Geometry & Topology

Volume 27 (2023)

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Geometry & Topology 27:9 (2023) 3699–3713
DOI: 10.2140/gt.2023.27.3699
Published: 5 December 2023

The derivative map for diffeomorphism of disks: an example

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We prove that the derivative map $d\colon \mathrm{Diff}_\partial(D^k) \to \Omega^k \mathrm{SO}_k$, defined by taking the derivative of a diffeomorphism, can induce a nontrivial map on homotopy groups. Specifically, for $k = 11$ we prove that the following homomorphism is nonzero:

$$d_*\colon \pi_5 \mathrm{Diff}_\partial(D^{11}) \to \pi_5 \Omega^{11} \mathrm{SO}_{11} \cong \pi_{16} \mathrm{SO}_{11}.$$

As a consequence we give a counterexample to a conjecture of Burghelea and Lashof by giving an example of a nontrivial vector bundle E over a sphere which is trivial as a topological $\mathbb{R}^k$–bundle (the rank of E is $k = 11$ and the base sphere is S^{17}).

The proof relies on a recent result of Burklund and Senger which determines the homotopy 17–spheres bounding 8–connected manifolds, the plumbing approach to the Gromoll filtration due to Antonelli, Burghelea and Kahn, and an explicit construction of low-codimension embeddings of certain homotopy spheres.

57R50, 57S05; 57R60

1 Introduction

The derivative map

$$d\colon \mathrm{Diff}_\partial(D^k) \to \mathrm{Map}((D^k, \partial D^k), (\mathrm{SO}_k, \mathrm{Id})) \simeq \Omega^k \mathrm{SO}_k, \quad f \mapsto (x \mapsto D_x f),$$

is a basic invariant of the diffeomorphism group of the k–disk; in fact the first-order approximation in the embedding calculus approach to the diffeomorphism group. While $d_\mathbb{Q}\colon \mathrm{Diff}_\partial(D^k)_\mathbb{Q} \to (\Omega^k \mathrm{SO}_k)_\mathbb{Q}$, the rationalisation of d, is nullhomotopic, as we explain in Section 3, much less is known about the derivative map d integrally. For example, to the best of our knowledge, it was not yet known whether the map induced by d on homotopy groups,

$$d_*\colon \pi_i \mathrm{Diff}_\partial(D^k) \to \pi_{i+k} \mathrm{SO}_k,$$

was ever nontrivial. Burghelea and Lashof showed that d_* vanishes for $i = 0, 1$. At odd primes p, they also showed that $d_* = 0$ provided $i < k-3$ and they made a conjecture equivalent to the claim that this holds for $p = 2$ as well [6, Conjecture, page 40]. Burghelea and Lashof also report A'Campo informing them about a proof that $d_* = 0$ for $i = 2$ (however, a written proof has not appeared).

Using smoothing theory, or an explicit geometric construction we introduce here, the map d_* admits an interpretation as describing the normal bundle of certain homotopy spheres embedded in euclidean space. Combining this interpretation with recent results of Burklund and Senger and the refined plumbing construction of Antonelli, Burghelea and Kahn, we obtain a counterexample to the conjecture of Burghelea and Lashof.

In more detail, in [3; 4] Antonelli, Burghelea and Kahn constructed families of diffeomorphisms of the disk using a pairing

$$\sigma\colon \pi_p \mathrm{SO}_{q-a} \otimes \pi_q \mathrm{SO}_{p-b} \to \pi_{a+b+1} \mathrm{Diff}_\partial(D^{p+q-a-b-1})$$

for $0 \le a \le q$ and $0 \le b \le p$, refining Milnor's plumbing pairing; see below. Now $\pi_8 \mathrm{SO}_6 \cong \mathbb{Z}/24$ (see [12, page 162]) and we have:

Theorem 1.1 *Let $\xi \in \pi_8 \mathrm{SO}_6 \cong \mathbb{Z}/24$ be a generator. The image of $\sigma(\xi, \xi)$ under the derivative map*

$$d_*\colon \pi_5 \mathrm{Diff}_\partial(D^{11}) \to \pi_{16} \mathrm{SO}_{11}$$

is nonzero.

Using Morlet's smoothing theory isomorphism, the derivative map d_* on π_k is identified with the boundary map

$$\partial\colon \pi_{n+k+1} \mathrm{PL}_k/O_k \to \pi_{n+k} \mathrm{SO}_k$$

of the fibration sequence $\mathrm{SO}_k \to \mathrm{SPL}_k \to \mathrm{PL}_k/O_k$ (and this allows for our interpretation of [6, Conjecture, page 40] in terms of the derivative map). We conclude that the map $\mathrm{SO}_{11} \to \mathrm{SPL}_{11}$ is not injective on π_{16}. More specifically, if $\tau_{11}\colon S^{11} \to B\mathrm{SO}_{11}$ represents the tangent bundle of the 11–sphere and $f\colon S^{17} \to S^{11}$ represents the unique nontrivial homotopy class (see [21, Proposition 5.11]), we have:

Corollary 1.2 *The pullback $f^*\tau_{11}$ is a nontrivial vector bundle which becomes trivial as an $\mathbb{R}^{11}$–bundle, even when considered as a bundle with structure group SPL_{11}.*

To the best of our knowledge, this is the first example of a nontrivial vector bundle over a sphere which is known to be trivial as a topological $\mathbb{R}^n$–bundle. Milnor famously gave examples of nontrivial vector bundles over Moore spaces, for example the Moore space $M(\mathbb{Z}/7,7) = S^7 \cup_7 D^8$, which are trivial as $\mathbb{R}^n$–bundles [17, Lemma 9.1]. These examples are stable bundles over 4–connected spaces and so the vector bundles are trivial as piecewise linear bundles too.

Acknowledgements

We would like to thank Alexander Kupers for helpful comments on an earlier draft, as well as two referees for helpful comments.

2 Proofs

In this section we give the proofs of Theorem 1.1 and Corollary 1.2. We first recall Gromoll's map $A = C \circ \lambda$ from [9],

$$A\colon \pi_{n-k}\mathrm{Diff}_\partial(D^k) \xrightarrow{\lambda} \pi_0\mathrm{Diff}_\partial(D^n) \xrightarrow{C} \Theta_{n+1},$$

where Θ_{n+1} is the group of homotopy $(n+1)$–spheres. The first map λ includes fibrewise diffeomorphisms of $D^{n-k} \times D^k$ into all diffeomorphisms, and the second C uses a diffeomorphism of $D^n \subset S^n$ as a datum to clutch two $(n+1)$–disks and make a homotopy sphere.[1]

Lemma 2.1 *For any $[\psi] \in \pi_{n-k}\mathrm{Diff}_\partial(D^k)$, the homotopy sphere $A([\psi]) \in \Theta_{n+1}$ admits an embedding into $\mathbb{R}^{n+k+1}$ whose normal bundle is classified* (*up to possible sign*) *by $d_*([\psi]) \in \pi_n\mathrm{SO}_k \cong \pi_{n+1}B\mathrm{SO}_k$.*

We will offer two proofs of this result; one by an explicit geometric construction and a more abstract one by the classification of smoothings through Rourke and Sanderson's theory of block bundles.

Next we recall that in [16, Section 1], Milnor constructed exotic spheres by plumbing linear disk bundles and taking the boundary sphere; this construction gives rise to a pairing

$$\sigma_M\colon \pi_p\mathrm{SO}_q \otimes \pi_q\mathrm{SO}_p \to \Theta_{p+q+1}.$$

[1]The map C is denoted by Σ in [8].

By [3, Proposition 3.1], the pairing of Antonelli, Burghelea and Kahn refines this pairing in the sense that we have a commutative diagram

$$\begin{tikzcd}
\pi_p \mathrm{SO}_{q-a} \otimes \pi_q \mathrm{SO}_{p-b} \arrow[r, "\sigma"] \arrow[d] & \pi_{a+b+1} \mathrm{Diff}_\partial(D^{p+q-a-b-1}) \arrow[d, "A"] \\
\pi_p \mathrm{SO}_q \otimes \pi_q \mathrm{SO}_p \arrow[r, "\sigma_M"] & \Theta_{p+q+1}
\end{tikzcd} \tag{1}$$

where the map on the left is the tensor product of the canonical stabilisations. We now consider the homotopy 17–sphere

$$\Sigma_{\xi,\xi} := A(\sigma(\xi,\xi)),$$

recalling that $\xi \in \pi_8 \mathrm{SO}_6 \cong \mathbb{Z}/24$ denotes a generator. By the commutativity of (1) and the definition of σ_M, $\Sigma_{\xi,\xi}$ is the boundary of an 8–connected compact 18–manifold and so by a recent result of Burklund and Senger [7, Theorem 1.4] its image under the normal invariant map $\Theta_{17} \to \mathrm{coker}(J_{17})$ must be either 0 or $[\eta\eta_4]$. We will show:

Lemma 2.2 *The homotopy sphere $\Sigma_{\xi,\xi}$ represents $[\eta\eta_4] \in \mathrm{coker}(J_{17})$.*

We deduce from this that every embedding $\Sigma_{\xi,\xi} \hookrightarrow S^{28}$ has a nontrivial normal bundle. Indeed, recall that the map $\Theta_{17} \to \mathrm{coker}(J_{17})$ is obtained by embedding a homotopy 17–sphere into some euclidean space with trivial normal bundle, and performing the Pontryagin–Thom collapse as to obtain an element in π^s_{17} which is well-defined modulo the image of J. Now, assuming by contradiction that $\Sigma_{\xi,\xi}$ embeds into S^{28} with trivial normal bundle, then $[\eta\eta_4]$ would have a representative in $\pi_{28} S^{11}$. However, this contradicts the computations of Toda [21, Theorem 12.17 and Proposition 12.20] on the stabilisation map $\pi_{28} S^{11} \to \pi^s_{17}$, which we display below:

$$\begin{tikzcd}[column sep=small]
\pi_{28} S^{11} \arrow[d] & = & \mathbb{Z}/2((\eta^2\rho)_{11}) \arrow[d] & \oplus & \mathbb{Z}/2((\mu_{17})_{11}) \arrow[d] & \oplus & \mathbb{Z}/2((\nu\kappa)_{11}) \arrow[d] & & \\
\pi^s_{17} & = & \mathbb{Z}/2(\eta^2\rho) & \oplus & \mathbb{Z}/2(\mu_{17}) & \oplus & \mathbb{Z}/2(\nu\kappa) & \oplus & \mathbb{Z}/2(\eta\eta_4)
\end{tikzcd}$$

Here the notation is such that an element $(\delta)_{11}$ stabilises to δ and the stable class $\eta^2\rho$ generates $\mathrm{im}(J_{17} \colon \pi_{17}(\mathrm{SO}) \to \pi^s_{17})$. Lemma 2.1 then implies that $d_*(\sigma(\xi,\xi)) \neq 0$, which concludes the proof of Theorem 1.1, modulo Lemmas 2.1 and 2.2.

To prove Corollary 1.2 we note that by Lemma 2.1 the normal bundle $\nu(\Sigma_{\xi,\xi} \subset S^{28})$ has clutching function $\pm d_*(\sigma(\xi,\xi)) \in \pi_{16}\mathrm{SO}_{11}$ and $d_*(\sigma(\xi,\xi)) \neq 0$ by Theorem 1.1.

Moreover, $d_*(\sigma(\xi,\xi))$ maps to $0 \in \pi_{16}\mathrm{SPL}_{11}$; this is explained following Theorem 1.1 using the exact sequence $\pi_{*+1}(\mathrm{PL}_{11}/O_{11}) \xrightarrow{d_*} \pi_*(\mathrm{SO}_{11}) \longrightarrow \pi_*(\mathrm{SPL}_{11})$. Now by Antonelli [2], the normal bundle of every homotopy 17–sphere embedded in euclidean space in codimension 12 is zero. Hence

$$\nu(\Sigma_{\xi,\xi} \subset S^{28}) \in \ker(\pi_{17}B\mathrm{SO}_{11} \to \pi_{17}B\mathrm{SO}_{12}) = \mathrm{im}(\pi_{17}S^{11} \to \pi_{17}B\mathrm{SO}_{11}),$$

where the last map is induced by the classifying map of the tangent bundle of the 11–sphere.

It remains to prove Lemmas 2.1 and 2.2.

Proof of Lemma 2.1 Choose a smooth map $\psi\colon D^{n-k} \times D^k \to D^k$ representing the class $[\psi] \in \pi_{n-k}(\mathrm{Diff}_\partial(D^k))$, ie for $x \in D^{n-k}$ we have that $\psi_x := \psi(x,-) \in \mathrm{Diff}_\partial(D^k)$, and $\psi_x = \mathrm{Id}_{D^k}$ for $x \in \partial D^{n-k}$. Then $\lambda([\psi])$ is represented by

$$\Psi\colon D^k \times D^{n-k} \to D^k \times D^{n-k}, \quad (x,y) \mapsto (x, \psi(x,y)),$$

and $A([\psi])$ is represented by the homotopy sphere Σ_Ψ^{n+1} obtained by gluing two copies of D^{n+1} along the boundary using the diffeomorphism Ψ. Note also that the image of $[\psi]$ under the derivative map is represented by $d\psi\colon D^{n-k} \times D^k \to \mathrm{Gl}_k(\mathbb{R})$ with $d\psi(x,y) = D_y\psi_x$.

For technical reasons, we actually assume without loss of generality that the maps are the identity maps in a neighbourhood of the boundaries.

We construct an explicit embedding $\iota_\psi\colon \Sigma_\Psi^{n+1} \hookrightarrow S^{n+k+1}$ of Σ_Ψ^{n+1}, compute the normal bundle of this embedding and show explicitly that it is obtained by clutching with $d\psi$.

As might be expected, given that all our data is on disks (and trivial near the boundary of the disks), we actually produce an interesting embedding ι_ψ of $D^{n+1} = D^{n-k} \times D^k \times [0,1]$ into $D^{n+k+1} = D^{n-k} \times D^k \times D^k \times [0,1]$, which has standard form near the boundary, and then obtain an embedding of Σ_Ψ^{n+1} by gluing with a standard embedding of D^{n+1} into D^{n+k+1} in the appropriate way.

The desired embedding ι_ψ is explicitly given by

$$\begin{aligned} \iota_\psi\colon D^{n-k} \times D^k \times [0,1] &\to D^{n-k} \times D^k \times D^k \times [0,1], \\ (x,y,t) &\mapsto (x, \alpha(t)y, \beta(t)\psi_x(y), t). \end{aligned}$$

Here, $\alpha, \beta\colon [0,1] \to [0,1]$ are smooth maps such that $\alpha(t) = 1$ for $t < 0.6$ and $\alpha(t) = 0$ for $t > 0.9$, and $\beta(t) = \alpha(1-t)$.

This is evidently a smooth embedding whose image we denote by S_ψ, and we let $\partial S_\psi = \iota_\psi(\partial D^{n+1})$. Then $\partial S_\psi \subset \partial D^{n+k+1}$: to see this, observe that if either the x– or the t–coordinate is in the boundary, then the first or fourth coordinate of the image point is so, too. For each $t \in [0,1]$, then $\alpha(t) = 1$ or $\beta(t) = 1$. If $y \in \partial D^k$, then therefore either the second or the third component of the image point is in the boundary (or both). As $\partial(D^{n-k} \times D^k \times [0,1])$ is the union of those points with at least one component in the boundary, this proves the claim.

We also note that the subset $\partial S_\psi \subset \partial D^{n+k+1}$ is in fact independent of ψ (as ψ is fixed to be the identity map near the boundary). Let us identify this image set ∂S_ψ with $S^n = \partial(D^{n-k} \times D^k \times [0,1])$ via the restriction of ι_{Id} to $\partial(D^{n-k} \times D^k \times [0,1])$.

Then $\iota_\psi|\colon \partial(D^{n-k} \times D^k \times [0,1]) \to \partial(D^{n-k} \times D^k \times D^k \times [0,1])$ is supported on the disk $D^{n-k} \times D^k \times \{1\}$, where it is given by Ψ. Therefore, we can glue two copies of $D^{n-k} \times D^k \times D^k \times [0,1]$ along the boundary by the identity map to obtain S^{n+k+1}, and the embeddings ι_ψ in one copy and ι_{Id} in the other glue together to form the desired embedding of Σ^{n+1}_Ψ into S^{n+k+1}.

Strictly speaking, one has to round the corners off to get an actual smooth embedding. This can easily be achieved, as ψ is the identity in a neighbourhood of the boundaries. We omit spelling out the somewhat cumbersome details.

It remains to compute the normal bundle of the embedding. To do this, we first compute the differential

$$D\iota_\psi\colon (D^{n-k} \times D^k \times [0,1]) \times (\mathbb{R}^{n-k} \oplus \mathbb{R}^k \oplus \mathbb{R}) \to S_\psi \times (\mathbb{R}^{n-k} \oplus \mathbb{R}^k \oplus \mathbb{R}^k \oplus \mathbb{R})$$

to be given in each fibre by

$$D_{(x,y,t)}\iota_\psi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha(t) & \alpha'(t)y \\ \beta(t)\,\partial_x\psi & \beta(t)\,d\psi & \beta'(t)\psi_x(y) \\ 0 & 0 & 1 \end{pmatrix}.$$

We obtain an explicit trivialisation of the normal bundle of this embedding via the fibrewise linear map covering ι_ψ,

$$\nu\colon D^{n-k} \times D^k \times [0,1] \times \mathbb{R}^k \to S_\psi \times (\mathbb{R}^{n-k} \oplus \mathbb{R}^k \oplus \mathbb{R}^k \oplus \mathbb{R}),$$

$$\nu_{(x,y,t)} = \begin{pmatrix} 0 \\ -\beta(t)(d\psi)^{-1} \\ \alpha(t) \\ 0 \end{pmatrix}.$$

To observe that this really describes the normal bundle, for dimension reasons we just have to check that the image of ν intersects the tangent bundle of S_ψ, ie the image of $D\iota_\psi$, trivially. It is clear that $\nu(v)$ can only be equal to a tangent vector of the form $(0, \alpha(t)w, \beta(t)\, d\psi(w), 0)$ for $w \in \mathbb{R}^k$. This implies $\alpha(t)v = \beta(t)\, d\psi(w)$ and $-\beta(t)v = \alpha(t)\, d\psi(w)$; the two equations imply $\alpha(t)^2 v = -\beta(t)^2 v$ and finally (as $\alpha(t)^2 + \beta(t)^2 > 0$) then $v = 0$ and then also $w = 0$. It follows that the image of ν represents the normal bundle of S_ψ in D^{n+k+1}.

For the other half-disk which produces the embedding of Σ^{n+1}_Ψ into S^{n+k+1}, we obtain a trivialisation of the normal bundle by the same recipe, replacing ψ by Id. We observe then that we obtain the global normal bundle by gluing these two explicitly chosen normal subbundles of TD^{n+k+1} along the boundary, where they coincide. The trivialisations differ precisely on the half-disk $\iota(D^{n-k} \times D^k \times \{1\})$, and there they differ by the derivative map $d\psi$. On the other half-disk, the two trivialisations coincide.

Consequently, the normal bundle of the embedding ι_ψ is obtained by clutching with $d\psi$, precisely as claimed, and the lemma is proved. $\square$

Remark 2.3 It is tempting to hope that the explicit geometric construction of d_* as the normal bundle of the embedding ι_ψ can be used to get some new information about d_*. On the other hand, the information obtained by the formulas in the proof given above seems rather limited. At least in the case where ψ lies in the image of σ, we present Conjecture 3.1 below on $d_* \circ \sigma$.

Proof of Lemma 2.2 Let A^{18}_8 denote the group of bordism classes relative boundary of 8–connected 18–manifolds with boundary a homotopy sphere, which are defined in [23, Section 17]. Specifically, elements of A^{18}_8 are represented by compact oriented 8–connected 18–manifolds W with boundary a homotopy sphere, and W_1 is bordant to W_2 if there is an h–cobordism Y between their boundaries such that the closed manifold $W_1 \cup Y \cup -W_2$ bounds an 8–connected 19–manifold. According to [23, Section 17] and [22, Theorem 2(5)], we have an isomorphism

$$A^{18}_8 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2, \quad [W] \mapsto (\Phi(\varphi_W), \varphi_W(\hat{\chi}_W)). \tag{2}$$

Here $\varphi_W \colon H_9(W;\mathbb{Z}) \to \mathbb{Z}/2$ is a quadratic refinement of the mod 2 intersection form defined as follows. By [22, Lemma 2], representing an integral homology class by an embedded sphere and taking its normal bundle gives rise to a quadratic map

$$\alpha_W \colon H_9(W;\mathbb{Z}) \to \pi_8 \mathrm{SO}_9.$$

Since the stabilisation map $S\colon \pi_8\mathrm{SO}_9 \to \pi_8\mathrm{SO} = \mathbb{Z}/2$ is split surjective with kernel $\mathbb{Z}/2$, from α_W we obtain a quadratic map $\varphi_W\colon H_9(W;\mathbb{Z}) \to \mathbb{Z}/2$ with values in $\mathbb{Z}/2 = \ker(S)$ by fixing a splitting of $\pi_8\mathrm{SO}_9$. The first component of (2) is the Arf invariant of φ_W and we next define the second component. Let $S\alpha_W\colon H_9(W;\mathbb{Z}) \to \mathbb{Z}/2 = \pi_8(\mathrm{SO})$ be the composition of α_W with the stabilisation map S above. Using [22, Lemma 2] again, we see that $S\alpha_W$ is a homomorphism. Define $\chi_W \in H_9(W;\mathbb{Z}/2)$ to be the Poincaré dual of

$$S\alpha_W \in \mathrm{Hom}(H_9(W;\mathbb{Z}), \mathbb{Z}/2) = H^9(W;\mathbb{Z}/2) \cong H^9(W, \partial W;\mathbb{Z}/2).$$

The second component of (2) is given by evaluating φ_W on any integral lift $\hat{\chi}_W$ of χ_W.

Let $S^3(\xi) \in \pi_8\mathrm{SO}_9$ be the image of $\xi \in \pi_8\mathrm{SO}_6$ under the inclusion $\mathrm{SO}_6 \to \mathrm{SO}_9$. By the commutativity of (1), $\Sigma_{\xi,\xi}$ is the boundary of the Milnor plumbing W of $S^3(\xi) \in \pi_8\mathrm{SO}_9$ with itself, and we compute $\varphi_W(\hat{\chi}_W)$ as follows: with $H_9(W;\mathbb{Z}) = \mathbb{Z}(x) \oplus \mathbb{Z}(y)$ the normal bundles obtained from representing x and y by embeddings are both given by $S^3(\xi)$. We conclude that $\varphi_W(x) = \varphi_W(y)$. Moreover, we may use that in this basis the intersection form λ_W of W has matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Furthermore, $S^3(\xi)$ stabilises to a generator of $\pi_8\mathrm{SO}$, by Lemma 2.4 below. Thus, $S\alpha_W$ maps both x and y to a generator and so we may take $\hat{\chi}_W = x + y$. We now compute that

$$\varphi_W(\hat{\chi}_W) = \varphi_W(x+y) = \underbrace{\varphi_W(x) + \varphi_W(y)}_{=0} + \rho_2(\lambda_W(x,y)) = 1,$$

where ρ_2 denotes reduction mod 2.

Now, taking the homotopy sphere on the boundary defines a homomorphism

$$\partial\colon A_8^{18} \to \Theta_{17}. \tag{3}$$

From the short exact sequence

$$0 \to bP_{18}(=\mathbb{Z}/2) \to \Theta_{17} \to \mathrm{coker}(J_{17}) \to 0$$

and [7, Theorem 1.4], we see that the image of the map ∂ from (3) consists of precisely 4 different elements, so the map ∂ is injective. Each of the bP–spheres is the boundary of a manifold P which satisfies $S\alpha_P = 0$ and therefore $\varphi_P(\hat{\chi}_P) = 0$ and it follows that the element W from above must map under ∂ to a non-bP–sphere, which then represents $[\eta\eta_4]$ in view of [7, Theorem 1.4]. □

Lemma 2.4 *The map* $\mathbb{Z}/24 \cong \pi_8\mathrm{SO}_6 \to \pi_8\mathrm{SO} \cong \mathbb{Z}/2$ *is surjective.*

Proof By [14, Theorem 1.4], $(\mathbb{Z}/2)^3 \cong \pi_8\mathrm{SO}_8 \to \pi_8\mathrm{SO} \cong \mathbb{Z}/2$ is onto and therefore has a kernel of 4 elements. (We refer to [12] for the computation of the relevant homotopy groups.) On the other hand, $(\mathbb{Z}/2)^2 \cong \pi_8\mathrm{SO}_7 \to \pi_8\mathrm{SO}_8$ is injective (its cokernel injects into $\pi_8(S^7) \cong \mathbb{Z}/2$) and so has an image of 4 elements. These two subgroups do not coincide: Since the maximal number of pointwise linearly independent vector fields on S^9 is 1 [1, Theorem 1.1], the tangent bundle of S^9 defines an element in $\pi_8\mathrm{SO}_8$ that is not in the image of $\pi_8\mathrm{SO}_7$ but maps to $0 \in \pi_8\mathrm{SO}$.

Therefore, $(\mathbb{Z}/2)^2 \cong \pi_8\mathrm{SO}_7 \to \pi_8\mathrm{SO}$ is surjective and has a kernel of precisely two elements; similarly the image of $\mathbb{Z}/24 \cong \pi_8\mathrm{SO}_6 \to \pi_8\mathrm{SO}_7 \cong (\mathbb{Z}/2)^2$ consists of precisely two elements (its cokernel injects into $\pi_8 S^6 \cong \mathbb{Z}/2$), and we are left to show that these two subgroups do not agree. To see this, we consider the element $a := (2\gamma)_7\eta_7$ where $(2\gamma)_7$ is a generator of $\mathbb{Z} \cong \pi_7\mathrm{SO}_7$ and $\eta_7\colon S^8 \to S^7$ is the nontrivial class: By [14, Theorem 1.4], $(2\gamma)_7$ stabilises to an element divisible by 2 and so a is in the kernel of the stabilisation; and it does not lift to $\pi_8\mathrm{SO}_6$ by the commutativity of the following diagram with exact rows:

$$\begin{tikzcd}
 & \pi_7\mathrm{SO}_7 \arrow[r] \arrow[d, "\eta_7"] & \pi_7 S^6 \arrow[r] \arrow[d, "\eta_7", "\cong"'] & \pi_6\mathrm{SO}_6\ (=0) \\
\pi_8\mathrm{SO}_6 \arrow[r] & \pi_8\mathrm{SO}_7 \arrow[r] & \pi_8 S^6 &
\end{tikzcd}$$

□

We conclude this section by giving the promised second proof of Lemma 2.1. To this end we recall from [18, Section 6] that a smoothing of S^{n+1} in S^{n+k+1} consists of a smooth manifold W and a PL homeomorphism $H\colon W \to S^{n+k+1}$, such that $\Sigma := H^{-1}(S^{n+1}) \subset W$ is a smooth submanifold, and such that H is concordant to the identity smoothing of S^{n+k+1}; and recall the group $\eth^k_{n+1}$ of concordance classes of such smoothings.[2] We note that up to diffeomorphism, W is a standard sphere mapping to S^{n+k+1} by a PL homeomorphism concordant to the identity, so that elements of $\eth^k_{n+1}$ are represented by PL homeomorphisms $H\colon S^{n+k+1} \to S^{n+k+1}$ which are concordant to the identity (ie orientation-preserving). Note also that Σ is a homotopy $(n+1)$–sphere, oriented through the PL homeomorphism $h := H|_\Sigma$, which is smoothly embedded into S^{n+k+1}.

[2] The group $\eth^k_{n+1}$ is denoted by Γ^k_{n+1} in [18]. We have used different notation, to avoid confusion with the notation Γ^{n+1}_k for the subgroups of the Gromoll filtration.

There are two obvious homomorphisms out of $\mathfrak{d}^k_{n+1}$,

$$\Theta_{n+1} \xleftarrow{F} \mathfrak{d}^k_{n+1} \xrightarrow{\nu} \pi_{n+1} B\mathrm{SO}_k,$$

the left one mapping the class of H to the diffeomorphism class of Σ, and the right one to the classifying map of the normal bundle of $\Sigma \subset S^{n+k+1}$ (where, as usual, we identify a homotopy sphere up to homotopy equivalence with a standard sphere using the given orientation). Then, Lemma 2.1 is clearly implied by the following result:

Lemma 2.5 *There exists a group homomorphism $B\colon \pi_{n-k}\mathrm{Diff}_\partial(D^k) \to \mathfrak{d}^k_{n+1}$ such that the following diagram commutes up to possible signs*:

$$\begin{tikzcd}
 & \pi_{n-k}\mathrm{Diff}_\partial(D^k) \arrow[r, "d_*"] \arrow[dl, "A"'] \arrow[d, "B"] & \pi_n\mathrm{SO}_k \arrow[d, "\cong"] \\
\Theta_{n+1} & \mathfrak{d}^k_{n+1} \arrow[l, "F"'] \arrow[r, "\nu"] & \pi_{n+1} B\mathrm{SO}_k
\end{tikzcd}$$

Proof We recall the homotopy equivalence

$$M_k\colon \mathrm{Diff}_\partial(D^k) \to \Omega^{k+1}\mathrm{PL}_k/\mathrm{SO}_k$$

of Morlet (see [5, Theorem 4.4]) and consider the diagram

$$\begin{tikzcd}
\pi_0\mathrm{Diff}_\partial(D^n) \arrow[dd, "(M_n)_*"] \arrow[ddd, bend right, "C"', "\cong"] & \pi_{n-k}\mathrm{Diff}_\partial(D^k) \arrow[l, "\lambda"'] \arrow[r, "d_*"] \arrow[d, "(M_k)_*"] & \pi_n\mathrm{SO}_k \arrow[d, equal] \\
 & \pi_{n+1}\mathrm{PL}_k/O_k \arrow[r, "\partial"] \arrow[dl, "S"'] \arrow[d, "\widetilde{i}"] & \pi_n\mathrm{SO}_k \arrow[d, equal] \\
\pi_{n+1}\mathrm{PL}/O & \pi_{n+1}\widetilde{\mathrm{PL}}_k/O_k \arrow[l, "\widetilde{S}"'] \arrow[r, "\partial"] & \pi_n\mathrm{SO}_k \\
\Theta_{n+1} \arrow[u, "\Psi"', "\cong"] & \mathfrak{d}^k_{n+1} \arrow[l, "F"'] \arrow[r, "\nu"] \arrow[u, "\cong"'] & \pi_{n+1} B\mathrm{SO}_k \arrow[u, "\cong"']
\end{tikzcd} \tag{4}$$

Here Ψ is the map which sends a homotopy sphere Σ to the element represented by the tangent PL microbundle of the mapping cylinder $\mathrm{cyl}(h\colon \Sigma \to S^{n+1})$ of an orientation-preserving PL homeomorphism h, along with its linear structure induced by the smooth structure of Σ on the Σ end of the cylinder and its canonical trivialisation at the S^{n+1} end. The map Ψ is an isomorphism by surgery theory; see eg [15, Theorem 6.48]. The map $\mathfrak{d}^k_{n+1} \to \pi_{n+1}\widetilde{\mathrm{PL}}_k/O_k$ is an isomorphism by [18, Corollary 6.7]: it is defined by sending the class of $(H,h)\colon (S^{n+k+1},\Sigma^{n+1})\to(S^{n+k+1},S^{n+1})$ to the normal block bundle ν_{cyl} of $\mathrm{cyl}(h)$ inside $\mathrm{cyl}(H)$ along with its linear reduction at the Σ^{n+1} end of the cylinder and its canonical trivialisation at the other end. Finally, the map $\widetilde{S}$ is obtained from the fact that the inclusion $\mathrm{PL}\to\widetilde{\mathrm{PL}}$ is an equivalence [19, Corollary 5.5(ii)]; that is, there is no essential difference between stable PL (micro)bundles and stable block bundles.

We claim that the lower left square of (4) is commutative up to sign. To see this, we may assume, increasing k if necessary, that the normal block bundle ν_{cyl} is given by a PL microbundle. Then, the sum of the two composites, applied to $[(H,h)]$, is represented by the direct sum microbundle $T\mathrm{cyl}(h)\oplus\nu_{\mathrm{cyl}}$ over $\mathrm{cyl}(h)$ along with its linear reduction at the front end and its canonical trivialisation at the other end. But now, we have an isomorphism $T\mathrm{cyl}(h)\oplus\nu_{\mathrm{cyl}}\cong T\mathrm{cyl}(H)|_{\mathrm{cyl}(h)}$ of microbundles which extends isomorphisms $T\Sigma\oplus\nu_{\Sigma\subset S^{n+k+1}}\cong TS^{n+k+1}|_{\Sigma}$ and $TS^{n+1}\oplus\nu_{S^{n+1}\subset S^{n+k+1}}\cong TS^{n+k+1}|_{S^{n+1}}$ of vector bundles.

Since H is PL isotopic to the identity (being an orientation-preserving PL homeomorphism of the sphere), we conclude that the sum of the two composite maps, applied to $[(H,h)]$, represents the zero element.

All other parts of this diagram commute up to possible signs: the commutativity of the squares on the right and of the triangle in the middle follows from the definitions. That $M_{n*}\circ\lambda=S\circ M_{k*}$ follows from [8, Lemma 2.5], and that $M_{n*}=\Psi\circ C$ is proven in [8, Lemma 2.7]. The lemma now follows by a diagram chase. $\square$

3 Concluding remarks

In this section we discuss some of the background to our results and state a conjecture about the map $d_*\circ\sigma$.

(1) The homotopy fibre of $d\colon \mathrm{Diff}_\partial(D^k)\to\Omega^k\mathrm{SO}_k$ is the H–space $\mathrm{Diff}^{\mathrm{fr}}_\partial(D^k)$ of framing-preserving diffeomorphisms. It is the loop space of the classifying space $B\mathrm{Diff}^{\mathrm{fr}}_\partial(D^k)$, which features in the recent work of Kupers and Randal-Williams [13] on the rational homotopy groups of $\mathrm{Diff}_\partial(D^k)$. We see that d is rationally trivial because the Alexander trick implies that d becomes nullhomotopic after composition with the natural map $\Omega^k\mathrm{SO}_k\to\Omega^k\mathrm{SPL}_k$. It is well known that $(\mathrm{SO}_k)_{\mathbb{Q}}$ is Eilenberg–Mac Lane, detected by the suspensions of the rational Pontryagin classes and rational Euler class. Since these classes are defined on $(\mathrm{SPL}_k)_{\mathbb{Q}}$, it follows that $(\mathrm{SO}_k)_{\mathbb{Q}}$ is a homotopy

retract of $(\mathrm{SPL}_k)_{\mathbb{Q}}$. If $\Omega^k_0 X$ denotes the connected component of the constant map, then it follows that $(\Omega^k_0\mathrm{SO}_k)_{\mathbb{Q}} \simeq \Omega^k_0(\mathrm{SO}_k)_{\mathbb{Q}}$ is a homotopy retract of $(\Omega^k_0\mathrm{SPL}_k)_{\mathbb{Q}}$, showing that the map $d\colon \mathrm{Diff}_\partial(D^k) \to \Omega^k_0\mathrm{SO}_k$ is rationally nullhomotopic.

(2) The proof of Theorem 1.1 relies on the fact that the normal bundle of any embedding $\Sigma_{\xi,\xi} \hookrightarrow S^{28}$ is nontrivial. Despite the elementary argument we give for this in Section 2, computing the normal bundle of an embedding of a homotopy sphere $g\colon \Sigma^{n+1} \hookrightarrow S^{n+k+1}$ is a subtle problem. Provided one is in the metastable range $n < 2k-4$, Haefliger [10] proved that the isotopy class of g depends only on the diffeomorphism type of Σ, so that, in particular, the normal bundle is independent of the choice of embedding. Hsiang, Levine and Sczarba [11] proved that the latter statement holds even for $n < 2k-2$, defined the homomorphism

$$\phi^k_{n+1}\colon \Theta_{n+1} \to \pi_{n+1}B\mathrm{SO}_k, \quad \Sigma \mapsto \nu(\Sigma \subset S^{n+k+1}), \quad \text{where } n < 2k-2,$$

and proved that $\phi^{13}_{16} \neq 0$; ie the exotic 16–sphere embeds into S^{29} with nontrivial normal bundle. Then Antonelli [2] made a systematic study of normal bundles of homotopy spheres in the metastable range, which includes the statement that $\phi^{11}_{17} \neq 0$.

(3) Concerning A'Campo's claim that d_* vanishes for $i = 2$, we note that, since $\phi^{13}_{16} \neq 0$, Lemma 2.1 entails that if A'Campo's claim holds, then the exotic 16–sphere does not lie in the image of the map $A\colon \pi_2\mathrm{Diff}_\partial(D^{13}) \to \Theta_{16} = \mathbb{Z}/2$. This is consistent with computations we have made for the refined plumbing pairing

$$\sigma\colon \pi_8\mathrm{SO}_6 \otimes \pi_7\mathrm{SO}_8 \to \pi_2\mathrm{Diff}_\partial(D^{13}),$$

which show that $A \circ \sigma = 0$, even though $\sigma_M\colon \pi_8\mathrm{SO}_7 \otimes \pi_7\mathrm{SO}_8 \to \Theta_{16}$ is nontrivial, a statement which can be deduced from [20, Satz 12.1].

(4) Finally, we present a conjectural description of the homomorphism

$$d_* \circ \sigma\colon \pi_p\mathrm{SO}_{q-a} \otimes \pi_q\mathrm{SO}_{p-b} \to \pi_{p+q}\mathrm{SO}_{p+q-a-b-1}$$

in purely homotopy-theoretic terms.

Let $h\colon \pi_i\mathrm{SO}_j \to \pi_i(S^{j-1})$ be the map induced by the canonical projection $\mathrm{SO}_j \to S^{j-1}$. For maps $f\colon W \to X$ and $f\colon Y \to Z$ let $f * g\colon W * Y \to X * Z$ be their *join*. Let $\partial\colon \pi_{m+1}(S^k) \to \pi_m\mathrm{SO}_k$ denote the boundary map in the homotopy long exact sequence of the fibration $\mathrm{SO}_k \to \mathrm{SO}_{k+1} \to S^k$. For compactness, we use the notation

$$p' := p-b \quad \text{and} \quad q' := q-a$$

and let $\xi_1 \in \pi_p \mathrm{SO}_{q'}$ and $\xi_2 \in \pi_q \mathrm{SO}_{p'}$. Then we have

$$h(\xi_1) \in \pi_p(S^{q'-1}), \quad h(\xi_2) \in \pi_q S^{p'-1} \quad \text{and} \quad h(\xi_1) * h(\xi_2) \in \pi_{p+q+1}(S^{p'+q'-1}),$$

so that $\partial\big(h(\xi_1) * h(\xi_2)\big) \in \pi_{p+q} \mathrm{SO}_{p'+q'-1}$.

In addition, we have the J–homomorphisms

$$J_{p,q'} \colon \pi_p \mathrm{SO}_{q'} \to \pi_{p+q'} S^{q'} \quad \text{and} \quad J_{q,p'} \colon \pi_q \mathrm{SO}_{p'} \to \pi_{q+p'} S^{p'},$$

and we can suspend in the target of each of these to get the homomorphisms

$$\Sigma^a \circ J_{p,q'} \colon \pi_p \mathrm{SO}_{q'} \to \pi_{p+q} S^q \quad \text{and} \quad \Sigma^b \circ J_{q,p'} \colon \pi_q \mathrm{SO}_{p'} \to \pi_{p+q} S^p.$$

We then take compositions with the maps induced by ξ_i for $i = 1, 2$ and the inclusions $i_{p'} \colon \mathrm{SO}_{p'} \to \mathrm{SO}_{p'+q'-1}$ and $i_{q'} \colon \mathrm{SO}_{q'} \to \mathrm{SO}_{p'+q'-1}$. Hence we have homomorphisms

$$\bar{\xi}_{2*} \colon \pi_p \mathrm{SO}_{q'} \xrightarrow{\Sigma^a \circ J_{p,q'}} \pi_{p+q} S^q \xrightarrow{\xi_{2*}} \pi_{p+q} \mathrm{SO}_{p'} \xrightarrow{i_{p'*}} \pi_{p+q} \mathrm{SO}_{p'+q'-1},$$
$$\bar{\xi}_{1*} \colon \pi_q \mathrm{SO}_{p'} \xrightarrow{\Sigma^b \circ J_{q,p'}} \pi_{p+q} S^p \xrightarrow{\xi_{1*}} \pi_{p+q} \mathrm{SO}_{q'} \xrightarrow{i_{q'*}} \pi_{p+q} \mathrm{SO}_{p'+q'-1}.$$

Conjecture 3.1 *Up to sign, the homomorphism*

$$d_* \circ \sigma \colon \pi_p \mathrm{SO}_{q'} \otimes \pi_q \mathrm{SO}_{p'} \to \pi_{p+q} \mathrm{SO}_{p'+q'-1}$$

is given by

$$d_*(\sigma(\xi_1, \xi_2)) = \partial\big(h(\xi_1) * h(\xi_2)\big) + \bar{\xi}_{1*}(\xi_2) + \bar{\xi}_{2*}(\xi_1).$$

We briefly discuss Conjecture 3.1 in light of Theorem 1.1 and Corollary 1.2. For $\xi \in \pi_8 \mathrm{SO}_6$ a generator, $h(\xi) \in \pi_8 S^5 \cong \pi_3^s$ is again a generator and we choose ξ so that $h(\xi) = \nu_5$. Hence Conjecture 3.1 gives $d_*(\sigma(\xi, \xi)) = \partial(\nu_5 * \nu_5) + 2\bar{\xi}_*(\xi)$. Now $\pi_{16} S^8 \cong (\mathbb{Z}/2)^4$, which entails that $2\bar{\xi}_*(\xi) = 0$ and the proof of Corollary 1.2 shows that $d_*(\sigma(\xi, \xi)) = \partial(\nu_{11}^2)$. Since $\nu_5 * \nu_5 = \nu_{11}^2$, Conjecture 3.1 is consistent with Theorem 1.1 and Corollary 1.2, with both giving the same nonzero expression for $d_* \circ \sigma \colon \pi_8 \mathrm{SO}_6 \otimes \pi_8 \mathrm{SO}_6 \to \pi_{16} \mathrm{SO}_{11}$.

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Received: 6 May 2021
Revised: 12 January 2022

GEOMETRY & TOPOLOGY
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Geometry & Topology (ISSN 1465-3060 printed, 1364-0380 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

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GEOMETRY & TOPOLOGY

Volume 27 Issue 9 (pages 3387–3831) 2023