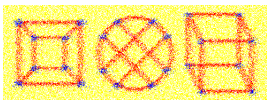

Journal of Graph Algorithms and Applications
http://jgaa.info/ vol. 26, no. 4, pp. 519–552 (2022)
DOI: 10.7155/jgaa.00606

# Monotonic Representations of Outerplanar Graphs as Edge Intersection Graphs of Paths on a Grid

*Eranda Çela*[1] *Elisabeth Gaar*[2]

[1]TU Graz, Steyrergasse 30, Graz A-8010, Austria
[2]Institute of Production and Logistics Management, Johannes Kepler University Linz, Altenberger Straße 69, Linz A-4040, Austria

| | | | |
|---|---|---|---|
| Submitted: | March 2022 | Reviewed: | August 2022 |
| Revised: | September 2022 | Accepted: | September 2022 |
| Final: | September 2022 | Published: | December 2022 |
| Article type: | Regular paper | Communicated by: | William Evans |

**Abstract.** In a representation of a graph $G$ as an edge intersection graph of paths on a grid (EPG) every vertex of $G$ is represented by a path on a grid and two paths share a grid edge iff the corresponding vertices are adjacent. In a monotonic EPG representation every path on the grid is ascending in both rows and columns. In a (monotonic) $B_k$-EPG representation every path on the grid has at most $k$ bends. The (monotonic) bend number $b(G)$ ($b^m(G)$) of a graph $G$ is the smallest natural number $k$ for which there exists a (monotonic) $B_k$-EPG representation of $G$.

In this paper we deal with the monotonic bend number of outerplanar graphs and show that $b^m(G) \leqslant 2$ holds for every outerplanar graph $G$. Moreover, we characterize the maximal outerplanar graphs and the cacti with (monotonic) bend number equal to 0, 1 and 2 in terms of forbidden induced subgraphs. As a byproduct we obtain low-degree polynomial time algorithms to construct (monotonic) EPG representations with the smallest possible number of bends for maximal outerplanar graphs and cacti.

**Keywords:** intersection graphs, paths on a grid, outerplanar graphs, cacti

## 1 Introduction and definitions

Edge intersection graphs of paths on a grid were introduced in 2009 by Golumbic, Lipshteyn and Stern [18]. A graph $G$ is called *an edge intersection graph of paths on a grid* (*EPG*) if there exists a rectangular grid with horizontal and vertical grid lines and a set of paths along the grid lines

The second author acknowledges support by the Austrian Science Fund (FWF): I 3199-N31.

*E-mail addresses:* cela@math.tugraz.at (Eranda Çela) elisabeth.gaar@jku.at (Elisabeth Gaar)

such that the vertices of $G$ correspond to the paths and two vertices are adjacent in $G$ if and only if the corresponding paths share a grid edge. In this case we say that the paths *intersect*. Such a representation of a graph is called an *EPG representation*.

An EPG representation is called a *$B_k$-EPG representation*, if every path has at most $k$ bends, for some $k \in \mathbb{N}$. A path on a grid is called *monotonic*, if it is ascending in both columns and rows. An EPG representation is called *monotonic* if every path is monotonic. A monotonic $B_k$-EPG representation is called a *$B_k^m$-EPG representation*. Furthermore a graph is called *$B_k$-EPG* or *$B_k^m$-EPG* if there is a $B_k$-EPG representation or a $B_k^m$-EPG representation, respectively. We denote by $B_k$ and $B_k^m$ the class of all graphs which are $B_k$-EPG and $B_k^m$-EPG, respectively. The *bend number* $b(G)$ and the *monotonic bend number* $b^m(G)$ of a graph $G$ are defined as the minimum $k \in \mathbb{N}$ and $\ell \in \mathbb{N}$ such that $G$ is in $B_k$ and $B_\ell^m$, respectively.

EPGs were initially motivated by applications in circuit layout design and chip manufacturing. For a detailed description of these applications and their relationship to EPGs we refer to [10, 18, 23]. Similar concepts which have been investigated in the literature include *edge intersection graphs of paths on a tree* (EPT), *vertex intersection graphs of paths on a tree* (VPT) and *vertex intersection graphs of paths on a grid* (VPG), see [1, 15, 16, 17].

Many research papers have dealt with (monotonic) EPG graphs since the introduction of the concept, see [2, 3, 4, 11, 13, 18, 19, 20, 21]. One of the well studied questions is the relationship between the classes $B_k$ and $B_k^m$, for $k \in \mathbb{N}$. Golumbic, Lipshteyn and Stern [18] showed that each graph is $B_k$-EPG and $B_\ell^m$-EPG for some $k, \ell \in \mathbb{N}$. Obviously $B_0 \subseteq B_1 \subseteq B_2 \subseteq \ldots$ and $B_0^m \subseteq B_1^m \subseteq B_2^m \subseteq \ldots$ hold. Heldt, Knauer and Ueckerdt [21] have shown that the first chain of inclusions above is strict, i.e. $B_k \subsetneqq B_{k+1}$ holds for every $k \geqslant 0$. Clearly $B_k^m \subseteq B_k$ holds for every $k$ and the relationships $B_0 = B_0^m$ and $B_0 \subseteq B_1^m$ are trivial. Golumbic, Lipshteyn and Stern conjectured in [18] that $B_1^m \subsetneqq B_1$ and this was confirmed in [11].

The recognition problem for the classes $B_k$ and $B_k^m$, $k \in \mathbb{N}$, has been investigated. It asks whether an input graph $G$ is $B_k$-EPG ($B_k^m$-EPG). The recognition problem for $B_0$ is solvable in linear time. This follows from the observation that a graph $G$ is $B_0$-EPG if and only if $G$ is an interval graph and from the fact that interval graphs can be recognized in linear time, see Booth and Lueker [7]. Heldt, Knauer and Ueckerdt [21] have proven that the recognition problem for $B_1$ is NP-complete. Cameron, Chaplick and Hoàng [11], have shown that the recognition problem for the class $B_1^m$ is NP-complete as well. Recently Pergel and Rzążewski [24] have settled the NP-completeness of the recognition problem also for the classes $B_2$ and $B_2^m$.

The computation of the (monotonic) bend number of a given graph is closely related to the recognition problem for the classes $B_k$ ($B_k^m$), $k \in \mathbb{N}$. The results on recognition problems mentioned above imply that the computation of the (monotonic) bend number $b(G)$ ($b^m(G)$) of a graph $G$ is a hard problem in general. Thus the identification of upper bounds on $b(G)$, $b^m(G)$ is a reasonable task. Biedl and Stern [4] have shown that $b(G) \leq 5$ holds for planar graphs. This upper bound was improved to 4 by Heldt, Knauer and Ueckerdt [20]. Moreover Heldt et al. [20] have also shown that $b(G) \leq 2$ holds if $G$ is outerplanar. This results holds also for Halin graphs as shown by Francis and Lahiri [14]. We strengthen the result of [20] and show that $b^m(G) \leq 2$ holds if $G$ is outerplanar.

Another question considered in the literature is the following: given a certain class of graphs and a natural number $k$, characterize the members of the class which belong to $B_k$ ($B_k^m$). Deniz, Nivelle, Ries and Schindl [12] provided a characterization of $B_1$-EPG split graphs for which there exists a $B_1$-EPG representation using only $L$-shaped paths on the grid. In this paper we provide characterizations of maximal outerplanar graphs and cacti with a given (monotonic) bend number.

The recognition problem for $B_k$ ($B_k^m$) and the characterization of $B_k$-EPG ($B_k^m$-EPG) graphs is

also relevant from an algorithmic point of view. Indeed several difficult problems from algorithmic graph theory turn out to be tractable on $B_k$-EPG graphs for a given $k \in \mathbb{N}$, see [6, 8, 9, 13].

**Organization of the paper.** In Section 2 we show that the monotonic bend number of outerplanar graphs is at most 2. In Section 3 we derive the (monotonic) bend number of the so-called $n$-sun graph, for $n \in \mathbb{N}$, $n \geqslant 3$. The results on $n$-sun graphs are relevant for the computation of the (monotonic) bend number of outerplanar graphs. In Section 4 and Section 5 we deal with maximal outerplanar graphs and cacti, respectively, and derive a full characterization for their membership in $B_0$, $B_1^m$, $B_1$ and $B_2^m$. We conclude with a summary and some open questions in Section 6.

**Terminology and notation.** The crossings of two grid lines are called *grid points*. The part of a grid line between two consecutive grid points is called a *grid edge* and the two consecutive grid points are the *vertices of the grid edge*. We distinguish between *horizontal grid edges* and *vertical grid edges*. A *path $P$ on a grid* is a sequence of grid points $\langle v_0, \ldots, v_i, \ldots, v_p \rangle =: P$ such that any two consecutive grid points are connected by a grid edge. The later are called *edges of the path $P$*. The grid point $v_0$ is called the *start point* of $P$ and $v_p$ is called the *end point* of $P$. A *bend point* $v_i$ of $P$, $i \notin \{0, p\}$, is a grid vertex such that one of the edges $(v_{i-1}, v_i)$ and $(v_i, v_{i+1})$ of $P$ is a horizontal grid edge and the other one is a vertical grid edge. The subpath $\langle v_{i-1}, v_i, v_{i+1} \rangle$ is called a *bend* on $P$. The part of a path $P$ between two consecutive bend points is called a *segment* of $P$. Also the parts of $P$ from the start point to the first bend point and from from the last bend point to the end point are called *segments*. A segment that consists of horizontal (vertical) edges is called *horizontal* (*vertical*) segment. We say that two paths on a grid *intersect*, if they have at least one common grid edge.

If a graph $G$ contains an induced subgraph isomorphic to a graph $H$, we consider that induced subgraph to be a *copy of $H$ in $G$* and say that *$G$ contains a copy of $H$*. Finally notice that when talking about an outerplanar graph $G$, we consider an arbitrary but fixed outerplanar embedding of $G$.

## 2 Outerplanar graphs are in $B_2^m$

Biedl and Stern [4] proved that every outerplanar graph is in $B_3$ and showed the existence of an outerplanar graph which does not belong to $B_1$. Furthermore, they conjectured that every outerplanar graph is in $B_2$. This conjecture was confirmed by Heldt, Knauer and Ueckerdt in [20] who proved that every graph with treewidth at most 2 is in $B_2$ and exploited the fact that every outerplanar graph has treewidth at most 2, see [5]. Thus 2 is a tight upper bound on the bend number of outerplanar graphs. We strengthen this result and show that all outerplanar graphs belong even to $B_2^m$.

To prove this result we construct a $B_2^m$-EPG representation of a given outerplanar graph $G = (V, E)$. Without loss of generality we assume $G$ to be connected (a $B_2^m$-EPG representation of an arbitrary graph can be obtained as the union of the disjoint $B_2^m$-EPG representations of its connected components). Our construction builds upon a so-called *nice labeling* of the vertices of $G$, which is a particular (not-unique) labeling of the vertices of an outerplanar graph obtained as follows. Consider an outerplanar embedding of $G$ and a new vertex $v_0 \notin V$. Let $n = |V|$ be the order of $G$. Consider some planar embedding of the planar graph $G' := (V', E')$ with $V' := V \cup \{v_0\}$ and $E' := E \cup \{\{v_0, i\} : i \in V\}$, where every edge in $E' \setminus E$ is drawn so as to start with a short straight line segment at $v_0$. Then label the vertices of $V$ by $v_1, v_2, \ldots, v_n$ such that the straight line segments of the edges $\{v_0, v_i\}$ are positioned around $v_0$ in counterclockwise order.

In the following we always consider an outerplanar graph with a nice labeling and identify its vertices with the corresponding labels.

We first observe that the nice labeling of a connected outerplanar graph fulfills two simple but useful properties, the *separation property* and the *path separation property*, defined below. Indeed, these properties are equivalent to each other.

**Definition 2.1** *Consider an outerplanar graph $G$ on $n$ vertices labeled by $(v_1, \dots, v_n)$. The labeling of the vertices is said to have the* separation property *iff for any edge $\{v_i, v_j\}$ in $G$ with $i < j$ the following holds: if $\{v_k, v_\ell\}$ is an edge in $G$ with $k, \ell \in \{1, 2, \dots, n\}$ and $\{k, \ell\} \cap \{i, j\} = \emptyset$, then either $i < k < j$ and $i < \ell < j$ hold, or $k \notin \{i, \dots, j\}$ and $\ell \notin \{i, \dots, j\}$ hold.*

*Analogously, the labeling of the vertices is said to have the* path separation property *iff for any path $P = (v_i = v_{p_1}, v_{p_2}, \dots, v_{p_r} = v_j)$ in $G$ with $i < j$ the following holds: if $\{v_k, v_\ell\}$ is an edge in $G$ with $k, \ell \in \{1, 2, \dots, n\}$ and $\{k, \ell\} \cap \{p_1, \dots, p_r\} = \emptyset$, then either $i < k < j$ and $i < \ell < j$ hold, or $k \notin \{i, \dots, j\}$ and $\ell \notin \{i, \dots, j\}$ hold.*

**Observation 2.2** *Let $G = (V, E)$ be a connected outerplanar graph with vertex set $V = \{1, 2, \dots, n\}$. Any nice labeling $(v_1, v_2, \dots, v_n)$ of the vertices has the separation property and the path separation property.*

**Proof:** Consider a planar embedding of $G'$ which gives rise to the nice labeling of the vertices of $G$ as described above. To prove the separation property consider an arbitrary edge $\{v_i, v_j\}$ in $G$ and any other edge $\{v_k, v_\ell\}$ in $G$ such that $k, \ell \in \{1, 2, \dots, n\}$ and $\{k, \ell\} \cap \{i, j\} = \emptyset$ hold. Then, $v_0, v_i, v_j$ close a cycle $C$ of length 3 in $G'$ and $v_k$ and $v_\ell$ have to be either both inside or both outside of $C$. In the first case $i < k < j$ and $i < \ell < j$ hold, in the second case $k \notin \{i, \dots, j\}$ and $\ell \notin \{i, \dots, j\}$ hold.

The path separation property can be shown by analogous arguments. □

Next we present Algorithm 2.1 which constructs a $B_2^m$-EPG representation of a given connected outerplanar graph $G$ with a given nice labeling of its vertices. The algorithm considers the vertices of $G$ one by one in a particular order. The subroutine Explore$(v, \dots)$ constructs a monotonic path on the grid with at most two horizontal segments and at most one vertical segment for every neighbor of $v$, for which no path has been constructed yet. The construction of the paths is done while maintaining the following invariant property at every call of Explore$(v_i, \dots)$ during the execution of the algorithm: the path $P_{v_i}$ on the grid corresponding to the green vertex $v_i$ which is currently being explored, has a *free* region $\mathcal{R}_{v_i}$ on the lower horizontal segment, i.e. a region where there is no intersection of $P_{v_i}$ with any of the paths constructed so far. The construction of the paths corresponding to the neighbors of $v_i$ in Explore$(v_i, \dots)$ is illustrated in Figure 1. There we highlight the free parts existing at the moment when Explore$(v_i, \dots)$ is called in light gray and the free parts of the paths constructed during Explore$(v_i, \dots)$ in dark gray.

The subroutine UpdateGraph$(v, \dots)$ deletes from $G$ the vertex $v$ which was explored at last together with its incident edges. Moreover, it deletes the edges connecting the neighbors of $v$ among each other, and also the edge connecting the last neighbor of $v$ to the green vertex with the second smallest label, if such an edge exist. Notice that the deleted edges are precisely the edges for which the corresponding intersections of paths on the grid have already been constructed during the last call of Explore$(v, \dots)$.

Next we show that Algorithm 2.1 is well-defined.

**Lemma 2.3** *Algorithm 2.1 is well-defined, i.e. the paths on the grid can be constructed as described in the algorithm and it terminates. Furthermore Algorithm 2.1 constructs exactly one path for each vertex of the graph.*

**Algorithm 2.1** Construct a $B_2^m$-EPG representation of a connected outerplanar graph

**Input:** A connected outerplanar graph $G = (V, E)$ with $n = |V|$ and $V = \{v_1, v_2, \ldots, v_n\}$ ordered according to a nice labeling
**Output:** A $B_2^m$-EPG representation of $G$

**procedure** B2M_OUTERPLANAR($G$)
  Set $color(v_i) :=$ *gray* for all $i \in \{1, 2, \ldots, n\}$
  Set $color(v_1) :=$ *green*, $ListGreen := \{v_1\}$
  Draw the path $P_{v_1}$ representing $v_1$ as a straight horizontal line on the grid
  Set $G' := G$, $V' := V$, $E' := E$
  **while** $ListGreen \neq \emptyset$ **do**
    Set $i := \min\{j : v_j \in ListGreen\}$
    Set $i^* := \inf\{j : v_j \in ListGreen, j \neq i\}$ (may be $\infty$)
    Set $\ell := \deg_{G'}(v_i)$
    Let $ListNeighbors := (v_{i_1}, v_{i_2}, \ldots, v_{i_\ell})$ be the list of the $\ell$ neighbors of $v_i$ in $G'$ such that $1 \leqslant i_1 < i_2 < \ldots < i_\ell \leqslant n$
    $color :=$ EXPLORE($v_i$, $E'$, *ListNeighbors*, $i^*$, *color*)
    $(V', E') :=$ UPDATEGRAPH($v_i$, $V'$, $E'$, *ListNeighbors*, $i^*$)
    $G' := (V', E')$
    $ListGreen := \{v_j \in V' : color(v_j) = green\}$
  **end while**
  **return**
**end procedure**

**procedure** EXPLORE($v_i$, $E'$, *ListNeighbors*, $i^*$, *color*)
  Set $color(v_{i_j}) :=$ *green* for all $j \in \{1, 2, \ldots, \ell\}$
  **for** $j = 1, \ldots, \ell - 1$ **do**
    **if** $j = 1$ **then**
      Construct the path $P_{v_{i_1}}$ as shown in Figure 1(a) ($P_{v_i}$ in the picture represents the lower horizontal segment of the already constructed path $P_{v_i}$)
    **else if** $\{v_{i_{j-1}}, v_{i_j}\} \notin E'$ **then**
      Construct the path $P_{v_{i_j}}$ as shown in Figure 1(b)
    **else**
      Construct the path $P_{v_{i_j}}$ as shown in Figure 1(c)
    **end if**
  **end for**
  **if** $i^* = \infty$ or $\{v_{i_\ell}, v_{i^*}\} \notin E'$ **then**
    Construct the path $P_{v_{i_\ell}}$ as shown in Figure 1(d), where the dotted line is drawn iff $\{v_{i_{\ell-1}}, v_{i_\ell}\} \in E'$ (If $i^* = \infty$, then $P_{v_{i^*}}$ is not there. Otherwise $P_{v_{i^*}}$ in the picture represents the lower horizontal segment of the already constructed path $P_{v_{i^*}}$)
  **else**
    Construct the path $P_{v_{i_\ell}}$ as shown in Figure 1(e), where the dotted line is drawn iff $\{v_{i_{\ell-1}}, v_{i_\ell}\} \in E'$
  **end if**
  **return** *color*
**end procedure**

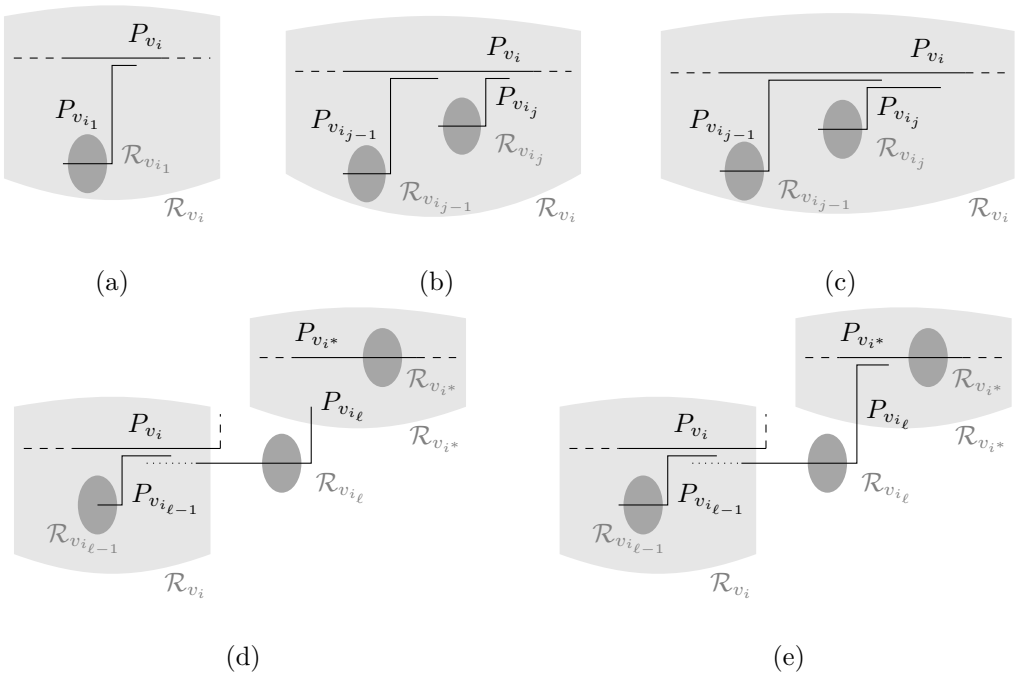



Figure 1: The constructions of the subroutine EXPLORE of Algorithm 2.1.

```
1: procedure UpdateGraph(v_i, V', E', ListNeighbors, i*)
2:     Set V' := V' \ {v_i}, E' := E' \ {{v_i, v_k}: {v_i, v_k} ∈ E'}
3:     for j = 1, ..., ℓ − 1 do
4:         if {v_{i_j}, v_{i_{j+1}}} ∈ E' then
5:             E' := E' \ {{v_{i_j}, v_{i_{j+1}}}}
6:         end if
7:     end for
8:     if {v_{i_ℓ}, v_{i*}} ∈ E' then
9:         E' := E' \ {{v_{i_ℓ}, v_{i*}}}
10:    end if
11:    return (V', E')
12: end procedure
```

**Proof:** Let $G = (V, E)$ be a connected outerplanar graph with vertex set $V = \{1, 2, \ldots, n\}$ and let $(v_1, v_2, \ldots, v_n)$ be a nice labeling of its vertices.

We first prove the first statement of this lemma. Observe first that whenever Algorithm 2.1 starts to explore a new vertex $v_i$ in line 11 all paths $P_v$ that correspond to some green vertex $v$ have a *free* part $\mathcal{R}_v$ on their lower horizontal segment, which is shared by no other path constructed so far. By construction these free parts are located in the grid in such a way that if $v_k$ and $v_j$ are two green vertices with $k < j$, then the right-most point of the free part of $P_{v_k}$ is located to the left and below the left-most point of the free part of $P_{v_j}$. Obviously, this property clearly holds when $v_1$ is explored and it is clearly maintained in the later constructions. In Figure 1 the free parts of the current iteration are highlighted in light gray and the free parts determined for the next iteration in dark gray. The paths $P_{v_i}$ and $P_{v_{i^*}}$ are located as depicted in Figure 1(d) and (e). Thus, EXPLORE($v_i$) and hence Algorithm 2.1 can really construct all the paths as described.

Observe further that the termination of the algorithm is clear: any explored vertex is deleted and there are only finitely many vertices in $G$.

Next we prove the second statement of the lemma. Algorithm 2.1 constructs a path $P_v$ corresponding to a vertex $v$ in the same call of EXPLORE as it colors $v$ green. Due to the connectivity of $G$ each vertex is colored green during the algorithm, so it constructs at least one path for each vertex. Thus, to prove the second statement of the lemma it is enough to show that whenever a vertex is colored green, is has not been colored green before. Equivalently, we show that whenever we explore a vertex $v_i$ in the subroutine EXPLORE all its current neighbors (i.e. all its neighbors in the current $G'$) are still colored *gray*. This is done in two steps. First we show that (i) all current neighbors of $v_i$ are contained in the set $\{v_1, v_2, \ldots, v_{i^*}\}$ whenever $i^* < \infty$. By construction $v_i$ and $v_{i^*}$ are the only green vertices in this set, so showing that (ii) $v_{i^*}$ is not a current neighbor of $v_i$ completes the proof.

Proof of (i). We show that if $i^* < \infty$ after executing line 10 of Algorithm 2.1, then *ListNeighbors* $\subseteq \{v_1, v_2, \ldots, v_{i^*}\}$. In other words and using the notation of line 10 we prove that $i_j \leqslant i^*$ holds for all neighbors $v_{i_j}$ of $v_i$ in $G'$. Indeed, assume by contradiction that there is a $j^*$ such that $i_{j^*} > i^*$. The vertex $v_{i^*}$ is colored green, hence it was colored green during the exploration of some vertex $x \neq v_i$ which was explored before $v_i$. The vertex $x$ itself was colored green during the exploration of some vertex $z$ and so on, until $v_1$, the very first vertex explored, is reached. Let $P = (v_1 = v_{p_1}, v_{p_2}, \ldots, v_{p_r} = v_{i^*})$ be a path in $G$ such that $v_{p_j}$ is colored green during the exploration of $v_{p_{j-1}}$ for every $2 \leqslant j \leqslant r$. Obviously $p_j \neq i$ and $p_j \neq i_{j^*}$ holds for every $1 \leqslant j \leqslant r$, because explored vertices are deleted, $v_i$ is explored in this iteration and and $v_{i_{j^*}}$ is in $G'$. But then, the path $P$ from $v_1$ to $v_{i^*}$ and the edge $\{v_i, v_{i_{j^*}}\}$ contradict the path separation property.

Proof of (ii). Next we prove $v_{i^*} \notin$ *ListNeighbors*. Assume for contradiction that $v_{i^*}$ is among the current neighbors of $v_i$ at some time during the algorithm, thus assume that $i_\ell = i^*$ holds. Consider the value of $i$ at the earliest occurrence of the above phenomenon. Let $P$ be the path from $v_1$ to $v_{i^*}$ as before. Analogously, let $Q = (v_1 = v_{q_1}, v_{q_2}, \ldots, v_{q_s} = v_i)$ be the path in $G$ such that $v_{q_j}$ is colored green during the exploration of $v_{q_{j-1}}$ for every $2 \leqslant j \leqslant s$. Both $P$ and $Q$ start in $v_1$ and end in different vertices. Moreover, for every vertex $x$ in $P$ or $Q$ there is exactly one vertex $y$ such that $x$ was colored green during the exploration of $y$ (because $i$ was chosen minimal with the corresponding property). Consequently there is an index $j'$ such that $p_j = q_j$ for all $j \leqslant j'$ and furthermore $p_j \neq q_k$ for all $j > j'$ and $k > j'$, i.e. the paths $P$ and $Q$ coincide for the first $j'$ vertices and then use different vertices.

First we observe that $p_j, q_k \notin \{i, i+1, \ldots, i^*\}$ for all $j < r$ and $k < s$, because otherwise the existence of the edge $\{v_i, v_{i^*}\}$ and the paths $(v_1 = v_{p_1}, v_{p_2}, \ldots, v_{p_{r-1}})$ and $(v_1 = v_{q_1}, v_{q_2}, \ldots, v_{q_{s-1}})$

would contradict the path separation property. Next due to the path separation property with the path $P$ from $v_1$ to $v_{i^*}$, the fact that $Q$ and $P$ use different vertices after $v_{p_{j'}} = v_{q_{j'}}$ and the fact that $q_r = i < i^*$ it follows that $q_k < i^*$. Therefore $q_k \leqslant i$ for all $j' < k \leqslant s$. Due to the separation property these indices have to be ascending, i.e. $q_k \leqslant q_{k+1}$ for all $j' < k \leqslant s-1$. Analogously, we have $i^* \leqslant p_{j+1} \leqslant p_j$ for all $j' < j \leqslant r-1$.

Clearly, when $v_{p_{j'}}$ is explored, both $v_{q_{j'+1}}$ and $v_{p_{j'+1}}$ are among the neighbors of $v_{p_{j'}}$, because they are colored green during its exploration. Furthermore $q_{j'+1} \leqslant i < i^* \leqslant p_{j'+1}$, hence $v_{q_{j'+1}}$ is explored before $v_{p_{j'+1}}$. By iteratively using this argument we get that all $v_{q_k}$, $j'+1 \leqslant k \leqslant s$, are explored before $v_{p_{j'+1}}$. However, when exploring $v_i = v_{q_s}$, $v_{i^*}$ had already been colored green. This is only possible if $p_{j'+1} = i^*$, and hence $v_{i^*}$ has already been colored green during the exploration of $v_{p_{j'}}$.

Now let us consider the exploration of $v_{q_{s-1}}$. At this point $q_{s-1}$ is the smallest green index. Furthermore the above arguments show that during the exploration of $v_{q_{s-1}}$ the vertex $v_{i^*}$ has already been colored green. Hence, the second lowest green index is less or equal to $i^*$. With (i) this implies that the index of any current neighbor of $v_{q_{s-1}}$ is less than or equal to $i^*$. Due to the separation property with the edge $\{v_i, v_{i^*}\}$, there can't be a neighbor of $v_{q_{s-1}}$ between $v_i$ and $v_{i^*}$, therefore $v_i$ is the current neighbor of $v_{q_{s-1}}$ with the largest index ( the actual $i_\ell$ in the algorithm). Note that $v_{i^*}$ can not be a neighbor of $v_{q_{s-1}}$ because $i$ was chosen minimal. Furthermore, $i^*$ has to be the green vertex with the second smallest index also during the exploration of $v_{q_{s-1}}$ (the actual $i^*$ in the algorithm), because if there would be another green vertex between $i$ and $i^*$, this vertex would remain green and during the exploration of $v_i$ this vertex and not $i^*$ would be the vertex with the second smallest index. But this implies that the edge between $v_i$ and $v_{i^*}$ is deleted in line 9 of UpdateGraph and therefore the edge $\{v_i, v_{i^*}\}$ is not present in $G'$ anymore when $v_i$ is explored, which contradicts the definition of $v_{i^*}$. □

Using Lemma 2.3 we obtain the following main result of this section.

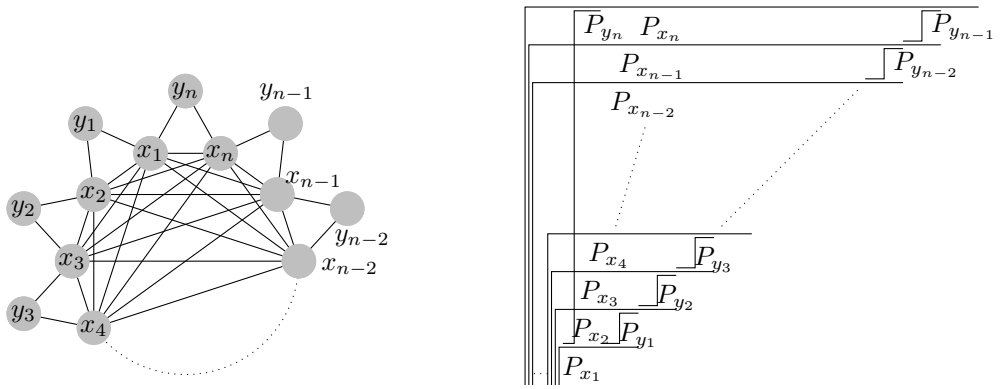
**Theorem 2.4** *Every outerplanar graph is in $B_2^m$.*

**Proof:** We show that Algorithm 2.1 constructs indeed a $B_2^m$-EPG representation of a connected outerplanar graph $G$ with a nice labeling of its vertices $V = \{1, 2, \ldots, n\}$. By Lemma 2.3 Algorithm 2.1 is well-defined and constructs exactly one path for each vertex. Clearly every path $P_v$ constructed by the algorithm hast at most two bends by construction.

What is left to show is that the paths $P_u$ and $P_v$ corresponding to two vertices $u$ and $v$ intersect iff $\{u, v\}$ is an edge in $G$. To prove this, consider the exploration of $v_i$ and the construction of $P_{v_{i_1}}, \ldots, P_{v_{i_\ell}}$. Clearly, the paths $P_{v_i}, P_{v_{i_1}}, \ldots, P_{v_{i_\ell}}$ intersect iff the corresponding edge is in the current graph $G'$ and is deleted in the subsequent update of $G'$. Furthermore none of the paths $P_{v_{i_1}}, \ldots, P_{v_{i_\ell}}$ intersects any path $P_v$ for $v \in V \setminus \{v_{i_1}, \ldots, v_{i_\ell}\}$, except for $v = v_i$. This is due to the fact that $P_{v_{i_1}}, \ldots, P_{v_{i_\ell}}$ are contained in the free part $\mathcal{R}_{v_i}$ of $P_{v_i}$. Hence during the exploration of $v_i$ there is a new intersection between two paths iff the corresponding edge is in the current graph $G'$, in which case that edge is deleted in the subsequent update of $G'$. These arguments hold in any exploration step and this completes the proof. □

A straightforward analysis of the time complexity of Algorithm 2.1 reveals that a $B_2^m$-EPG representation of an $n$-vertex outerplanar graph can be constructed in $O(n)$ time.

Finally, observe that 2 is a tight upper bound on the monotonic bend number of outerplanar graphs, because 2 is a tight upper bound on the bend number of outerplanar graphs as mentioned at the beginning of this section.




(a) (b)

Figure 2: (a) The graph $S_n$. (b) A $B_2^m$-EPG representation of $S_n$.

## 3 The (monotonic) bend number of the $n$-sun

The $n$-sun graph $S_n$ is a graph of order $2n$ defined as follows.

**Definition 3.1** *Let $n \in \mathbb{N}$, $n \geqslant 3$. The $n$-sun graph $S_n = (V, E)$ is the graph with vertex set $V = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ and edge set $E = E_1 \cup E_2$, where $E_1 = \{\{x_i, x_j\} \mid 1 \leqslant i < j \leqslant n\}$ and $E_2 = \{\{x_i, y_i\}, \{x_{i+1}, y_i\} \mid 1 \leqslant i < n\} \cup \{\{x_1, y_n\}, \{x_n, y_n\}\}$. The vertices $\{x_1, x_2, \ldots, x_n\}$ are called* central vertices *of $S_n$ and the edges between them, i.e. the edges in $E_1$, are called* central edges *of $S_n$.*

A picture of $S_3$ can be found in Figure 9 and $S_n$ is depicted in Figure 2 (a). Golumbic, Lipshteyn and Stern [18] have shown that $S_3$ is in $B_1$ and that $S_n$ is not in $B_1$ for every $n \geqslant 4$. Cameron, Chaplick and Hoàng [11] have shown that $S_3$ is not in $B_0$ and not in $B_1^m$. It can be easily checked that Figure 2(b) depicts a $B_2^m$-EPG representation of $S_n$ for any $n \geqslant 3$. This implies the following theorem.

**Theorem 3.2** *The $n$-sun $S_n$ is in $B_2^m$ for all $n \geqslant 3$.*

It is easily seen that the $B_2^m$-representation of $S_n$ in Figure 2(b) can be constructed in $O(n)$ time. We summarize the results for the $n$-sun as follows.

**Corollary 3.3** *The (monotonic) bend number of $S_n$, $n \geqslant 3$, is given as follows.*

$$b(S_n) = \begin{cases} 1 & \text{for } n = 3 \\ 2 & \text{for } n \geqslant 4 \end{cases} \qquad b^m(S_n) = 2 \text{ for } n \geqslant 3$$

Next, we recall the following definitions used in the investigation of $S_n$.

**Definition 3.4 (Golumbic, Lipshteyn and Stern [18])** *Consider some graph $G$ with a $B_1$-EPG representation and some grid edge $e$. The set of all paths which use $e$ in the $B_1$-EPG representation is called an* edge clique*. Consider a copy of the claw graph $K_{1,3}$ in the grid, i.e. a grid point together with three (arbitrarily selected but fixed) grid edges which have that grid point as a vertex. The set of all paths that use 2 edges of this copy of the claw $K_{1,3}$ is called a* claw

clique. *The grid point of degree* 3 *in the claw* $K_{1,3}$ *is called* central vertex of the claw *and* central grid point of the claw clique.

Clearly, the vertices of $G$ corresponding to the paths of an edge clique form a clique. Also, the vertices of $G$ corresponding to the paths of a claw clique form a clique. In fact, a converse statement is also true.

**Lemma 3.5 (Golumbic, Lipshteyn, Stern [18])** *Let $G$ be a $B_1$-EPG graph. Then in every $B_1$-EPG representation of $G$, the paths corresponding to the vertices of a maximal clique in $G$ form either an edge clique or a claw clique.*

In general we say that *a set $X$ of vertices in $G$ corresponds to an edge clique (a claw clique)* in a $B_1$-EPG representation of $G$ iff the paths corresponding to the vertices of $X$ build an edge clique (a claw clique) in the $B_1$-EPG representation.

We close this section with a simple but useful observation on $S_3$, that will be used in Section 4.

**Observation 3.6 (Biedl, Stern [4])** *In every $B_1$-EPG representation of the graph $S_3$ the set of the central vertices $\{x_1, x_2, x_3\}$ corresponds to a claw clique. The paths corresponding to different vertices contain different pairs of edges of the claw, hence the central (grid) point of the claw clique is a bend point for exactly two of the three paths corresponding to $\{x_1, x_2, x_3\}$. In this case we say that these two* paths are bent in the claw.

# 4 The (monotonic) bend number of maximal outerplanar graphs

In this section we determine the bend number and the monotonic bend number of maximal outerplanar graphs.

**Definition 4.1** *A graph $G$ is called maximal outerplanar if (a) $G$ is outerplanar and (b) joining any two non-adjacent vertices of $G$ by an additional edge yields a graph which is not outerplanar anymore.*

Notice that maximal outerplanar graphs are closely related to triangulations. Indeed, it is easy to see that maximal outerplanar graphs are exactly those outerplanar graphs, where the boundary of the outer face is a Hamiltonian cycle (thus containing all vertices) and every inner face is a triangle. We make use of the following auxiliary graph.

**Definition 4.2** *Let $G$ be a maximal outerplanar graph with an arbitrary but fixed outerplanar embedding. Then the almost-dual graph $\widehat{G}$ of $G$ has a vertex for every inner face of $G$. Two vertices of $\widehat{G}$ are adjacent if and only if the corresponding inner faces of $G$ share an edge in $G$.*

Clearly, the almost-dual $\widehat{G}$ of $G$ is an induced subgraph of the planar dual of $G$. Figure 3(a) represents a maximal outerplanar graph with 8 vertices and its almost-dual graph. Observe that in this case the almost-dual graph is a path. Indeed, the following observation is easy to see.

**Observation 4.3** *The almost-dual graph of any maximal outerplanar graph $G$ is a tree with maximum degree at most* 3. *This tree is a path iff $G$ does not contain $S_3$ as an induced subgraph, that is, $G$ is $S_3$-free.*

Finally, we use the following notation. For any vertex $\hat{v}$ of $\widehat{G}$ denote by $T_{\hat{v}}$ the subgraph of $G$ induced by the vertices on the border of the triangular face of $G$ associated to $\hat{v}$ and by $V(T_{\hat{v}})$ the corresponding set of vertices.

**Algorithm 4.1** Construct a $B_0$-EPG representation of an $S_3$-free maximal outerplanar graph $G$

**Input:** An $S_3$-free maximal outerplanar graph $G = (V, E)$ and its almost-dual path $\widehat{G}$
**Output:** A $B_0$-EPG representation of $G$

**procedure** $B_0$_$S_3$FREE_MAX_OUTERPLANAR($G$, $\widehat{G}$)
  Let $\hat{v}_1, \ldots, \hat{v}_\ell$ be the consecutive vertices of the path $\widehat{G}$
  Label one horizontal line of the grid with 1 and $\ell+1$ vertical lines of the grid with $1, 2, \ldots, \ell+1$
  Let $(i, 1)$ be the grid point where the one horizontal and the $i$-th vertical grid line intersect, $1 \leqslant i \leqslant \ell+1$
  **for** $v \in V$ **do**
    Draw the path $P_v$ from $(a, 1)$ to $(b+1, 1)$, where $\hat{v}_a$ is the first and $\hat{v}_b$ is the last vertex in the path $\widehat{G}$ such that $v \in V(T_{\hat{v}_a})$ and $v \in V(T_{\hat{v}_b})$
  **end for**
**end procedure**

## 4.1 Maximal Outerplanar Graphs in $B_0$

In this section we characterize maximal outerplanar graphs which belong to $B_0$. Cameron, Chaplick, Hoàng [11] showed that $S_3$ is not in $B_0$. Thus, any maximal outerplanar graph which is not $S_3$-free is not in $B_0$. We show by construction that the non-trivial converse of this statement is also true: any $S_3$-free maximal outerplanar graph is in $B_0$. We first prove that Algorithm 4.1 constructs a $B_0$-EPG representation of an $S_3$-free maximal outerplanar graph.

**Lemma 4.4** *Let $G$ be an $S_3$-free maximal outerplanar graph and let $\widehat{G}$ be its almost-dual graph. Then Algorithm 4.1 constructs a $B_0$-EPG representation of $G$.*

**Proof:** First notice that due to Observation 4.3 $\widehat{G}$ is a path as required by the input of Algorithm 4.1. Next observe that since $G$ is maximal outerplanar, each of its vertices is contained in at least one triangle and thus in at least one vertex of the path $\widehat{G}$. Hence Algorithm 4.1 draws exactly one path $P_v$ for every vertex $v$ of $G$. In order to prove the lemma we show that (i) for each vertex $w \in V(G)$ the set of vertices $\hat{v}$ of $\widehat{G}$, for which $w \in V(T_{\hat{v}})$ holds, build a subpath of $\widehat{G}$ and (ii) the paths on the grid $P_u$ and $P_v$ representing two vertices $u$ and $v$ of $G$ intersect iff $\{u, v\}$ is an edge in $G$.

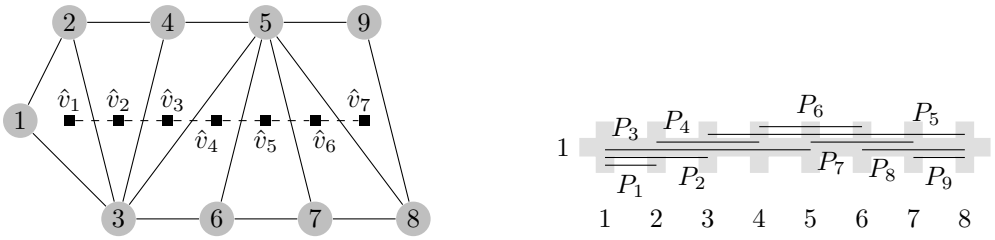
Proof of (i). Assume that (i) does not hold, and consider a $w \in V(G)$, for which there exist $j$ and $k \geqslant j+2$ with $w \in V(T_{\hat{v}_j})$ and $w \in V(T_{\hat{v}_k})$, but $w \notin V(T_{\hat{v}_i})$ for any $j+1 \leqslant i \leqslant k-1$. It is easy to see that for any two consecutive vertices $\hat{v}_i$ and $\hat{v}_{i+1}$ on the path $\widehat{G}$, the triangles $T_{\hat{v}_i}$ and $T_{\hat{v}_{i+1}}$ share two vertices in $G$, and every other vertex of $G$ is only in $T_{\hat{v}_r}$ with either $r \leqslant i$ or $r \geqslant i+1$. Thus, because $w \in V(T_{\hat{v}_j})$ and $w \notin V(T_{\hat{v}_{j+1}})$, $w$ can only be in $T_{\hat{v}_r}$ with $r \leqslant j$. Furthermore, because $w \in V(T_{\hat{v}_k})$ and $w \notin V(T_{\hat{v}_{k-1}})$, $w$ can only be in $T_{\hat{v}_r}$ with $r \geqslant k$, a contradiction.

Proof of (ii). Consider an edge $\{u, v\}$ of $G$. Clearly $\{u, v\}$ is part of at least one triangle $T_{\hat{v}_i}$ for some $i \in \{1, 2, \ldots, \ell\}$. This implies that $P_u$ and $P_v$ share the grid edge connecting $(i, 1)$ and $(i+1, 1)$. Hence $P_u$ and $P_v$ intersect.

Assume now that $P_v$ and $P_u$ intersect, so $P_u$ and $P_v$ share some grid edge $e$ from $(k, 1)$ to $(k+1, 1)$ for some $k \in \{1, 2, \ldots, \ell\}$. By construction, this implies that $v$ is in $V(T_{\hat{v}_a})$ for some $a \leqslant k$ and in $V(T_{\hat{v}_b})$ for some $b \geqslant k$. Then, (i) implies that $v \in V(T_{\hat{v}_k})$. Analogously, $u \in V(T_{\hat{v}_k})$ holds. Hence $u$ and $v$ are adjacent. □

With the help of Lemma 4.4 we show that the following theorem holds.

**Theorem 4.5** *Let $G$ be a maximal outerplanar graph. Then $G$ is in $B_0$ if and only if $G$ is $S_3$-free.*




(a) (b)

Figure 3: (a) A graph $G$ and its almost-dual $\widehat{G}$ with the vertices $\hat{v}_i$ for $1 \leqslant i \leqslant 7$. (b) A $B_0$-EPG representation of $G$ constructed by Algorithm 4.1.

**Proof:** Let $G$ be a maximal outerplanar graph. If $G$ is not $S_3$-free, then $G$ is not in $B_0$ because of Cameron, Chaplick, Hoàng [11]. If $G$ is $S_3$-free, then Algorithm 4.1 constructs a $B_0$-EPG representation as stated in Lemma 4.4. Hence, $G$ is in $B_0$. □

4.5 and Observation 4.3 imply that it can be decided in $O(n)$ time whether a maximal outerplanar graph $G$ is in $B_0$. Further, it is not difficult to see that in the positive case the construction of a $B_0$-EPG representation in Algorithm 4.1 can be done in $O(n)$ time for a graph of order $n$.

## 4.2 Maximal Outerplanar Graphs in $B_1$

In this section we characterize the maximal outerplanar graphs which are $B_1^m$-EPG and $B_1$-EPG, respectively. It turns out that for maximal outerplanar graphs $B_0$-EPG and $B_1^m$-EPG coincide, see Corollary 4.6. Thus, surprisingly, allowing two more shapes of paths in the EPG representation does not increase the class of graphs which can be represented.

**Corollary 4.6** *Let $G$ be a maximal outerplanar graph. Then $G$ is in $B_1^m$ if and only if $G$ is $S_3$-free.*

**Proof:** Since $S_3$ is not in $B_1^m$ (as shown by Cameron, Chaplick, Hoàng [11]), a graph which is not $S_3$-free is not in $B_1^m$. Further, by 4.5 every $S_3$-free maximal outerplanar graph is in $B_0$ and hence also in $B_1^m$. □

Since $S_3$ is in $B_1$ (Golumbic, Lipshteyn, Stern [18]) but not in $B_0$ (Cameron, Chaplick, Hoàng [11]) the class of $B_1$-EPG maximal outerplanar graphs is strictly larger than the class of $B_1^m$-EPG maximal outerplanar graphs. In the sequel we characterize the class of $B_1$-EPG maximal outerplanar graphs. To this end we need the concept of the *reduced graph* of a maximal outerplanar graph. Recall the definition of central vertices and central edges of $S_3$ from Definition 3.1.

**Definition 4.7** *Let $G$ be a maximal outerplanar graph. The reduced graph $\widetilde{G}$ of $G$ is defined in the following way. For every copy of $S_3$ in $G$, the central vertices and the central edges of $S_3$ are colored green. Then all non-colored vertices and non-colored edges are removed from $G$. The resulting graph is the reduced graph $\widetilde{G}$.*

The next lemma addresses the structural relationship of maximal outerplanar graphs and their reduced and almost-dual graphs.

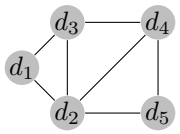



Figure 4: The graph $M_1$.

**Lemma 4.8** *Let $G$ be a maximal outerplanar graph with reduced graph $\widetilde{G}$ and almost-dual graph $\widehat{G}$. Then (i) the copies of $S_3$ in $G$, (ii) the triangles in $\widetilde{G}$ and (iii) the vertices of degree 3 in $\widehat{G}$ are in a one-to-one correspondence.*

**Proof:** First we prove the one-to-one correspondence between (ii) the triangles in $\widetilde{G}$ and (i) the copies of $S_3$ in $G$. Clearly each copy of $S_3$ in $G$ corresponds to a triangle in $\widetilde{G}$.

Now let $\{v_1, v_2, v_3\}$ be the vertices of an arbitrary triangle $T$ in $\widetilde{G}$. Notice that there are exactly two possibilities for the order in which edges are colored: either all the edges of $T$ were colored at once, or the three edges were colored at three different times as central edges belonging to three different copies of $S_3$ in $G$.

If all edges of $T$ were colored at once, then $T$ corresponds to the central vertices of a copy of $S_3$ in $G$. If $\{v_1, v_2\}$, $\{v_2, v_3\}$, $\{v_1, v_3\}$ were colored at three different times, there exists vertices $v_4$, $v_5$ and $v_6$ such that $\{v_1, v_2, v_4\}$, $\{v_2, v_3, v_5\}$ and $\{v_1, v_3, v_6\}$ are sets of central vertices of some copy of $S_3$ in $G$, respectively. Since $G$ is outerplanar the vertices $v_4$, $v_5$ and $v_6$ are pairwise distinct and none of them can be adjacent to any of the two others. Therefore, the subgraph of $G$ induced by the vertices $\{v_i | 1 \leqslant i \leqslant 6\}$ is a copy of $S_3$ in $G$ and hence the vertices of $T$ form the set of the central vertices of this copy of $S_3$. So (ii) and (i) are in a one-to-one correspondence.

To see the bijection between (iii) and (i), recall that any vertex of degree 3 in $\widehat{G}$ represents a (triangular) inner face of $G$ such that each edge on its boundary is shared with the boundary of another triangular inner face. Thus each vertex of degree 3 in $\widehat{G}$ corresponds to a copy of $S_3$ in $G$, and clearly, also vice-versa. □

Objects corresponding to each other in terms of the bijections given in the proof of Lemma 4.8 are referred to as *corresponding objects*.

Now we are able to prove the following structural result for $B_1$-EPG representations of maximal outerplanar graphs.

**Corollary 4.9** *Let $G$ be a maximal outerplanar graph and $\widetilde{G}$ its reduced graph. In every $B_1$-EPG representation of $G$ (if there is any) the vertices of any triangle in $\widetilde{G}$ correspond to a claw clique.*

**Proof:** According to Lemma 4.8 the vertices of every triangle in $\widetilde{G}$ are the central vertices of a copy of $S_3$ in $G$. Observation 3.6 implies that this set of central vertices corresponds to a claw clique. □

Next we consider some maximal outerplanar graphs which are not $B_1$-EPG.

**Lemma 4.10** *Let $G$ be a maximal outerplanar graph and $\widetilde{G}$ its reduced graph. If the graph $M_1$ depicted in Figure 4 or the graph $M_1^\ell$ depicted in Figure 5 is an induced subgraph of $\widetilde{G}$ for some $\ell \geqslant 0$, then $G$ is not in $B_1$.*

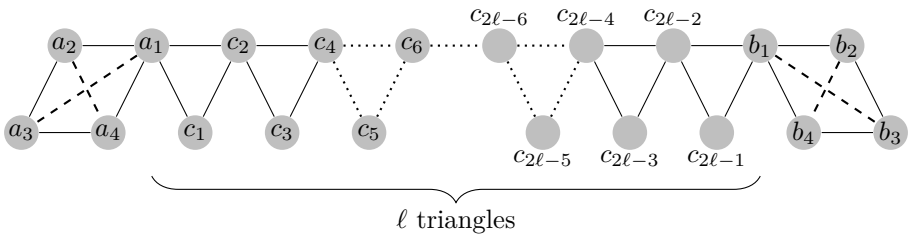



Figure 5: The graph $M_1^\ell$ with $\ell \geqslant 0$ consecutive triangles between the vertices $a_1$ and $b_1$. For $\ell = 0$ there are no vertices $c$ and the vertices $a_1$ and $b_1$ coincide. For all values of $\ell$ either $\{a_1, a_3\}$ or $\{a_2, a_4\}$ and either $\{b_1, b_3\}$ or $\{b_2, b_4\}$ is an edge in $M_1^\ell$.

**Proof:** Assume by contradiction that the maximal outerplanar graph $G$ is not $M$-free while being $B_1$-EPG and consider a $B_1$-EPG representation of $G$.

Consider first the case where $\widetilde{G}$ contains a copy of $M_1^\ell$ for some $\ell \geqslant 0$ (see Figure 5). Since exactly one of $\{a_1, a_3\}$ and $\{a_2, a_4\}$ is an edge in the copy of $M_1^\ell$ in $\widetilde{G}$ the vertices $\{a_1, a_2, a_3, a_4\}$ form two triangles in $\widetilde{G}$. According to Corollary 4.9 the vertices of any of these two triangles corresponds to a claw clique and due to Observation 3.6 in each claw clique two of the three paths are bent. It is easily observed that these two claw cliques must have different central grid points and there cannot be a path which is bent in both claw cliques. Thus any path corresponding to some vertex in $\{a_1, a_2, a_3, a_4\}$ is bent in exactly one of the these two claw cliques. Analogously, any path corresponding to a vertex in $\{b_1, b_2, b_3, b_4\}$ is bent in exactly one of the two claw cliques corresponding to the two triangles formed by $\{b_1, b_2, b_3, b_4\}$.

If $\ell = 0$, then $a_1 = b_1$ and the vertices of two triangles in $\widetilde{G}$ which share only one vertex correspond to claw cliques with different central grid points. Thus, the vertex $a_1 = b_1$ has to be bent in two claw cliques with different central grid points, a contradiction to $G$ being $B_1$-EPG. Assume now $\ell \geqslant 1$. We set $c_0 := a_1$ and $c_{2\ell} := b_1$ for notational consistency. Due to Corollary 4.9 the vertices $\{c_0, c_1, c_2\}$ of the first triangle correspond to a claw clique $K_1$. It is easily observed that the central grid point of $K_1$ is different from the central grid points of the claw cliques corresponding to the two triangles formed by $\{a_1, a_2, a_3, a_4\}$. Then, the path corresponding to $a_1 = c_0$ is bent in one of the two claw cliques corresponding to the two triangles formed by $\{a_1, a_2, a_3, a_4\}$, so it cannot be bent in $K_1$. Hence, the paths corresponding to $c_1$ and $c_2$ are bent in the claw clique $K_1$. Let $K_i$ be the claw clique corresponding to the vertices $\{c_{2i-2}, c_{2i-1}, c_{2i}\}$ of the $i$-th triangle for $1 \leqslant i \leqslant \ell$. By induction we get that the paths corresponding to the vertices $c_{2i-1}$ and $c_{2i}$ are bent in $K_i$ for $1 \leqslant i \leqslant \ell$. Thus the path corresponding to $c_{2\ell} = b_1$ has to be bent in the claw clique $K_\ell$ as well as in one of the claw cliques corresponding to the two triangles formed by $\{b_1, b_2, b_3, b_4\}$, and this is a contradiction.

Consider now the case where $M_1$ is an induced subgraph of $\widetilde{G}$ (see Figure 4). By analogous arguments as for $\{a_1, a_2, a_3, a_4\}$ the path corresponding to any vertex in $\{d_1, d_2, d_3, d_4\}$ is bent in exactly one of the two claw cliques corresponding to the two triangles formed by $\{d_1, d_2, d_3, d_4\}$. But then two of the paths corresponding to $\{d_2, d_4, d_5\}$ have to be bent in the claw clique corresponding to $\{d_2, d_4, d_5\}$ while $d_2$ and $d_4$ have to be bent also in another claw clique with a different central grid point. This contradicts the definition of a $B_1$-EPG representation. □

Lemma 4.10 describes a class of maximal outerplanar graphs which are not $B_1$-EPG. In fact this class contains all maximal outerplanar graphs which are not $B_1$-EPG as stated in in 4.23, the

main result in this section. The following definition allows us to simplify notation.

**Definition 4.11** *A maximal outerplanar graph $G$ is called* $M$-free *if its reduced graph $\widetilde{G}$ contains neither $M_1$ nor $M_1^\ell$, for any $\ell \geqslant 0$, as an induced subgraph.*

Before we can consider the construction of a $B_1$-EPG representation of a $B_1$-EPG maximal outerplanar graph $G$, we first need the following definitions.

**Definition 4.12** *Let $G$ be a maximal outerplanar graph with almost-dual graph $\widehat{G}$ and reduced graph $\widetilde{G}$. We denote by $\widehat{V}_3$ the set of vertices of degree 3 in the almost-dual graph $\widehat{G}$, so $\widehat{V}_3 = \{\hat{v} \in V(\widehat{G})\colon \hat{v}$ has degree 3 in $\widehat{G}\}$.*

*Two distinct triangles in $\widetilde{G}$ which share an edge are called* neighbored (to each other)*. Two distinct triangles in $\widetilde{G}$ which share a vertex, but not an edge, are called* touching (each other)*. A* sequence of touching triangles *(*STT*) $T_1, \ldots, T_k$ for some $k \in \mathbb{N}$ is a sequence of triangles $T_i$ in $\widetilde{G}$ such that the triangles $T_i$ and $T_{i+1}$ are touching for each $1 \leqslant i \leqslant k-1$. The* surrounding of a pair of triangles $T$ *and* $T'$ neighbored to each other *in $\widetilde{G}$ is the set of all triangles $T^*$ in $\widetilde{G}$, such that $T$ or $T'$ can be reached from $T^*$ over an STT, i.e. there is an STT $T_1, \ldots, T_k$ with $T_1 = T^*$ and $T_k \in \{T, T'\}$.*

*Furthermore we translate the definitions related to triangles in $\widetilde{G}$ also to the corresponding vertices of degree 3 in $\widehat{G}$. More precisely two vertices $\hat{v}$, $\hat{v}'$ of degree 3 in $\widehat{G}$ that correspond to two neighbored (touching) triangles in $\widetilde{G}$ with respect to the bijection of Lemma 4.8 are called* neighbored *(*touching*). A vertex $\hat{v}^*$ of degree 3 in $\widehat{G}$ is said to be* in the surrounding of two neighbored vertices $\hat{v}$, $\hat{v}'$ *of degree 3 in $\widehat{G}$ if the corresponding triangle $T_{\hat{v}^*}$ is in the surrounding of the neighbored triangles $T_{\hat{v}}$, $T_{\hat{v}'}$.*

*Finally a* cycle of touching or neighbored triangles *(*CTNT*) is defined as a sequence of triangles $T_1, \ldots, T_k$ in $\widetilde{G}$, $k \in \mathbb{N}$, such that for each $1 \leqslant i \leqslant k-1$, $T_i$ and $T_{i+1}$ are either neighbored or touching triangles, and also $T_1$ and $T_k$ are either neighbored or touching triangles. A CTNT $T_1, \ldots, T_k$ is called* reduced *if for all $1 \leqslant i < j \leqslant k$ the triangles $T_i$ and $T_j$ are only neighbored or touching if either $j = i+1$, or $i = 1$ and $j = k$. In other words a CTNT is reduced iff triangles of that cycle which are non-consecutive are neither touching each other nor neighbored to each other.*

Next we investigate the structure of $M$-free maximal outerplanar graphs.

**Lemma 4.13** *Let $G$ be an $M$-free maximal outerplanar graph with almost-dual graph $\widehat{G}$ and reduced graph $\widetilde{G}$. Then the vertices in $\widehat{V}_3$ can be partitioned such that each vertex is either $(A)$ neighbored to exactly one other vertex of degree 3 in $\widehat{G}$, or $(B)$ not neighbored, but in the surrounding of exactly one pair of neighbored vertices in $\widehat{G}$, or $(C)$ not in the surrounding of any pair of neighbored vertices in $\widehat{G}$.*

**Proof:** Due to Lemma 4.8 we can consider triangles in $\widetilde{G}$ instead of considering vertices of degree 3 in $\widehat{G}$.

Since $M_1$ is not an induced subgraph of $\widetilde{G}$, every triangle in $\widetilde{G}$ is neighbored to at most one other triangle. Moreover if $T_1$ and $T_2$ are neighbored triangles in $\widetilde{G}$ with vertex sets $V(T_1)$ and $V(T_2)$, respectively, then none of the vertices in $V(T_1) \cup V(T_2)$ can be a vertex of some other pair of neighbored triangles, because otherwise $M_1^0$ would be an induced subgraph of $\widetilde{G}$. To summarize, neighbored triangles in $\widetilde{G}$ appear in pairs of two and two different pairs of neighbored triangles share neither edges nor vertices.

Further, it is easy to see that each triangle of $\widetilde{G}$ is in the surrounding of at most one pair of neighbored triangles. Indeed, if a triangle $T$ of $\widetilde{G}$ was in the surrounding of two different pairs of neighbored triangles $(T_1, T_2)$ and $(T_3, T_4)$, then the graph induced by $T$, $T_1$, $T_2$, $T_3$, $T_4$ and the two STTs from $T$ to the two pairs $(T_1, T_2)$, $(T_3, T_4)$ would contain a copy of $M_1^\ell$ for some $\ell \geqslant 0$, a contradiction. Hence, the surroundings of different pairs of neighbored triangles are disjoint and the result follows. □

For an $M$-free graph $G$ we denote the sets of vertices of degree 3 in $\widehat{G}$ in the parts $(A)$, $(B)$ and $(C)$ of the partition of Lemma 4.13 by $A$, $B$ and $C$, respectively.

Finally we are able to present Algorithm 4.2 which recognizes whether an input maximal outerplanar graph $G$ is $M$-free and computes a particular assignment of pairs of vertices of $G$ to vertices in $\widehat{V}_3$ in the positive case. The particular assignment is called *assigned*. It maps $\widehat{V}_3$ to $V(G) \times V(G)$ such that both vertices of the pair *assigned*$(\hat{v})$ are central vertices of the copy of $S_3$ in $G$ corresponding to $\hat{v}$ in the sense of Lemma 4.8. Moreover, these two vertices are selected carefully, such that no vertex of $V$ is assigned to more than one $\hat{v} \in \widehat{V}_3$. This assignment will be used to construct a $B_1$-EPG representations of a $B_1$-EPG maximal outerplanar graph in Algorithm 4.3. In particular, the path corresponding to a vertex $a \in V(G)$ will bend for representing $\hat{v}$ iff $a \in V(T_{\hat{v}})$ and $a$ is assigned to $\hat{v}$.

Next we address two important arrays used by the algorithm: *level*$(\hat{v})$, for $\hat{v} \in \widehat{V}_3$, and $\Delta(a)$, for $a \in V(G)$. Their meaning is as follows. $\Delta(a) = \hat{v} \in \widehat{V}_3$ iff $a$ belongs to *assigned*$(\hat{v})$. The meaning of *level*$(\hat{v})$ is a bit more complicated: in the case that the input graph $G$ is $M$-free for every pair $(\hat{v}, \hat{v}')$ of neighbored vertices in $\widehat{V}_3$ we have *level*$(\hat{v})$ = *level*$(\hat{v}') = 0$. For a vertex $\hat{v}$ in the surrounding of some pair of neighbored vertices $(\hat{w}, \hat{w}')$ (i.e. $\hat{v} \in B$ according to Lemma 4.13) the quantity *level*$(\hat{v}) =: k - 1$ determines the length of a STT $\hat{v}_1, \ldots, \hat{v}_k$ in $\widehat{V}_3$ starting at $\hat{w}$ or $\hat{w}'$ and ending at $\hat{v} = \hat{v}_k$. Thus in the execution of line 25 of Algorithm 4.2 the vertex $\hat{v}''$ coincides with $\hat{v}_{i-1}$ when $\hat{v}$ coincides with $\hat{v}_i$ for every $i \in \{2, \ldots, k\}$.

For each vertex $\hat{v}$ which does not belong to the surrounding of some pair of neighbored vertices in $\widehat{V}_3$ (i.e. $\hat{v} \in C$ according to Lemma 4.13) the following holds. There is a STT $\hat{v}_1, \ldots, \hat{v}_\ell$ in $\widehat{V}_3$ such that *level*$(\hat{v}_1) = 0$ holds, $\hat{v} = \hat{v}_\ell$ and in the execution of line 25 of the algorithm the vertex $\hat{v}''$ coincides with $\hat{v}_{j-1}$ when $\hat{v}$ coincides with $\hat{v}_j$ for every $j \in \{2, \ldots, \ell\}$.

The following Lemma 4.14 states some properties of the assignment constructed in Algorithm 4.2.

**Lemma 4.14** *Let $G$ be a maximal outerplanar graph with almost-dual $\widehat{G}$ and reduced graph $\widetilde{G}$. Then Algorithm 4.2 is well-defined. At termination it either outputs "$G$ is not $M$-free" or it returns an assignment that assigns two vertices $x_{\hat{v}}$, $y_{\hat{v}}$ of $G$ to every vertex $\hat{v}$ in $\widehat{V}_3$ such that (1) $x_{\hat{v}}$, $y_{\hat{v}}$ are central vertices of the copy of $S_3$ in $G$ corresponding to $\hat{v}$, and (2) no vertex of $G$ is assigned to more than one vertex $\hat{v}$ of $\widehat{G}$.*

**Proof:** We first show that Algorithm 4.2 is well-defined, i.e. that it can be executed as described and that it terminates. Observe that by the definition of neighbored vertices $|V(T_{\hat{v}}) \cap V(T_{\hat{v}'})| = 2$ holds, so $a$, $b$, $c$ and $d$ can be defined as described in line 7 and 8. Furthermore, *level*$(\Delta(x)) < \infty$ if and only if $\Delta(x) \neq NULL$, and $\Delta(x) \in \widehat{V}_3$ holds for all $x \in V(G)$. So $\hat{v}'' \in \widehat{V}_3$ holds in line 20 and the sum in line 25 is well-defined.

Next we show that the algorithm always terminates correctly. This clearly holds if Algorithm 4.2 outputs "$G$ is not $M$-free". Assume now that the algorithm does not output "$G$ is not $M$-free". The algorithm starts with all vertices of $\widehat{V}_3$ being labeled *unserved* and having no assigned vertices (line 2). Then the algorithm iteratively *serves the vertices* $\hat{v}$ of $\widehat{V}_3$, i.e. it labels them *served* and

**Algorithm 4.2** Construct an assignment *assigned*: $\widehat{V}_3 \to V(G) \times V(G)$

**Input:** A maximal outerplanar graph $G$, its reduced graph $\widetilde{G}$ and its almost-dual $\widehat{G}$
**Output:** *assigned*: $\widehat{V}_3 \to V(G) \times V(G)$ or "$G$ is not $M$-free"

**procedure** ASSIGNMENT_MAX_OUTERPLANAR($G$, $\widehat{G}$, $\widetilde{G}$)
  Set *assigned*($\hat{v}$) := $\emptyset$ and *label*($\hat{v}$) := *unserved* for each $\hat{v} \in \widehat{V}_3$
  Set $\Delta(a) := NULL$ for all $a \in V(G)$
  Set *level*($\hat{v}$) := $\infty$ for each $\hat{v} \in \widehat{V}_3$ and *level*($NULL$) := $\infty$
  **for** $\hat{v} \in \widehat{V}_3$ **do**
    **if** *label*($\hat{v}$) = *unserved* and $\exists \hat{v}' \in \widehat{V}_3$: $\hat{v}$ is neighbored to $\hat{v}'$ **then**
      Set $\{a, b\} := V(T_{\hat{v}}) \cap V(T_{\hat{v}'})$
      Set $\{c\} := V(T_{\hat{v}}) \setminus \{a, b\}$ and $\{d\} := V(T_{\hat{v}'}) \setminus \{a, b\}$
      **if** $\exists x \in V(T_{\hat{v}}) \cup V(T_{\hat{v}'})$ with $\Delta(x) \neq NULL$ **then**
        Output "$G$ is not $M$-free" and STOP
      **end if**
      Set *assigned*($\hat{v}$) := $\{a, c\}$ and *label*($\hat{v}$) := *served*
      Set *assigned*($\hat{v}'$) := $\{b, d\}$ and *label*($\hat{v}'$) := *served*
      Set *level*($\hat{v}$) := 0 and *level*($\hat{v}'$) := 0
      Set $\Delta(a) := \hat{v}$, $\Delta(c) := \hat{v}$, $\Delta(b) := \hat{v}'$, $\Delta(d) := \hat{v}'$
    **end if**
  **end for**
  **while** $\exists \hat{v} \in \widehat{V}_3$: *label*($\hat{v}$) = *unserved* **do**
    **while** $\exists \hat{v} \in \widehat{V}_3$: *label*($\hat{v}$) = *unserved* and
        $\min\{level(\Delta(x))\colon x \in V(T_{\hat{v}})\} \neq \infty$ **do**
      Set $a := \arg\min\{level(\Delta(x))\colon x \in V(T_{\hat{v}})\}$ and $\hat{v}'' := \Delta(a)$
      **if** $\exists x \in V(T_{\hat{v}}) \setminus \{a\}$ with $\Delta(x) \neq NULL$ **then**
        Output "$G$ is not $M$-free" and STOP
      **end if**
      Set *assigned*($\hat{v}$) := $V(T_{\hat{v}}) \setminus \{a\}$ and *label*($\hat{v}$) := *served*
      Set *level*($\hat{v}$) := *level*($\hat{v}''$) + 1
      **for** $x \in V(T_{\hat{v}}) \setminus \{a\}$ **do** $\Delta(x) := \hat{v}$ **end for**
    **end while**
    **if** $\exists \hat{v} \in \widehat{V}_3$: *label*($\hat{v}$) = *unserved* **then**
      Let $a \in V(T_{\hat{v}})$ be a random vertex of $V(T_{\hat{v}})$
      Set *assigned*($\hat{v}$) := $V(T_{\hat{v}}) \setminus \{a\}$ and *label*($\hat{v}$) := *served*
      Set *level*($\hat{v}$) := 0
      **for** $x \in V(T_{\hat{v}}) \setminus \{a\}$ **do** $\Delta(x) := \hat{v}$ **end for**
    **end if**
  **end while**
  **return** *assigned*
**end procedure**

assigns them the vertices $x_{\hat{v}}$, $y_{\hat{v}}$ from $G$. Since the while loop in lines 18 to 34 is executed as long as there are unserved vertices in $\widehat{V}_3$, the algorithm terminates after a finite number of steps. Moreover, it has assigned a pair of vertices $x_{\hat{v}}$, $y_{\hat{v}}$ from $G$ to every vertex $\hat{v}$ in $\widehat{V}_3$ at termination.

It is easy to see that the vertices $x_{\hat{v}}$, $y_{\hat{v}}$ assigned to $\hat{v} \in \widehat{V}_3$ fulfill property (1) by construction. In order to see that also (2) is fulfilled, observe that $\Delta(x) = \hat{v}$ iff $x \in \mathit{assigned}(\hat{v})$ for all $x$ and all $\hat{v}$. Moreover, both the equality and the inclusion hold immediately after the vertex $\hat{v}$ is served. Finally, $\Delta(x) = NULL$ holds whenever a vertex $x$ of $G$ is assigned to a vertex $\hat{v}$ in $\widehat{V}_3$ in lines 12, 13 or 24. Otherwise the condition of the if statement in lines 9 or 21 would have been fulfilled and the algorithm would have already terminated with "$G$ is not $M$-free". Hence, a vertex $x$ of $G$ can only be assigned to at most one vertex $\hat{v}$ of $\widehat{V}_3$. □

In order to examine Algorithm 4.2 in more detail, we refer to the execution of lines 5 to 17 as step one, the first execution of lines 19 to 27 as step two and to the remaining executions of lines 18 to 34 as step three. We start our investigation by considering step one in the next two lemmata.

**Lemma 4.15** *Let $G$ be a maximal outerplanar graph with almost-dual $\widehat{G}$ and reduced graph $\widetilde{G}$. Then in step one Algorithm 4.2 either outputs "$G$ is not $M$-free" or it serves exactly the vertices in $\widehat{V}_3$ which are neighbored.*

**Proof:** If Algorithm 4.2 does not output "$G$ is not $M$-free" in step one, then it loops through all vertices of $\widehat{V}_3$ and serves all vertices which are unserved and neighbored to another vertex, so clearly exactly neighbored vertices are served in step one. Moreover, if the algorithm outputs "$G$ is not $M$-free" in step one, it does so in line 10 and at this point there are still unserved vertices which belong to a pair of neighbored vertices (see line 6). Thus in this case the algorithm does not serve all neighbored vertices in $\widehat{V}_3$. □

**Lemma 4.16** *Let $G$ be a maximal outerplanar graph with almost-dual $\widehat{G}$ and reduced graph $\widetilde{G}$. Then Algorithm 4.2 outputs "$G$ is not $M$-free" in line 10 iff $\widetilde{G}$ contains a copy of $M_1$ or $M_1^0$.*

**Proof:** Consider first the case that $\widetilde{G}$ contains a copy of $M_1$ (see Figure 4). Let $\hat{v}_1$, $\hat{v}_2$ and $\hat{v}_3$ be the vertices in $\widehat{V}_3$ that induce an $M_1$ in $\widetilde{G}$. We show that Algorithm 4.2 outputs "$G$ is not $M$-free" in line 10 in this case. Assume by contradiction that this is not the case. Thus, the algorithm completes the for loop which starts at line 5. Assume that $\hat{v}_1$ is served by the algorithm before $\hat{v}_2$ and $\hat{v}_3$. The other cases ($\hat{v}_2$ or $\hat{v}_3$ are served first) can be handled analogously. Consider the execution of line 6 with $\hat{v} = \hat{v}_1$ or $\hat{v}' = \hat{v}_1$ prior to serving $\hat{v}_1$. There are two cases: either $\{\hat{v}, \hat{v}'\} \neq \{\hat{v}_1, \hat{v}_2\}$ or $\{\hat{v}, \hat{v}'\} = \{\hat{v}_1, \hat{v}_2\}$. In the first case the condition in line 9 is fulfilled at the latest by the execution of this line with $\hat{v} = \hat{v}_2$ or $\hat{v}' = \hat{v}_2$ and the algorithm would stop with "$G$ is not $M$-free" in line 10, a contradiction. In the latter case the condition in line 9 is fulfilled at the latest by the execution of this line with $\hat{v} = \hat{v}_3$ or $\hat{v}' = \hat{v}_3$ and the algorithm would stop with "$G$ is not $M$-free" in line 10, again a contradiction.

Consider now the case that $\widetilde{G}$ contains a copy of $M_1^0$ but no copy of $M_1$. Let $\hat{v}_1$, $\hat{v}_2$, $\hat{v}_3$ and $\hat{v}_4$ be vertices of $\widehat{V}_3$ that induce an $M_1^0$ such that $\hat{v}_1$ and $\hat{v}_2$ are neighbored to each other and $\hat{v}_3$ and $\hat{v}_4$ are neighbored to each other. We show by contradiction that Algorithm 4.2 outputs "$G$ is not $M$-free" in line 10.

Notice that since $\widetilde{G}$ does not contain a copy of $M_1$, each neighbored triangle is neighbored to exactly one other triangle. Assume w.l.o.g. that during the algorithm $\hat{v}_1$ and $\hat{v}_2$ coincide with $\hat{v}$ and $\hat{v}'$ in line 6 and then, later, $\hat{v}_3$ and $\hat{v}_4$ coincide with $\hat{v}$ and $\hat{v}'$ at that line. At that point one of the vertices of $G$ contained in the triangles corresponding to $\hat{v}_1$ and $\hat{v}_2$ is among $x \in V(T_{\hat{v}}) \cup V(T_{\hat{v}'})$. So with the same arguments as above Algorithm 4.2 outputs "$G$ is not $M$-free" in line 10.

We complete the proof by showing that $\widetilde{G}$ contains a copy of $M_1$ or $M_1^0$ if Algorithm 4.2 outputs "$G$ is not $M$-free" in line 10.

Let $\hat{v}_1$ and $\hat{v}_2$ be the vertices $\hat{v}$ and $\hat{v}'$ in line 6 at the last execution of this line before the algorithm stops. Clearly $\hat{v}_1$ and $\hat{v}_2$ are neighbored. Since Algorithm 4.2 stops, there has to be a vertex $x \in V(T_{\hat{v}}) \cup V(T_{\hat{v}'})$ such that $\Delta(x)$ was set to either $\hat{v}$ or $\hat{v}'$ in line 15 in a previous iteration. Let $v_3$ and $v_4$ be the vertices $\hat{v}$ or $\hat{v}'$ at that previous iteration. Clearly, $x$ is contained in $T_{\hat{v}_1}$ or $T_{\hat{v}_2}$ and also contained in $T_{\hat{v}_3}$ and $T_{\hat{v}_4}$, so at least one of $v_1$ and $v_2$ is neighbored to or touching at least one of $v_3$ and $v_4$. If the later vertices are neighbored, then $\widetilde{G}$ contains a copy of $M_1$. Otherwise $\widetilde{G}$ contains a copy of $M_1^0$. □

Notice that Lemma 4.15 and Lemma 4.16 imply that if $G$ is $M$-free, then in step one Algorithm 4.2 serves exactly the vertices in $A$ (see Lemma 4.13). In the next two lemmata we consider step two of the algorithm in more detail.

**Lemma 4.17** *Let $G$ be a maximal outerplanar graph with almost-dual $\widehat{G}$ and reduced graph $\widetilde{G}$. Assume that Algorithm 4.2 serves a vertex $\hat{v}$ in $\widehat{V}_3$ in step two. Then $\hat{v}$ is not neighbored, but it is in the surrounding of neighbored vertices.*

**Proof:** Clearly, $\hat{v}$ is not neighbored, otherwise it would have been already served at step one (see Lemma 4.15). We show that $\hat{v}$ is in the surrounding of neighbored vertices. Consider the vertex $\hat{v}''$ in line 20 when $\hat{v}$ is served in step two. Then $\hat{v}$ and $\hat{v}''$ are touching. If $\hat{v}''$ is a neighbored vertex, then $\hat{v}$ is in the surrounding of $\hat{v}''$ and we are done. Otherwise, $\hat{v}''$ has been served in step two and the above argument can be inductively repeated with $\hat{v}''$ in the role of $\hat{v}$. Since $level(\hat{v}'') < level(\hat{v})$, a neighbored vertex, i.e. a vertex with level equal to 0, is reached after a finite number of repetitions. Therefore $\hat{v}$ is in the surrounding of neighbored vertices, but not neighbored. □

**Lemma 4.18** *Let $G$ be a maximal outerplanar graph with almost-dual $\widehat{G}$ and reduced graph $\widetilde{G}$. Assume that Algorithm 4.2 has not terminated with "$G$ is not $M$-free" in step one. Then in step two it either outputs "$G$ is not $M$-free" or it serves exactly the vertices in $\widehat{V}_3$ which are not neighbored, but in the surrounding of neighbored vertices.*

**Proof:** Assume first that Algorithm 4.2 does not output "$G$ is not $M$-free" in step two. Thus the algorithm has completed step two, i.e. it has completed the first run of the while loop in lines 19 to 27. Lemma 4.17 implies that each vertex served in step two is not neighbored, but in the surrounding of neighbored vertices. Now we consider a vertex $\hat{v}$, that is not neighbored, but in the surrounding of neighbored vertices. Assume by contradiction that it is not served in step two. Clearly, $\hat{v}$ has not been served in step one, due to Lemma 4.15. Thus $\hat{v}$ is not served yet at the end of step two. Consider an STT $T_{\hat{v}_1}, \ldots, T_{\hat{v}_k}$ such that $\hat{v} = \hat{v}_k$ and $\hat{v}_1$ belongs to a neighbored pair of vertices. Obviously, $\hat{v}_1$ was already served at the beginning of step two. Observe that $k \geqslant i \geqslant 2$ holds for the smallest index $i \in \{1, 2, \ldots, k\}$ such that the vertex $\hat{v}_i$ is not served at the end of step two. Thus, at the end of step two $\hat{v}_{i-1}$ is a served vertex, $\hat{v}_{i-1}$ and $\hat{v}_i$ are touching and $\hat{v}_i$ is not served, implying that the while-condition in line 19 is fulfilled, a contradiction to the completion of step two.

On the other hand, if Algorithm 4.2 outputs "$G$ is not $M$-free" in step two, this is because the if condition in line 21 is fulfilled, meaning that there is an unserved vertex $\hat{v}$ touching a served vertex $\hat{v}''$ (see line 20). According to Lemma 4.15 and Lemma 4.17 $\hat{v}''$ is neighbored or lies in the surrounding of a neighbored vertex. Consequently, vertex $\hat{v}$ also lies in the surrounding of a neighbored vertex, while being unserved. □

Under the premise that Algorithm 4.2 correctly recognizes whether a graph is $M$-free, Lemma 4.18 implies that if $G$ is $M$-free, then in step two Algorithm 4.2 serves exactly the vertices in $B$ (see Lemma 4.13). Furthermore, under this premise, it follows from Lemma 4.14, Lemma 4.15 and Lemma 4.18 that if $G$ is $M$-free, then in step three Algorithm 4.2 serves exactly the vertices in $C$ (see Lemma 4.13). The next two lemmata are used to establish this premise.

**Lemma 4.19** *Let $G$ be an $M$-free maximal outerplanar graph, with almost-dual graph $\widehat{G}$ and reduced graph $\widetilde{G}$. Then $\widetilde{G}$ does not contain a reduced CTNT $T_1, \ldots, T_k$, for any $k \geqslant 4$. Moreover, if a reduced CTNT of length 3 is contained in $\widehat{G}$, then all the triangles in that CTNT share a common vertex.*

**Proof:** We prove Lemma 4.19 by contradiction. Assume that $C = (T_1, \ldots, T_k)$ is a reduced CTNT in $\widetilde{G}$ for some $k \geqslant 4$. Due to Lemma 4.13 touching triangles can be in the surrounding of at most one pair of neighbored triangles. Therefore at most two consecutive triangles can be neighbored in a CTNT, all other consecutive triangles are touching. For each $1 \leqslant i \leqslant k-1$ let $v_i$ be a vertex shared by $T_i$ and $T_{i+1}$ and let $v_k$ be a vertex shared by $T_k$ and $T_1$. Clearly, $v_1, \ldots, v_k$ are pairwise distinct (otherwise $C$ would not be reduced) and form a cycle $C'$ of length $k \geqslant 4$ in $\widetilde{G}$ and hence also in $G$. Since $G$ is maximal outerplanar, all the vertices of the triangles $T_1, \ldots, T_k$ which are not in $C'$ lie on the outside of $C'$ and all faces of $G$ within $C'$ are triangles. Now we distinguish two cases: (i) there are two consecutive neighbored triangles in $C$ and (ii) there are no two consecutive neighbored triangles in $C$.

In Case (i) let $i^*$ such that $v_{i^*}$ is the vertex of $C'$ which is shared by the neighbored triangles and let $v'_{i^*}$ be the other vertex shared by the neighbored triangles. Notice that $v'_{i^*}$ is not in $C'$. It is easy to see that if we replace $v_{i^*}$ by $v'_{i^*}$ in $C'$ we get another cycle $C''$ in $G$ such that $v_{i^*}$ is in the interior of the cycle $C''$, which is a contradiction to the outerplanarity of $G$.

In Case (ii) all faces of $G$ within the cycle $C'$ are triangles and all of them are center triangles of some copy of $S_3$ in $G$, thus all of them are present also in $\widetilde{G}$. But this implies that a copy of $M_1$ is contained in $\widetilde{G}$, a contradiction.

Thus there are no reduced CTNT of length $k$ with $k \geqslant 4$ in $\widetilde{G}$.

Now consider a reduced CTNT $C = (T_1, T_2, T_3)$ of length 3 in $\widetilde{G}$. We distinguish again the cases (i) and (ii).

In Case (i) let w.l.o.g. $T_1$ and $T_2$ be two neighbored triangles in $C$. Then $T_3$ must share a vertex $a$ with $T_1$ and a vertex $b$ with $T_2$. If $a \neq b$ and $|\{a, b\} \cap V(T_1) \cap V(T_2)| \in \{0, 2\}$, then $G$ would not be outerplanar, a contradiction. If $a \neq b$ and $|\{a, b\} \cap V(T_1) \cap V(T_2)| = 1$, then $\widetilde{G}$ would contain a copy of $M_1$, which contradicts $G$ being $M$-free. So $a = b$ is shared by all triangles of $C$.

In Case (ii), if there are no neighbored triangles in $C$, then $T_1$, $T_2$ and $T_3$ are pairwise touching. Again, if they would not all three touch at a common vertex, then $\widetilde{G}$ would contain a copy of $M_1$, contradicting $G$ being $M$-free. □

**Lemma 4.20** *Let $G$ be a maximal outerplanar graph with almost-dual $\widehat{G}$ and reduced graph $\widetilde{G}$. Then Algorithm 4.2 outputs "$G$ is not $M$-free" in line 22 iff $\widetilde{G}$ contains a copy of $M_1^\ell$ for some $\ell \geqslant 1$ and no copy of $M_1$ and $M_1^0$.*

**Proof:** First, let $G$ be such that $\widetilde{G}$ contains a copy of $M_1^\ell$ for some $\ell \geqslant 1$ and no copy of $M_1$ and $M_1^0$ (see Figure 4 and Figure 5). We show that Algorithm 4.2 outputs "$G$ is not $M$-free" in line 22. Let us denote by $\hat{a}_1$ and $\hat{a}_2$ ($\hat{b}_1$ and $\hat{b}_2$) the vertices in $\widehat{V}_3$ corresponding to the triangles with vertices $\{a_1, a_2, a_3, a_4\}$ ($\{b_1, b_2, b_3, b_4\}$), and by $\hat{v}_i$ the vertex in $\widehat{V}_3$ corresponding to the triangle

with vertices $\{c_{2i-2}, c_{2i-1}, c_{2i}\}$, $1 \leqslant i \leqslant \ell$, in an arbitrary but fixed copy of $M_1^\ell$ in $\widetilde{G}$. Due to Lemma 4.15 and Lemma 4.16 the pairs of neighbored vertices $\hat{a}_1$, $\hat{a}_2$ and $\hat{b}_1$, $\hat{b}_2$ have been served in step one and the algorithm has not terminated with "$G$ is not $M$-free" in step one. According to Lemma 4.15 and Lemma 4.18 Algorithm 4.2 either outputs "$G$ is not $M$-free" in step two or all vertices $\hat{v}_i$ are served at the end of step two.

There is noting to show if the algorithm stops with "$G$ is not $M$-free"in step two. So assume w.l.o.g. that all vertices $\hat{v}_i$, $1 \leqslant i \leqslant \ell$, are served at the end of step 2 and consider the moment when line 24 is executed for the last such vertex $\hat{v}_i$ with $\hat{v} = \hat{v}_i$. It can be shown inductively that the vertices $c_{2i-2}$ and $c_{2i}$ of $V(T_{\hat{v}_i})$ have already been assigned to other vertices. Hence $\Delta(c_{2i-2}) \neq NULL$ and $\Delta(c_{2i}) \neq NULL$, implying that the if condition in line 21 is fulfilled. Thus the algorithm stops with "$G$ is not $M$-free" in line 22.

Next we assume that Algorithm 4.2 outputs "$G$ is not $M$-free" in line 22. If $\widetilde{G}$ would contain a copy of $M_1$ or $M_1^0$, then Algorithm 4.2 would output "$G$ is not $M$-free" already in line 10 (according to Lemma 4.16). We complete the proof by showing that $\widetilde{G}$ contains a copy of $M_1^\ell$ for some $\ell \geqslant 1$.

Let $\hat{v}$ be the vertex for which the if condition in line 21 is fulfilled right before termination with "$G$ is not $M$-free" in line 22. Consider two vertices $a \neq x \in V(T_{\hat{v}})$, with $\Delta(a) \neq NULL$ and $\Delta(x) \neq NULL$ (see lines 20 and 21). Thus $a$ and $x$ have been assigned to some already served vertices, say $\hat{a}$ and $\hat{x}$ in $\widehat{V}_3$. Notice that $\hat{a} \neq \hat{x}$, otherwise $\hat{a} = \hat{x}$ and $\hat{v}$ would be neighbored to each other and $\hat{v}$ would have already been served in step one due to Lemma 4.15. Notice, moreover, that $\hat{a}$ and $\hat{v}$ can not be neighbored due to the outerplanarity of $G$. Furthermore, $\hat{a}$ and $\hat{x}$ do not touch, otherwise $\widetilde{G}$ would contain a copy of $M_1$. Hence $T_{\hat{a}}$ and $T_{\hat{x}}$ do not share a vertex.

The vertices $\hat{a}$ and $\hat{x}$ have been served in step one or two, and hence each of them is neighbored or in the surrounding of some neighbored vertices (see Lemma 4.15 and Lemma 4.17). Hence there exist two STT $S_1$ and $S_2$ such that (i) $\hat{a}$ is the vertex preceding $\hat{v}$ in $S_1$, (ii) $\hat{x}$ the vertex preceding $\hat{v}$ in $S_2$, and both $S_1$ and $S_2$ (iii) end in $\hat{v}$, (iv) start in a neighbored triangle and (v) contain as few triangles as possible. If the only vertex in $S_1$ touching or being neighbored to some vertex different from $\hat{v}$ in $S_2$ is $\hat{v}$, then the union of $S_1$ and $S_2$ together with the vertices, to which the start vertices of $S_1$ and $S_2$ are neighbored, build an $M_1^\ell$ with $\ell \geqslant 1$.

Otherwise, consider the subsequence $S_3$ of $S_1$ which ends at $\hat{v}$ and starts at the last vertex $\hat{y} \neq \hat{v}$ which touches or is neighbored to some vertex different from $\hat{v}$ in $S_2$. Then $S_3$ and $S_2$ contain a CTNT. Since $\hat{a}$ and $\hat{x}$ do not touch and are not neighbored to each other there is a reduced CTNT of the above CTNT containing $\hat{v}$, $\hat{a}$, $\hat{x}$ and at least one other vertex. The existence of such a reduced CTNT of length at least 4 implies that $G$ is not $M$-free (see Lemma 4.19). The latter statement together with the absence of copies of $M_1$ and $M_1^0$ in $\widetilde{G}$ implies the existence of a copy of $M_1^\ell$, for some $\ell \geqslant 1$, in $\widetilde{G}$. □

Lemma 4.14 is one of many ingredients to prove the following theorem.

**Theorem 4.21** *Let $G$ be a maximal outerplanar graph. Algorithm 4.2 correctly decides whether $G$ is $M$-free.*

**Proof:** As an immediate consequence of Lemma 4.16 and Lemma 4.20 Algorithm 4.2 outputs "$G$ is not $M$-free" if and only if $G$ is not $M$-free. □

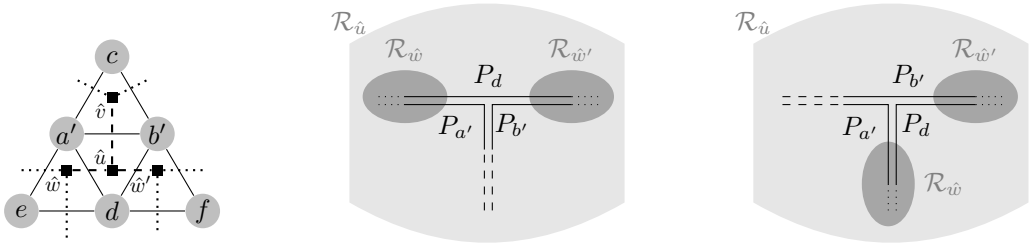
It can be shown that Algorithm 4.2 can be implemented to run in $O(n^2)$ time, where $n$ is the order of the input graph $G$. Thus, it is possible to decide in $O(n^2)$ time whether a maximal outerplanar graph $G$ of order $n$ is $M$-free. Moreover, in the positive case an assignment with the properties stated in Lemma 4.14 can be constructed in $O(n^2)$ time. Algorithm 4.3 uses such an assignment to construct a $B_1$-EPG representation of a maximal outerplanar $M$-free graph $G$.

**Algorithm 4.3** Construct a $B_1$-EPG representation for a maximal outerplanar $M$-free graph $G$

**Input:** A maximal outerplanar $M$-free graph $G$, its almost-dual $\widehat{G}$, its reduced graph $\widetilde{G}$ and an assignment *assigned*: $\widehat{V}_3 \to V(G) \times V(G)$ as computed by Algorithm 4.2

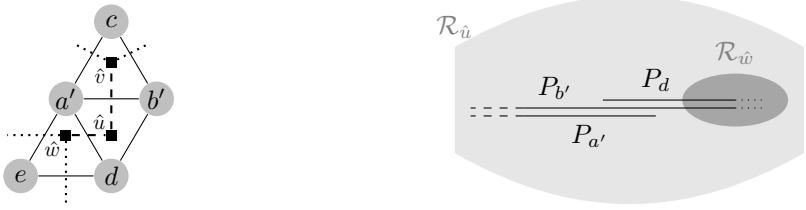
**Output:** A $B_1$-EPG representation of $G$

**procedure** B1_Mfree_Max_Outerplanar($G$, $\widehat{G}$, $\widetilde{G}$, *assigned*)
  Choose an arbitrary vertex $\hat{v}$ of degree 1 from $V(\widehat{G})$
  Let $\hat{u}$ be the neighbor of $\hat{v}$ in $V(\widehat{G})$
  Let $\{a, b\} := V(T_{\hat{u}}) \cap V(T_{\hat{v}})$ and $\{c\} := V(T_{\hat{v}}) \setminus \{a, b\}$
  Draw the paths $P_a$, $P_b$ and $P_c$ and set $\mathcal{R}_{\hat{u}}$ as depicted in Figure 8(c)
  Set $ToConstruct := \{(\hat{u}, \hat{v}, \mathcal{R}_{\hat{u}})\}$
  **while** $ToConstruct \neq \emptyset$ **do**
    Let $(\hat{u}, \hat{v}, \mathcal{R}_{\hat{u}}) \in ToConstruct$
    Set $ToConstruct := ToConstruct \setminus \{(\hat{u}, \hat{v}, \mathcal{R}_{\hat{u}})\}$
    Let $\{a, b\} := V(T_{\hat{u}}) \cap V(T_{\hat{v}})$
    Let $\{c\} := V(T_{\hat{v}}) \setminus \{a, b\}$ and $\{d\} := V(T_{\hat{u}}) \setminus \{a, b\}$
    **if** $\deg(\hat{u}) = 3$ **then**
      Let $a' \in assigned(\hat{u}) \cap \{a, b\}$ and $\{b'\} = \{a, b\} \setminus \{a'\}$
      Let $\hat{w} \in V(\widehat{G})$ and $e \in V(G) \setminus \{b'\}$ such that $\{a', d, e\} = V(T_{\hat{w}})$
      Let $\hat{w}' \in V(\widehat{G})$ and $f \in V(G) \setminus \{a'\}$ such that $\{b', d, f\} = V(T_{\hat{w}'})$
      **if** $assigned(\hat{u}) = \{a', b'\}$ **then**
        Extend the paths $P_{a'}$ and $P_{b'}$, draw the path $P_d$ in the region $\mathcal{R}_{\hat{u}}$ and set $\mathcal{R}_{\hat{w}}$, $\mathcal{R}_{\hat{w}'}$ as depicted in Figure 6(b)
      **else** ($assigned(\hat{u}) = \{a', d\}$)
        Extend the paths $P_{a'}$ and $P_{b'}$, draw the path $P_d$ in the region $\mathcal{R}_{\hat{u}}$ and set $\mathcal{R}_{\hat{w}}$, $\mathcal{R}_{\hat{w}'}$ as depicted in Figure 6(c)
      **end if**
      $ToConstruct := ToConstruct \cup \{(\hat{w}, \hat{u}, \mathcal{R}_{\hat{w}}), (\hat{w}', \hat{u}, \mathcal{R}_{\hat{w}'})\}$
    **else if** $\deg(\hat{u}) = 2$ **then**
      Let $\hat{w} \neq \hat{v}$ be the other neighbor of $\hat{u}$ in $\widehat{G}$
      Let $\{a'\} := V(T_{\hat{w}}) \cap \{a, b\}$ and $\{b'\} := \{a, b\} \setminus \{a'\}$
      Let $e$ such that $\{a', d, e\} = V(T_{\hat{w}})$
      Extend the paths $P_{a'}$ and $P_{b'}$, draw the path $P_d$ in the region $\mathcal{R}_{\hat{u}}$ and set $\mathcal{R}_{\hat{w}}$ as depicted in Figure 7(b)
      $ToConstruct := ToConstruct \cup \{(\hat{w}, \hat{u}, \mathcal{R}_{\hat{w}})\}$
    **else** $\deg(\hat{u}) = 1$
      Extend the paths $P_a$ and $P_b$ and draw the path $P_d$ in the region $\mathcal{R}_{\hat{u}}$ as depicted in Figure 8(b)
    **end if**
  **end while**
**end procedure**




(a) (b) (c)

Figure 6: (a) A part of a graph $G$ with $\widehat{G}$ and (b), (c) how Algorithm 4.3 constructs its $B_1$-EPG representation. Dotted edges may or may not exist.




(a) (b)

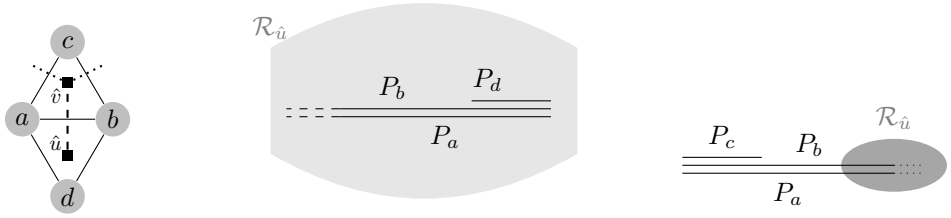
Figure 7: (a) A part of a graph $G$ with $\widehat{G}$ and (b) how Algorithm 4.3 constructs its $B_1$-EPG representation. Dotted edges may or may not exist.

The next result establishes the correctness of Algorithm 4.3.

**Lemma 4.22** *Let $G$ be a maximal outerplanar $M$-free graph with almost-dual $\widehat{G}$ and reduced graph $\widetilde{G}$. Then Algorithm 4.3 constructs a $B_1$-EPG representation of $G$.*

**Proof:** We first prove that all paths can be constructed as described in Algorithm 4.3. Towards that end, note that we have the following invariants throughout the algorithm. For each $(\hat{u}, \hat{v}, \mathcal{R}_{\hat{u}}) \in$ *ToConstruct* the vertices $\hat{u}$ and $\hat{v}$ are adjacent in $\widehat{G}$ and their corresponding triangles $T_{\hat{u}}$ and $T_{\hat{v}}$ share two vertices $a$ and $b$ in $G$. Furthermore the paths of the three vertices of $T_{\hat{v}}$ have already been constructed. Moreover $\mathcal{R}_{\hat{u}}$ indicates a region of the $B_1$-EPG representation, where the paths $P_a$ and $P_b$ intersect and where they can be extended such that no other paths are in this region so far. This implies that all the paths can really be constructed as described by Algorithm 4.3.

Next we prove that Algorithm 4.3 constructs exactly one path for each vertex of $G$. After the execution of the operations in lines 10 to 30 of the while loop for the triple $(\hat{u}, \hat{v}, \mathcal{R}_{\hat{u}})$ all paths corresponding to vertices of $T_{\hat{u}}$ have been constructed and all neighbors of $\hat{u}$ have been added to *ToConstruct*. This implies that for each vertex of $G$ the path representing it on the grid has been constructed at this time, thus the algorithm constructs one path per vertex. Further, the algorithm constructs new paths only for those vertices the paths of which have not been constructed yet. Thus the algorithm does not construct more than one path per vertex.




(a) (b) (c)

Figure 8: (a) A part of a graph $G$ with $\widehat{G}$ and (b) how Algorithm 4.3 constructs its $B_1$-EPG representation. Dotted edges may or may not exist. (c) The starting construction of Algorithm 4.3.

Next we show that two paths $P_a$ and $P_b$ intersect iff the vertices $a$ and $b$ are adjacent. When constructing the paths, it is easy to see that if $P_a$ and $P_b$ intersect, then their corresponding vertices are adjacent. On the other hand, if $a$ and $b$ are adjacent vertices of $G$, then there is a vertex $\hat{v}$ of $\widehat{G}$ such that $\{a, b\} \subseteq V(T_{\hat{v}})$. Algorithm 4.3 constructs the paths $P_a$ and $P_b$ as intersecting paths, when $\hat{v}$ or one of its neighbors containing both $a$ and $b$ is considered in lines 8 to 30.

Finally we show that every path has at most one bend. By construction a path $P_a$ corresponding to a vertex $a \in V(G)$ only bends if $a \in V(T_{\hat{v}})$ and $a$ was assigned to $\hat{v}$ in the assignment given by Algorithm 4.2. Due to Lemma 4.14 each vertex of $G$ is assigned to at most one vertex of $\widehat{G}$, hence each path $P_a$ bends at most once. □

Lemma 4.22 implies the following characterization of $B_1$-EPG maximal outerplanar graphs.

**Theorem 4.23** *Let $G$ be a maximal outerplanar graph. Then $G$ is $B_1$-EPG if and only if $G$ is $M$-free.*

**Proof:** If $G$ is not $M$-free, then $G$ is not $B_1$-EPG by Lemma 4.10. Otherwise, Algorithm 4.3 constructs a $B_1$-EPG representation according to Lemma 4.22, so $G$ is $B_1$-EPG. □

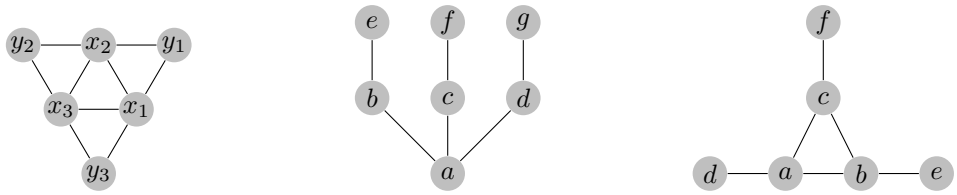
4.23 and 4.21 imply that it can be decided in $O(n^2)$ time whether a given maximal outerplanar graph $G$ of order $n$ is $B_1$-EPG. Furthermore, Algorithm 4.3 can be implemented to run in $O(n)$ time for an input graph $G$ of order $n$. Thus, a $B_1$-EPG representation of a $B_1$-EPG maximal outerplanar graph $G$ of order $n$ can be constructed in $O(n^2)$ time.

Recalling that all outerplanar graphs are in $B_2^m$ (see 2.4), we obtain the following full characterization of the maximal outerplanar graphs belonging to $B_0$, $B_1^m$, $B_1$ and $B_2^m$.

**Corollary 4.24** *The (monotonic) bend number of a maximal outerplanar graph $G$ is given as follows.*

$$b(G) = \begin{cases} 0 & \text{if } G \text{ is } S_3\text{-free} \\ 1 & \text{if } G \text{ is } M\text{-free but not } S_3\text{-free} \\ 2 & \text{otherwise} \end{cases}$$

$$b^m(G) = \begin{cases} 0 & \text{if } G \text{ is } S_3\text{-free} \\ 2 & \text{otherwise} \end{cases}$$




(a) (b) (c)

Figure 9: (a) The graph $S_3$. (b) The graph $M_2$. (c) The graph $M_3$.

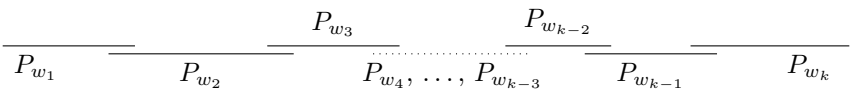



Figure 10: The $B_0$-EPG representation of the path $G'$ of Algorithm 5.1.

# 5 The (monotonic) bend number of cacti

**Definition 5.1** *A graph is called a cactus if it is connected and any two simple cycles in it have at most one vertex in common.*

It is easy to see that every cactus is outerplanar, so due to 2.4 cacti are in $B_2^m$. Notice that cacti are in some sense the opposite to maximal outerplanar graphs: in a cactus edges which connect non-consecutive vertices belonging to the same simple cycle are not allowed, whereas in maximal outerplanar graphs there have to be as many edges as possible connecting such vertices.

## 5.1 Cacti in $B_0$

Consider first some simple cacti which are not in $B_0$: cycles $C_r$ for $r \geqslant 4$, and the graphs $M_2$ and $M_3$ depicted in Figure 9. Indeed, as shown in [22] (cf. Introduction) an interval graph (a) does not contain an induced cycle of length more than three and (b) in any triple of pairwise distinct and pairwise non-adjacent vertices, there exists at least one vertex which is a neighbor to any path connecting the two other vertices. It can be easily seen that none of the graphs mentioned above fulfills both (a) and (b), hence none of these graphs is an interval graph. Therefore, they are not contained in $B_0$ (and equivalently in $B_0^m$), because a graph $G$ is in $B_0$ (and equivalently in $B_0^m$) iff $G$ is an interval graph, as mentioned in Section 1.

**Definition 5.2** *A cactus is called* $MC$-free *if it contains neither $M_2$, nor $M_3$, nor $C_r$, for any $r \geqslant 4$, as an induced subgraph.*

Since $C_r$, for any $r \geqslant 4$, $M_2$ and $M_3$ are not in $B_0$, a cactus that is not $MC$-free is not in $B_0$. The next theorem shows that the converse is also true. Its constructive proof is based on Algorithm 5.1.

**Theorem 5.3** *A cactus $G$ is in $B_0$ if and only if $G$ is $MC$-free.*

**Algorithm 5.1** Construct a $B_0$-EPG representation of an $MC$-free cactus

**Input:** An $MC$-free cactus $G = (V(G), E(G))$ and the set $\delta_G(v)$ of edges incident to $v$ in $G$, $\forall v \in V(G)$
**Output:** A $B_0$-EPG representation of $G$

**procedure** B0_MCfree_cactus($G$)
  Set $\bar{\Delta} := \langle\rangle$ and $\Delta' := \langle\rangle$, where $\langle\rangle$ denotes an empty list
  Set $\bar{G} := G$
  **while** $G$ contains a triangle with vertex set $\{v_1, v_2, v_3\}$ **do**
    Determine a vertex $v$ of degree two in $\{v_1,v_2,v_3\}$
    Set $V(\bar{G}) := V(\bar{G}) \setminus \{v\}$ and $E(\bar{G}) := E(\bar{G}) \setminus \{\delta_G(v)\}$
    Set $\bar{\Delta} := \langle\bar{\Delta}, v\rangle$
  **end while**
  **for** $v \in V(\bar{G})$ **do**
    If $v$ is a leaf in $\bar{G}$, set $\Delta' := \langle\Delta', v\rangle$
  **end for**
  Set $V(G') := V(\bar{G}) \setminus \Delta'$, $E(G') := E(\bar{G}) \setminus (\cup_{v\in\Delta'}\delta_{\bar{G}}(v))$, where $\delta_{\bar{G}}(v)$ is the set of edges incident with $v$ in $\bar{G}$
  Set $G' = (V(G'), E(G'))$
  Set $k := |V(G')|$
  **for** $i = 1, \ldots, k$ **do**
    Construct the path $P_{w_i}$ corresponding to the $i$-vertex $w_i$ in $G'$ as shown in Figure 10
  **end for**
  **for** $v \in \Delta'$ **do**
    Let $w \in V(\bar{G})$ such that $\{(w, v)\} = \delta_{\bar{G}}(v)$
    Construct a path $P_v$ representing $v$ as a subpath of $P_w$ which is not contained in any other path constructed already
  **end for**
  **for** $v \in \bar{\Delta}$ **do**
    Construct a path $P_v$ representing $v$ as a common edge of the (already constructed) paths $P_{v_1}$, $P_{v_2}$ for $\{(v, v_1), (v, v_2)\} = \delta_G(v)$
  **end for**
**end procedure**

**Proof:** As mentioned in Section 5.1 a graph which is not $MC$-free is not in $B_0$. The sufficient condition of 5.3 is proven by construction, more precisely, by showing that Algorithm 5.1 correctly constructs a $B_0$-EPG representation of an $MC$-free cactus $G$.

Notice that if $G$ contains any cycles, then those are triangles. Furthermore notice that if the vertex set $\{v_1, v_2, v_3\}$ induces a triangle in $G$, then there is no vertex $v_4 \in V(G) \setminus \{v_1, v_2, v_3\}$ adjacent to at least two (assume w.l.o.g. $v_1$ and $v_2$) vertices of $\{v_1, v_2, v_3\}$, because otherwise the simple cycles $\{v_1, v_2, v_3\}$ and $\{v_1, v_2, v_4\}$ would share an edge and contradict the property of $G$ being a cactus. Hence if the vertex set $\{v_1, v_2, v_3\}$ induces a triangle in $G$, then at least one vertex in $\{v_1, v_2, v_3\}$ has degree 2, otherwise $G$ would contain $M_3$ as an induced subgraph. Assume w.l.o.g. that $v_3$ has degree 2.

If the graph $G \setminus v_3$ obtained from $G$ by deleting the vertex $v_3$ and its incident edges $\{v_1, v_3\}$, $\{v_2, v_3\}$ had a $B_0$-EPG representation, the latter could be extended to a $B_0$-EPG representation of $G$ by simply representing $v_3$ by just one grid edge in the $B_0$-EPG representation of $G \setminus v_3$ shared by the paths $P_{v_1}$ and $P_{v_2}$ corresponding to $v_1$ and $v_2$, respectively. Obviously, this argument holds for any triangle in $G$. Following this idea, Algorithm 5.1 first constructs the graph $\bar{G}$ obtained by removing from $G$ one vertex of degree 2 together with its incident edges from every triangle in $G$. This is done in lines 4-8. In lines 9-21 the algorithm constructs a $B_0$-EPG representation of $\bar{G}$, which is then extended to a $B_0$-EPG representation of $G$ in lines 22-24 as explained above.

Next consider the construction of a $B_0$-EPG representation of $\bar{G}$. Observe first that $\bar{G}$ is a cactus which does not contain an induced $M_2$ or $C_r$, for any $r \geqslant 3$. Thus $\bar{G}$ is a tree. More precisely $\bar{G}$ is a so-called caterpillar, i.e. the graph $G'$ obtained from $\bar{G}$ by deleting all its leaves and the edges incident to them is a path. Indeed, there is no vertex $v$ with degree more than 2 in $G'$, because if a vertex $v$ with 3 neighbors $v_1$, $v_2$, $v_3$ would exists in $G'$, then each of $v_1$, $v_2$, $v_3$ would have degree at least 2 in $\bar{G}$ and $\bar{G}$ would contain $M_2$ as an induced subgraph.

Let $G'$ be the path $(w_1, w_2, \ldots, w_k)$ for some $k \leqslant |V(G)|$ and $w_i \in V(G)$, $1 \leqslant i \leqslant k$. It is straightforward to construct a $B_0$-EPG representation of $(w_1, w_2, \ldots, w_k)$ as depicted in Figure 10. Moreover this $B_0$-EPG representation of $G'$ can be easily extended to a $B_0$-EPG representation of $\bar{G}$ in the following way. For any leaf $v$ of $\bar{G}$ consider its neighbor $w_\ell$ in $\bar{G}$. Since $w_\ell$ belongs to $G'$, there is a path $P_{w_\ell}$ representing $w_\ell$ in the $B_0$-EPG representation of $G'$. Construct a subpath $P_v$ of $P_{w_\ell}$ to represent $v$, such that $P_v$ does not intersect any other path constructed so far. Obviously this constriction yields a $B_0$-EPG representation of $\bar{G}$. Following these ideas the algorithm first constructs the path $G'$ in lines 9-13, then it constructs the $B_0$-EPG representation of $G'$ in lines 14-17 and finally it extends the latter to a $B_0$-EPG representation of $\bar{G}$ in lines 18-21. □

A standard analysis of the time complexity of Algorithm 5.1 reveals that a $B_0$-EPG representation a cactus of order $n$ belonging to $B_0$ can be constructed in $O(n \log(n))$ time. The existence of a faster construction of a $B_0$-EPG representation of such a cactus remains an open problem.

## 5.2 Cacti in $B_1$

In this section we show that every cactus belongs to $B_1$ by demonstrating that Algorithm 5.2 constructs a $B_1^m$-EPG representation of an arbitrary cactus.

**Theorem 5.4** *Every cactus is in $B_1^m$.*

**Proof:** We show that Algorithm 5.2 correctly constructs a $B_1^m$-EPG representation for an arbitrary input cactus $G$. Note that during the algorithm each vertex has a color $col(v)$ and that the colors of

**Algorithm 5.2** Construct a $B_1^m$-EPG representation of a cactus $G$

**Input:** A cactus $G = (V(G), E(G))$ with $n = |V(G)|$ and the set $\delta(v)$ of edges incident with $v$ in $G$, $\forall v \in V(G)$
**Output:** A $B_1^m$-EPG representation of $G$

**procedure** B1M_CACTUS($G$)
  Set $col(v) := gray$ for all $v \in V(G)$
  Select an arbitrary vertex $v_0 \in V(G)$
  Set $col(v_0) := green$
  Draw the path $P_{v_0}$ representing $v_0$ as a straight horizontal line on the grid
  Set the free part $\mathcal{R}_{v_0}$ of $P_{v_0}$ to be the whole grid
  **while** There is a green vertex in $V(G)$ **do**
    Select a vertex $v \in V(G)$ with $col(v) = green$
    Determine all $k_v$ cycles $C_1(v)$, ..., $C_{k_v}(v)$ in $G$, which contain $v$ and only gray vertices except for $v$ ($k_v \in \mathbb{N} \cup \{0\}$)
    Determine $C_0(v) := \{u \in V(G) \colon \{u, v\} \in \delta(v), col(u) = gray\} \setminus \cup_{i=1}^{k_v} V(C_i(v))$
    **for** $u \in C_0(v)$ **do**
      Construct the path $P_u$ representing $u$ such that it lies in the free part $\mathcal{R}_v$ of $P_v$ and only intersects $P_v$, but no other already constructed path, as shown in Figure 11(a)
      Set $col(u) := green$
    **end for**
    **for** $i = 1, \ldots, k_v$ **do**
      Let $\langle v, u_1, u_2, \ldots, u_{\ell-1}, v\rangle$ be the vertices of $C_i(v)$ in the order they are visited in the cycle $C_i(v)$
      **for** $j = 1, \ldots, \ell - 1$ **do**
        Construct the grid path $P_{u_j}$ representing $u_j$ such that
        (1) $P_{u_j}$ lies on the free part $\mathcal{R}_v$ of $P_v$
        (2) $P_{u_j}$ intersects $P_{u_{j-1}}$
        (3)$P_{u_j}$ intersects $P_v$ if $j = \ell - 1$
        (4) $P_{u_j}$ does not intersect other already constructed paths
        as shown in Figure 11(b), Figure 11(c) and Figure 11(d) for $\ell = 3$, $\ell = 4$ and $\ell \geqslant 5$, respectively
        Set $col(u_j) := green$
      **end for**
    **end for**
    Set $col(v) := red$
  **end while**
**end procedure**

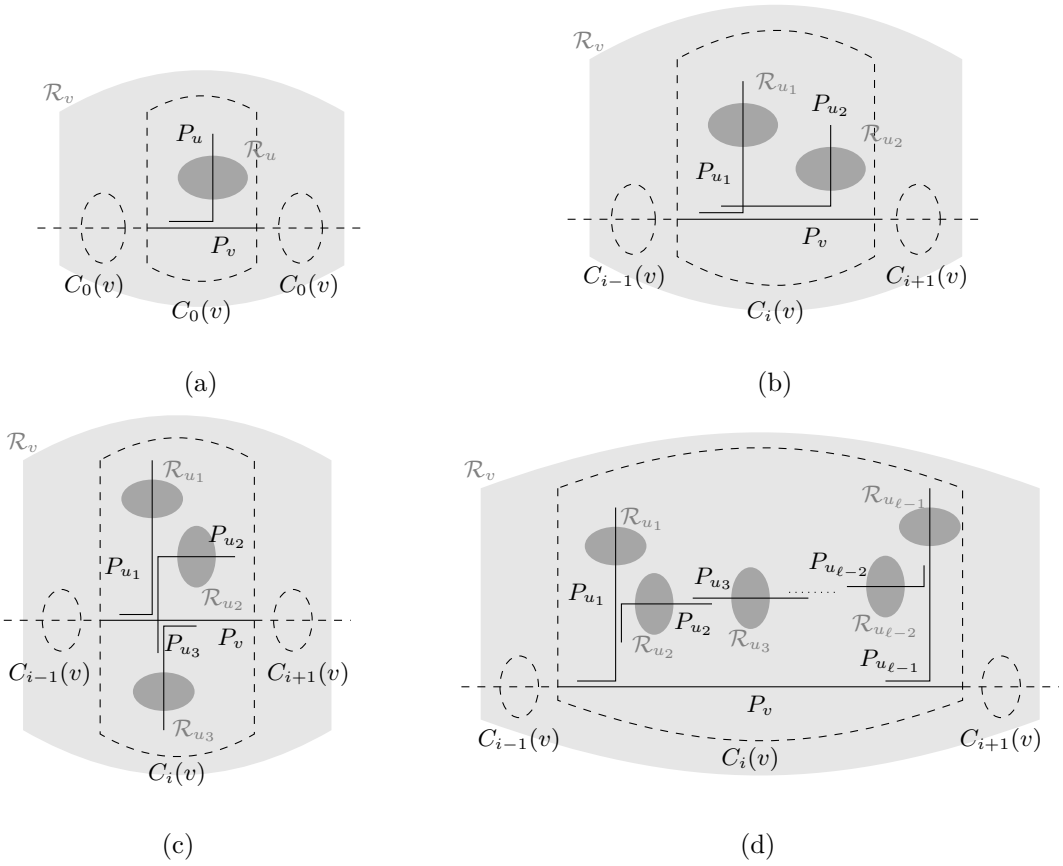



Figure 11: The construction of a $B_1^m$-EPG representation of a cactus with free parts of the paths. These constructions are used if $\mathcal{R}_v$ lies on a horizontal line of $P_v$. If $\mathcal{R}_v$ lies on a vertical line of $P_v$, then the constructions have to be rotated by $90^o$ clockwise and then flipped vertically.

the vertices change during the algorithm. At the beginning all vertices are colored gray. A vertex $v$ is gray iff the corresponding grid path $P_v$ has not been constructed yet. A vertex $v$ turns green as soon as $P_v$ has been constructed. Finally, $v$ is colored red as soon as the paths corresponding to all neighbors of $v$ have been constructed in the $B_1^m$-EPG representation.

Essentially the algorithm consists of a while loop which is repeated as long as there exist green vertices. After selecting a green vertex $v$ (line 8), the algorithm *explores* $v$ in the lines 9-27. Then, after the exploration, $v$ turns red in line 27. The algorithm terminates when all green vertices have turned red, hence after repeating the while loop $|V|$ times.

Notice that whenever a path $P_v$ is constructed, also a free region $\mathcal{R}_v$ is indicated, in which a part of one of the two segments of $P_v$ is contained and in which no other paths have been constructed yet.

Observe further that the following properties IA1, IA2 and IA3 always hold when the exploration of some vertex is completed, i.e. at line 27. We refer to these three properties as the *invariants of the algorithm*.

(IA1) For any green or red vertex $u$ there is an $\ell \in \mathbb{N} \cup \{0\}$ and a sequence $S_u = \langle v_0, v_1, \ldots, v_\ell \rangle$ of red vertices such that $u$ has turned green during the exploration of $v_\ell$, and if $\ell \geqslant 1$, $v_j$ has turned green during the exploration of $v_{j-1}$ for all $j \in \{1, \ldots, \ell\}$.

(IA2) Moreover, if $\ell \geqslant 1$, then any two vertices $v_j$, $v_{j+1}$, $0 \leqslant j \leqslant \ell - 1$, in $S_u$ are connected by a path of non-gray vertices all of which have been gray when the exploration of $v_j$ started.

(IA3) In particular, for any green or red vertex $u$ there is a path $Q_{v_0,u}$ consisting of non-gray vertices connecting $v_0$ and $u$ in $G$.

The proof of the theorem is completed by proving the next four claims.

(i) Consider a green vertex $v$ currently selected in line 8 of Algorithm 5.2 and a cycle $C$ which contains $v$. Let $V(C)$ denote the set of vertices in $C$. Then either all or none of the vertices in $V(C) \setminus \{v\}$ are gray.

(ii) For every vertex $u$ of $G$ the algorithm constructs exactly one path on the grid representing $u$.

(iii) All constructed paths are monotonic and have at most one bend.

(iv) Two paths $P_u$ and $P_w$ on the grid intersect iff the corresponding vertices $u$ and $w$ are adjacent in $G$.

Proof of (i). This fact is trivially true if $v = v_0$ (i.e. $v$ is the first vertex explored at all). Assume now that $v \neq v_0$ and that there is a cycle $C$ in $G$ containing $v$ as well as some gray vertex $u$ and some non-gray vertex $w \neq v$. Then $C$ does not contain $v_0$ (otherwise all vertices in $C$ would have been colored green during the exploration of $v_0$). IA3 implies that there is a path $Q_{v_0,v}$ connecting $v_0$ and $v$ and consisting of non-gray vertices, and that there is also a path $Q_{v_0,w}$ connecting $v_0$ and $w$ and consisting of non-gray vertices. But then $Q_{v_0,v}$, $Q_{v_0,w}$ and a path $Q_{v,w}$ connecting $v$ and $w$ along $C$ close a cycle $C'$. Moreover $C$ and $C'$ intersect along path $Q_{v,w}$ which contains at least two vertices, and this is a contradiction to $G$ being a cactus.

Proof of (ii). The construction of at least one path $P_u$ representing $u$, for every vertex $u$ of $G$, follows from the connectivity of $G$. Indeed consider a path $Q_{v_0,u} = (v_0, v_1, \ldots, v_\ell = u)$ connecting $v_0$ and $u$ in $G$. The vertex $v_1$ is colored green during the exploration of $v_0$. Since the algorithm does not terminate before the exploration of all green vertices, at some point $v_1$ will be explored

and then $v_2$ will turn green at the latest. By repeating this argument along the vertices of $Q_{v_0,u}$ we conclude that $u$ will turn green at some point, meaning that a grid path $P_u$ representing $u$ is constructed.

A path corresponding to a vertex is constructed only if a vertex turns green. Furthermore, each vertex turns green only once, because during the exploration of vertex $v$ the sets $V(C_i(v))$, $0 \leqslant i \leqslant k_v$ have pairwise empty intersection since $G$ is a cactus. Hence during the exploration of a vertex (only) gray vertices turn green at most once. Hence for each vertex exactly one corresponding path is constructed.

Proof of (iii). This claim follows directly from the construction: in Figure 11 all depicted paths are monotonic and have at most one bend. Moreover a $90^o$ clockwise rotation of the paths depicted in Figure 11 followed by a vertical flipping yields monotonic paths with at most one bend.

Proof of (iv). We first consider an edge $\{u, w\}$ in $G$ and show that $P_u$ intersects $P_w$. Assume w.l.o.g. that $w$ was explored before $u$ by the algorithm.

If $u$ was gray at the moment when $w$ is explored, then $u$ belongs to some $C_i(w)$, $0 \leqslant i \leqslant k_w$, and by construction $P_u$ intersects $P_w$.

Assume now that $u$ is not gray when $w$ is explored. Let $v_u$ ($v_w$) be the vertex during the exploration of which $u$ ($w$) turned green. If $v_u = v_w$, then $u$, $w$, $v_u$ are contained in one of the cycles $C_i(v_u)$, $1 \leqslant i \leqslant k_{v_u}$ and $P_u$, $P_v$ intersect by construction.

Next we show that $v_u \neq v_w$ cannot happen. Indeed, if $v_u \neq v_w$, then consider the sequences $S_{v_u}$ and $S_{v_w}$ described in IA1. Notice that both sequences start with $v_0$ and denote by $x$ the last common vertex in $S_{v_u}$ and $S_{v_w}$. Clearly $x \neq v_u$ or $x \neq v_w$ by assumption. IA2 implies the existence of two paths $Q_{u,x}$, $Q_{w,x}$ in $G$ joining $x$ with $u$ and $w$, respectively, and (except for $x$) consisting of gray vertices at the moment of the exploration of $x$. But then $Q_{u,x}$, $Q_{w,x}$ and $\{u, w\}$ would close a cycle with all vertices but $x$ being gray when $x$ is selected for exploration. This would imply that both $u$ and $w$ turn green during the exploration of $x$, hence $v_u = v_w = x$ would hold.

To summarize, in all possible cases whenever $u$ and $w$ are adjacent in $G$, the paths $P_u$ and $P_v$ intersect.

Finally, we show that two vertices $u$ and $w$ of $G$ are adjacent, whenever the corresponding grid paths $P_u$, $P_w$ intersect. Assume w.l.o.g. that $P_w$ was constructed by the algorithm before $P_u$. Observe that by construction all grid paths constructed after the completion of the exploration of $w$ do not intersect $P_w$. Thus $P_u$ has been constructed during the exploration of $w$. But then, by construction $P_w$ and $P_u$ intersect iff $w$ and $u$ are neighbors. □

A standard time complexity analysis of Algorithm 5.2 reveals that a $B_1^m$-EPG representation of a cactus $G$ of order $n$ can be constructed in $O(n^2)$ time. The existence of a faster algorithm remains an open question.

Finally, we summarize the results of this section as follows.

**Corollary 5.5** *The (monotonic) bend number of a cactus $G$ is given as follows.*

$$b(G) = b^m(G) = \begin{cases} 0 & \text{if } G \text{ contains no copy of } M_2,\ M_3,\ C_r, \text{ for } r \geqslant 4, \\ 1 & \text{otherwise} \end{cases}$$

# 6 Conclusions and open problems

In this paper we focused on EPG representations and on the (monotonic) bend number of outerplanar graphs. In particular, we dealt with two subclasses of outerplanar graphs: the maximal

outerplanar graphs and the cacti. The main contribution of the paper is the full characterization of graphs with (monotonic) bend number equal to 0, 1 or 2 in the subclasses mentioned above. All presented proofs are constructive and lead to efficient algorithms for the construction of the corresponding EPG representations.

It was already known from [4, 20] that 2 is the best possible upper bound on the bend number of outerplanar graphs. In this paper we showed that 2 is also an upper bound on the monotonic bend number of outerplanar graphs, i.e. $b^m(G) \leqslant 2$ holds for every outerplanar graph $G$. The result of [4] implies the existence of an outerplanar graph which is not in $B_1^m$, so we conclude that 2 is a best possible upper bound on the monotonic bend number of outerplanar graphs. Thus, the best possible upper bounds on the monotonic bend number and on the bend number coincide for outerplanar graphs.

In [20] the inequality $b(G) \leqslant 2$ for the class of graphs $G$ of treewidth at most 2 is proven, the outerplanar graphs being a proper subclass of the later. So far no upper bound is known on $b^m(G)$ for graphs $G$ with treewidth bounded by 2. In particular, it is an open question whether the coincidence mentioned above extends to the class of graphs with treewidth bounded by 2.

Another challenging open question concerns the monotonic bend number of planar graphs. As shown in [20], $b(G) \leqslant 4$ holds for every planar graph. Also the existence of a planar graph with bend number at least 3 is shown in [20]. So the best possible upper bound on the bend number of planar graphs is 3 or 4. With respect to the upper bound on the monotonic bend number of planar graphs we only now that it is at least 3.

Our precise results on the (monotonic) bend number of maximal outerplanar graphs can be summarized as follows. We distinguished two types of maximal outerplanar graphs: (I) maximal outerplanar graphs which do not contain $S_3$ as an induced subgraph, and (II) maximal outerplanar graphs which do contain $S_3$ as an induced subgraph. We showed that $b(G) = b^m(G) = 0$ holds for every graph $G$ of type I. Thus for maximal outerplanar graphs the classes $B_0$ and $B_1^m$ coincide, hence allowing two more shapes of paths in the EPG representation does not increase the class of maximal outerplanar graphs, which can be represented. For graphs of type II we showed that $b^m(G) = 2$ holds, while $b^m(G) = b(G)$ does not necessarily hold. More precisely, for graphs $G$ of type II the equality $b(G) = 1$ holds iff $G$ is $M$-free (see Definition 4.11 and 4.23). Otherwise $b(G) = 2$ holds.

Our results on cacti can be summarized as follows. For cacti the monotonic bend number and the bend number coincide and are bounded by 1, i.e. and $b(G) = b_1^m(G) \leqslant 1$ holds for every cactus $G$. Furthermore $b(G) = b^m(G) = 0$ holds iff the cactus $G$ is $MC$-free (see Definition 5.2). Otherwise $b(G) = b^m(G) = 1$ holds.

Observe that the two investigated subclasses of outerplanar graphs, the maximal outerplanar graphs and the cacti, behave quite differently in terms of the (monotonic) bend number. This is not surprising given the major structural differences of these two graph classes.

The full characterization of outerplanar graphs with a (monotonic) bend number equal to 1 or 2 remains a challenging open question. In particular, the characterization of Halin graphs and series-parallel graphs with a (monotonic) bend number equal to 1 or 2 are interesting open questions. Note that both graphs classes belong to the planar graphs and 2 is an upper bound on the bend number for both of them. In [14] it was shown that $b^m(G) \leqslant 2$ for every Halin graph $G$. In the case of series-parallel graphs the upper bound of 2 follows form the results in [20] and the fact that the treewidth of series-parallel graphs is bounded by 2, see [5].

## References


[1] A. Asinowski, E. Cohen, M. Golumbic, V. Limouzy, M. Lipshteyn, and M. Stern. Vertex intersection graphs of paths on a grid. *Journal of Graph Algorithms and Applications*, 16(2):129–150, 2012. doi:10.7155/jgaa.00253.

[2] A. Asinowski and B. Ries. Some properties of edge intersection graphs of single-bend paths on a grid. *Discrete Mathematics*, 312(2):427–440, 2012. doi:10.1016/j.disc.2011.10.005.

[3] A. Asinowski and A. Suk. Edge intersection graphs of systems of paths on a grid with a bounded number of bends. *Discrete Applied Mathematics*, 157(14):3174–3180, 2009. doi:10.1016/j.dam.2009.06.015.

[4] T. Biedl and M. Stern. Edge-intersection graphs of $k$-bend paths in grids. In *Computing and Combinatorics*, pages 86–95. Springer, 2009. doi:10.1007/978-3-642-02882-3_10.

[5] H. L. Bodlaender. A partial $k$-arboretum of graphs with bounded treewidth. *Theoretical Computer Science*, 209(1–2):1 – 45, 1998. doi:10.1016/S0304-3975(97)00228-4.

[6] F. Bonomo, M. a. P. a. Mazzoleni, and M. Stein. Clique coloring $B_1$-EPG graphs. *Discrete Mathematics*, 340(5):1008–1011, 2017. doi:10.1016/j.disc.2017.01.019.

[7] K. S. Booth and G. S. Lueker. Testing for the consecutive ones property, interval graphs, and graph planarity using pq-tree algorithms. *Journal of Computer and System Sciences*, 13(3):335–379, 1976. doi:10.1016/S0022-0000(76)80045-1.

[8] M. Bougeret, S. Bessy, D. Gonçalves, and C. Paul. On independent set on B1-EPG graphs. In *Approximation and online algorithms*, volume 9499 of *Lecture Notes in Comput. Sci.*, pages 158–169. Springer, Cham, 2015. doi:10.1007/978-3-319-28684-6_14.

[9] N. Bousquet and M. Heinrich. Computing maximum cliques in $B_2$-EPG graphs. In *Graph-theoretic concepts in computer science*, volume 10520 of *Lecture Notes in Comput. Sci.*, pages 140–152. Springer, Cham, 2017. doi:10.1007/978-3-319-68705-6_11.

[10] M. L. Brady and M. Sarrafzadeh. Stretching a knock-knee layout for multilayer wiring. *IEEE Transactions on Computers*, 39(1):148–151, Jan 1990. doi:10.1109/12.46293.

[11] K. Cameron, S. Chaplick, and C. T. Hoàng. Edge intersection graphs of L-shaped paths in grids. *Electronic Notes in Discrete Mathematics*, 44:363–369, 2013. doi:10.1016/j.dam.2015.01.039.

[12] Z. Deniz, S. Nivelle, B. Ries, and D. Schindl. On split $B_1$-EPG graphs. In *LATIN 2018: Theoretical Informatics*, volume 10807 of *Lecture Notes in Comput. Sci.*, pages 361–375. Springer, Cham, 2018. doi:10.1007/978-3-319-77404-6_27.

[13] D. Epstein, M. C. Golumbic, and G. Morgenstern. Approximation algorithms for $B_1$-EPG graphs. In *Algorithms and Data Structures*, pages 328–340. Springer, 2013. doi:10.1007/978-3-642-40104-6_29.

[14] M. C. Francis and A. Lahiri. VPG and EPG bend-numbers of Halin graphs. *Discrete Appl. Math.*, 215:95–105, 2016. doi:10.1016/j.dam.2016.07.007.

[15] F. Gavril. A recognition algorithm for the intersection graphs of paths in trees. *Discrete Mathematics*, 23(3):211 – 227, 1978. doi:10.1016/0012-365X(78)90003-1.

[16] M. C. Golumbic and R. E. Jamison. Edge and vertex intersection of paths in a tree. *Discrete Mathematics*, 55(2):151 – 159, 1985. doi:10.1016/0012-365X(85)90043-3.

[17] M. C. Golumbic and R. E. Jamison. The edge intersection graphs of paths in a tree. *Journal of Combinatorial Theory, Series B*, 38(1):8 – 22, 1985. doi:10.1016/0095-8956(85)90088-7.

[18] M. C. Golumbic, M. Lipshteyn, and M. Stern. Edge intersection graphs of single bend paths on a grid. *Networks*, 54(3):130–138, 2009. doi:10.1002/net.20305.

[19] M. C. Golumbic, M. Lipshteyn, and M. Stern. Single bend paths on a grid have strong helly number 4: errata atque emendationes ad "Edge intersection graphs of single bend paths on a grid". *Networks*, 62(2):161–163, 2013. doi:10.1002/net.21509.

[20] D. Heldt, K. Knauer, and T. Ueckerdt. On the bend-number of planar and outerplanar graphs. In *LATIN 2012: Theoretical Informatics*, pages 458–469. Springer, 2012. doi:10.1007/978-3-642-29344-3_39.

[21] D. Heldt, K. Knauer, and T. Ueckerdt. Edge-intersection graphs of grid paths: The bend-number. *Discrete Applied Mathematics*, 167(0):144 – 162, 2014. doi:10.1016/j.dam.2013.10.035.

[22] C. Lekkeikerker and J. Boland. Representation of a finite graph by a set of intervals on the real line. *Fundamenta Mathematicae*, 51:45–64, 1962. doi:10.4064/fm-51-1-45-64.

[23] P. Molitor. A survey on wiring. *Journal of Information Processing and Cybernetics*, 27(1):3–19, Apr. 1991. doi:10.5555/108118.108119.

[24] M. Pergel and P. Rzążewski. On edge intersection graphs of paths with 2 bends. In *Graph-theoretic concepts in computer science*, volume 9941 of *Lecture Notes in Comput. Sci.*, pages 207–219. Springer, Berlin, 2016. doi:10.1007/978-3-662-53536-3_18.