Lagrangian Rabinowitz Floer Homology and its Application to Powered Flyby Orbits in the Restricted Three Body Problem

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Chapter 1

Introduction

The history of modern space exploration already goes back over a hundred years, at least when it comes to its theoretical consideration. In fact, the first widely recognised scientific book about rockets and the planning of space missions was published in 1923 under the name "Die Rakete zu den Planetenräumen" (The Rocket into Planetary Space) [Obe23]. It was written by one of the fathers of modern space flight Hermann Oberth. Before this time celestial mechanics was of course already widely studied, but the questions behind those considerations were always focused on understanding the movement of the celestial bodies. Oberth, however, was one of the first persons to use our knowledge about classical mechanics to explore how a human built space ship could move among these celestial bodies. With these thoughts he was very much ahead of his time and unfortunately had to face a lot of harsh opposition because of it. Originally the book "Die Rakete zu den Planetenräumen" was his PhD thesis, but in 1922 it was rejected by the University of Heidelberg for being about an unachievable fantasy. In the beginning Oberth also had a lot of difficulties to raise funding to build the rockets he had designed in his work, because of the negative opinions of the established professors of his time. Back then the idea of space flight seemed so strange to some scientists that they even tried to prove that space flights are theoretically impossible. Oberth, however, was convinced of his work and continued with his research until the success of the world wide space programs proved that his ideas were right. But there is also a dark side to this story: The advancements in the technology of rockets are not just used for the exploration of space, but are also heavily used in military weapons. The most famous one to use the work of Oberth to build large scale rockets was, in fact, his former student Wernher von Braun as an attempt to construct a powerful weapon for Nazi Germany in the second world war. This effort culminated in the development of the V-2, which Oberth also helped develop in 1941, and that would cost the lives of thousands of people. This serves as a good reminder that even a purely theoretic work can cause a lot of damage if it is used for malicious purposes. It is always an exciting prospect if one's theoretical work is considered to be used in the real world, but it is the goal of the one, who intents to use it, that determines whether we should support the endeavour or even try to prevent it.

The main objects we are investigating in this thesis - namely the consecutive collision orbits - can also trace back their origins to the work of Oberth. A consecutive collision orbit (c.c. orbit) is a trajectory that starts and ends in the centre of gravity of the planet it circles around, i.e. if we were to send a space ship on this course it would collide with the planet twice. In most cases this is of course neither desirable nor physically possible, since it is very unlikely that the space ship would survive even the first collision. The actual reason why we are interested in consecutive collision orbits is based on the fact that close to every such c.c. orbit there is another orbit that barley misses the planet. Those almost collision orbits are what we are really looking for. One might ask the question: Why are we not just directly searching for an almost collision orbit? The problem is that for those orbits it is not easy to find a sharp mathematical definition one can easily work with, where on the other hand a collision has a very clear description as a Lagrangian subspace $\{collision\ pt.\} \times \mathbb{R}^n \subset \mathbb{R}^{2n}$. When designing a space mission the almost collision orbits can fulfil a lot of different purposes, for example if one wants to take detailed pictures of the surface of a planet or wants to probe the upper atmosphere using an orbit that comes very close to the planet is a valid strategy. The most prominent applications of almost collisions are, however, as powered flyby orbits. The manoeuvre of a powered flyby was first thought of by Hermann Oberth and is based on a well known concept in astronautics called the *Oberth effect*. In his 1923 book he explains the effect roughly in the following way: Given a rocket with mass m and velocity v. If we use a short boost of the rocket engine, we directly change the velocity by Δv via conservation of momentum. The kinetic energy before the boost was $E_{\text{kin}}^{\text{old}} = \frac{1}{2}mv^2$ and after the boost is

$$E_{\text{kin}}^{\text{new}} = \frac{1}{2}m(v + \Delta v)^2 = \frac{1}{2}mv^2 + mv\Delta v + O\big((\Delta v)^2\big).$$

Here we assume that we only used a negligible small part of the fuel and in return increase our velocity only by a very small amount. Hence, the dominant term in the change of kinetic energy is

$$E_{\text{kin}}^{\text{new}} - E_{\text{kin}}^{\text{old}} = mv\Delta v.$$

This means that the gain of kinetic energy is directly proportional to the speed at which we perform the boost. With a powered flyby manoeuvre one can use this effect to gain kinetic energy for a space ship as efficient as possible: In a gravitational potential the velocity of an object is the biggest if the potential energy is at its lowest, i.e. when the object is close to the centre of gravity. Therefore a powered flyby orbit tries to get as close to the planet as possible and then boost exactly at the point of minimal distance, which will result in the most effective gain in kinetic energy. Further, not just the existence of these consecutive collision orbits is of importance, but also the orientation of the "belly" of the orbits. This might be the case when one is planing the communication of the space ship with the control station on earth to make sure that during the communication phase there is no planet in the way blocking the signals. This is the motivation behind studying symmetric c.c orbits, where the symmetry allows us to gain more information about the "orientation" of the orbit.

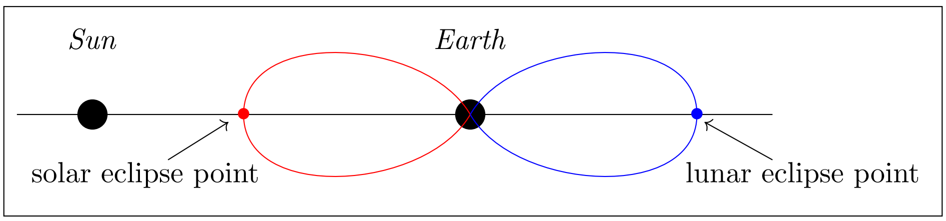
The main result of this thesis can be summarized in the following theorem:

Theorem A
In the restricted three body problem there are for all energies below the first Lagrange point

- *infinitely many symmetric consecutive collision orbits or at least one periodic symmetric consecutive collision orbit all intersecting the symmetry axis in a solar eclipse point and*
- *infinitely many symmetric consecutive collision orbits or at least one periodic symmetric consecutive collision orbit all intersecting the symmetry axis in a lunar eclipse point,*

where for a generic choice of energy and mass periodic symmetric c.c. orbits can not appear.

Note that the symmetry we want to consider here is given by the reflection of the space coordinate at the axis between earth and sun combined with the reflection of the momentum coordinate at the axis perpendicular to the first axis. A sketch of the position space picture for two different orbits (one red, the other blue) would look like:



Note that this is only a rough sketch and the actual orbits might be much more complicated then the ones drawn above. We want to remark that the existence of infinitely many or one periodic consecutive collision orbit in the three body problem was already discussed in [FZ19], but by studying symmetric c.c. orbits we are able to derive additional information about the "orientation" of the orbits and can exclude the case of a periodic c.c. orbit for a generic choice of energy and mass. Our main tool to achieve this theorem is an equivariant version of Rabinowitz Floer homology, where the dimension of the homology will give a lower bound for the number of symmetric c.c. orbits. To compute this homology we uses the following connection between G-equivariant Rabinowitz Floer homology and the Tate homology of the finite group G:

Theorem B
Let G be a finite group and a symmetry of the Hamiltonian system (M, ω, H) with Lagrangian L, which acts free. Assume that $L \cap H^{-1}(0)$ is a connected submanifold of dimension at least 1. Further, let the system be displaceable and assume that the absolute value of the Maslov index for non-constant chords is bounded from below by $dim\big(L \cap H^{-1}(0)\big)$. Then the G-equivariant Lagrangian RF-homology is equal to the Tate homology of G, i.e.

$$RFH_*^G(M, H, L) = TH_*(G, \mathbb{Z}_2).$$

The idea behind the connection of these seemingly very different homologies is that the ordinary Rabinowitz Floer homology is just zero in every degree for a displaceable Hamiltonian system, i.e. if there is an additional symmetry of the system we can view the Rabinowitz Floer complex as a complete resolution of the trivial $\mathbb{Z}_2[G]$-module $\mathbb{Z}_2$. The condition on the Maslov index ensures that we can define the augmentation map and can be seen as the Lagrangian analogue to the notion of *dynamically convex*. From this we see that the Rabinowitz Floer complex can also be used as the underlying long exact sequence for the Tate homology.

Before starting with the main body of this thesis, we first want to give a short overview of the content: We start in the second chapter with the introduction of some fundamental notions concerning Hamiltonian systems and Liouville domains, which we will need throughout this work. The third chapter introduces the restricted three body problem and the concept of its regularization. At first we discuss the Levi-Civita regularization for the planar case in great detail, before we then consider its generalization to the spatial case, namely the Kustaanheimo-Stiefel regularization. Since we later on want to use techniques from Floer homology we prove for both regularizations that we can view the energy hypersurface as the boundary of a Liouville domain for all energies below the first critical energy value. The fourth chapter is all about introducing the Floer homology tools we need in order to answer the main question of this thesis. We start the chapter by giving the definition of the Rabinowitz action functional for the case of Lagrangian boundary conditions. Afterwards we invest some time into explaining the Maslov index for Hamiltonian chords, since it is the grading of the Lagrangian Rabinowitz Floer homology we will define later on and therefore plays an important roll in its calculation. The next important ingredient for the Lagrangian Rabinowitz Floer homology are the gradient flow lines. The definition is analogue to the ones in other Floer homologies, but since the critical points in our case have Lagrangian boundary conditions, the proof of compactness needs some more detailed explanations. In order to properly define the Lagrangian Rabinowitz Floer homology we also need to introduce the concept of gradient flow lines with cascades and their breaking. With this we then finally have all the ingredients to define the Lagrangian Rabinowitz Floer homology. Afterwards we introduce the notion of leaf-wise intersection points in the Lagrangian boundary setting and use it to prove that for a displaceable energy hypersurface the Lagrangian Rabinowitz Floer homology vanishes. Since a vanishing homology is not sufficient to prove the existence of infinitely many c.c. orbits (c.f. Theorem A), this shows us the necessity to define an equivariant version of the Lagrangian RFH. The definition of the G-equivariant Lagrangian RFH for G being a compact Lie group acting free on the trajectories is very straight forward. The only technical difficulty that arises is that we now need transversality for a generic choice of G-equivariant almost complex structures. However, the usual proof for not necessarily G-equivariant almost complex structures still works if we adapt one auxiliary theorem to the G-equivariant setting. The final step in this chapter is to relate the G-equivariant Lagrangian RFH to the Tate homology of G in the case that G is finite, or the positive part of the G-equivariant Lagrangian RFH to the group homology of G in the general case. This will allow for a more convenient computation than via the definition directly. In the last chapter we take on the main goal of this thesis, i.e. proving the existence of infinitely many symmetric consecutive collision orbits in the restricted three body problem. We start with proving the claim for the planar case. Here we use the $\mathbb{Z}_2$-equivariant Lagrangian RFH as a lower bound for the number of symmetric c.c. orbits and the calculation of the Tate homology of $\mathbb{Z}_2$ then shows that this number is infinitely large. However, since the homology counts the orbits with multiplicity there is also the possibility that all these orbits are just iterations of the same periodic c.c. orbit. So to conclude the planar case we prove that for a generic choice of energy and mass there are no periodic symmetric c.c. orbits. The strategy of the spatial case is now very similar, but instead of the finite group $\mathbb{Z}_2$ we now have S^1 as the symmetry group. The only difficulty is that we don't know if the Liouville vector field is transverse to the energy hypersurface at all points, but only for those which also lie on the hypersurface defined by another conserved quantity. This first requires some technical discussion before we can use the same strategy as in the planar case.

Chapter 2

The Basics: Hamiltonian Dynamics and Liouville Domains

The goal of this chapter is to repeat some of the basic principles of Hamiltonian dynamics and introduce the concept of a Liouville domain and its completion, i.e. the setting in which later on we want to define our Floer theoretic tools.

2.1 Introduction to Hamiltonian Mechanics

Having a mathematical description of classical mechanical systems is of course very important and can be used to study a large number of different problems (for example questions in celestial mechanics as we will see in the later chapters). Hamiltonian mechanics is one of the ways to do so: The behaviour of every particle (we will only work with point masses for now) in classical mechanics is completely described by its position and (canonical) momentum. The space of all positions and momenta is called the phase space and is represented via the cotangent bundle of a given manifold Q, where the coordinates coming from the underlying manifold Q describe the position and the fibre coordinates describe the (canonical) momentum. The time evolution of every such particle is determined by the energy function of the given physical system, which is usually called the Hamiltonian function. Note that in the literature there are a couple of different sign conventions used when it comes to the definition of a Hamiltonian system. For us the canonical symplectic form on a $2n$-dimensional cotangent bundle is locally given by

$$\omega_0 = \sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}q_i, \tag{2.1.1}$$

where p_i are the fibre coordinates and q_i are the coordinates of the underlying manifold. In general we call a symplectic manifold exact if $\omega = \mathrm{d}\lambda$ for a one form λ.

Definition 2.1.1 *Given a symplectic manifold (M, ω) with smooth function H. We define the Hamiltonian vector field X_H as the unique vector field that satisfies*

$$\mathrm{d}H = -i_{X_H}\omega, \tag{2.1.2}$$

where i_{X_H} inserts the vector field into the first slot.

Assume we are now given a classical mechanical system with possible positions described by a manifold Q and the corresponding momenta at a given point q_0 as a vector in $T^*_{q_0}Q$. The change of the position and the momentum of a particle in time is then determined by the Hamiltonian (function) of the given system H via the flow of the Hamiltonian vector field X_H.

Remark 2.1.2 Even though every system in classical mechanics (without friction) can be described via a Hamiltonian function on a suitable cotangent bundle with the canonical symplectic form, we will always consider more general symplectic manifold when possible. The reason is that if a given Hamiltonian system (T^*Q, ω_0, H) has a symmetry, we can reduce the dimension of the problem using a method called symplectic reduction. This will change the phasespace from T^*Q into a general symplectic manifold of smaller dimension and hence it is useful to consider more general symplectic manifolds.

One very well known fact in classical mechanics is the conservation of energy:

Proposition 2.1.3 *In a Hamiltonian system (M, ω, H), where the Hamiltonian function H is not explicitly time dependent, H is constant along the flow of the system.*

Proof: The proof just consists of using the Cartan formula and the fact that a symplectic form is antisymmetric. $\Phi_{X_H,t}$ should denote the flow of X_H.

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi^*_{X_H,t}H = \Phi^*_{X_H,t}(\mathcal{L}_{X_H}H) = \Phi^*_{X_H,t}(i_{X_H}\,\mathrm{d}H + 0) = \Phi^*_{X_H,t}(\omega(X_H, X_H)) = 0 \qquad \square$$

Now let us look at an example for such a system.

Example 2.1.4 (The Harmonic Oscillator) In this example we want to study the one dimensional harmonic oscillator. The position space is just $\mathbb{R}$, i.e. the suitable phase space is $T^*\mathbb{R}$ together with the canonical symplectic form $\mathrm{d}p \wedge \mathrm{d}q$. The Hamiltonian that describes this system is

$$H(q,p) = \frac{p^2}{2m} + \frac{k}{2}q^2 \tag{2.1.3}$$

with positive constants m and k. First calculate

$$\mathrm{d}H = \frac{p}{m}\,\mathrm{d}p + kq\,\mathrm{d}q,$$

from which we get the Hamiltonian vector field

$$X_H(q,p) = \frac{p}{m}\frac{\partial}{\partial q} - kq\frac{\partial}{\partial p}.$$

For more convenience we identify $\frac{\partial}{\partial q}$ with $\begin{pmatrix}1\\0\end{pmatrix}$ and $\frac{\partial}{\partial p}$ with $\begin{pmatrix}0\\1\end{pmatrix}$. By slight abuse of notation we denote the flow by $\begin{pmatrix}q(t)\\p(t)\end{pmatrix}$ and so the equation determining the flow becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}q(t)\\p(t)\end{pmatrix} = \begin{pmatrix}\frac{1}{m}p(t)\\-kq(t)\end{pmatrix}.$$

This ODE can then be transformed into

$$\begin{cases}\ddot{q}(t) = -\frac{k}{m}q(t)\\ p(t) = m\dot{q}(t)\end{cases}.$$

The solutions to these equations then give us the Hamiltonian flow of the system

$$q(t) = Q_0\cos\left(\sqrt{\frac{k}{m}}t\right) + \frac{P_0}{\sqrt{mk}}\sin\left(\sqrt{\frac{k}{m}}t\right)$$
$$p(t) = P_0\cos\left(\sqrt{\frac{k}{m}}t\right) - \sqrt{km}Q_0\sin\left(\sqrt{\frac{k}{m}}t\right),$$

where Q_0 is the starting position and P_0 is the starting momentum.

2.2 Liouville Domains and Their Completions

In this section we want to take a closer look at Liouville domains and define the completion of them. But first we start with introducing contact manifolds.

Definition 2.2.1 *Given a $2n-1$ dimensional manifold Σ and a one form λ. We call the pair (Σ, λ) a contact manifold if $\lambda \wedge (\mathrm{d}\lambda)^{n-1}$ is a volume form. The Reeb vector field on (Σ, λ) is the unique vector field fulfilling the conditions*

$$i_R \,\mathrm{d}\lambda = 0, \quad \lambda(R) = 1. \tag{2.2.1}$$

Remark 2.2.2 In the literature people sometimes use a weaker definition of a contact manifold, namely, that the manifold admits a maximally non-integrable field of hyperplanes. In this setting our definition from above would be considered a cooriented contact manifold, where the hyperplane field is given by $\zeta = \ker(\lambda)$ and carries the coorientation induced by λ.

Now to the definition of a Liouville domain:

Definition 2.2.3 *A Liouville domain is a compact exact symplectic manifold (M, λ) with the property that the Liouville vector field X, defined by $i_X \,\mathrm{d}\lambda = \lambda$, is transverse to the boundary and outward pointing.*

Note that every compact exact symplectic manifold needs to have a nonempty boundary, i.e. the conditions on the Liouville vector field always make sense. In this context it is also important to remark that for us the tangent space at the boundary also contains the direction transverse to the boundary. To see that this gives us a well-defined tangent space one can use the definition of a tangent vector via derivations of germs of smooth functions, which naturally extends to the boundary points. The special property of a Liouville domain is the fact that it has an exact symplectic form, that becomes a contact form when restricted to the boundary:

Proposition 2.2.4 *Let (M, λ) be a Liouville domain with Liouville vector field X and boundary Σ, then $\left(\Sigma, \lambda\big|_{\Sigma}\right)$ is a contact manifold.*

Proof: (c.f. Prop. 2.6.1 [FVK18]) To see this we need to show that $\lambda \wedge (\mathrm{d}\lambda)^{n-1}$ is a volume form when restricted to Σ. So given an $x \in \Sigma$ we choose a basis $\{v_1, \ldots, v_{2n-1}\}$ of $T_x\Sigma$. Now compute

$$\begin{aligned} \lambda \wedge (\mathrm{d}\lambda)^{n-1} =& i_X\omega \wedge \omega^{n-1}(v_1, \ldots, v_{2n-1}) \\ =& \frac{1}{n}\omega^n(X_x, v_1, \ldots, v_{2n-1}), \end{aligned}$$

where we call $\mathrm{d}\lambda = \omega$. Since X is per assumption transverse to Σ, it follows that $\{X_x, v_1, \ldots, v_{2n-1}\}$ is a basis of T_xM. Because ω is non-degenerate, we see that

$$\omega^n(X_x, v_1, \ldots, v_{2n-1}) \neq 0,$$

which implies that $\lambda\big|_{\Sigma}$ is a contact form on Σ. ☐

But Liouville domains are not just relevant form a purely mathematical standpoint, they also show up in the study of classical mechanical systems:

Example 2.2.5 Given an n-dimensional compact manifold Q with corresponding phase space T^*Q and the canonical one form λ_0, s.t. $\mathrm{d}\lambda_0 = \omega_0$. A mechanical Hamiltonian is a Hamiltonian function of the form

$$H(q, p) = \frac{1}{2}|p|_g^2 + V(q), \tag{2.2.2}$$

where g is a cometric on Q. Now consider an energy value $E_0 > \max V$. This implies that the energy hypersurface does not intersect the zero section. The canonical one form is locally given by $p_i \,\mathrm{d}q_i$, i.e. the corresponding Liouville vector field is locally given by $X = \sum_{i=1}^n p_i \frac{\partial}{\partial p_i}$. The Liouville vector field is transverse to the boundary $H^{-1}(E_0)$ and outward pointing if we have $\mathrm{d}H(X) > 0$ on it. So let us verify this fact:

$$\mathrm{d}H = \sum_{i,j=1}^n g_{ij} p_i \,\mathrm{d}p_j + \sum_{i=1}^n \frac{\partial V}{\partial q_i} \,\mathrm{d}q_i$$

and therefore we have on $H^{-1}(E_0)$

$$\mathrm{d}H\left(\sum_{i=1}^n p_i \frac{\partial}{\partial p_i}\right) = \sum_{i=1}^n g_{ij} p_i p_j = |p|_g^2 > 0$$

Note that it is not hard to see that E_0 is a regular value and since Q is compact the energy hypersurface has to bound a compact submanifold where the boundary is exactly the hypersurface.

There is now a way to enlarge a Liouville domain to a noncompact exact symplectic manifold:

Proposition 2.2.6 *Let (W, λ) be a Liouville domain with Liouville vector field X and boundary $\left(\Sigma, \alpha := \lambda\big|_\Sigma\right)$. Then the manifold $\widehat{W} := W \cup_\Sigma \Sigma \times [1, \infty)$ forms together with*

$$\omega = \begin{cases} \mathrm{d}\lambda & \text{on } W \\ \mathrm{d}(r\alpha) & \text{on } \Sigma \times [1, \infty) \end{cases} \tag{2.2.3}$$

an exact symplectic manifold.

PROOF: The first part is to show that ω is a smooth two form: Note that for the Liouville vector field X on W we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi_X^t\right)^* \lambda = \left(\Phi_X^t\right)^* \mathcal{L}_X \lambda = \left(\Phi_X^t\right)^* (i_X \,\mathrm{d}\lambda + \mathrm{d}(i_X i_X \,\mathrm{d}\lambda)) = \left(\Phi_X^t\right)^* \lambda$$

and therefore $\left(\Phi_X^t\right)^* \lambda = e^t \lambda$. Since by assumption X is transverse to Σ, we can define the following diffeomorphism

$$\Psi : \Sigma \times (1 - \delta, 1] \to U \qquad (y, r) \mapsto \Phi_X^{\ln(r)}(y),$$

where $\delta > 0$ and U is an open subset of W with $\Sigma \subset U$. Under this diffeomorphism the one form λ becomes

$$(\Psi^* \lambda)(v_y, r) = \left(\Phi_X^{\ln(r)}\right)^* \lambda\ (v_y) = r\lambda(v_y) = r\alpha(v_y).$$

We can extend this diffeomorphism to $U \cup_\Sigma \Sigma \times [1, \infty)$ by setting it to the identity on $\Sigma \times [1, \infty)$. On $\Sigma \times [1 - \delta, \infty)$ the one form $r\alpha$ is obviously smooth and if we push it forward with the extension of Ψ we get the smooth one form

$$\widehat{\lambda} = \begin{cases} \lambda & \text{on } W \\ r\alpha & \text{on } \Sigma \times [1, \infty) \end{cases}$$

on $U \cup_\Sigma \Sigma \times [1, \infty)$. One can then extend this form to the whole symplectization in the obvious way and see that $\mathrm{d}\widehat{\lambda} = \omega$, which shows that ω is a smooth two form.

The second part is to show that ω is symplectic, i.e. we have to argue that $\omega^n > 0$. On W this is true by assumption, on $\Sigma \times [1, \infty)$ consider

$$\mathrm{d}(r\alpha) \wedge \ldots \wedge \mathrm{d}(r\alpha) = n \cdot \mathrm{d}r \wedge \alpha \wedge \mathrm{d}\alpha \wedge \ldots \wedge \mathrm{d}\alpha.$$

Since α is a contact structure on Σ we know that $\alpha \wedge \mathrm{d}\alpha \wedge \ldots \wedge \mathrm{d}\alpha > 0$ and $n\,\mathrm{d}r$ is of course also a volume form on $[1, \infty)$. This proves the proposition. $\square$

Definition 2.2.7 *Let (W, λ) be a Liouville domain with Liouville vector field X and boundary $\left(\Sigma, \alpha := \lambda\big|_\Sigma\right)$. Then we call the manifold $\widehat{W} = W \cup_\Sigma \Sigma \times [1, \infty)$ together with the symplectic structure*

$$\omega = \begin{cases} \mathrm{d}\lambda & \text{on } W \\ \mathrm{d}(r\alpha) & \text{on } \Sigma \times [1, \infty) \end{cases} \tag{2.2.4}$$

the completion of the Liouville domain (W, λ).

The completion of a Liouville domain will be our preferred setting in the later chapters in which we want to define the Rabinowitz Floer homology.

Chapter 3

The Restricted Three Body Problem

The physical setting we want to work in is the following: Given three massive bodies described by point masses, the only relevant force is gravity and the whole system can be described using classical mechanics. In this setting one would like to understand how these three bodies move in the superposition of their gravitational fields depending on the starting configuration. Even though the two body problem is rather easy to solve analytically, adding another body to that system will make it impossible to find an analytic solution in general. So to make the problem a bit more manageable, assume that two of our bodies are much heavier than the third, such that the gravitational influence of the third one on the first two is negligible (this is why we call it "restricted"). We further assume that the two heavy masses move around their shared centre of gravity in a circle. Viewing the three body problem under those additional assumptions is called the *the circular restricted three body problem.* Since the movement of the two big masses is already part of our assumptions, only the behaviour of the small body is left to be determined. Let us call the position of one of the big bodies s for sun and the other position e for earth, but note that the masses μ and $1-\mu$ of these two bodies don't need to be similar to the one of the actual sun and earth. Then the Hamiltonian for this system is given by

$$H(q,p,t) = \frac{1}{2}\|p\|^2 - \frac{\mu}{\|q-s(t)\|} - \frac{1-\mu}{\|q-e(t)\|}, \tag{3.0.1}$$

where as usually $q \in \mathbb{R}^n$ stands for the space coordinates, $p \in \mathbb{R}^n$ stands for the momentum coordinates and the total mass is normalised to 1.

One problem with this Hamiltonian is that it is time dependent because of the movement of earth and sun. In order to get a time independent Hamiltonian we consider a rotating coordinate system where earth and sun stay at fix places. Hence, the Hamiltonian becomes autonomous, but it also receives additional terms for the centrifugal force and the Coriolis force. Note that at this point the assumption of *circular* is crucial. If we declare the plane in which earth and sun are moving in to be the x-y plane, then the new Hamiltonian is

$$H(q,p) = \frac{1}{2}\|p\|^2 + p_1q_2 - p_2q_1 - \frac{\mu}{\|q-s\|} - \frac{1-\mu}{\|q-e\|}, \tag{3.0.2}$$

where e and s are now fixed coordinates. Even though the Hamiltonian is now autonomous, there is still one important issue we have to deal with, namely the collisions. The way to deal with this problem is to regularize the system for a given energy E_0, which means one compactifies the corresponding energy hypersurface in such a way that we can still recover the original dynamics up to parametrization. For more convenience we only regularize around the earth for energies below the first Lagrange point and shift the coordinate system such that the earth is at the origin.

$$H(q,p) = \frac{1}{2}\|p\|^2 + p_1(q_2+e_2) - p_2(q_1+e_1) - \frac{\mu}{\|q-s+e\|} - \frac{1-\mu}{\|q\|}, \tag{3.0.3}$$

Then the first step is always to reparametrize the time variable to get a Hamiltonian that is well-defined on the complete phase space, i.e. we set

$$t_{\text{new}} = \int \frac{1}{\|q\|} \, \mathrm{d}t_{\text{old}}.$$

In terms of the Hamiltonian function this corresponds to redefining it as

$$H_{\text{new}} = \|q\|(H_{\text{old}} - E_0),$$

where E_0 is the energy we want to regularize at. Note that the new Hamiltonian is now also defined at $q = 0$, but not yet smooth. The second step is then to find a way to compactify the hypersurface $H_{\text{new}}^{-1}(0)$, or to be more precise the part of the hypersurface that corresponds to the bounded states. The reason why there is still an issue with the compactness after removing the singularity is that by multiplying with $\|q\|$ we allowed the momentum at $q = 0$ to grow arbitrary large while still staying on the hypersurface $H_{\text{new}}^{-1}(0)$. There are now different approaches to compactify the hypersurface. The first one we want to introduce is the Moser regularization. It will not be the regularization we use later on, but since it is one of the most used we at least would like to sketch how it works: The idea is to switching the role of position and momentum coordinates while preserving the symplectic structure, such that the momentum is now described by the base coordinates of the cotangent bundle, and then apply the inverse stereographic projection to compactify the momentum that at $q = 0$ escapes to infinity (similar to the compactification of the complex plane). To be more precise, we of course take the cotangent lift of the stereographic projection to also properly transform the position coordinates. The final step of this regularization is to make H_{new} into a smooth function including $q = 0$ and without changing the energy hypersurface for energy zero. This is achieved by defining

$$H_M = \frac{1}{2}(H_{\text{new}} + 1)^2 \tag{3.0.4}$$

and considering the energy hypersurface $H_M^{-1}\left(\frac{1}{2}\right)$. In the following two sections we discuss the main regularizations used throughout this thesis.

3.1 Levi-Civita Regularization

The Levi-Civita regularization is originally a technique to regularize two body collisions in the planar Kepler problem. To be able to also use it in the three body setting, we need to assume that the starting energy and the initial conditions are chosen in such a way that we are bound to one of the two massive bodies. Like in the above discussion we will take the earth. The crucial ingredient of the Levi-Civita regularization is the map

$$l : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\} \;\; ; \; z \mapsto z^2$$

and its cotangent lift

$$\mathfrak{L} : T^*(\mathbb{C} \setminus \{0\}) \to T^*(\mathbb{C} \setminus \{0\}) \;\; ; \; (z, w) \mapsto \left(l(z), \frac{w}{\overline{l'(z)}} \right).$$

Remember that we shifted our coordinate system, such that e lies now in the origin. To regularize the Hamiltonian system $(T^*\big(\mathbb{R}^2 \setminus \{0\}\big), H_{\text{new}}, \omega = \mathrm{d}\lambda)$ we first identify $T^*\big(\mathbb{R}^2 \setminus \{0\}\big)$ with $T^*(\mathbb{C} \setminus \{0\})$ and then pull it back using the map $\mathfrak{L}$ to get the new system

$$(T^*(\mathbb{C} \setminus \{0\}), K := \mathfrak{L}^* H_{\text{new}}, \widehat{\omega} := \mathfrak{L}^* \omega = \mathrm{d}\mathfrak{L}^* \lambda), \tag{3.1.1}$$

where

$$\mathfrak{L}^*\lambda = z_1\,\mathrm{d}w_1 - w_1\,\mathrm{d}z_1 + z_2\,\mathrm{d}w_2 - w_2\,\mathrm{d}z_2 \tag{3.1.2}$$

with the non-standard notation of $z = z_1 + iz_2$ and $z_1, z_2 \in \mathbb{R}$. Note that $X_K = \mathfrak{L}^* X_{H_{\text{new}}}$ because

$$\mathfrak{L}^* i_{X_{H_{\text{new}}}}\omega = \mathfrak{L}^*\,\mathrm{d}H_{\text{new}}$$

which is equivalent to

$$i_{\mathfrak{L}^* X_{H_{\text{new}}}}\widehat{\omega} = \mathrm{d}K. \tag{3.1.3}$$

So if we have a curve γ satisfying $\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) = X_K(\gamma(t))$, we can recover the Hamiltonian dynamics on the original space by applying the map $\mathfrak{L}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{L}(\gamma(t)) = \mathrm{d}\mathfrak{L}\left(\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)\right) = \mathrm{d}\mathfrak{L}(X_K(\gamma(t))) = X_{H_{\text{new}}}(\mathfrak{L}(\gamma(t)))$$

The advantage of the system $(T^*(\mathbb{C}\setminus\{0\}), K, \widehat{\omega})$ is now that it extends to $T^*\mathbb{C}$. If we now choose our coordinate system in such a way that the two big masses lie on the real axis, the Levi-Civita regularised Hamiltonian at energy E is given by

$$\begin{aligned} K(z,w) &= (\mathfrak{L}^* H_{\text{new}})(z,w) \\ &= \frac{1}{8}|w|^2 - E|z|^2 - (1-\mu) - \frac{\mu|z|^2}{|z^2 + e - s|} + \frac{|z|^2}{2}(w_1 z_2 - w_2 z_1) - \frac{w_1 z_2 + w_2 z_1}{2} e_1, \end{aligned} \tag{3.1.4}$$

where e_1 is the x component of e. Since we later on want to use Rabinowitz Floer homology in this setting, we want to show that we can interpret the energy hypersurface as the boundary of a Liouville domain using a strategy similar to [FVK18, Chapter 4.2].

Proposition 3.1.1 *The bounded part of the energy hypersurface for the planar circular restricted three body problem in Levi-Civita regularization is for all energies below the first critical energy value star-shaped, i.e. we can view it as the boundary of a Liouville domain.*

PROOF: Fortunately this was already proven by [AFKP12] in the case of Moser regularisation, so we only need to transfer the result to the Levi-Civita setting. Since our underlying manifold is $T^*\mathbb{C}$, it is clear that the open area bounded by the compact energy hypersurface is an exact symplectic manifold. Hence, it is left to show that the corresponding Liouville vector field is transverse to the hypersurface. Let us call the Hamiltonian we get through Moser regularization H_M and the bounded part of the energy hypersurface Σ_M. If X_M is the Liouville vector field in this setting, then we know from [AFKP12] that

$$\mathrm{d}H_M(X_M)\big|_{\Sigma_M} > 0$$

and since this is a smooth function on a compact manifold, there is a $c_0 > 0$ such that

$$\mathrm{d}H_M(X_M)\big|_{\Sigma_M} > c_0.$$

Now we want to define a map ψ that transforms the energy hypersurface in Levi-Civita regularization to the hypersurface in Moser regularization. This map is a concatenation of the map $\mathfrak{L}$ from above, the symplectic switch of fibre and base coordinates `sw` and the inverse stereographic projection at the north pole P_N^{-1}:

$$\psi : T^*(\mathbb{C}\setminus\{0\}) \to T^*S^2 \setminus \left(S^2 \cup T_N^*S^2\right)$$

The inverse stereographic projection and the switch of fibre and base coordinates are now diffeomorphisms and $\mathfrak{L}$ is a local diffeomorphism, hence ψ is also a local diffeomorphism. If we pull back the symplectic two form and the Liouville one form with this map, we get for the symplectic two form

$$\begin{aligned}
\psi^*\omega_{\text{Moser}} &= (P_N^{-1} \circ \texttt{sw} \circ \mathfrak{L})^*\omega_{\text{Moser}} \\
&= \mathfrak{L}^* \operatorname{Re}(\mathrm{d}q \wedge \overline{\mathrm{d}p}) \\
&= \operatorname{Re}\left(2z\,\mathrm{d}z \wedge \overline{\left(\frac{1}{2\bar{z}}\,\mathrm{d}w - \frac{w}{2\bar{z}^2}\,\mathrm{d}\bar{z} \right)} \right) \\
&= \operatorname{Re}(\mathrm{d}z \wedge \overline{\mathrm{d}w}) \\
&= \mathrm{d}z_1 \wedge \mathrm{d}w_1 + \mathrm{d}z_2 \wedge \mathrm{d}w_2,
\end{aligned}$$

and for the Liouville one form

$$\begin{aligned}
\widehat{\mathfrak{L}}^*\lambda_{\text{Moser}} &= (P_N^{-1} \circ \texttt{sw} \circ \mathfrak{L})^*\lambda_{\text{Moser}} \\
&= \mathfrak{L}^* \operatorname{Re}(q\overline{\mathrm{d}p}) \\
&= \operatorname{Re}\left(z^2 \overline{\mathrm{d}\left(\frac{w}{2\bar{z}} \right)} \right) \\
&= \frac{1}{2} \operatorname{Re}(z\overline{\mathrm{d}w} - \overline{w}\,\mathrm{d}z) \\
&= \frac{1}{2}(z_1\,\mathrm{d}w_1 - w_1\,\mathrm{d}z_1 + z_2\,\mathrm{d}w_2 - w_2\,\mathrm{d}z_2).
\end{aligned}$$

From this we see that if we pull back the Liouville vector field in Moser regularization X_M with ψ we get the Liouville vector field in the Levi-Civita setting X_{LC}. By construction of ψ it is also easy to see that (c.f. equation (3.0.4))

$$\psi^* H_M = \frac{1}{2}(K+1)^2$$

and hence

$$\mathrm{d}(\psi^* H_M) = \mathrm{d}K,$$

see [AFKP12, chapter 3] for more details about the construction of the Moser Hamiltonian. With this we can then conclude that

$$\mathrm{d}K(X_{LC}) = \mathrm{d}(\psi^* H_M)(\psi^* X_M) = \psi^*(\mathrm{d}H_M)(\psi^* X_M) = \psi^*(\mathrm{d}H_M(X_M)) > c_0$$

on $\Sigma_{LC} \cap T^*(\mathbb{C}\setminus\{0\})$. This is a dense subset of Σ_{LC} and $\mathrm{d}K(X_{LC})$ is a smooth function, i.e. $\mathrm{d}K(X_{LC})$ needs to be bigger than zero on all of Σ_{LC}. This proves the proposition. □

3.2 The Kustaanheimo-Stiefel Regularization

The Levi-Civita regularization described in the previous section can only be used in the planar setting since it relies on the complex numbers. But by using a generalization of the complex numbers to higher dimensions, namely the quaternions, one can also use a very similar approach to regularize the spatial restricted three body problem. This procedure is then called the Kustaanheimo-Stiefel regularization.

First let us repeat the basics concerning the quaternions: To construct the quaternions out of the real numbers one introduces three new elements i, j and k with the property that

$$i^2 = j^2 = k^2 = ijk = -1. \tag{3.2.1}$$

These elements together with $1 \in \mathbb{R}$ generate the set of quaternions $\mathbb{H}$, i.e. every element can be uniquely expressed in the form $x_0 + x_1 i + x_2 j + x_3 k$ with $x_0, \ldots, x_3 \in \mathbb{R}$. The addition is defined via

$$x + y = x_0 + y_0 + (x_1 + y_1)i + (x_2 + y_2)j + (x_3 + y_3)k \tag{3.2.2}$$

and the multiplication arises as the combination of the multiplication in $\mathbb{R}$ and the rules (3.2.1) while we demand that the distributive property with respect to the above defined addition holds. One can show that these operations turn the quaternions into a skew field. As for the complex numbers we can also interpret $\mathbb{H}$ as a real vector space, this time as $\mathbb{R}^4$. With this interpretation the addition is just the usual vector addition and the multiplication can be expressed as

$$\begin{pmatrix} x_0 \\ \vec{x} \end{pmatrix} \cdot \begin{pmatrix} y_0 \\ \vec{y} \end{pmatrix} = \begin{pmatrix} x_0 y_0 - \langle \vec{x}, \vec{y} \rangle \\ x_0 \cdot \vec{y} + y_0 \cdot \vec{x} + \vec{x} \times \vec{y} \end{pmatrix}, \tag{3.2.3}$$

where $\vec{x}, \vec{y} \in \mathbb{R}^3$. The conjugation of a quaternion is defined analogously to the complex conjugation as

$$\bar{x} = x_0 - x_1 i - x_2 j - x_3 k \tag{3.2.4}$$

and also analogously to the complex numbers we define the map

$$\mathrm{Re} : \mathbb{H} \to \mathbb{R}, \quad x_0 + x_1 i + x_2 j + x_3 k \mapsto x_0. \tag{3.2.5}$$

To perform the Kustaanheimo-Stiefel regularization we define the following map:

$$\mathrm{KS} : \mathbb{H} \setminus \{0\} \times \mathbb{H} \to \mathbb{IH} \times \mathbb{H}, \quad (z, w) \mapsto \left(\bar{z} i z, \frac{\bar{z} i w}{2|z|^2} \right), \tag{3.2.6}$$

where $\mathbb{IH}$ stands for the imaginary part of the quaternions, i.e. the part we identify with $\{0\} \times \mathbb{R}^3$. Now consider $BL(z, w) := \mathrm{Re}(\bar{z} i w)$ as a function on $\mathbb{H} \times \mathbb{H}$. One can show that $BL^{-1}(0) \setminus \{0\}$ is a smooth manifold and note that $BL^{-1}(0)$ is precisely the condition needed to restrict the image of $\mathrm{KS}\big|_{BL^{-1}(0)}$ to $\mathbb{IH} \times \mathbb{IH} \cong T^*\mathbb{R}^3$, which is of course needed when we want to pull back the Hamiltonian of the spatial restricted three body problem. Restricting this set $BL^{-1}(0)$ such that KS is actually a well-defined smooth function on it we get $\Sigma_1 := BL^{-1}(0) \cap (\mathbb{H} \setminus \{0\} \times \mathbb{H})$. The corresponding regularization procedure is now very similar to the one from Levi-Civita: First we pull back the Hamiltonian of the spatial restricted three body problem H_{S3BP} using the map $\mathrm{KS}\big|_{\Sigma_1}$. Afterwards we multiply with $|z|^2$ (which again corresponds to a rescaling of the time variable) to remove the singularity at zero. The regularized Hamiltonian then reads

$$\widehat{K}(z, w) = |z|^2 \Big(\mathrm{KS}\big|_{\Sigma_1}^* H_{R3BP} \Big)(z, w). \tag{3.2.7}$$

It is not hard to see that this Hamiltonian extends to a smooth function on $\mathbb{H} \times \mathbb{H}$ and forms together with the symplectic form $\mathrm{Re}(\mathrm{d}\bar{w} \wedge \mathrm{d}z) = \sum_{i=0}^{3} \mathrm{d}w_i \wedge \mathrm{d}z_i$ the new Hamiltonian system

$$(\mathbb{H} \times \mathbb{H}, \mathrm{Re}(\mathrm{d}\bar{w} \wedge \mathrm{d}z), \widehat{K}). \tag{3.2.8}$$

As in the case of the Levi-Civita regularization we would like the resulting compactified energy hypersurface to be the boundary of a Liouville domain. Unfortunately it is not clear how to show this, since we gained an extra dimension through our regularization procedure that has no analogue in the spatial Moser regularization. So for now we are forced to settle for a weaker result, but with the advantage that we can still use the same proof strategy as for Levi-Civita regularization. Note that we still only consider that part of the energy hypersurface, which lies around the earth below the first critical energy value.

Lemma 3.2.1 *Combining the Kustaanheimo-Stiefel mapping with the symplectic switch of fibre and base coordinate and the inverse stereographic projection in dimension three we get a smooth map* Ψ *that induces a diffeomorphism*

$$\psi : \Sigma_1/S^1 \to T^*S^3 \setminus \big(S^3 \cup T^*_N S^3\big). \tag{3.2.9}$$

PROOF: Our first step is to show that Ψ is injective up to the S^1 action. So assume that there are $(z_1, w_1), (z_2, w_2) \in \Sigma_1$ with $\Psi(z_1, w_1) = \Psi(z_2, w_2)$. Since the switch of coordinates and the inverse stereographic projection are bijective this implies that $\bar{z}_1 i z_1 = \bar{z}_2 i z_2$. From this we see that $|z_1| = |z_2|$, which means there is an $\eta \in \mathbb{H}$ with $z_2 = \eta \cdot z_1$ and $|\eta| = 1$, i.e. we can write $\bar{z}_2 i z_2 = \bar{z}_1 \bar{\eta} i \eta z_1$ and get $i = \bar{\eta} i \eta$. If we now look at the second component of this equation, we see that

$$\eta_0^2 + \eta_1^2 - \eta_2^2 - \eta_3^2 = [\bar{\eta} i \eta]_{i=1} = 1 = |\eta| = \eta_0^2 + \eta_1^2 + \eta_2^2 + \eta_3^2. \tag{3.2.10}$$

This implies that $\eta_2 = \eta_3 = 0$ and $\eta_0^2 + \eta_1^2 = 1$, i.e. $\eta = e^{i\varphi} \in \mathbb{H}$. $\Psi(z_1, w_1) = \Psi(z_2, w_2)$ further implies that $\frac{\bar{z}_1 i w_1}{|z_1|^2} = \frac{\bar{z}_2 i w_2}{|z_2|^2}$, which leads to $|w_1| = |w_2|$. With the same line of argument as above we get the existence of a $\xi \in \mathbb{H}$, such that $w_2 = \xi \cdot w_1$ and therefore

$$i = \eta i \xi \quad \Rightarrow \quad 1 = e^{-i\varphi}\xi. \tag{3.2.11}$$

With this we can finally conclude that $(z_2, w_2) = (e^{i\varphi} z_1, e^{i\varphi} w_1)$, i.e. Ψ is injective up to the S^1 action. This shows that the induced map ψ is injective. The surjectivity is ensured by restricting the codomain to $T^*S^3 \setminus \big(S^3 \cup T^*_N S^3\big)$. □

Lemma 3.2.2 *The map* $\psi : \Sigma_1/S^1 \to T^*S^3 \setminus \big(S^3 \cup T^*_N S^3\big)$ *defined above pulls back the canonical one form on* T^*S^3 *to a one form on* Σ_1/S^1 *that in our usual local coordinates looks like* $\frac{1}{2}\operatorname{Re}(z\overline{\mathrm{d}w} - \overline{w}\,\mathrm{d}z)$*, i.e.*

$$\psi^* \sum_{i=1}^{3} p_i \,\mathrm{d}q_i = \sum_{i=0}^{3} \frac{1}{2}(w_i \,\mathrm{d}z_i - z_i \,\mathrm{d}w_i). \tag{3.2.12}$$

Note that we only need to define the one form on representatives, since $\frac{1}{2}\operatorname{Re}(z\overline{\mathrm{d}w} - \overline{w}\,\mathrm{d}z)$ is invariant under the S^1 action. The proof is just a very long but not so complicated computation, hence we relocated the proof to the Appendix A.1.

Proposition 3.2.3 *The Liouville vector field of* $(\mathbb{H} \times \mathbb{H}, \bar{w}\,\mathrm{d}z)$ *is transverse to* $\widehat{K}^{-1}(0)$ *and outward pointing at all points that lie in* $\widehat{K}^{-1}(0) \cap BL^{-1}(0)$ *for all energies below the first critical energy value.*

PROOF: Since we know that in the Moser regularization of the three dimensional circular restricted three body problem the energy hypersurface is fibrewise star-shaped for all energies below the first critical energy value ([CJK20, Theorem 1]), we can recover the same result for

$$\left(\Sigma_1/S^1, \omega = \mathrm{d}\left(\frac{1}{2}\operatorname{Re}(z\overline{\mathrm{d}w} - \overline{w}\,\mathrm{d}z\right), \widehat{K} = \psi^* H_M\right)$$

by using the previous two lemmas. Because the Hamiltonian $\widehat{K}$ and $\frac{1}{2}\operatorname{Re}(z\overline{\mathrm{d}w} - \overline{w}\,\mathrm{d}z)$ - and therefore also the corresponding Liouville vector field X_L - are invariant under the action of S^1, we can just extend $\mathrm{d}\widehat{K}(X_L)$ smoothly to Σ_1 as an S^1 invariant function and hence maintain $\mathrm{d}\widehat{K}(X_L) > c_0$ on $\Sigma_1 \cap \widehat{K}^{-1}(0)$ for a positive constant c_0 (we refer to the proof of Proposition 3.1.1 to see why such a constant exists). By the smoothness of $\mathrm{d}\widehat{K}(X_L)$ we get that the Liouville vector field is transverse to $\widehat{K}^{-1}(0)$ and outward pointing at all points that lie in $\widehat{K}^{-1}(0) \cap BL^{-1}(0)$. □

In Chapter 5.2 we will see that the setting described in this proposition is actually strong enough, so that we can still apply our techniques from symplectic topology.

Chapter 4

Lagrangian Rabinowitz Floer Homology

In this chapter we now want to construct our main tool, which will allow us to take on questions concerning consecutive collision orbits in the restricted three body problem. The general setting we want to work in is that of an exact symplectic Hamiltonian system $(M, \omega = \mathrm{d}\lambda, H)$, where the energy hypersurface $\Sigma := H^{-1}(0)$ is a contact manifold with respect to $\lambda\big|_\Sigma$ and the corresponding Reeb vector field R fulfils $R = X_H\big|_\Sigma$. We further assume the existence of a Lagrangian submanifold L on which the one form λ becomes exact, i.e. $\lambda\big|_L = \mathrm{d}l$ for a $l \in C^\infty(L)$. As for all Floer theories the starting point is a suitable action functional and its corresponding critical points.

4.1 Rabinowitz Action Functional and Critical Points

The difference between Rabinowitz Floer homology (RFH) and the other homologies in symplectic topology is that the complex is generated by tupels consisting of both a trajectory $x \in C^\infty([0,1], M)$ and a Lagrange multiplier/period $\tau \in \mathbb{R}$. The action functional one uses for these tuples is called the Rabinowitz action functional and is given by

$$\mathscr{A}^H(x,\tau) := \int_0^1 x^*\lambda - \tau \int_0^1 H(x(t))\,\mathrm{d}t. \tag{4.1.1}$$

The original definition goes back to the work of Rabinowitz [Rab78], the idea to use it to define a Floer homology originated in the work of Cieliebak and Frauenfelder [CF09]. Note that this action functional is designed to handle periodic orbits. In this thesis, however, we want to consider $x \in C^\infty([0,1], M)$ to be a trajectory starting at and ending inside the given Lagrangian submanifold L. We define the set containing these paths as $P(M,L) := \{x \in C^\infty([0,1], M) \mid x(0), x(1) \in L\}$. For this setting one needs to take care of the possible boundary terms and hence the action functional has to read (c.f. [Mer14, Chapter 1.2])

$$\mathscr{A}^H(x,\tau) := \int_0^1 x^*\lambda + l(x(0)) - l(x(1)) - \tau \int_0^1 H(x(t))\,\mathrm{d}t. \tag{4.1.2}$$

The critical points of this functional satisfy the equations

$$\begin{cases} \partial_t x(t) = \tau X_H(x(t)), & t \in [0,1] \\ H(x(t)) = 0 \end{cases}. \tag{4.1.3}$$

The first equation arises in the same way as in Floer homology, now just with an additional constant τ. From there we can deduce that $x\left(\frac{1}{\tau}t\right)$ is a solution of the usual Hamilton equation and this is the reason why we interpret τ as the period of the trajectory x. The second equation is now special to the Rabinowitz action functional. If one differentiates the functional with respect to τ the resulting equation is

$$\int_0^1 H(x(t))\,\mathrm{d}t = 0. \tag{4.1.4}$$

But the first equation tells us that x is just a time reparametrization of a Hamiltonian trajectory, i.e. $H(x(t))$ is constant in time. Therefore the second equation reduces to

$$H(x(t)) = 0 \tag{4.1.5}$$

and it ensures that only those Hamiltonian trajectories with energy zero contribute to the set of critical points. Since in the usual Floer homology we don't have the extra parameter τ, the period of the critical points is always equal to one, but on the other hand the trajectories are not restricted to only one energy level.

As we want to define a Floer homology corresponding to this Rabinowitz action functional, it is important for the functional to be at least Morse-Bott. Since the constant trajectories $(x_0, 0)$ with $x_0 \in H^{-1}(0) \cap L$ are always critical points of the action functional, it, in fact, can never be just Morse. But if we consider the functional on $P(M, L)$, those are the only critical points that prevent $\mathscr{A}^H$ from being Morse:

Theorem 4.1.1 *For a generic choice of $H \in C_c^\infty(M)$ the Rabinowitz action functional is Morse-Bott and the critical set consists of $H^{-1}(0) \cap L$ together with an isolated collection of points.*

The proof of this theorem is almost the same as the one for [CF09, Theorem B.1.], except that the nonconstant trajectories in $P(M, L)$ don't have the S^1-symmetry by time shift and therefore we can achieve the Morse condition on those.

To be able to properly define the Rabinowitz Floer homology, we need to be able to assign a grading to the critical points. This is the task we will take on in the next section.

4.2 Maslov Index

Given a critical point (x, τ) of $\mathscr{A}^H$. The way we want to assign an index to this point is by first identifying it with the Hamiltonian trajectory $x\left(\frac{1}{\tau}t\right)$ and associate to this trajectory a path of Lagrangian subspaces. The grading is then given by the Maslov index of the path of Lagrangians.

So first let us start by defining the Maslov index (this part is a condensed version of [FVK18, Chapter 10.1-10.3]): The space we are working on in this context is a manifold called the Lagrangian Grassmannian denoted by $\Lambda = \Lambda(n)$ and consists of all the Lagrangian subspaces in $\mathbb{C}^n$. This set indeed is a smooth manifold, since we can identify it with $U(n)/O(n)$, i.e. the Lagrangian Grassmannian is even a homogeneous space. For a given $L_0 \in \Lambda$ we define the submanifolds

$$\Lambda_{L_0}^k = \Lambda_{L_0}^k(n) := \{L \in \Lambda(n) : \dim(L \cap L_0) = k\} \tag{4.2.1}$$

for all $0 \leq k \leq n$. Now choose two Lagrangians $L_1, L_2 \in \Lambda$ such that $L_1 \cap L_2 = \{0\}$, i.e. $L_1 \in \Lambda_{L_2}^0$. Then we define a map from $\Lambda_{L_2}^0$ to the quadratic forms on L_1denoted by $S^2(L_1)$: Given an $L \in \Lambda_{L_2}^0$, then there is for every $v \in L_1$ a unique $w_v \in L_2$ such that $v + w_v \in L$, because L_1 and L_2 are two subspaces with half of the full dimension and $L_1 \cap L_2 = \{0\}$. With this we define the quadratic form as

$$Q_L^{L_1,L_2}(v) = \omega(v, w_v) \tag{4.2.2}$$

for all $v \in L_1$ and ω the symplectic structure on $\mathbb{C}^n$. The map from $\Lambda^0_{L_2}$ to $S^2(L_1)$ is then simply given by

$$L \mapsto Q_L^{L_1,L_2}. \tag{4.2.3}$$

If we are now given a path of Lagrangians $\lambda : [0,1] \to \Lambda$ and take a $L_0 \in \Lambda$ as fixed, then we define the crossing form

$$C(\lambda, L_0, t) := \left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} Q^{L_0,L_2}_{\lambda(t)} \right)\bigg|_{\lambda(t)\cap L_0} \tag{4.2.4}$$

for all $t \in [0,1]$, where L_2 is an arbitrary element in $\Lambda^0_{L_0}$. Note that by [FVK18, Lemma 10.2.3.] this form is independent of the chosen L_2. After a perturbation with fixed endpoints we can assume that for any path $\lambda : [0,1] \to \Lambda$ the crossing form $C(\lambda, L_0, t)$ is non singular for all $t \in \lambda^{-1}\left(\overline{\Lambda^1_{L_0}}\right)$, where the completion of $\Lambda^1_{L_0}$ is simply $\bigcup\limits_{k=1}^{n} \Lambda^k_{L_0}$. So suppose it is true for $\lambda : [0,1] \to \Lambda$, then we define its Maslov index with respect to the fixed L_0 as

$$\mu_{L_0}(\lambda) := \frac{1}{2}\mathrm{sign}C(\lambda, L_0, 0) + \sum_{t \in \lambda^{-1}\left(\overline{\Lambda^1_{L_0}}\right)} \mathrm{sign}C(\lambda, L_0, t) + \frac{1}{2}\mathrm{sign}C(\lambda, L_0, 1). \tag{4.2.5}$$

Here sign stands for the signature of a quadratic form. This definition is due to Robbin and Salamon [RS93] and has the following useful properties:

(i) **Invariance**: Two paths $\lambda_1, \lambda_2 : [0,1] \to \Lambda$, that are homotopic with fixed endpoint, also have the same Maslov index $\mu_{L_0}(\lambda_1) = \mu(\lambda_2)$

(ii) **Concatenation**: Assume that $\lambda_1, \lambda_2 : [0,1] \to \Lambda$ satisfy $\lambda_1(1) = \lambda_2(0)$, then the concatenation $\lambda_1 \# \lambda_2$ fulfils

$$\mu_{L_0}(\lambda_1 \# \lambda_2) = \mu_{L_0}(\lambda_1) + \mu_{L_0}(\lambda_2). \tag{4.2.6}$$

Before continuing with defining the grading for the Hamiltonian trajectories we want to give a more geometric interpretation of the Maslov index defined above. For this assume that we have a path λ of Lagrangian subspaces, such that $\lambda(t) \notin \Lambda^k$ for $k \geq 2$. In fact, after a perturbation with fixed endpoints we can always achieve this for $t \in (0,1)$. The Maslov index of a path λ can then be interpreted as the intersection number of λ with the submanifold $\Lambda^1 \subset \Lambda$. Note that the codimension of the $\Lambda^k \subset \Lambda$ is given by

$$\mathrm{codim}(\Lambda^k, \Lambda) = \frac{k \cdot (k+1)}{2}. \tag{4.2.7}$$

For a proof of this formula see for example [FVK18, Proposition 10.2.1]. Therefore $\Lambda^1 \subset \Lambda$ has codimension one and hence, the image of a given path of Lagrangian subspaces and the submanifold Λ^1 are of complementary dimension. To be able to properly define an intersection number we need to know which intersections we have to count positively and which with a negative sign. Unfortunately Λ^1 is in general not orientable, but it is at least always coorientable. To define this coorientation we first need the following lemma (c.f. [FVK18, Lemma 10.2.4]

Lemma 4.2.1 *Assume that $L_0 \in \Lambda$, $k \in \{0, \dots, n\}$ and $L \in \Lambda^k_{L_0}$, then*

$$T_L \Lambda^k_{L_0} = \left\{ \widehat{L} \in T_L \Lambda \ : \ Q^{\widehat{L}}\big|_{L \cap L_0} = 0 \right\}. \tag{4.2.8}$$

Here we define

$$Q^{\widehat{L}} := \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} Q^{L_0, L_2}_{L(t)}, \tag{4.2.9}$$

such that $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} L(t) = \widehat{L}$ and L_2 an arbitrary element in $\Lambda^0_{L_0}$ on which $Q^{\widehat{L}}$ does not depend. One can show that the induced map

$$T_{L_0}\Lambda \to S^2(L_0), \quad \widehat{L} \mapsto Q^{\widehat{L}} \tag{4.2.10}$$

is a vector space isomorphism (see [FVK18, Chapter 10.2] for more details). If we now have a $\widehat{L} \in T_L\Lambda \setminus T_L\Lambda^1$ with $L \in \Lambda^1$, Lemma 4.2.1 implies that $Q^{\widehat{L}}\big|_{L\cap L_0} \neq 0$. This enables us to define the coorientation in the following way: We call a vector $[\widehat{L}] \in T_L\Lambda / T_L\Lambda^1$ positively oriented if and only if for a representative $\widehat{L} \in [\widehat{L}]$ the form $Q^{\widehat{L}}\big|_{L\cap L_0}$ is positive. Since by assumption $\widehat{L} \in T_L\Lambda \setminus T_L\Lambda^1$, we know that the quadratic form can not be zero and because again by assumption $L \in \Lambda^1_{L_0}$ we only consider the quadratic form on a one dimensional subspace. Hence, the quadratic form $Q^{\widehat{L}}\big|_{L\cap L_0}$ is either always positive or always negative. Having set up the coorientation, we can go ahead and define

$$\nu(t) := \begin{cases} 1 & [\partial_t \lambda(t)] \text{ positive} \\ -1 & \text{else} \end{cases}. \tag{4.2.11}$$

The Maslov index in terms of an intersection number with Λ^1 is then given by

$$\mu_{L_0}(\lambda) = \sum_{t \in \lambda^{-1}\left(\Lambda^1_{L_0}\right)} \nu(t). \tag{4.2.12}$$

In a more general setting where the path λ of Lagrangian subspaces does intersect $\Lambda^k_{L_0}$ for $k \geq 2$, counting the intersections is more complicated and therefore the signature of the quadratic form appears in the formula (4.2.5).

Now we want to understand how to assign a Maslov index to a Hamiltonian trajectory starting and ending in a given Lagrangian submanifold: First we need to assume that the given Hamiltonian trajectory γ is contractible with respect to the Lagrangian submanifold L inside of the $2n$-dimensional manifold M, i.e. γ can be contracted to a point in L while the endpoints stay inside of L throughout the contraction. This implies that we can find a filling

$$\bar{\gamma} : (\overline{\mathbb{D}^+}, [-1, 1]) \to (M, L) \tag{4.2.13}$$

such that $\bar{\gamma}(e^{i\pi t}) = \gamma(t)$. Here $\mathbb{D}^+$ should denote the unit complex disk intersected with the upper half plane and $[-1, 1]$ is the part of $\overline{\mathbb{D}^+}$ that is mapped to L. Since D^+ is contractible there exists a trivialisation

$$\mathfrak{T} : \bar{\gamma}^* TM \to \mathbb{D}^+ \times \mathbb{R}^{2n}. \tag{4.2.14}$$

In fact, we can even assume for $\mathfrak{T}$ to be symplectic. The second step is to associate to our given trajectory a path of symplectic matrices. We know that $\mathrm{d}\phi^t_H(\gamma(0))$ is a symplectic matrix for every

$t \in [0,1]$ in the vector space $T_{\gamma(t)}M$. To make this into a path in $Sp(n)$ we use the trivialisation from above, where we denote by a slight abuse of notation the linear part of the trivialization at the point x by $\mathfrak{T}_x$. Hence, we define

$$\Psi(t) := \mathfrak{T}_{\gamma(t)}\, \mathrm{d}\phi_H^t(\gamma(0))\mathfrak{T}_{\gamma(t)}^{-1} \tag{4.2.15}$$

as the path in $Sp(n)$ corresponding to γ. Note that since $\mathfrak{T}$ is a symplectic trivialisation we can here even achieve that $\mathfrak{T}_{\bar{\gamma}(s)}\big(T_{\bar{\gamma}(0)}L\big)$ is a constant Lagrangian subspace in $\mathbb{R}^{2n}$ for $s \in [-1,1] \subseteq \overline{\mathbb{D}^+}$. With this set-up it is now very straight forward to associate to the trajectory γ a path of Lagrangian subspaces and therefore ultimately a Maslov index. As path of Lagrangian subspaces we take $\Psi(t)\mathfrak{T}_{\bar{\gamma}(0)}\big(T_{\bar{\gamma}(0)}L\big)$ and define the Maslov index of γ to be the Maslov index of this very path of Lagrangians in accordance with (4.2.5). Note that this index depends on the given Lagrangian and it might in general also depend on the filling and the trivialization. Now let us look at an example

Example 4.2.2 Consider as manifold $\mathbb{C}^n$ with the Hamiltonian $\widetilde{H}(z) = \left(\sum_i^n |z_i|^2\right) - 1$ and the Lagrangian submanifold $L = \mathbb{R}^n \subset \mathbb{C}^n$. We want to assign to the Hamiltonian trajectories that lie on the energy hypersurface $\Sigma = \widetilde{H}^{-1}(0)$ a Maslov index. The Hamiltonian vector field in this case is

$$X_{\widetilde{H}}(x,y) = 2\begin{pmatrix} -y \\ x \end{pmatrix} \tag{4.2.16}$$

with $x, y \in \mathbb{R}^n$. For a given $z = \binom{x}{y} \in \Sigma \cap L$ the corresponding chord is then simply

$$\phi(z,t) = \begin{pmatrix} \cos(2t)\mathbb{1}_n & -\sin(2t)\mathbb{1}_n \\ \sin(2t)\mathbb{1}_n & \cos(2t)\mathbb{1}_n \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{for } t \in \Big[0, \frac{m}{2}\Big]. \tag{4.2.17}$$

To compute the Maslov index for those chords we use our Lagrangian L as base point and the following path of Lagrangian subspaces:

$$\Gamma(t) := \Omega(t)L \quad \text{with} \quad \Omega(t) = \begin{pmatrix} \cos(2t)\mathbb{1}_n & \sin(2t)\mathbb{1}_n \\ -\sin(2t)\mathbb{1}_n & \cos(2t)\mathbb{1}_n \end{pmatrix} \tag{4.2.18}$$

For the quadratic form take as Lagrangian subspaces $\mathbb{R}^n$, $i\mathbb{R}^n \subset \mathbb{C}^n$. To calculate this form we first need to find for every $\binom{x}{0}$ in $\mathbb{R}^n$ a $\binom{0}{b}$ in $i\mathbb{R}^n$ such that

$$\begin{pmatrix} x \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ b \end{pmatrix} = \begin{pmatrix} \cos(2t)c_0 \\ -\sin(t)c_0 \end{pmatrix}. \tag{4.2.19}$$

for an arbitrary constant c_0. This is fulfilled by $z = -\tan(2t)$. Hence, the quadratic form is given by

$$\begin{aligned} Q_{\Gamma(t)}\left(\begin{pmatrix} x \\ 0 \end{pmatrix}\right) &= \omega\left(\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \tan 2t \end{pmatrix}\right) \\ &= \langle x, -\tan(2t)x\rangle. \end{aligned}$$

Therefore the crossing form is

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=\frac{m}{2}\pi} Q_{\Gamma(t)}\left(\begin{pmatrix} x \\ 0 \end{pmatrix}\right)\bigg|_L = \left\langle x, \frac{-2}{\cos^2(2t)}\mathbb{1}_n x\right\rangle. \tag{4.2.20}$$

Now to the intersection points: One can easily see that $\Gamma(t)$ only intersects L for $t = \frac{m}{2}\pi$ with $m \in \mathbb{Z}$, i.e. in the case where $\Gamma(t) = L$. This implies two things: First, only the trajectories with period (here the time from start to finish) equal to $\frac{m}{2}\pi$ with $m \in \mathbb{Z}$ are the ones starting and ending in the Lagrangian L. Second, the signature of the crossing form is $-n$ for $m \geq 0$ and n for $m < 0$. If γ is now a trajectory with endpoints in L and period $\frac{m}{2}\pi$, then its Maslov index is

$$\mu_L(\gamma) = \frac{1}{2}n + (m-1)\cdot n + \frac{1}{2}n = nm \tag{4.2.21}$$

To be able to use the Maslov index as a well-defined grading for the trajectories in the definition of the Rabinowitz Floer homology, we need it to be independent of the chosen filling. The setting, in which we want to apply the RFH later on, is just the $\mathbb{R}^n$ where we don't need a trivialization. But to maintain a certain generality, we want to give at least the idea of how to deal with this problem in a more general setting. The additional condition one then needs to require is that the Maslov index of the Lagrangian submanifold L vanishes, which then gives us the independence of the filling. So first let us discuss what the Maslov index of a Lagrangian is: If we take an element $[A] \in \pi_2(M, L)$ where $A \subset M$ with $\partial A \subset L$, we can find a symplectic trivialization

$$\mathfrak{T} : TA \to \mathbb{D} \times \mathbb{R}^{2n}. \tag{4.2.22}$$

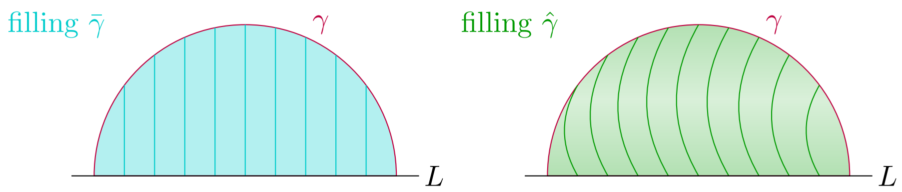
Note that here it is in general not possible to set in the trivialization the boundary to the constant Lagrangian $\mathbb{R}^n \subset \mathbb{R}^{2n}$. If we pick now an $f : \partial\mathbb{D} \to \partial A$, we can construct a loop of Lagrangian subspaces via

$$s \mapsto \mathfrak{T}_{f(e^{is})} T_{f(e^{is})} L \mathfrak{T}^{-1}_{f(e^{is})}. \tag{4.2.23}$$

For this loop we then calculate the Maslov index according to equation (4.2.5). Assigning this way a number to every element in $\pi_2(M, L)$ yields a map

$$\mu : \pi_2(M, L) \to \mathbb{Z}. \tag{4.2.24}$$

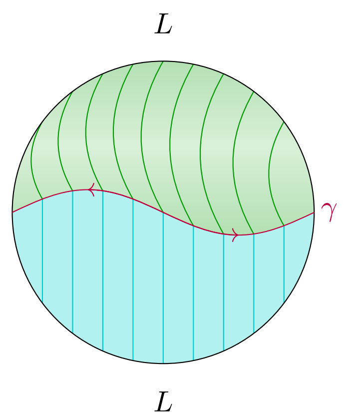
By a slight abuse of notation we call this map the Maslov index of the Lagrangian L. Now we want to understand why the vanishing of the Maslov index, i.e. the map $\mu : \pi_2(M, L) \to \mathbb{Z}$ being the zero map, implies that we have independence of the filling. So given a trajectory γ in M with two fillings $\bar{\gamma}$ and $\hat{\gamma}$.



The idea is now to calculate the difference of their corresponding Maslov indices using the concatenation property of the Maslov index, i.e.

$$\mu(\bar{\gamma}) - \mu(\hat{\gamma}) = \mu(\bar{\gamma}\#\hat{\gamma}^-). \tag{4.2.25}$$

Here $\bar{\gamma}\#\hat{\gamma}^-$ means that we invert the orientation of γ in the filling $\hat{\gamma}$ and concatenate the paths around the fillings in such a way that we first go along one of the Lagrangian boundaries then back and forth along γ and finally through the other Lagrangian boundary. Note that also for $\mu(\bar{\gamma})$ and $\mu(\hat{\gamma})$ we can assume that Lagrangian boundary to be a part of the path since under the trivialisation it will be constant anyway.



It is easy to see that the path along γ in $\bar{\gamma}\#\hat{\gamma}^-$ does not contribute to the Maslov index and hence $\mu(\bar{\gamma}\#\hat{\gamma}^-)$ is equal to the Maslov index of the loop we get when just going back and forth on the boundaries of $\mu(\bar{\gamma})$ and $\mu(\hat{\gamma})$. But this path is now a representative in $\pi_2(M,L)$ and by assumption its Maslov index is zero. Hence, we get that

$$\mu(\bar{\gamma}) - \mu(\hat{\gamma}) = 0. \tag{4.2.26}$$

So from here on out we assume in this chapter that the Maslov index of the given Lagrangian submanifolds vanishes.

Later on we also want to incorporate symmetries of a Hamiltonian system in its Rabinowitz Floer homology. So assume we have two trajectories γ_1 and γ_2 and G is a Lie group that is a symmetry of the underlying Hamiltonian system, such that the Lagrangian L is invariant under the action of G. Then we have:

Proposition 4.2.3 *If in the above setting the Maslov index of L vanishes and there is a $g \in G$ such that $g \triangleright \gamma_1 = \gamma_2$, then the Maslov index of γ_1 and γ_2 coincides*

$$\mu(\gamma_1) = \mu(\gamma_2). \tag{4.2.27}$$

Proof: If $\bar{\gamma}_1$ is a filling of γ_1, then $\phi_g(\bar{\gamma}_1)$ is a filling of γ_2 and since the Maslov index of L vanishes we can choose any filling we want. For the corresponding trivialisations $\mathfrak{T}$ of $T\bar{\gamma}_1$ and $\mathfrak{T}'$ of $T\bar{\gamma}_2 = T\phi_g(\bar{\gamma}_1)$ we get the relation

$$\mathfrak{T}'_{\gamma_2(t)} = \mathfrak{T}_{\gamma_1(t)}\,\mathrm{d}\phi_g^{-1}$$

for all $t \in [0,1]$. Since G is a symmetry of the Hamiltonian system we have that $\phi_H^t(g \triangleright x) = g \triangleright \phi_H^t(x)$, which implies

$$\mathrm{d}\phi_g \circ \mathrm{d}\phi_H^t \circ \mathrm{d}\phi_g^{-1} = \mathrm{d}\phi_H^t(\phi_g(\,\cdot\,)).$$

Combining these results we get

$$\begin{aligned}
\mu(\gamma_2) :&= \mu\Big(t \mapsto \mathfrak{T}'_{\gamma_2(t)}\,\mathrm{d}\phi_H^t(\gamma_2(0))\Big(\mathfrak{T}'_{\gamma_2(t)}\Big)^{-1}\Big)\\
&= \mu\Big(t \mapsto \mathfrak{T}'_{\gamma_2(t)}\,\mathrm{d}\phi_H^t(g \triangleright \gamma_1(0))\Big(\mathfrak{T}'_{\gamma_2(t)}\Big)^{-1}\Big)\\
&= \mu\Big(t \mapsto \mathfrak{T}'_{\gamma_2(t)}\,\mathrm{d}\phi_g\,\mathrm{d}\phi_H^t(\gamma_1(0))\,\mathrm{d}\phi_g^{-1}\Big(\mathfrak{T}'_{\gamma_2(t)}\Big)^{-1}\Big)\\
&= \mu\Big(t \mapsto \mathfrak{T}_{\gamma_1(t)}\,\mathrm{d}\phi_g^{-1}\,\mathrm{d}\phi_g\,\mathrm{d}\phi_H^t(\gamma_1(0))\,\mathrm{d}\phi_g^{-1}\,\mathrm{d}\phi_g\big(\mathfrak{T}_{\gamma_1(t)}\big)^{-1}\Big)\\
&= \mu\Big(t \mapsto \mathfrak{T}_{\gamma_1(t)}\,\mathrm{d}\phi_H^t(\gamma_1(0))\big(\mathfrak{T}_{\gamma_1(t)}\big)^{-1}\Big)\\
&= \mu(\gamma_1).
\end{aligned}$$

□

4.3 Gradient Flow Lines and Compactness

Another crucial ingredient for the Rabinowitz Floer Homology are the gradient flow lines of $\mathscr{A}^H$. To define them we first need to fix a metric on the space $P(M,L) \times \mathbb{R}$. So for $(x,\tau) \in P(M,L) \times \mathbb{R}$ and $(\xi_1, \hat{\tau}_1)$, $(\xi_2, \hat{\tau}_2) \in T_{(x,\tau)}(P(M,L) \times \mathbb{R})$ set

$$g_J((\xi_1, \hat{\tau}_1), (\xi_2, \hat{\tau}_2)) = \int_0^1 \omega(\xi_1, J(x)\xi_2)\,\mathrm{d}t + \hat{\tau}_1 \cdot \hat{\tau}_2, \tag{4.3.1}$$

where J is an almost complex structure compatible with ω. The gradient of the Rabinowitz action functional is with respect to this metric given by

$$\nabla \mathscr{A}_H(x;\tau) = \begin{pmatrix} -J(x)(\partial_t x - \tau X_H(x)) \\ -\int\limits_0^1 H(x)\,\mathrm{d}t \end{pmatrix}. \tag{4.3.2}$$

The gradient flow lines of $\nabla \mathscr{A}_H$ are therefore solutions $(u,\tau) : \mathbb{R} \to P(M,L) \times \mathbb{R}$ to the following equations

$$\begin{cases} \partial_s u + J(u)(\partial_t u - \tau X_H(u)) = 0 \\ \partial_s \tau + \int\limits_0^1 H(u)\,\mathrm{d}t = 0. \end{cases} \tag{4.3.3}$$

We can assign to every gradient flow line an energy:

$$\begin{aligned} E((u,\tau)) :&= \int\limits_{-\infty}^{+\infty} |\,\partial_s u|_J^2 \,\mathrm{d}s \\ &= \int\limits_{-\infty}^{+\infty} |\,\partial_t u - \tau X_H(u)|_J^2 \,\mathrm{d}s \end{aligned} \tag{4.3.4}$$

Morally this energy measures how far the gradient flow line is on average away from a Hamiltonian trajectory. With this interpretation the following theorem seams quite natural.

Theorem 4.3.1 *Let $\hat{u} = (u,\tau)$ be a gradient flow line. If $E(\hat{u}) < +\infty$, then there exist $\gamma^{\pm} \in P(M,L)$ such that*

$$\lim_{s\to\pm\infty} u(s,\,\cdot\,) = \gamma^{\pm} \quad \textit{and} \quad \lim_{s\to\pm\infty} \partial_s u(s,\,\cdot\,) = 0$$

in C^∞. The energy can then be calculated via

$$E(\hat{u}) = \mathscr{A}^H(\gamma^-) - \mathscr{A}^H(\gamma^+). \tag{4.3.5}$$

The proof of this theorem is an adaptation of the arguments found in [ADE14, Chapter 6.5b]. For $-\infty < a < b < \infty$ we denote by $\widetilde{\mathcal{M}}(H,J)_a^b$ the set of all parametrized gradient flow lines with respect to H and J with finite energy such that $a \leq \mathscr{A}^H(u(s)) \leq b$ for all $s \in \mathbb{R}$.

Our next step is now to show that the moduli space $\widetilde{\mathcal{M}}(H,J)_a^b$ is compact, as this is an important ingredient to show the well-definedness of the Rabinowitz Floer homology later on. First we prove compactness with respect to the C^∞_{loc}-topology. The usual idea to achieve this is to find an L^∞ bound on the gradient flow lines and their first derivatives, then use Arzelà–Ascoli to get compactness with respect to the C^0_{loc}-topology and finally employ a bootstrapping argument to get to higher regularity.

To establish the L^∞ bound we follow the usual approach of proving (c.f. [CF09, Chapter 3.1.])

- the L^∞ boundedness for the chords,
- the L^∞ boundedness for the Lagrange multiplier and its derivative,
- the L^∞ boundedness for the derivative of the chords.

Since this procedure is already well understood in the case of periodic orbits, we will mainly focus on the issues that arise because of the Lagrangian boundary condition.

To be able to find a universal bound for our chords u, we have to make some further assumptions: Assume that our Hamiltonian system $(M, \omega = \mathrm{d}\lambda, H)$ is the completion of a Liouville domain $(\widetilde{M}, \lambda)$ (c.f. Definition 2.2.7) and the Hamiltonian is constant outside of a compact set K with $\widetilde{M} \subset K$. Further assume that the almost complex structure J is *SFT like outside of* K:

Definition 4.3.2 *Let (M, ω) be the completion of a Liouville domain and $\widetilde{M} \subset K$ a compact set of M. We call J SFT like outside of K if J is ω-compatible and fulfils*

$$J(x)R(x) = \partial_r\big|_x \tag{4.3.6}$$

for all $x \in M \setminus K$, where R is the usual extension of the Reeb vector field to the positive part of the symplectization and ∂_r stands for the unit vector field in $\mathbb{R}$ direction.

This definition is of course motivated by the assumptions placed on J in symplectic field theory (see [EGH00, Chapter 1.4]). Before showing the existence of an L^∞ bound for the chords with Lagrangian boundary condition, let us first repeat some known but useful results.

Proposition 4.3.3 *Let (M, ω) be the completion of a Liouville domain $(\widetilde{M}, \lambda)$ with boundary Σ and let K be a compact set of M with $\widetilde{M} \subsetneq K$. Then we can always find an almost complex structure that is SFT like outside of K.*

PROOF: Note that $M \setminus \widetilde{M}^\circ$ is given by the positive part of the symplectization, i.e. $[0, \infty) \times \Sigma$. We know that $(\ker \lambda, \mathrm{d}\lambda|_{\ker \lambda})$ defines a symplectic vector bundle and so we choose a compatible almost complex structure $\tilde{J}$ on it. Since for the tangent space of the symplectization

$$T_{(r,x)}([0, \infty) \times \Sigma) = \mathbb{R} \times T_x\Sigma$$

holds, we can extend the Reeb vector field to all of $[0, \infty) \times \Sigma$ by setting $R(r, x) = (r, R(x))$. The symplectic form on $[0, \infty) \times \Sigma$ is given by $\mathrm{d}(r\lambda)$, so we get a symplectic splitting of the form:

$$T_{(r,x)}([0, \infty) \times \Sigma) = \langle \partial_r \rangle \oplus \langle R \rangle \oplus \ker \lambda$$

Now we extend $\tilde{J}$ to the whole tangent space of the symplectization by defining this new J as:

$$JR := \partial_r \quad , \quad J\,\partial_r := -R \quad \text{and} \quad J\big|_{\ker \lambda} := \tilde{J}$$

This definition now also extends to the negative part of the symplectization and is still an almost complex structure on there. To finally get a compatible almost complex structure on all of M, we first choose a compatible almost complex structure J_0 on $\widetilde{M}$. From [MS17, Chapter 4] we know that the set of compatible almost complex structures (w.r.t. a given symplectic structure) is path connected, i.e. we can then find a homotopy $\mathcal{J}_s$ from J to J_0 on the symplectization. Then we define on the symplectization the almost complex structure $\widehat{J}$, which is equal to J for $r > 0$, equal to J_0 for $r < -1$ and equal to $\mathcal{J}_{-r}$ for $r \in [-1, 0]$. This $\widehat{J}$ fulfils now all the requirements. $\square$

With this notion of an SFT like J we have all the tools to discuss the L^∞ bound of the chords. But before we can prove the main theorem, we first need the following lemma.

Lemma 4.3.4 *Let M be again the completion of a Liouville domain with the same assumptions as above. If the almost complex structure J is SFT like outside of a compact set K, then the solutions of the J-holomorphic curve equation*

$$\partial_s u + J\,\partial_t u = 0 \tag{4.3.7}$$

fulfil

$$\Delta(\pi \circ u) \geq 0 \tag{4.3.8}$$

on $M \setminus K$, where π is the projection on the $\mathbb{R}$-component of the symplectization.

PROOF: First note that

$$\mathrm{d} \circ (\mathrm{d}(\pi \circ u) \circ i) = \mathrm{d}(-\partial_t(\pi \circ u)\,\mathrm{d}s + \partial_s(\pi \circ u)\,\mathrm{d}t) = \Delta(\pi \circ u)\,\mathrm{d}s \wedge \mathrm{d}t.$$

Next we consider

$$\mathrm{d}(\pi \circ u) \circ i = -\,\mathrm{d}\pi(u) \cdot \partial_t u\,\mathrm{d}s + \mathrm{d}\pi(u) \cdot \partial_s u\,\mathrm{d}t.$$

Since the symplectic form on the symplectization is given by $\omega = \mathrm{d}r \wedge \lambda + r\,\mathrm{d}\lambda$, the Reeb vector field R on the boundary of our Liouville domain fulfils $i_R\omega = \mathrm{d}r$. Now note that the projection π on the $\mathbb{R}$ component is the same map as the coordinate chart for the $\mathbb{R}$ component r. Hence we get

$$\mathrm{d}(\pi \circ u) \circ i = -\omega(R, \partial_t u)\,\mathrm{d}s + \omega(R, \partial_s u)\,\mathrm{d}t$$

Then we use that u is a J-holomorphic curve and the ω-compatibility of J:

$$\mathrm{d}(\pi \circ u) \circ i = \omega(JR, \partial_s u)\,\mathrm{d}s + \omega(JR, \partial_t u)\,\mathrm{d}t$$

Remember that per definition J maps the Reeb vector field onto the unit vector in $\mathbb{R}$ direction, i.e. $JR = \partial_r$. In combination with the special form of ω discussed above, this leads to

$$\mathrm{d}(\pi \circ u) \circ i = \lambda(\partial_s u))\,\mathrm{d}s + \lambda(\partial_t u)\,\mathrm{d}t = u^*\lambda.$$

This means now:

$$\Delta(\pi \circ u)\,\mathrm{d}s \wedge \mathrm{d}t = \mathrm{d}u^*\lambda = u^*d\lambda = \omega(\partial_s u, \partial_t u) = \omega(\partial_s u, J\,\partial_s u) \geq 0$$

Here we again used the ω-compatibility together with the J-holomorphic curve equation. □

The first step is now to prove that there exists a bound for the chords u:

Theorem 4.3.5 *So let $(M, \omega = \mathrm{d}\lambda)$ be the completion of a Liouville domain with a Hamiltonian H such that $\partial\widetilde{M} = H^{-1}(0)$, the support of $\mathrm{d}H$ is inside of a compact set K and the Hamiltonian vector field coincides with the Reeb vector field on $\partial\widetilde{M}$. Let further be L an exact Lagrangian submanifold, such that l with $\mathrm{d}l = \lambda$ has support in K and let there be an almost complex structure J, which is ω-compatible and SFT like outside of K. Then every solution of the Floer equation*

$$\partial_s u + J(u)(\partial_t u - \tau X_H(u)) = 0 \tag{4.3.9}$$

with $u(s, \cdot) \in P(M, L)$ $\forall s \in \mathbb{R}$ and of finite energy can not leave the compact set K and therefore these solutions are all uniformly bounded.

PROOF: By contradiction assume that there exists such a solution u that takes values outside of K. Per definition H is constant there and hence the Floer equation becomes the J-holomorphic curve equation:

$$0 = \partial_s u + J(u)(\partial_t u - \tau X_H(u)) = \partial_s u + J(u)(\partial_t u - 0)$$

Since J was chosen to be SFT like outside of K, Lemma 4.3.4 tells us that

$$\Delta(\pi \circ u) \geq 0.$$

Now the usual approach in Floer homology is to use Hopf's strong maximum principle from the theory of elliptic partial differential equations (see [GT01] Theorem 3.5 for a modern reference) to infer that $\pi \circ u$ is not allowed to reach its maximum outside of K, since the maximum principle says that a function with $\Delta f \geq 0$ can only reach its maximum in the interior if it is constant. But because u needs to return back to $\partial\widetilde{M}$ as $s \to \infty$, $\pi \circ u$ can not be constant and needs to reach its maximum. In the case of loops this yields the desired contradiction, since $\mathbb{R} \times S^1$ has no boundary. For the case of chords with Lagrangian boundary this is not enough, because $\pi \circ u$ is defined on $\mathbb{R} \times [0, 1]$, i.e. on a set with boundary. The idea is now to extend $\pi \circ u$ from the strip to the cylinder by reflecting it at the boundary. As cylinder we take $\mathbb{R} \times [0, 2]$ where we identify 0 with 2 in the second component. On the $\mathbb{R} \times [0, 1]$ part of the cylinder the $\pi \circ u$ is defined as usual and on the $\mathbb{R} \times [1, 2]$ part we set $(\pi \circ u)(s, t) = (\pi \circ u)(s, 2 - t)$. To show that this extension is still C^2, consider:

$$\begin{aligned}
\partial_t(\pi \circ u)\big|_{t=0,1} &= \mathrm{d}\pi(\partial_t u\big|_{t=0,1}) \\
&= -\omega(R, \partial_t u\big|_{t=0,1}) \\
&= -\omega(R, J\,\partial_s u\big|_{t=0,1}) \\
&= \omega(JR, \partial_s u\big|_{t=0,1}) \\
&= \omega(\partial_r, \partial_s u\big|_{t=0,1}) \\
&= \lambda(\partial_s u\big|_{t=0,1})
\end{aligned}$$

The arguments used in the above computation are the same as in Lemma 4.3.4 and note that for our contradiction we only need to make this considerations outside of K. Per definition $u\big|_{t=0,1}$ is in L and also per definition is $\lambda = \mathrm{d}l = 0$ on L outside of K. Hence $\partial_t(\pi \circ u)\big|_{t=0,1} = 0$ outside of K and the extension is C^2. Note that for extension by reflection the second derivative is always well-defined if the original function already was C^2, since

$$\partial_t^2\big|_{t=2} f(2-t) = (-1)^2 f''(2-t)\big|_{t=2} = f''(t)\big|_{t=0}.$$

Obviously we have $\Delta(\pi \circ u) \geq 0$ for the extension and since it is now defined on a set without boundary we finally arrive at our contradiction. □

After having established the bound for the chords u proving the boundedness of the Lagrange multiplier is not much different then in the case of loops, so we will just refer to the original work on RFH [CF09, Chapter 3.1] for the proof.

The final step is to show the boundedness of the derivatives of the chords u. The idea is the same as in the ordinary Floer homology: We show that exploding derivatives of the solution to the Floer equation give rise to holomorphic bubbles, which can not exist due to our assumptions. The difference when one studies chords with boundary conditions instead of loops is that in addition to bubbles we also have to discuss the bubbling of holomorphic disks. Since the case of holomorphic bubbles is already well known, we will only focus on the holomorphic disks.

The first step is always the rescaling with the help of the following lemma by Helmut Hofer:

Lemma 4.3.6 *Assume (X, d) is a complete metric space, $f : X \to \mathbb{R}_{\geq 0}$ is continuous. Then for every $x \in X$ and $\delta > 0$, there exists an $z \in X$ and $0 < \epsilon \leq \delta$ s.t.*

$$d(x, z) < 2\delta, \quad \sup_{y \in B_\epsilon(z)} f(y) \leq 2f(z), \quad \textit{and} \quad \epsilon f(z) \geq \delta f(x). \tag{4.3.10}$$

Note that here $B_\epsilon(z)$ denotes the closed ball.

For the proof we refer to [HV92, Lemma 3.3]. The idea is now to assume that there is a sequence $(s_k, t_k) \in \mathbb{C}$ and a sequence of solutions to the Floer equation u_k such that $\lim_{k\to\infty} |\,\partial_s u_k(s_k, t_k)| \to \infty$. Then we use the above lemma to set up the right rescaling in the following known way:

Lemma 4.3.7 *There exists a sequence (R_k) in $(0, +\infty)$ with $R_k \to +\infty$, a sequence (J_k) of ω-compatible almost complex structures on (M, ω), a sequence $z_k' = (s_k', t_k') \in \mathbb{R} \times [0, 1]$ and a sequence $v_k \in C^\infty(B_{R_k}(0), M)$ such that*

$$\lim_{k\to\infty} ||\,\partial_\sigma v_k + J_k\,\partial_\tau v_k||_\infty = 0 \quad \textit{and} \quad 1 \le ||\,\partial_\sigma v_k||_\infty \le 2. \tag{4.3.11}$$

Note that here $B_{R_k}(0) \subseteq \mathbb{R} \times |\,\partial_s u_k(z_k')| \cdot [-t_k', 1 - t_k']$, so $B_{R_k}(0)$ might only be a part of a circle, cut of by the boundary.

Proof: Since $\mathbb{R} \times [0, 1]$ is still a complete metric space, the argument is almost the same as in the Floer homology case: Set $z_k = (s_k, t_k) \in \mathbb{R} \times [0, 1]$. Now apply Hofer's lemma from above with

$$x = z_k, \quad \delta_k := \frac{1}{\sqrt{m_k}} \quad \text{and} \quad f := |\,\partial_s u_k|.$$

Hence, there exists $z_k' = (s_k', t_k') \in [0, 1] \times \mathbb{R}$ and $0 < \epsilon_k \le \delta_k$ such that

$$\sup_{z \in B_{\epsilon_k}(z_k')} |\,\partial_s u_k(z)| \le 2|\,\partial_s u_k(z_k')|,$$

where $B_{\epsilon_k}(z_k') \subseteq \mathbb{R} \times [0, 1]$ and

$$\epsilon_k|\,\partial_s u_k(z_k')| \ge \delta_k|\,\partial_s u_k(z_k)| = \sqrt{m_k}.$$

Rescale $m_k' := |\,\partial_s u_k(z_k')|$, set $R_k := \epsilon_k m_k'$ and

$$v_k(\sigma, \tau) := u_k\left(\frac{\sigma}{m_k'} + s_k', \frac{\tau}{m_k'} + t_k'\right),$$

for all $(\sigma, \tau) \in B_{R_k}(0) \subseteq \mathbb{R} \times m_k'[-t_k', 1 - t_k']$. Since u_k is a solution to the Floer equation, v_k satisfies the equation

$$\partial_\sigma v_k(\sigma, \tau) + J\,\partial_\tau v_k(\sigma, \tau) = \frac{1}{m_k'} J\tau X_H(v_k(\sigma, \tau)).$$

As $m_k' \ge m_k \;\; \forall_k$ and as $\mathrm{d}H$ has by assumption compact support, it follows that

$$\begin{aligned} ||\,\partial_\sigma v_k||_\infty &= \sup_{z \in B_{R_k}(0)} |\,\partial_\sigma v_k(z)| \\ &= \frac{1}{m_k'} \sup_{z \in B_{\epsilon_k}(z_k')} |\,\partial_s u_k(z)| \\ &\le \frac{2}{m_k'} |\,\partial_s u_k(z_k')| \\ &= 2, \end{aligned}$$

and

$$m_k'\,\partial_\sigma v_k(0, 0) = \partial_\sigma u_k(z_k').$$

This concludes the proof. □

The next proposition will give a good candidate for the holomorphic plane we need to construct the bubble.

Proposition 4.3.8 *Let (v_k) be the sequence constructed in Lemma 4.3.7. Then there exists $v_\infty \in C^\infty(\Omega, M)$ such that*

$$v_k \xrightarrow{C^0_{\mathrm{loc}}} v_\infty \tag{4.3.12}$$

up to a subsequence, where Ω depends on the asymptotic behaviour of the sequence $(m'_k \cdot t'_k)$ from the above lemma and we define $t_\infty := \lim\limits_{k\to\infty} t'_k$.

i) If t_∞ is not at the boundary, i.e. $t_\infty \notin \{0,1\}$, then we have $\Omega = \mathbb{C}$.

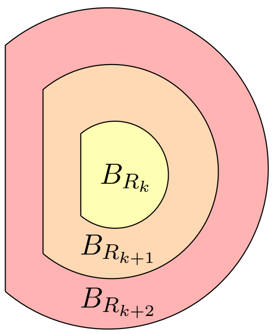
ii) If t_∞ lies at the boundary and $\lim\limits_{k\to\infty} m'_k \cdot t'_k = \infty$, then we also have $\Omega = \mathbb{C}$.

iii) If t_∞ lies at the boundary and $\lim\limits_{k\to\infty} m'_k \cdot t'_k = c_0 \in \mathbb{R}^+_0$, then we get $\Omega = \mathcal{H} := \{z \in \mathbb{C} \mid \operatorname{Im}(z) \geq 0\}$.

Here we assumed for simplicity that if t_∞ lies in the boundary then $t_\infty = 0$. For $t_\infty = 1$ consider instead $\lim\limits_{k\to\infty} m'_k \cdot (1 - t'_k)$.

Proof: In the first case where $t_\infty \notin \{0,1\}$, we can always find a subsequence of (v_k) such that the corresponding $B_{R_k}(0)$'s never touch the boundary. This is because t_∞ lies in the interior and we can always find an open neighbourhood around it, hence as $t'_k \to t_\infty$ and $0 < \epsilon_k \leq \frac{1}{\sqrt{m_k}} \to 0$ we can get this subsequence by starting at a later index (see Lemma 4.3.7). The rest of the proposition is in this case then exactly the same as in the usual Floer homology approach. But instead of using that M is compact, one has to use the fact that all u_k's (and therefore all v_k's) have to stay inside of the compact set K (see Theorem 4.3.5).

In the second case t_∞ is a boundary point. The problem with t_∞ lying at the boundary is that the $B_{R_k}(0)$'s constructed in the previous lemma are disks that have been cute off by the boundary some where between $\tau = -1$ and $\tau = 0$. Since the point of the cut off depends for every v_k on m'_k and t'_k, it is in general not clear if $B_{R_k}(0) \subseteq B_{R_{k+1}}(0)$. For the case of $\lim\limits_{k\to\infty} m'_k \cdot t'_k = \infty$ we do know that $B_{R_k}(0) \subseteq B_{R_{k+1}}(0)$ after potentially switching to a subsequence, since the cut off for $t_\infty = 0$ happens at $-m'_k \cdot t'_k$. From there on the proof work exactly the same as in the first case, but now the exhausting sequence for $\mathbb{C}$ is given by disks with a cut off.



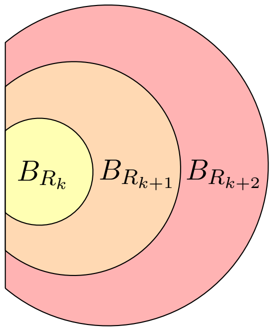
In the third case t_∞ is also a boundary point, but with $\lim\limits_{k\to\infty} m'_k \cdot t'_k = c_0 \in \mathbb{R}^+_0$. This means that the disks $B_{R_k}(0)$ with a cut off are not an exhausting sequence for $\mathbb{C}$ any more, remember that the cut off for v_k happens at the point $\tau = m'_k \cdot t'_k$. Now we want to make another variable transformation, such that the domains of the transformed v_k's form an exhausting sequence for $\mathcal{H}$. Hence, define:

$$\hat{v}_k(\sigma, \tau) := v_k(\sigma, \tau - m'_k \cdot t'_k)$$

Then the cut off for all $\hat{v}_k$'s happens at $\tau = 0$, i.e. the domains are now of the type

$$\{z \in (B_{R_k}(0) - m'_k \cdot t'_k) \subset \mathbb{C} \mid \operatorname{Im}(z) \geq 0\}$$

and since $\lim\limits_{k\to\infty} R_k = \infty$ we can find a subsequence of $\hat{v}_k$, such that the domains all lie inside of each other with respect to the increasing index.



Therefore we found an exhausting sequence for $\mathcal{H}$ and the argument from case one now also holds in this case, but with $\mathcal{H}$ instead of $\mathbb{C}$. So we get the existence of a $\hat{v}_\infty \in C^0(\mathcal{H}, M)$ with $\hat{v}_k \xrightarrow{C^0_{\mathrm{loc}}} \hat{v}_\infty$. $\square$

The next step on our way to prove the uniformly boundedness of the derivatives for the solutions to the Floer equation is to show that v_∞ (with $\Omega = \mathbb{C}$) is in fact a non constant holomorphic sphere and $\hat{v}_\infty$ (with $\Omega = \mathcal{H}$) is in fact a non constant holomorphic disk:

Proposition 4.3.9 *Let v_∞ and $\hat{v}_\infty$ be the maps constructed in the previous proposition. Then they are both not just continuous but smooth and their sequences converge also in C^∞_{loc}. They are further non constant and fulfil the J holomorphic curve equation*

$$\partial_\sigma w + J\,\partial_\tau w = 0, \tag{4.3.13}$$

with either $w = v_\infty$ or $w = \hat{v}_\infty$.

Proof: For v_∞ the argument is the same as in usual Floer homology. For $\hat{v}_\infty$ we will need to use the regularity result from [MS12, Propostion B.4.9] instead of the Calderon-Zygmund inequality. As before fix sequences $(z_j) \subseteq \mathcal{H}$ and $r_j \subseteq (0, +\infty)$ such that $\hat{v}_\infty\big|_{B_{2r_j}(z_j)}$ is contained in a chart U_j of M and $\bigcup\limits_{j\in\mathbb{N}} B_{r_j}(z_j) = \mathcal{H}$. Note that here $B_r(z)$ denotes a ball in $\mathcal{H}$, which doesn't need to be a ball in $\mathbb{C}$. Since we already know that $\hat{v}_k$ converges to the continuous $\hat{v}_\infty$ for every $j \in \mathbb{N}$ we find a K_j such that $\hat{v}_k\big|_{B_{r_j}(z_j)} \subseteq U_j$ for all $k \geq K_j$. Then we consider the restrictions

$$\hat{v}_k\big|_{B_{r_j}(z_j)} \in W^{1,p}(B_{r_j}(z_j), \mathbb{R}^{2n}),$$

but to keep the formulas readable we will identify the restriction with $\hat{v}_k$ in this proof. The fact that the image of $\bar{v}_k$ is for now completely contained in a chart allows us to see it as an element in $C^\infty_c(\mathcal{H}, \mathbb{R}^{2n})$ instead of in $C^\infty_c(\mathcal{H}, M)$. Now we use [MS12, Propostion B.4.9] with $B_{r_j}(z_j) \subset$

$B_{2r_j}(z_j) \subset \mathcal{H}$ being the open sets and every $\hat{v}_k$ is $W^{1,p}(B_{2r_j}(z_j))$. Note that J and X_H are smooth and therefore $J(\hat{v}_k)$ and $X_H(\hat{v}_k)$ are $W^{1,p}(B_{2r_j}(z_j))$. Then the proposition tells us that

$$||\hat{v}_k|_{W^2(B_{r_j}(z_j)} \leq c\Big(||\,\partial_\sigma \hat{v}_k + J\,\partial_\tau \hat{v}_k||_{W^1(B_{2r_j}(z_j))} + ||\hat{v}_k||_{W^1(B_{2r_j}(z_j))}\Big). \qquad (\star)$$

Since the v_k's are solutions to the Floer equation we have

$$||\,\partial_\sigma \hat{v}_k + J\,\partial_\tau \hat{v}_k||_{W^1(B_{2r_j}(z_j))} = ||\frac{1}{m_k'} J\tau X_H(\hat{v}_k)||_{W^1(B_{2r_j}(z_j))}.$$

Further, every $\hat{v}_k$ can take only values in the compact set $K \subset M$ by step one at the beginning of this chapter and hence the first part of the right hand side of the equation $(\star)$ only depends on J, τ and X_H, note that we already know that τ is uniformly bounded. The second part is also uniformly bounded, because we already established a uniform bound on the v_k's and Lemma 4.3.7 gives us a uniform bound on the derivatives. So we can finally conclude that $||\hat{v}_k||_{W^2(B_{r_j}(z_j)}$ is uniformly bounded and therefore in particular $||\,\partial_\sigma \hat{v}_k||_{W^1(B_{r_j}(z_j)}$ and $||\,\partial_\sigma \hat{v}_k||_{W^1(B_{r_j}(z_j)}$. As usually we use now Morrey's inequality to argue that on $B_{r_j}(z_j)$ $\partial_\sigma \hat{v}_k$ and $\partial_\tau \hat{v}_k$ belong to the Hölder class $C^{0,\alpha}$ for all $0 < \alpha < 1$. Thus by Arzelá-Ascoli $\partial_\sigma \hat{v}_k$ and $\partial_\tau \hat{v}_k$ admit subsequences which converge and so $\hat{v}_\infty \in C^1(B_{r_j}(z_j), \mathbb{R})$ for all $j \in \mathbb{N}$, i.e. $\hat{v}_\infty \in C^1(\mathcal{H}, M)$. Now repeat the argument for higher derivatives and then by induction one gets that $\hat{v}_\infty \in C^\infty(\mathcal{H}, M)$ and

$$\hat{v}_k \xrightarrow{C^\infty_{\text{loc}}} \hat{v}_\infty$$

modulo subsequence. With this we can also conclude that

$$\partial_\sigma \hat{v}_\infty + J\,\partial_\tau \hat{v}_\infty = 0$$

and $\hat{v}_\infty$ is nonconstant, because

$$||\,\partial_\sigma \hat{v}_\infty||_\infty = \lim_{k\to\infty} ||\,\partial_\sigma v_k||_\infty \geq 1$$

by Lemma 4.3.7. □

In the case that we now have a J-holomorphic plane the remaining steps are the same as in standard Floer homology. Remember that even for a non compact manifold the solutions of our Floer equation have to stay inside of a compact set. If we, however, get a J-holomorphic half plane, then we will have to deal with some new difficulties.

In the following we want to discuss the removal of singularities in the case where we turn a J-holomorphic half plane into a J-holomorphic disk: The first step is to remove the origin from $\mathcal{H}$ and map the remaining part conformally to a strip with boundary, i.e. consider

$$\varphi : \mathcal{H} \setminus \{0\} \to \mathbb{R} \times [0, \pi], \quad \varphi(z) := \log(z) = log\Big(re^{i\theta}\Big) = \log(r) + i\theta, \qquad (4.3.14)$$

where we name $s = \log(r)$, $t = \theta$ and identify $s + it$ with (s, t). With this map we define then

$$w : \mathbb{R} \times [0, \pi] \to M, \quad w := \hat{v}_\infty \circ \varphi^{-1}. \qquad (4.3.15)$$

The important question is now of course: Does w still satisfy the J holomorphic curve equation?

$$\begin{aligned}\partial_s w(s,t) + J\,\partial_t w &= \partial_\sigma v_\infty(e^s\cos(t), e^s\sin(t)) \cdot e^s\cos(t) + J\,\partial_\tau v_\infty(e^s\cos(t), e^s\sin(t)) \cdot e^s\cos(t) \\ &\quad + \partial_\tau v_\infty(e^s\cos(t), e^s\sin(t)) \cdot e^s\sin(t) - J\,\partial_\sigma v_\infty(e^s\cos(t), e^s\sin(t)) \cdot e^s\sin(t) \\ &= 0\end{aligned}$$

Removal of singularities means now to compactify the image of w by adding a constant asymptotic for $s \to +\infty$ and $s \to -\infty$. The definition of w was already engineered so that we have a constant negative asymptotic

$$\lim_{s \to -\infty} w(s, \cdot\,) = v_\infty(0,0), \tag{4.3.16}$$

since $\lim_{s \to -\infty} e^s e^{it} = 0 \in \mathbb{C}$ for $(s,t) \in \mathbb{R} \times [0,\pi]$. To prove the existence of the constant positive asymptotic, first define an energy density

$$e(s,t) := ||\, \partial_s w(s,t)||^2. \tag{4.3.17}$$

Usually the norm is just taken with respect to the metric $\omega(\,\cdot\,, J\,\cdot\,)$, but in our case we will need to take a different metric to account for our boundary conditions. The problem with having a boundary is that the subharmonic type inequalities we want to use later on, are only valid in the interior, which is the same problem we already had in step one. The solution to this is now again to reflect the function $e(s,t)$ at the boundary, but of course we need this extension to still be at least C^2, i.e. we need $\partial_t e(s,t)$ to be zero for $t \in \{0,1\}$. Unfortunately there is no reason for $e(s,t)$ to fulfil this requirement in general, if we use the metric $\omega(\,\cdot\,, J\,\cdot\,)$. The good news is though that we can always find an equivalent metric such that $e(s,t)$ does fulfil this condition:

Lemma 4.3.10 (c.f. [Fra00]) *Let (M,J) be an almost complex manifold and $L \subset M$ be a totally real submanifold with $2\dim L = \dim M$. Then there exists a Riemannian metric g on M such that*

i) $g(J(p)v, J(p)w) = g(v,w)$ for $p \in M$ and $v, w \in T_pM$,

ii) $J(p)T_pL$ is the orthogonal complement of T_pL for every $p \in L$,

iii) L is totally geodesic with respect to g.

If one is interested in the proof, we recommend looking at [RS01, Lemma D.1]. With this new metric we define now

$$e(s,t) := g(\partial_s w(s,t), \partial_s w(s,t)). \tag{4.3.18}$$

Then consider the following computations for $t \in \{0,1\}$ from [RS01, Proof of Lemma B.1]:

$$\begin{aligned}
\partial_t e(s,t) &= \partial_t g(\partial_s w(s,t), \partial_s w(s,t)) \\
&= g(\nabla_t\, \partial_s w(s,t), \partial_s w(s,t)) + g(\partial_s w(s,t), \nabla_t\, \partial_s w(s,t)) \\
&= 2 \cdot g(\partial_s w(s,t), \nabla_t\, \partial_s w(s,t)),
\end{aligned}$$

where ∇ is the Levi-Civita connection. Define $\xi := \partial_s w$ and $\eta := \partial_t u$ and note that we can view them either as vector field along the path $u_s(t)$ depending on a parameter s or vice versa, so if we write $\nabla_t \eta$ we differentiate the s-dependent vector field η along the path $w_s(t)$ and the analogue for ∇_s and the path $w_t(s)$. With this it is now easy to see that

$$\nabla_t \xi = \Big(\partial_t\, \partial_s u(s,t) + \partial_t u(s,t)\, \partial_s u(s,t) \Gamma^k_{ij}\Big)\big|_{u(s,t)} = \nabla_s \eta, \tag{4.3.19}$$

since we have a Levi-Civita connection. So we can calculate further

$$\begin{aligned}
2 \cdot g(\partial_s w(s,t), \nabla_t\, \partial_s w(s,t)) &= 2 \cdot g(\partial_s w(s,t), \nabla_s\, \partial_t w(s,t)) \\
&= 2 \cdot g(\partial_s w(s,t), \nabla_s (J(w)\, \partial_s w(s,t))) \\
&= 2 \cdot g(\partial_s w(s,t), (\nabla_\xi J)(w)\, \partial_s w(s,t)) + 2 \cdot g(\partial_s w(s,t), J(w) \nabla_s\, \partial_s w(s,t)),
\end{aligned}$$

where we used that w satisfies the J-holomorphic curve equation. Remember that for $t \in 0,1$ we are in a Lagrangian submanifold, which is always totally real and hence we can apply the above lemma. The first term is zero because $\nabla_\xi J$ is skew symmetric with respect to g:

Let $x \in \mathfrak{X}(M)$ be given, since J is bijective we can find a y such that $Jy = x$. Then we have on one hand

$$\begin{aligned}\nabla_\xi g(Jx, Jy) &= g(\nabla_\xi(Jx), Jy) + g(Jx, \nabla_\xi(Jy))\\ &= g((\nabla_\xi J)x, Jy) + g(J\nabla_\xi x, Jy) + g(Jx, (\nabla_\xi J)y) + g(Jx, J\nabla_\xi y)\end{aligned}$$

and on the other hand

$$\begin{aligned}\nabla_\xi g(Jx, Jy) &= \nabla_\xi g(x, y)\\ &= g(\nabla_\xi x, y) + g(x, \nabla_\xi y),\end{aligned}$$

remember that by Lemma 4.3.10 the metric g was chosen such that $g(J(p)v, J(p)w) = g(v, w)$. From the above two equations follows

$$g((\nabla_\xi J)x, Jy) + g(Jx, (\nabla_\xi J)y) = 0.$$

Then we use $Jy = x$ and

$$0 = \nabla_\xi(-\mathbb{1}) = \nabla_\xi(J \cdot J) = (\nabla_\xi J) \cdot J + J\nabla_\xi J$$

to get

$$\begin{aligned}0 &= g((\nabla_\xi J)x, Jy) + g(Jx, (\nabla_\xi J)y)\\ &= 2g((\nabla_\xi J)x, x).\end{aligned}$$

Since x was arbitrary, this proves that J is skew symmetric with respect to g. For the second summand remember that we are in the situation $t \in \{0, 1\}$ and that the Lagrangian submanifold L is totally geodesic by Lemma 4.3.10. Restricted to L it is obvious that $\nabla_s \partial_s u(s,t) \in T_{u(s,t)}L$, but since L is totally geodesic this stays true even when considered on M. By Lemma 4.3.10 $J(p)T_pL$ is the orthogonal complement of T_pL for every $p \in L$, so in particular is $J\nabla_s \partial_s u(s,t)$ orthogonal to $\nabla_s \partial_s u(s,t)$. So we can finally conclude that

$$\partial_t e(s,t) = 0 \quad \text{for } t \in \{0, 1\}. \tag{4.3.20}$$

Now one can proceed as usual to prove the necessary inequalities for the extended e and use them to prove the existence of a constant positive asymptotic. In the proof of the last part one again needs to use an bootstrapping argument, which works in our case exactly the same way as in Proposition 4.3.9. To see that this positive asymptotic is unique we use the exponential decay property of the gradient flow lines in the same way as in Floer homology. Note that the exponential decay is a general property of the gradient flow lines of the action functional, which doesn't depend on the specific chords we restrict the functional to.

Finally, to get our desired contradiction we pull back w with the concatenation of the Cayley transformation with the map φ^{-1}. So w becomes a nonconstant J-holomorphic disk with boundary and the boundary will be mapped to L. For the contradiction consider

$$0 < \int_{\mathbb{D}} w^* \,\mathrm{d}\lambda = \int_{\partial\mathbb{D}} w^*\lambda = \int_{\partial\mathbb{D}} w^*\big(\lambda\big|_L\big) = \int_{S^1} w^* \,\mathrm{d}l = \int_{S^1} \mathrm{d}(w^* l) = 0$$

Here we used that w is a J-holomorphic curve, so $0 < \int_{\mathbb{D}} w^* \, \mathrm{d}\lambda$, and the fact that λ has a primitive when restricted to the Lagrangian submanifold. This contradiction tells us now that there needs to be an L^∞ bound on the derivatives $\partial_s u$ and by the Floer equation also on the $\partial_t u$.

With the L^∞ bounds in place we can use Arzelà–Ascoli to prove compactness of $\widetilde{\mathcal{M}}(H,J)^b_a$ with respect to C^0_{loc}. To finally achieve compactness in C^∞_{loc} one uses a bootstrapping argument similar to Proposition 4.3.9. Note that C^∞_{loc}-compactness is the best we can achieve for $\widetilde{\mathcal{M}}(H,J)^b_a$ and C^∞-compactness is in general simply not true.

In this section we so far only studied parametrized gradient flow lines, but usually if one is interested in defining a boundary operator via the count of gradient flow lines the unparametrized flow lines for fixed asymptotics are of more importance. Let $\hat{u}$ be a parametrized gradient flow line with asymptotics γ^+ and γ^- that are isolated critical points of $\mathcal{A}_H$. Then we know that $(T^*\hat{u})(s,t) := \hat{u}(s+T,t)$ as a map in s and t is still a gradient flow line with the same asymptotics for all $T \in \mathbb{R}$, i.e. we have a well-defined action of the Lie group $\mathbb{R}$ on $\widetilde{\mathcal{M}}(H,J)^b_a(\gamma^-,\gamma^+)$. The quotient $\widetilde{\mathcal{M}}(H,J)^b_a(\gamma^-,\gamma^+)/\mathbb{R}$ is then the set of unparametrized gradient flow lines with negative asymptotic γ^- and positive asymptotic γ^+ and we denote it by $\mathcal{M}(H,J)^b_a(\gamma^-,\gamma^+)$ or $\mathcal{M}(\gamma^-,\gamma^+)^b_a$ for short. Before we can talk about the compactness of this set, we of course need to introduce a topology. We want to do this by stating what being a convergent sequence means: Given a sequence $([\hat{u}_n])_{n\in\mathbb{N}}$ in $\mathcal{M}(\gamma^-,\gamma^+)^b_a$. We say it converges to a $[\hat{u}_*] \in \mathcal{M}(\gamma^-,\gamma^+)^b_a$ if there is a sequence $(r_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that for given representatives we have:

$$r_n^* \hat{u}_n \xrightarrow{C^\infty_{\text{loc}}} \hat{u}_*$$

This is a well-defined notion since if we are given different representatives we just need to choose a different sequence $(r_n)_{n\in\mathbb{N}}$ that compensates the different time shifts. The problem that arises for these sets of gradient flow lines with fixed asymptotics is that they are not compact any more, since the above definition of convergence allows the sequences of flow lines to change their asymptotics in the limit. To compactify them again we need to introduce broken gradient flow lines.

Definition 4.3.11 *A $(m-1)$-fold broken gradient flow line in $\mathcal{M}(\gamma^-,\gamma^+)^b_a$ is a tuple $(v_1,\dots,v_m)$ such that $v_i \in \mathcal{M}(\gamma^{i-1},\gamma^i)^b_a$ for all $i \in \{1,\dots,m\}$ where $\gamma^0,\dots,\gamma^m$ are pairwise different critical points with $\gamma^0 = \gamma^-$ and $\gamma^m = \gamma^+$.*

$\overline{\mathcal{M}}(\gamma^-,\gamma^+)^b_a$ denotes the set of gradient flow lines including the broken ones. To complete the definition of a topology on this set, we have to clarify what it means for a sequence of gradient flow lines to converge to a broken gradient flow line:

Definition 4.3.12 (Floer-Gromov convergence) *Given a sequence $(w_n)_{n\in\mathbb{N}}$ in $\mathcal{M}(\gamma^-,\gamma^+)^b_a$. We say it converges to a broken gradient flow line $(v_1,\dots,v_m)$ if there are sequences $\left(r^i_n\right)_{n\in\mathbb{N}}$ such that for all $i \in \{1,\dots,m\}$*

$$\left(r^i_n\right)^* w_n \xrightarrow{C^\infty_{\text{loc}}} v_i.$$

Proving that $\overline{\mathcal{M}}(\gamma^-,\gamma^+)^b_a$ is compact with respect to this topology is now a standard procedure as we already showed the C^∞_{loc} compactness (c.f. [Sal90, Proof of Lemma 4.2]).

4.4 The Lagrangian Rabinowitz Floer Homology

The idea behind the Lagrangian Rabinowitz Floer complex (c.f. [Mer14]) is the same as for all other Floer type homologies: One takes the critical points of the action functional $\mathscr{A}_H$ as a formal basis of a $\mathbb{Z}_2$vector space, that becomes together with the Maslov index a graded vector space. The boundary operator is again given by counting gradient flow lines between the critical points of index difference one. To show that this approach will give us a well-defined complex is the main goal of this section.

Theorem 4.1.1 tells us that for a generic choice of Hamiltonian H the critical points of $\mathscr{A}_H$ with periods $\tau \in [a,b]$ are a finite collection of isolated points combined with $H^{-1}(0) \cap L$. Since we can only use a isolated critical points in the definition of the Rabinowitz Floer complex, we first need to choose a Morse function f on $H^{-1}(0) \cap L$. We then define $CRF_\bullet(H,L,f)^a_b$to be $\mathbb{Z}_2$ vector spaces spanned by the isolated critical points of $\mathscr{A}_H$ and the critical points of the Morse function f. The grading of this vector space is given by a combination of the Maslov index with the Morse index: Let x be a formal basis vector of $CRF_\bullet(H,L,f)^a_b$ and since this means it represents a Hamiltonian trajectory we can assign it a Maslov index according to Section 4.2. Further, x is either an isolated critical point, then we set its Maslov index to be the grading, or a critical point of the chosen Morse function f, then we set $\mu(x) = \mu_{\text{Maslov}}(x) + \mu_{\text{Morse}}(x)$ to be the grading. In the case when the action functional is Morse one defines the boundary operator on this vector space simply by counting the gradient flow lines between critical points. But since the Rabinowitz action functional is never Morse we need to count gradient flow lines with cascades, c.f. [Fra04]:

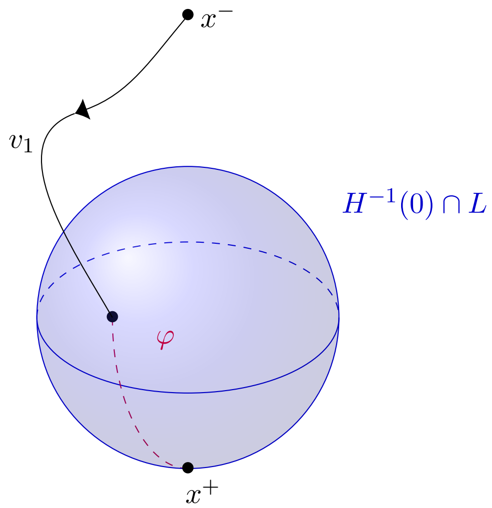
Definition 4.4.1 *Fix a $k \in \mathbb{N}$. Let x^+ and x^- be critical points of $\mathscr{A}_H$ and of the Morse function f, if it lies in $H^{-1}(0) \cap L$. A gradient flow line with k cascades between x^- and x^+ is a tuple $(v_1, \ldots, v_k, T_1, \ldots, T_{k-1})$ of gradient flow lines $v_j \in \mathcal{M}(H,J)^b_a$ and real numbers $T_j \geq 0$ such that*

i) $\lim_{s \to -\infty} v_1(s) \in C_{x^-}$ *and* $\lim_{s \to \infty} v_k(s) \in C_{x^+}$, *where here $C_{x^\pm}$ is either $H^{-1}(0) \cap L$ or $\{x^\pm\}$,*

ii) $\lim_{s \to -\infty} v_{j+1}(s) = \varphi^{T_j}\left(\lim_{s \to \infty} v_j(s)\right)$ *for $j = 1, \ldots, k-1$,*

where φ^t is the negative gradient flow of f. Gradient flow lines with 0 cascades are defined to be unparametrized flow lines of $-\nabla f$ in which case both x^+ and x^- need to lie in $H^{-1}(0) \cap L$. We denote by $\mathcal{M}_{cas}(x^-, x^+)^b_a$ the set of cascades from x^- to x^+.

Note that the asymptotic endpoints of the cascades uniquely determine the flow line of $-\nabla f$ between these points and therefore one should think of them as being part of the cascade. In the case where $H^{-1}(0) \cap L$ is just the two sphere a cascade might look like:



The boundary operator on $CRF_\bullet(H,L,f)^a_b$ is now defined via

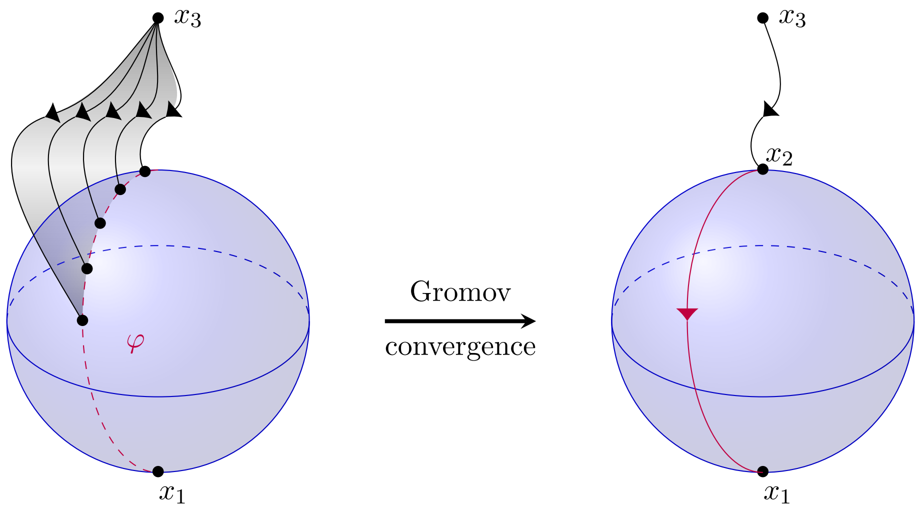
$$\partial : CRF_\bullet(H,L,f)^a_b \to CRF_{\bullet-1}(H,L,f)^a_b, \quad \partial x = \sum_{\substack{y \in \mathrm{Crit}(H,f) \\ \mu(y) = \mu(x)-1}} \#_2 \mathcal{M}_{\text{cas}}(x,y) \cdot y \tag{4.4.1}$$

So we sum over all formal basis vectors whose index is smaller by exactly one and multiply each of them by the number of gradient flow lines with cascades modulo two. To prove that ∂ is a well-defined boundary operator, we need the following theorem about the moduli spaces $\mathcal{M}_{\mathrm{cas}}(x,y)$ (c.f. [Mer14, Theorem 2.23]).

Theorem 4.4.2 *For a generic choice of J and a generic Morse–Smale metric on $H^{-1}(0) \cap L$ the moduli spaces $\mathcal{M}_{cas}(x,y)$ are smooth manifolds of finite dimension with*

$$\dim \mathcal{M}_{cas}(x,y) = \mu(x) - \mu(y) - 1. \tag{4.4.2}$$

Note that the topology of these moduli spaces is again given through defining what the convergent sequences are. For gradient flow lines with cascades $(v_1, \ldots, v_k, T_1, \ldots, T_{k-1})$ we define convergence component wise, where the convergence of the v_i is given via the Floer-Gromov convergence and the convergence of the T_i is just the usual one in $\mathbb{R}$. In the special case of a flow line with zero cascades we use the notion of Floer-Gromov convergence form Morse homology. As for the gradient flow lines without the notion of cascades the moduli spaces $\mathcal{M}_{\mathrm{cas}}(x,y)$ need to be compactified by introducing broken flow lines. The definition of broken gradient flow lines with cascades follows the same idea as before, but now we need to consider tuples of tuples.



After this compactification we then see that $\overline{\mathcal{M}}_{\mathrm{cas}}(x,y)$ is a compact manifold with $\dim \overline{\mathcal{M}}_{\mathrm{cas}}(x,y) = \mu(x)-\mu(y)-1$. So in the case of index difference one the moduli space $\dim \overline{\mathcal{M}}_{\mathrm{cas}}(x,y) = \mu(x)-\mu(y)-1$ is a zero dimensional compact manifold, i.e. a finite set, and since breaking can not occur in this situation we have $\overline{\mathcal{M}}_{\mathrm{cas}}(x,y) = \mathcal{M}_{\mathrm{cas}}(x,y)$. This shows the well-definedness of the boundary operator ∂. The proof that $\partial^2 = 0$ now follows the usual approach of showing that the formula of ∂^2 is equivalent to counting 1-fold broken gradient flow lines, i.e. elements in $\partial\overline{\mathcal{M}}_{\mathrm{cas}}(x,z)$ for $\mu(z) = \mu(x)-2$ (by some standard gluing argument). Since one dimensional compact manifolds are diffeomorphic to lines and circles, the number of boundary points is always even and therefore $\partial^2 = 0$.

Remark 4.4.3 Since for a generic choice of Hamiltonian the action functional $\mathcal{A}_H$ is Morse except for the constant trajectories, we considered the only critical manifold to be $H^{-1}(0)\cap L$. But the above definitions and arguments work the same way if a functional also has different critical manifolds.

Definition 4.4.4 (Lagrangian Rabinowitz Floer homology) *Let $(M, \omega = \mathrm{d}\lambda)$ be the completion of a Liouville domain with a Hamiltonian H such that $\partial\widetilde{M} = H^{-1}(0)$, the support of $\mathrm{d}H$ is inside of a compact set K the Hamiltonian vector field coincides with the Reeb vector field on $\partial\widetilde{M}$ and $\mathcal{A}_H$ is Morse-Bott. Let further be L an exact Lagrangian submanifold, such that l with $\mathrm{d}l = \lambda$*

has support in K and let there be an almost complex structure J, which is ω-compatible and SFT like outside of K. Then define

$$RFH_i(H,L)^a_b := \frac{\ker(\partial : CRF_i(H,L,f)^a_b \to CRF_{i-1}(H,L,f)^a_b)}{im(\partial : CRF_{i+1}(H,L,f)^a_b \to CRF_i(H,L,f)^a_b)} \tag{4.4.3}$$

and set

$$RFH_\bullet(H,L)^a_b = \bigoplus_{i=-\infty}^{\infty} RFH_i(H,L)^a_b. \tag{4.4.4}$$

The Lagrangian Rabinowitz Floer homology is then

$$RFH_\bullet(H,L) := \varinjlim_{a\to-\infty} \varprojlim_{b\to\infty} RFH_\bullet(H,L)^a_b. \tag{4.4.5}$$

Note that the direct limit is taken with respect to the projection $RFH_\bullet(H,L)^a_b \to RFH_\bullet(H,L)^{a'}_b$ with $a < a'$ and the inverse limit with respect to the inclusions $RFH_\bullet(H,L)^a_b \to RFH_\bullet(H,L)^a_{b'}$ with $b > b'$.

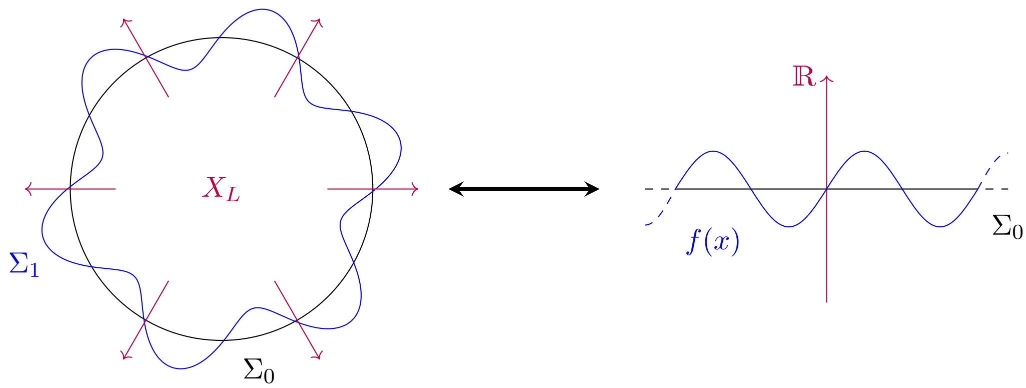
Remark 4.4.5 A standart argument in Floer theory shows that $RFH_\bullet(H,L)$ as defined above is up to a canonical isomorphism independent of the almost complex structure J, the Morse function f and the Morse-Smale metric chosen on $H^{-1}(0)\cap L$. Therefore it is justified to not include this information in our notation of the Lagrangian Rabinowitz Floer homology.

One of the key properties of the Lagrangian RFH is the far reaching invariance:

Theorem 4.4.6 (c.f. Theorem 1.1 in [CF09]) *Given the setting needed for a well-defined Lagrangian Rabinowitz Floer homology. If there is a family H_s for $0 \le s \le 1$ of smooth functions such that $H_s^{-1}(0)$ can be interpreted as the boundary of a Liouville domain that completes to our given manifold M for all s, then $RFH_\bullet(H_0,L)$ and $RFH_\bullet(H_1,L)$ are canonically isomorphic.*

The proof of this theorem follows by the same arguments as in [CF09, Chapter 3.2.].

Remark 4.4.7 Given two Hamiltonians H_0 and H_1 homotopic in the way described above, then their respective energy hypersurfaces $\Sigma_0 = H_0^{-1}(0)$ and $\Sigma_1 = H_1^{-1}(0)$ do not need to coincide and we can not view Σ_0, Σ_1 as the boundary of the same Liouville domain. But for the above theorem we assumed that the underlying manifold, i.e. the completion of the Liouville domain, does not change and hence we want to give an idea why the completion of the Liouville domains corresponding to Σ_0 and Σ_1 respectively do, in fact, coincide: Given the contact manifold (Σ_0, α_0) we define its symplectization as $(\Sigma_0 \times \mathbb{R}, e^r\alpha_0) \subset M$. Since Σ_1 is by assumption still transverse to the same Liouville vector field X_L, there is a smooth function $f : \Sigma_0 \to \mathbb{R}$ such that $\Sigma_1 = \mathrm{Graph}(f)$.



This implies that for all $\xi = (f(x), v_x) \in T\Sigma_1 \subset \mathbb{R} \times T\Sigma_0$

$$\alpha_1(\xi) = \lambda\big|_{\Sigma_1}(\xi) = e^{f(x)}\alpha_0(v_x) \tag{4.4.6}$$

If we now take the symplectization with respect to (Σ_1, α_1), the corresponding one form is $e^{r'}\alpha_1 = e^{r'+f}\alpha_0$. The function f was defined in such a way that

$$\Sigma_1 \times \mathbb{R} \in (p, r') = (x, r' - f(x)) = (x, r) \ni \Sigma_0 \times \mathbb{R}, \tag{4.4.7}$$

where we view $\Sigma_1 \times \mathbb{R}, \Sigma_0 \times \mathbb{R} \subset M$. Hence the one forms coincide and we can choose a Liouville domain with boundary Σ_1 that still completes to (M, λ).

After having established the basics of Lagrangian RFH let us now introduce a useful tool that will allow us to further study this homology. The first part of this exposition follows [Ruc23].

Definition 4.4.8 (Leaf-wise Intersection Points) *Given the above assumptions we call a point $p \in \Sigma$ a leaf-wise intersection point with respect to a compactly supported Hamiltonian diffeomorphism φ_F (or a Hamiltonian F) if there is a time τ such that $\phi_R^\tau(p)$ lies inside of $\varphi_F^{-1}(L)$, where ϕ_R^t is the flow of the Reeb vector field.*

Leaf-wise intersections were first considered by Moser in [Mos78] and can actually be defined in a more general setting. We chose the above set up, because it allows us to interpret the leaf-wise intersection points in terms of critical points of a perturbed Rabinowitz action functional, which was first done in [AF10a]. One important difference between the considerations in those papers and the above definition is that they were interested in intersections of Hamiltonian trajectories with the leaves of the Reeb vector field in both the starting and end point. In our setting, however, we are interested in Reeb chords starting in the Lagrangian manifold L and ending in another Lagrangian, which is defined by displacing L with a Hamiltonian diffeomorphism. These kind of leaf-wise intersection points were first considered by Will Merry in [Mer14]. There he also explained how to interpret them in terms of perturbed Lagrangian Rabinowitz Floer homology: Further assume that $\beta : [0,1] \to \mathbb{R}$ is a smooth function with support in $\left(0, \frac{1}{2}\right)$ and

$$\int_0^1 \beta(t)\,\mathrm{d}t = 1, \tag{4.4.8}$$

and let F be a smooth time dependent function such that $F(\,\cdot\,, t) = 0$ for all $t \in \left[0, \frac{1}{2}\right]$. Denote by $P(M, L)$ all the chords $x : [0,1] \to M$, which start and end in L. Then define the perturbed Rabinowitz action functional

$$\mathcal{A}_F^H : P(M, L) \times \mathbb{R} \to \mathbb{R}$$

as

$$\mathcal{A}_F^H(x, \tau) := \int_0^1 x^*\lambda + l(x(0)) - l(x(1)) - \tau \int_0^1 \beta(t) H(x(t))\,\mathrm{d}t - \int_0^1 F(x(t), t)\,\mathrm{d}t. \tag{4.4.9}$$

A critical point of this functional is now a pair (x, τ) with

$$\begin{aligned} \partial_t x(t) &= \tau\beta(t) X_H(x(t)) + X_F(x(t)), \\ \int_0^1 \beta(t) H(x(t))\,\mathrm{d}t &= 0. \end{aligned} \tag{4.4.10}$$

Since $\beta(t)H$ and F have disjoint support the first in $\left(0, \frac{1}{2}\right)$ and the second in $\left(\frac{1}{2}, 1\right)$, it is not difficult to show that for every leaf-wise intersection point w.r.t. φ_F there is a corresponding critical point of the above action functional (see [AF10a, Propostion 2.4] for an explicit calculation).

Theorem 4.4.9 (c.f. Theorem 2.9. [AF10a]) *Let $\hat{u}_n = (u_n, \tau_n)$ be a sequence of gradient flow lines for which there exists $a < b$ such that*

$$a \leq \mathcal{A}_F^H(\hat{u}(s)) \leq b \quad \forall s \in \mathbb{R}. \tag{4.4.11}$$

Then for every reparametrisation sequence $\sigma_n \in \mathbb{R}$ the sequence $\hat{u}_n(\,\cdot\, + \sigma_n)$ has a subsequence which converges in C^∞_{loc}.

The proof of this theorem follows the same procedure as our discussions in Chapter 4.3. This then enables us to define a perturbed Lagrangian Rabinowitz Floer homology $RFH_\bullet(H, L, F)$ in the same way as above. Note that we can again extend the definition to the Morse-Bott case if we use gradient flow lines with cascades. If we choose a smooth homotopy F_s from F to 0, we can use the standard arguments from Floer homology to show that

$$RFH_\bullet(H, L, F) \cong RFH_\bullet(H, L, 0) \cong RFH_\bullet(H, L). \tag{4.4.12}$$

For more details see [Mer12, Section 6]. By using this fact we can now prove the well known vanishing property of RFH in the Lagrangian case:

Theorem 4.4.10 *If there is a compactly supported Hamiltonian diffeomorphism φ_F such that $H^{-1}(0) \cap \varphi_F(L) = \emptyset$, then*

$$RFH_\bullet(H, L) = 0. \tag{4.4.13}$$

On one hand this property can be a powerful tool to prove the existence of certain chords since if the Lagrangian submanifold at question intersects the energy hypersurface, we always have non zero contributions to the chain complex. If there are no further non constant contributions, then the Lagrangian RFH is equal to the Morse homology of $L \cap H^{-1}(0)$ which can not vanish completely. This contradiction always guarantees the existence of a non constant chord. But on the other hand this also limits the number of chords the Lagrangian RFH can detect in most of the applications. For example all physical systems with a compact hypersurface in $\mathbb{R}^n$ and a Lagrangian subspace have $RFH_\bullet(H, L) = 0$ because of this property. In order to overcome this limitation we will introduce the equivariant Lagrangian Rabinowitz Floer homology in the next section.

4.5 Equivariant Lagrangian Rabinowitz Floer Homology

One of the key features of a homology is that it does not change under certain deformations of the underlying mathematical structure. In our case this property allows us to compute the Rabinowitz Floer homology for a complicated Hamiltonian function by finding a homotopic Hamiltonian for which the calculations are much easier. A too far reaching invariance property on the other hand can drastically reduce the information that the homology can tell us about a given system, as we have seen at the end of the last section. So the idea is now to introduce a modified version of the Lagrangian Rabinowitz Floer homology that takes the symmetry of a given Hamiltonian system into account and by doing so restricts its invariance property to homotopies that respect this symmetry. As we will see later on these additional constraints on the allowed homotopies remove the systems with zero chords from the homotopy class of physically relevant systems, which enables us to prove much stronger existence results.

So first let us discuss how one can define the G-equivariant RFH for a general compact Lie group: The underlying idea is based on [FS16]. Let (M, λ, H, L) fulfil all the necessary requirements for a well-defined Lagrangian RFH (we allow $\mathcal{A}_H$ to be Morse-Bott also on non constant chords) and let G be a compact Lie group that is a symmetry of the Hamiltonian H and acts free on the solutions of the Hamiltonian equation. Further assume that L is invariant under the action of G and we have a compatible almost complex structure J that is equivariant w.r.t. G in the following sense:

$$\mathrm{d}\phi_g J(x) = J(g \triangleright x)\,\mathrm{d}\phi_g \quad \text{for all } g \in G \tag{4.5.1}$$

To construct the complex for the G-equivariant RFH, we first consider the critical manifolds corresponding to the Maslov index μ, we shall call them crit_μ. By assumption G acts free on these manifolds and because G is also assumed to be compact, the action is even proper. Hence crit_μ/G is again a well-defined manifold and we can choose a Morse function on it. To define the complex $CRF^G_\bullet(M, H, L, J)^a_b$ we take as formal basis the critical points of all the corresponding Morse functions and the index is given by the Maslov index plus the Morse index. Note that since G is a symmetry of our Hamiltonian, by Proposition 4.2.3 all the critical points of the Rabinowitz action functional, which are identified by the group action, have the same Maslov index. As differential of this complex we define:

$$\partial[x] = \sum_{\substack{\mathrm{index}([y])= \\ \mathrm{index}([x])-1}} \#_2\Big\{u \mid x \xrightarrow{u} \tilde{y} \text{ gradient flow, } \tilde{y} \in [y]\Big\}[y] \tag{4.5.2}$$

Here $[x]$ denotes the equivalence class of chords under the G action and because we are in the Morse-Bott situation gradient flow lines implicitly means with cascades. Again the fact that G is a symmetry of our Hamiltonian and J is G-equivariant allows us to take any representative of the class $[x]$ in the definition of the differential. If we take a different $\tilde{x} = g \triangleright x$, then $g \triangleright \cdot$ is a bijection between $\Big\{u \mid x \xrightarrow{u} \tilde{y} \text{ gradient flow, } \tilde{y} \in [y]\Big\}$ and $\Big\{u \mid \tilde{x} \xrightarrow{u} \tilde{y} \text{ gradient flow, } \tilde{y} \in [y]\Big\}$. It is further known that there are only finitely many gradient flow lines between a given critical point in crit_μ and the G-orbit of a critical point with index difference one, as long as we choose a J that ensures transversality. One can see this by viewing the Morse functions on the crit_μ/G's as G-invariant Morse-Bott functions on crit_μ. The G-equivariant Lagrangian RFH is then as always defined via

$$RFH^G_i(H, L)^a_b := \frac{\ker\big(\partial : CRF^G_i(H, L, f)^a_b \to CRF^G_{i-1}(H, L, f)^a_b\big)}{\mathrm{im}\big(\partial : CRF^G_{i+1}(H, L, f)^a_b \to CRF^G_i(H, L, f)^a_b\big)}. \tag{4.5.3}$$

A very integral part of the well-definedness is the assumption that J is G-equivariant, but the question is: Is it always possible to find such an almost complex structure that still guarantees the transversality?

Theorem 4.5.1 *For a generic G-equivariant J and a generic Morse–Smale metric on $H^{-1}(0) \cap L$ the moduli spaces $\mathcal{M}_{cas}(x, y)$ are smooth manifolds of finite dimension with*

$$\dim \mathcal{M}_{cas}(x, y) = \mu(x) - \mu(y) - 1. \tag{4.5.4}$$

The overall strategy of the proof is still the same as in all Floer theories (cf. [FHS95]) with some small adjustments at the right place. One technical but important definition is that of a regular point in $\mathbb{R}^2$ for a solution u of the perturbed J-holomorphic curve equation. The fact that these points are dense allows us usually to construct the contradiction "$0 \neq 0$" in the proof of the transversality. But in the case of G-equivariant Raboniwitz Floer homology we need to consider G-equivariant almost complex structures and in order to show transversality with this extra condition one needs a notion of regular point that also acknowledges the existence of our Lie group action.

Definition 4.5.2 *Given the smooth action ϕ of the Lie group G as above and u a solution of the perturbed J-holomorphic curve equation. A point $(s,t) \in \mathbb{R}^2$ is called G-regular w.r.t. u if*

i) $\partial_s u(s,t) \neq 0$

ii) $u(s,t) \neq \phi_g(x^{\pm}(t))$

iii) $u(s,t) \notin \phi_g(u(\mathbb{R} \setminus \{s\}, t))$

for all $g \in G$. We call $R^G(u)$ the set of all the G-regular points of u.

For the next proposition we will need the following two lemmas:

Lemma 4.5.3 (cf. Lemma 4.1 in [FHS95]) *Let $u : B_\epsilon \to \mathbb{C}^n$ be a C^l-solution of*

$$\partial_s u + J(t,u)(\partial_t u - X(t,u)) = 0 \tag{4.5.5}$$

and assume that $\partial_s u \not\equiv 0$. Then the set of points $(s,t) \in B_\epsilon$ with $\partial_s u(s,t) = 0$ is discrete.

Lemma 4.5.4 (cf. Lemma 4.2 in [FHS95]) *Let $u, v : B_\epsilon \to \mathbb{C}^n$ be C^l-solutions of (4.5.5) with $X = 0$ such that*

$$u(0) = v(0), \qquad \mathrm{d}u(0) \neq 0, \qquad \mathrm{d}v(0) \neq 0.$$

Moreover, assume that there exists a constant $0 < \delta < \epsilon$ such that for every $(s,t) \in B_\delta$ there exists an $s' \in \mathbb{R}$ such that $(s',t) \in B_\epsilon$ and $u(s,t) = v(s',t)$. Then $v(z) = u(z)$ for $|z| < \epsilon$.

Proposition 4.5.5 ($R^G(u)$ is open and dense) *Let $u : \mathbb{R} \to M$ be a C^l-solution of the Floer equation that fulfils*

i) $\lim_{s \to \pm\infty} u(s,t) = x^{\pm}(t)$

ii) $\lim_{s \to \pm\infty} \partial_s u(s,t) = 0$

such that $\partial_s u \not\equiv 0$. If further the Lie group acts free and is compact, then the set of G-regular points of u is open and dense in $\mathbb{R}^2$.

Proof: [1] We first reduce the proposition to the case $X_{H_t} = 0$: Let $\psi_t : M \to M$ be the time dependent flow generated by the vector field X_{H_t}. Since H_t is constant outside a compact set, ψ_t is compactly supported for every t. It's not hard to see that then $v(s,t) := \psi_t^{-1}(u(s,t))$ satisfies

$$\partial_s v + \psi_t^* J_t(v)\, \partial_t v = 0,$$

where $\psi^* J_t$ is still G-equivariant since G is a symmetry of H_t. For the same reason we also have $R^G(u) = R^G(v)$.

Part one: $R^G(v)$ is open. Assume by contradiction that there exists a point $(s,t) \in R^G(v)$, which can be approximated by a sequence $(s_\nu, t_\nu) \notin R^G(v)$. Then $\partial_s v(s_\nu, t_\nu) \neq 0$ and $v(s_\nu, t_\nu) \neq \phi_g(x^{\pm})$ $\forall g \in G$ for ν large enough, since G is compact. Note that $\lim_{s \to \pm\infty} = \psi_t^{-1}(x^{\pm}(t)) = x^{\pm}$ is

[1]The proof is an adaptation of [FHS95, Theorem 4.3] and we follow the original proof very closely.

constant. Since $(s_\nu, t_\nu) \notin R^G(v)$ it follows that there exists a sequence $(g_\nu \in G)_{\nu\in\mathbb{N}}$ and $(s'_\nu)_{\nu\in\mathbb{N}} \in \mathbb{R}$ with $s_\nu = s'_\nu$ such that

$$v(s_\nu, t_\nu) = \phi_{g_\nu}(v(s'_\nu, t_\nu)).$$

If the sequence $(s_\nu)_{n\in\mathbb{N}}$ is unbounded, then up to passing to a subsequence we may assume that $s'_\nu \to \pm\infty$. Because G is compact we can get by again passing to a subsequence a convergent subsequence $(g_\nu)_{\nu\in\mathbb{N}}$ with limit g_*. So we get

$$\phi_{g_\nu}(v(s'_\nu, t_\nu)) \to \phi_{g_*}(x^\pm),$$

which implies $v(s,t) = \phi_{g_*}(x^\pm)$ in contradiction to $(s,t) \in R^G(u)$. Note that this is true since

$$\phi : G \times \mathrm{im}(v) \to M$$

is uniformly continuous in $G \times \mathrm{im}(v)$, because G and $\mathrm{im}(v)$ are compact. Hence the sequence s'_ν is bounded and we may assume without loss of generality that $s'_\nu \to s'$ and $g_\nu \to g_*$. Then

$$u(s,t) = \phi_{g_*}(v(s,t))$$

and since $(s,t) \in R^G(v)$ we must have $s' = s$. Hence $(s'_\nu)_{\nu\in\mathbb{N}}$ and $(s_\nu)_{\nu\in\mathbb{N}}$ both converge to s and this contradicts the fact that $\partial_s v(s,t) \neq 0$, i.e. it follows that $R^G(v)$ is open.

Part two: $R^G(v)$ is dense. From Lemma 4.5.3 we know that the set $C(v)$ of all points $(s,t) \in \mathbb{R}^2$ with $\partial_s v(s,t) = 0$ is discrete. Therefore it suffices to prove that one can approximate every point in $\mathbb{R}^2 \setminus C(v)$ by a sequence in $R^G(v)$. A point $(s,t) \notin C(v)$ can now be approximated by a sequence $(s_\nu, t) \in \mathbb{R}^2 \setminus C(v)$ with $v(s_\nu, t) \neq \phi_g(x^\pm(t))$ for any $g \in G$. Otherwise

$$\partial_s v(s,t) = \partial_s \phi_g(x^\pm(t)) = 0,$$

which is a contradiction. Hence we reduced the problem to points $(s_0, t_0) \in \mathbb{R} \times [0,1]$ with

$$\partial_s v(s_0, t_0) \neq 0 \quad \text{and} \quad v(s_0, t_0) \neq \phi_g(x^\pm(t_0)) \ \forall g \in G.$$

Now assume by contradiction that for such a point

$$B_\epsilon(s_0, t_0) \cap R(v) = \emptyset$$

for some $\epsilon > 0$. Now take an ϵ so small and a $T > 0$ so large that the following holds:

i) $v(s,t) \notin \phi_g(v(B_\epsilon(s_0, t_0))$ for all $g \in G$, $|s| > T$ and $|t - t_0| < \epsilon$.

ii) If $|t - t_0| \leq \epsilon$ the map

$$\phi_{(\,\cdot\,)}(v(\,\cdot\,, t_0)) : B_\epsilon(e) \times [s_0 - \epsilon, s_0 + \epsilon] \to M, (g,s) \mapsto \phi_g(v(s, t_0))$$

is an immersion in $B_\epsilon(e)$ for fixed $s \in [s_0 - \epsilon, s_0 + \epsilon]$ and in $[s_0 - \epsilon, s + \epsilon]$ for fixed $g \in B_\epsilon(e)$.

Here $B_\epsilon(e)$ stand for the preimage of an ϵ ball in $\mathbb{R}^n$ under a chart of G around the unit e. The first part of the choice is possible since $v(s_0, t_0) \neq \phi_g(x^\pm(t_0)) \ \forall g \in G$ and G is compact. The second part is possible, because $\partial_s v(s, t_0) \neq 0$ in a neighbourhood around s_0 and G acts freely on M. Now by Lemma 4.5.3 the set $C(v) \cap [-T,T] \times [0,1]$ is finite. Varying the point (s_0, t_0) slightly we may assume that $v(s_0, t_0) \neq v(s,t)$ whenever $(s,t) \in C(v) \cap [-T,T] \times [0,1]$. By shrinking $\epsilon > 0$ we may obtain

iii) $v(B_\epsilon(s_0, t_0)) \cap v(C(v) \cap [-T,T] \times [0,1]) = \emptyset$.

Now from *ii)* it follows that $\partial_s v(s,t) \neq 0$ and from *i)* that $v(s,t) \neq \phi_g(x^{\pm}(t))$ for all $g \in G$ and all $(s,t) \in B_\epsilon(s_0,t_0)$. Therefore the condition $v(B_\epsilon(s_0,t_0)) \cap R^G(v) = \emptyset$ implies that for all $(s,t) \in B_\epsilon(s_0,t_0)$ there exists an $s' \in \mathbb{R}$ and $g \in G$ such that $v(s',t) = \phi_g(v(s',t))$ and $s \neq s'$. In view of *iii)* we get $\partial_s v(s',t)) \neq 0$ and therefore also

$$\mathrm{d}\phi_g\big(\partial_s v(s',t)\big) = \phi_g(v(s',t)) \neq 0$$

for any such point s' and all $g \in G$. In view of *i)* we have $|s'| \leq T$. Hence there can only exist finitely many of such points s' for each pair (s,t), because otherwise we again could find a subsequence such that both $g_\nu \to g_*$ and $s_\nu \to s_*$ with $\partial_s \phi_{g_*}(v(s_*,t)) = 0$. Since $\mathrm{d}\phi_g$ is bijective at every point for all $g \in G$ this implies $\partial_s v(s_*,t) = 0$, which is a contradiction. If there are $g_1, g_2 \in G$ with

$$v(s,t) = \phi_{g_1}(v(s',t)) = \phi_{g_2}(v(s',t)),$$

then the freeness of the group action implies $g_1 = g_2$, i.e. for every (s,t) there is only a finite amount of pairs (s',g). Hence let $(s_1,g_1),\ldots,(s_N,g_N) \in [-T,T] \times G$ be the point with

$$v(s_0,t_0) = \phi_{g_1}(v(s_1,t_0)) = \ldots = \phi_{g_N}(v(s_N,t_0))$$

<u>Claim:</u> For every constant $r > 0$ there exists a $\delta > 0$ s.t.

$$v(B_{2\delta}(s_0,t_0)) \subseteq \bigcup_{j=1}^{N} \phi_{g_j}(v(B_r(s_j,t_0))).$$

<u>Proof of Claim:</u> Assume the contrary, then there exists a sequence $(s_\nu,t_\nu) \to (s_0,t_0)$ such that

$$v(s_\nu,t_\nu) \notin \phi_{g_j}(v(B_r(s_j,t_0))) \quad \text{but} \quad v(s_\nu,t_\nu) \in v(B_{2\delta}(s_0,t_0))$$

for all $\nu \in \mathbb{N}$ and every $j \geq 1$. We can of course assume that $2\delta < \epsilon$, i.e. there exists a sequence (g_ν, s'_ν) with $s'_\nu \neq s_\nu$ such that

$$v(s_\nu,t_\nu) = \phi_{g_\nu}(v(s'_\nu,t_\nu)).$$

By *i)* we know that $|s'_\nu| < T$ and hence there is an accumulation point (g',s'). *ii)* shows that either $|s'_\nu - s_\nu| \geq \epsilon$ or $g_\nu \notin B_\epsilon(e)$, i.e. $s' \neq s$ or $g' \neq e$ and since

$$v(s'_\nu,t_\nu) \notin \phi_{g_j}(v(B_r(s_j,t_0)))$$

for all $j \geq 1$ it also does not coincide with any of the pairs $(g_1,s_1)\ldots(g_N,s_N)$. So we found a new pair $(g',s') \neq (e,s)$ with

$$v(s,t) = \phi_{g'}(v(s',t)).$$

This contradiction proves our claim.
Now we define

$$\Sigma_j = \big\{(s,t) \in \mathrm{cl}(B_\delta(s_0,t_0)) \big| v(s,t) \in \mathrm{cl}\big(\phi_{g_j}(v(B_r(s_j,t_0)))\big)\big\}$$

for $j = 1,\ldots,N$. These sets are closed with

$$\mathrm{cl}(B_\delta(s_0,t_0)) = \Sigma_1 \cup \ldots \cup \Sigma_k,$$

hence at least one of the sets Σ_j has a nonempty interior. We assume without loss of generality that $\mathrm{int}(\Sigma_1) \neq \emptyset$. Now choose $\rho > 0$ so small that $B_\rho(s_0,t_0) \subseteq \Sigma_1 \subseteq B_\epsilon(s_0,t_0)$ and note that

$$B_\rho(s_0,t_0) \cap B_r(s_1,t_0) = \emptyset$$

if $r > 0$ was chosen sufficiently small. On the other hand we can deduce from the definition of Σ_1 that for every $(s,t) \in B_\rho(s_0,t_0)$ there exists an $s' \in \mathbb{R}$ such that $(s',t) \in B_r(s_1,t_0)$ and $v(s,t) = \phi_{g_1}(v(s',t))$. To be able to apply Lemma 4.5.4, we consider the equation

$$v(0+s_0, 0+t_0) = \phi_{g_1}(v(0+s_1, 0+t_0)).$$

Further choose $\epsilon = r$ and $\delta = \rho$ to fulfil all the conditions from Lemma 4.5.4, i.e. we can conclude that

$$v(s+s_0, t+t_0) = \phi_{g_1}(v(s+s_1, t+t_0))$$

in a neighbourhood around zero. By the unique continuation property of J-holomorphic curves we get that this equation holds on all of $\mathbb{R}^2$, which implies

$$v(s,t) = \lim_{k\to\pm\infty} \phi_{g_1}(v(s-k(s_0-s_1),t)) = \phi_{g_1}(x^\pm)$$

for all s and t where for the formula we without loss of generality assumed $s_0 > s_1$. Hence, u is constant, which contradicts our assumption that $\partial_s u \not\equiv 0$. $\square$

With all that we discussed above it is now not difficult to see that the same results concerning well-definedness and invariance under homotopy also hold for the equivariant version of Lagrangian RFH, since the equivariant complex just arises via taking quotients with respect to a free and proper Lie group action. We want to summarise this facts in the following theorem:

Theorem 4.5.6 *Let $(M, \omega = \mathrm{d}\lambda)$ be the completion of a Liouville domain with a Hamiltonian H such that $\partial\widetilde{M} = H^{-1}(0)$, the support of $\mathrm{d}H$ is inside of a compact set K the Hamiltonian vector field coincides with the Reeb vector field on $\partial\widetilde{M}$ and $\mathcal{A}_H$ is Morse-Bott. Let further be L an exact Lagrangian submanifold, such that l with $\mathrm{d}l = \lambda\big|_L$ has support in K and let there be an almost complex structure J, which is ω-compatible and SFT like outside of K. In addition assume that we have a compact Lie group G that is a symmetry of the Hamiltonian system, acts free on the solutions of the Hamiltonian equation, leaves the Lagrangian L invariant and the J is G-equivariant. Then the G-equivariant Lagrangian Rabinowitz Floer homology*

$$RFH^G_\bullet(H,L) := \varinjlim_{a\to-\infty} \varprojlim_{b\to\infty} RFH^G_\bullet(H,L)^a_b \tag{4.5.6}$$

is well-defined and if there is a family H_s for $0 \le s \le 1$ of smooth G-invariant functions such that $H_s^{-1}(0)$ can be interpreted as the boundary of a Liouville domain that completes to our given manifold M for all s, then $RFH^G_\bullet(H_0,L)$ and $RFH^G_\bullet(H_1,L)$ are canonically isomorphic.

The proof of this theorem just consists of the same proofs as discussed in the previous sections, where we consider now equivalence classes of critical points with respect to G and count gradient flow lines between those as described above. Note that in the same spirit we can also define the perturbed G-equivariant Lagrangian Rabinowitz Floer homology by requiring the perturbation F to be G-invariant.

4.6 Equivariant RFH and Tate homology

By introducing a variation of the Lagrangian Rabinowitz Floer homology in the previous section that is more restrictive, when it comes to the homotopies that leave the homology invariant we hope to get more information than via the usual approach. But this restriction of the available homotopies on the other hand makes it harder to compute the homology itself. On one hand the vanishing theorem did limit the amount of information we can get from the homology, but on the other hand it also made it very easy to compute it. In this section our goal is now to find a substitution for those tools, which allowed us to easily compute the Lagrangian RFH with an abstract argument. The way we want to achieve this is by relating the equivariant RFH to the Tate homology. This section is a more detailed version of [Ruc23, Chapter 5].

First let us repeat some preliminary definitions:

Definition 4.6.1 *Given a ring R and an R-module M, a resolution of M is an exact sequence of R-modules*

$$\cdots \longrightarrow F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\epsilon} M \longrightarrow 0. \tag{4.6.1}$$

We call this a projective resolution if every F_i is a projective module.

Definition 4.6.2 [2] *A complete resolution (for a finite group G) is an acyclic complex $F = (F_i)_{i\in\mathbb{Z}}$ of projective $\mathbb{Z}_2[G]$-modules, together with a map $\epsilon : F_0 \to \mathbb{Z}_2$ such that $\epsilon : F_+ \to \mathbb{Z}_2$ is a resolution in the usual sense, where $F_+ := (F_i)_{i\geq 0}$.*

The following property of complete resolutions is crucial for the well-definedness of Tate homology:

Proposition 4.6.3 *If $\epsilon : F \to \mathbb{Z}_2$. and $\epsilon' : F' \to \mathbb{Z}_2$ are complete resolutions, then there exists a unique homotopy class of augmentation-preserving maps from F to F'. These maps are homotopy equivalences.*

See [Bro82][Proposition 3.3] for the proof.

Definition 4.6.4 [3] *Let $F = (F_i)_{i\in\mathbb{Z}}$ be a complete resolution for the finite group G. The Tate homology of G with coefficients in a G-module M is defined by*

$$TH_*(G, M) := H_*(F \otimes_G M). \tag{4.6.2}$$

The way one can think of Tate homology is as an extension of the usual group homology to negative gradings by attaching the group cohomology in a suitable way. From this view point it is clear that we have to extend the usual projective resolution for the group homology

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \xrightarrow{\epsilon} M \longrightarrow 0. \tag{4.6.3}$$

to a complete resolution:

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r] & F_1 \arrow[r] & F_0 \arrow[rr] \arrow[dr, "\epsilon"] & & F_{-1} \arrow[r] & F_{-2} \arrow[r] & \cdots \\
& & & & M \arrow[ur, "\eta"] & & &
\end{tikzcd} \tag{4.6.4}$$

Note that proposition 4.6.3 tells us that the Tate homology is unique up to isomorphism and therefore well-defined. This also means that if we want to compute the Tate homology of a group G we can choose any acyclic complex to do so, as long as the complex is a complete resolution. This fact is the main idea behind the connection between G-equivariant RFH and Tate homology:

[2] See [Bro82][§VI.3], note that we use $\mathbb{Z}_2$ instead of $\mathbb{Z}$.
[3] We took the definition from [Bro82][§VI.4], but it goes back to [Tat52].

Theorem 4.6.5 *Let G be a finite group and a symmetry of the Hamiltonian system (M, ω, H) with Lagrangian L, such that the induced group action on the trajectories is free. Assume that $L \cap H^{-1}(0)$ is at least a one dimensional connected submanifold and that the system (M, ω, H, L) fulfils all the requirements needed for equivariant Lagrangian Rabinowitz Floer homology. Further, let the system be displaceable and the Maslov index μ for the non-constant chords of (M, ω, H) fulfil*

$$|\mu(x)| > dim\big(L \cap H^{-1}(0)\big). \tag{4.6.5}$$

Then the G-equivariant Lagrangian RFH is equal to the Tate homology (with $\mathbb{Z}_2$ coefficients) of G, i.e.

$$RFH_*^G(M, H, L) = TH_*(G, \mathbb{Z}_2). \tag{4.6.6}$$

Remark 4.6.6 The statement is also true for equivariant Rabinowitz Floer homology for loops instead of chords between a Lagrangian, if one changes the condition (4.6.5) to $|\mu| > \dim H^{-1}(0)$. The proof works exactly the same.

Proof (Proof of Theorem 4.6.5): The idea of this proof is to show that the Lagrangian Rabinowitz Floer complex for displaceable systems is a complete resolution for G. The displaceability guarantees that our Floer complex is acyclic and by definition consists of $\mathbb{Z}_2$ vector spaces. Since G acts by assumption on the generators of the complex, we can view each space in the complex as free $\mathbb{Z}_2[G]$ module. We know that $L \cap H^{-1}(0)$ is at least one dimensional and by condition (4.6.5) this means that

$$CRF_1(M, H, L) \xrightarrow{\partial_1} CRF_0(M, H, L)$$

is equal to the corresponding part of the Morse complex

$$CM_1(L \cap H^{-1}(0), f) \xrightarrow{\partial_1} CM_0(L \cap H^{-1}(0), f),$$

where f is a G-invariant Morse function on $L \cap H^{-1}(0)$. Note that here we assume the grading to be $\mu = \mu_{Maslov} + \mu_{Morse}$. If one would like to use the signature index instead of the Morse index, one would need to include an index shift in the statement of the theorem. Our goal is now to find an augmentation $\epsilon : CRF_0(M, H, L) = CM_0(L \cap H^{-1}(0), f) \to \mathbb{Z}_2$, such that $\epsilon : (CRF_i(M, H, L))_{i \geq 0} \to \mathbb{Z}_2$ is a resolution of $\mathbb{Z}_2$. It is well known that for every connected manifold the zeroth homology is one dimensional as long as it is not empty. This means that the image of ∂_1 is a codimension one subspace inside $CM_0(L \cap H^{-1}(0), f)$ and therefore we can define ϵ as the linear map that maps all elements in the image of ∂_1 to zero and the basis vector in the complement to one. With this definition it is clear that $\mathrm{im}(\partial_1) = \ker(\epsilon)$ and that ϵ is surjective, i.e.

$$\ldots \xrightarrow{\partial_2} CRF_1(M, H, L) \xrightarrow{\partial_1} CRF_0(M, H, L) \xrightarrow{\epsilon} \mathbb{Z}_2 \longrightarrow 0$$

is a resolution of $\mathbb{Z}_2$ and $(CRF_i(M, H, L))_{i \in \mathbb{Z}}$ is a complete resolution. The corresponding Tate homology is then $TH_*(G, \mathbb{Z}_2) := H_*(CRF_*(M, H, L) \otimes_G \mathbb{Z}_2)$, where we consider $\mathbb{Z}_2$ as a trivial G-module. Tensoring over G to the $\mathbb{Z}_2[G]$ modules $CRF_i(M, H, L)$ the trivial G-module $\mathbb{Z}_2$ corresponds to taking the quotient of the $CRF_i(M, H, L)$'s with respect to the G action. The new boundary operators between the modules $CRF_i(M, H, L) \otimes_G \mathbb{Z}_2$ are just the old boundary operators applied to the first part of the tensor product, i.e. $\widetilde{\partial}(x \otimes_G 1) := (\partial x) \otimes_G 1$. A quick calculation shows that these new boundary operators are exactly the same as the ones we defined for the G-equivariant RF-homology. Hence the two complexes $(CRF_*(M, H, L) \otimes_G \mathbb{Z}_2, \widetilde{\partial})$ and $(CRF_*^G(M, H, L), \partial_G)$ are the same and therefore also their respective homologies. This concludes the proof of the theorem. □

The Tate homology is only really defined for finite groups, so a generalization to general compact Lie groups does not necessarily make sense. But we can at least always define the group homology and generalize the theorem for the positive degrees of the Rabinowitz Floer homology. To emphasise that this phenomenon is a property of the general RFH not just of the Lagrangian one, we phrase the theorem in the following way:

Theorem 4.6.7 *Let (M, ω, H) be a Hamiltonian system with connected energy hypersurface $H^{-1}(0)$ and G a compact Lie group that is a symmetry of this Hamiltonian system. Assume that this set-up fulfils all the requirements for a well-defined G-equivariant RFH and that the system is dynamically convex. If we can displace the energy hypersurface away from itself with a compactly supported Hamiltonian diffeomorphism, than*

$$RFH^{G}_{*\geq 0}(M, H) = H_*(G, \mathbb{Z}_2). \tag{4.6.7}$$

PROOF: The proof just consists of repeating the arguments as for Theorem 4.6.5. Note that the projective resolution we constructed in the proof of Theorem 4.6.5 is now used to define the group homology. □

Chapter 5

About the Existence of Symmetric Consecutive Collision Orbits

In this chapter we now take on the main goal of this thesis, proving the existence of infinitely many symmetric consecutive collision orbits in the setting of the circular restricted three body problem. A consecutive collision orbit (c.c. orbit) is a Hamiltonian trajectory that starts with a collision and after some time comes back to a collision. We specifically want to study these orbits in the circular restricted three body problem for energies below the first critical energy value, i.e. the trajectories of this system are bound to one of the masses. One usually chooses the coordinates in such a way that this mass is fixed at the origin. The collision points are then those points in the phase space $\mathbb{R}^{2n}$ with position coordinate equal to zero and arbitrary momentum, i.e. $\{0\} \times \mathbb{R}^n$ is the set of all collisions. Note that we can only consider these points as part of the phase space after applying one of the regularization procedures discussed in Chapter 3. As symplectic structure on $\mathbb{R}^{2n}$ we take the one induced by the canonical one form on $T^*\mathbb{R}$, hence we can view consecutive collision orbits as chords starting and ending in the Lagrangian subspace $\{0\} \times \mathbb{R}^n$. In [FZ19] the authors then use Lagrangian Rabinowitz Floer homology in the setting of Moser regularization to prove that there is either one periodic c.c. orbit or infinitely many. We now want to take it a step further and prove this result with the additional condition that the c.c. orbits fulfil a certain symmetry condition and as we will see later it is beneficial to do this in the setting of Levi-Civita or Kustaanheimo-Stiefel regularization.

5.1 The Planar Case

For the planar restricted three body problem the phase space is $\mathbb{R}^4$ which we can identify with $\mathbb{C}^2$. The symmetry we want to consider is given by the anti-symplectic involution

$$R : T^*\mathbb{C} \to T^*\mathbb{C} \ \ ; \ q \mapsto \overline{q} \ , \ p \mapsto -\overline{p}, \tag{5.1.1}$$

which leaves the Hamiltonian invariant. When we apply the Levi-Civita regularization this anti-symplectic involution corresponds then to two involutions

$$\widehat{R}_1 : z \mapsto -\overline{z} \ \ ; \ w \mapsto \overline{w} \tag{5.1.2}$$

$$\widehat{R}_2 : z \mapsto \overline{z} \ \ ; \ w \mapsto -\overline{w}. \tag{5.1.3}$$

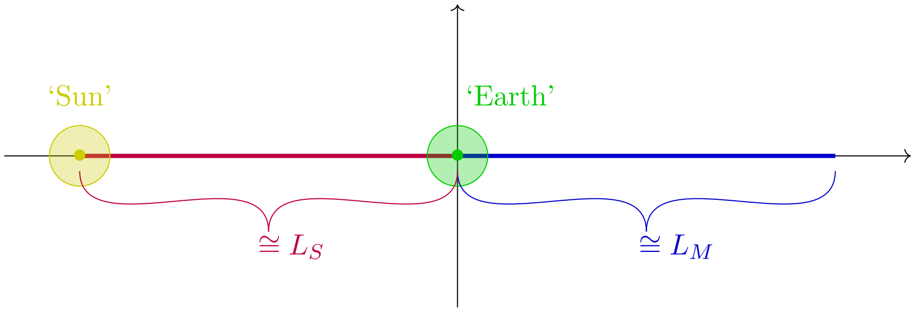
To see this remember that the Levi-Civita map is given by

$$(z, w) \mapsto \left(z^2, \frac{wz}{2|z|^2}\right). \tag{5.1.4}$$

Their corresponding fix point sets are

$$L_S := \mathrm{Fix}(\widehat{R}_1) = i\mathbb{R} \times \mathbb{R} \quad \text{and} \quad L_M := \mathrm{Fix}(\widehat{R}_2) = \mathbb{R} \times i\mathbb{R}. \tag{5.1.5}$$

A symmetric c.c. orbit is now a chord from the collision Lagrangian $L_{\mathrm{col}} = \{0\} \times \mathbb{R}^2$ to one of the above fix point sets. The interpretation of these fix point Lagrangians is the following: The position coordinates of points in L_S lie in $i\mathbb{R}$ and to get from the regularized setting to the real physical system we need to square the position variables. This implies that points in L_S represent phase space points in the physical system with position coordinates lying in the left hand side of the x-axis. Note that in Chapter 3 we chose the coordinates such that the two masses lie on the x-axis as well, i.e. points in L_S represent the positions that are on the direct line between the two masses. A sketch of the position space would look like the following picture:



The constraint to lie between the two masses and not further away comes from the fact that we only consider energies below the first critical energy value. If we would view the mass at the centre as the earth, the other mass as the sun and the third body as the moon, L_S would correspond to points of a solar eclipse. Following the same line of reasoning L_M corresponds to points of a lunar eclipse. To understand why trajectories starting in Lcol and ending in L_S or L_M are symmetric consecutive collision orbits consider the following computation: Let $x(t)$ be a trajectory of the Hamiltonian K with $x(0) \in L_{\mathrm{col}}$ and $x(1) \in L_S$ (or equivalently $x(1) \in L_M$). Then $\widehat{R}_1(x(1-t))$ is also a trajectory of K since

$$\begin{aligned} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0} \widehat{R}_1(x(1-t)) &= \mathrm{d}\widehat{R}_1 \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0} x(1-t) \\ &= -\,\mathrm{d}\widehat{R}_1 \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{1-t_0} x(t) \\ &= -\,\mathrm{d}\widehat{R}_1 X_K(x(t))\big|_{1-t_0} \\ &= X_K\Big(\widehat{R}_1(x(t))\Big)\Big|_{1-t_0} \\ &= X_K\Big(\widehat{R}_1(x(1-t))\Big)\Big|_{t_0}. \end{aligned} \tag{5.1.6}$$

Further, the requirement $x(1) \in L_S$ guarantees that the concatenation $x \circ \widehat{R}_1(x(1-\,\cdot\,))$ is still a smooth solution of K. This means, if we find a chord from L_{col} to L_S, we also automatically find a symmetric consecutive collision orbit.

The way we want to prove the existence of these curves is by interpreting them as leaf-wise intersection points (see Definition 4.4.8). The next part is based on the same considerations made in [Ruc23]. To do this identification we need Hamiltonians F_S or F_M such that

$$\varphi_{F_S}(L_S) = L_{\mathrm{col}} \quad \text{and} \quad \varphi_{F_M}(L_M) = L_{\mathrm{col}}, \tag{5.1.7}$$

where φ_{F_S} or φ_{F_M} is the Hamiltonian flow. So define

$$F_S(z,w) = \pi(z_2^2 + w_2^2) \quad \text{and} \quad F_M(z,w) = \pi(z_1^2 + w_1^2), \tag{5.1.8}$$

where (z_1, z_2) are the space coordinates and (w_1, w_2) are the momentum coordinates. The Hamiltonian vector field of F_S is then

$$X_{F_S} = \frac{\pi}{2} \cdot w_2 \frac{\partial}{\partial z_2} - \frac{\pi}{2} \cdot z_2 \frac{\partial}{\partial w_2} \tag{5.1.9}$$

and the corresponding flow is given by

$$\varphi_{F_S}(z, w, t) = \begin{pmatrix} z_1 \\ \cos\left(\frac{\pi}{2} t\right) z_2 + \sin\left(\frac{\pi}{2} t\right) w_2 \\ w_1 \\ \cos\left(\frac{\pi}{2} t\right) w_2 - \sin\left(\frac{\pi}{2} t\right) z_2 \end{pmatrix}. \tag{5.1.10}$$

With this it is not hard to see that $\varphi_{F_S}(L_S, t = 1) = L_{\text{col}}$, for L_M the calculations are almost the same. Note that both F_S and F_M are invariant under the $\mathbb{Z}_2$ action $z \mapsto -z$ and $w \mapsto -w$. To be able to interpret our chords as leaf-wise intersection points, we will have to turn F_S or F_M respectively into a compactly supported function. For this choose a compact set $\widetilde{K}$ such that $\Sigma \subseteq \widetilde{K}$ and a corresponding bump function $\chi_{\widetilde{K}}$, which is 1 on $\widetilde{K}$ and compactly supported. Define

$$\widehat{F}_S(z, w) = \chi_{\widetilde{K}}(z^2, w^2) \cdot F_S(z, w) \quad \text{and} \quad \widehat{F}_M(z, w) = \chi_{\widetilde{K}}(z^2, w^2) \cdot F_M(z, w),$$

so that we don't loose the invariance with respect to the $\mathbb{Z}_2$ action of our Hamiltonian functions. By again choosing a suitable bump function $\zeta : [0, 1] \to \mathbb{R}$ we finally get the desired form of the Hamiltonian functions, i.e.

$$\widehat{F}^t_S(z, w) := \zeta(t) \widehat{F}_S(z, w) \quad \text{and} \quad \widehat{F}^t_M(z, w) := \zeta(t) \widehat{F}_M(z, w)$$

with $\widehat{F}^t_S(z, w) = 0 = \widehat{F}^t_M(z, w)$ for all $t \in \left[0, \frac{1}{2}\right]$ and $\zeta(t) = 1$ in a neighbourhood around 1. As a next step we want to show via some standard computations that the critical points of $\mathcal{A}_K^{\widehat{F}^t_S}$ that start and end in L_{col} are in one to one correspondence with the critical points of $\mathcal{A}_K$ that start in L_{col} and end in $\varphi^{-1}_{\widehat{F}^t_S}(L_{\text{col}})$. So first let (y, η) be a critical point of $\mathcal{A}_K^{\widehat{F}^t_S}$ with $y(0), y(1) \in L_{\text{col}}$. Then as discussed in Section 4.4 in $\left[0, \frac{1}{2}\right]$ we have

$$\partial_t y = \eta \beta(t) X_K(y(t)). \tag{5.1.11}$$

By Picard-Lindelöf there is also a unique solution of

$$\begin{cases} \partial_t x = \eta X_K(x(t)) \\ x(0) = y(0) \end{cases} \tag{5.1.12}$$

in $[0, 1]$. Then consider $\widetilde{x}(t) := x\left(\int\limits_0^t \beta(T)\, \mathrm{d}T\right)$ and see that

$$\partial_t \widetilde{x} = \eta \beta(t) X_K(\widetilde{x}) \tag{5.1.13}$$

in $\left[0, \frac{1}{2}\right]$. By uniqueness this means that $y(t) = \widetilde{x}(t)$ and therefore

$$x(1) = x\left(\int\limits_0^{\frac{1}{2}} \beta(T)\, \mathrm{d}T\right) = y\left(\frac{1}{2}\right) \in \varphi^{-1}_{\widehat{F}^t_S}(L_{\text{col}}) \tag{5.1.14}$$

On the other hand let (x,τ) be a critical point of $\mathcal{A}_K$ with $x(0)\in L_{\mathrm{col}}$, $x(1)\in\varphi^{-1}_{\widehat{F}^t_S}(L_{\mathrm{col}})$. Then take the usual cut off function $\beta(t)$ and define

$$y(t) := \begin{cases} x\left(\int\limits_0^t \beta(T)\,\mathrm{d}T\right) & t\in\left[0,\frac{1}{2}\right] \\ \varphi^t_{\widehat{F}^t_S}(x(1)) & t\in\left[\frac{1}{2},1\right] \end{cases}, \tag{5.1.15}$$

which is clearly a critical point of $\mathcal{A}_K^{\widehat{F}^t_S}$. This means for every leaf-wise intersection point there exists a corresponding symmetric c.c. orbit.

The logical next step is now to calculate the homology $RFH(M,L_{\mathrm{col}},K,\widehat{F}^t_S)$ via

$$RFH(M,L_{\mathrm{col}},K,\widehat{F}^t_S)\cong RFH(M,L_{\mathrm{col}},K) \tag{5.1.16}$$

(see Equation 4.4.12). But since our underlying manifold is $\mathbb{C}^2$ and our energy hypersurface is compact, we can use Theorem 4.4.10 to see that the resulting homology is zero given the fact that we can define the Lagrangian RFH in this setting. $RFH(M,L_{\mathrm{col}},K)=0$ does of course not give us enough information about the system to prove that there are infinitely many leaf-wise intersection point. For such a situation we discussed in Section 4.6 the concept of the equivariant Lagrangian RFH and hence we will use this homology to prove the following result.

Theorem 5.1.1 *For the planar circular restricted three body problem in Levi-Civita regularization there are for all energies below the first critical energy value infinitely many leaf-wise intersection points with respect to $\widehat{F}^t_S$ and infinitely many with respect to $\widehat{F}^t_M$ all of them starting and ending in L_{col} and the count is with multiplicity.*

Proof: The first step we have to take is to prove that our setting fulfils the necessary requirements for a well-defined equivariant Lagrangian RFH. Since we want to use equivariant Lagrangian RFH we need to specify a symmetry and make sure that all the structures in this proof respect this symmetry. Our choice is the $\mathbb{Z}_2$ symmetry $(z,w)\mapsto(-z,-w)$ that was introduced into the system by applying the Levi-Civita regularization. Proposition 3.1.1 tells us that below the first critical energy value the energy hypersurface $K^{-1}(0)$ is the boundary of a Liouville domain and because of continuity there is even a neighbourhood in $C^\infty(\mathbb{C}^2,\mathbb{R})$ around K such that for all functions in it the same statement holds. Note that after performing the Levi-Civita regularization both the resulting Hamiltonian and the contact form become invariant under the action of $\mathbb{Z}_2$, hence, we can interpret $\mathcal{A}_K$ as a functional on $C^\infty\big([0,1],\big(\mathbb{C}^2\setminus\{0\}\big)/\mathbb{Z}_2\big)$. Theorem 4.1.1 then tells us that for a generic choice of $H\in C^\infty_c\big(\big(\mathbb{C}^2\setminus\{0\}\big)/\mathbb{Z}_2\big)$ the functional $\mathcal{A}_K$ is Morse-Bott. Therefore we can find a sequence $(H_n)_{n\in\mathbb{N}}$ of $\mathbb{Z}_2$-invariant functions in $C^\infty_c\big(\mathbb{C}^2\setminus\{0\}\big)$ that converge to the given Hamiltonian K such that for every H_n the functional $\mathcal{A}_{H_n}$ is Morse-Bott. Note that we can cut off K to a constant function away from the hypersurface we want to consider and view K as an element in $C^\infty_c\big(\mathbb{C}^2\setminus\{0\}\big)$. By choosing a suitable rescaling function for every H_n we still have that the Hamiltonian vector field coincides with the Reeb vector field on every hypersurface $H_n^{-1}(0)$. In the construction of $\widehat{F}^t_S$ and $\widehat{F}^t_M$ we already made sure that they are invariant under the action of $\mathbb{Z}_2$. We have seen in Chapter 3.1 that the Liouville one form is given by $w_1\,\mathrm{d}z_1 - z_1\,\mathrm{d}w_1 + w_2\,\mathrm{d}z_2 - z_2\,\mathrm{d}w_2$, hence, on the Lagrangian subspace $L_{\mathrm{col}}=\{0\}\times\mathbb{R}^2\subset\mathbb{R}^4$ the one form fulfils $\lambda=0=\mathrm{d}0$. Theorem 4.5.1 tells us that a generic $\mathbb{Z}_2$-equivariant almost complex structure will guarantee transversality and by Proposition 4.3.3 we can always make it into an SFT like almost complex structure outside of a compact set. Since both the Reeb vector field and the Liouville one form are invariant under the action of $\mathbb{Z}_2$ the SFT like condition does not interfere with the equivariance condition. Finally, it is obvious that L_{col} is invariant under $\mathbb{Z}_2$.

The second step is now to calculate the $\mathbb{Z}_2$-equivariant Lagrangian RFH for all the H_n. To do so we first want to find a simpler Hamiltonian function that is homotopic to all H_n. Since the $H_n^{-1}(0)$

are still star-shaped hypersurfaces (c.f. Proposition 3.1.1) we can easily find a homotopy from each $H_n^{-1}(0)$ to S^3 that stays star-shaped throughout the deformation, or between H_n and

$$\widetilde{H} : \mathbb{C}^2 \to \mathbb{R}, \quad z \mapsto \left(\sum_{i=1}^{2} |z_i|^2 \right) - 1,$$

respectively. Again since we can consider $\widetilde{H}$ and H_n to be in $C^\infty\big((\mathbb{C}^2 \setminus \{0\})/\mathbb{Z}_2\big)$ this homotopy can be chosen to be $\mathbb{Z}_2$ invariant. For the Hamiltonian $\widetilde{H}$ it is now much easier to prove the conditions on the Maslov indices required by Theorem 4.6.5, in fact, we already computed the Maslov indices for the trajectories of the system in Example 4.2.2. There we saw that the non-constant trajectories γ of $\widetilde{H}$ have periods $\frac{m}{2}\pi$ for $m \in \mathbb{Z} \setminus \{0\}$ with Maslov index $\mu(\gamma) = 2m$ (since we are in $\mathbb{C}^2$). On the other hand $\widetilde{H}^{-1}(0) \cap \mathbb{R}^2$ is S^1, i.e. $|\mu| > \dim\Big(\mathbb{R}^2 \cap \widetilde{H}^{-1}(0)\Big)$. Hence, we can use Theorem 4.6.5 to calculate $RFH^{\mathbb{Z}_2}\Big(\widetilde{H}, \mathbb{R}^2\Big)$ and with the invariance of RFH we get

$$RFH^{\mathbb{Z}_2}(H_n, L_{\text{col}}) \cong RFH^{\mathbb{Z}_2}\Big(\widetilde{H}, \mathbb{R}^2\Big) \cong TH_*(G = \mathbb{Z}_2, \mathbb{Z}_2).$$

The calculation of the Tate homology of $\mathbb{Z}_2$ is now straight forward: The classifying space of $\mathbb{Z}_2$ is $\mathbb{RP}^\infty$ and therefore the group homology and cohomology of $\mathbb{Z}_2$ is $\mathbb{Z}_2$ in every degree. By [Bro82][§VI.4] we know that the Tate homology of a group is just its group homology in positive degrees and its group cohomology in the negative degrees. This implies that $RFH^{\mathbb{Z}_2}(H_n, L_{\text{col}})$ is also $\mathbb{Z}_2$ in every degree. Now we can use the equivariant version of Equation 4.4.12 to show that

$$RFH^{\mathbb{Z}_2}\Big(H_n, L_{\text{col}}, \widehat{F}_S^t\Big) \cong RFH^{\mathbb{Z}_2}(H_n, L_{\text{col}}) \cong TH_*(G = \mathbb{Z}_2, \mathbb{Z}_2).$$

Note that both $\widehat{F}_S^t$ and $\widehat{F}_M^t$ are $\mathbb{Z}_2$ invariant. This proves the claim of the theorem for the Hamiltonian functions H_n.

The last step of the proof is to transfer the result for the sequence H_n to the Hamiltonian of the restricted three body problem K. Having proved the claim of the theorem for the Hamiltonian functions H_n implies that for every Maslov index $\mu \in \mathbb{Z}$ there is a trajectory x_n^μ of H_n with $x_n^\mu(0) \in L_{\text{col}}$ and $x_n^\mu(1) \in L_S$. By definition the H_n converge to K in C^∞ and therefore the Hamiltonian vector fields X_{H_n} also converge in C^∞ to X_K. Note that here the Hamiltonian vector fields are just smooth maps on $\mathbb{R}^4$ consisting of partial derivatives of the Hamiltonian. So if there is a subsequence of $(x_n^\mu)_{n\in\mathbb{N}}$ that converges to a x_*, then $\frac{\mathrm{d}}{\mathrm{d}t}x_*(t) = \tau_* X_K(x_*(t))$ with $x_*(0) \in L_{\text{col}}$ and $x_*(1) \in L_S$. To show that such a convergent subsequence exists we use the theorem of Arzelà–Ascoli: It is obvious that the $|(x_n^\mu)|$ are uniformly bounded since they all lie on compact hypersurfaces that converge to $K^{-1}(0)$. With

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x_n^\mu(t)\| = |\tau_n|\|X_{H_n}(x_n^\mu(t))\|$$

we can then also uniformly bound the derivative, if the periods τ_n are bounded. We know that if x_n^μ is a critical point of $\mathcal{A}_{H_n}$ with period τ_n then $\mathcal{A}_{H_n}(x_n^\mu) = \tau_n$. Hence, our goal is now to show that the actions of the x_n^μ are bounded:

First we prove the claim with a simplifying assumption, such that the underlying idea becomes clear without getting too technical. So fix a $N \in \mathbb{N}$ such that $\max\{|H_n - H_N|\} < \epsilon_0$ for all $n \geq N$. For a given $n \geq N$ we define a homotopy between H_N and H_n via

$$H_s = s \cdot (H_n - H_N) + H_N.$$

Now assume that we have a homotopy of Hamiltonian trajectories (x_s, τ_s) with $(x_0, \tau_0) = (x_N^\mu, \tau_N)$ and $(x_1, \tau_1) = (x_n^\mu, \tau_n)$. Then consider the function

$$s \mapsto \mathcal{A}_{H_s}(x_s, \tau_s).$$

This function satisfies the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{A}_{H_s}(x_s, \tau_s) = -\tau_s \cdot \int_0^1 H_s'(x_s(t)) = -\mathcal{A}_{H_s}(x_s, \tau_s) \cdot \int_0^1 H_s'(x_s(t))\,\mathrm{d}t$$

with the initial condition $\mathcal{A}_{H_0}(x_0, \tau_0) = \mathcal{A}_{H_N}(x_N^\mu, \tau_N)$. Here H_s' stand for the derivative of H_s with respect to s. From usual ODE theory we know that therefore

$$\mathcal{A}_{H_s}(x_s, \tau_s) = \mathcal{A}_{H_N}(x_N^\mu, \tau_N) \cdot e^{-\int_0^s \int_0^1 H_\sigma'(x_\sigma(t))\,\mathrm{d}t\,\mathrm{d}\sigma}.$$

In fact, this is just a special case of a more general result on the growth of spectral numbers [AF10b]. Now consider:

$$\begin{aligned} -\int_0^s \int_0^1 H_\sigma'(x_\sigma(t))\,\mathrm{d}t\,\mathrm{d}\sigma &= -\int_0^s \int_0^1 (H_n(x_\sigma(t)) - H_N(x_\sigma(t)))\,\mathrm{d}t\,\mathrm{d}\sigma \\ &\leq \int_0^s \int_0^1 \big|H_n(x_\sigma(t)) - H_N(x_\sigma(t))\big|\,\mathrm{d}t\,\mathrm{d}\sigma \\ &\leq \max_{\mathbb{C}^2}\{|H_n - H_N|\} \cdot s \end{aligned}$$

For $s = 1$ this implies

$$|\mathcal{A}_{H_n}(x_n^\mu, \tau_n)| \leq |\mathcal{A}_{H_N}(x_N^\mu, \tau_N)| \cdot \exp\Big(\max_{\mathbb{C}^2}\{|H_n - H_N|\}\Big).$$

Now we can use the ϵ_0 defined above to get a universal bound for all $n \geq N$:

$$|\mathcal{A}_{H_n}(x_n^\mu, \tau_n)| \leq |\mathcal{A}_{H_N}(x_N^\mu, \tau_N)| \cdot e^{\epsilon_0}$$

Hence, we found a universal bound for the periods τ_n and can therefore use Arzelà–Ascoli to get a subsequence of $(x_n^\mu)_{n\in\mathbb{N}}$ that converges to x_* with period τ_*.

In general the existence of the homotopy (x_s, τ_s) we used above is not given. Therefore one has to use a much more involved argument via spectral numbers like in [AF10b, chapter 5], especially consider Theorem 5.5 and it's corollary.

Note that the Maslov index of x_* does not need to be precisely μ, but it can only differ by a maximum of ± 1. This is because the Maslov index only changes in the limit, if the end point of the path associated to x_n^μ converge to a point in the Maslov pseudocycle but is in the complement for all $n \in \mathbb{N}$. Hence, by Equation 4.2.5 the Maslov index in this case changes by $\frac{1}{2}\mathrm{sign}C(\lambda, L_{\mathrm{col}}, 1)$ which is at most $\pm\frac{1}{2}\dim(L_{\mathrm{col}})$, i.e. ± 1. Note that the construction of (x_*, τ_*) can be done for any choice of Maslov index $\mu \in \mathbb{Z}$ and since the resulting trajectory x_* of K has then Maslov index $\mu(x_*) \in [\mu - 1, \mu + 1]$ this construction will give rise to infinitely many different trajectories of K all of them of course still starting in L_{col} and ending in L_S. This now proves the claim also for K.

Finally note that one can repeat the whole proof with L_M instead of L_S. □

Combining the result of the theorem with the considerations made before we immediately see that there need to be infinitely many symmetric consecutive collision orbits. But since we count these orbits with multiplicity, they could all be part of just one periodic symmetric consecutive collision orbit. However, the next theorem tells us that periodic symmetric consecutive collision orbits usually do not exist and can be perturbed away.

Theorem 5.1.2 *In the planar circular restricted three body problem for energies below the first critical energy value there are no periodic symmetric consecutive collision orbits with respect to the symmetries corresponding to L_S and L_M for a generic choice of energy and mass.*

PROOF: In the proof we will work in Levi-Civita regularization and we prove the statement for the symmetry corresponding to L_M, the case of L_S works exactly the same. First let us fix some notation: We know that for a given mass μ there are only two possible periodic symmetric consecutive collision orbits, one with positive starting momentum and one with negative starting momentum. We denote by

$$\gamma(\mu,t) = \begin{pmatrix} x(\mu,t) \\ y(\mu,t) \\ p_x(\mu,t) \\ p_y(\mu,t) \end{pmatrix}$$

the solution of the Hamiltonian equation at mass μ and time t with initial condition $x(\mu,0) = y(\mu,0) = p_x(\mu,0) = 0$, $p_y(\mu,0) = 2\sqrt{2}\sqrt{1-\mu}$. Now let us fix an energy E and assume that there is a set of masses μ with an accumulation point, such that for all μ in this set we have a periodic symmetric consecutive collision orbit. In this set we can without loss of generality assume to find a sequence $(\mu_n)_{n\in\mathbb{N}}$ such that $p_y(\mu_n,0) = +2\sqrt{2}\sqrt{1-\mu_n}$ for all $n \in \mathbb{N}$ (otherwise we just carry an additional minus sign with us). The corresponding orbits are given by $\gamma(\mu_n,t)$ with period τ_n. Since we are below the first critical energy value, i.e. the energy hypersurface is of restricted contact type, we can refer to the considerations made in the previous theorem to see that the periods of these orbits are bounded. With the natural bound of the mass by 1 we get that the sequence $(\mu_n,\tau_n)_{n\in\mathbb{N}}$ converges to the limit (μ_*,τ_*) - up to passing to a subsequence. Note that $\gamma(\mu,t)$ is a continuously differentiable function, because it's the solution of an ODE with smooth dependence on the initial conditions. This means we can conclude that

$$\gamma(\mu_*,\tau_*) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2\sqrt{2}\sqrt{1-\mu_*} \end{pmatrix} = \gamma(\mu_*,0).$$

So at mass μ_* we have a periodic symmetric c.c. orbit with also positive starting momentum and period τ_* and since

$$\partial_t y(\mu_*,\tau_*) = \frac{1}{4} p_y(\mu_*,\tau_*) \neq 0$$

we can use the implicit function theorem to see that on an open neighbourhood of μ_* there exists a differentiable function $\tau(\mu)$ such that $y(\mu,\tau(\mu)) = 0$ for all μ in the open neighbourhood. Again by continuity we have that $(\tau(\mu_n))_{n\in\mathbb{N}}$ converges to τ_*. From this we can conclude that for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for every $n \geq N$ we have

$$|\tau(\mu_n) - \tau_n| < \epsilon.$$

We know that the orbits continuously depend on the mass and fulfil the Hamiltonian equation for a given energy. This implies that $\tau(\mu_n)$ and τ_n can not be arbitrarily close while $\gamma(\mu_n, \tau(\mu_n)) \neq \gamma(\mu_n, \tau_n)$. Note that $\tau(\mu)$ is defined such that $y(\mu_n, \tau(\mu_n)) = 0$ and τ_n such that $\gamma(\mu_n, \tau_n) \in L_M \cap L_{\text{col}}$. So there needs to exist a $M \in \mathbb{N}$ such that for all $n \geq M$ $\tau(\mu_n) = \tau_n$. Now define the functions $f(\mu) := x(\mu, \tau(\mu))$ and $g(\mu) := p_x(\mu, \tau(\mu))$. Since the Hamiltonian equation for the restricted three body problem is real analytic and

$$\frac{\mathrm{d}}{\mathrm{d}\mu}\tau(\mu) = \frac{\partial_\mu y(\mu, \tau(\mu))}{\partial_t y(\mu, \tau(\mu))}$$

is also real analytic for $\mu \in [0, 1]$, the Cauchy–Kovalevskaya theorem tells us that the two functions f and g are real analytic, for now at least in the neighbourhood around μ_* where τ is defined. But f and g are also zero at an accumulation point, which means that they are constant zero since they are real analytic. Let's say the original neighbourhood for τ is $(\mu_* - \delta_0, \mu_* + \delta_0)$ for a $\delta_0 > 0$. By continuity we also have at the boundary point

$$\gamma(\mu_* - \delta, \tau(\mu_* - \delta)) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ p_0 \end{pmatrix},$$

where $\tau(\mu_* - \delta)$ should be understood as the result of a limit and $p_0 > 0$ the appropriate momentum. Then take $y(\mu_* - \delta, \tau(\mu_* - \delta)) = 0$ as a new starting point to get a $\tilde{\tau}$ via the implicit function theorem and again by continuity we get that

$$\forall_{\epsilon > 0} \exists_{N \in \mathbb{N}} \forall_{N \leq n} : |\tau(\mu_* - \delta_0 + \epsilon_n) - \tilde{\tau}(\mu_* - \delta_0 + \epsilon_n)| < \epsilon,$$

where $(\epsilon_n)_{n \in \mathbb{N}}$ is a sequence for positive numbers converging to zero. With the same argument as above we can conclude that $\tau(\mu_* - \delta_0 + \epsilon_n) = \tilde{\tau}(\mu_* - \delta_0 + \epsilon_n)$ for all $n \in \mathbb{N}$ above a certain index and because of the uniqueness in the implicit function theorem $\tau = \tilde{\tau}$ of their shared domain. Since the equation

$$\frac{\mathrm{d}}{\mathrm{d}\mu}\tilde{\tau}(\mu) = \frac{\partial_\mu y(\mu, \tilde{\tau}(\mu))}{\partial_t y(\mu, \tilde{\tau}(\mu))}$$

is still real analytic, $\tilde{\tau}$ is a well-defined real analytic extension of τ. Repeating this line of argument enough times we will eventually get that $f(\mu)$ and $g(\mu)$ are real analytic functions on all of $[0, \mu_* + \delta_0)$ and like we already mentioned both are zero at an accumulation point, i.e. $f \equiv 0 \equiv g$ on this set. This implies that $\gamma(\mu = 0, t)$ is a periodic symmetric c.c. orbit, where $\mu = 0$ means that we are in the rotating Kepler problem. But in the rotating Kepler problem periodic orbits can only have a period equal to a rational multiple of the rotation period of the system itself and since the period of an orbit in the Kepler problem can be expressed like

$$T = \sqrt{\frac{\pi}{2E^3}},$$

for a generic choice of an energy E there can not be any periodic c.c. orbits. This gives us the desired contradiction, i.e. for a generic energy we can not have an accumulation of masses for which we have periodic symmetric consecutive collision orbits. □

If we now combine the Theorem 5.1.1 and Theorem 5.1.2 we get the following corollary.

Corollary 5.1.3 *In the setting of the planar circular restricted three body problem there are*

- *infinitely many symmetric consecutive collision orbits intersecting their symmetry axis on the straight line between the second and the main body and*
- *infinitely many symmetric consecutive collision orbits intersecting their symmetry axis on the extension of this line to the opposite side of the main body*

for a generic choice of mass and energy, where the energy is in addition below the first critical energy value.

5.2 The Spatial Case

For the spatial restricted three body problem the phase space is $\mathbb{R}^6$ and the symmetry we want to consider is given by the anti-symplectic involution

$$\phi : \mathbb{R}^6 \to \mathbb{R}^6; \quad \left(\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}, \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \right) \mapsto \left(\begin{pmatrix} q_1 \\ -q_2 \\ -q_3 \end{pmatrix}, \begin{pmatrix} -p_1 \\ p_2 \\ p_3 \end{pmatrix} \right). \tag{5.2.1}$$

After Kustaanheimo-Stiefel regularization the phase space becomes $\mathbb{H} \times \mathbb{H}$ and this anti-symplectic involution again splits into two different ones

$$\hat{\phi}_1 : \mathbb{H} \times \mathbb{H} \to \mathbb{H} \times \mathbb{H}; \quad \left(\begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{pmatrix}, \begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \end{pmatrix} \right) \mapsto \left(\begin{pmatrix} -z_0 \\ -z_1 \\ z_2 \\ z_3 \end{pmatrix}, \begin{pmatrix} w_0 \\ w_1 \\ -w_2 \\ -w_3 \end{pmatrix} \right) \tag{5.2.2}$$

and

$$\hat{\phi}_2 : \mathbb{H} \times \mathbb{H} \to \mathbb{H} \times \mathbb{H}; \quad \left(\begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{pmatrix}, \begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \end{pmatrix} \right) \mapsto \left(\begin{pmatrix} z_0 \\ z_1 \\ -z_2 \\ -z_3 \end{pmatrix}, \begin{pmatrix} -w_0 \\ -w_1 \\ w_2 \\ w_3 \end{pmatrix} \right). \tag{5.2.3}$$

We again call their corresponding fixed point Lagrangians

$$L_S := \mathrm{Fix}(\hat{\phi}_1) \quad \text{and} \quad L_M := \mathrm{Fix}(\hat{\phi}_2), \tag{5.2.4}$$

where the collision Lagrangian is now given by $L_{\mathrm{col}} := \{0\} \times \mathbb{H}$. As in the planar case the symmetric consecutive collision orbits can be described by the trajectories of $\widehat{K}$ starting in L_{col} and ending in either L_S or L_M, which we can see with exactly the same argument as in Equation 5.1.6. To identify them with leaf-wise intersection points we again define corresponding Hamiltonian diffeomorphisms φ_{F_S} and φ_{F_M} such that

$$\varphi_{F_S}(L_S) = L_{\mathrm{col}} \quad \text{and} \quad \varphi_{F_M}(L_M) = L_{\mathrm{col}}. \tag{5.2.5}$$

More explicitly we define

$$F_s(z, w) = \pi(z_2^2 + z_3^2 + w_2^2 + w_3^2) \quad \text{and} \quad F_M(z, w) = \pi(z_0^2 + z_1^2 + w_0^2 + w_1^2), \tag{5.2.6}$$

where the Hamiltonian flow is given by

$$\varphi_{F_S}(z, w, t) = \left(\begin{pmatrix} z_0 \\ z_1 \\ \cos(\frac{\pi}{2}t)z_2 + \sin(\frac{\pi}{2}t)w_2 \\ \cos(\frac{\pi}{2}t)z_3 + \sin(\frac{\pi}{2}t)w_3 \end{pmatrix}, \begin{pmatrix} w_0 \\ w_1 \\ \cos(\frac{\pi}{2}t)w_2 + \sin(\frac{\pi}{2}t)z_2 \\ \cos(\frac{\pi}{2}t)w_3 + \sin(\frac{\pi}{2}t)z_3 \end{pmatrix} \right) \tag{5.2.7}$$

and analogously for F_M. With the same line of argument as in the previous section one can now show that the leaf-wise intersection points of $(\widehat{K}, L_{\mathrm{col}})$ with respect to F_S or F_M are in one to one correspondence with the trajectories of $\widehat{K}$ starting in L_{col} and ending in L_S or L_M. To prove the existence of infinitely many of these leaf-wise intersection points we again want to use an appropriate version of RFH, but to be able to do so we first need to solve a couple of issues: The first one is that Proposition 3.2.3 only gives us transversality at the points in $\widehat{K}^{-1}(0) \cap BL^{-1}(0)$, so the question arises if that is still enough to secure the well-definedness of RFH.

Theorem 5.2.1 *Let $(M, \omega = \mathrm{d}\lambda)$ be the completion of a Liouville domain with a Hamiltonian H and an additional invariant B such that $\{H, B\} = 0$ and the support of $\mathrm{d}H$ is inside of a compact set K_0. Further assume that the Liouville vector field is transverse to the energy hypersurface $H^{-1}(0)$ at the points in $H^{-1}(0) \cap B^{-1}(0)$ and the Hamiltonian vector field satisfies the conditions $\lambda(X_H) = 1$. Let further be L an exact Lagrangian subspace with $L \subseteq B^{-1}(0)$, such that l with $\mathrm{d}l = \lambda$ on L has support in K_0. Finally let there be an almost complex structure J, which is ω-compatible and SFT like outside of K_0. Then for every sequence (v_n, τ_n) of gradient flow lines of $\nabla \mathscr{A}^H$ for which there exists $a < b$ such that*

$$a \leq \mathscr{A}^H(v_n(s), \tau_n) \leq b \qquad \forall\ n \in \mathbb{N},\ s \in \mathbb{R}, \tag{5.2.8}$$

there exists a subsequence n_i of n and a gradient flow line (v, τ) of $\nabla \mathscr{A}^H$ such that (v_{n_i}, τ_{n_i}) converges in the C^∞_{loc}-topology to (v, τ).

This theorem is a generalization of the usual compactness result in Rabinowitz Floer homology (see [CF09, Theorem 3.1.]), the general proof strategy will therefore also follow the ideas in [CF09, Chapter 3]. Hence, before talking about the proof of the theorem we first need to establish the following lemma.

Lemma 5.2.2 *If under the above assumptions there exists an $\epsilon > 0$, such that*

$$\|\nabla \mathscr{A}^H(x, \tau)\| < \epsilon, \tag{5.2.9}$$

then we have

$$|\tau| \leq c(|\mathscr{A}^H(x, \tau)| + 1) \tag{5.2.10}$$

PROOF: First choose $\delta > 0$ such that

$$\lambda(X_H)\Big|_{H^{-1}([-\delta,\delta]) \cap B^{-1}([-\delta,\delta])} \geq \frac{1}{2},$$

this is possible since by assumption $\lambda(X_H)\big|_{H^{-1}(0) \cap B^{-1}(0)} = 1$ and $\lambda(X_H)$ is a smooth function. Since $H^{-1}([-\delta, \delta]) \cap B^{-1}([-\delta, \delta])$ is a bounded set there is a constant $c_1 > 0$ such that

$$\left\| \lambda(X_H)\Big|_{H^{-1}([-\delta,\delta]) \cap B^{-1}([-\delta,\delta])} \right\|_\infty \leq c_1.$$

Define $\delta_0 := \max\{\delta, \frac{1}{4}\}$. The first step is to show that if we have a gradient flow line (u, τ) with $u(t) \in H^{-1}\big([-\delta, \delta]\big) \cap B^{-1}([-\delta, \delta]\big)$ for all $t \in [0, 1]$, then there exists $c_2 > 0$, such that

$$|\tau| \leq c_2\big(|\mathscr{A}^H(x, \tau)| + \|\nabla \mathscr{A}^H(x, \tau)\|\big).$$

To see this consider the following computation:

$$\begin{aligned}
|\mathscr{A}^H(x,\tau)| &= \left| -\int_0^1 x^*\lambda + \tau \int_0^1 H(x(t))\,\mathrm{d}t \right| \\
&\geq \left| \tau \int_0^1 \lambda(x(t)) X_H(x(t))\,\mathrm{d}t \right| - \left| \int_0^1 \lambda(x(t))(\partial_t x(t) - \tau X_H(x(t)))\,\mathrm{d}t \right| - \left| \tau \int_0^1 H(x(t))\,\mathrm{d}t \right| \\
&\geq \frac{1}{2}|\tau| - c_1 \| \partial_t x(t) - X_H(x(t)) \|_{L_1} - \frac{1}{4}|\tau| \\
&\geq \frac{|\tau|}{4} - c_1 \|\nabla \mathscr{A}^H(x,\tau)\|
\end{aligned}$$

The second part of the proof now consists of showing that if for a trajectory (x,τ) there exists a $t_0 \in [0,1]$ s.t $x(t_0) \notin H^{-1}([-\delta,\delta]) \cap B^{-1}([-\delta,\delta])$, then there is an $\epsilon_0 > 0$ with

$$\|\nabla \mathscr{A}^H(x,\tau)\| \geq \epsilon \tag{5.2.11}$$

Case 1 $x(t_0) \notin H^{-1}([-\delta_0,\delta_0])$, then the argument works as usually. See the proof of [CF09, Proposition 3.2.] for more details.

Case 2 Assume $x(t_0) \notin B^{-1}([-\delta_0,\delta_0])$. Since every gradient flow line by assumption starts and ends in $L \subseteq B^{-1}(0)$, we can always find a $t_1,\ t_2 \in [0,1]$ such that

$$|B(x(t_1))| = \delta_0 \quad \text{and} \quad |B(x(t_2))| = \frac{\delta_0}{2}$$

with

$$\frac{\delta_0}{2} \leq |B(x(t))| \leq \delta_0 \quad \text{for all } t \in [t_1,t_2]$$

Now compute

$$\begin{aligned}
\frac{\delta_0}{2} &= |B(x(t_1))| - |B(x(t_2))| \\
&\leq |B(x(t_1)) - B(x(t_2))| \\
&\leq \left| \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t} B(x(t))\,\mathrm{d}t \right| \\
&= \left| \int_{t_1}^{t_2} \mathrm{d}B(x(t))\,\partial_t x(t)\,\mathrm{d}t \right| \\
&= \left| \int_{t_1}^{t_2} \mathrm{d}B(x(t))(\partial_t x(t) - X_H(x(t)))\,\mathrm{d}t \right|
\end{aligned}$$

The last equality holds, because $\mathfrak{L}_{X_H} B = 0$. Now set $c_0 := \max_{x \in B^{-1}([-\delta_0,\delta_0])} \|\nabla_t B(x)\|_t$, then further estimate:

$$\left| \int_{t_1}^{t_2} \mathrm{d}B(x(t))(\partial_t x(t) - X_H(x(t)))\,\mathrm{d}t \right|$$

$$
\begin{aligned}
&= \int_{t_1}^{t_2} \langle \nabla_t B(x(t)), \partial_t x(t) - X_H(x(t)) \rangle \, \mathrm{d}t \\
&\leq \int_{t_1}^{t_2} \| \nabla_t B(x(t)) \|_t \| \, \partial_t x(t) - X_H(x(t)) \|_t \, \mathrm{d}t \\
&\leq c_0 \int_{t_1}^{t_2} \| \, \partial_t x(t) - X_H(x(t)) \|_t \, \mathrm{d}t \\
&\leq c_0 \| \, \partial_t x - X_H(x) \|_{L^2} \\
&\leq c_0 \| \nabla \mathscr{A}^H(x, \tau) \|
\end{aligned}
$$

This means that

$$\| \nabla \mathscr{A}^H(x, \tau) \| \geq \frac{\delta_0}{2c_0}.$$

Now choose ϵ_0 to be the minimum of all the bounds for $\|\nabla \mathscr{A}^H(x,\tau)\|$ one finds in case 1 and case 2, then set $\epsilon := \min\{\epsilon_0, 1\}$. If we have a trajectory (x, τ) such that $\|\nabla \mathscr{A}^H(x,\tau)\| < \epsilon$, then we know from the second step that $x(t) \in H^{-1}([-\delta_0, \delta_0]) \cap B^{-1}([-\delta_0, \delta_0])$. With the first step we finally get

$$|\tau| \leq c_2 \big(|\mathscr{A}^H(x,\tau)| + \|\nabla \mathscr{A}^H(x,\tau)\| \big) \leq c_2 \big(|\mathscr{A}^H(x,\tau)| + 1 \big) \qquad \square$$

Proof (of Theorem 5.2.1): The proof follows the usual approach of showing the existence of uniform bounds for

- the chords v,
- the Lagrange multiplier τ,
- the derivatives of the chords v.

The L^∞ bound of the chords is as usually established by combining the fact that the Hamiltonian is constant outside of a compact subset, which turns the Floer equation into the holomorphic curve equation, and that we chose our almost complex structure to be SFT like outside of a compact set. For the bound of the Lagrange multiplier one can take Lemma 5.2.2 and follow the argument in [CF09, Chapter 3]. After having established the L^∞ bounds of the chord and the Lagrange multiplier the bound of the derivatives works as in the usual case. Arzelà–Ascoli and bootstrapping gives us the desired result. $\square$

The second issue is again that one can displace the Lagrangian L_{col} away from the energy hypersurface, which results in $RFH(\mathbb{H}^2, L_{\mathrm{col}}, \widehat{K}) = 0$. The solution is as in the previous section to use the equivariant RFH instead. The additional symmetry introduced by the regularization is in this case an action by S^1 via

$$S^1 \times \mathbb{H}^2 \to \mathbb{H}^2; \; e^{i\varphi} \times (z, w) \mapsto (e^{i\varphi} \cdot z, e^{i\varphi} \cdot w). \tag{5.2.12}$$

Theorem 5.2.3 *For the spatial circular restricted three body problem in Kustaanheimo-Stiefel regularization there are for all energies below the first critical energy value infinitely many S^1 families of leaf-wise intersection points with respect to F_S and infinitely many S^1 families with respect to F_M all of them starting and ending in L_{col} and the count is with multiplicity.*

PROOF: The key idea of this proof is to use the invariance property of the RFH to find a system in which it is much easier to calculate the homology. But before we can do this we first need to show that the S^1-equivariant RFH is well-defined in the setting of this theorem. Since $BL(z,w) = \mathrm{Re}(\bar{z}iw)$ it is easy to see that $L_{\mathrm{col}} \subset BL^{-1}(0)$ and that L_{col} is invariant under the S^1 action. From Lemma 3.2.2 we see that $\lambda\big|_{L_{\mathrm{col}}} = 0$ and Proposition 3.2.3 gives the necessary transversality of the Liouville vector field. The other conditions for the well-definedness can be achieved with the same arguments as in the proof of Theorem 5.1.1. Note that from here on we assume that $\mathcal{A}_{\widehat{K}}$ is Morse-Bott, if this is not the case then one can approximate the Hamiltonian by a sequence of smooth functions for which the action functional is Morse-Bott and use the same techniques as in the proof of Theorem 5.1.1.

Now the first identification we use is

$$RFH^{S^1}(\mathbb{H}^2, L_{\mathrm{col}}, \widehat{K}, F_S) \cong RFH^{S^1}(\mathbb{H}^2, L_{\mathrm{col}}, \widehat{K})$$

and then we homotope $\widehat{K}$ to $\widetilde{H}$ with

$$\widetilde{H} : \mathbb{R}^8 \to \mathbb{R}, \quad x \mapsto \left(\sum_{i=1}^{8} |x_i|^2\right)^2 - 1$$

and get

$$RFH^{S^1}(\mathbb{H}^2, L_{\mathrm{col}}, \widehat{K}) \cong RFH^{S^1}(\mathbb{R}^8, L_{\mathrm{col}}, \widetilde{H}).$$

To finish the proof we now need to compute this last homology. In Example 4.2.21 we already computed all the trajectories starting and ending in $L_{\mathrm{col}} = \{0\} \times \mathbb{R}^4$ and saw that such a trajectory $\left(x, \tau = \frac{m}{2}\pi\right)$ has Maslov index $\mu(x) = -4m$. Every chord is uniquely defined by its starting point on $\widetilde{H}^{-1}(0) \cap L_{\mathrm{col}} = S^3$ and its period τ, hence these critical points form infinitely many copies of S^3, where every copy of S^3 carries one of the Maslov indices $\mu = -4m$. In the next step we have to divide out the S^1 action, i.e. the resulting manifolds are $S^3/S^1 = S^2$. If we choose as Morse function the hight function on S^2, we see that the RFH coincides with the Morse Homology of S^2 for the indices $4m+0$ to $4m+2$ with $m \in \mathbb{Z}$. Since the index difference between each sphere is four, but the dimension of them is two, there is simply no critical point with index $4m+3$. From this we can then finally conclude that:

$$RFH^{S^1}(\mathbb{R}^8, L_{\mathrm{col}}, \widetilde{H}) = \begin{cases} \mathbb{Z}_2 & \text{if } i \text{ is even} \\ 0 & \text{if } i \text{ is odd} \end{cases}$$

This proves the claim of the theorem. □

Remark 5.2.4 Note that we also could have used the Lagrangian version of Theorem 4.6.7. Indeed, the group homology of S^1 is exactly

$$H_*(S^1, \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{if } i \text{ is even} \\ 0 & \text{if } i \text{ is odd} \end{cases}. \tag{5.2.13}$$

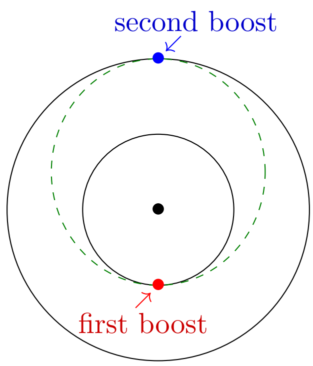
Now let us discuss what this implies for the existence of consecutive collision orbits in the spatial restricted three body problem: First note that the S^1 families of orbits in Kustaanheimo-Stiefel regularization each correspond to only one distinct orbit in the actual physical system. Applying the Kustaanheimo-Stiefel mapping to L_S and L_M reveals that as in the planar case the trajectories with endpoints in L_S correspond to orbits that intersect their symmetry axis in a solar eclipse point and the trajectories with endpoints in L_M to orbits that intersect their symmetry axis in a lunar eclipse point. However, we do not get more information about the existing orbits than from the planar case since all these infinitely many orbits we found could just be confined to the plane.

Chapter 6

Outlook: Towards a Mathematical Theory of Fuel

Fuel consumption is one of the key parameters that determine both the cost and the possible destinations of a space mission and therefore reducing the amount of fuel necessary is an important ingredient to further improve our capabilities when it comes to space exploration. The classic approach to space mission design tries to find the most efficient trajectory in a given N-body problem by dividing the position space into N spheres of influence, solving the Kepler problem for each of the N masses and then patch the resulting orbits together in such a way that they satisfy the boundary conditions set by the mission. This approach is known as the *patched-conic approximation*, see [BMW71] for more details. Even though this approximation has been very successful in the past, there are a lot of important multi body systems that are not very well described by this method. An example of such a system is the 4-body problem consisting of the sun, earth, moon and a space ship. Of course we have to acknowledge the fact that the Apollo missions have been very successful despite using only a patch conic approximation, but since then people were able to find much more efficient trajectories from the earth to the moon by using a patched three body approximation. Another situation where a patched three body approximation enables a significant increase in fuel efficiency, is a multi-moon orbiter tour of Jupiter's moons (see [KMR99]). The goal is to very closely study the moons of Jupiter like Europa and Ganymede within a single space mission by making multiple transitions from an orbit around one moon to an orbit around another one and the exploration capacity of such a mission of course heavily depends on finding fuel efficient transition orbits.

The trajectories we studied in this thesis are mathematically very similar to those transition orbits. To see this let us consider the easier case of the Kepler problem and take as transition orbit the Hohmann transfer:



The goal is to find a Hamiltonian trajectory that connects the inner orbit to the outer one. To achieve this the space ship performs a boost at the location of the red dot, then follows its resulting

Hamiltonian trajectory (dashed line) without using any fuel until it reaches the blue dot where it again performs a boost to enter the outer orbit. The first boost can also be interpreted as a translation in the affine Lagrangian subspace $L_1 := \{Q_1\} \times \mathbb{R}^2$, where Q_1 stands for the position of the first boost. In the same way the second boost is a translation in the affine Lagrangian subspace $L_2 := \{Q_2\} \times \mathbb{R}^2$. With this interpretation the Hohmann transfer is just a Hamiltonian chord between two different Lagrangian manifolds.

The idea is now to use a version of Lagrangian Rabinowitz Floer homology in the setting of the restricted three body problem to show the existence of Hamiltonian trajectories between two different boost coordinates. With this information one can then plan a space mission by patching together the solutions of the restricted three body problem between the different affine Lagrangian subspaces. If we have a trajectory that ends in a Lagrangian $\{Q_k\} \times \mathbb{R}^2$ and is followed up with a trajectory starting in the Lagrangian $\{Q_k\} \times \mathbb{R}^2$, the fuel needed to perform this manoeuvre is proportional to the translation in $\{Q_k\} \times \mathbb{R}^2$ to get from the end point of the first trajectory to the starting point of the second trajectory. To find an fuel efficient route one then has to vary the considered Lagrangians, such that the sum of the norms of the translations is minimal. In this context it is also important to note that there is much to be gained from considering the spatial restricted three body problem and to look for orbits that do not lie in the ecliptic. In the last section of the previous chapter we discussed the application of Lagrangian RFH in the spatial case, but we were unfortunately not able to find a way to detect specifically spatial orbits that do not lie in the ecliptic. The reason why these spatial orbits are important to fuel efficient mission design is, that changing the plane of a stable orbit around a planet or moon requires a lot of fuel. If for example we want to fly our space ship in a periodic orbit around Jupiter's moon Europa that passes over the north and south pole, but to get to the moon we only considered planar trajectories, we would need to perform a plane change manoeuvre of 90°. This would cost 1.4 times the fuel one needs to accelerate the space ship from zero to the current orbiting velocity (c.f. [BMW71, Chapter 3.4]) , i.e. if one can find spatial trajectories, such that the initial orbit around Europa is already passing over the poles, it could save a lot of fuel.

Appendix A

An Additional Proof

A.1 Proof of Lemma 3.2.2

First we pull back the canonical one form on T^*S^3 with the usual combination of change of position and momentum and the stereographic projection. This leaves us with the one form $\sum_{i=0}^{3} -q_i \,\mathrm{d}p_i$ on $T^*\mathbb{R}^3$. Then we pull back this form with the Kustaanheimo-Stiefel map to Σ_0 and since this means sending q to $\bar{z}iz$ and p to $\frac{\bar{z}iw}{2|z|^2}$ the pulled back one form is of the form

$$\sum_{i=0}^{3} -F_i(z)\,\mathrm{d}w_i + \sum_{i,j=0}^{3} -G_{ij}(z)w_i\,\mathrm{d}z_j.$$

Our strategy is now to order the computation by the individual F_i's and G_{ij}'s. Note that we start very detailed, but as we do more and more of the same type of computation it will get less and less detailed.

$$\begin{aligned}
F_0(z) &= \frac{z_0^2 + z_1^2 - z_2^2 - z_3^2}{2|z|^2} \cdot z_0 + \frac{2z_1z_2 - 2z_0z_3}{2|z|^2} \cdot (-z_3) + \frac{2z_0z_2 + z_1z_3}{2|z|^2} \cdot z_2 \\
&= \frac{z_0^3 + z_0z_1^2 - z_0z_2^2 - z_0z_3^2 + 2z_0z_3^2 + 2z_0z_2^2}{2|z|^2} \\
&= z_0 \cdot \frac{z_0^2 + z_1^2 + z_2^2 + z_3^2}{2|z|^2} \\
&= \frac{z_0}{2}
\end{aligned}$$

$$\begin{aligned}
F_1(z) &= \frac{z_0^2 + z_1^2 - z_2^2 - z_3^2}{2|z|^2} \cdot z_1 + \frac{2z_1z_2 - 2z_0z_3}{2|z|^2} \cdot (z_2) + \frac{2z_0z_2 + z_1z_3}{2|z|^2} \cdot z_3 \\
&= \frac{z_1z_0^2 + z_1^3 - z_1z_2^2 - z_1z_3^2 + 2z_1z_3^2 + 2z_1z_2^2}{2|z|^2} \\
&= z_1 \cdot \frac{z_0^2 + z_1^2 + z_2^2 + z_3^2}{2|z|^2} \\
&= \frac{z_1}{2}
\end{aligned}$$

$$\begin{aligned}
F_2(z) &= \frac{z_0^2+z_1^2-z_2^2-z_3^2}{2|z|^2}\cdot(-z_2) + \frac{2z_1z_2-2z_0z_3}{2|z|^2}\cdot(z_1) + \frac{2z_0z_2+z_1z_3}{2|z|^2}\cdot z_0 \\
&= z_2\cdot\frac{z_0^2+z_1^2+z_2^2+z_3^2}{2|z|^2} \\
&= \frac{z_2}{2}
\end{aligned}$$

$$\begin{aligned}
F_3(z) &= \frac{z_0^2+z_1^2-z_2^2-z_3^2}{2|z|^2}\cdot(-z_3) + \frac{2z_1z_2-2z_0z_3}{2|z|^2}\cdot(-z_0) + \frac{2z_0z_2+z_1z_3}{2|z|^2}\cdot z_1 \\
&= z_3\cdot\frac{z_0^2+z_1^2+z_2^2+z_3^2}{2|z|^2} \\
&= \frac{z_3}{2}
\end{aligned}$$

$$\begin{aligned}
G_{00}(z) &= \frac{z_0^2+z_1^2-z_2^2-z_3^2}{2|z|^2} - \frac{z_0^2+z_1^2-z_2^2-z_3^2}{|z|^4}\cdot z_0^2 - \frac{2z_1z_2-2z_0z_3}{|z|^4}\cdot(-z_3z_0) \\
&\quad - \frac{2z_0z_2+z_1z_3}{|z|^4}\cdot z_2z_0 \\
&= \frac{z_0^2+z_1^2-z_2^2-z_3^2}{2|z|^2} - \frac{z_0^2}{|z|^2} \\
&= -\frac{1}{2} + \frac{z_1^2}{|z|^2}
\end{aligned}$$

$$\begin{aligned}
G_{10}(z) &= -\frac{z_0^2+z_1^2-z_2^2-z_3^2}{|z|^4}\cdot z_0z_1 - \frac{2z_1z_2-2z_0z_3}{|z|^4}\cdot(-z_2z_0) \\
&\quad - \frac{2z_0z_2+z_1z_3}{|z|^4}\cdot z_3z_0 \\
&= -\frac{z_0z_1}{|z|^2}
\end{aligned}$$

$$\begin{aligned}
G_{20}(z) &= \frac{2z_0z_2+z_1z_3}{2|z|^2} - \frac{z_0^2+z_1^2-z_2^2-z_3^2}{|z|^4}\cdot(-z_0z_2) - \frac{2z_1z_2-2z_0z_3}{|z|^4}\cdot(z_1z_0) \\
&\quad - \frac{2z_0z_2+z_1z_3}{|z|^4}\cdot z_0^2 \\
&= \frac{2z_0z_2+z_1z_3}{2|z|^2} - \frac{z_0z_2}{|z|^2} \\
&= \frac{z_1z_3}{|z|^2}
\end{aligned}$$

$$G_{30}(z) = \frac{-2z_1z_2+2z_0z_3}{2|z|^2} - \frac{z_0z_3}{|z|^2} = -\frac{z_1z_2}{|z|^2}$$

$$G_{01}(z) = -\frac{z_1z_0}{|z|^2}$$

$$G_{11}(z) = \frac{z_0^2 + z_1^2 - z_2^2 - z_3^2}{2|z|^2} - \frac{z_1^2}{|z|^2} = -\frac{1}{2} + \frac{z_0^2}{|z|^2}$$

$$G_{21}(z) = \frac{2z_1z_2 - 2z_0z_3}{2|z|^2} - \frac{z_1z_2}{|z|^2} = -\frac{z_0z_3}{|z|^2}$$

$$G_{31}(z) = \frac{2z_0z_2 + z_1z_3}{2|z|^2} - \frac{z_1z_3}{|z|^2} = \frac{z_0z_2}{|z|^2}$$

$$G_{02}(z) = \frac{2z_0z_2 + z_1z_3}{2|z|^2} - \frac{z_2z_0}{|z|^2} = \frac{z_1z_3}{|z|^2}$$

$$G_{12}(z) = \frac{2z_1z_2 - 2z_0z_3}{2|z|^2} - \frac{z_1z_2}{|z|^2} = -\frac{z_0z_3}{|z|^2}$$

$$G_{22}(z) = -\frac{z_0^2 + z_1^2 - z_2^2 - z_3^2}{2|z|^2} - \frac{z_2^2}{|z|^2} = -\frac{1}{2} + \frac{z_3^2}{|z|^2}$$

$$G_{32}(z) = -\frac{z_3z_2}{|z|^2}$$

$$G_{12}(z) = -\frac{2z_1z_2 - 2z_0z_3}{2|z|^2} - \frac{z_0z_3}{|z|^2} = -\frac{z_1z_2}{|z|^2}$$

$$G_{02}(z) = \frac{2z_0z_2 + z_1z_3}{2|z|^2} - \frac{z_1z_3}{|z|^2} = \frac{z_0z_0}{|z|^2}$$

$$G_{23}(z) = -\frac{z_3z_2}{|z|^2}$$

$$G_{33}(z) = -\frac{z_0^2 + z_1^2 - z_2^2 - z_3^2}{2|z|^2} - \frac{z_3^2}{|z|^2} = -\frac{1}{2} + \frac{z_2^2}{|z|^2}$$

If we now consider $\sum_{i=0}^{3} G_{ij}(z)w_i \,\mathrm{d}z_j$ we see that

$$\sum_{i=0}^{3} G_{ij}(z)w_i \,\mathrm{d}z_j = -\frac{1}{2}w_j \,\mathrm{d}z_j \pm \frac{z_{j\pm1}}{|z|^2}(z_1w_0 - z_0w_1 + z_3w_2 - z_2w_3)\,\mathrm{d}z_j.$$

Since Σ_0 is precisely defined by $z_1w_0 - z_0w_1 + z_3w_2 - z_2w_3 = 0$, this then finally proves the theorem. □

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