$ being any finite combination of confocal centered ellipses and hyperbolae is integrable.

Proof. Consider confocal centered ellipses in the form of

$$\frac{z_1^2}{a^2} + \frac{z_2^2}{a^2 - c^2} = 1, \quad a > c,$$

and confocal centered hyperbolae in the form of

$$\frac{z_1^2}{b^2} - \frac{z_2^2}{c^2 - b^2} = 1, \quad 0 < b < c.$$

For both cases, we obtain the same condition on the coefficients k_1 and k_2 such as

$$k_1 - k_2 = c^2,$$

thus there exists a common first integral given by

$$G(z_1, z_2, w_1, w_2) := \frac{c^2}{1 + c^2}(2fz_1^2 + w_1^2) + \frac{1}{1 + c^2}(z_1w_2 - z_2w_1)^2. \quad \square$$

We note that by a direct limiting procedure we also get the integrability of the Hooke billiard with a line or any finite combination of parallel/perpendicular lines as an integrable boundary.

Corollary 2. *The attractive/repulsive Hooke billiard*

$$(\mathbb{C}, g_{flat}, -f|z|^2, \mathcal{B})$$

with $\mathcal{B}$ being any finite combination of parallel/perpendicular lines is integrable.

Proof. When we take a limit $e \rightarrow \infty$ in the first integral (3), we obtain the form:

$$w_1^2 + 2fz_1^2$$

which is invariant under reflections against a line which is parallel to z_1 - or z_2 -axis, as well as for any finite combinations of lines which are parallel to the z_1 - or z_2 -axis. $\square$

Remark 3. The Hooke billiard also allows other (degenerate) integrable reflection walls. As a more or less trivial example, any finite combination of lines passing through the center is integrable with the first integral $(z_1 w_2 - z_2 w_1)^2$.

3.2. Integrable Kepler billiards

As opposed to the study of Hooke billiards, the study of Kepler billiards, on the other hand, seems to be rather recent. In [4], L. Boltzmann considered the billiard system of a central force problem in the plane, which includes the Kepler problem, with a line as a wall of reflection. He asserted that any billiard system thus obtained is ergodic. Recently Gallavotti-Jauslin [11] disproved this assertion in the case that the central force problem is the Kepler problem, by actually showing the integrability of the corresponding billiard system. This integrability is revisited together with an in-detailed analysis on its integrable dynamics in Felder [8]. An alternative proof of this integrability based on projective invariance of the Kepler problem is provided in [37]. We also note that prior to Gallavotti-Jauslin, Panov explained the equivalence of the Hooke billiard in a centered ellipse and the Kepler billiard in an ellipse focused at the origin and thus established the integrability of the latter in a brief note [26].

We now revisit the first integral of Gallavotti-Jauslin G using the complex square mapping, thus we provide yet another alternative proof for the integrability of Gallavotti-Jauslin [11] as well as some extensions.

Lemma 2. *The additional first integral*

$$G(z_1, z_2, w_1, w_2) := \frac{a^2 e^2}{1 + a^2 e^2} (2f z_1^2 + w_1^2) + \frac{1}{1 + a^2 e^2} (z_1 w_2 - z_2 w_1)^2$$

given in Theorem 3 of the Hooke billiard $(\mathbb{C}, g_{flat}, -f|z|^2, \mathcal{B})$ is transformed under the complex square mapping $\mathbb{C} \setminus O \rightarrow \mathbb{C} \setminus O, z \mapsto z^2$, after being multiplied by $(1 + a^2 e^2)$, into Gallavotti-Jauslin's first integral

$$A(p_1, p_2, q_1, q_2) := (p_1 q_2 - p_2 q_1)^2 - 2\tilde{a} \left((-p_1 q_2 + p_2 q_1) p_2 - \frac{s q_1}{\sqrt{q_1^2 + q_2^2}} \right)$$

in which $\tilde{a} = a^2 e^2 / 2$, on the $-f$ -energy hypersurface of the Kepler problem $(\mathbb{C} \setminus O, g_{flat}, \frac{s}{|q|})$.

Proof. We observe that $(z_1 w_2 - z_2 w_1)^2$ is mapped into the squared angular momentum $(q_1 p_2 - q_2 p_1)^2$. Indeed, we see that

$$\begin{aligned}
z_1 w_2 - z_2 w_1 &= \frac{(z_1^2 - z_2^2)(z_1 w_2 + z_2 w_1) - 2z_1 z_2(z_1 w_1 - z_2 w_2)}{z_1^2 + z_2^2} \\
&= q_1 p_2 - q_2 p_1.
\end{aligned}$$

We now consider the other term $2fz_1^2 + w_1^2$. With the relations

$$q_1 = z_1^2 - z_2^2, \quad q_2 = 2z_1 z_2,$$

we get

$$\begin{aligned}
z_1^2 &= \frac{z_1^2 - z_2^2 + \sqrt{(z_1^2 - z_2^2)^2 + 4z_1^2 z_2^2}}{2} \\
&= \frac{q_1 + \sqrt{q_1^2 + q_2^2}}{2},
\end{aligned}$$

and

$$\begin{aligned}
z_2^2 &= z_1^2 - q_1 \\
&= \frac{-q_1 + \sqrt{q_1^2 + q_2^2}}{2}.
\end{aligned}$$

From these and $w_1 = z_1 p_1 + z_2 p_2$, we get

$$\begin{aligned}
w_1^2 &= (z_1 p_1 + z_2 p_2)^2 \\
&= z_1^2 p_1^2 + 2z_1 z_2 p_1 p_2 + z_2^2 p_2^2 \\
&= \frac{1}{2} \left(p_1^2 q_1 - p_2^2 q_1 + 2p_1 p_2 q_2 + p_1^2 \sqrt{q_1^2 + q_2^2} + p_2^2 \sqrt{q_1^2 + q_2^2} \right).
\end{aligned}$$

From these we see that $2fz_1^2 + w_1^2$ is mapped into

$$f \left(q_1 + \sqrt{q_1^2 + q_2^2} \right) + \frac{1}{2} \left(p_1^2 q_1 - p_2^2 q_1 + 2p_1 p_2 q_2 + p_1^2 \sqrt{q_1^2 + q_2^2} + p_2^2 \sqrt{q_1^2 + q_2^2} \right).$$

After fixing the energy of the Hooke problem to s and transforming the resulting system via the complex square mapping we get the energy constraint

$$(p_1^2 + p_2^2)/2 - s/(\sqrt{q_1^2 + q_2^2}) + f = 0.$$

From which we deduce that

$$\begin{aligned}
&f \left(q_1 + \sqrt{q_1^2 + q_2^2} \right) + \frac{1}{2} \left(p_1^2 q_1 - p_2^2 q_1 + 2p_1 p_2 q_2 + p_1^2 \sqrt{q_1^2 + q_2^2} + p_2^2 \sqrt{q_1^2 + q_2^2} \right) \\
&= - \left((-p_1 q_2 + p_2 q_1) p_2 - s \cdot \frac{q_1}{\sqrt{q_1^2 + q_2^2}} - s \right).
\end{aligned}$$

Therefore, the additional first integral

$$G(z_1, z_2, w_1, w_2) = a^2 e^2 (2f z_1^2 + w_1^2) + (z_1 w_2 - z_2 w_1)^2$$

is transformed into the form

$$A(p_1, p_2, q_1, q_2) := (p_1 q_2 - p_2 q_1)^2 - 2\tilde{a} \left((-p_1 q_2 + p_2 q_1) p_2 - \frac{s q_1}{\sqrt{q_1^2 + q_2^2}} \right),$$

where $\tilde{a} = a^2 e^2 / 2$ (which is the distance from the center to one of the foci in the case when the transformed curve is a focused ellipse or hyperbola), on the $-f$ -energy hypersurface of the Kepler problem. $\square$

We say a conic section is *focused*, when the origin is a focus of it. Using the duality between the Kepler billiard system and the Hooke billiard system given in Theorem 2, we deduce various integrable Kepler billiards from Theorem 3, that we summarize in the following theorem:

Theorem 4. *The Kepler system $(\mathbb{C}, g_{flat}, \frac{s}{|q|}, \mathbb{R})$ admits any focused conic sections, degenerate cases allowed, as integrable reflection walls. These include*

1. *any focused parabola*
2. *any focused ellipse*
3. *any focused hyperbola*
4. *any line.*

The additional first integral is given in Lemma 2.

Proof. We discuss case by case.

1. Any lines are integrable reflection walls for the Hooke potential. By rotation-invariance it is enough to consider the case of a line given by the expression $z_1 = c, c \in \mathbb{R} \setminus \{0\}$, which is transformed by $z \mapsto z^2 = q$ into the parabola

$$q_1 = -\frac{q_2^2}{4c^2} + c^2,$$

focused at the origin.

2. Consider a centered ellipse given by

$$\frac{z_1^2}{a^2} + \frac{z_2^2}{b^2} = 1, \quad a, b > 0 \tag{5}$$

We parametrize this elliptic curve as

$$z_1 = a \cos u, \quad z_2 = b \sin u$$

with a parameter $u \in [0, 2\pi)$. Then the image of this curve by the conformal mapping $z \mapsto z^2 = q$ is given by

$$q_1 = a^2 \cos^2 u - b^2 \sin^2 u, \quad q_2 = 2ab \sin u \cos u,$$

which describes the focused ellipse

$$\frac{(q_1 - (a^2 - b^2)/2)^2}{(a^2 + b^2)^2/4} + \frac{q_2^2}{a^2 b^2} = 1. \quad (6)$$

3. Consider a centered hyperbola given by

$$\frac{z_1^2}{a^2} - \frac{z_2^2}{b^2} = 1, \quad a, b > 0, a \neq b, \quad (7)$$

parametrized as

$$z_1 = a \cosh u, \quad z_2 = b \sinh u$$

with parameter $u \in (-\pi, \pi)$ for one branch and $u \in (-\pi, \pi)$ for the other branch. Then the image of this curve by the conformal mapping $z \mapsto z^2 = q$ is given by

$$q_1 = a^2 \cosh^2 u - b^2 \sinh^2 u, \quad q_2 = 2ab \sinh u \cosh u.$$

We thus get that the transformed curve satisfies

$$\frac{(q_1 - (a^2 + b^2)/2)^2}{(a^2 - b^2)^2/4} - \frac{q_2^2}{a^2 b^2} = 1, \quad (8)$$

which describes a focused hyperbola. Indeed this image is seen to be a branch of this hyperbola. The pre-image of the other branch of this hyperbola is the confocal centered hyperbola given by

$$\frac{z_1^2}{b^2} - \frac{z_2^2}{a^2} = 1. \quad (9)$$

To see this, it is enough to exchange the roles of a and b in the above reasoning. Since the pre-image of the focused hyperbola consists of two confocal hyperbolae, we may thus conclude with Corollary 1.

4. Finally, a rectangular hyperbola given by

$$\frac{z_1^2}{a^2} - \frac{z_2^2}{a^2} = 1$$

is transformed by the conformal mapping $z \mapsto z^2 = q$ into the line

$$q_1 = a^2. \quad \square$$

Corollary 3. *The Kepler system $(\mathbb{C}, g_{flat}, \frac{s}{|q|}, \mathbb{R})$ admits any finite combination of confocal focused ellipses and hyperbolae as an integral reflection wall.*

Proof. It suffices to see that confocal centered ellipses/hyperbolae are transformed into confocal focused ellipses/hyperbolae by the complex square mapping $z \mapsto z^2$. This can be easily checked from the forms of a transformed focused ellipse (6) and a transformed focused hyperbola (8) by setting $b^2 = a^2 - c^2$ for ellipses and $b^2 = c^2 - a^2$ for hyperbolae. $\square$

Similarly, from Corollary 2 we obtain integrable Kepler billiards with any finite combination of focused parabolae with collinear major axis.

Corollary 4. *The Kepler system $(\mathbb{C}, g_{flat}, \frac{s}{|q|}, \mathbb{R})$ admits any finite combination of focused parabolae with collinear major axes as integrable reflection wall.*

Proof. The argument follows directly from Theorem 4, Case 1 and Corollary 2. $\square$

3.3. From Hooke/Kepler billiards to free billiards

We now discuss the classical case of free billiards based on our discussions on integrable Hooke/Kepler billiards, by setting $f = 0$ in the Hooke billiards, or $s = 0$ in the Kepler billiards. The following proposition now becomes a direct corollary.

Corollary 5. *Free billiards admit conic section reflection walls as integrable reflection walls.*

We now link the additional first integral given by (3) to the well-known Joachimsthal first integral, as follows: From Theorem 3, in the case of $f = 0$, we have the additional first integral

$$(a^2 - b^2)w_1^2 + (z_1w_2 - z_2w_1)^2$$

for the free billiard with a centered elliptic integrable reflection wall given by

$$\frac{z_1^2}{a^2} + \frac{z_2^2}{b^2} = 1. \quad (10)$$

By dividing this by a^2b^2 , we get

$$\left(\frac{1}{b^2} - \frac{1}{a^2}\right)w_1^2 + \frac{(z_1w_2 - z_2w_1)^2}{a^2b^2}$$

$$\begin{aligned}
&= \frac{w_1^2}{b^2} - \frac{w_1^2}{a^2} + \frac{z_1^2 w_2^2 - 2z_1 z_2 w_1 w_2 + z_2^2 w_1^2}{a^2 b^2} \\
&= \frac{w_1^2}{b^2} - \frac{w_1^2}{a^2} + \frac{z_2^2}{b^2} \cdot \frac{w_1^2}{a^2} + \frac{z_1^2}{a^2} \cdot \frac{w_2^2}{b^2} - \frac{2z_1 z_2 w_1 w_2}{a^2 b^2} \\
&= \frac{w_1^2 + w_2^2}{b^2} - \left(1 - \frac{z_2^2}{b^2}\right) \frac{w_1^2}{a^2} - \left(1 - \frac{z_1^2}{a^2}\right) \frac{w_2^2}{b^2} - \frac{2z_1 z_2 w_1 w_2}{a^2 b^2} \\
&= \frac{1}{b^2} - \left(\frac{z_1^2 w_1^2}{a^4} + \frac{z_2 w_2}{b^4} + \frac{2z_1 z_2 w_1 w_2}{a^2 b^2}\right) \\
&= \frac{1}{b^2} - \left(\frac{z_1 w_1}{a^2} + \frac{z_2 w_2}{b^2}\right)^2.
\end{aligned}$$

In the fourth equation, we used the equation of a centered ellipse (10). In which we recognize the classical Joachimsthal first integral

$$\frac{z_1 w_1}{a^2} + \frac{z_2 w_2}{b^2}$$

of the free billiard with an elliptic boundary.

3.4. Conjectures related to the Birkhoff conjecture

From Theorem 3 and in view of the Birkhoff-Poritsky's conjecture [27], we make the following conjectures for Hooke and Kepler billiards.

Conjecture 1. *The only Hooke billiards with smooth connected reflection walls which are integrable on all regular energy hypersurfaces are those with a branch of a centered conic section or a line.*

Conjecture 2. *The only Kepler billiards with smooth connected reflection walls which are integrable on all regular energy hypersurfaces are those with a focused conic section or a line.*

4. Integrable stark-type billiards

4.1. Separability and integrability of stark-type billiards

In this section, we investigate some two degrees of freedom mechanical systems which are separable after the complex square mapping, and identify integrable reflection walls for such systems. We consider some special class of systems with force function given in the form of

$$\frac{s}{|q|} + R(q), \quad R \in C^\infty(\mathbb{R}^2 \setminus O, \mathbb{R}),$$

so that the Kepler problem is further modified by the additional influence from $R(q)$. The Hamiltonian of such a system is

$$H = \frac{|p|^2}{2} - \frac{s}{|q|} - R(q_1, q_2). \quad (11)$$

On its fixed energy hypersurface $\{H + f = 0\}$ we may again transform the system by the complex square mapping after a proper time change as described in Theorem 1 which then leads to the system

$$\hat{H} = \frac{|w|^2}{2} - s + f(z_1^2 + z_2^2) - (z_1^2 + z_2^2)R(z_1^2 - z_2^2, 2z_1z_2).$$

Now the transformed Hamiltonian $\hat{H}$ is separable in (z_1, z_2) coordinates if and only if the term

$$(z_1^2 + z_2^2)R(z_1^2 - z_2^2, 2z_1z_2)$$

is separable in (z_1, z_2) coordinates. When the function $R(q)$ satisfies this separability condition, we call such systems (11) *Stark-type systems*. By using the separability of Stark-type systems, we obtain infinitely many integrable Stark-type billiard systems as we state in the following theorem:

Theorem 5. *There exists infinitely many functions $R \in C^\infty(\mathbb{R}^2 \setminus O, \mathbb{R})$ such that the system*

$$H = \frac{|p|^2}{2} - \frac{s}{|q|} - R(q_1, q_2)$$

allows any focused parabola with the q_1 -axis as the main axis as an integrable reflection wall.

Proof. Assume that the system H is of Stark-type, so that the transformed Hamiltonian is separable, *i.e.*

$$\hat{H} = \frac{|w|^2}{2} + s - f(z_1^2 + z_2^2) + (z_1^2 + z_2^2)R(z_1^2 - z_2^2, 2z_1z_2) = \hat{H}_1(z_1, w_1) + \hat{H}_2(z_2, w_2).$$

From its separability, this system has the additional first integral $\hat{H}_1(z_1, w_1)$, which is invariant under the reflections against a line which is parallel to the z_1 - or z_2 -axis. Now since any line which is parallel to the z_1 - or z_2 -axis is transformed into a focused parabola in the form of

$$q_1 = -\frac{q_2^2}{4c^2} + c^2$$

or

$$q_1 = \frac{q_2^2}{4c^2} - c^2$$

by the mapping $z \mapsto z^2$, by Theorem 1, the original system allows any focused parabola with the q_1 -axis as the main axis as an integrable reflection wall.

We are just left to show that there exist infinitely many Stark-type systems. We assume that the function R depending only on $z_1^2 - z_2^2$ and $2z_1z_2$ satisfies

$$(z_1^2 + z_2^2)R(z_1^2 - z_2^2, 2z_1z_2) = g_1(z_1) + g_2(z_2)$$

for some smooth even functions $g_1, g_2 \in C^\infty(\mathbb{R}, \mathbb{R})$, i.e. $g_1(-z_1) = g_1(z_1)$ and $g_2(-z_2) = g_2(z_2)$ for all $z_1, z_2 \in \mathbb{R}$. We then define $R(z_1^2 - z_2^2, 2z_1z_2) := \frac{g_1(z_1) + g_2(z_2)}{z_1^2 + z_2^2}$, and we may then solve R as a function of $q_1 = z_1^2 - z_2^2$ and $q_2 = 2z_1z_2$, which is possible since $\frac{g_1(z_1) + g_2(z_2)}{z_1^2 + z_2^2}$ is invariant under reflections with respect to the origin *i.e.*

$$\frac{g_1(z_1) + g_2(z_2)}{z_1^2 + z_2^2} = \frac{g_1(-z_1) + g_2(-z_2)}{(-z_1)^2 + (-z_2)^2}. \quad \square$$

This theorem is an analogue of [5, Theorem 3.1] in the setting of mechanical billiards.

4.2. Examples of stark-type billiard systems

In the following, we discuss some concrete examples of Stark-type systems.

Stark problem Firstly we consider the Stark problem by setting $R(q) = gq_1$. The Stark problem can be interpreted as a planar system consisting of gravitational potential and an external constant force field. The Hamiltonian of this problem is given by

$$H = \frac{|p|^2}{2} - gq_1 - \frac{s}{|q|},$$

which on its energy hypersurface $\{H + f = 0\}$ is then transformed into the system

$$\hat{H} = \frac{|w|^2}{2} - g(z_1^4 - z_2^4) + f(z_1^2 + z_2^2) - s,$$

which is separable in (z_1, z_2) coordinates. From this we get

Corollary 6. *The Stark problem $(\mathbb{R}^2 \setminus O, g_{flat}, \frac{s}{|q|} + gq_1)$ admits any focused parabola with the q_1 -axis as the main axis as an integrable reflection wall. In particular, by setting respectively $s = 0$ we get that any focused parabola with the q_1 -axis as the main axis is an integrable reflection wall in a uniform gravitational field along the q_1 -direction.*

Note that this argument on the integrability of the Stark problem using conformal transformation provides an alternative proof of the theorem of Korsch-Lang [19].

Frozen-Hill's problem with centrifugal correction Setting $V = gq_1^2 + gq_2^2/4$ gives rise to the so-called frozen-Hill's problem with centrifugal correction [5]. The Hamiltonian of this system is given by

$$H = \frac{|p|^2}{2} - \frac{s}{|q|} - gq_1^2 - \frac{g}{4}q_2^2.$$

Similarly as in the case of the Stark problem, on its energy-hypersurface $\{H + f = 0\}$ we transform the system into

$$\hat{H} = \frac{|p|^2}{2} - s - (z_1^2 + z_2^2)(g(z_1^2 - z_2^2)^2 + gz_1^2z_2^2 - f),$$

which can be written as

$$\hat{H} = \frac{|p|^2}{2} - s - g(z_1^6 + z_2^6) + f(z_1^2 + z_2^2)$$

which is separable in (z_1, z_2) coordinates. We thus get

Corollary 7. *The frozen Hill's problem with centrifugal corrections $(\mathbb{R}^2 \setminus O, g_{flat}, \frac{s}{|q|} + gq_1^2 + \frac{g}{4}q_2^2)$ admits any focused parabola with the q_1 -axis as the main axis as an integrable reflection wall.*

5. Integrable mechanical billiards of the two-center problem

We now consider the two-center problem in the plane $\mathbb{C}$ with the two centers at $-1, 1 \in \mathbb{C}$. The Hamiltonian of this system with mass factors m_1, m_2 (which can take both signs), is given by

$$H = \frac{|p|^2}{2} - \frac{m_1}{|q-1|} - \frac{m_2}{|q+1|}.$$

A classical way to show the integrability of this system uses its separability in elliptic-hyperbolic coordinates [33]. Set $r_1 = |q-1|, r_2 = |q+1|$ and define the elliptic-hyperbolic coordinates as

$$\xi = \frac{r_1 + r_2}{2}, \quad \eta = \frac{r_1 - r_2}{2}.$$

In this coordinate system, the curves $\xi = \text{const.}$ and $\eta = \text{const.}$ describe, respectively, ellipses and branches of hyperbolae in the plane. Note that confocal ellipses in general

intersect a branch of confocal hyperbola in 0 or 2 points and thus the change of coordinates $q \mapsto (\xi, \eta)$ is in general a 2-to-1 transformation. The above Hamiltonian is then transformed into

$$H(p_\xi, p_\eta, \xi, \eta) = \frac{1}{\xi^2 - \eta^2} \left(\frac{1}{2}(\xi^2 - 1)^2 p_\xi^2 - (m_1 + m_2)\xi + \frac{1}{2}(\eta^2 - 1)^2 p_\eta^2 + (m_1 - m_2)\eta \right),$$

in which (p_ξ, p_η) are the conjugate coordinate to (ξ, η) respectively. By fixing $H = -f$ and changing the time by multiplying the Hamiltonian $H + f$ by $(\xi^2 - \eta^2)$ on the zero energy surface, we obtain the new Hamiltonian

$$K(p_\xi, p_\eta, \xi, \eta) = \frac{1}{2}(\xi^2 - 1)^2 p_\xi^2 - (m_1 + m_2)\xi + \frac{1}{2}(\eta^2 - 1)^2 p_\eta^2 + (m_1 - m_2)\eta + f(\xi^2 - \eta^2)$$

which is separable. This shows its integrability. The curves $\xi = \text{const.}$ and $\eta = \text{const.}$ actually give integrable reflection walls of the two-center problem, as we shall establish below. Note that the elliptic-hyperbolic coordinate system is not conformal, therefore we prefer to use the following approach.

The conformal mapping that we are going to use for our purpose is the following one by Birkhoff [2]:

$$z \mapsto q = \frac{z + z^{-1}}{2}, \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C},$$

in real coordinates we have

$$q_1 = z_1 + \frac{z_1}{z_1^2 + z_2^2}, \quad q_2 = z_2 - \frac{z_2}{z_1^2 + z_2^2}.$$

It is conjugate to the complex square mapping by a Möbius Transformation [35], [6].

We use the cotangent lift of this mapping, given by the expression

$$q = \frac{z + z^{-1}}{2}, \quad p = \frac{2w}{1 - \bar{z}^{-2}},$$

to pull the shifted Hamiltonian $K = H - f$ back to the expression

$$\frac{2|z|^4|w|^2}{|z + 1|^2|z - 1|^2} - \frac{2m_1|z|}{|z - 1|^2} - \frac{2m_2|z|}{|z + 1|^2} + f.$$

By changing time on the zero-energy hypersurface, we obtain the new Hamiltonian

$$\hat{K} = \frac{|w|^2}{2} - \frac{m_1|z + 1|^2}{2|z|^3} - \frac{m_2|z - 1|^2}{2|z|^3} + f \frac{|z - 1|^2|z + 1|^2}{4|z|^4} = 0.$$

Proposition 3. *The mapping $z \mapsto \frac{z + z^{-1}}{2}$ pulls confocal ellipses back to two centered circles, and pulls confocal hyperbolae to a pair of lines passing through the center.*

Proof. The family of confocal ellipses with foci at the Kepler centers $(\pm 1, 0)$ is given by the equation

$$\frac{q_1^2}{b^2 + 1} + \frac{q_2^2}{b^2} - 1 = 0, \quad (12)$$

with $b > 0$ as a parameter.

With the conformal mapping we use, the LHS of the above equation is transformed into

$$\begin{aligned} & \frac{(b^2 z_1^2 + (b^2 + 1)z_2^2)((z_1^2 + z_2^2)^2 + 1) + (2b^2 z_1^2 - 2(b^2 + 1)z_2^2)(z_1^2 + z_2^2)}{4b^2(b^2 + 1)(z_1^2 + z_2^2)^2} - 1 \\ &= \frac{(b^2 z_1^2 + (b^2 + 1)z_2^2)((z_1^2 + z_2^2)^2 - 2(2b^2 + 1)(z_1^2 + z_2^2) + 1)}{4b^2(b^2 + 1)(z_1^2 + z_2^2)^2}, \end{aligned}$$

and thus the transformed equation is equivalent to

$$(z_1^2 + z_2^2)^2 - 2(2b^2 + 1)(z_1^2 + z_2^2) + 1 = 0$$

which, seen as a quadratic equation of $z_1^2 + z_2^2$, has two positive solutions, giving rise to two centered circles.

For confocal hyperbolae, we set b in (12) as a purely imaginary number such that $b^2 + 1 > 0$, then the equation

$$(z_1^2 + z_2^2)^2 - 2(2b^2 + 1)(z_1^2 + z_2^2) + 1 = 0$$

has no real-valued solutions and we get that the transformed equation of (12) is equivalent to

$$(b^2 z_1^2 + (b^2 + 1)z_2^2) = 0$$

which describes a pair of lines passing through the origin. Note that they are the two asymptotes of the confocal hyperbola

$$\frac{z_1^2}{b^2 + 1} + \frac{z_2^2}{b^2} - 1 = 0. \quad \square$$

The separability of the (properly-transformed) two-center Hamiltonian in the elliptic-hyperbolic coordinates is thus equivalent to the separability of $\hat{K}$ in polar coordinates. We now verify the latter.

We set $z = re^{i\theta}$, and denote the conjugate momenta by p_θ, p_r respectively. Explicitly we have $w = p_r \mathbf{e}_r + \frac{p_\theta}{r} \mathbf{e}_\theta$. The transformed Hamiltonian $\hat{K}$ into the polar coordinates $(p_r, p_\theta, r, \theta)$ with zero energy zero becomes

$$\hat{K} = \frac{1}{2} \left(p_r^2 + \frac{p_\theta^2}{r^2} - \frac{2(m_1 - m_2) \cos \theta}{r^2} - \frac{(m_1 + m_2)(r^2 + 1)}{r^3} + 2f \frac{r^4 + r^2 + 1 - 4r^2 \cos^2 \theta}{r^4} \right) = 0.$$

By multiplying this by $2r^2$, we obtain

$$r^2 p_r^2 + p_\theta^2 - 2(m_1 - m_2) \cos \theta - \frac{(m_1 + m_2)(r^2 + 1)}{r} + 2f \frac{r^4 + r^2 + 1}{r^2} - 8f \cos^2 \theta = 0,$$

which is now seen to be separable. From this we have the following additional first integral

$$r^2 p_r^2 - \frac{(m_1 + m_2)(r^2 + 1)}{r} + 2f \frac{r^4 + r^2 + 1}{r^2},$$

showing the integrability of the system.

In the next lemma, we establish the integrability of centered circular reflection walls and centered line reflection walls in this system.

Lemma 3. *Any finite combination of centered circles and lines passing through the origin are integrable reflection walls for the system $\hat{K}$ (at its zero-energy level).*

Proof. It is sufficient to check the invariance of p_r^2 before and after the reflection against the reflection walls. For centered circular reflection walls, the θ -component $r\dot{\theta}$ of the conjugate momenta w is preserved and the sign of the r -component $\dot{r}$ is switched after the reflection. At a line passing through the origin, the r -component is preserved and the sign of the θ -component is switched after the reflection. Hence, in both cases, the value $p_r^2 = \dot{r}^2$ is unchanged before and after the reflection against these reflection walls. $\square$

We now deduce the following theorem for billiards defined with the two-center problems:

Theorem 6. *The two-center problem in the plane admits any finite combination of confocal ellipses and confocal hyperbolae with foci at the Kepler centers as an integrable reflection wall.*

Proof. This follows directly from Proposition 3 and Lemma 3. $\square$

Note that the case with a confocal ellipse is a particular case of [20, Sec. 4, Thm. 2].

Remark 4. By letting one mass parameter in the two-center problem be zero, we obtain the Kepler billiards with any finite combination of confocal focused ellipses/hyperbolae with the Kepler center as a focus as an integrable reflection wall directly from the theorem above. This thus provides an alternative proof of Corollary 3.

Remark 5. By letting one mass parameter be zero and sending it to infinity, we obtain the integrable Kepler billiards with a focused parabola as the limiting cases from focused ellipses or hyperbolae. Additionally we may also deduce the same result for focused parabolae with collinear major axes as the limiting case from combinations of focused ellipses/hyperbolae. This argument provides an alternative proof of Corollary 4.

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