Shooting for collinear periodic orbits in the Helium model

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Abstract. The frozen-planet periodic orbit of the classical collinear Helium model with negative energy is shown to exist by a simple shooting argument. This simplifies the approach established in Cieliebak et al. (Ann Inst H Poincaré Anal Non Linéaire 40:379–455, 2022). With this argument, it also follows that the algebraic count of the number of such orbits with a given negative energy is 1, as recently established in Cieliebak et al. (Nondegeneracy and integral count of frozen-planet orbits in helium, 2022. arXiv:2209.12634). The same argument also leads to the existence of other collinear periodic orbits of the classical collinear Helium model.

Mathematics Subject Classification. 70F07, 70F16, 35B10.

1. Introduction

In the classical description, the Helium is modeled as a charged three-particle problem with a nucleus composed with two protons and two nucleons, together with two moving electrons. The nucleus, with charge $+2$, is much heavier than the electrons, and is considered to be fixed in the space. The two electrons q_1, q_2 are moving and both have charge -1. In this problem, the masses of the microscopic particles are much smaller as compared to their charges, and therefore will be neglected when appropriate.

Special collinear periodic orbits, along which all the particles move on a fixed line, play important role in the understanding of the Helium model in the semiclassical theory of quantum mechanics. The physicists have studied their existence and dynamical properties numerically. For all of these, we refer to the physical references [1,2] for more details. These studies give rise to interesting mathematical questions which should be further rigorously investigated. Note that due to the collinearity and the presence of attractive forces, in this note periodic orbits are understood so that double collisions are allowed and are regularized.

We now consider the collinear Helium model and identify the ambient line with $\mathbb{R}$ while putting the nucleus at 0. The electrons have positions $q_1, q_2 \in \mathbb{R}$ (Fig. 1).

According to whether both electra lie on the same side of the nucleus or not, there are two regimes to be considered:

1. (Zee) Both electrons lie on the same side of the nucleus. We assume in this case $0 < q_1 < q_2$.
2. (eZe) The lie on the different sides of the nucleus. We assume in this case $q_1 < 0 < q_2$.

The aim of this note is to explain how shooting argument can provide us the existence of various types of periodic orbits in these regimes. In the (Zee) regime, the existence of frozen-planet orbits has been recently established in [3] with an involved non-local analysis argument. In Sect. 2, we provide an alternative proof of this existence. In Sect. 3, we explain how the same method leads in a rather simple way the existence of plenty of periodic orbits in the (eZe) regime.
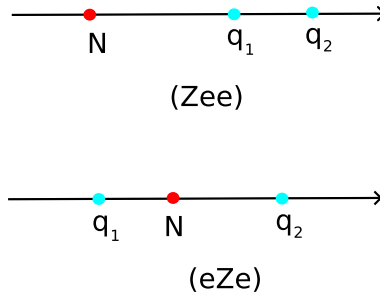
FIG. 1. The (Zee) and (eZe) regimes

2. (Zee): The frozen-planet orbits

In the (Zee)-regime, the Hamiltonian of the system is given by

$$H(p_1, p_2, q_1, q_2) = \frac{p_1^2}{2} + \frac{p_2^2}{2} - \frac{2}{q_1} - \frac{2}{q_2} + \frac{1}{q_2 - q_1},$$

which is defined on the exact symplectic manifold

$$(T^*Q, \lambda = p_1 dq_1 + p_2 dq_2)$$

in which

$$Q := \{(q_1, q_2) \in \mathbb{R}^2, q_1 > q_2 > 0\}.$$

To allow an analysis of collisional orbits in this model, we regularize the collisions of q_1 with $q_0 = 0$ by the usual Levi-Civita regularization [4] restricted to a line, performed on the energy level $\{H = E\}$. See also [5].

This means to pull-back the time-reparametrized system on $\{H - E = 0\}$ given by the Hamiltonian $q_1(H - E)$ by the mapping

$$(z_1, w_1, q_2, p_2) \mapsto (q_1 = z_1^2, p_1 = w_1/(2z_1), q_2, p_2).$$

This procedure leads to the following regularized system with Hamiltonian

$$K(w_1, p_2, z_1, q_2) = \frac{w_1^2}{8} + \frac{z_1^2 p_2^2}{2} + z_1^2 \left(-E - \frac{2}{q_2} + \frac{1}{q_2 - z_1^2} \right) - 2 = 0, \tag{1}$$

which can now be extended to define on the exact symplectic manifold

$$(T^*\tilde{Q}, \lambda = w_1 dz_1 + p_2 dq_2)$$

in which

$$\tilde{Q} := \{(z_1, q_2) \in \mathbb{R}^2, q_2 > z_1^2\}.$$

The exact symplectic manifold $T^*\tilde{Q}$ is equipped with two commuting anti-symplectic involutions

$$\rho_1 := (z_1, w_1, p_2, q_2) \mapsto (-z_1, w_1, -p_2, q_2)$$

and

$$\rho_2 : (z_1, w_1, p_2, q_2) \mapsto (z_1, -w_1, -p_2, q_2).$$

The invariant loci of these two anti-symplectic involutions are, respectively,

- $L_1 := \{z_1 = 0, p_2 = 0\}$, which is an exact Lagrangian submanifold and corresponds to the status that the inner electron is at a collision with the nucleus while the outer electron brakes (having zero velocity); and

- $L_2 := \{w_1 = 0, p_2 = 0\}$, which is also an exact Lagrangian submanifold and corresponds to the status that both electrons brake.

Note that since points with $(z_1, w_1) = (0, 0)$ cannot lie in $\{K = 0\}$, we have

$$L_1 \cap L_2 \cap \{K = 0\} = \emptyset.$$

Now we switch back to the original, non-regularized problem and consider an orbit $\gamma(t), t \in [0, \tau)$ such that

$$p_1(0) = 0, \lim_{t \to \tau - 0} q_1(t) = 0, \quad p_2(0) = 0, \lim_{t \to \tau - 0} p_2(t) = 0, q_2([0, \tau]) > 0.$$

The time t refers to the original time variable. Along such an orbit, the inner electron q_1 moves with initial velocity zero and tends to collide with the nucleus at 0 at time instant $t = \tau$, while there is no collision of q_1 with 0 in the time interval $[0, \tau)$. The outer electron q_2 moves with zero initial velocity and brakes again at time instant $t = \tau$.

By the above regularization procedure, such an orbit is lifted and extended to an orbit chord $\tilde{\gamma}$ connecting L_2 to L_1 in the regularized system in $\{K = 0\}$. It follows from the Hamiltonian equations associated to K, that there hold

$$w_1(0) = 0, z_1(0) \neq 0, q_2(0) > z_1(0)^2, p_2(0) = 0,$$
$$w_1'(0) \neq 0, z_1'(0) = 0, q_2'(0) = 0$$

at L_2, and

$$z_1(\tau) = 0, w_1(\tau) = 4, q_2(\tau) > 0, p_2(\tau) = 0,$$
$$z_1'(\tau) \neq 0, w_1'(\tau) = 0, q_2'(\tau) = 0, p_2'(\tau) = 0$$

at L_1. We have used $'$ to denote the time derivative with respect to the regularized time s, related to the original time by the relationship

$$\mathrm{d}t = z_1^2(s)\mathrm{d}s.$$

We equip $T^*\tilde{Q}$ with its standard Euclidean structure. Then one readily sees that the velocities of the orbit chord $\tilde{\gamma}$ are orthogonal to L_1 and L_2 at its end points, respectively. They are reflected, respectively, by ρ_1 and ρ_2 again to vectors which are orthogonal to L_1 and L_2. Therefore, after reflecting the chord $\tilde{\gamma}$ under ρ_1, ρ_2 and $\rho_1 \circ \rho_2$ and patching these chords, we get from $\tilde{\gamma}$ a periodic orbit of the regularized system. Along such a periodic orbit, q_1 collides and bounces at the nucleus 0, then move away from the nucleus until it brakes and move again toward the nucleus, while q_2 librates far-away from the nucleus. A periodic orbit of this kind is called a *frozen-planet orbit* of Helium [1], [2]. The reason for this terminology is that in observation the libration of q_2 is insignificant as compared to the motion of q_1, and appear to be somehow frozen in a region far-away from the nucleus (Fig. 2).

The following theorem on the existence of a frozen-planet orbit has been recently established in [3] using intensive techniques of calculus of variations and non-local analysis:

Theorem 2.1. *There exists a* frozen-planet orbit *for any negative energy E.*

Here we provide an alternative, elementary proof based on the shooting argument.

Proof. We use a shooting argument to show the existence of a $\gamma(t), t \in [0, \tau)$ in the initial system such that

$$p_1(0) = 0, \lim_{t \to \tau - 0} q_1(t) = 0, p_2(0) = 0, \lim_{t \to \tau - 0} p_2(t) = 0.$$

The conclusion then follows from the previous discussions.
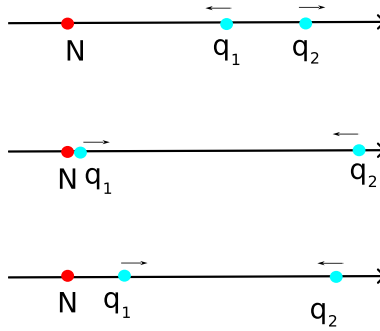
FIG. 2. Motion along a frozen-planet orbit

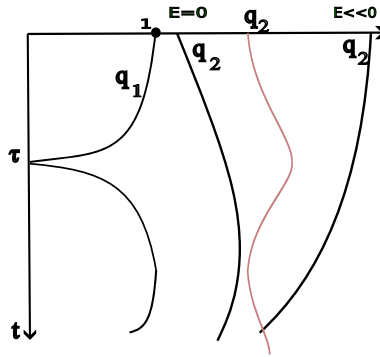FIG. 3. An illustration for the shooting argument

The equations of the motion which we shall use for our analysis are

$$\ddot{q}_1 = -\frac{2}{q_1^2} - \frac{1}{(q_2 - q_1)^2},$$
$$\ddot{q}_2 = -\frac{2}{q_2^2} + \frac{1}{(q_2 - q_1)^2}.$$

(2)

The right-hand side of the last equation is the force on q_2, whose sign depends on the ratio q_1/q_2. In particular, it follows from this equation that $\ddot{q}_2 < 0$ when $q_1/q_2 < 2 + \sqrt{2}$.

By normalization, we set

$$q_1(0) = 1, q_2(0) = x.$$

We call a frozen-planet orbit with this normalization a normalized frozen-planet orbit.

We release q_1, q_2 with this initial configuration, with zero initial velocities, so $\dot{q}_1(0) = \dot{q}_2(0) = 0$. In the Hamiltonian formalism, this corresponds to the condition $p_1(0) = p_2(0) = 0$ (Fig. 3).

The force on q_1 is composed of two parts, namely the attraction from the nucleus and the repulsion from q_2. Now by ignoring the force from q_2 on q_1, we get that τ is smaller than the (first) collision time of q_1 with 0 when it is only attracted by the nucleus at 0. By the theory of the Kepler problem, it is not hard to make a precise computation, but it is enough for us to know from this standard theory that this latter quantity is finite. Consequently we obtain $\tau < \infty$.

Alternatively, this finiteness $\tau < \infty$ follows directly from the fact that in the time interval $[0, \tau]$ we always have $\ddot{q}_1(t) < 0$, thus the particle q_1 moves monotonically from 1 to 0. Consequently the velocity

$\dot{q}_1(t) < 0$ and $|\dot{q}_1(t)|$ increases with time and tends to infinity when $t \to \tau - 0$, i.e., when q_1 moves toward the nucleus.

Now τ is determined implicitly by the equation

$$\int_0^\tau (-\dot{q}_1(t)) \mathrm{d}t = 1.$$

In the regularized system, the corresponding equation reads

$$\int_0^S (-z_1'(s)) \mathrm{d}s = 1, \tag{3}$$

with

$$\tau = \int_0^S z_1(s)^2 \mathrm{d}s \tag{4}$$

and $s = S$ corresponds to $t = \tau$. We have used $'$ to denote the derivative with respect to τ.

By continuous dependence of solutions of ODE on parameters, $z_1(s)$ depends continuously on E. Moreover, $z_1'(s)$ remains finite, with norm bounded from below when s is close to S in the regularized system. It thus follows from (3) that S depends also continuously on E. Consequently by (4), τ depends continuously on E as well. Consequently, we obtain that the mapping

$$\mathbb{R} \to \mathbb{R}, E \mapsto \dot{q}_2(\tau(E))$$

is continuous.

Since the initial velocities are zero, the energy is related to the normalized initial configuration by the relation

$$E = -\frac{2}{x} - 2 + \frac{1}{x-1}. \tag{5}$$

Thus,

$$\frac{\mathrm{d}E}{\mathrm{d}x} = \frac{2}{x^2} - \frac{1}{(x-1)^2}. \tag{6}$$

From (6), we deduce that the monotonicity of E with respect to x changes at the critical point $x = q_2^* := 2 + \sqrt{2}$: we obtain $\dfrac{\mathrm{d}E}{\mathrm{d}x} < 0$ when $1 < x < q_2^*$ and $\dfrac{\mathrm{d}E}{\mathrm{d}x} > 0$ when $x > q_2^*$.

When $x > q_2^*$, the combined force on q_2 attracts it toward the nucleus. Moreover, since $0 < q_1 < q_2$, the force on q_1 is always stronger than the force on q_2, so there holds

$$\dot{q}_1(t) < \dot{q}_2(t) < 0, t \in (0, \tau).$$

So when q_1 reaches the position $1 - \delta, \delta \in (0, 1)$ we have at the same instant $q_2 \in (x - \delta, x)$. By considering the ratio q_1/q_2 when $q_1 = 1 - \delta, \delta \in (0, 1)$, we obtain that in the time interval $(0, \tau)$ there holds

$$\frac{q_1}{q_2} < \frac{1-\delta}{x-\delta} < \frac{1-\delta}{2+\sqrt{2}-\delta} < \frac{1}{2+\sqrt{2}},$$

which combined with (2) thus leads to

$$\ddot{q}_2(t) < 0, t \in (0, \tau).$$

We thus get that no triple collision happens in the time interval $[0, \tau]$, and $\dot{q}_2(\tau) < 0$.

It can be shown that $\dot{q}_2(\tau) > 0$ when x is sufficiently close to 1, corresponding to the case that E is sufficiently large. We shall not discuss this and defer the discussion to Proposition 2.2 as we aim to show

the existence of an orbit with negative energy E. For this purpose, we now consider the case $E = 0$ and we aim to show that $\dot{q}_2(\tau) > 0$ in this case.

When $E = 0$, we have in this case[1]

$$x = (1 + \sqrt{17})/4 \approx 1.2808 < q_2^*.$$

We observe from the equations of motion (2) that there always holds

$$|\ddot{q}_1(t)| > |\ddot{q}_2(t)|$$

and thus

$$|\dot{q}_1(t)| > |\dot{q}_2(t)|$$

for $t \in [0, \tau)$.

The initial acceleration of q_1 is

$$\ddot{q}_1(0) = -\frac{1}{(x-1)^2} - \frac{2}{x^2} \approx -13.902.$$

The initial acceleration of q_2

$$\ddot{q}_2(0) = \frac{1}{(x-1)^2} - \frac{2}{x^2} \approx 11.464.$$

Since $|\ddot{q}_1(t)| > |\ddot{q}_2(t)|$, in the time interval $[0, \tau)$, when q_1 moves from 1 to $1 - \sigma$, $\sigma \in (0, 1)$, the particle q_2 will not reach the point $x + \sigma$, i.e., will stay in the interval $(1 - \sigma, x + \sigma)$.

We consider the following function describing the force on q_2:

$$F(q_2, q_1) := -\frac{2}{q_2^2} + \frac{1}{(q_2 - q_1)^2}.$$

As long as $q_1 < q_2$, this function is monotonically increasing in the q_1 variable. Its partial derivative with respect to q_2 is

$$\frac{\partial F(q_2, q_1)}{\partial q_2} = \frac{4}{q_2^3} - \frac{2}{(q_2 - q_1)^3}$$

which is negative, so $F(q_2, q_1)$ is monotonically decreasing in the q_2-variable, as long as

$$1 < q_2/q_1 < 2^{2/3} + 2^{1/3} + 2 \approx 4.8473.$$

Thus, for $q_2 \in (1 - \sigma, x + \sigma)$ with

$$\frac{x + \sigma}{1 - \sigma} < 2^{2/3} + 2^{1/3} + 2$$

there holds

$$F(q_2, 1 - \sigma) > F(x + \sigma, 1 - \sigma).$$

The equation

$$F(x + \sigma, 1 - \sigma) = 0$$

has only one real root at

$$\sigma_0 = -\frac{3\sqrt{17}}{28} + \frac{13}{28} + \frac{5\sqrt{2}}{28} + \frac{\sqrt{34}}{28} \approx 0.48332.$$

Consequently, we have

$$F(q_2, q_1) \geq 0$$

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[1]In this argument, we shall need to compute numerical values of quantities, which can be done with a standard calculator or a scientific computation software.

till q_1 to move from 1 to the position $1 - \sigma_0 \approx 0.51668$: This follows from the estimates

$$\frac{q_2}{q_1} < \frac{x + \sigma}{1 - \sigma} \leq \frac{x + \sigma_0}{1 - \sigma_0} \approx 3.4141 < 2^{2/3} + 2^{1/3} + 2 \approx 4.8473,$$

and thus

$$F(q_2, q_1) \geq F(q_2, 1 - \sigma_0) \geq F(x + \sigma_0, 1 - \sigma_0) \geq 0. \tag{7}$$

Therefore, the velocity $\dot{q}_2(t)$ does not decrease, and therefore q_2 keeps moving away from the nucleus before q_1 reaches the position $1 - \sigma_0 \approx 0.51668$.

We now estimate the velocity $\dot{q}_2$ at the instant $t_1 \in (0, \tau)$ that q_1 reaches the position 0.9. The acceleration of q_1 is given by

$$\ddot{q}_1(t) = -\frac{2}{q_1(t)^2} - \frac{1}{(q_2(t) - q_1(t))^2}.$$

In the time interval $[0, t_1]$, we have that $\dot{q}_1(t) < 0$ and $\dot{q}_2(t) - \dot{q}_1(t) > 0$. Thus, when q_1 moves from 1 to 0.9, we have

$$|\ddot{q}_1(t)| \leq \frac{2}{0.9^2} + \frac{1}{(x - 1)^2} \approx 15.152.$$

From the relationship among travel distance, velocity and acceleration of a particle we get

$$\int_0^{t_1} \int_0^t \ddot{q}(\tilde{t}) \, \mathrm{d}\tilde{t} \mathrm{d}t = -0.1,$$

we have now

$$\int_0^{t_1} \left(\frac{2}{0.9^2} + \frac{1}{(x - 1)^2} \right) t \mathrm{d}t \geq 0.1.$$

The positive solution t_1^* of the equation

$$\left(\frac{2}{0.9^2} + \frac{1}{(x - 1)^2} \right) \cdot \frac{t^2}{2} = 0.1$$

is thus a lower bound for t_1. Solving the above equation, we get

$$t_1 \geq t_1^* := \frac{9\sqrt{10}\sqrt{\left(13 - 3\sqrt{17}\right)\left(406 - 75\sqrt{17}\right)}}{8120 - 1500\sqrt{17}} \approx 0.11491.$$

Next we have that by (7), when q_1 moves from 1 to 0.9, a lower bound for $\ddot{q}_2(t)$ is given by

$$F(x + 0.1, 0.9) = -2\left(\frac{7}{20} + \frac{\sqrt{17}}{4} \right)^{-2} + \left(-\frac{11}{20} + \frac{\sqrt{17}}{4} \right)^{-2} \approx 3.2768.$$

Therefore, we have that the velocity

$$\dot{q}_2(t_1) = \int_0^{t_1} \ddot{q}_2(t)\mathrm{d}t$$

has the lower bound

$$t_1^* \cdot F(x + 0.1, 0.9) \approx 0.37653.$$

Now since $\dot{q}_2$ does not decrease till q_1 reach $1 - \sigma_0$, this lower bound for $\dot{q}_2(t_1)$ is also a lower bound for $\dot{q}_2(\tau')$ at the instant τ' that q_1 reaches the position $1 - \sigma_0$.

We check now how much at most can $\dot{q}_2$ decrease till the instant τ when q_1 collides with the nucleus. For this purpose, we first estimate $\tau - \tau'$. This elapsed time has an upper bound, which is the elapsed time of q_1 from $1 - \sigma_0$ to 0 in the Coulomb system

$$\ddot{q}_1 = -\frac{2}{q_1^2},$$

which describes the motion of q_1 with the pure Coulomb attraction from the nucleus, while the repulsion from q_2 is dropped.

In this pure Coulomb problem, $\ddot{q}_1$ is negative and decreases in the time interval $t \in [0, \tau)$, as well as the velocity $\dot{q}_1$. This means q_1 travels faster and faster toward the nucleus. Therefore, this elapsed time has a (rather loose) upper bound $(1 - \tau_0)T$ for $T = \frac{\sqrt{6}\pi}{12}$ the half-period of the pure Coulomb motion. This upper bound is approximately 0.33133.

We now derive a lower bound for $\ddot{q}_2(t)$ in the time interval $[\tau', \tau]$. We assume that $\dot{q}_2$ changes sign, and reaches the point 1 for the first time at the time instant $\tau'' \in (\tau', \tau]$. Then for $t \in [\tau', \tau'']$, we have $q_2(t) \geq 1$, and

$$\ddot{q}_2(t) \geq -\frac{1}{q_2^2(t)} \geq -1,$$

which is obtained by comparing with the situation that q_1 stays colliding with the nucleus. The velocity $\dot{q}_2$ has to be zero at the instant of brake τ_b, which by assumption is in the time interval (τ', τ''). Thus, we have the estimation

$$\dot{q}_2(t) \geq -1 \cdot (1 - \sigma_0)T \approx -0.33133, t \in [\tau_b, \tau''].$$

But then in the time interval $[\tau_b, \tau'']$, the particle q_2 can travel in the negative direction at most for a distance of $(1 - \sigma_0)^2 T^2 \approx 0.10978$. On the other hand, it has to travel at least from x to 1, which has distance $(x - 1) \approx 0.28080$. Contradiction. So whenever q_2 changes its direction of motion or not, it cannot reach the point 1 in the time interval $(0, \tau]$, and thus the estimate

$$\ddot{q}_2 \geq -1$$

holds in the time interval $[\tau', \tau]$.

We then have

$$\dot{q}_2(t) \geq \dot{q}_2(t_1) - (1 - \sigma_0) \cdot T \approx 0.04520 > 0, \qquad t \in [t_1, \tau].$$

Consequently, when $E = 0$, the particle q_2 does not change its direction of motion in the time interval $(0, \tau)$ and thus $\dot{q}_2(\tau) > 0$.

Previously we have shown that when $x > q_2^*$, we have $\dot{q}_2(\tau) < 0$. Thus by continuity, there exists an x in the interval $((1 + \sqrt{17})/4, q_2^*]$ such that the corresponding orbit has $\dot{q}_2(\tau) = 0$, and thus gives rise to a normalized frozen-planet orbit. Moreover, this orbit has negative energy, as it follows from (5) that $E < 0$ for any $x \in ((1 + \sqrt{17})/4, q_2^*]$.

Finally, it is classically known that the system has a rescaling symmetry: If $q(t) = (q_1(t), q_2(t))$ is a solution of the system with energy E, then $\lambda^2 q(\lambda^{-3} t)$ is also a solution, now with energy $\lambda^{-2} E$. With this observation, we conclude that there exists a frozen-planet orbit with any negative energy. $\qquad \square$

The following result has been deduced in [6], for which we also provide a short proof.

Proposition 2.2. *For any negative energy E, the algebraic count of the number of frozen-planet orbit is* 1.

Proof. We continue with the previous proof, and consider the case that x is sufficiently close to 1, with $x = 1 + \nu$ for $\nu > 0$ small. This corresponds to the case that $E >> 0$ is large by (5). We consider the motion of q_2 which move away from the position x.

Remind that the system can be rescaled and if $q(t) = (q_1(t), q_2(t))$ is a solution with velocity $\dot{q}(t) = (\dot{q}_1(t), \dot{q}_2(t))$, so is $\lambda^2 q(\lambda^{-3} t)$ with velocity $\lambda^{-1} q(\lambda^{-3} t)$.

We rescale the system by the factor $1/\nu$, and consider the motion from the initial condition

$$q_2(0) - q_1(0) = 1, \dot{q}_2(0) = \dot{q}_1(0) = 0.$$

Note that now the nucleus is at the distance $1/\nu$ to $q_1(0)$. For the motion of q_2 before its first break time, it follows from (2) that $\ddot{q}_2(t)$ is bounded in the interval

$$\left[-\frac{2}{(1 + 1/\nu)^2} + \frac{1}{(q_2(t) - q_1(t))^2}, \frac{1}{(q_2(t) - q_1(t))^2} \right).$$

So provided ν is sufficiently small, the motions of q_1 and q_2 are $O(\nu^2)$-perturbations of their repulsive two-body motions on a fixed time interval. Consequently, we have $\dot{q}_2(t) > 0$ for $t \in (0, 1]$ for ν sufficiently small. We set $v_2 = \dot{q}_2(1) > 0$.

Now we rescale back by setting $\lambda = \nu^{1/2}$. Then we get back to the initial conditions we first considered in the proof of Theorem 2.1. With the rescaling we get that $\dot{q}_2(t) > 0$ for $t \in (0, \nu^{3/2}]$ with $\dot{q}_2(\nu^{3/2}) = \nu^{-1/2} v_2 \sim \nu^{-1/2}$. Therefore for ν sufficiently small, there is a time instant, namely $\nu^{3/2}$ on which $\dot{q}_2$ can be arbitrarily large.

We estimate the force on q_2 from the time instant $\nu^{3/2}$ till it breaks in a fairly rough way, as

$$F(q_2, q_1) := -\frac{2}{q_2^2} + \frac{1}{(q_2 - q_1)^2} \geq -2.$$

With this we conclude that the time that q_2 does not break in the time interval $[0, \nu^{-1/2} v_2/2]$.

On the other hand, by ignoring the repulsion of q_2 on q_1 we conclude that the first collision time τ of q_1 is uniformly bounded by the half-period T of the pure Coulomb motion of q_1 attracted by the nucleus from 1 to 0. As this quantity is independent of ν we conclude that $\dot{q}_2(\tau) > 0$ provided ν sufficiently small, corresponding to the case that x is sufficiently close to 1.

Combined with the conclusion from the previous proof that $\dot{q}_2(\tau) < 0$ when x is sufficiently large, we get that the algebraic count of zeros of the function $\mathbb{R} \mapsto \mathbb{R}, E \mapsto \dot{q}_2(\tau)$ is 1. Thus, the algebraic count of the frozen-planet orbit is 1 for any negative energy. $\qquad\square$

3. (eZe) Various periodic orbits

The Hamiltonian of the system is now

$$H = \frac{p_1^2 + p_2^2}{2} + \frac{2}{q_1} - \frac{2}{q_2} + \frac{1}{q_2 - q_1}.$$

in which $q_1 < 0, q_2 > 0$.

On the energy level $\{H = E\}$ a simultaneous regularization of non-simultaneous double collisions is given by application of Levi-Civita regularizations to each of the particles: We set

$$q_1 = -z_1^2, q_2 = z_2^2, p_1 = -\frac{w_1}{2z_1}, p_2 = \frac{w_2}{2z_2}$$

in which one accounts the negativity of q_1 for the choice of the transformation.

After a proper time change with the factor $-q_1 q_2$, we get the Hamiltonian

$$\frac{-q_1 q_2 p_1^2 - q_1 q_2 p_2^2}{2} - 2q_2 + 2q_1 - \frac{q_1 q_2}{q_2 - q_1} + E q_1 q_2 = 0,$$

and further the regularized Hamiltonian

$$\frac{z_2^2 w_1^2 + z_1^2 w_2^2}{8} - 2z_2^2 - 2z_1^2 + \frac{z_1^2 z_2^2}{z_2^2 + z_1^2} - E z_1^2 z_2^2 = 0. \tag{8}$$

Note that the set of triple collisions $\{q_1 = q_2 = 0\}$ is now transformed into $\{z_1 = z_2 = 0\}$ which remains singular. Only non-simultaneous double collisions of q_1 and q_2 with 0 are regularized in the regularized system (8).

A collinear periodic orbit in this setting is called of type (n, m) for $n, m \in \mathbb{N}_+$ if during one period q_1 goes n-times from brake to collision and back, and q_2 goes m-times so.

To show the existence of these orbits, we consider to shoot from the singular set $\{q_1 = 0, q_2 > 0, \dot{q}_2 = 0\}$ to the singular set $\{q_1 < 0, \dot{q}_1 = 0, q_2 = 0\}$. After regularization, these sets correspond, respectively, to the sets

$$\{z_1 = 0, z_2 > 0, w_2 = 0\}$$

and

$$\{z_1 > 0, w_1 = 0, z_2 = 0\},$$

which are (open) Legendrian submanifolds in the regularized energy hypersurface. As in previous discussion, a regularized orbit connecting them without passing through triple collisions can again get reflected and be patched to a closed orbit in the regularized system free from triple collisions.

To be able to consider shooting in the non-regularized system between singular sets, we rewrite the Hamiltonian of the system as

$$H = H_{Kep,1} + \frac{p_2^2}{2} - \frac{2}{q_2} + \frac{1}{q_2 - q_1}.$$

with

$$H_{Kep,1} = \frac{p_1^2}{2} + \frac{2}{q_1}$$

being the Coulomb energy of q_1: This is a quantity which thanks to the regularization [4] is well-defined at collisions of q_1 with 0, despite the singularity at $q_1 = 0$. See also the discussion in [5,7] for this point.

Next we show the existence of periodic orbits of type $(1, 2n - 1)$. As normalization, we may normalize so that

$$H_{Kep,1}(0) = -1.$$

The parameter we take is the total energy E of the system, which uniquely determines the position $q_2(0) = x > 0$ that we will take as a parameter in the shooting argument. Note that this implies $E < -1$ by the expression of the Hamiltonian, with $E \to -1$ when $x \to +\infty$.

We assume that there are no triple collisions. For any $n \in \mathbb{N}_+$, the n-th collision times and brake times of q_1 and q_2 with 0 depend continuously on E: This follows from the same type of argument as in the previous proof for the (Zee) case.

Theorem 3.1. *There exists a collinear periodic orbit of type $(1, 2n - 1)$ or of type $(2n - 1, 1)$ for $n \in N_+$ with any negative energy.*

Proof. We treat the case for periodic orbits of type $(1, 2n - 1)$. The other case is completely similar. Again we search for an orbit starting from the status that $q_1(0)$ collides with the nucleus and $q_2(0) > 0$ brakes, $\dot{q}_2(0) = 0$, to the status that q_1 achieve its n-th brake while q_2 collides with the nucleus.

We set

$$\tau = (\text{1-st collision time of } q_2) - (\text{n-th brake time of } q_1)$$

which depends continuously on E, and we wish to show that there exists an energy $E = E_0$ with $\tau(E_0) = 0$. This follows from the following simple consideration, which we indicate in an intuitive way:

When $x = q_2(0)$ is sufficiently close to 0, the energy E is small and thus q_2 collides first with 0 before q_1 brakes. Thus, for all n we may have $\tau < 0$ when E is sufficiently small. When $x = q_2(0)$ is sufficiently large, $E < 0$ is sufficiently close to -1, the point $x = q_2(0) >> 1$ is sufficiently far-away from 0 and q_2

can be made to move to 0 after the n-th brake of q_1. Indeed the time for q_1 to reach its n-th brake has an upper bound, which is $(n-1/2)$ times the Coulomb period of q_1 with the pure attraction from the nucleus. Now consider the motion of q_2 from x to 1. An upper bound of $|\ddot{q}_2|$ is obtained from the pure Coulomb attracting force from the nucleus on q_2 when q_2 reaches 1. With this upper bound, we conclude that when x is sufficiently large, $q_2 > 1$ holds till the n-th brake of q_1 happen. Thus when E is sufficiently close to 1, we have $\tau > 0$.

The conclusion follows by continuity of τ on E as well as the symmetric considerations in the regularized system to patch chords into a periodic orbit similar to the argument from the previous section.

This argument thus gives a periodic orbit of the prescribed type for an energy $E \leq -1$. The same rescaling argument as in the previous section thus shows the existence of an orbit of prescribed type for any negative energy. □

It is a rather different story to analyze periodic orbits containing, or close to total triple collisions. Various periodic orbits have been identified with the help of a symbolic dynamics of triple-collision orbits [2]. These have been nicely discussed in [8, Chap. 42] and references therein. See also [9]. As this requires rather different techniques we shall not discuss these orbits in this note.

Acknowledgements

Declarations

Conflict of interest The authors declare no competing interests.

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