

Quantum evolution with random phase scattering

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Abstract

We consider the quantum evolution of a fermion–hole pair in a d -dimensional gas of non-interacting fermions in the presence of random phase scattering. This system is mapped onto an effective Ising model, which enables us to show rigorously that the probability of recombining the fermion and the hole decays exponentially with the distance of their initial spatial separation. In contrast, without random phase scattering the recombination probability decays like a power law, which is reflected by an infinite mean square displacement. The effective Ising model is studied within a saddle point approximation and yields a finite mean square displacement that depends on the evolution time and on the spectral properties of the deterministic part of the evolution operator.

Keywords: random phase scattering, localization of fermion–hole pairs, effective Ising model

1. Introduction

The creation of an electron–hole pair and its subsequent recombination is a fundamental process in quantum physics with many applications in different fields, such as quantum electrodynamics [1, 2] and semiconductor physics [3, 4] with applications in electronics, photoelectric devices [5] and light-emitting diodes [6]. Although there exist phenomenological descriptions of this process by classical decay models based on diffusion [7, 8], for a deeper understanding a quantum approach is required. We will focus here on a fermion–hole pair in a d -dimensional system of non-interacting fermions. The pair can be created either by photons through the photoelectric effect [9] or by phonons through electron–phonon coupling [10] in a real material. Then the question is, whether this pair recombines after some evolution by emitting a photon/phonon or the fermion and the hole remain localized near the place where they were created initially (see figure 1). Both possibilities can be studied by measuring the return probability to the initial quantum state. This probability depends on the spatial separation of

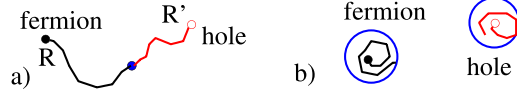


Figure 1. (a) There is a recombination of a fermion created at \mathbf{R} and a hole created at \mathbf{R}' at the blue dot. (b) The localization of a fermion and a hole near their points of creation implies an exponentially small probability for recombination. The localization radius (or decay length) is indicated by the blue circles. In this sketch we consider $|\mathbf{R} - \mathbf{R}'|$ much larger than the localization radius.

the fermion and the hole. Assuming that the hole is created at the site \mathbf{R} and the fermion at the site \mathbf{R}' , we can define the probability $P_{\mathbf{R}\mathbf{R}'}$ that the system returns to the initial state over the finite time interval τ . Although it is plausible that this probability decreases with increasing distance $|\mathbf{R} - \mathbf{R}'|$, the law of change with the distance depends on the interaction with the environment. For instance, on a periodic lattice this probability has a long range behavior, which depends on the dimensionality of the underlying space. In the following we will focus on the effect of random phase scattering on the spatial decay of this probability. In other words, is it possible to control the spatial fermion–hole separation to avoid their recombination?

To analyze the evolution and calculate physical quantities, the standard procedure would be to diagonalize the Hamiltonian H of the evolution operator $e^{-iH\tau/\hbar}$. For a translational invariant system this can be achieved through a Fourier transformation. However, in a realistic system the Hamiltonian H is not translational invariant but subject to some disorder. In this case the corresponding random Hamiltonian cannot be diagonalized by a Fourier transformation. To mimic the effect of disorder in the evolution of a quantum state we ‘scramble’ a translational invariant $e^{-iH\tau/\hbar}$ with a random phase factor $e^{i\alpha}$ by using the evolution operator $U = e^{i\alpha} e^{-iH\tau/\hbar}$. This choice was inspired by the random unitary gate models that have been discussed in the context of quantum circuits [11–13]. The following analysis is also inspired by previous studies of the invariant measure of transport in systems with random chiral Hamiltonians [14]. Although this seems to be an entirely different problem, there are some striking similarities that are reflected by their graphical representations.

2. Summary of the main results

The central result of this work is that the random phase scattering of non-interacting fermions is equivalent to scattering on discrete Ising spins or on a continuous (real) Ising field. This is a consequence of a geometric restriction in the graphical representation due to the Fermi statistics. In section 4 we derive the final result of equation (10), which states that the return probability becomes an integral with respect to the Ising field ϕ as

$$P_{\mathbf{R}'\mathbf{R}} = \langle |(\phi + h)_{\mathbf{R}\mathbf{R}'}^{-1}|^2 \rangle_\phi \quad \text{with} \quad \langle \dots \rangle_\phi = \frac{1}{\mathcal{N}_\phi} \int e^{-\frac{1}{2} \sum_{\mathbf{r}} \phi_{\mathbf{r}}^2} |\det(\phi + h)|^2 \dots \prod_{\mathbf{r}} d\phi_{\mathbf{r}}.$$

This integral enables us to deform the Ising field integration such that the poles of the integrand of the return probability $P_{\mathbf{R}'\mathbf{R}}$ are avoided. This implies an exponential decay with respect to $|\mathbf{R} - \mathbf{R}'|$. For an explicit evaluation of the decay we employ a saddle point approximation of the Ising field. This provides for $h = e^{-iH\tau}$

$$P_{\mathbf{R}'\mathbf{R}} = \langle |(\phi + h)_{\mathbf{R}\mathbf{R}'}^{-1}|^2 \rangle_\phi \approx |(\phi_0 + h)_{\mathbf{R}-\mathbf{R}'}^{-1}|^2,$$

where ϕ_0 is determined by a saddle point equation. The corresponding mean square displacement reads

$$[R_\nu^2] = \frac{\tau^2}{2} \int_{\mathbf{k}} \frac{(\partial_{k_\nu} \epsilon_{\mathbf{k}})^2}{[1 + \phi_0^2 + 2\phi_0 \cos(E_0 + \epsilon_{\mathbf{k}}\tau)]^2} / \int_{\mathbf{k}} \frac{1}{1 + \phi_0^2 + 2\phi_0 \cos(E_0 + \epsilon_{\mathbf{k}}\tau)},$$

where $\epsilon_{\mathbf{k}}$ is the dispersion of the Hamiltonian H and E_0 is related to the Fermi energy. Moreover, in the absence of random phase scattering we have $\phi_0 = 1$, which implies for $E_0 + \epsilon_{\mathbf{k}}\tau \leq \bar{E}$

$$[R_\nu^2] \sim \frac{2\pi\tau}{3d\mu} (\pi - \bar{E})^{-2}$$

when $\bar{E} < \pi$, and $[R_\nu^2]$ is infinite for $\bar{E} \geq \pi$.

3. Quantum evolution with random phase scattering

A system of non-interacting fermions with the Hamiltonian H evolves during a fixed time step τ with the unitary evolution operator $U_\tau = e^{i\alpha} e^{-iH\tau}$. Here and subsequently we have chosen the scale of physical quantities such that $\hbar = 1$. The phases $\{\alpha_{\mathbf{r}}\}$ are randomly distributed on $[-\pi, \pi)$, independently on different lattice sites \mathbf{r} . U_τ acts on the $2^{|\Lambda|}$ dimensional Hilbert space, spanned by the fermionic states $|\{n_{\mathbf{r}}\}\rangle$ with occupation numbers $n_{\mathbf{r}} = 0$ or $n_{\mathbf{r}} = 1$ on a lattice site $\mathbf{r} \in \Lambda$. $|\Lambda|$ is the number of lattice sites and $|\{n(0)\}\rangle \equiv |\{n_{\mathbf{r}}(0)\}\rangle$ is the initial state, in which the fermionic system is prepared at the beginning. Then we consider the situation in which a fermion–hole pair is created by the anti-commuting fermion operators $c_{\mathbf{R}'}^\dagger$ and $c_{\mathbf{R}}$ at time $t = 0$ at different sites \mathbf{R}, \mathbf{R}' . To determine the spatial correlation of the fermion–hole pair after the time τ , where the quantum state is the initial state again, we define the return probability

$$\langle |\langle \{n(0)\} | U_\tau c_{\mathbf{R}'}^\dagger c_{\mathbf{R}} | \{n(0)\} \rangle|^2 \rangle_\alpha. \quad (1)$$

It describes the probability of a process that starts from the initial state $|\{n(0)\}\rangle$, immediately followed by the creation a fermion–hole pair at time $t = 0$ at sites \mathbf{R} and \mathbf{R}' . After a unitary evolution over the time τ , represented by the random unitary operator U_τ , the probability for the overlap with the initial state is determined. This probability is eventually averaged with respect to the random phases of U_τ . To avoid the specific definition of the initial state, we (i) sum over the return probabilities of all basis states and (ii) calculate the ratio of the return probabilities with and without the fermion–hole pair

$$P_{\mathbf{R}\mathbf{R}'} := \frac{\langle |Tr(U_\tau c_{\mathbf{R}'}^\dagger c_{\mathbf{R}})|^2 \rangle_\alpha}{\langle |Tr U_\tau|^2 \rangle_\alpha}, \quad (2)$$

where Tr is the trace of $2^{|\Lambda|} \times 2^{|\Lambda|}$ matrices. For $\tau = 0$ we have

$$\langle \{n\} | c_{\mathbf{R}'}^\dagger c_{\mathbf{R}} | \{n\} \rangle = \delta_{\mathbf{R}\mathbf{R}'} \delta_{n_{\mathbf{R}}, 1},$$

such that only the particle number operator $c_{\mathbf{R}}^\dagger c_{\mathbf{R}}$ contributes to the trace, while the spatially separated fermion–hole pair does not. For $\tau > 0$, though, the evolution $U_\tau c_{\mathbf{R}'}^\dagger c_{\mathbf{R}} | \{n\} \rangle$ can create some overlap with $|\{n\}\rangle$, which contributes to the trace. Thus, the return probability $P_{\mathbf{R}\mathbf{R}'}$ ($\mathbf{R}' \neq \mathbf{R}$) is a measure for how effective the evolution with U_τ can move the fermion–hole pair to the same site. It is plausible that this is less likely the larger the distance $|\mathbf{R} - \mathbf{R}'|$ is and that it increases with increasing time τ . Therefore, $P_{\mathbf{R}\mathbf{R}'}$ decays with this distance and may increase with τ .

Besides creating a fermion and a hole simultaneously, we can also create a hole at site \mathbf{R} and time $t = 0$, let this hole evolve for the time τ and then annihilate it. The probability for this annihilation process reads

$$P'_{\mathbf{R}\mathbf{R}'} := \frac{\langle |Tr(c_{\mathbf{R}}^\dagger U_\tau c_{\mathbf{R}})|^2 \rangle_\alpha}{\langle |Tr U_\tau|^2 \rangle_\alpha}. \quad (3)$$

In the following we drop the index τ for simplicity and use $U_\tau \equiv U$. Assuming that $|\{\tilde{n}_{\mathbf{q}}\}\rangle \equiv |\{\tilde{n}\}\rangle$ are eigenstates of U for a special realization of the phases $\{\alpha_{\mathbf{r}}\}$, we obtain

$$\begin{aligned} \tilde{C}_{\mathbf{Q},t} &:= Tr(U^{(1-t)} c_{\mathbf{Q}}^\dagger U^t c_{\mathbf{Q}}) \\ &= \sum_{\{\tilde{n}_{\mathbf{q}}\}, \{\tilde{n}'_{\mathbf{q}}\}} \langle \{\tilde{n}\} | U^{(1-t)} | \{\tilde{n}\} \rangle \langle \{\tilde{n}\} | c_{\mathbf{Q}}^\dagger | \{\tilde{n}'\} \rangle \langle \{\tilde{n}'\} | U^t | \{\tilde{n}'\} \rangle \langle \{\tilde{n}'\} | c_{\mathbf{Q}} | \{\tilde{n}\} \rangle \end{aligned} \quad (4)$$

for $t = 0, 1$. With $\langle \{\tilde{n}\} | c_{\mathbf{Q}}^\dagger | \{\tilde{n}'\} \rangle \langle \{\tilde{n}'\} | c_{\mathbf{Q}} | \{\tilde{n}\} \rangle = \delta_{\tilde{n}'_{\mathbf{Q}}, 0} \delta_{\tilde{n}_{\mathbf{Q}}, 1} \prod_{\mathbf{q} \neq \mathbf{Q}} \delta_{\tilde{n}'_{\mathbf{q}}, \tilde{n}_{\mathbf{q}}}$ we get

$$\tilde{C}_{\mathbf{Q},t} = \sum_{\{\tilde{n}_{\mathbf{q}}\}, \{\tilde{n}'_{\mathbf{q}}\}} \langle \{\tilde{n}\} | U^{(1-t)} | \{\tilde{n}\} \rangle \langle \{\tilde{n}'\} | U^t | \{\tilde{n}'\} \rangle \delta_{\tilde{n}'_{\mathbf{Q}}, 0} \delta_{\tilde{n}_{\mathbf{Q}}, 1} \prod_{\mathbf{q} \neq \mathbf{Q}} \delta_{\tilde{n}'_{\mathbf{q}}, \tilde{n}_{\mathbf{q}}}.$$

Since U is diagonal in this basis with $\langle \{\tilde{n}\} | U | \{\tilde{n}\} \rangle = \prod_{\mathbf{q}} \langle \tilde{n}_{\mathbf{q}} | e^{-iE_{\mathbf{q}} \tilde{n}_{\mathbf{q}}} | \tilde{n}_{\mathbf{q}} \rangle = \prod_{\mathbf{q}} e^{-iE_{\mathbf{q}} \tilde{n}_{\mathbf{q}}}$, we obtain a product of diagonal matrix elements

$$\langle \{\tilde{n}\} | U^{(1-t)} | \{\tilde{n}\} \rangle \langle \{\tilde{n}'\} | U^t | \{\tilde{n}'\} \rangle = \prod_{\mathbf{q}} e^{-iE_{\mathbf{q}} \tilde{n}_{\mathbf{q}} (1-t)} e^{-iE_{\mathbf{q}} \tilde{n}'_{\mathbf{q}} t}.$$

Inserting this into equation (4), we get for the sum due to the Kronecker deltas

$$\tilde{C}_{\mathbf{Q},t} = Tr(U^{(1-t)} c_{\mathbf{Q}}^\dagger U^t c_{\mathbf{Q}}) = e^{-iE_{\mathbf{Q}} (1-t)} \prod_{\mathbf{q} \neq \mathbf{Q}} (1 + e^{-iE_{\mathbf{q}}}) = \frac{e^{iE_{\mathbf{Q}} t}}{1 + e^{iE_{\mathbf{Q}}}} \prod_{\mathbf{q}} (1 + e^{-iE_{\mathbf{q}}}). \quad (5)$$

Finally, we return to the real-space representation to obtain $e^{-iE_{\mathbf{q}}} \rightarrow \hat{U}_{\mathbf{r}\mathbf{r}'}$, where \hat{U} is a $|\Lambda| \times |\Lambda|$ matrix on the lattice, and

$$Tr(U c_{\mathbf{R}'}^\dagger c_{\mathbf{R}}) = (\mathbf{1} + \hat{U}^\dagger)_{\mathbf{R}\mathbf{R}'}^{-1}, \det(\mathbf{1} + \hat{U}), \quad Tr(c_{\mathbf{R}'}^\dagger U c_{\mathbf{R}}) = (\mathbf{1} + \hat{U})_{\mathbf{R}\mathbf{R}'}^{-1}, \det(\mathbf{1} + \hat{U}), \quad (6)$$

where \det is the corresponding determinant. Hence the return probabilities become

$$P_{\mathbf{R}\mathbf{R}'} = \frac{\langle |(\mathbf{1} + \hat{U}^\dagger)_{\mathbf{R}\mathbf{R}'}^{-1}, \det(\mathbf{1} + \hat{U})|^2 \rangle_\alpha}{\langle |\det(\mathbf{1} + \hat{U})|^2 \rangle_\alpha}, \quad P'_{\mathbf{R}\mathbf{R}'} = \frac{\langle |(\mathbf{1} + \hat{U})_{\mathbf{R}\mathbf{R}'}^{-1}, \det(\mathbf{1} + \hat{U})|^2 \rangle_\alpha}{\langle |\det(\mathbf{1} + \hat{U})|^2 \rangle_\alpha} \quad (7)$$

due to equations (2) and (3). The identity $|\langle (\mathbf{1} + \hat{U}^\dagger)_{\mathbf{R}\mathbf{R}'}^{-1} |^2 = |\langle (\mathbf{1} + \hat{U})_{\mathbf{R}'\mathbf{R}}^{-1} |^2$ implies that $P'_{\mathbf{R}\mathbf{R}'} = P_{\mathbf{R}'\mathbf{R}}$.

4. Functional integral representation

For the further treatment of the return probability $P'_{\mathbf{R}\mathbf{R}'} = P_{\mathbf{R}'\mathbf{R}}$ in equation (7) it is convenient to separate the random phase factor and the deterministic evolution of \hat{U} as $\hat{U}_{\mathbf{r}\mathbf{r}'} = e^{i\alpha_{\mathbf{r}}} h_{\mathbf{r}\mathbf{r}'}$. Then we employ a Grassmann functional integral to write

$$\begin{aligned}
P_{\mathbf{R}'\mathbf{R}} &= \frac{1}{\mathcal{N}} \int_{\varphi} \exp \begin{vmatrix} \varphi_1 & \mathbf{1} + \mathbf{e}^{i\alpha} h & 0 \\ \varphi_2 & 0 & \mathbf{1} + h^\dagger \mathbf{e}^{-i\alpha} \end{vmatrix} \begin{vmatrix} \varphi'_1 & \varphi_{1\mathbf{R}} \varphi'_{1\mathbf{R}'} \varphi_{2\mathbf{R}'} \varphi'_{2\mathbf{R}} \\ \varphi'_2 & \end{vmatrix}_{\alpha} \\
&= \frac{1}{\mathcal{N}} \langle \text{adj}_{\mathbf{R}\mathbf{R}'}(\mathbf{1} + \mathbf{e}^{i\alpha} h) \text{adj}_{\mathbf{R}'\mathbf{R}}(\mathbf{1} + h^\dagger \mathbf{e}^{-i\alpha}) \rangle_{\alpha} = \frac{1}{\mathcal{N}} \langle |\det(\mathbf{1} + \mathbf{e}^{i\alpha} h)|^2 |(\mathbf{1} + \mathbf{e}^{i\alpha} h)_{\mathbf{R}\mathbf{R}'}^{-1}|^2 \rangle_{\alpha},
\end{aligned} \tag{8}$$

where the adjugate matrix $\text{adj}_{\mathbf{R}\mathbf{R}'}(\phi + h)$ obeys the relation $\text{adj}_{\mathbf{R}\mathbf{R}'}(\phi + h) = \det(\phi + h)(\phi + h)_{\mathbf{R}\mathbf{R}'}^{-1}$, and the normalization reads

$$\mathcal{N} = \int_{\varphi} \exp \begin{vmatrix} \varphi_1 & \mathbf{1} + \mathbf{e}^{i\alpha} h & 0 \\ \varphi_2 & 0 & \mathbf{1} + h^\dagger \mathbf{e}^{-i\alpha} \end{vmatrix} \begin{vmatrix} \varphi'_1 \\ \varphi'_2 \end{vmatrix}_{\alpha} = \langle |\det(\mathbf{1} + \mathbf{e}^{i\alpha} h)|^2 \rangle_{\alpha}.$$

We note that the kernel of the quadratic form has zero modes (i.e. eigenmodes of $\mathbf{1} + \mathbf{e}^{i\alpha} h$ with vanishing eigenvalue) because the eigenvalues of the random unitary matrices $\mathbf{e}^{i\alpha} h$ and $h^\dagger \mathbf{e}^{-i\alpha}$ are randomly distributed on the unit circle in the complex plane. These zero modes depend on the realization of the random phase.

In the integral (8) we pull out the phase factors by rescaling the Grassmann fields to obtain

$$\begin{aligned}
P_{\mathbf{R}'\mathbf{R}} &= \frac{1}{\mathcal{N}} \int_{\varphi} \exp \begin{vmatrix} \varphi_1 & \mathbf{e}^{-i\alpha} + h & 0 \\ \varphi_2 & 0 & \mathbf{e}^{i\alpha} + h^\dagger \end{vmatrix} \begin{vmatrix} \varphi'_1 & \varphi_{1\mathbf{R}} \varphi'_{1\mathbf{R}'} \varphi_{2\mathbf{R}'} \varphi'_{2\mathbf{R}} \\ \varphi'_2 & \end{vmatrix}_{\alpha} \\
&= \frac{1}{\mathcal{N}} \int_{\varphi} \prod_{\mathbf{r}} \langle (1 + \mathbf{e}^{-i\alpha_{\mathbf{r}}} \varphi_{1\mathbf{r}} \varphi'_{1\mathbf{r}}) (1 + \mathbf{e}^{i\alpha_{\mathbf{r}}} \varphi_{2\mathbf{r}} \varphi'_{2\mathbf{r}}) \rangle_{\alpha} \\
&\quad \times \exp \begin{vmatrix} \varphi_1 & h & 0 \\ \varphi_2 & 0 & h^\dagger \end{vmatrix} \begin{vmatrix} \varphi'_1 & \varphi_{1\mathbf{R}} \varphi'_{1\mathbf{R}'} \varphi_{2\mathbf{R}'} \varphi'_{2\mathbf{R}} \\ \varphi'_2 & \end{vmatrix}_{\alpha},
\end{aligned}$$

which gives after phase averaging

$$= \frac{1}{\mathcal{N}} \int_{\varphi} \prod_{\mathbf{r}} (1 + \varphi_{1\mathbf{r}} \varphi'_{1\mathbf{r}} \varphi_{2\mathbf{r}} \varphi'_{2\mathbf{r}}) \exp \begin{vmatrix} \varphi_1 & h & 0 \\ \varphi_2 & 0 & h^\dagger \end{vmatrix} \begin{vmatrix} \varphi'_1 & \varphi_{1\mathbf{R}} \varphi'_{1\mathbf{R}'} \varphi_{2\mathbf{R}'} \varphi'_{2\mathbf{R}} \\ \varphi'_2 & \end{vmatrix}_{\alpha} \tag{9}$$

We get the same result when we replace the phase factor by an Ising spin $\{S_{\mathbf{r}} = \pm 1\}$ or by a real Gaussian field $\phi_{\mathbf{r}}$ which will be called Ising field in the following. For the latter we write

$$\begin{aligned}
P_{\mathbf{R}'\mathbf{R}} &= \frac{1}{\mathcal{N}_{\phi}} \int e^{-\frac{1}{2} \sum_{\mathbf{r}} \phi_{\mathbf{r}}^2} \int_{\varphi} \exp \begin{vmatrix} \varphi_1 & \phi + h & 0 \\ \varphi_2 & 0 & \phi + h^\dagger \end{vmatrix} \begin{vmatrix} \varphi'_1 & \varphi_{1\mathbf{R}} \varphi'_{1\mathbf{R}'} \varphi_{2\mathbf{R}'} \varphi'_{2\mathbf{R}} \\ \varphi'_2 & \end{vmatrix} \prod_{\mathbf{r}} d\phi_{\mathbf{r}} \\
&= \frac{1}{\mathcal{N}_{\phi}} \int e^{-\frac{1}{2} \sum_{\mathbf{r}} \phi_{\mathbf{r}}^2} \text{adj}_{\mathbf{R}\mathbf{R}'}(\phi + h) \text{adj}_{\mathbf{R}'\mathbf{R}}(S + h^\dagger) \prod_{\mathbf{r}} d\phi_{\mathbf{r}} \\
&= \frac{1}{\mathcal{N}_{\phi}} \int e^{-\frac{1}{2} \sum_{\mathbf{r}} \phi_{\mathbf{r}}^2} |\text{adj}_{\mathbf{R}\mathbf{R}'}(\phi + h)|^2 \prod_{\mathbf{r}} d\phi_{\mathbf{r}} = \langle |(\phi + h)_{\mathbf{R}\mathbf{R}'}^{-1}|^2 \rangle_{\phi} \text{ with} \\
\langle \dots \rangle_{\phi} &= \frac{1}{\mathcal{N}_{\phi}} \int e^{-\frac{1}{2} \sum_{\mathbf{r}} \phi_{\mathbf{r}}^2} |\det(\phi + h)|^2 \dots \prod_{\mathbf{r}} d\phi_{\mathbf{r}}.
\end{aligned} \tag{10}$$

This result is reminiscent of the average two-particle Green's function with respect to a Gaussian distribution of $\phi_{\mathbf{r}}$, multiplied by the determinant term $|\det(\phi + h)|^2$. There are two important differences in comparison to the average two-particle Green's function $\langle |(V + H_0)_{\mathbf{R}\mathbf{R}'}^{-1}|^2 \rangle$ of Anderson localization, though. The first is that the determinant can be written as a product of the eigenvalues of $\phi + h$. This cancels poles of $|(\phi + h)_{\mathbf{R}\mathbf{R}'}^{-1}|^2$, implying that the poles of the Green's functions are not relevant for the $\phi_{\mathbf{r}}$ integration. In other words, the adjugate matrix $\text{adj}_{\mathbf{R}\mathbf{R}'}(\phi + h)$ does not have any pole for $|\phi_{\mathbf{r}}| < \infty$, and the integration with respect

to $\phi_{\mathbf{r}}$ can be deformed in any finite area of the complex plane. This reflects an exponential decay with a finite decay length of the return probability $P_{\mathbf{R}'\mathbf{R}}$. The second difference is that h is unitary and its eigenvalues are located on the unit circle of the complex plane, while the Hamiltonian H_0 in the Anderson localization problem is a Hermitian matrix with eigenvalues on the real axis.

The deformation of the $\phi_{\mathbf{r}}$ integration provides a rigorous but only qualitative result regarding the decay of $P_{\mathbf{R}'\mathbf{R}}$. For a quantitative result of the decay we must perform the integration explicitly. We will do that approximately within a saddle-point integration in section 5.2 and appendix.

5. Discussion

First, we note that the expansion of the integrand of equation (8) and a subsequent Grassmann and phase integration yields graphs with four-vertices, where two edges $h_{\mathbf{r}\mathbf{r}'}$ from φ_1 and two Hermitian conjugate edges $h_{\mathbf{r}'\mathbf{r}}^\dagger$ from φ_2 are connected. This condition is enforced by the Grassmann property, which requires a product of $\varphi_{1\mathbf{r}}\varphi_{1\mathbf{r}'}^\dagger\varphi_{2\mathbf{r}}\varphi_{2\mathbf{r}'}^\dagger$ at each site \mathbf{r} . Moreover, the random phase factors glue h and h^\dagger at these products to form a four-vertex and to prevent a two-vertex. The geometric property of the four-vertex enables us either to form loops of edges or to connect the sites \mathbf{R} and \mathbf{R}' by a string of both types of edges. Both, the h edges as well as the h^\dagger edges form loops and an \mathbf{R} – \mathbf{R}' string separately. This is a consequence of the diagonal kernel of the quadratic form in equation (8). Moreover, each loop carries a factor -1 from the Grassmann field. Two typical examples are depicted in figure 2 with the same formation of the nine black edges but with different formations of the nine red edges. In the left example a loop and a double string are separated by a special choice of red edges, while in the right example there is only one connected graph.

This type of graphs is known from the invariant measure of chiral random Hamiltonians [14]. There is a crucial difference though that is related to the zero mode: in contrast to the random phase scattering $e^{i\alpha_{\mathbf{r}}}h_{\mathbf{r}\mathbf{r}'}$, the scattering of the chiral model is $e^{i\alpha_{\mathbf{r}}}h_{\mathbf{r}\mathbf{r}'}\sum_{\mathbf{r}''}h_{\mathbf{r}'\mathbf{r}''}^\dagger e^{-i\alpha_{\mathbf{r}''}}$. For the latter we have a uniform zero mode

$$\sum_{\mathbf{r}'} \left[\delta_{\mathbf{r}\mathbf{r}'} - e^{i\alpha_{\mathbf{r}}}h_{\mathbf{r}\mathbf{r}'} \sum_{\mathbf{r}''} h_{\mathbf{r}'\mathbf{r}''}^\dagger e^{-i\alpha_{\mathbf{r}''}} \right] = 0 \quad (11)$$

for any realization of the random phase.

5.1. Hopping expansion

In order to get a better understanding of the behavior of the return probability $P_{\mathbf{R}'\mathbf{R}}$ we return to the expression of equation (7) with random phases and simplify it by neglecting the determinants. This leads to the product of the conjugate one-particle Green's functions of only two individual particles:

$$\langle (1 + h e^{i\alpha})_{\mathbf{R}\mathbf{R}'}^{-1} (1 + e^{-i\alpha} h^\dagger)_{\mathbf{R}'\mathbf{R}}^{-1} \rangle_\alpha = \langle (e^{-i\alpha} + h)_{\mathbf{R}\mathbf{R}'}^{-1} (e^{i\alpha} + h^\dagger)_{\mathbf{R}'\mathbf{R}}^{-1} \rangle_\alpha.$$

A hopping expansion of the inverse matrices in powers of the evolution operator $e^{i\alpha}h$ and its Hermitian conjugate can be written as a truncated geometric series

$$(1 + e^{i\alpha}h)_{\mathbf{R}\mathbf{R}'}^{-1} (1 + h^\dagger e^{-i\alpha})_{\mathbf{R}'\mathbf{R}}^{-1} = \sum_{l,m=0}^{N-1} (e^{i\alpha}h)_{\mathbf{R}\mathbf{R}'}^l (h^\dagger e^{-i\alpha})_{\mathbf{R}'\mathbf{R}}^m,$$

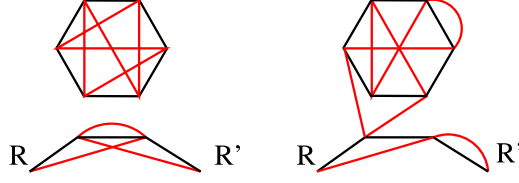


Figure 2. Two typical graphs representing contributions to the functional integral of the return probability $P_{\mathbf{R}'\mathbf{R}}$ in equation (8). Black (red) edges represent h (h^\dagger). Both edges form (i) a loop and (ii) a string connecting the sites \mathbf{R} and \mathbf{R}' of the fermion and the hole. The strings are contributions to the inverse matrix elements $(\mathbf{1} + e^{i\alpha}h)_{\mathbf{R}\mathbf{R}'}^{-1}$, and $(\mathbf{1} + h^\dagger e^{-i\alpha})_{\mathbf{R}\mathbf{R}'}^{-1}$, respectively, while loops are contributions to the determinants. There are only four-vertices, except for the endpoints \mathbf{R} and \mathbf{R}' , which are connecting to two black and two red edges. Other edge crossings are not connected by vertices.

where the truncation with $N < \infty$ is necessary because it is not clear whether the series converges. Since after phase averaging only $l = m$ survives, we can ignore terms with $l \neq m$ here. This gives

$$\begin{aligned} \sum_{l=0}^{N-1} (he^{i\alpha})_{\mathbf{R}\mathbf{R}'}^l (e^{-i\alpha}h^\dagger)_{\mathbf{R}'\mathbf{R}}^l &= \delta_{\mathbf{R}\mathbf{R}'} + h_{\mathbf{R}\mathbf{R}'}h_{\mathbf{R}'\mathbf{R}}^\dagger + \sum_{\mathbf{r}_1, \mathbf{r}_1'} h_{\mathbf{R}\mathbf{r}_1}h_{\mathbf{r}_1\mathbf{R}'}h_{\mathbf{R}'\mathbf{r}_1'}^\dagger h_{\mathbf{r}_1'\mathbf{R}}^\dagger e^{i\alpha_{\mathbf{r}_1} - i\alpha_{\mathbf{r}_1'}} \\ &+ \dots + \sum_{\mathbf{r}_1, \mathbf{r}_1', \mathbf{r}_2, \mathbf{r}_2', \dots, \mathbf{r}_{N-1}, \mathbf{r}_{N-1}'} h_{\mathbf{R}\mathbf{r}_1}h_{\mathbf{r}_1\mathbf{r}_2} \dots h_{\mathbf{r}_{N-1}\mathbf{R}'}h_{\mathbf{R}'\mathbf{r}_{N-1}'}^\dagger h_{\mathbf{r}_{N-1}'\mathbf{r}_{N-2}'}^\dagger \dots h_{\mathbf{r}_1'\mathbf{R}}^\dagger \prod_{j=1}^{N-1} e^{i\alpha_{\mathbf{r}_j} - i\alpha_{\mathbf{r}_j'}}. \end{aligned} \quad (12)$$

Now we can average over the random phases to obtain

$$\begin{aligned} \langle (\mathbf{1} + he^{i\alpha})_{\mathbf{R}\mathbf{R}'}^{-1} (\mathbf{1} + e^{-i\alpha}h^\dagger)_{\mathbf{R}'\mathbf{R}}^{-1} \rangle_\alpha &= \delta_{\mathbf{R}\mathbf{R}'} + h_{\mathbf{R}\mathbf{R}'}h_{\mathbf{R}'\mathbf{R}}^\dagger + \sum_{\mathbf{r}_1} h_{\mathbf{R}\mathbf{r}_1}h_{\mathbf{r}_1\mathbf{R}'}h_{\mathbf{R}'\mathbf{r}_1}^\dagger h_{\mathbf{r}_1\mathbf{R}}^\dagger \\ &+ \dots + \sum_{\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_{N-1}} \sum_{\pi_{N-1}} h_{\mathbf{R}\mathbf{r}_1}h_{\mathbf{r}_1\mathbf{r}_2} \dots h_{\mathbf{r}_{N-1}\mathbf{R}'}h_{\mathbf{R}'\pi(\mathbf{r}_{N-1})}^\dagger h_{\pi(\mathbf{r}_{N-1})\pi(\mathbf{r}_{N-2})}^\dagger \dots h_{\pi(\mathbf{r}_1)\mathbf{R}}^\dagger, \end{aligned} \quad (13)$$

where we sum with respect to all permutations π_{N-1} of all non-degenerate sites of $\{\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_{N-1}\}$. Although this is a compact expression, it is difficult to perform the sum over the permutations and to calculate the corresponding values. Nevertheless, as an important special case the identity $\pi_{N-1} = id$ can be calculated. It is a contribution of an unrestricted random walk on the lattice. This represents a long range correlation in the form of diffusion. However, it will be destroyed by the determinant factor in equation (8), as mentioned in the previous section, where the Ising field representation leads to an exponential decay. In other words, the coupling of many fermions to the random phase scattering supports localization by avoiding singularities that appear in the case of two particles. For a quantitative result of the exponential decay we study the mean square displacement within a saddle point approximation in the next section.

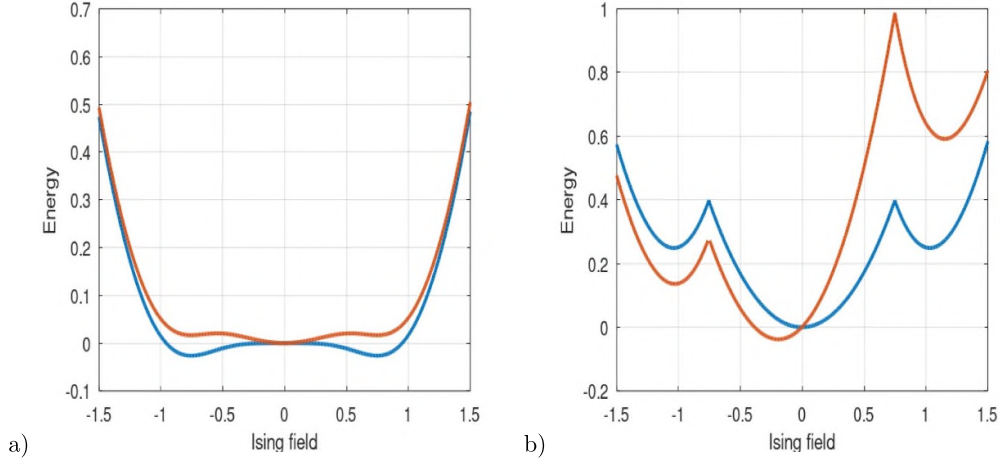


Figure 3. The Ising energy $\mathcal{E}(\phi)$ as defined in equation (25) for a constant density of states. (a) $\mathcal{E}(\phi)$ is plotted at the symmetry point $E_0 = \pi/2$ for the band width $a = 1.9$ (blue curve) and $a = 2.1$ (red curve). It indicates a jump of the Ising field from two degenerate nonzero values to $\phi_{0,min} = 0$. (b) $\mathcal{E}(\phi)$ is plotted with band width $a = 7.5$ for a symmetric band $E_0 = 0$ (red curve) and a band that is shifted by $E_0 = \pi/2$ (blue curve).

5.2. Saddle point approximation

The return probability in equation (10) is treated within the saddle point integration of the Ising field ϕ (see [appendix](#)). This yields

$$P_{\mathbf{R}'\mathbf{R}} = \langle |(\phi + h)_{\mathbf{R}\mathbf{R}'}^{-1}|^2 \rangle_\phi \approx |(\phi_0 + h)_{\mathbf{R}-\mathbf{R}'}^{-1}|^2, \quad (14)$$

where we have neglected the fluctuations $\delta\phi_{\mathbf{r}}$ around the saddle point ϕ_0 . This approximation enables us to factorize the return probability as

$$P_{\mathbf{R}'\mathbf{R}} \approx |C_{\mathbf{R}-\mathbf{R}'}|^2, \quad C_{\mathbf{R}-\mathbf{R}'} = (\phi_0 + h)_{\mathbf{R}-\mathbf{R}'}^{-1},$$

where $C_{\mathbf{R}-\mathbf{R}'}$ can be represented by its Fourier transform

$$\tilde{C}_{\mathbf{k}} = \frac{1}{\phi_0 + e^{-iE_{\mathbf{k}}}} \quad (15)$$

with the eigenvalue $E_{\mathbf{k}}$ of the translational invariant matrix $H\tau$. Thus, the effect of the random phase scattering is associated only with the value of ϕ_0 , where the latter is determined by $E_{\mathbf{k}}$ via the saddle point equation (24). Moreover, $\phi_0 = 1$ represents the absence of random phase scattering.

The results for the Ising field ϕ_0 of see [appendix](#) can be interpreted in terms of the magnetic properties of the classical Ising model [15]. The asymmetric shift $\cos E_0$ plays the role of an external magnetic field and ϕ_0 corresponds to the magnetization [16]. Thus, the effective Ising model has a unique Ising field $\phi_0 > 0$ or $\phi_0 < 0$ when $\cos E_0 \neq 0$, while for $\cos E_0 = 0$ there are either two degenerate solutions with opposite signs of ϕ_0 (ferromagnetic phase) or a single solution with $\phi_0 = 0$ (paramagnetic phase). In contrast to the classical Ising model with a continuous transition though, figure 3 indicates a jump of ϕ_0 for our effective Ising model.

5.3. Mean square displacement

The mean square displacement provides a measure for the correlation length. It is defined as

$$[R_\nu^2] := \frac{\sum_{\mathbf{R}'} (\mathbf{R}_\nu - \mathbf{R}'_\nu)^2 P_{\mathbf{R}'\mathbf{R}}}{\sum_{\mathbf{R}} P_{\mathbf{R}'\mathbf{R}}} = \frac{-\partial_{q_\nu}^2 \tilde{P}_{\mathbf{q}}}{\tilde{P}_0} \Big|_{\mathbf{q}=0}, \quad (16)$$

where $\tilde{P}_{\mathbf{q}}$ is the Fourier transform of the translational-invariant $P_{\mathbf{R}'\mathbf{R}} \equiv P_{\mathbf{R}'-\mathbf{R}}$. Now we study $P_{\mathbf{R}\mathbf{R}'} = |C_{\mathbf{R}\mathbf{R}'}|^2$ with the help of the saddle point integration. In this case the mean square displacement reads

$$[R_\nu^2] = \frac{\sum_{\mathbf{R}} R_\nu^2 |C_{\mathbf{R}}|^2}{\sum_{\mathbf{R}} |C_{\mathbf{R}}|^2}, \quad (17)$$

where the numerator is

$$\sum_{\mathbf{R}} R_\nu^2 e^{i\mathbf{q}\cdot\mathbf{R}} |C_{\mathbf{R}}|^2 \Big|_{\mathbf{q}=0} = -\partial_{q_\nu}^2 \sum_{\mathbf{R}} e^{i\mathbf{q}\cdot\mathbf{R}} |C_{\mathbf{R}}|^2 \Big|_{\mathbf{q}=0} = -\partial_{q_\nu}^2 \sum_{\mathbf{R}} \int_{\mathbf{k}} \int_{\mathbf{k}'} e^{i(\mathbf{q}-\mathbf{k}-\mathbf{k}')\cdot\mathbf{R}} \tilde{C}_{\mathbf{k}} \tilde{C}_{\mathbf{k}'}^* \Big|_{\mathbf{q}=0}$$

with the Fourier transform $\tilde{C}_{\mathbf{k}}$ of equation (15). Then the \mathbf{R} summation can be performed and leads to a Kronecker delta, which gives for equation (17)

$$[R_\nu^2] = -\partial_{q_\nu}^2 \int_{\mathbf{k}} \tilde{C}_{\mathbf{k}} \tilde{C}_{\mathbf{q}-\mathbf{k}}^* \Big|_{\mathbf{q}=0} / \int_{\mathbf{k}} \tilde{C}_{\mathbf{k}} \tilde{C}_{-\mathbf{k}}^*.$$

Now we assume that $C_{-\mathbf{k}} = C_{\mathbf{k}}$ to obtain eventually

$$[R_\nu^2] = \int_{\mathbf{k}} |\partial_{k_\nu} \tilde{C}_{\mathbf{k}}|^2 / \int_{\mathbf{k}} |\tilde{C}_{\mathbf{k}}|^2, \quad (18)$$

which becomes with equation (15)

$$[R_\nu^2] = \int_{\mathbf{k}} \frac{(\partial_{k_\nu} E_{\mathbf{k}})^2}{(1 + \phi_0^2 + 2\phi_0 \cos E_{\mathbf{k}})^2} / \int_{\mathbf{k}} \frac{1}{1 + \phi_0^2 + 2\phi_0 \cos E_{\mathbf{k}}}. \quad (19)$$

A special case is the one in which the energy in equation (23) is symmetric with respect to $\phi_0 \rightarrow -\phi_0$. Then there exists a critical value τ_c of the evolution time τ : if τ exceeds τ_c the saddle point is always $\phi_{0;\min} = 0$, as indicated in figure 3(a). This implies for equation (15) that $\tilde{C}_{\mathbf{k}} = e^{iE_{\mathbf{k}}}$, which yields for the corresponding mean square displacement

$$[R_\nu^2] = \int_{\mathbf{k}} (\partial_{k_\nu} E_{\mathbf{k}})^2 = \tau^2 \int_{\mathbf{k}} (\partial_{k_\nu} \epsilon_{\mathbf{k}})^2. \quad (20)$$

$\epsilon_{\mathbf{k}}$ is the dispersion of the Hamiltonian H . Thus, the mean square displacement increases with the squared evolution time τ . For $\tau < \tau_c$, on the other hand, or when the energy is asymmetric with respect to $\phi_0 \rightarrow -\phi_0$, we have $\phi_0 \neq 0$.

In the absence of random phase scattering we have $\phi_0 = 1$ directly from equation (8). Then for the special case $E_{\mathbf{k}} = k^2 \tau / 2\mu$ ($0 \leq k \leq \lambda$) the mean square displacement on a d -dimensional lattice reads

$$[R_\nu^2] = \frac{\tau}{d\mu} \int_0^{\bar{E}} \frac{E^{d/2}}{(1 + \cos E)^2} dE / \int_0^{\bar{E}} \frac{E^{d/2-1}}{1 + \cos E} dE \quad (21)$$

with the integration cut-off $\bar{E} = \lambda^2 \tau / 2\mu$. This is a finite expression for $\bar{E} < \pi$, which diverges with a power law as

$$[R_\nu^2] \sim \frac{2\pi\tau}{3d\mu} (\pi - \bar{E})^{-2} \quad (22)$$

when we approach $\bar{E} = \pi$ from below. For $\bar{E} \geq \pi$ the mean square displacement is always infinite without random phase scattering. This result has the form of a diffusion relation with time τ and a divergent diffusion coefficient for $\bar{E} \rightarrow \pi$ when we ignore the fact that \bar{E} also depends on τ . A possible interpretation is that the fermion–hole pair is subject to diffusion due to its interaction with the other fermions of the system. The divergence, on the other hand, reflects a long range correlation of the fermion and the hole that reflects the pole of $\tilde{C}_k = 1/(1 + e^{-iE_k})$.

A more detailed analysis, especially for the evaluation of τ_c , requires specific expressions of the dispersion ϵ_k . This would exceed the goal of this work to present a generic approach for the effect of disorder on the recombination of fermion–hole pairs.

6. Conclusions and outlook

The probability $P_{\mathbf{R},\mathbf{R}'}$, which describes the probability to return to the initial quantum state after the creation of a fermion at site \mathbf{R} and a hole at site \mathbf{R}' and their evolution, decays always exponentially with the distance $|\mathbf{R} - \mathbf{R}'|$ in the presence of random phase scattering. To obtain this rigorous result a mapping of the random phase model onto an Ising-like model was essential. This was supplemented by an approximative calculation of this decay, based on a saddle point integration of the effective Ising model, to get some quantitative insight into the decay. The latter calculation is instructive, since it demonstrates how the solution of the saddle point equation avoids the singularities of the underlying fermion model. In the absence of random phase scattering, one of these singularities leads to a non-exponential decay for a sufficiently long evolution of the state with the fermion–hole pair. This is reflected by an infinite mean square displacement of the fermion and the hole.

In our approximation we have not included the Gaussian fluctuations around the saddle point solution. It would be interesting to include them and to determine their effect on the decay of the return probability. In this context it would also be useful to understand the effect of these fluctuations on the transition from $\phi_0 \neq 0$ to $\phi_0 = 0$ at the symmetry point under an increasing evolution time. Another extension of our approach is the application to the return probability of a system under periodically repeated projective measurements [17, 18] or under randomly repeated projective measurements [19]. Then the effect of random phase scattering on the resulting monitored evolution could also be described by the effective Ising field model. Even more interesting but also more challenging would be the extension of the approach to the transition probability for the monitored evolution under randomly repeated projective measurements [20].

Data availability statement

No new data were created or analysed in this study.

Appendix. Saddle point integration

We approximate the integral

$$\langle \dots \rangle_\phi = \frac{1}{\mathcal{N}_\phi} \int e^{-\frac{1}{2} \sum_r \phi_r^2} |\det(\phi + h)|^2 \dots \prod_r d\phi_r$$

by using a saddle-point integration. Then we determine the maximal contribution to the integrand by assuming a uniform ϕ and write $\phi_r = \phi + \delta\phi_r$. This enables us to approximate the

integral in terms of Gaussian fluctuation around the uniform ϕ with respect to $\delta\phi_r$, where ϕ must be fixed as ϕ_0 at the minimum of the Ising energy

$$\mathcal{E}(\phi) = \frac{1}{2}\phi^2 - \int_{-\infty}^{\infty} \log(1 + \phi^2 + 2\phi \cos E) \rho(E) dE \quad (23)$$

with the density of states $\rho(E)$. The integrand is singular for $\phi = 1$, $E = \pi$ and for $\phi = -1$, $E = 0$. These singularities yield large values for the energy. Therefore, they do not represent lowest energy contributions of the saddle point. This is also reflected in the curves of figure 3(b). Moreover, the saddle-point solution ϕ_0 must satisfy

$$\mathcal{E}'(\phi) = \phi - 2 \int_{-\infty}^{\infty} \frac{\phi + \cos E}{1 + \phi^2 + 2\phi \cos E} \rho(E) dE = 0. \quad (24)$$

For a constant density of states $\rho(E)$ on the interval $[-a/2 + E_0, a/2 + E_0]$ we get

$$\mathcal{E}(\phi) = \frac{1}{2}\phi^2 - \frac{1}{a} \int_{-a/2+E_0}^{a/2+E_0} \log(1 + \phi^2 + 2\phi \cos E) dE. \quad (25)$$

A special case is $E_0 = \pi/2$, where we have

$$\mathcal{E}(\phi) = \frac{1}{2}\phi^2 - \frac{1}{a} \int_{-a/2}^{a/2} \log(1 + \phi^2 - 2\phi \sin E) dE$$

with the symmetry relation $\mathcal{E}(\phi) = \mathcal{E}(-\phi)$. This would also hold when the density of states is symmetric with respect to $E = \pi/2$ in equation (23). The Ising energy is plotted for several values of E_0 and the band width a in figure 3.

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