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Homogenisation of the Stokes equations for evolving microstructure

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Abstract

We consider the homogenisation of the quasi-stationary Stokes equations in a porous medium that evolves over time. The evolution is a priori given. At the interface of the pore space and the solid part, we prescribe an inhomogeneous Dirichlet boundary condition, which enables a no-slip boundary condition at the evolving boundary. We pass rigorously to the homogenisation limit employing the two-scale transformation method. In order to derive uniform a priori estimates, we show a Korn-type inequality for the two-scale transformation method. The homogenisation result is a new version of Darcy's law. It features a time- and space-dependent permeability tensor, which accounts for the local pore structure, and a macroscopic inhomogeneous divergence condition, which induces a new source term for the pressure. In the case of a no-slip boundary condition at the interface, this source term relates to the change of the local pore volume.

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Keywords: Darcy's law; Stokes equations; Homogenisation; Evolving microstructure; Two-scale transformation method

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1. Introduction

Effective fluid flow in completely saturated porous media can be described by Darcy's law. Based on results of experiments, it was formulated by Henry Darcy in [1]. This empirical law can be justified by means of homogenisation theory on the basis of general laws of fluid dynamics. Using the formal two-scale expansion, Darcy's law was derived from the Stokes equations (see for example [2–4]). For a periodically perforated porous medium with disconnected solid matrix, L. Tartar proved the convergence of the homogenisation for the following ε-scaled problem in [5]. He considered the Stokes equations in a periodic porous medium Ω with fluid phase Ω_ε (of period ε) with homogeneous Dirichlet boundary conditions

$$-\varepsilon^2 \nu \Delta v_\varepsilon + \nabla p_\varepsilon = f \text{ in } \Omega_\varepsilon, \qquad \operatorname{div}(v_\varepsilon) = 0 \text{ in } \Omega_\varepsilon, \qquad v_\varepsilon = 0 \text{ on } \partial\Omega_\varepsilon, \tag{1}$$

where v_ε and p_ε denote the velocity and pressure of the fluid, ν the viscosity and f the density of the forces acting on the fluid. He proved that the extension of v_ε to Ω by 0 converges weakly to v in $L^2(\Omega)$ and that P_ε, which is an extension of the pressure p_ε to the solid part of Ω, converges strongly in $L^2(\Omega)/\mathbb{R}$ to p, where v and p are the unique solutions of

$$v = \frac{1}{\nu} K(f - \nabla p) \text{ in } \Omega, \qquad \operatorname{div}(v) = 0 \text{ in } \Omega, \qquad v \cdot n = 0 \text{ on } \partial\Omega, \tag{2}$$

and K is a positive definite symmetric permeability tensor, which can be computed by means of solutions of the Stokes problems on a reference cell. The main task in the proof of the convergence is the derivation of an ε-independent bound for p_ε. At this point, Tartar had to assume that the solid part of a cell is strictly contained inside the cell. Extending the ideas of L. Tartar, G. Allaire could omit this assumption and proved the convergence for more general domains in [6].

The homogenisation of the instationary Navier–Stokes equations was considered in [7] leading to a Darcy law. A different ε-scaling of the time-derivative term leads to a Darcy-type law with memory, which is an integro–differential equation (see [8,9]). Moreover, by scaling the viscosity, a non-linear Darcy law was derived in [4,10]. In the case of disconnected solid obstacles, one can also consider different ε-scalings of the obstacles compared to the cell size see, which lead to different limit equations as for instance Brinkman flow (see [11,12]).

The purpose of the present paper is to extend problem (1) to a domain $\Omega_\varepsilon(t)$ whose microstructure is evolving in time and to prove the convergence of the homogenisation for this new setting. The case of an evolving microstructure is motivated by many different physical, chemical and biological applications. For example, for dissolution and precipitation in a porous medium, a precipitate layer may be added to or be dissolved from the pore walls, implying that the overall solid part (and, implicitly, the void space) is evolving unless there is a local balance between precipitation and dissolution, see e.g. [13]. In [13–20], such processes are modelled as free boundaries by means of a level-set function or phase-field approaches. However, these models are only formally upscaled by asymptotic two-scale expansions. For fixed microstructure, related (advection–) diffusion problems were homogenised rigorously in [21–23].

For given evolving microstructure evolution, rigorous homogenisation results are presented in [24–28]. There, the equations are transformed to a fixed microstructure and the resulting substitute problems were homogenised. For a general class of transformations, it was shown in

[29] that the homogenisation and the transformation commutes, which justifies this transformation approach. This approach was also used in [30,31] for the rigorous homogenisation of an reaction–diffusion problem with free boundary where the evolution of the domain is coupled with the unknown concentration.

The homogenisation of fluid flow in evolving porous media is also important for problems in poroelasticity. The first linear theory was developed by Biot (cf. [32,33]). Starting with a description of the microporomechanics by equations of elasticity and fluid flow, effective equations can also be derived by means of homogenisation (cf. [34], [35]). However, in order to homogenise rigorously, the Stokes problem was linearised by assuming that the fluid domain is constant in time (cf. [36]). Recently, the corresponding non-linear model received considerable attention (cf. [37–39]). However, these works passed to the homogenisation limit only formally. In this paper, we provide a rigorous homogenisation result for the decoupled Stokes-problem, which is a step towards the homogenisation of the fully coupled fluid–structure interaction problem.

In the present paper, we consider the rigorous homogenisation of the quasi-stationary Stokes equations for ε-scaled domains $\Omega_\varepsilon(t)$ that are evolving in time. The evolution of the domain is a priori given. Thus, we consider the Stokes problem

$$-\varepsilon^2 \nu \operatorname{div}(2e(v_\varepsilon)) + \nabla p_\varepsilon = f_\varepsilon \qquad \text{in } \Omega_\varepsilon(t), \tag{3a}$$

$$\operatorname{div}(v_\varepsilon) = 0 \qquad \text{in } \Omega_\varepsilon(t), \tag{3b}$$

$$v_\varepsilon = v_{\Gamma_\varepsilon} \qquad \text{on } \Gamma_\varepsilon(t), \tag{3c}$$

$$p_\varepsilon n - \varepsilon^2 \nu 2e(v_\varepsilon)^\top n = p_{b,\varepsilon} n \qquad \text{on } \partial\Omega_\varepsilon(t) \setminus \Gamma_\varepsilon(t) \tag{3d}$$

for a force term f_ε on a time-dependent spatial domain $\Omega_\varepsilon(t)$ with $t \in S$ for the time interval S. Thereby, $e(v_\varepsilon) := (\nabla v_\varepsilon + \nabla v_\varepsilon^\top)/2$ denotes the symmetric gradient of v_ε, which we use noting that in the standard derivation of the Stokes equation from the momentum balance equation, the continuity equation and the axioms of Newtonian fluids originally imply a symmetric stress tensor. Indeed, the incompressibility condition allows us to replace $2e(v_\varepsilon)$ by ∇v_ε in the strong formulation. However, for the weak formulation, this substitution can be done only for certain boundary values as for instance homogeneous Dirichlet boundary values, which is not the case in our model. For the boundary condition, we distinguish between the interface of pore and solid space, which is denoted by $\Gamma_\varepsilon(t)$, and the remaining boundary. At $\Gamma_\varepsilon(t)$, we use a Dirichlet boundary condition for the fluid velocity with boundary values v_{Γ_ε}. This (inhomogeneous) Dirichlet boundary condition (3c) is motivated by the no-slip boundary condition and allows a fluid velocity equal to the velocity of the boundary's deformation, which can be modelled by v_{Γ_ε}. By using a normal stress boundary condition with an outer unit normal vector n and a normal boundary stress $p_{b,\varepsilon}$ at $\partial\Omega_\varepsilon(t) \setminus \Gamma_\varepsilon(t)$, we allow fluid in- and outflow at the boundary of the porous medium. Thus, even if the total pore volume changes, there is no incompatibility with the fluid incompressibility, so that we can consider this case as well. In [40], the homogenisation of Stokes flow with such a normal stress boundary condition at the outer boundary is considered for the case of a rigid domain and a homogeneous Dirichlet boundary condition at the pore interface.

We prove that the extension of $v_\varepsilon(t)$ by 0 to Ω converges weakly in $L^2(\Omega)$ for a.e. $t \in S$ and the extension of the pressure $p_\varepsilon(t)$ by a cell-wise mean value converges strongly in $L^2(\Omega)$ for a.e. $t \in S$ to the unique solution $(v(t), p(t))$ of the following Darcy law:

$$v(t,x) = \frac{1}{\nu} K(t,x)(f(t,x) - \nabla p(t,x)) \qquad \text{in } S \times \Omega, \tag{4a}$$

$$\operatorname{div}(v(t,x)) = -\int_{\Gamma_x(t)} v_\Gamma(t,x,y)\cdot n\,\mathrm{d}y \quad \left(= -\frac{\mathrm{d}}{\mathrm{d}t}\Theta(t,x)\right) \qquad \text{in } S\times\Omega, \tag{4b}$$

$$p = p_b \qquad \text{on } S\times\partial\Omega, \tag{4c}$$

where v_Γ is the two-scale limit of v_{Γ_ε} and $\Gamma_x(t)$ is the interface of the pore and solid part in the reference cell at the macroscopic position $x\in\Omega$ at time $t\in S$. If v_{Γ_ε} is the velocity of the boundary deformation, the right-hand side of (4b) can be simplified to $-\frac{\mathrm{d}}{\mathrm{d}t}\Theta(t,x)$, where Θ is the porosity of the medium.

Compared to the Darcy law (2), the permeability tensor K now depends on time and space taking into account the shape of a pore $Y_x^*(t)$ at the point $x\in\Omega$ at the time $t\in S$. Moreover, the microscopic incompressibility condition together with the inhomogeneous Dirichlet boundary condition gives rise to the macroscopic inhomogeneous divergence condition (4b). Combined with (4a), this gives an additional source or sink term for the pressure p. In the case that v_{Γ_ε} is the velocity of the boundary deformation, this term captures the suction and compression effects arising from the change of porosity.

For the homogenisation of (3), we use the *two-scale transformation method*. We transform the problem to a substitute problem onto a periodic reference domain Ω_ε, where we pass to the limit $\varepsilon\to 0$ using two-scale convergence. Then, we transform the resulting limit problem back (cf. Fig. 1). This method was proposed for the homogenisation of a diffusion problem in [24].

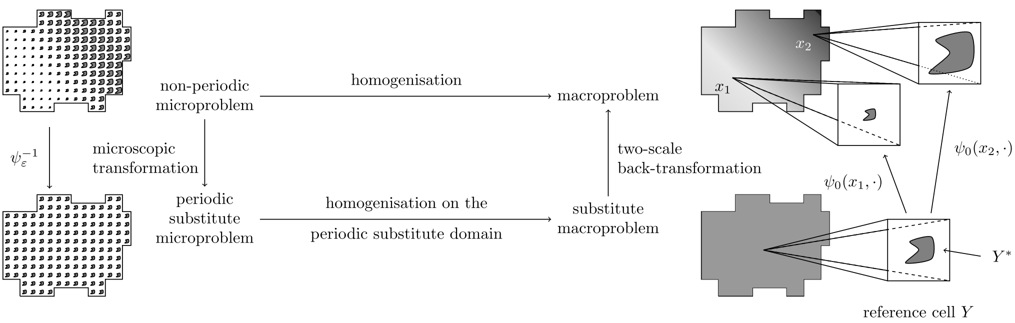

Fig. 1. Two-scale transformation method.

It was applied in several works – in the same sense that the homogenisation of the substitute problem was proven – (cf. [25–28,30,31]). In [29], a rigorous two-scale convergence concept for this transformation method was developed and it was proven that the homogenisation of the substitute problem is equivalent to the homogenisation of the actual problem (cf. Fig. 1). Thus, the homogenisation result for the periodic substitute problem can be rigorously transferred to a homogenisation result of the actual problem.

After the transformation onto the periodic reference domain, we derive uniform a priori estimates for the velocity field v_ε and the pressure field p_ε. However, the transformation of the equation induces coefficients in the symmetric-gradient term. Therefore, we derive a uniform Korn-type inequality for the two-scale transformation method, which allows to estimate the transformed symmetric gradient from below.

Since the extension of the inhomogeneous Dirichlet boundary condition inside the domain is not necessarily divergence-free, we cannot estimate the velocity directly without estimating the pressure as is done in the existing works on the homogenisation of Stokes flow. Instead, we employ a family of ε-scaled operators $\operatorname{div}_\varepsilon^{-1}$, which are right-inverse to the corresponding divergences, using the restriction operator that was introduced in [5] and developed further in [6]. Employing this Korn-type inequality and the $\operatorname{div}_\varepsilon^{-1}$-operator, we can deduce an ε-independent estimate on the velocity and the pressure during the existence proof without having an a priori estimate on the velocity at hand. For the case of a rectangular macroscopic domain with periodic Dirichlet boundary conditions, such a family of operators was constructed by V. V. Zhikov through different means (cf. [41]).

Having obtained these uniform a priori estimates, we can pass to the limit $\varepsilon \to 0$ in the reference configuration. There, we prove the strong convergence of the extension of the pressure. Then, we use two-scale compactness results in order to derive a microscopic incompressibility condition and a macroscopic inhomogeneous divergence condition. In the next step, we pass to the limit $\varepsilon \to 0$ in (3a) for divergence-free functions and reconstruct a microscopic pressure. The intermediate result is a two-pressure Stokes system in the cylindrical two-scale domain.

Afterwards, we transform this two-pressure Stokes system back into the reference configuration. Since the system contains not only microscopic, but also macroscopic derivatives, it does not yield a transformation-independent result directly. The same problem occurs also in the formal back-transformation after the homogenisation of diffusion and elasticity equations in a periodic reference domain (cf. [24,27]). However, recently some two-scale transformation rules for gradients have been shown in [29]. These allow for diffusion and elasticity equations, a back-transformation yielding a transformation-independent homogenisation result. By developing these ideas further, we back-transform the two-pressure Stokes system into a transformation-independent system in the actual evolving two-scale domain. Moreover, the results of [29] directly transfer the convergence of the substitute velocity field into the convergence of the actual velocity field. Since the extension of the pressure and the transformation to the periodic substitute domain do not commute, the strong convergence of the substitute pressure can not be transferred directly to the actual pressure. However, with computations similar to those of [29], we can overcome this lack of commutativity.

In the last step, we separate the microscopic from the macroscopic scale and thereby derive *Darcy's law for evolving microstructure*. This Darcy's law differs from the standard Darcy's law by its time- and space-dependent permeability tensor, which corresponds to the time- and space-dependent microscopic porosity. Moreover, it contains a new source term for the pressure equation. This term captures the suction and compression effects arising from the change of the porosity.

The paper is organized as follows: in Section 2, we transform the Stokes problems onto the periodic domain Ω_ε by the two-scale transformation method. In Section 3, we derive a uniform Korn inequality for the two-scale transformation method and a family of ε-scaled operators $\operatorname{div}_\varepsilon^{-1}$, which are right-inverse to the corresponding divergences. Using these results, we give uniform a priori estimates, which allow us to pass to the homogenisation limit. In Section 4, we pass to the limit $\varepsilon \to 0$ in the reference configuration and derive a two-pressure Stokes equation. Finally, in Section 5, we transform this two-scale limit problem back to the actual domain and derive (4), which we call Darcy's law for evolving microstructure.

Notation 1.1. In the following, let $C \in \mathbb{R}$ be a generic constant independent of $t \in S$ and $\varepsilon > 0$. Moreover, let $1 \leq p_s \leq \infty$ and $C(t)$ be a generic time-dependent function that is independent of ε, where $C \in L^{p_s}(S)$. For an open set $U \subset \mathbb{R}^N$, we write $(f, g)_U$ for the scalar product of $f, g \in L^2(U)$ and define $\|f\|_U := (f, f)_U^{\frac{1}{2}}$. If $G \subset \partial U$ is Lipschitz regular, we define $H_G^1(U)$ as the closure of smooth functions that are zero in a neighbourhood of G. If U is a subset of $(0,1)^N$, we denote the subset of $H^1_G(U)$ of Y-periodic functions by $H^1_{G\#}(U)$ and, similarly, we use the index $_\#$ to indicate the Y-periodicity for smooth functions.

For a (weakly) differentiable scalar- or vector-valued function u, we write Du for its Jacobian matrix, i.e. $(Du)_{ij} := \partial_{x_j} u_i$ and $\nabla u = Du^\top$. We define the divergence for a matrix-valued function A by its columns, i.e. $\operatorname{div}(A) = \operatorname{div}((A_{ji})_{j=1}^N)_{i=1}^N$. Thus, we get $(\operatorname{div}(\nabla u))_i = \operatorname{div}(\nabla u_i)$ for $u : \mathbb{R}^N \to \mathbb{R}^N$. Due to this choice of the matrix-valued divergence, the transformation of the Stokes equations resembles the transformation of scalar-valued problems.

2. The Stokes problem on the microscopic scale

Let $S = (0, T)$ for $T > 0$ be the finite time interval. For $N \in \mathbb{N}$ with $N \geq 2$, let $\Omega \subset \mathbb{R}^N$ be a bounded and connected domain that can be represented as a finite union of axis-parallel cuboids with corner coordinates in $\mathbb{Q}^N$, representing the macroscopic domain of the porous medium. Thus, there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that $\Omega = \operatorname{int}\left(\bigcup_{k\in I_{\varepsilon_n}} k + \varepsilon_n \overline{Y}\right)$ for every $n \in \mathbb{N}$, where $I_\varepsilon := \{k \in \varepsilon\mathbb{Z}^N \mid \operatorname{int}(k + \varepsilon Y) \subset \Omega\}$ for $\varepsilon > 0$ and $Y := (0,1)^N$ is the microscopic reference cell. We consider in the following such a fixed sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $0 < \varepsilon_n \leq 1$ for every $n \in \mathbb{N}$ and write shortly ε. Let $Y^* \subset Y$ be open and $Y^{\mathrm{s}} := \operatorname{int}(Y \setminus Y^*)$. The set Y^* is the pore part and the set Y^{s} is the solid part of the reference cell. We denote the interface of them by $\Gamma := [0,1]^N \cap \partial\Big(\bigcup_{k\in\mathbb{Z}^N} k + \overline{Y^*}\Big) \cap \partial\Big(\bigcup_{k\in\mathbb{Z}^N} k + \overline{Y^{\mathrm{s}}}\Big)$. We assume that:

1. Y^* and Y^{s} have positive measure,
2. Y^* is a connected set with Lipschitz boundary,
3. $\partial Y^* \cap \{x_i = 0\} + e_i = \partial Y^* \cap \{x_i = 1\}$ for every $i \in \{1, \dots, N\}$,
4. $Y^*_\# := \operatorname{int}\Big(\bigcup_{k\in\mathbb{Z}^N} k + \overline{Y^*}\Big)$ is connected and has a C^1-boundary.

We define $C^\infty_{\Gamma\#}(Y^*) := \{\varphi \in C^\infty_\#(\overline{Y}) \mid \operatorname{supp}(\varphi) \subset Y^*_\#\}$ and $H^1_{\Gamma\#}(Y^*)$ as its closure with respect to the $H^1(Y)$-norm. In the following, we identify $H^1_{\Gamma\#}(Y^*)$ with $\{v \in H^1_\#(Y) \mid v|_{Y\setminus Y^*} = 0\}$ if the function has to be evaluated on $Y \setminus Y^*$.

We define Ω_ε, which represents the pore part in the reference configuration, by $\Omega_\varepsilon := \operatorname{int}\left(\bigcup_{k\in I_\varepsilon} k + \varepsilon\overline{Y^*}\right)$, the corresponding solid part by $\Omega^{\mathrm{s}}_\varepsilon := \operatorname{int}\left(\Omega \setminus \Omega_\varepsilon\right)$ and the interface of these by $\Gamma_\varepsilon = \partial\Omega_\varepsilon \cap \partial\Omega^{\mathrm{s}}_\varepsilon$. Note that Ω_ε is connected and Γ_ε as well as the remaining part of the boundary $\partial\Omega_\varepsilon \setminus \Gamma_\varepsilon$ are Lipschitz regular by their construction.

We assume that, for a.e. $t \in S$, the evolving domain $\Omega_\varepsilon(t)$ and the evolving surface Γ_ε can be described by locally periodic transformations $\psi_\varepsilon \in L^\infty(S; C^2(\overline{\Omega})^N)$. That means $\Omega_\varepsilon(t) = \psi_\varepsilon(t, \Omega_\varepsilon)$, $\Omega^{\mathrm{s}}_\varepsilon(t) = \psi_\varepsilon(t, \Omega^{\mathrm{s}}_\varepsilon)$, $\Gamma_\varepsilon(t) = \psi_\varepsilon(t, \Gamma_\varepsilon)$ for a.e. $t \in S$, where ψ_ε satisfy Assumption 2.1.

We denote the Jacobians of ψ_ε with respect to x by $\Psi_\varepsilon(t,x) = D\psi_\varepsilon(t,x)$ and $J_\varepsilon(t,x) := \det(\Psi_\varepsilon(t,x))$.

Assumption 2.1. *We assume that*

1. $\psi_\varepsilon(t,\cdot)$ *is a given* C^2*-diffeomorphism from* $\overline{\Omega}$ *onto* $\overline{\Omega}$ *for a.e.* $t \in S$ *with inverse* $\psi_\varepsilon^{-1}(t,\cdot)$ *for* $\psi_\varepsilon, \psi_\varepsilon^{-1} \in L^\infty(S; C^2(\overline{\Omega})^N)$
2. *there exists* $c_J > 0$ *such that* $J_\varepsilon(t) \geq c_J$ *for a.e.* $t \in S$,

3. *there exists* $C > 0$ *such that* $\varepsilon^{i-1} \left\| \check{\psi}_\varepsilon \right\|_{L^\infty(S;C^i(\overline{\Omega}))} \leq C$ *for* $i \in \{0, 1, 2\}$, *where* $\check{\psi}_\varepsilon(t, x) := \psi_\varepsilon(t, x) - x$ *are the corresponding displacement mappings,*
4. *there exists* $\psi_0 \in L^\infty(S \times \Omega; C^2(\overline{Y})^N)$ *such that*
   (a) $\psi_0(t, x, \cdot) : \overline{Y} \to \overline{Y}$ *are* C^2*-diffeomorphisms for a.e.* $(t, x) \in S \times \Omega$ *with inverses* $\psi_0^{-1}(t, x, \cdot)$ *for* $\psi_0, \psi_0^{-1} \in L^\infty(S \times \Omega; C^2(\overline{Y})^N)$,
   (b) *the corresponding displacement mapping* $\check{\psi}_0(t, x, y) := \psi_0(t, x, y) - y$, *can be extended* Y*-periodically such that* $\check{\psi} \in L^\infty(S \times \Omega; C^2_\#(\overline{Y})^N)$,
   (c) $\varepsilon^{|\alpha|-1} D_{x_\alpha} \check{\psi}_\varepsilon(t)$ *two-scale converges strongly with respect to every* L^p*-norm for* $p \in (1, \infty)$ *to* $D_{y_\alpha} \check{\psi}_0(t)$ *for every multiindex* $\alpha \in \{0, 1, 2\}^N$ *with* $|\alpha| \leq 2$.

For the inverses of the transformations, we denote the corresponding displacement mappings by $\check{\psi}_\varepsilon^{-1}(t, x) = \psi_\varepsilon^{-1}(t, x) - x$ and $\check{\psi}_0^{-1}(t, x, y) = \psi_0^{-1}(t, x, y) - y$. While the actual ε-scaled domain $\Omega_\varepsilon(t)$ at time $t \in S$ is given by $\Omega_\varepsilon(t) = \psi_\varepsilon(\Omega_\varepsilon)$, one gets analogously the local reference cell $Y_x^*(t)$ at time $t \in S$ and macroscopic position $x \in \Omega$ via $Y_x^*(t) := \psi_0(t, x, Y^*)$. We denote the interfaces in these local reference cells by $\Gamma(t, x) := \psi_0(t, x, \Gamma)$. The sets $Y_x^*(t)$ are independent of the chosen diffeomorphism and depend only on the sets Ω_ε. Indeed, one has that $\chi_{\Omega_\varepsilon(t)}(x)$ two-scale converges to $\chi_{Y_x^*(t)}(y)$. For a measurable set U, χ_U denotes the indicator function, i.e. $\chi_U(x) = 1$ if $x \in U$ and $\chi_U(x) = 0$ if $x \notin U$.

The domains $\Omega_\varepsilon(t)$ and their evolution are assumed to be a priori given by the C^2-diffeomorphisms ψ_ε, which have to satisfy only Assumption 2.1. This Assumption is formulated in a very generic setting. In particular, it is not required that ψ_ε can be written by an ε-scaled two-scale asymptotic expansion. Instead, the asymptotic behaviour is characterised by means of two-scale convergence. Indeed, a domain evolution justifying these assumptions arises also in free boundary problems (see [30,31]).

An evolution of the domains that satisfies Assumption 2.1 can be obtained for example from the following model:

Example 2.2. Let $\Theta : [0, T] \times \overline{\Omega} \to (0, 1)$ be a smooth function with $D\Theta$ small enough. For instance, Θ can describe the local porosity. Let $\psi_0 : (0, 1) \times \overline{Y} \to \overline{Y}$ such that, for fixed first argument, $\psi_0(\Theta, \cdot)$ is a family of diffeomorphisms and the corresponding displacement mapping $\bar{\psi}_0(\Theta, y) = \psi_0(\Theta, y) - y$ can be extended to a Y-periodic function. For instance, $\psi_0(\Theta, Y^*)$ could give a cell with porosity Θ. Then,

$$\psi_\varepsilon(t, x) := x + \varepsilon \tilde{\psi}_0\left(\Theta(t, x), \frac{x}{\varepsilon}\right), \quad \psi_0(t, x, y) := \tilde{\psi}_0(\Theta(t, x), y)$$

fulfil Assumption 2.1.

Assumption 2.3. *Let* $p_s \in [1, \infty]$ *be fixed. We assume on the data that:*

1. f_ε *is a sequence in* $L^{p_s}(S; L^2(\Omega)^N)$ *such that* $\|f_\varepsilon(t)\|_{L^2(\Omega)} \leq C(t)$ *for a.e.* $t \in S$ *for* $C \in L^{p_s}(S)$
2. *there exists* $f \in L^{p_s}(S; L^2(\Omega)^N)$ *such that* $f_\varepsilon(t)$ *two-scale converges weakly with respect to the* L^2*-norm to* $f(t)$ *for a.e.* $t \in S$,
3. $p_{b,\varepsilon}$ *is a sequence in* $L^{p_s}(S; H^1(\Omega))$ *such that* $\left\|p_{b,\varepsilon}(t)\right\|_{L^2(\Omega)} \leq C(t)$ *for a.e.* $t \in S$ *for* $C \in L^{p_s}(S)$,

4. *there exists* $(p_b, p_{b,1}) \in L^{p_s}(S; H^1(\Omega)) \times L^{p_s}(S; L^2(\Omega; H^1_\#(Y)/\mathbb{R}))$ *such that* $\nabla p_{b,\varepsilon}(t)$ *two-scale converges weakly with respect to the* L^2*-norm to* $\nabla_x p_b(t) + \nabla_y p_{b,1}(t)$ *for a.e.* $t \in S$,
5. v_{Γ_ε} *is a sequence in* $L^{p_s}(S; H^1(\Omega)^N)$ *such that* $\frac{1}{\varepsilon}\left\|v_{\Gamma_\varepsilon}(t)\right\|_{L^2(\Omega)} \leq C(t)$ *for a.e.* $t \in S$ *and* $\left\|\nabla v_{\Gamma_\varepsilon}(t)\right\|_{L^2(\Omega)} \leq C(t)$ *for a.e.* $t \in S$ *for* $C \in L^{p_s}(S)$,
6. *there exists* $v_\Gamma \in L^{p_s}(S; L^2(\Omega; H^1_\#(Y)^N))$ *such that* $\frac{1}{\varepsilon} v_{\Gamma_\varepsilon}(t)$ *two-scale converges weakly with respect to the* L^2*-norm to* $v_\Gamma(t)$ *for a.e.* $t \in S$ *and* $\nabla v_{\Gamma_\varepsilon}(t)$ *two-scale converges weakly with respect to the* L^2*-norm to* $\nabla_y v_\Gamma(t)$ *for a.e.* $t \in S$,
7. *if* v_{Γ_ε} *should be the velocity of the boundary deformation, i.e.* $v_{\Gamma_\varepsilon}(t,x) = \partial_t \psi_\varepsilon(t, \psi_\varepsilon^{-1}(t,x))$, *we assume that* ψ_ε *is a sequence in* $W^{1,p_s}(S; H^1(\Omega)^N)$ *such that* $\frac{1}{\varepsilon}\|\partial_t \psi_\varepsilon(t)\|_{L^2(\Omega)} \leq C(t)$ *for a.e.* $t \in S$ *for* $C \in L^{p_s}(S)$*. Moreover, we assume that* $\psi_0 \in W^{1,p_s}(S; L^2(\Omega; H^1_\#(Y)^N))$ *such that* $\varepsilon^{|\alpha|-1} D_{x_\alpha} \partial_t \psi_\varepsilon(t)$ *two-scale converges weakly with respect to the* L^2*-norm to* $D_{y_\alpha} \partial_t \psi_0(t)$ *for a.e.* $t \in S$ *and every multiindex* $\alpha \in \{0,1\}^N$ *with* $|\alpha| \leq 1$.

Since we consider the stationary Stokes equations, time becomes only a parameter. Therefore, we have formulated the previous assumptions in a way that allows us to consider the equation and the homogenisation process pointwise in time for a.e. $t \in S$. However, we have assumed for all used quantities the measurability with respect to time. Thus, we can show that the solutions of the ε-scaled problem can be uniformly bounded for a.e. $t \in S$ by a $L^{p_s}(S)$ bound for a fixed $p_s \in [1, \infty]$, which allows us to translate the two-scale convergence into the time-dependent two-scale convergence, which is used in parabolic problems. This allows a coupling of this Stokes problem with such processes in future works.

In order to derive a weak form for (3), we substitute the boundary values and define $w_\varepsilon(t) := v_\varepsilon(t) - v_{\Gamma_\varepsilon}(t)$ and $q_\varepsilon(t) := p_\varepsilon(t) - p_{b,\varepsilon}(t)$. Then, we multiply (3a) by test functions φ which are 0 on $\Gamma_\varepsilon(t)$ and integrate over $\Omega_\varepsilon(t)$. After integration by parts and using the boundary conditions (3c)–(3d), we get (5a). In addition, we multiply (3b) with a test function $\phi \in L^2(\Omega_\varepsilon(t))$ and integrate over $\Omega_\varepsilon(t)$. Then, we obtain the following weak form:

Find $(w_\varepsilon, q_\varepsilon) \in L^{p_s}(S; H^1_{\Gamma_\varepsilon(t)}(\Omega_\varepsilon(t))^N) \times L^{p_s}(S; L^2(\Omega_\varepsilon(t)))$ such that, for a.e. $t \in S$,

$$\int\limits_{\Omega_\varepsilon(t)} \nu\varepsilon^2 2e(w_\varepsilon(t,x)) : \nabla\varphi(x) - q_\varepsilon(t,x)\operatorname{div}(\varphi(x))\,\mathrm{d}x = \int\limits_{\Omega_\varepsilon(t)} f_\varepsilon(t,x)\cdot\varphi(x)\,\mathrm{d}x - \int\limits_{\Omega_\varepsilon(t)} \nu\varepsilon^2 2e(v_{\Gamma_\varepsilon}(t,x)) : \nabla\varphi(x) + \nabla p_{b,\varepsilon}(t,x)\cdot\varphi(x)\,\mathrm{d}x, \tag{5a}$$

$$\int\limits_{\Omega_\varepsilon(t)} \operatorname{div}(w_\varepsilon(t,x))\phi(x)\,\mathrm{d}x = -\int\limits_{\Omega_\varepsilon(t)} \operatorname{div}(v_{\Gamma_\varepsilon}(t,x))\phi(x)\,\mathrm{d}x \tag{5b}$$

for every $(\varphi, \phi) \in H^1_{\Gamma_\varepsilon(t)}(\Omega_\varepsilon(t))^N \times L^2(\Omega_\varepsilon(t))$.

2.1. *Transformation on the periodic reference domain*

We transform the given data on the reference configuration by

$$\hat{f}_\varepsilon(t,x) := f(t, \psi_\varepsilon(t,x)),\ \hat{p}_{b,\varepsilon}(t,x) := p_{b,\varepsilon}(t, \psi_\varepsilon(t,x)),\ \hat{v}_{\Gamma_\varepsilon}(t,x) := v_{\Gamma_\varepsilon}(t, \psi_\varepsilon(t,x)) \tag{6}$$

and define $A_\varepsilon = J_\varepsilon \Psi_\varepsilon^{-1}$ as well as the transformed symmetric gradient $\hat{e}_{\varepsilon,t}(v) := (\Psi_\varepsilon^{-\top}(t)\nabla v + (\Psi_\varepsilon^{-\top}(t)\nabla v)^\top)/2$. Then, we transform (7a)–(7b) onto the periodic reference domain and obtain the following problem:

Find $(\hat{w}_\varepsilon, \hat{q}_\varepsilon) \in L^{p_s}(S; H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N) \times L^{p_s}(S; L^2(\Omega_\varepsilon))$ such that for a.e. $t \in S$

$$\int_{\Omega_\varepsilon} \nu\varepsilon^2 A_\varepsilon(t,x)2\hat{e}_{\varepsilon,t}(\hat{w}_\varepsilon(t,x)) : \nabla\varphi(x) - \hat{q}_\varepsilon(t,x)\operatorname{div}(A_\varepsilon(t,x)\varphi(x))\,\mathrm{d}x$$
$$= \int_{\Omega_\varepsilon} J_\varepsilon(t,x)\hat{f}_\varepsilon(t,x)\cdot\varphi(x)\,\mathrm{d}x - \int_{\Omega_\varepsilon} \nu\varepsilon^2 A_\varepsilon(t,x)2\hat{e}_{\varepsilon,t}(\hat{v}_{\Gamma_\varepsilon}(t,x)) : \nabla\varphi(x)\,\mathrm{d}x \tag{7a}$$
$$- \int_{\Omega_\varepsilon} A_\varepsilon^\top(t,x)\nabla\hat{p}_{b,\varepsilon}(t,x)\cdot\varphi(x)\,\mathrm{d}x,$$

$$\int_{\Omega_\varepsilon} \operatorname{div}(A_\varepsilon(t,x)\hat{w}_\varepsilon(t,x))\phi(x)\,\mathrm{d}x = -\int_{\Omega_\varepsilon} \operatorname{div}(A_\varepsilon(t,x)\hat{v}_{\Gamma_\varepsilon}(t,x))\phi(x)\,\mathrm{d}x \tag{7b}$$

for every $(\varphi, \phi) \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N \times L^2(\Omega_\varepsilon)$.

The strong form of this transformed Stokes problem (7) is given by

$$-\operatorname{div}(\varepsilon^2\nu A_\varepsilon 2\hat{e}_{\varepsilon,t}(\hat{v}_\varepsilon)) + A_\varepsilon^\top\nabla\hat{p}_\varepsilon = J_\varepsilon\hat{f}_\varepsilon \qquad \text{in } \Omega_\varepsilon, \tag{8a}$$
$$\operatorname{div}(A_\varepsilon\hat{v}_\varepsilon)) = 0 \qquad \text{in } \Omega_\varepsilon, \tag{8b}$$
$$\hat{v}_\varepsilon = \hat{v}_{\Gamma_\varepsilon} \qquad \text{on } \Gamma_\varepsilon, \tag{8c}$$
$$A_\varepsilon^\top n\hat{p}_\varepsilon - \varepsilon^2\nu 2\hat{e}_{\varepsilon,t}(\hat{w}_\varepsilon)^\top A_\varepsilon^\top n = A_\varepsilon^\top n p_b \qquad \text{on } \partial\Omega_\varepsilon \setminus \Gamma_\varepsilon. \tag{8d}$$

Lemma 2.4. *Problem (5) is equivalent to (7), in the sense that $(w_\varepsilon, q_\varepsilon)$ solves (5) if and only if $(\hat{w}_\varepsilon, \hat{q}_\varepsilon)$ solves (7), where the solutions can be transformed by $\hat{w}_\varepsilon(t,x) = w_\varepsilon(t, \psi_\varepsilon(t,x))$ and $\hat{q}_\varepsilon(t,x) = q_\varepsilon(t, \psi_\varepsilon(t,x))$.*

Proof. Using the product rule, we can transform between (5) and (7). □

3. Existence and uniform a priori estimates

In this section, we show the following existence and uniqueness result for (7) and derive an ε-independent bound for the solution.

Theorem 3.1. *For $\varepsilon > 0$, there exists a unique solution $(\hat{w}_\varepsilon, \hat{q}_\varepsilon) \in L^p(S; H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N) \times L^p(S; L^2(\Omega_\varepsilon))$ of (7) and a $C \in L^{p_s}(S)$ for a.e. $t \in S$ such that*

$$\left\|\hat{w}_\varepsilon(t)\right\|_{\Omega_\varepsilon} + \varepsilon\left\|\nabla\hat{w}_\varepsilon(t)\right\|_{\Omega_\varepsilon} + \left\|\hat{q}_\varepsilon(t)\right\|_{\Omega_\varepsilon} \le C(t). \tag{9}$$

We want to clarify (9) regarding the question if (9) holds only for every $t \in S \setminus S'_\varepsilon$ with $|S'_\varepsilon| = 0$ or for every $t \in S \setminus S'$ with $|S'| = 0$ and S' independent of ε. Actually, we need the later and a-priori stronger condition for the homogenisation because we pass to the limit using the two-scale convergence for fixed t. However, these two conditions are in fact equivalent. Assume that S'_ε depends on ε. Since ε is a countable sequence, we can choose $S' := \bigcup_{n\in\mathbb{N}} S'_{\varepsilon_n}$ and get $|S'| = 0$. Thus, the estimate holds also for an ε-independent zero-set S'.

For the proof of Theorem 3.1, we use the following generic saddle-point formulation, where V and Q are Hilbert spaces and $a \in \mathcal{L}(V; V')$ and $b \in \mathcal{L}(V; Q')$:

Given $f \in V'$ and $g \in Q'$, find a solution $(v, p) \in V \times Q$ such that:

$$a(v,\varphi) + b(\varphi, p) = \langle f, \varphi\rangle_{V'\times V} \qquad \text{for all } \varphi \in V, \tag{10a}$$

$$b(v,\phi) = \langle g, \phi\rangle_{Q'\times Q} \qquad \text{for all } \phi \in Q. \tag{10b}$$

The existence and uniqueness of a solution and a corresponding estimate for such saddle-point problems are given by the following well-known lemma. A proof is given in [42] Theorem 4.2.3, for example.

Lemma 3.2. *If there exist constants $\alpha, \beta > 0$ such that*

$$a(w,w) \geq \alpha \|w\|_V^2 \quad \textit{for all } w \in V, \tag{11}$$

$$\inf_{u\in Q\setminus\{0\}} \sup_{w\in V\setminus\{0\}} \frac{|b(w,u)|}{\|w\|_V \|u\|_Q} \geq \beta, \tag{12}$$

then the saddle-point problem (10) has a unique solution $(v, p) \in V \times Q$. Furthermore, the following estimates hold for the solution:

$$\|v\|_V \leq \frac{1}{\alpha}\|f\|_{V'} + \frac{2\|a\|_{\mathcal{L}(V;V')}}{\alpha\beta}\|g\|_{Q'}, \tag{13}$$

$$\|p\|_Q \leq \frac{2\|a\|_{\mathcal{L}(V;V')}}{\alpha\beta}\|f\|_{V'} + \frac{2\|a\|^2_{\mathcal{L}(V;V')}}{\alpha\beta^2}\|g\|_{Q'}. \tag{14}$$

The following Lemma enables us to add time as a parameter in Lemma 3.2. More precisely, we use it later to show that $(w_\varepsilon, q_\varepsilon)$ is measurable with respect to time.

Lemma 3.3. *For the spaces*

$$A := \{a \in \mathcal{L}(V;V') \mid a(v,v) \geq \alpha \|v\|_V^2 \ \textit{for all } v \in V \textit{ and } \alpha > 0\},$$

$$B := \left\{ b \in \mathcal{L}(V;Q') \mid \inf_{p\in Q\setminus\{0\}} \sup_{v\in V\setminus\{0\}} \frac{|b(v,p)|}{\|v\|_V \|p\|_Q} \geq \beta \textit{ for } \beta > 0 \right\}$$

of bilinear forms, the unique solution of the corresponding saddle-point problem (10) given by Lemma 3.2 depends continuously on the data $(a, b, f, g) \in A \times B \times V' \times Q'$.

Proof. Lemma 3.3 can be proven by computations which are similar to those in standard proofs of the estimates of Lemma 3.2. □

In order to derive the uniform estimate (9), we employ (13) and (14). Hence, we equip $H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N$ with a proper norm and derive a uniform coercivity and a uniform inf–sup estimate for the bilinear forms.

First, we show some uniform estimates for the coefficients (cf. Lemma 3.4). Then, we derive a family of ε-scaled Korn-type inequalities for the two-scale transformation method (cf. Lemma 3.6). These Korn-type inequalities allow us to estimate the transformed symmetric gradients $\hat{e}_{\varepsilon,t}(\hat{w}_\varepsilon)$ uniformly from below, which implies the uniform coercivity for the first bilinear form. In order to show the uniform inf–sup estimate for the other bilinear form, we construct a family of ε-scaled operators $\mathrm{div}_\varepsilon^{-1}$, which are right inverses to the divergence operator (cf. Lemma 3.12).

Lemma 3.4. *There exists a constant $C > 0$ such that*

$$\|J_\varepsilon\|_{L^\infty(S;C(\overline{\Omega_\varepsilon}))} + \|\Psi_\varepsilon\|_{L^\infty(S;C(\overline{\Omega_\varepsilon}))} + \left\|\Psi_\varepsilon^{-1}\right\|_{L^\infty(S;C(\overline{\Omega_\varepsilon}))} \le C,$$

$$\varepsilon\left\|\partial_{x_i} J_\varepsilon\right\|_{L^\infty(S;C(\overline{\Omega_\varepsilon}))} + \varepsilon\left\|\partial_{x_i} J_\varepsilon^{-1}\right\|_{L^\infty(S;C(\overline{\Omega_\varepsilon}))} \le C,$$

$$\varepsilon\left\|\partial_{x_i} \Psi_\varepsilon\right\|_{L^\infty(S;C(\overline{\Omega_\varepsilon}))} + \varepsilon\left\|\partial_{x_i} A_\varepsilon\right\|_{L^\infty(S;C(\overline{\Omega_\varepsilon}))} + \varepsilon\left\|\partial_{x_i} \Psi_\varepsilon^{-1}\right\|_{L^\infty(S;C(\overline{\Omega_\varepsilon}))} \le C$$

for every $i \in \{0, \dots, N\}$.

Proof. We note that $\Psi_\varepsilon = \mathbb{1} + D\check{\psi}_\varepsilon$. Then the uniform estimate of $D\check{\psi}_\varepsilon$ given by Assumption 2.1 shows that $\|\Psi_\varepsilon\|_{L^\infty(S;C(\overline{\Omega_\varepsilon}))} \le C$.

Since J_ε and the entries of Ψ_ε^{-1} are polynomials with respect to the entries of Ψ_ε and J_ε^{-1}, the uniform bound of $J_\varepsilon \ge c_J > 0$ from below (cf. Assumption 2.1) and $\|\Psi_\varepsilon\|_{L^\infty(S;C(\overline{\Omega_\varepsilon}))} \le C$ implies $\|J_\varepsilon\|_{L^\infty(S;C(\overline{\Omega_\varepsilon}))} \le C$ and $\left\|\Psi_\varepsilon^{-1}\right\|_{L^\infty(S;C(\overline{\Omega_\varepsilon}))} \le C$.

After rewriting $\partial_{x_i}\Psi_\varepsilon = \partial_{x_i} D\psi_\varepsilon = \partial_{x_i}(\mathbb{1} + D\check{\psi}_\varepsilon) = \partial_{x_i} D\check{\psi}_\varepsilon$, Assumption 2.1 shows $\varepsilon\left\|\partial_{x_i}\Psi_\varepsilon\right\|_{L^\infty(S;C(\overline{\Omega_\varepsilon}))} \le C$ for every $i \in \{0, \dots, N\}$.

We note that A_ε is the adjugate matrix of $\Psi_\varepsilon = \mathbb{1} + D\psi_\varepsilon$. Thus, all of its entries are minors of Ψ_ε. We rewrite the x_i-derivative of these minors with the product rule into the sum of products, where each product has $(n-2)$-factors which are entries of $\Psi_\varepsilon(t)$ and one factor which is a entry of $\partial_{x_i}\Psi_\varepsilon$. Then, the estimates $\|\Psi_\varepsilon\|_{L^\infty(S;C(\overline{\Omega_\varepsilon}))} \le C$ and $\varepsilon\left\|\partial_{x_i}\Psi_\varepsilon\right\|_{L^\infty(S;C(\overline{\Omega_\varepsilon}))} \le C$ give $\varepsilon\left\|\partial_{x_i} A_\varepsilon\right\|_{L^\infty(S;C(\overline{\Omega_\varepsilon}))} \le C$.

We obtain $\varepsilon\left\|\partial_{x_i} J_\varepsilon\right\|_{L^\infty(S;C(\overline{\Omega_\varepsilon}))} \le C$ by the same argumentation as for the estimate of $\varepsilon\left\|\partial_{x_i} A_\varepsilon\right\|_{L^\infty(S;C(\overline{\Omega_\varepsilon}))}$.

Using the chain rule, we rewrite $\partial_{x_i} J_\varepsilon^{-1} = -J_\varepsilon^{-2}\partial_{x_i} J_\varepsilon$. Then, the uniform bound $J_\varepsilon \ge c_J > 0$ from below and the estimate $\varepsilon\left\|\partial_{x_i} J_\varepsilon\right\|_{L^\infty(S;C(\overline{\Omega_\varepsilon}))} \le C$ imply the estimate $\varepsilon\left\|\partial_{x_i} J_\varepsilon^{-1}\right\|_{L^\infty(S;C(\overline{\Omega_\varepsilon}))} \le C$.

We rewrite $\Psi_\varepsilon^{-1} = J_\varepsilon^{-1} A_\varepsilon$. Then, we obtain $\varepsilon\left\|\partial_{x_i}\Psi_\varepsilon^{-1}\right\|_{L^\infty(S;C(\overline{\Omega_\varepsilon}))} \le C$ with the product rule and the previous estimates. □

3.1. Korn-type inequality for the two-scale transformation method

In order to derive the Korn-type inequalities for the two-scale transformation method, we need the following ε-scaled Poincaré inequality for periodic domains.

Lemma 3.5. *There exists a constant* $C_P \in \mathbb{R}$ *such that*

$$\|v\|_{\Omega_\varepsilon} \leq \varepsilon C_P \|\nabla v\|_{\Omega_\varepsilon}$$

for every $v \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N$.

Proof. Lemma 3.5 is a standard result and can be proven by covering Ω_ε with ε-scaled copies of Y^* and scaling them on Y^*. Then, applying the Poincaré inequality for piecewise zero boundary values there and scaling back yields the estimate. □

Lemma 3.6. *There exists a constant* $\alpha \in \mathbb{R}$ *independent of* ε, *such that*

$$\alpha \|\nabla v\|^2_{\Omega_\varepsilon} \leq \left\|\hat{e}_{\varepsilon,t}(v)\right\|^2_{\Omega_\varepsilon} \tag{15}$$

for $\varepsilon > 0$ *and all* $v \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N$.

Proof. Lemma 3.6 follows directly from Lemma 3.7, since the estimates of Lemma 3.4 show that the prerequisites of Lemma 3.7 are fulfilled. □

Lemma 3.7. *Let* $c, C > 0$. *Then, there exists an* ε*-independent constant* $\alpha > 0$ *such that*

$$\alpha \|\nabla v\|^2_{\Omega_\varepsilon} \leq \left\|M_\varepsilon \nabla v + (M_\varepsilon \nabla v)^\top\right\|^2_{\Omega_\varepsilon}$$

for any $v \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N$ *and for every* $M_\varepsilon \in C^{0,1}(\overline{\Omega_\varepsilon})$ *with*

$$\|M_\varepsilon\|_{C(\overline{\Omega_\varepsilon})} + \varepsilon \|M_\varepsilon\|_{C^{0,1}(\overline{\Omega_\varepsilon})} \leq C,$$
$$\det(M_\varepsilon(x)) \geq c > 0 \text{ for every } x \in \overline{\Omega_\varepsilon}.$$

Proof. Since $\left\|M_\varepsilon \nabla v + (M_\varepsilon \nabla v)^\top\right\|^2_{\Omega_\varepsilon} = \sum_{k \in I_\varepsilon} \left\|M_\varepsilon \nabla v + (M_\varepsilon \nabla v)^\top\right\|^2_{k+\varepsilon Y^*}$ and $\|\nabla v\|^2_{\Omega_\varepsilon} = \sum_{k \in I_\varepsilon} \|\nabla v\|^2_{k+\varepsilon Y^*}$, we can reduce the problem on the reference cell. After transforming $k + \varepsilon Y^*$ on Y^*, it is sufficient to show

$$\alpha \|\nabla v\|^2_{Y^*} \leq \left\|M \nabla v + (M \nabla v)^\top\right\|^2_{Y^*} \tag{16}$$

for any $v \in H^1_\Gamma(Y^*)^N$, $\varepsilon > 0$ and $k \in I_\varepsilon$ where $M(x) := M_\varepsilon(k + \varepsilon x)$. From the Lipschitz estimate of M_ε and the transformation $x \mapsto k + \varepsilon x$, we can conclude that $\|M\|_{C^{0,1}(\overline{Y^*})} \leq C$. The uniform bound of the determinant from below remains preserved under the transformation. Hence $M \in \mathcal{M} := \{M \in C^{0,1}(\overline{Y^*})^{N \times N} \mid \|M\|_{C^{0,1}(\overline{Y^*})} \leq C \text{ and } \det(M) \geq c\}$.

The uniform Lipschitz continuity of $\mathcal{M}$ implies the equicontinuity of $\mathcal{M}$ and since $\mathcal{M}$ is also pointwise bounded, we obtain by the theorem of Arzelà–Ascoli that $\mathcal{M}$ is relatively compact in $C(\overline{Y^*})^{N \times N}$. Then, we apply Lemma 3.8 on the closure of $\mathcal{M}$ in $C(\overline{Y^*})^{N \times N}$ and we obtain Lemma (3.7). □

Lemma 3.8. *Let $U \subset \mathbb{R}^N$ be an open connected Lipschitz domain and $G \subset \partial U$ open with $|G| > 0$. Let $\mathcal{M}$ be a compact subset of $C(\overline{U})^{N\times N}$ and assume there exists $c > 0$ such that $\det(M) \geq c$ for every $M \in \mathcal{M}$. Then, there exists $\alpha > 0$ such that*

$$\alpha \|\nabla v\|_U \leq \left\|(M\nabla v + (M\nabla v)^\top\right\|_U$$

for every $M \in \mathcal{M}$ and $v \in H^1_G(U)^N$.

For the case that the set $\mathcal{M}$ in Lemma 3.8 consists of Hölder continuous functions the result is proven in [43].

Proof. Let $M \in \mathcal{M}$, then Lemma 3.9 gives a constant α_M such that

$$\alpha_M \|\nabla v\|_U \leq \left\|M\nabla v + (M\nabla v)^\top\right\|_U \tag{17}$$

for every $v \in H^1_G(U)^N$. We obtain for $B \in \mathcal{M}$

$$\left\|M\nabla v + (M\nabla v)^\top - (B\nabla v + (B\nabla v)^\top)\right\|_U \leq 2\|M - B\|_{C(\overline{U})} \|\nabla v\|_U ,$$

which implies

$$\left\|M\nabla v + (M\nabla v)^\top\right\|_U \leq \left\|B\nabla v + (B\nabla v)^\top\right\|_U + 2\|M - B\|_{C(\overline{U})} \|\nabla v\|_U . \tag{18}$$

Combining (17) and (18) gives for any $B \in B_{\alpha_M/4}(M)$

$$\frac{1}{2}\alpha_M \|\nabla v\|_U \leq \left\|B\nabla v + (B\nabla v)^\top\right\|_U . \tag{19}$$

Then, we cover $\mathcal{M}$ by $\bigcup\limits_{M\in\mathcal{M}} B_{\alpha_M/4}(M)$ and, since $\mathcal{M}$ is compact, there exists a finite set $\mathcal{I}$ such that for every $B \in \mathcal{M}$ there exists $M \in \mathcal{I}$ with $B \in B_{\alpha_M/4}(M)$. We choose $\alpha = \min\limits_{M\in\mathcal{I}} \alpha_M/2$ and obtain from (19)

$$\alpha \|\nabla v\|_U \leq \frac{\alpha_M}{2} \|\nabla v\|_U \leq \left\|(B\nabla v + (B\nabla v)^\top\right\|_U$$

for every $B \in \mathcal{M}$ and $v \in H^1_G(U)^N$. □

Lemma 3.9. *Let $U \subset \mathbb{R}^N$ be an open connected Lipschitz domain and $G \subset \partial U$ open with $|G| > 0$. Let $A : \overline{U} \to \mathbb{R}^{n\times n}$ be a continuous mapping with $\det(A) \geq c > 0$. Then, there is a constant $\alpha > 0$ such that*

$$\alpha \|\nabla v\|_U \leq \left\|(A\nabla v + (A\nabla v)^\top\right\|_U$$

for all $v \in H^1_G(U)^N$.

Proof. Lemma 3.9 is proven in Theorem [44]. □

3.2. Right-inverse divergence operator

In order to construct explicitly the operators $\operatorname{div}_\varepsilon^{-1} : L^2(\Omega) \to H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N$, we use the following right-inverse divergence operator (see Lemma 3.10) and the restriction operator (see Lemma 3.11), which was originally introduced in [5] and developed further in [6].

Lemma 3.10. *Let U be a bounded domain. Then, there exists a continuous linear operator $\operatorname{div}^{-1} : L^2(U) \to H^1(U)^N$ such that $\operatorname{div} \circ \operatorname{div}^{-1} = id_{L^2(U)}$.*

Proof. See for instance [45, Exercise III.3.1]. □

Lemma 3.11. *There exists a linear continuous operator $R_\varepsilon : H^1(\Omega)^N \to H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N$ such that*

1. *$u \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)$ implies $R_\varepsilon u = u$ in Ω_ε*
2. $\operatorname{div}(R_\varepsilon u) = \operatorname{div}(u) + \sum_{k \in I_\varepsilon} \chi_{k+\varepsilon Y^*} \frac{1}{|\varepsilon Y^*|} \int_{k+\varepsilon Y^s} \operatorname{div}(u)$,
3. *there exists a constant C such that*

$$\|R_\varepsilon u\|_{\Omega_\varepsilon} + \varepsilon \|\nabla(R_\varepsilon u)\|_{\Omega_\varepsilon} \leq C(\|u\|_\Omega + \varepsilon \|\nabla u\|_\Omega)$$

for every $u \in H^1(\Omega)^N$.

Proof. In [6] this restriction operator is explicitly constructed from $H^1_0(\Omega)^N$ to $H^1_0(\Omega_\varepsilon)^N$. Indeed, the construction is done locally so that the same construction yields an operator $R_\varepsilon : H^1(\Omega)^N \to H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N$. □

Lemma 3.12. *There exists a linear continuous operator $\operatorname{div}_\varepsilon^{-1} : L^2(\Omega_\varepsilon) \to H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N$, which is right inverse to the divergence, i.e. $\operatorname{div} \circ \operatorname{div}_\varepsilon^{-1} = id_{L^2(\Omega_\varepsilon)}$, such that*

$$\left\|\operatorname{div}_\varepsilon^{-1}(f)\right\|_{\Omega_\varepsilon} + \varepsilon \left\|\nabla \operatorname{div}_\varepsilon^{-1}(f)\right\|_{\Omega_\varepsilon} \leq \|f\|_{L^2(\Omega_\varepsilon)}$$

for every $f \in L^2(\Omega_\varepsilon)$.

Proof. By Lemma 3.10 there exists a linear continuous operator $\operatorname{div}^{-1} : L^2(\Omega) \to H^1(\Omega)^N$ such that $\operatorname{div} \circ \operatorname{div}^{-1} = \mathrm{id}_{L^2(U)}$. Using this operator and the restriction operator R_ε of Lemma 3.11, we can define

$$\operatorname{div}_\varepsilon^{-1}(f) := R_\varepsilon(\operatorname{div}^{-1}(\widetilde{f})),$$

where $\widetilde{f}$ denotes the extension of $f \in L^2(\Omega_\varepsilon)$ by 0 to Ω.

The explicit formula for $\operatorname{div} \circ R_\varepsilon$ from Lemma 3.11 shows

$$\begin{aligned}\operatorname{div}(\operatorname{div}_\varepsilon^{-1}(f)) &= \operatorname{div}(R_\varepsilon(\operatorname{div}^{-1}(\widetilde{f}))) = \\ &= \operatorname{div}(\operatorname{div}^{-1}(\widetilde{f})) + \sum_{k \in I_\varepsilon} \chi_{k+\varepsilon Y^*} \frac{1}{|\varepsilon Y^*|} \int_{k+\varepsilon Y^s} \operatorname{div}(\operatorname{div}^{-1}(\widetilde{f}(x)))\,\mathrm{d}x\end{aligned}$$

$$= \widetilde{f} + \sum_{k\in I_\varepsilon} \chi_{k+\varepsilon Y^*} \frac{1}{|\varepsilon Y^*|} \int\limits_{k+\varepsilon Y^{\mathrm{s}}} \widetilde{f}(x)\,\mathrm{d}x = f.$$

Moreover, using the estimate of Lemma 3.11, we obtain, for $\varepsilon \le 1$,

$$\left\|\operatorname{div}_\varepsilon^{-1}(f)\right\|_{\Omega_\varepsilon} + \varepsilon \left\|\nabla \operatorname{div}_\varepsilon^{-1}(f)\right\|_{\Omega_\varepsilon} \le C\Big(\left\|\operatorname{div}^{-1}(\widetilde{f})\right\|_{\Omega} + \varepsilon \left\|\nabla \operatorname{div}^{-1}(\widetilde{f})\right\|_{\Omega}\Big)$$
$$\le C \left\|\nabla \operatorname{div}^{-1}(\widetilde{f})\right\|_{H^1(\Omega)} \le C \,\|\widetilde{f}\|_{\Omega} = C\,\|f\|_{\Omega_\varepsilon}. \quad \square$$

3.3. Estimates of the data

Lemma 3.13. *There exists a constant $C \in L^{p_s}(S)$ such that*

$$\left\|\hat{f}_\varepsilon(t)\right\|_{L^2(\Omega_\varepsilon)} + \left\|\hat{p}_{b,\varepsilon}(t)\right\|_{L^2(\Omega_\varepsilon)} + \left\|\nabla \hat{p}_{b,\varepsilon}(t)\right\|_{L^2(\Omega_\varepsilon)} \le C(t),$$
$$\frac{1}{\varepsilon}\left\|\hat{v}_{\Gamma_\varepsilon}(t)\right\|_{L^2(\Omega_\varepsilon)} + \left\|\nabla \hat{v}_{\Gamma_\varepsilon}(t)\right\|_{L^2(\Omega_\varepsilon)} \le C(t)$$

for a.e. $t \in S$.

Proof. We note that $D\psi_\varepsilon^{-1}(t,x) = \Psi_\varepsilon^{-1}(t, \psi_\varepsilon^{-1}(t,x))$. Hence, we get

$$\left\|\hat{f}_\varepsilon(t)\right\|_{\Omega_\varepsilon}^2 = \int\limits_{\Omega_\varepsilon} f_\varepsilon(t, \psi_\varepsilon(t,x)))^2\,\mathrm{d}x = \int\limits_{\Omega_\varepsilon(t)} \det(\Psi_\varepsilon^{-1}(t, \psi_\varepsilon^{-1}(t,x)))\, f_\varepsilon(t,x)^2\,\mathrm{d}x$$
$$\le \int\limits_{\Omega_\varepsilon(t)} J_\varepsilon^{-1}(t, \psi_\varepsilon^{-1}(t,x))\, f_\varepsilon(t,x)^2\,\mathrm{d}x \le c_J^{-1} \int\limits_{\Omega_\varepsilon(t)} f_\varepsilon(t,x)^2\,\mathrm{d}x \le C\,\|f_\varepsilon(t)\|_{\Omega_\varepsilon}^2.$$

Then, the uniform bound of $f_\varepsilon(t)$ given by Assumption 2.3 implies the uniform bound of $\left\|\hat{f}(t)\right\|_{L^2(\Omega_\varepsilon)} \le C(t)$ for a.e. $t \in S$. By similar computation, we obtain $\left\|\hat{p}_{b,\varepsilon}(t)\right\|_{L^2(\Omega_\varepsilon))} + \frac{1}{\varepsilon}\left\|\hat{v}_{\Gamma_\varepsilon}(t)\right\|_{L^2(\Omega_\varepsilon)} \le C(t)$. In order to estimate the gradient of $\hat{p}_{b,\varepsilon}$, we use the chain rule and rewrite $\nabla \hat{p}_{b,\varepsilon}(t,x) = \Psi_\varepsilon^\top(t,x)\nabla p_{b,\varepsilon}(t, \psi_\varepsilon(t,x))$. Then, the uniform estimates of Lemma 3.4 yield

$$\left\|\nabla \hat{p}_{b,\varepsilon}(t)\right\|_{\Omega_\varepsilon}^2 = \int\limits_{\Omega_\varepsilon(t)} J_\varepsilon^{-1}(t, \psi_\varepsilon^{-1}(t,x))(\Psi_\varepsilon^\top(t, \psi_\varepsilon^{-1}(t,x))\nabla p_{b,\varepsilon}(t,x))^2\,\mathrm{d}x$$
$$\le C \int\limits_{\Omega_\varepsilon(t)} J_\varepsilon^{-1}(t, \psi_\varepsilon^{-1}(t,x))\nabla p_{b,\varepsilon}^2(t,x)\,\mathrm{d}x \le C \int\limits_{\Omega_\varepsilon(t)} \nabla p_{b,\varepsilon}^2(t,x)\,\mathrm{d}x \le C\left\|\nabla p_{b,\varepsilon}(t)\right\|_{\Omega_\varepsilon(t)}^2.$$

The uniform bound of $\left\|\nabla p_{b,\varepsilon}(t)\right\|_{\Omega}$ given by Assumption 2.3 implies the uniform bound of $\left\|\nabla \hat{p}_{b,\varepsilon}(t)\right\|_{L^2(\Omega_\varepsilon} \le C(t)$ for a.e. $t \in S$. By similar computation, we obtain $\left\|\nabla \hat{v}_{\Gamma_\varepsilon}(t)\right\|_{L^2(\Omega_\varepsilon)} \le C(t)$. $\square$

Proof of Theorem 3.1. First, we show that there exists a solution $(\hat{w}_\varepsilon(t), \hat{q}_\varepsilon(t)) \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N \times L^2(\Omega_\varepsilon)$ of (7) for a.e. $t \in S$. Due to the Poincaré inequality from Lemma 3.5, $\|\cdot\|_{V_\varepsilon} : H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N \to \mathbb{R},\ v \mapsto \|v\|_{V_\varepsilon} := \varepsilon \|\nabla v\|_{\Omega_\varepsilon}$ defines a norm on $H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N$. We define $a_\varepsilon(t)$ and $b_\varepsilon(t)$ for a.e. $t \in S$ by

$$\begin{aligned} a_\varepsilon(t)(v, w) &= (\varepsilon^2 \nu 2 A_\varepsilon(t)\hat{e}_{\varepsilon,t}(v), \nabla w)_{\Omega_\varepsilon} && \text{for } v, w \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N, \\ b_\varepsilon(t)(v, p) &= (\operatorname{div}(A_\varepsilon(t)v), p)_{\Omega_\varepsilon} && \text{for } v \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N,\ p \in L^2(\Omega_\varepsilon). \end{aligned}$$

Using the uniform estimates of Lemma 3.4 and the Korn-type inequality for the two-scale transformation method (see Lemma 3.6), we obtain the following uniform coercivity and continuity estimate for $a_\varepsilon(t)$:

$$\begin{aligned} a_\varepsilon(t)(w, w) &= (\varepsilon^2 \nu A_\varepsilon(t) 2\hat{e}_{\varepsilon,t}(w), \nabla w)_{\Omega_\varepsilon} = (\varepsilon^2 \nu J_\varepsilon(t) 2\hat{e}_{\varepsilon,t}(w), \Psi_\varepsilon^{-\top}(t)\nabla w)_{\Omega_\varepsilon} \\ &= (\varepsilon^2 \nu J_\varepsilon(t) 2\hat{e}_{\varepsilon,t}(w), (\Psi_\varepsilon^{-\top}\nabla w + (\Psi_\varepsilon^{-\top}\nabla w)^\top)/2)_{\Omega_\varepsilon} \\ &= (\varepsilon^2 \nu J_\varepsilon(t) 2\hat{e}_{\varepsilon,t}(w), \hat{e}_{\varepsilon,t}(w))_{\Omega_\varepsilon} \\ &\geq \varepsilon^2 \nu c_J 2 \left\|\hat{e}_{\varepsilon,t}(w)\right\|^2_{\Omega_\varepsilon} \geq \varepsilon^2 \nu c_J 2\alpha \|\nabla w\|^2_{\Omega_\varepsilon} \geq C \|w\|^2_{V_\varepsilon}, \end{aligned} \tag{20}$$

$$a_\varepsilon(t)(v, w) = (\varepsilon^2 \nu J_\varepsilon(t)\Psi_\varepsilon^{-1}(t)(\Psi_\varepsilon^{-\top}\nabla v + (\Psi_\varepsilon^{-\top}\nabla v), \nabla w)_{\Omega_\varepsilon} \leq C \|v\|_{V_\varepsilon} \|w\|_{V_\varepsilon}. \tag{21}$$

In order to give a uniform estimate of the inf–sup constant, we choose an arbitrary $\phi \in L^2(\Omega_\varepsilon)$. Then, Lemma 3.12 gives a constant $C \in \mathbb{R}$, independent of ε and ϕ, and a function $\hat{v} \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N$ such that

$$\operatorname{div}(\hat{v}) = \phi, \quad \varepsilon \left\|\hat{v}\right\|_{\Omega_\varepsilon} \leq C \|\phi\|_{\Omega_\varepsilon}. \tag{22}$$

We define $v := J_\varepsilon^{-1}(t)\Psi_\varepsilon(t)\hat{v} \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N$. Using the product rule, the estimates from Lemma 3.4 and the ε-scaled Poincaré inequality (cf. Lemma 3.5), we obtain

$$\begin{aligned} \|v\|_{V_\varepsilon} &= \varepsilon \left\| D(J_\varepsilon^{-1}(t)\Psi_\varepsilon(t)\hat{v}) \right\|_{\Omega_\varepsilon} \leq \varepsilon \left\| D(J_\varepsilon^{-1}(t)\Psi_\varepsilon(t))\hat{v} \right\|_{\Omega_\varepsilon} + \varepsilon \left\| J_\varepsilon^{-1}(t)\Psi_\varepsilon(t) D\hat{v} \right\|_{\Omega_\varepsilon} \\ &\leq C \left\|\hat{v}\right\|_{\Omega_\varepsilon} + C\varepsilon \left\|D\hat{v}\right\|_{\Omega_\varepsilon} \leq C \left\|\hat{v}\right\|_{V_\varepsilon} \leq C \|\phi\|_{\Omega_\varepsilon}. \end{aligned}$$

With this choice of v, we see that

$$\sup_{w \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N \setminus \{0\}} \frac{|b_\varepsilon(t)(w, \phi)|}{\|w\|_{V_\varepsilon}} \geq \frac{(\phi, \phi)_{\Omega_\varepsilon}}{\|v\|_{V_\varepsilon}} \geq \frac{\|\phi\|^2_{\Omega_\varepsilon}}{C \|\phi\|_{\Omega_\varepsilon}} = C \|\phi\|_{\Omega_\varepsilon} \tag{23}$$

for a.e. fixed $t \in S$.

In order to show the continuity of the bilinear form $b_\varepsilon(t)$, we use the product rule, the Piola identity ($\operatorname{div}(A_\varepsilon(t)) = 0$) and the ε-scaled Poincaré inequality,

$$\begin{aligned} b_\varepsilon(t)(v, p) &= (\operatorname{div}(A_\varepsilon(t)v), p) = (\operatorname{div}(A_\varepsilon(t))v + A_\varepsilon(t) : \nabla v, p) \\ &\leq C \|\nabla v\|_{\Omega_\varepsilon} \|p\|_{\Omega_\varepsilon} \leq \varepsilon^{-1} C \|v\|_{V_\varepsilon} \|p\|_{\Omega_\varepsilon}. \end{aligned} \tag{24}$$

Note that a more precise estimate like for the inf–sup constant does not yield an ε-independent constant for $b_\varepsilon(t)$. However, the norm of b_ε does not appear in the right-hand sides of (13) and (14). Nevertheless, the continuity and the coercivity constant of $a_\varepsilon(t)$ as well as the inf–sup constant of $b_\varepsilon(t)$, which occur in (13) and (14), do not depend on ε or t.

Now, we estimate the right-hand sides of (7). For the first summand of the right-hand side of (7a), we obtain with Lemma 3.4, Lemma 3.13 and the ε-scaled Poincaré inequality

$$\begin{aligned}
&\left\| J_\varepsilon(t)\hat{f}_\varepsilon(t) - A_\varepsilon^\top(t)\nabla \hat{p}_{b,\varepsilon} \right\|_{V_\varepsilon'} \\
&= \sup_{v\in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N\setminus\{0\}} \frac{\int_{\Omega_\varepsilon} (J_\varepsilon(t,x)\hat{f}_\varepsilon(t,x) - A_\varepsilon^\top(t,x)\nabla \hat{p}_{b,\varepsilon}(t,x))\cdot v(x)\,\mathrm{d}x}{\|v\|_{V_\varepsilon}} \\
&\le \sup_{v\in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N\setminus\{0\}} \frac{C(t)\,\|v\|_{\Omega_\varepsilon}}{\|v\|_{V_\varepsilon}} \le \sup_{v\in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N\setminus\{0\}} \frac{C(t)\,\|v\|_{V_\varepsilon}}{\|v\|_{V_\varepsilon}} \le C(t), \qquad (25)
\end{aligned}$$

where $C \in L^{p_s}(S)$. We rewrite the second summand of (7a) and obtain from the continuity estimate (21) of $a_\varepsilon(t)$

$$\left\| -a_\varepsilon(t)(\hat{v}_{\Gamma_\varepsilon}(t),\cdot)\right\|_{V_\varepsilon'} \le C\left\|\hat{v}_{\Gamma_\varepsilon}(t)\right\|_{V_\varepsilon} \le \varepsilon C. \qquad (26)$$

Later, this term will also vanish during the homogenisation because it is of order $O(\varepsilon)$. We can estimate the right-hand side of (7b) with the continuity estimate (24) of $b_\varepsilon(t)$ and Lemma 3.13 by

$$\left\| -b_\varepsilon(t)(\hat{v}_{\Gamma_\varepsilon}(t),\cdot)\right\|_{L^2(\Omega_\varepsilon)'} \le \varepsilon^{-1} C\left\|\hat{v}_{\Gamma_\varepsilon}(t)\right\|_{V_\varepsilon} \le \varepsilon^{-1}\varepsilon C(t) \le C(t) \qquad (27)$$

for $C \in L^p(S)$. Using Lemma 3.2 with the estimates (20), (21), (23), (25), (26) and (27) yields, for $\varepsilon > 0$ small enough and a.e. $t \in S$, the existence of a unique solution $(\hat{w}_\varepsilon(t), \hat{q}_\varepsilon(t)) \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N \times L^2(\Omega_\varepsilon)$ of (7) such that

$$\left\|\hat{w}_\varepsilon(t)\right\|_{V_\varepsilon} + \left\|\hat{q}_\varepsilon(t)\right\|_{\Omega_\varepsilon} \le C(t) \qquad (28)$$

for $C \in L^p(S)$.

By the definition of the norm of V_ε and Lemma 3.5, we can estimate further

$$\left\|\hat{w}_\varepsilon(t)\right\|_{\Omega_\varepsilon} + \varepsilon\left\|\nabla\hat{w}_\varepsilon(t)\right\|_{\Omega_\varepsilon} + \left\|\hat{q}_\varepsilon(t)\right\|_{\Omega_\varepsilon} \le C(t) \qquad (29)$$

for $\varepsilon > 0$ small enough, a.e. $t \in S$ and $C \in L^p(S)$.

By Lemma 3.3, we get the continuity of the solution with respect to $a_\varepsilon(t)$ and $b_\varepsilon(t)$ and the right-hand sides. Moreover, $a_\varepsilon(t)$ and $b_\varepsilon(t)$ and the right-hand sides are measurable in time. Thus, $\hat{w}_\varepsilon : S \to H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N$ and $\hat{q}_\varepsilon : S \to L^2(\Omega_\varepsilon)$ are a composition of a continuous and a measurable function and hence measurable. With (29) we get the p_s integrability, i.e. $\hat{w}_\varepsilon \in L^{p_s}(S, H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N)$ and $\hat{q}_\varepsilon \in L^{p_s}(S; L^2(\Omega_\varepsilon))$. □

Remark 1. Another ansatz for the proof of Theorem 3.1 would be to substitute $\hat{w}_\varepsilon$, so that we obtain a homogeneous divergence condition. Then, we could use the Lemma of Lax–Milgram for functions v with $\operatorname{div}(A_\varepsilon(t)v) = 0$. Using the same preliminary work, we could prove the same uniform bounds for the velocity and the pressure. But the difficulty with this ansatz is that the measurability with respect to time cannot be concluded directly, since the space of functions v satisfying $\operatorname{div}(A_\varepsilon(t)v) = 0$ depends on time.

4. Homogenisation in the periodic reference domain

In this section, we pass to the limit $\varepsilon \to 0$ in (7) using the notion of two-scale convergence, which was introduced in [46,47] (see also [48]) and derive the following two-pressure Stokes system (30) as two-scale limit problem.

Find $(\hat{w}_0, \hat{q}, \hat{q}_1) \in L^{p_s}(S; L^2(\Omega; H^1_{\Gamma\#}(Y^*))) \times L^{p_s}(S; H^1_0(\Omega)) \times L^{p_s}(S; L^2(\Omega; L^2_0(Y^*)))$ such that

$$\int_\Omega \int_{Y^*} \nu A_0(t,x,y)\Psi_0^{-\top}(t,x,y)\nabla_y \hat{w}_0(t,x,y) : \nabla_y \varphi(x,y)\,\mathrm{d}y\,\mathrm{d}x$$
$$+ \int_\Omega \int_{Y^*} A_0^\top(t,x,y)\nabla_x \hat{q}(t,x)\cdot\varphi(x,y) + \hat{q}_1(t,x,y)\operatorname{div}_y(A_0(t,x,y)\varphi(x,y))\,\mathrm{d}y\,\mathrm{d}x$$
$$= \int_\Omega \int_{Y^*} (J_0(t,x,y)f(t,x) - A_0^\top(t,x,y)\nabla_x p_b(t,x))\cdot\varphi(x,y)\,\mathrm{d}y\,\mathrm{d}x \tag{30a}$$

$$\int_\Omega \int_{Y^*} \operatorname{div}_y(A_0(t,x,y)\hat{w}_0(t,x,y))\theta_1(x,y)\,\mathrm{d}y\,\mathrm{d}x = 0 \tag{30b}$$

$$\int_\Omega \operatorname{div}_x\left(\int_{Y^*} A_0(t,x,y)\hat{w}_0(t,x,y)\,\mathrm{d}y\right)\theta_0(x)\,\mathrm{d}x$$
$$= -\int_\Omega \int_{Y^*} \operatorname{div}_y\left(A_0(t,x,y)\hat{v}_\Gamma(t,x,y)\right)\mathrm{d}y\,\theta_0(x)\,\mathrm{d}x \tag{30c}$$

for a.e. $t \in S$ and every $(\varphi, \theta_0, \theta_1) \in L^2(\Omega; H^1_{\Gamma\#}(Y^*)^N) \times H^1_0(\Omega) \times L^2(\Omega; L^2_0(Y^*))$.

Definition 4.1. Let $p, q \in (1,\infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. We say that a sequence u_ε two-scale converges weakly to $u_0 \in L^p(\Omega \times Y)$ if

$$\lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon(x)\varphi\left(x, \frac{x}{\varepsilon}\right)\mathrm{d}x = \int_\Omega \int_Y u_0(x,y)\varphi(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

for every $\varphi \in L^q(\Omega; C_\#(Y))$. If additionally $\lim_{\varepsilon\to 0} \|v_\varepsilon\|_{L^p(\Omega)} = \|v_0\|_{L^p(\Omega\times Y)}$, we say u_ε two-scale converges strongly to u_0.

The following lemma is one of the fundamental compactness results in the notion of two-scale convergence.

Lemma 4.2. *Let $p \in (1, \infty)$ and let u_ε be a bounded sequence in $L^p(\Omega)$. Then, there exists a subsequence and $u_0 \in L^p(\Omega \times Y)$ such that this subsequence two-scale converges weakly to u_0.*

The following result allows us to handle the coefficients in the homogenisation.

Lemma 4.3. *Let $1 < p, q, r < \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Let u_ε be a sequence in $L^p(\Omega)$ which two-scale converges strongly to $u_0 \in L^p(\Omega \times Y)$ and let v_ε be a sequence in $L^q(\Omega)$ which two-scale converges weakly (resp. strongly) to $v_0 \in L^q(\Omega \times Y)$. Then, $u_\varepsilon v_\varepsilon$ is a sequence of functions in $L^r(\Omega)$ which two-scale converges weakly (resp. strongly) to $u_0 v_0 \in L^r(\Omega \times Y)$.*

In order to translate the two-scale convergence of the data and the solution between the actual and the transformed configuration, we use the following transformation result of [29].

Lemma 4.4. *Let $p \in (1, \infty)$ and $t \in S$. Let u_ε be a sequence in $L^p(\Omega)$ and $\hat{u}_\varepsilon = u_\varepsilon \circ \psi_\varepsilon(t)$ with ψ_ε as in Assumption 2.1. Then, for a.e. $t \in S$, u_ε two-scale converges weakly / strongly with respect to the L^p-norm to $u_0 \in L^p(\Omega \times Y)$ if and only if $\hat{u}_\varepsilon$ two-scale converges weakly / strongly with respect to the L^p-norm to $\hat{u}_0 \in L^p(\Omega \times Y)$. Moreover, $\hat{u}_0(x, y) = u_0(x, \psi_0(t, x, y))$ holds and, equivalently, $u_0(x, y) = \hat{u}_0(x, \psi_0^{-1}(t, x, y))$.*

Proof. For a proof of Lemma 4.4 see [29] Theorem 3.8 and Theorem 3.14. Note, that there the deformations ψ_ε are only defined on $\Omega_\varepsilon \subset \Omega$, which is why the transformation result has to deal with the extension of the functions by 0. However, the results there holds for $\Omega_\varepsilon = \Omega$, which proves Lemma 4.4. □

Note that Lemma 4.4 translates the two-scale convergence of $\chi_{\Omega_\varepsilon}$ to χ_{Y^*} into the two-scale convergence of $\chi_{\Omega_\varepsilon(t)}$ to $\chi_{Y_x^*(t)}$.

Further two-scale transformation results for weakly differentiable functions can be found in [29] Theorem 3.8.

4.1. Two-scale convergence of the transformed data

Lemma 4.5. *Let $\hat{f}_\varepsilon$, $\hat{p}_{b,\varepsilon}$ and $\hat{v}_{\Gamma_\varepsilon}$ be defined by (6). Then,*

$$\begin{aligned} &\hat{f}_\varepsilon(t) \to f(t) && \text{weakly in the two-scale sense,} \\ &\nabla \hat{p}_{b,\varepsilon}(t) \to \nabla_x p_b(t) + \nabla_y \hat{p}_{b,1}(t) && \text{weakly in the two-scale sense,} \\ &\frac{1}{\varepsilon} \hat{v}_{\Gamma_\varepsilon}(t) \to \hat{v}_\Gamma(t) && \text{weakly in the two-scale sense,} \\ &\nabla \hat{v}_{\Gamma_\varepsilon}(t) \to \nabla_y \hat{v}_\Gamma(t) && \text{weakly in the two-scale sense} \end{aligned}$$

for a.e. $t \in S$, where f, p_b and v_Γ are the two-scale limits given in Assumption 2.3 and $\hat{p}_{b,1}(t, x, y) = p_{b,1}(t, x, \psi_0(t, x, y)) + \check{\psi}_0(t, x, y) \cdot \nabla_x p_b(t, x)$ and $\hat{v}_\Gamma(t, x, y) = v_\Gamma(t, x, \psi_0(t, x, y))$ for a.e. $(x, y) \in \Omega_\varepsilon \times Y^*$.

Proof. The two-scale convergence of $\hat{f}_\varepsilon(t)$ and of $\frac{1}{\varepsilon} \hat{v}_{\Gamma_\varepsilon}$ follows from Lemma 4.4. The two-scale convergence of $\nabla \hat{p}_{b,\varepsilon}(t)$ follows from [29, Theorem 3.10] and the two-scale convergence of $\nabla \hat{v}_{\Gamma_\varepsilon}$ from [29, Theorem 3.9]. □

Moreover, we need the two-scale convergence of the transformation coefficients, which are given by the following result.

Lemma 4.6. *Let ψ_ε and ψ_0 be given by Assumption 2.1. Then,*

$$\begin{aligned}&\Psi_\varepsilon(t) \to \Psi_0(t),\ \Psi_\varepsilon^{-1}(t) \to \Psi_0^{-1}(t),\ J_\varepsilon(t) \to J_0(t), J_\varepsilon^{-1}(t) \to J_0^{-1}(t),\\ &A_\varepsilon(t) \to A_0(t),\ \varepsilon D_x A_\varepsilon(t) \to D_y A_0(t),\ \varepsilon D_x A_\varepsilon^{-1}(t) \to D_y A_0(t)\end{aligned}$$

strongly in the two-scale sense, for a.e. $t \in S$, where the strong two-scale convergence holds with respect to every L^p-norm for $p \in (1, \infty)$ and $A_0 := J_0 \Psi_0^{-1}$.

Proof. Using the results of [29], it remains to prove that $\varepsilon D_x A_\varepsilon(t) \to D_y A_0(t)$ and $\varepsilon D_x A_\varepsilon^{-1}(t) \to D_y A_0^{-1}(t)$ strongly in the two-scale sense. Therefore, we note that $\varepsilon D_x \Psi_\varepsilon(t) = \varepsilon D_x \nabla \psi_\varepsilon^\top(t)$ two-scale converges strongly to $D_y \Psi_0(t) = D_y \nabla_y \psi_0^\top(t)$ by Assumption 2.1.

We rewrite $D_x A_\varepsilon(t)$ into the sum of polynomials like in the proof of Lemma 3.4. Then, we pass to the limit $\varepsilon \to 0$ using the two-scale convergence of $\varepsilon D_x \Psi_\varepsilon(t)$ and $\Psi_\varepsilon(t)$.

By the same argumentation, we obtain the strong two-scale convergence of $\varepsilon D_x J_\varepsilon(t)$ to $D_y J_0(t)$. Then, we rewrite $A_\varepsilon^{-1}(t) = J_\varepsilon^{-1}(t)\Psi_\varepsilon(t)$. Using the quotient rule and the previously proven two-scale convergences, we obtain the strong two-scale convergence of $\varepsilon D_x A_\varepsilon^{-1}(t)$ to $D_y A_0^{-1}(t)$. □

4.2. Homogenisation of the transformed Stokes equations

Theorem 4.7. *Let $\hat{w}_\varepsilon$ and $\hat{q}_\varepsilon$ be the solution of (7). Let $\hat{Q}_\varepsilon$ be the extension of $\hat{q}_\varepsilon$ as defined in Lemma 4.8 and $\widetilde{\hat{w}}_\varepsilon \in L^{p_s}(S; H^1(\Omega))$ be the extension of $\hat{w}_\varepsilon$ by 0 on $\Omega \setminus \Omega_\varepsilon$. Then, $\widetilde{\hat{w}}_\varepsilon(t)$ two-scale converges to $\hat{w}_0(t)$ and $\hat{Q}_\varepsilon(t)$ converges strongly in $L^2(\Omega)$ to $\hat{q}(t)$, for a.e. $t \in S$, where $(\hat{w}_0, \hat{q}, \hat{q}_1) \in L^{p_s}(S; L^2(\Omega; H^1_{\Gamma\#}(Y^*)^N)) \times L^{p_s}(S; H^1_0(\Omega)) \times L^{p_s}(S; L^2(\Omega; L^2_0(Y^*)))$ is the unique solution of (30).*

In order to pass to the limit $\varepsilon \to 0$ in (7a), we test it by $A_\varepsilon^{-1}(t)\varphi$ and obtain

$$\begin{aligned}&(\varepsilon^2 \nu A_\varepsilon(t)\hat{e}_{\varepsilon,t}(\hat{w}_\varepsilon(t)), \nabla(A_\varepsilon^{-1}(t)\varphi))_{\Omega_\varepsilon} - (\hat{Q}_\varepsilon(t), \operatorname{div}(\varphi))_{\Omega_\varepsilon}\\ &= (\Psi_\varepsilon^\top(t)\hat{f}_\varepsilon(t) - \nabla \hat{p}_{b,\varepsilon}, \varphi)_{\Omega_\varepsilon} - (\varepsilon^2 \nu A_\varepsilon(t)\hat{e}_{\varepsilon,t}(\hat{v}_{\Gamma_\varepsilon}(t)), \nabla(A_\varepsilon^{-1}(t)\varphi))_{\Omega_\varepsilon}. \qquad (31)\end{aligned}$$

Since $A_\varepsilon^{-1}(t)$ is invertible (7a) can be replaced by (31).

First, we prove the strong convergence of $\hat{Q}_\varepsilon(t)$. Thereto, we transfer the argumentation of [6] on our weak form with the different function spaces.

Lemma 4.8. *Let $\hat{q}_\varepsilon$ be the second part of the solution of (7) and $\hat{Q}_\varepsilon$ be the extension of $\hat{q}_\varepsilon$ on Ω defined by*

$$\hat{Q}_\varepsilon(t,x) := \begin{cases} \hat{q}_\varepsilon(t,x) & \text{if } x \in \Omega_\varepsilon, \\ \frac{1}{|\varepsilon Y^*|} \int\limits_{k+\varepsilon Y^*} \hat{q}_\varepsilon(t,x) & \text{if } x \in k + \varepsilon Y^{\mathrm{s}} \text{ for } k \in I_\varepsilon. \end{cases} \qquad (32)$$

Then, for a.e. $t \in S$, there exists $\hat{q}(t) \in L^2(\Omega)$ and a subsequence of $\hat{Q}_\varepsilon(t)$ which converges strongly in $L^2(\Omega)$ to $q(t)$.

Proof. Using the restriction operator from Lemma 3.11, we define $F_\varepsilon(t) \in (H^1(\Omega)^N)'$ by

$$\langle F_\varepsilon(t), \varphi\rangle_{H^1(\Omega)', H^1(\Omega)} := \int_{\Omega_\varepsilon} \hat{q}_\varepsilon(t,x) \operatorname{div}(R_\varepsilon \varphi(x)) \,\mathrm{d}x. \tag{33}$$

From (31), we obtain

$$\int_\Omega \hat{q}_\varepsilon(t) \operatorname{div}(R_\varepsilon\varphi)\,\mathrm{d}x = (\varepsilon^2 \nu A_\varepsilon(t) 2\hat{e}_{\varepsilon,t}(\hat{w}_\varepsilon(t)), \nabla(A_\varepsilon^{-1}(t) R_\varepsilon\varphi))_{\Omega_\varepsilon}$$
$$-(\Psi_\varepsilon^\top(t)\hat{f}_\varepsilon(t) - \nabla \hat{p}_{b,\varepsilon}(t), R_\varepsilon\varphi)_{\Omega_\varepsilon} - (\varepsilon^2\nu A_\varepsilon(t) 2\hat{e}_{\varepsilon,t}(\hat{v}_{\Gamma_\varepsilon}(t)), \nabla(A_\varepsilon^{-1}(t)R_\varepsilon\varphi))_{\Omega_\varepsilon}.$$

Thus, we can estimate $F_\varepsilon(t)$ using the estimates of $\varepsilon\nabla\hat{w}_\varepsilon(t)$ (cf. (9)), the coefficients (cf. Lemma 3.4) and the data (cf. Assumption 2.3) as well as the product rule by

$$\begin{aligned}|\langle F_\varepsilon(t), \varphi\rangle_{H^1(\Omega)', H^1(\Omega)}| &\le C\varepsilon \left\|\nabla(A_\varepsilon^{-1}(t)R_\varepsilon\varphi)\right\|_{\Omega_\varepsilon} + C\left\|R_\varepsilon\varphi\right\|_{\Omega_\varepsilon} + \varepsilon^2\left\|\nabla(A_\varepsilon^{-1}(t)R_\varepsilon\varphi)\right\|_{\Omega_\varepsilon}\\ &\le C(\varepsilon+\varepsilon^2)\left\|\nabla R_\varepsilon\varphi\right\|_{\Omega_\varepsilon} + C(1+\varepsilon)\left\|R_\varepsilon\varphi\right\|_{\Omega_\varepsilon}.\end{aligned}$$

Then, the estimates of Lemma 3.11 imply, for $\varepsilon \le 1$,

$$|\langle F_\varepsilon(t), \varphi\rangle_{H^1(\Omega)', H^1(\Omega)}| \le C(\|\varphi\|_\Omega + \varepsilon\|\nabla\varphi\|_\Omega) \tag{34}$$

and in particular $\|F_\varepsilon(t)\|_{H^1(\Omega)'} \le C$.

Because $\operatorname{div}(R_\varepsilon\varphi) = 0$ if $\operatorname{div}(\varphi) = 0$, we obtain

$$\langle F_\varepsilon(t), \varphi\rangle_{H^1(\Omega)', H^1(\Omega)} = \int_{\Omega_\varepsilon} \hat{q}_\varepsilon(t,x)\operatorname{div}(R_\varepsilon\varphi(x))\,\mathrm{d}x = \int_{\Omega_\varepsilon}\hat{q}_\varepsilon(t,x)\operatorname{div}(\varphi(x))\,\mathrm{d}x = 0$$

for every $\varphi \in H^1(\Omega)$ with $\operatorname{div}(\varphi) = 0$. Since div has closed range (cf. Lemma 3.10), the closed-range theorem implies that there exists $\hat{Q}_\varepsilon(t) \in L^2(\Omega)$ such that

$$\int_\Omega \hat{Q}_\varepsilon(t)\operatorname{div}(\varphi)\,\mathrm{d}x = \langle F_\varepsilon(t), \varphi\rangle_{H^1(\Omega)', H^1(\Omega)} = \int_{\Omega_\varepsilon}\hat{q}_\varepsilon(t,x)\operatorname{div}(R_\varepsilon\varphi(x))\,\mathrm{d}x. \tag{35}$$

Moreover, we obtain the uniform boundedness of $\left\|\hat{Q}_\varepsilon(t)\right\|_{L^2(\Omega)}$ with Lemma 3.12 and (34) by

$$\begin{aligned}\left\|\hat{Q}_\varepsilon(t)\right\|^2_{L^2(\Omega)} &= \int_\Omega \hat{Q}_\varepsilon(t)\operatorname{div}(\operatorname{div}^{-1}(\hat{Q}_\varepsilon(t))) = |\langle F_\varepsilon(t), \operatorname{div}^{-1}(\hat{Q}_\varepsilon(t))\rangle_{H^1(\Omega)', H^1(\Omega)}|\\ &\le C\Big(\left\|\operatorname{div}^{-1}(\hat{Q}_\varepsilon(t))\right\|_\Omega + \varepsilon\left\|\nabla\operatorname{div}^{-1}(\hat{Q}_\varepsilon(t))\right\|_\Omega\Big) \le C\left\|\hat{Q}_\varepsilon(t)\right\|_{L^2(\Omega)}.\end{aligned} \tag{36}$$

In order to identify $\hat{Q}_\varepsilon(t)$ with $\hat{q}_\varepsilon(t)$ on Ω_ε, we note that $R_\varepsilon(\widetilde{\varphi}) = \varphi$ for every $\varphi \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)$, where $\widetilde{\varphi}$ is the extension of φ by 0 to Ω. Then, we obtain from (35)

$$\int_\Omega \hat{Q}_\varepsilon(t)\,\mathrm{div}(\widetilde{\varphi})\,\mathrm{d}x = \int_{\Omega_\varepsilon} \hat{Q}_\varepsilon(t)\,\mathrm{div}(\varphi)\,\mathrm{d}x = \int_{\Omega_\varepsilon} \hat{q}_\varepsilon(t)\,\mathrm{div}(\varphi)\,\mathrm{d}x.$$

Lemma 3.12 gives the existence of a function $\varphi \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)$ with $\mathrm{div}(\varphi) = \hat{Q}_\varepsilon(t) - \hat{q}_\varepsilon(t)$. Testing with this φ implies $\hat{Q}_\varepsilon(t) = \hat{q}_\varepsilon(t)$ on Ω_ε.

In order to show the strong convergence of $\hat{Q}_\varepsilon(t)$, we note that the boundedness of $\hat{Q}_\varepsilon(t)$ in $L^2(\Omega)$ allows us to pass to a subsequence, which we still denote by $\hat{Q}_\varepsilon(t)$, such that $\hat{Q}_\varepsilon(t)$ converges weakly to a function $\hat{q}(t) \in L^2(\Omega)$. For the same subsequence $\varphi_\varepsilon = \mathrm{div}^{-1}(\hat{Q}_\varepsilon(t))$ converges weakly to $\varphi = \mathrm{div}^{-1}(\hat{q}(t))$ in $H^1(\Omega)$, where div^{-1} is given by Lemma 3.10. Then, we obtain from (35) and (34)

$$(\hat{Q}_\varepsilon(t), \mathrm{div}(\varphi_\varepsilon - \varphi))_{\Omega_\varepsilon} \le C(\|\varphi_\varepsilon - \varphi\|_\Omega + \varepsilon\,\|\nabla(\varphi_\varepsilon - \varphi)\|_\Omega).$$

The compact embedding of $H^1(\Omega)$ into $L^2(\Omega)$ implies that $\|\varphi_\varepsilon - \varphi\|_\Omega \to 0$ and since $\|\nabla(\varphi_\varepsilon - \varphi)\|_\Omega$ is bounded, we obtain

$$(\hat{Q}_\varepsilon(t), \hat{Q}_\varepsilon(t) - \hat{q}(t)) = (\hat{Q}_\varepsilon(t), \mathrm{div}(\varphi_\varepsilon - \varphi))_{\Omega_\varepsilon} \to 0.$$

By using additionally the weak convergence of $\hat{Q}_\varepsilon(t)$ to $\hat{q}(t)$, we obtain

$$\left\| \hat{Q}_\varepsilon(t) - \hat{q}(t) \right\|^2_\Omega = (\hat{Q}_\varepsilon(t), \hat{Q}_\varepsilon(t) - \hat{q}(t))_\Omega - (\hat{q}(t), \hat{Q}_\varepsilon(t) - \hat{q}(t))_\Omega \to 0,$$

which shows the strong convergence of $\hat{Q}_\varepsilon(t)$.

The explicit formula (32) of $\hat{Q}_\varepsilon(t)$ can be directly transferred from [6]. □

In the second step, we pass to the limit $\varepsilon \to 0$ in the divergence condition (7b) and derive the microscopic incompressibility condition (30b) and the macroscopic inhomogeneous divergence condition (30c).

Lemma 4.9. *Let $\hat{w}_\varepsilon \in L^{p_s}(S; H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N)$ be the first part of the solution of (7) and $\widetilde{\hat{w}_\varepsilon}$ defined as in Theorem 4.7. Then, there exists, for a.e. $t \in S$, a subsequence $\widetilde{\hat{w}_\varepsilon}(t)$ and $\hat{w}_0(t) \in L^2(\Omega; H^1_{\Gamma\#}(Y^*)^N)$ such that, for this subsequence, $\widetilde{\hat{w}_\varepsilon}(t)$ and $\varepsilon\nabla\widetilde{\hat{w}_\varepsilon}(t)$ two-scale converge to $\hat{w}_0(t)$ and $\nabla_y \hat{w}_0(t)$, respectively. Furthermore, $\hat{w}_0(t)$ satisfies (30b) and (30c).*

Proof. The uniform estimate (9) implies that $\widetilde{\hat{w}_\varepsilon}(t)$ and $\varepsilon\nabla\widetilde{\hat{w}_\varepsilon}(t)$ are bounded as well. Then, by a standard two-scale compactness result there exists, for a.e. $t \in S$, a subsequence and a function $\hat{w}_0(t) \in L^2(\Omega; H^1_\#(Y)^N)$ such that for this subsequence $\widetilde{\hat{w}_\varepsilon}(t) \to w_0(t)$ and $\varepsilon\nabla\widetilde{\hat{w}_\varepsilon}(t) \to \nabla_y w_0(t)$ in the two-scale sense. Using arbitrary two-scale test functions $\varphi \in C(\overline{\Omega}; C^\infty_\#(Y)^N)$ which are 0 on $\Omega \times Y^*$ shows $w_0(t) = 0$ in $\Omega \times Y^s$, which means $w_0(t) \in L^2(\Omega; H^1_{\Gamma\#}(Y^*)^N)$.

By applying the estimates of Lemma 3.4 and Lemma 3.13 on the inhomogeneous divergence condition (7b), we obtain

$$\left\| \operatorname{div}(A_\varepsilon(t)\hat{w}_\varepsilon(t)) \right\|_{L^2(\Omega_\varepsilon)} = \left\| \operatorname{div}(A_\varepsilon(t)\hat{v}_{\Gamma_\varepsilon}(t)) \right\|_{L^2(\Omega_\varepsilon))} \leq C(t)$$

for a.e. $t \in S$ for $C \in L^{p_s}(S)$. Using this estimate, the Y-periodicity of $\hat{w}_0$, $A_0(t)$ and the two-scale test functions as well as the strong two-scale convergence of the coefficients given by Lemma 4.6, we can conclude for $\theta \in D(\Omega; C^\infty_\#(Y))$ and for a.e. $t \in S$:

$$\begin{aligned}
&\int_\Omega \int_{Y^*} \operatorname{div}_y(A_0(t,x,y)\hat{w}_0(t,x,y))\theta(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&= -\int_\Omega \int_{Y^*} A_0(t,x,y)\hat{w}_0(t,x,y)\cdot\nabla_y\theta(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&= -\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} A_\varepsilon(t,x)\hat{w}_\varepsilon(t,x)\cdot\left(\varepsilon\nabla_x\theta\left(x,\frac{x}{\varepsilon}\right)+\nabla_y\theta\left(x,\frac{x}{\varepsilon}\right)\right)\mathrm{d}x \\
&= \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} \varepsilon\operatorname{div}(A_\varepsilon(t,x)\hat{w}_\varepsilon(t,x))\theta\left(x,\frac{x}{\varepsilon}\right)\mathrm{d}x = 0.
\end{aligned}$$

By the density of $D(\Omega; C^\infty_\#(Y))$ in $L^2(\Omega; L^2(Y^*))$, we obtain the microscopic incompressibility condition (30b).

In order to derive the macroscopic inhomogeneous divergence condition, we test (7b) with $\theta \in D(\Omega)$ and pass to the limit $\varepsilon \to 0$. Using the two-scale convergence of $A_\varepsilon(t)$ and $\hat{v}_{\Gamma_\varepsilon}(t)$ and their derivatives yields

$$\begin{aligned}
&\int_\Omega \operatorname{div}_x\Big(\int_{Y^*} A_0(t,x,y)\hat{w}_0(t,x,y)\,\mathrm{d}y\Big)\theta(x)\,\mathrm{d}x \\
&= -\int_\Omega\int_{Y^*} A_0(t,x,y)\hat{w}_0(t,x,y)\,\mathrm{d}y\cdot\nabla_x\theta(x)\,\mathrm{d}x \\
&= -\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} A_\varepsilon(t,x,y)\hat{w}_\varepsilon(t,x)\cdot\nabla_x\theta(x)\,\mathrm{d}x \\
&= \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} \operatorname{div}_x(A_\varepsilon(t,x)\hat{w}_\varepsilon(t,x))\theta(x)\,\mathrm{d}x \\
&= -\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} \operatorname{div}_x(A_\varepsilon(t,x)\hat{v}_{\Gamma_\varepsilon}(t,x))\theta(x)\,\mathrm{d}x \\
&= -\int_\Omega\int_{Y^*} \operatorname{div}_y\left(A_0(t,x,y)\hat{v}_\Gamma(t,x,y)\right)\mathrm{d}y\;\theta(x)\,\mathrm{d}x.
\end{aligned}$$

By the density of $D(\Omega)$ in $H^1_0(\Omega)$, we can deduce the macroscopic inhomogeneous divergence condition (30c). □

In the third step, we can pass to the limit $\varepsilon \to 0$ in (31).

Proof of Theorem 4.7. Employing Lemma 4.9 and Lemma 4.8, there exists, for a.e. $t \in S$, a subsequence, $\hat{q}(t) \in L^2(\Omega)$ and $\hat{w}(t) \in L^2(\Omega; H^1_{\Gamma\#}(Y^*)^N)$ such that, for this subsequence, $\widehat{Q}_\varepsilon(t)$ converges strongly to $\hat{q}(t)$, $\widetilde{w_\varepsilon}(t)$ and $\varepsilon\nabla\widetilde{w_\varepsilon}(t)$ two-scale converge weakly to $\hat{w}_0(t)$ and $\nabla_y \hat{w}_0(t)$, respectively. We consider this subsequence in the following.

Let $\varphi \in C^\infty(\overline{\Omega}; C^\infty_{\Gamma\#}(Y^*)^N)$ such that $\operatorname{div}_y(\varphi) = 0$. Then, we test (31) with $\varphi(x, \frac{x}{\varepsilon})$ and pass to the limit $\varepsilon \to 0$, which yields

$$\begin{aligned}\int_\Omega\int_{Y^*} \nu A_0(t,x,y) 2\hat{e}_{y,t}(\hat{w}_0(t,x,y)) : \nabla_y(A_0^{-1}(t,x,y)\varphi(x,y))\,\mathrm{d}y\,\mathrm{d}x \\ - \int_\Omega\int_{Y^*} \hat{q}(t,x)\operatorname{div}_x(\varphi(x,y))\,\mathrm{d}y\,\mathrm{d}x = \int_\Omega\int_{Y^*} \Psi_0^\top(t,x,y) f(t,x)\cdot\varphi(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ - \int_\Omega\int_{Y^*} (\nabla_x p_b(t,x) + \nabla_y \hat{p}_{b,1}(t,x,y))\cdot\varphi(x,y)\,\mathrm{d}y\,\mathrm{d}x\end{aligned} \tag{37}$$

for any $\varphi \in C^\infty(\overline{\Omega}; C^\infty_{\Gamma\#}(Y^*))^N$ such that $\operatorname{div}_y(\varphi) = 0$. By density, (37) holds for every $\varphi \in C^\infty(\overline{\Omega}; H^1_{\Gamma\#}(Y^*)^N)$ with $\operatorname{div}_y(\varphi) = 0$.

Moreover, the following integration by parts shows that we can omit $\nabla_y \hat{p}_{b,1}(t)$

$$\int_\Omega\int_{Y^*} \nabla_y \hat{p}_{b,1}(t,x,y)\cdot\varphi(x,y)\,\mathrm{d}y\,\mathrm{d}x = -\int_\Omega\int_{Y^*} \hat{p}_{b,1}(t)\operatorname{div}_y(\varphi(x,y))\,\mathrm{d}y\,\mathrm{d}x = 0.$$

The boundary term in this integration by parts vanishes since $\hat{p}_{b,1}(t)$ is Y-periodic. Now, we choose $\varphi = \phi\phi_i$ in (37), for $\phi \in C^\infty(\overline{\Omega})$ and $\phi_i \in H^1_{\Gamma\#}(Y^*)^N$, with $\phi_i = 0$ in Y^{s}, $\operatorname{div}_y(\phi_i) = 0$ and $\int_{Y^*} \phi_i(y) = e_i$ for $i \in \{1, \dots, N\}$, which can be constructed by the Stokes operator similar to the proof of [46, Lemma 2.10]. We obtain

$$\int_\Omega -\hat{q}(t,x)\partial_{x_i}\phi(x) - G_i(t,x)\phi(x)\,\mathrm{d}x = 0$$

for

$$\begin{aligned}G_i(t,x) = -\int_{Y^*} \nu A_0(t,x,y) 2\hat{e}_{y,t}(\hat{w}_0(t,x,y)) : \nabla_y(A_0^{-1}(t,x,y)\phi_i(y))\,\mathrm{d}y \\ + \int_\Omega\int_{Y^*} (\Psi_0^\top(t,x,y) f(t,x) - \nabla_x p_b(t,x))\cdot\phi_i(y)\,\mathrm{d}y.\end{aligned}$$

Since $G_i(t) \in L^2(\Omega)$ it follows that $\hat{q}(t) \in H^1_0(\Omega)$. Thus, in (37), we can integrate the pressure term by parts and by a density argument the resulting equation holds for all test function in $\varphi \in L^2(\Omega; H^1_{\Gamma\#}(Y^*)^N)$ with $\operatorname{div}_y(\varphi) = 0$.

Employing the Bogovskiĭ-operator on Y^*, we obtain that $\operatorname{div}_y : L^2(\Omega; H^1_{\Gamma\#}(Y^*)^N) \to L^2(\Omega; L^2_0(Y^*))$ is surjective because $L^2(\Omega; H^1_{\Gamma\#}(Y^*)) \supset L^2(\Omega; H^1_0(Y^*))$. Then, the closed-range theorem gives $\hat{q}_1(t) \in L^2(\Omega; L^2_0(Y^*))$ such that

$$\begin{aligned}\int_\Omega \int_{Y^*} \nu A_0(t,x,y) 2\hat{e}_{y,t}(\hat{w}_0(t,x,y)) : \nabla_y (A_0^{-1}(t,x,y)\varphi(x,y)) \,\mathrm{d}y\,\mathrm{d}x \\ + \int_\Omega \int_{Y^*} \nabla_x \hat{q}(t,x) \cdot \varphi(x,y) - \hat{q}_1(t,x,y) \operatorname{div}_y(\varphi(x,y)) \,\mathrm{d}y\,\mathrm{d}x \\ = \int_\Omega \int_{Y^*} (\Psi_0^\top(t,x,y) f(t,x) - \nabla_x p_b(t,x)) \cdot \varphi(x,y) \,\mathrm{d}y\,\mathrm{d}x\end{aligned} \tag{38}$$

for all $\varphi \in L^2(\Omega; H^1_{\Gamma\#}(Y^*)^N)$. Since $A_0^{-1}(t)$ is invertible (38) is equivalent to

$$\begin{aligned}\int_\Omega \int_{Y^*} \nu A_0(t,x,y) 2\hat{e}_{y,t}(\hat{w}_0(t,x,y)) : \nabla_y \varphi(x,y) \,\mathrm{d}y\,\mathrm{d}x \\ + \int_\Omega \int_{Y^*} A_0^\top(t,x,y) \nabla_x \hat{q}(t,x) \cdot \varphi(x,y) - \hat{q}_1(t) \operatorname{div}_y(A_0(t,x,y)\varphi(x,y)) \,\mathrm{d}y\,\mathrm{d}x \\ = \int_\Omega \int_{Y^*} (J_0(t,x,y) f(t,x) - A_0^\top(t,x,y) \nabla p_b(t,x)) \cdot \varphi(x,y) \,\mathrm{d}y\,\mathrm{d}x.\end{aligned} \tag{39}$$

Furthermore, (30b)–(30c) follow from Lemma 4.9.

In order to simplify (39) to (30a), we show that the microscopic incompressibility condition (30b), the Y-periodicity and the zero boundary values of φ on Γ imply that

$$\int_{Y^*} \nu A_0(t,x,y) (\Psi_0^{-\top}(t,x,y) \nabla_y \hat{w}(t,x,y))^\top : \nabla_y \hat{\varphi}(y) \,\mathrm{d}y = 0 \tag{40}$$

for all $\hat{\varphi} \in H^1_{\Gamma\#}(Y^*)^N$ and a.e. $(t,x) \in S \times \Omega$. For fixed $(t,x) \in S \times \Omega$, we transform the left-hand side of (40) in the actual two-scale coordinates by means of ψ_0^{-1} for a.e. $x \in \Omega$, which yields

$$\begin{aligned}\int_{Y^*} \nu (\nabla_y w(t,x,y))^\top : \nabla_y \varphi(y) \,\mathrm{d}y \\ = \int_{Y^*_x(t)} \nu A_0(t,x,y) (\Psi_0^{-\top}(t,x,y) \nabla_y \hat{w}(t,x,y))^\top : \nabla_y \hat{\varphi}(y) \,\mathrm{d}y,\end{aligned} \tag{41}$$

for $w_0(t,x,y) = \hat{w}_0(t,x,\psi_0^{-1}(t,x,y))$ and $\varphi(x) = \hat{\varphi}(x, \psi_0(t,x,y))$. Transforming the microscopic incompressibility condition (30b) shows that w_0 is divergence-free

$$\operatorname{div}_y(w_0(t,x,y)) = 0. \tag{42}$$

We note that the divergence-free smooth functions $\{u \in C^\infty_{\Gamma(t,x)\#}(Y_x^*(t))^N \mid \operatorname{div}_y(u) = 0\}$ are dense in the divergence-free H^1-functions $\{u \in H^1_{\Gamma(t,x)\#}(Y_x^*(t))^N \mid \operatorname{div}_y u = 0\}$ with respect to the H^1-norm (see [45, Chapter III.4]). Thus, we can choose a sequence $(u_n)_{n\in\mathbb{N}}$ in $C^\infty_{\Gamma(t,x)\#}(Y_x^*(t))^N$ with $\operatorname{div}_y(u_n) = 0$, which converges to w_0 with respect to the $H^1(Y_x^*(t))$-norm. After integration by parts, we get

$$\begin{aligned}\int\limits_{Y_x^*(t)} (\nabla_y w_0(x,y))^\top : \nabla_y \varphi(y)\,\mathrm{d}y &= \lim_{n\to\infty} \int\limits_{Y_x^*(t)} \nabla_y (u_n(y))^\top : \nabla_y\varphi(y)\,\mathrm{d}y \\ &= -\lim_{n\to\infty} \int\limits_{Y_x^*(t)} \operatorname{div}_y(\nabla_y(u_n(y))^\top)\cdot\varphi(y)\,\mathrm{d}y = 0,\end{aligned} \tag{43}$$

where the boundary integral from the integration by parts vanishes due to the zero boundary values of φ at $\Gamma(t,x)$ and the Y-periodicity of w_0 and φ. The last equality of (43) follows from

$$\begin{aligned}(\operatorname{div}_y(\nabla_y u_n(y))^\top)_i &= \sum_{j=1}^n \partial_{y_j}((\nabla u_n(y))^\top)_{ji} = \sum_{j=1}^n \partial_{y_j}\partial_{y_i}(u_n)_j(y) \\ &= \partial_{y_i}\sum_{j=1}^n \partial_{y_j}(u_n)_j(y) = \partial_{y_i}\operatorname{div}_y(u_n(y)) = 0.\end{aligned}$$

Combining (41) and (43) shows (40) and, hence, (39) can be simplified to (30a).

Thus, we have shown that the limit $\hat{w}_0, \hat{q}$ solves (30). Since (30) has a unique solution (cf. Lemma 4.11), the convergence holds for the whole sequence. □

In order to show the existence and uniqueness of the solution of (30), we derive the following inf–sup estimate for the div-conditions.

Lemma 4.10. *There exists a constant $C \in \mathbb{R}$ such that*

$$\sup_{v\in L^2(\Omega;H^1_{\Gamma\#}(Y^*)^N)\setminus\{0\}} \frac{(A_0(t)v, \nabla_x\phi_0)_{\Omega\times Y^*} - (\operatorname{div}_y(A_0(t)v), \phi_1)_{\Omega\times Y^*}}{\|v\|_{L^2(\Omega;H^1_{\Gamma\#}(Y^*))}\,\|(\phi_0,\phi_1)\|_{H^1_0(\Omega)\times L^2(\Omega;L^2_{0\#}(Y^*))}} \geq \beta \tag{44}$$

for a.e. $t \in S$ and any $(\phi_0, \phi_1) \in H^1_0(\Omega) \times L^2(\Omega; L^2_{0\#}(Y^*))$.

Proof. Let $(\phi_0, \phi_1) \in H^1_0(\Omega) \times L^2(\Omega; L^2_0(Y^*))$. From the Bogovskiĭ-operator, we obtain $u \in L^2(\Omega; H^1_{\Gamma\#}(Y^*)^N)$ such that

$$\operatorname{div}_y(u) = \phi_1, \quad \|u\|_{L^2(\Omega;H^1_0(Y^*))} \leq C\,\|\phi_1\|_{L^2(\Omega;L^2_{0\#}(Y^*))}$$

for a constant C which only depends on Ω and Y^* and not on ϕ_1.

We define the functions $v_1, \ldots, v_n \in H^1_{\Gamma\#}(Y^*)^N$ as the solutions of the following Stokes problems: Find $(v_i, p_i) \in H^1_{\Gamma\#}(Y^*)^N \times L^2_0(Y^*)$ such that

$$(\nabla v_i, \nabla \varphi)_{Y^*} - (p_i, \operatorname{div}(\varphi))_{Y^*} = (e_i, \varphi)_{Y^*},$$
$$(\operatorname{div}(v_i), \phi)_{Y^*} = 0$$

for any $(\varphi, \phi) \in H^1_{\Gamma\#}(Y^*)^N \times L^2_0(\Omega)$. Choosing $\varphi = v_j$ gives

$$A := \begin{pmatrix} \vdots & & \vdots \\ \int\limits_{Y^*} v_1(y)\,\mathrm{d}y & \cdots & \int\limits_{Y^*} v_n(y)\,\mathrm{d}y \\ \vdots & & \vdots \end{pmatrix} = \begin{pmatrix} (\nabla v_1, \nabla v_1)_{Y^*} & \cdots & (\nabla v_1, \nabla v_n)_{Y^*} \\ \vdots & & \vdots \\ (\nabla v_n, \nabla v_1)_{Y^*} & \cdots & (\nabla v_n, \nabla v_n)_{Y^*} \end{pmatrix}. \quad (45)$$

Since A is the permeability tensor from the usual Darcy law, it is symmetric and positive definite (see for instance [4, Chapter 7, Proposition 2.2]). This guarantees that the following boundary-value problem is well-defined: Find a solution $w \in H^1_0(\Omega)$ such that

$$(A\nabla w, \nabla\varphi)_\Omega = (\nabla\phi_0, \nabla\varphi)_\Omega + \Big(\int\limits_{Y^*} u(\cdot, y)\,\mathrm{d}y, \nabla\varphi\Big)_\Omega \quad (46)$$

for any $\varphi \in H^1_0(\Omega)$.

By the Theorem of Lax–Milgram, we obtain unique solutions $w \in H^1_0(\Omega)$, which can be estimated by

$$\|w\|_{H^1_0(\Omega)} \le C(\|\phi_0\|_{H^1_0(\Omega)} + \|u\|_{\Omega\times Y^*}).$$

We define $\bar{v}(x, y) := A_0^{-1}(t, x, y)\left(\sum_{i=1}^n v_i(y)\,\partial_{x_i} w(x) - u(x, y)\right)$ and estimate

$$\begin{aligned} \|\bar{v}\|_{L^2(\Omega; H^1_{\Gamma\#}(Y^*))} &\le C(\|w\|_{H^1_0(\Omega)} + \|u\|_{L^2(\Omega; H^1_{\Gamma\#}(Y^*))}) \\ &\le C(\|\phi_0\|_{H^1_0(\Omega)} + \|\phi_1\|_{L^2(\Omega; L^2_{0\#}(Y^*))}) \end{aligned}$$

for $C \in \mathbb{R}$ independent of t. Then, we obtain

$$(A_0(t)\bar{v}, \nabla\phi_0)_{\Omega\times Y^*} = \Big(A\nabla w - \int\limits_{Y^*} u(\cdot, y)\,\mathrm{d}y, \nabla\phi_0\Big)_\Omega = (\nabla\phi_0, \nabla\phi_0)_\Omega,$$

$$\operatorname{div}_y(A_0(t)\bar{v}) = \sum_{i=1}^n \operatorname{div}_y(v_i(y))\partial_{x_i} w(x) - \operatorname{div}_y(u(x, y)) = -\phi_1(x).$$

Using this explicitly constructed $\bar{v}$, we can deduce (44) for $C > 0$, which is independent of t. □

Lemma 4.11. *There exists a unique solution* $(\hat{w}_0, \hat{q}, \hat{q}_1) \in L^p(S; L^2(\Omega; H^1_{\Gamma\#}(Y^*)^N)) \times L^p(S; H^1_0(\Omega)) \times L^p(S; L^2(\Omega; L^2_{0\#}(Y^*)))$ *of* (30).

Proof. Note that the existence of a solution for a.e. $t \in S$ is, up to the measurability with respect to time, already secured by the homogenisation process. However, it remains to prove the uniqueness. We rewrite (30) in the setting of the generic saddle-point formulation of Lemma 3.2. We define the following time-dependent bilinear forms:

$$a_t : L^2(\Omega; H^1_{\Gamma\#}(Y^*)^N) \times L^2(\Omega; H^1_{\Gamma\#}(Y^*)^N) \to \mathbb{R},$$
$$(v, w) \mapsto (\nu A_0(t)\Psi_0^{-\top}(t)\nabla_y v, \nabla_y w)_{\Omega\times Y^*}, \tag{47}$$
$$b_t : L^2(\Omega; H^1_{\Gamma\#}(Y^*)^N) \times (H^1_0(\Omega) \times L^2(\Omega; L^2_0(Y^*))) \to \mathbb{R},$$
$$(v, (p_0, p_1)) \mapsto (A_0^\top(t)\nabla_x p_0, v)_{\Omega\times Y^*} - (p_1, \operatorname{div}_y(A_0(t)v))_{\Omega\times Y^*}. \tag{48}$$

Using the time-independent boundedness of the transformation $\psi_0(t)$, the boundedness of $J_0 \geq c_J$ from below and the Poincaré inequality for $H^1_{\Gamma\#}(Y^*)$, we obtain

$$\begin{aligned}
a_t(v, v) &= (\nu J_0(t)\Psi_0^{-\top}(t)\nabla_y v, \Psi_0^{-\top}(t)\nabla_y v)_{\Omega\times Y^*} \geq \nu c_J \left\|\Psi_0^{-\top}(t)\nabla_y v\right\|^2_{L^2(\Omega; H^1_{\Gamma\#}(Y^*))} \\
&\geq \nu c_J \left\|\Psi_0^\top\right\|^{-2}_{L^\infty(S\times\Omega\times Y^*)} \left\|\nabla_y v\right\|^2_{\Omega\times Y^*} \geq C\,\|v\|^2_{L^2(\Omega; H^1_{\Gamma\#}(Y^*))}, \\
|a_t(v, w)| &\leq \left\|\sqrt{J_0}\Psi_0^{-\top}\right\|^2_{L^\infty(S\times\Omega\times Y^*)} \left\|\nabla_y v\right\|_{\Omega\times Y^*} \left\|\nabla_y w\right\|_{\Omega\times Y^*} \\
&\leq C\,\|w\|_{L^2(\Omega; H^1_{\Gamma\#}(Y^*))}\,\|v\|_{L^2(\Omega; H^1_{\Gamma\#}(Y^*))}
\end{aligned}$$

for any $v, w \in L^2(\Omega; H^1_{\Gamma\#}(Y^*)^N)$ for a time-independent constant C.

Let $(v, (p_0, p_1)) \in L^2(\Omega; H^1_{\Gamma\#}(Y^*)^N) \times (H^1_0(\Omega) \times L^2(\Omega; L^2_{0\#}(Y^*)))$. Using the product rule, the Poincaré inequalities for $H^1_0(\Omega)$ and $H^1_{\Gamma\#}(Y^*)$ as well as the Piola identity, we get

$$\begin{aligned}
|b_t(v, (p_0, p_1))| &\leq |(A_0^\top(t)\nabla_x p_0, v)_{\Omega\times Y^*}| + |(p_1, \operatorname{div}_y(A_0^\top(t)v))_{\Omega\times Y^*}| \\
&= |(A_0^\top(t)\nabla_x p_0, v)_{\Omega\times Y^*}| + |(p_1, A_0^\top(t) : \nabla v))_{\Omega\times Y^*}| \\
&\leq C\,\|\nabla p_0\|_\Omega\,\|v\|_{\Omega\times Y^*} + C\,\|p_1\|_{\Omega\times Y^*}\left\|\nabla_y v\right\|_{\Omega\times Y^*} \\
&\leq C(\|p_0\|_{H^1_0(\Omega)} + \|p_1\|_{\Omega\times Y^*})\,\|v\|_{L^2(\Omega; H^1_{\Gamma\#}(Y^*))}
\end{aligned}$$

for a time-independent constant C.

From Lemma 4.10, we get a time-independent inf–sup constant for b_t.

Since the right-hand sides of (30) can be bounded pointwise for a.e. $t \in S$ by $C \in L^{p_s}(S)$, we can infer for a.e. $t \in S$ with Lemma 3.2 the existence and uniqueness of a solution $(\hat{w}(t), \hat{q}(t), \hat{q}_1(t)) \in L^2(\Omega; H^1_{\Gamma\#}(Y^*)^N) \times H^1_0(\Omega) \times L^2(\Omega; L^2_0(Y^*))$ such that

$$\left\|\hat{w}(t)\right\|_{L^2(\Omega; H^1_{\Gamma\#}(Y^*))} + \left\|\hat{q}(t)\right\|_{H^1(\Omega)} + \left\|\hat{q}_1(t)\right\|_{L^2(\Omega; L^2_{0\#}(Y^*))} \leq C(t) \tag{49}$$

for $C \in L^p(S)$.

Using the same argumentation as in the proof of Theorem 3.1, we obtain additionally the measurability of $(\hat{w}_0, \hat{q}, \hat{q}_1)$ with respect to time. □

5. The limit problem in the evolving domain

5.1. Back-transformation of the limit problem

We transform the two-pressure Stokes problem (30) from the cylindrical substitute domain into the actual two-scale domain. The result is the two-pressure Stokes problem (53), which does not depend on the transformation ψ_0. The two-scale transformation method (cf. [29]) transforms the two-scale convergence results from the substitute problem (cf. Theorem 4.7) into two-scale convergence results for the untransformed setting (cf. Theorem 5.1).

Moreover, the homogenisation of the Stokes problem yields not only the two-scale convergence for the pressure but a strong convergence for an appropriate extension of it (cf. Theorem 4.8). Using the two-scale transformation method, we can transform the strong convergence of $\hat{Q}_\varepsilon$ back and obtain the strong convergence for the back-transformed extension of the pressure $Q'_\varepsilon = \hat{Q}_\varepsilon \circ \psi_\varepsilon^{-1}$ (cf. Theorem 5.1). Indeed, Q'_ε is some extension of the pressure of the original problem but this extension is not transformation-independent. In particular, if $\psi_\varepsilon(k+\varepsilon Y) \neq k+\varepsilon Y$ for $k \in I_\varepsilon$, it can be easily seen that Q'_ε is not constant on $\psi_\varepsilon(k+\varepsilon Y^{\mathrm{s}}) \neq (k+\varepsilon Y) \cap \Omega^{\mathrm{s}}_\varepsilon(t)$. Nevertheless, it can be used in order to show the strong convergence for the extension of the pressure Q_ε that we have chosen in Theorem 5.4. This extension is the transformation-independent counterpart of the extension of Lemma 4.8 for the untransformed setting.

In the last step, we separate the y-dependency in the two-pressure Stokes problem (53) and derive the Darcy law for evolving microstructure (4).

Theorem 5.1. *Let $(w_\varepsilon, q_\varepsilon) \in L^{p_s}(S; H^1_{\Gamma_\varepsilon(t)}(\Omega_\varepsilon(t))^N) \times L^{p_s}(S; L^2(\Omega_\varepsilon(t))$ be the solution of (5) and let $\widetilde{w_\varepsilon} \in L^{p_s}(S; H^1(\Omega)^N)$ be the extensions by 0 and $Q'_\varepsilon \in L^{p_s}(S; L^2(\Omega))$ the extension of $q_\varepsilon(t)$ defined by $Q'_\varepsilon(t,x) := \hat{Q}_\varepsilon(t, \psi_\varepsilon(t,x))$, where $\hat{Q}_\varepsilon$ is given by (32). Then, for a.e. $t \in S$,*

$$\widetilde{w_\varepsilon}(t) \to \widetilde{w_0}(t) \qquad \text{two-scale converge weakly}, \tag{50}$$

$$\varepsilon\widetilde{\nabla w_\varepsilon}(t) \to \nabla_y \widetilde{w_0}(t) \qquad \text{two-scale converge weakly}, \tag{51}$$

$$Q'_\varepsilon(t) \to q(t) \qquad \text{converges strongly in } L^2(\Omega), \tag{52}$$

where $\widetilde{w_0}$ is the extension of w_0 by 0 on $S \times \Omega \times Y$ and (w_0, q, q_1) the solution of (53).

The transformation-independent two-pressure Stokes problem in the actual two-scale domain is given by: Find $(w_0, q, q_1) \in L^{p_s}(S; L^2(\Omega; H^1_{\Gamma(t,x)}(Y^*_x(t))^N)) \times L^{p_s}(S; H^1_0(\Omega)) \times L^{p_s}(S; L^2(\Omega; L^2_0(Y^*_x(t))))$ such that

$$\begin{aligned}&\int_\Omega \int_{Y^*_x(t)} \nu \nabla_y w_0(t,x,y) : \nabla_y \varphi(x,y) + \nabla_x q(t,x) \cdot \varphi(x,y) \,\mathrm{d}y\,\mathrm{d}x \\ &- \int_\Omega \int_{Y^*_x(t)} q_1(t,x,y) \operatorname{div}(\varphi(x,y)) \,\mathrm{d}y\,\mathrm{d}x = \int_\Omega \int_{Y^*_x(t)} (f(t,x) - \nabla_x p_b(t,x)) \cdot \varphi(x,y) \,\mathrm{d}y\,\mathrm{d}x,\end{aligned} \tag{53a}$$

$$\int_\Omega \int_{Y^*_x(t)} \operatorname{div}_y(w_0(t,x,y))\, \phi_1(x,y) \,\mathrm{d}y\,\mathrm{d}x = 0, \tag{53b}$$

$$\int_\Omega \operatorname{div}_x \Big(\int_{Y_x^*(t)} w_0(t,x,y)\,\mathrm{d}y \Big) \phi_0(x)\,\mathrm{d}x = -\int_\Omega \int_{Y_x^*(t)} \operatorname{div}_y(v_\Gamma(t,x,y))\,\mathrm{d}y\,\phi_0(x)\,\mathrm{d}x \tag{53c}$$

for every $(\varphi, \phi_0, \phi_1) \in L^2(\Omega; H^1_{\Gamma(t,x)\#}(Y_x^*(t))^N) \times H^1_0(\Omega) \times L^2(\Omega; L^2_0(Y_x^*(t)))$.

Proof. Let $t \in S$ be fixed and let $\widetilde{w_\varepsilon}$, $\widetilde{Q'_\varepsilon}$ be defined as in Theorem 5.1. Then, we obtain from Lemma 2.4 that $\widetilde{w_\varepsilon}(t, \psi_\varepsilon(t,x)) = \hat{w}_\varepsilon(t,x)$ for a.e. $x \in \Omega_\varepsilon$. The two-scale transformation rule (cf. Lemma 4.4) shows that $\widetilde{w_\varepsilon}(t)$ and $\widetilde{\nabla w_\varepsilon}(t)$ two-scale converge to $\widetilde{w_0}(t)$ with $\widetilde{w_0}(t,x,y) = \hat{w}_0(t,x,\psi_0^{-1}(t,x,y))$ and $\nabla_y \widetilde{w_0}(t,x,y)$, respectively, where $\hat{w}_0$ is the two-scale limit of $\widetilde{\hat{w}_\varepsilon}$ given by Theorem 4.7. Moreover, the strong convergence of $\hat{Q}_\varepsilon(t)$ to $\hat{q}(t)$ in $L^2(\Omega)$ gives the strong two-scale convergence of $\hat{Q}_\varepsilon(t)$. Lemma 4.4 transforms this into the strong two-scale convergence of $Q'_\varepsilon(t)$ to $\hat{q}(t) = q(t) \in L^2(\Omega)$ and, since $q(t)$ is independent of y, it implies the strong convergence in $L^2(\Omega)$.

It remains to derive the transformation-independent limit problem (53) in its actual coordinates. We test (30a) with $\hat{\varphi}(x,y) = \varphi(x, \psi_0(t,x,y))$ and transform the Y^*-integral with $\psi_0^{-1}(t,x,\cdot)$. Then, we obtain

$$\begin{aligned} &\int_\Omega \int_{Y_x^*(t)} \nu \nabla_y w_0(t,x,y) : \nabla_y \varphi(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &\quad + \int_\Omega \int_{Y_x^*(t)} \Psi_0^\top(t,x,\psi_0^{-1}(t,x,y))(\nabla_x q(t,x) + \nabla_x p_b(t,x)) \cdot \varphi(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &- \int_\Omega \int_{Y_x^*(t)} \hat{q}_1(t,x,\psi_0^{-1}(t,x,y)) \operatorname{div}(\varphi(x,y))\,\mathrm{d}y\,\mathrm{d}x = \int_\Omega \int_{Y_x^*(t)} f(t,x) \cdot \varphi(x,y)\,\mathrm{d}y\,\mathrm{d}x. \end{aligned} \tag{54}$$

Note that the transformation coefficients vanish in front of the y-derivatives because of the product rule. In order to remove them in front of the x-gradient, we note

$$\Psi_0^{-\top}(t,x,\psi_0^{-1}(t,x,y)) = \nabla_y \psi_0^{-1}(t,x,y) = \mathbb{1} + \nabla_y \check{\psi}_0^{-1}(t,x,y). \tag{55}$$

Thus, we can rewrite the macroscopic pressure terms and obtain after integration by parts

$$\begin{aligned} &\int_\Omega \int_{Y_x^*(t)} \Psi_0^{-\top}(t,x,\psi_0^{-1}(t,x,y))(\nabla_x q(t,x) + \nabla_x p_b(t,x)) \cdot \varphi(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_\Omega \int_{Y_x^*(t)} (\nabla_x q(t,x) + \nabla_x p_b(t,x)) \cdot \varphi(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &\quad + \int_\Omega \int_{Y_x^*(t)} \nabla_y \check{\psi}_0^{-1}(t,x,y)(\nabla_x q(t,x) + \nabla_x p_b(t,x)) \cdot \varphi(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_\Omega \int_{Y_x^*(t)} (\nabla_x q(t,x) + \nabla_x p_b(t,x)) \cdot \varphi(x,y)\,\mathrm{d}y\,\mathrm{d}x \end{aligned} \tag{56}$$

$$-\int_{\Omega}\int_{Y_x^*(t)} \check{\psi}_0^{-1}(t,x,y)(\nabla_x q(t,x) + \nabla_x p_b(t,x))\,\mathrm{div}_y(\varphi(x,y))\,\mathrm{d}y\,\mathrm{d}x.$$

The boundary integral, which arises in the integration by parts in (56), vanishes on $\partial Y^* \cap \partial Y$ because all the terms are Y-periodic and on $\partial Y^* \setminus \partial Y$ because $\varphi = 0$ there. As the last term of (56) has only a microscopic contribution, we can add it to the microscopic pressure. We define

$$q_1(t,x,y) := \hat{q}_1(t,x,\psi_0^{-1}(t,x,y)) + \check{\psi}_0^{-1}(t,x,y)\cdot(\nabla_x q(t,x) + \nabla_x p_b(t,x))$$

so that the pressure terms of (54) transform to the pressure terms in (53a).

By a similar transformation of (30a) and (30b), we get

$$\int_{\Omega}\int_{Y_x^*(t)} \mathrm{div}_y(w_0(t,x,y))\phi_1(x,y)\,\mathrm{d}y\,\mathrm{d}x = 0, \tag{57}$$

which shows (53b) and

$$\begin{aligned}&\int_{\Omega}\mathrm{div}_x\Big(\int_{Y_x^*(t)} \Psi_0^{-\top}(t,x,\psi_0^{-1}(t,x,y))w_0(t,x,y)\,\mathrm{d}y\Big)\phi_0(x)\,\mathrm{d}x\\ &\quad= -\int_{\Omega}\int_{Y_x^*(t)} \mathrm{div}_y(v_\Gamma(t,x,y))\,\mathrm{d}y\;\phi_0(x)\,\mathrm{d}x.\end{aligned} \tag{58}$$

Using again (55) and integration by parts, we can rewrite (58)

$$\begin{aligned}&-\int_{\Omega}\mathrm{div}_x\Big(\int_{Y_x^*(t)} \Psi_0^{-\top}(t,x,\psi_0^{-1}(t,x,y))w_0(t,x,y)\,\mathrm{d}y\Big)\varphi(x)\,\mathrm{d}x\\ &= -\int_{\Omega}\mathrm{div}_x\Big(\int_{\Omega}\int_{Y_x^*(t)} w_0(t,x,y) + \check{\psi}_0^{-1}(t,x,y)\,\mathrm{div}_y(w_0(t,x,y))\,\mathrm{d}y\Big)\varphi(x)\,\mathrm{d}x.\end{aligned} \tag{59}$$

The second summand on the right-hand side of (59) vanishes because of the microscopic incompressibility condition (57). Thus, we have rewritten the left-hand side of (58) into the left-hand side of (53c). □

For the case of a no-slip boundary condition at the interface Γ_ε, in which $v_{\Gamma_\varepsilon}(t,x) = \psi_\varepsilon(t,\psi_\varepsilon^{-1}(t,x))$ models the boundary deformation, we can simplify the right-hand side of the macroscopic inhomogeneous divergence condition (53c) in the two-pressure Stokes system.

Corollary 5.2. *If v_{Γ_ε} is the velocity of the boundary deformation, i.e. $v_{\Gamma_\varepsilon}(t,x) = \partial_t\psi_\varepsilon(t, \psi_\varepsilon^{-1}(t,x,y))$, the right-hand side of (30c), and equivalently the right-hand side of (53c), can be rewritten as*

$$-\int_{\Omega}\int_{Y^*} \mathrm{div}_y(A_0(t)\hat{v}_\Gamma(t,x,y))\,\mathrm{d}y\,\varphi_0(x)\,\mathrm{d}x = -\int_{\Omega}\partial_t|Y_x^*(t)|\,\varphi_0(x)\,\mathrm{d}x \tag{60}$$

Proof. First, we note that $v_{\Gamma_\varepsilon}(t,x) = \partial_t \psi_\varepsilon(t, \psi_\varepsilon^{-1}(t,x))$ yields $\hat{v}_{\Gamma_\varepsilon} = \partial_t \psi_\varepsilon$, which implies $\hat{v}_\Gamma = \partial_t \psi_0$. Thus, we can rewrite the right-hand side of (53c) by

$$-\int_\Omega \int_{Y^*} \operatorname{div}_y(A_0(t)\hat{v}_\Gamma(t,x,y))\,\mathrm{d}y\,\varphi_0(x)\,\mathrm{d}x = -\int_\Omega \int_{Y^*} \operatorname{div}_y(A_0(t)\partial_t\psi_0(t,x,y))\,\mathrm{d}y\,\varphi_0(x)\,\mathrm{d}x.$$

The Piola identity implies $\operatorname{div}_y(J_0\Psi_0^{-1}\partial_t\psi_0) = \partial_t J_0$, which gives

$$\begin{aligned}-\int_\Omega \int_{Y^*} \operatorname{div}_y(A_0(t)\partial_t\psi_0(t,x,y))\,\mathrm{d}y\,\varphi_0(x)\,\mathrm{d}x &= -\int_\Omega \int_{Y^*} \partial_t J_0(t,x,y)\,\mathrm{d}y\,\varphi_0(x)\,\mathrm{d}x \\ &= -\int_\Omega \partial_t \int_{Y_x^*(t)} \mathrm{d}y\,\varphi_0(x)\,\mathrm{d}x = -\int_\Omega \partial_t |Y_x^*(t)|\,\varphi_0(x)\,\mathrm{d}x. \quad \square\end{aligned}$$

In the next step, we consider the limit $\varepsilon \to 0$ of the actual fluid velocity v_ε, i.e. we add the Dirichlet boundary values to w_ε. We extend v_ε on Ω by 0, which is not regularity preserving but conforms with the physical model that no fluid flow happens in the solid phase.

Corollary 5.3. *Let $v_\varepsilon := w_\varepsilon - v_{\Gamma_\varepsilon} \in L^{p_s}(S; H^1_{\Gamma_\varepsilon(t)}(\Omega_\varepsilon(t))^N)$, where w_ε is the solution of (5). Let $\widetilde{v_\varepsilon}$ and $\widetilde{\nabla v_\varepsilon}$ be the extension by zero on $\Omega \times Y$. Then, $\widetilde{v_\varepsilon}(t)$ and $\varepsilon\widetilde{\nabla v_\varepsilon}(t)$ two-scale converge to the extension by 0 of $w_0(t)$ and $\nabla_y w_0(t)$, respectively, where w_0 is the solution of (53).*

Proof. We note that $\widetilde{v_\varepsilon}(t) - \widetilde{w_\varepsilon}(t) = \chi_{\Omega_\varepsilon(t)} v_{\Gamma_\varepsilon}$ and $\widetilde{\nabla v_\varepsilon}(t) - \widetilde{\nabla w_\varepsilon}(t) = \chi_{\Omega_\varepsilon(t)} \nabla v_{\Gamma_\varepsilon}$. Since $\left\| v_{\Gamma_\varepsilon}(t) \right\|_\Omega + \varepsilon \left\| \nabla v_{\Gamma_\varepsilon}(t) \right\|_\Omega \leq \varepsilon C(t)$ for a.e. $t \in S$ for $C \in L^{p_s}(S)$, we can identify the two-scale limit of $\widetilde{v_\varepsilon}$ and $\varepsilon\widetilde{\nabla v_\varepsilon}$ with the two-scale limits of $\widetilde{w_\varepsilon}$ and $\varepsilon\widetilde{\nabla w_\varepsilon}$, respectively, which are given by Theorem 5.1 gives the desired two-scale convergence. $\square$

Theorem 5.4. *Assume that $|k + \varepsilon Y \cap \Omega_\varepsilon(t)| \geq c$ for every $\varepsilon > 0$ and $k \in I_\varepsilon$ with a time- and space-independent constant $c > 0$. Let*

$$Q_\varepsilon(t,x) := \begin{cases} q_\varepsilon(t,x) & \text{if } x \in \Omega_\varepsilon(t), \\ \frac{1}{|k+\varepsilon Y \cap \Omega_\varepsilon(t)|} \int\limits_{k+\varepsilon Y \cap \Omega_\varepsilon(t)} q_\varepsilon(t,z)\,\mathrm{d}z & \text{if } x \in k + \varepsilon Y \cap \Omega_\varepsilon^s(t) \text{ for } k \in I_\varepsilon, \end{cases} \tag{61}$$

where q_ε is the second part of the solution of (5). Then, Theorem 5.1 holds for $Q_\varepsilon(t)$ instead of $Q'_\varepsilon(t)$, i.e. for a.e. $t \in S$, $Q_\varepsilon(t)$ converges strongly in $L^2(\Omega)$ to $q(t)$, where $q \in L^{p_s}(S; H^1(\Omega))$ is the second part of the solution of (53).

Proof. In the following, we use the unfolding operator $\mathcal{T}_\varepsilon : L^p(\Omega) \to L^p(\Omega \times Y)$, which was introduced in [49,50], see also [51]. We use the notation $[x]_Y = \sum_{i=1}^N e_i \lfloor x_i \rfloor$ for $x \in \mathbb{R}^N$ and e_i the Euclidean unit vectors. It allows us to translate between strong two-scale convergence and strong convergence in $L^p(\Omega \times Y)$ (cf. [29]). Thus, $Q_\varepsilon(t)$ two-scale converges strongly to $q(t)$ if and only if $\mathcal{T}_\varepsilon(Q_\varepsilon(t))$ converges strongly in $L^2(\Omega \times Y)$ to $q(t)$. Let $\widetilde{q}_\varepsilon(t)$ be the extension of $q_\varepsilon(t)$ by 0 to Ω. With the definition of $\mathcal{T}_\varepsilon$, we can rewrite

$$\mathcal{T}_\varepsilon(Q_\varepsilon(t))(x,y) = \begin{cases} q_\varepsilon(t, \varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y) & \text{if } \varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y \in \Omega_\varepsilon(t), \\ \frac{1}{|\Omega_\varepsilon(t) \cap (\varepsilon[\frac{x}{\varepsilon}]_Y + \varepsilon Y)|} \int\limits_{\varepsilon[\frac{x}{\varepsilon}]_Y + \varepsilon Y} \widetilde{q}_\varepsilon(t,z)\,\mathrm{d}z & \text{if } \varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y \in \Omega^{\mathrm{s}}_\varepsilon(t) \end{cases}$$

$$= \mathcal{T}_\varepsilon(\widetilde{q}_\varepsilon(t))(x,y) + \mathcal{T}_\varepsilon(\chi_{\Omega^{\mathrm{s}}_\varepsilon(t)})(x,y) \frac{1}{\left|\Omega_\varepsilon(t) \cap \left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon Y\right)\right|} \int\limits_Y \mathcal{T}_\varepsilon(\widetilde{q}_\varepsilon(t))(x,z)\,\mathrm{d}z$$

$$= \mathcal{T}_\varepsilon(\widetilde{q}_\varepsilon(t))(x,y) + \mathcal{T}_\varepsilon(\chi_{\Omega^{\mathrm{s}}_\varepsilon(t)})(x,y) \frac{1}{\int\limits_Y \mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon(t)})(x,z)\,\mathrm{d}z} \int\limits_Y \mathcal{T}_\varepsilon(\widetilde{q}_\varepsilon(t))(x,z)\,\mathrm{d}z. \tag{62}$$

In order to pass to the limit $\varepsilon \to 0$, we rewrite $\mathcal{T}_\varepsilon(\widetilde{q}_\varepsilon(t)) = \mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon(t)})\mathcal{T}_\varepsilon(Q'_\varepsilon(t))$. We translate the strong two-scale converges of $Q'_\varepsilon(t)$ and $\chi_{\Omega_\varepsilon(t)}$ in

$$\mathcal{T}_\varepsilon(Q'_\varepsilon(t)) \to q(t) \qquad \text{in } L^2(\Omega \times Y), \tag{63}$$

$$\mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon(t)}) \to \chi_{Y^*_x(t)} \qquad \text{in } L^p(\Omega \times Y) \text{ for every } p \in (1,\infty), \tag{64}$$

Due to the fact that $\left\|\mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon(t)})\right\|_{L^\infty(\Omega\times Y)}$ is uniformly bounded, we can combine (63)–(64) to

$$\mathcal{T}_\varepsilon(\widetilde{q}_\varepsilon(t)) = \mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon(t)})\mathcal{T}_\varepsilon(Q'_\varepsilon(t)) \to \chi_{Y^*_x(t)} q(t) \text{ in } L^2(\Omega \times Y). \tag{65}$$

Using the Hölder inequality on (64) and (65) with respect to Y yields

$$\int\limits_Y \mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon(t)})(x,y)\,\mathrm{d}y \to \int\limits_Y \chi_{Y^*_x(t)}(y)(x,y)\,\mathrm{d}y \qquad \text{in } L^p(\Omega) \text{ for every } p \in (1,\infty), \tag{66}$$

$$\int\limits_Y \mathcal{T}_\varepsilon(\widetilde{q}_\varepsilon(t))(x,y)\,\mathrm{d}y \to \int\limits_Y \chi_{Y^*_x(t)}(y)\,\mathrm{d}y\, q(t) \qquad \text{in } L^2(\Omega). \tag{67}$$

Since $\int\limits_Y \mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon(t)})(\cdot_x, z)\,\mathrm{d}z \geq c > 0$ is uniformly bounded from below, (66) yields together with (67)

$$\frac{\int\limits_Y \mathcal{T}_\varepsilon(\widetilde{q}_\varepsilon(t))(x,z)\,\mathrm{d}z}{\int\limits_Y \mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon(t)})(x,z)\,\mathrm{d}z} \to \frac{\int\limits_Y \chi_{Y^*_x(t)}(z)\,\mathrm{d}z q(t)}{\int\limits_Y \chi_{Y^*_x(t)}(z)\,\mathrm{d}z} = q(t) \text{ in } L^2(\Omega). \tag{68}$$

Moreover, (64) gives $\mathcal{T}_\varepsilon(\chi_{\Omega^{\mathrm{s}}_\varepsilon(t)}) \to \chi_{Y^{\mathrm{s}}_x(t)}$ in $L^p(\Omega \times Y)$ for every $p \in (1,\infty)$. Due to fact that $\left\|\mathcal{T}_\varepsilon(\chi_{\Omega^{\mathrm{s}}_\varepsilon(t)})\right\|_{L^\infty(\Omega\times Y)}$ is uniformly bounded, we obtain with (68)

$$\mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon^s(t)}) \frac{\int_Y \mathcal{T}_\varepsilon(\widetilde{q}_\varepsilon(t))(\cdot_x, z)\,\mathrm{d}z}{\int_Y \mathcal{T}_\varepsilon(\chi_{\Omega_\varepsilon(t)})(\cdot_x, z)\,\mathrm{d}z} \to \chi_{Y_x^s(t)} q(t) \text{ in } L^2(\Omega \times Y). \tag{69}$$

By combining (62), (65) and (69), we can pass to the limit $\varepsilon \to 0$ for $\mathcal{T}_\varepsilon(Q_\varepsilon(t))$ and obtain:

$$\mathcal{T}_\varepsilon(Q_\varepsilon(t)) \to \chi_{Y_x^*(t)} q(t) + \chi_{Y_x^s(t)} q(t) = q(t) \text{ in } L^2(\Omega \times Y). \tag{70}$$

Therefore, $Q_\varepsilon(t)$ two-scale converges strongly to $q(t)$ and, since $q(t)$ is independent of y, $Q_\varepsilon(t)$ converges strongly in $L^2(\Omega)$. □

5.2. *The Darcy law for evolving microstructure*

In the last step, we derive the Darcy law (4) by separating the y-dependence in (53). It contains the time- and space-dependent permeability tensor $K \in L^\infty(S \times \Omega)^{N \times N}$, which can be computed explicitly by

$$K(t,x)_{ij} = \int_{Y_x^*(t)} \nabla_y u_i(t,x,y) : \nabla_y u_j(t,x,y)\,\mathrm{d}y = \int_{Y_x^*(t)} u_i(t,x,y) \cdot e_j\,\mathrm{d}y, \tag{71}$$

where $u_i \in L^\infty(S \times \Omega; H^1_{\Gamma\#}(Y_x^*(t))^N)$ are the unique solution of the local Stokes problems on the cell domains $Y_x^*(t)$,

$$-\Delta_y u_i(t,x,y) - \nabla_y \pi_i(t,x,y) = e_i \quad \text{in } Y_x^*(t), \tag{72a}$$

$$\operatorname{div}_y(u_i(t,x,y)) = 0 \quad \text{in } Y_x^*(t), \tag{72b}$$

$$u_i(t,x,y) = 0 \quad \text{on } \partial\Gamma_x(t), \tag{72c}$$

$$y \mapsto \pi(t,x,y), u_i(t,x,y) \quad \text{is } Y\text{-periodic}. \tag{72d}$$

The corresponding weak formulation of (4) consists of the following Dirichlet boundary-value problem (73) for the pressure and the explicit equation for the fluid velocity (74), where $p = q + p_b$: Find $q \in L^{p_s}(S; H_0^1(\Omega))$ such that, for a.e. $t \in S$,

$$\int_\Omega \frac{1}{\nu} K(t,x) \nabla q(t,x) \cdot \nabla \varphi(x)\,\mathrm{d}x = \int_\Omega \frac{1}{\nu} K(t,x)(f(t,x) - \nabla p_b(t,x)) \cdot \nabla \varphi(x)\,\mathrm{d}x \\ - \int_\Omega \int_{Y_x^*(t)} \operatorname{div}_y(v_\Gamma(t,x))\,\mathrm{d}y \varphi(x)\,\mathrm{d}y\,\mathrm{d}x \tag{73}$$

for every $\varphi \in H_0^1(\Omega)$, where $K \in L^\infty(S \times \Omega)^{N \times N}$ is defined by (71) and let

$$v(t) = \frac{1}{\nu} K(t) \left(f(t) - \nabla p(t)\right), \tag{74}$$

where $p = q + p_b$.

In the case of a no-slip boundary conditions, i.e. $\hat{v}_\Gamma = \partial_t \psi_\varepsilon$, Corollary 5.2 simplifies the last term of (73) to

$$-\int_\Omega \int_{Y_x^*(t)} \operatorname{div}_y(v_\Gamma(t,x,y))\,\mathrm{d}y\,\varphi(x)\,\mathrm{d}y = -\int_\Omega \partial_t |Y_x^*(t)|\,\varphi(x)\,\mathrm{d}x.$$

Theorem 5.5. *Let $(\widetilde{v_\varepsilon}, Q_\varepsilon)$ be defined by Corollary 5.3 and Theorem 5.4, respectively. Then, for a.e. $t \in S$, $Q_\varepsilon(t)$ converges strongly in $L^2(\Omega)$ to $q(t)$, where $q \in L^{p_s}(S; H_0^1(\Omega))$ is the unique solution of (73). Moreover, $\widetilde{v_\varepsilon}(t)$ converges weakly in $L^2(\Omega)$ to $v(t)$, where $v \in L^{p_s}(S; L^2(\Omega)^N)$ is given by (74).*

Proof. The linearity of (53a) gives

$$w_0(t,x,y) = \frac{1}{\nu}\sum_{i=1}^{N}(f_i(t,x) - \partial_{x_i}(q(t,x) + p_b(t,x)))u_i(t,x,y), \tag{75}$$

$$q_1(t,x,y) = \frac{1}{\nu}\sum_{i=1}^{N}(\partial_{x_i}(q(t,x) + p_b(t,x)) - f_i(t,x))\pi_i(t,x,y), \tag{76}$$

where (u_i, π_i) is the solution of (72) for $i = \{1, \dots, N\}$. Taking the integral over $Y_x^*(t)$ gives (74) for $v(t,x) := \int_{Y_x^*(t)} w_0(t,x,y)\,\mathrm{d}y$. Moreover, with (53c), we obtain the inhomogeneous divergence condition $\operatorname{div}(v) = -\int_{Y_x^*(t)} \operatorname{div}(v_\Gamma(t,x,y))\,\mathrm{d}y$ and in the case of Corollary 5.2, we can simplify it to $\operatorname{div}(v) = -\partial_t|Y_x^*(t)|$. Combining this inhomogeneous divergence condition with (74) yields (73). □

By stating separately the inhomogeneous divergence condition $\operatorname{div}(v) = -\int_{Y_x^*(t)} \operatorname{div}(v_\Gamma(t,x,y))\,\mathrm{d}y$ $(= -\partial_t|Y_x^*(t)|)$, which we have derived in the proof of Theorem 5.5, we obtain the strong formulation of the Darcy law for evolving microstructure (4). It differs in three points from the Darcy law for fixed microstructure (2). The first is the time- and space-dependent permeability tensor, which arises from the time- and space-dependent (evolving) microstructure. The second and most interesting difference is the macroscopic inhomogeneous divergence condition, which arises from the homogenisation of the inhomogeneous Dirichlet boundary condition. The last difference is the Dirichlet boundary condition in (4) which is caused from the homogenisation of the pressure boundary condition.

Remark 2. Instead of doing the homogenisation for a.e. $t \in S$ separately, the two-scale convergence with respect to the $L^{p_s}(S; L^2(\Omega))$-norm for $1 < p_s < \infty$ could have been used. In this case, the data do not have to be bounded pointwise in time. Instead, it suffices if they are bounded in $L^{p_s}(S)$.

6. Conclusion

We derived a Darcy law for porous media with evolving microstructure by means of homogenisation. The microscopic evolution is a priori known and features a time- and space-dependent permeability tensor, which accounts for the local pore structure. Moreover, we considered an inhomogeneous Dirichlet boundary condition at the fluid–solid interface, modelling a no-slip boundary condition for moving interfaces or general compression and suction effects caused by the moving interface. Combined with the microscopic incompressibility condition, this inhomogeneous boundary condition becomes a macroscopic inhomogeneous divergence condition for the fluid in the homogenisation process. Thus, a new source term for the pressure occurs. In the case of the inhomogeneous Dirichlet boundary condition modelling a no-slip boundary condition, this source term relates to the change of the local pore volume.

In order to perform the homogenisation on the evolving domain, we applied the two-scale transformation method. The homogenisation in the periodic substitute domain required the derivation of a new Korn-type inequality for the two-scale transformation method. Moreover, we showed new two-scale transformation rules for the divergence operator and thus obtained a transformation-independent limit problem.

In various applications, the evolution of the microstructure is a priori not known and is affected by different processes as for example by dissolution or precipitation processes. This results in highly non-linear systems of reaction–advection–diffusion processes coupled with Stokes flow and evolving microstructure. The homogenisation of reaction–diffusion processes coupled with an a priori unknown evolving microstructure is considered in [31,30]. The homogenisation of the Stokes flow presented here constitutes a framework for the addition of advective transportation in such models.

Data availability

No data was used for the research described in the article.

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References

[1] H. Darcy, Les fontaines publiques de la ville de Dijon, 1856.
[2] J.B. Keller, Darcy's law for flow in porous media and the two-space method, in: Nonlinear Partial Differential Equations in Engineering and Applied Scienc, Dekker, New York, 1980, pp. 429–443.
[3] J.-L. Lions, Some Methods in the Mathematical Analysis of Systems and Their Control, Science Press, Gordon & Breach Science Publishers, New York, 1981.
[4] E. Sánchez-Palencia, Nonhomogeneous Media and Vibration Theory, Lecture Notes in Physics, vol. 127, Springer-Verlag, Berlin-New York, 1980.
[5] L. Tartar, Incompressible fluid flow in a porous medium - convergence of the homogenization process, Appendix of [4], 1979.
[6] G. Allaire, Homogenization of the Stokes flow in a connected porous medium, Asymptot. Anal. 2 (1989) 203–222.
[7] A. Mikelić, Homogenization of nonstationary Navier-Stokes equations in a domain with a grained boundary, Ann. Mat. Pura Appl. (4) 158 (1991) 167–179, https://doi.org/10.1007/BF01759303.
[8] G. Allaire, Homogenization of the unsteady Stokes equations in porous media, in: C. Bandle (Ed.), Progress in Partial Differential Equations: Calculus of Variations, Applications, 1992.
[9] A. Mikelić, Mathematical derivation of the Darcy-type law with memory effects, governing transient flow through porous media, Glas. Mat. Ser. III 29 (49) (1994) 57–77.
[10] A. Mikelić, Effets inertiels pour un écoulement stationnaire visqueux incompressible dans un milieu poreux, C. R. Math. Acad. Sci. Paris Sér. I Math. 320 (1995) 1289–1294.

[11] G. Allaire, Homogenization of the Navier-Stokes equations in open sets perforated with tiny holes. I. and II, Arch. Rational Mech. Anal. 113 (1990) 209–259, https://doi.org/10.1007/BF00375065, 261–298.
[12] E. Feireisl, Y. Namlyeyeva, v. Nečasová, Homogenization of the evolutionary Navier-Stokes system, Manuscr. Math. 149 (2016) 251–274, https://doi.org/10.1007/s00229-015-0778-y.
[13] T.L. van Noorden, Crystal precipitation and dissolution in a porous medium: effective equations and numerical experiments, Multiscale Model. Simul. 7 (2008) 1220–1236, https://doi.org/10.1137/080722096.
[14] N. Ray, T. van Noorden, F. Frank, P. Knabner, Multiscale modeling of colloid and fluid dynamics in porous media including an evolving microstructure, Transp. Porous Media 95 (2012) 669–696, https://doi.org/10.1007/s11242-012-0068-z.
[15] N. Ray, T. Elbinger, P. Knabner, Upscaling the flow and transport in an evolving porous medium with general interaction potentials, SIAM J. Appl. Math. 75 (2015) 2170–2192, https://doi.org/10.1137/140990292.
[16] C. Bringedal, I. Berre, I.S. Pop, F.A. Radu, Upscaling of non-isothermal reactive porous media flow with changing porosity, Transp. Porous Media 114 (2016) 371–393, https://doi.org/10.1007/s11242-015-0530-9.
[17] M. Redeker, C. Rohde, I.S. Pop, Upscaling of a tri-phase phase-field model for precipitation in porous media, IMA J. Appl. Math. 81 (2016) 898–939, https://doi.org/10.1093/imamat/hxw023.
[18] R. Schulz, P. Knabner, Derivation and analysis of an effective model for biofilm growth in evolving porous media, Math. Methods Appl. Sci. 40 (2017) 2930–2948, https://doi.org/10.1002/mma.4211.
[19] R. Schulz, P. Knabner, An effective model for biofilm growth made by chemotactical bacteria in evolving porous media, SIAM J. Appl. Math. 77 (2017) 1653–1677, https://doi.org/10.1137/16M108817X.
[20] C. Bringedal, L. Von Wolff, I.S. Pop, Phase field modeling of precipitation and dissolution processes in porous media: upscaling and numerical experiments, Multiscale Model. Simul. 18 (2020) 1076–1112, https://doi.org/10.1137/19M1239003.
[21] C. Conca, J.I. Díaz, C. Timofte, Effective chemical processes in porous media, Math. Models Methods Appl. Sci. 13 (2003) 1437–1462, https://doi.org/10.1142/S0218202503002982.
[22] A. Mikelić, M. Primicerio, Modeling and homogenizing a problem of absorption/desorption in porous media, Math. Models Methods Appl. Sci. 16 (2006) 1751–1781, https://doi.org/10.1142/S0218202506001704.
[23] A. Meirmanov, R. Zimin, Compactness result for periodic structures and its application to the homogenization of a diffusion-convection equation, Electron. J. Differ. Equ. 2011 (115) (2011) 1–11.
[24] M.A. Peter, Homogenisation in domains with evolving microstructure, C. R., Méc. 335 (2007) 357–362, https://doi.org/10.1016/j.crme.2007.05.024.
[25] M.A. Peter, Homogenisation of a chemical degradation mechanism inducing an evolving microstructure, C. R., Méc. 335 (11) (2007) 679–684, https://doi.org/10.1016/j.crme.2007.09.003.
[26] M.A. Peter, Coupled reaction-diffusion processes inducing an evolution of the microstructure: analysis and homogenization, Nonlinear Anal. 70 (2009) 806–821, https://doi.org/10.1016/j.na.2008.01.011.
[27] M. Eden, A. Muntean, Homogenization of a fully coupled thermoelasticity problem for a highly heterogeneous medium with a priori known phase transformations, Math. Methods Appl. Sci. 40 (2017) 3955–3972, https://doi.org/10.1002/mma.4276.
[28] M. Gahn, M. Neuss-Radu, I.S. Pop, Homogenization of a reaction-diffusion-advection problem in an evolving micro-domain and including nonlinear boundary conditions, J. Differ. Equ. 289 (2021) 95–127, https://doi.org/10.1016/j.jde.2021.04.013.
[29] D. Wiedemann, The two-scale-transformation method, Asymptot. Anal. 131 (2023) 59–82, https://doi.org/10.3233/ASY-221766.
[30] M. Gahn, I.S. Pop, Homogenization of a mineral dissolution and precipitation model involving free boundaries at the micro scale, J. Differ. Equ. 343 (2023) 90–151, https://doi.org/10.1016/j.jde.2022.10.006.
[31] D. Wiedemann, M.A. Peter, Homogenisation of local colloid evolution induced by reaction and diffusion, Nonlinear Anal. 227 (2023) 32, https://doi.org/10.1016/j.na.2022.113168.
[32] M.A. Biot, Theory of elasticity and consolidation for a porous anisotropic solid, J. Appl. Phys. 26 (1955) 182–185.
[33] M.A. Biot, General solutions of the equations of elasticity and consolidation for a porous material, J. Appl. Mech. 23 (1956) 91–96.
[34] T. Lévy, Propagation of waves in a fluid-saturated porous elastic solid, Int. J. Eng. Sci. 17 (1979) 1005–1014, https://doi.org/10.1016/0020-7225(79)90022-3.
[35] R. Burridge, J.B. Keller, Poroelasticity equations derived from microstructure, J. Acoust. Soc. Am. 70 (1981) 1140–1146, https://doi.org/10.1121/1.386945.
[36] A. Meirmanov, Mathematical Models for Poroelastic Flows, Atlantis Studies in Differential Equations, vol. 1, Atlantis Press, Paris, 2014.
[37] D.L. Brown, P. Popov, Y. Efendiev, Effective equations for fluid-structure interaction with applications to poroelasticity, Appl. Anal. 93 (2014) 771–790, https://doi.org/10.1080/00036811.2013.839780.

[38] J. Collis, D.L. Brown, M.E. Hubbard, R.D. O'Dea, Effective equations governing an active poroelastic medium, Proc. R. Soc. A 473 (2017) 20160755, https://doi.org/10.1098/rspa.2016.0755.
[39] L. Miller, R. Penta, Homogenized balance equations for nonlinear poroelastic composites, Appl. Sci. 11 (2021), https://doi.org/10.3390/app11146611.
[40] J. Fabricius, E. Miroshnikova, P. Wall, Homogenization of the Stokes equation with mixed boundary condition in a porous medium, Cogent Math. 4 (2017) 1327502, https://doi.org/10.1080/23311835.2017.1327502.
[41] V.V. Zhikov, On the homogenization of the system of Stokes equations in a porous medium, Russ. Acad. Sci., Dokl., Math. 49 (1994) 144–147.
[42] D. Boffi, F. Brezzi, M. Fortin, Mixed Finite Element Methods and Applications, Springer Series in Computational Mathematics, vol. 44, Springer, Heidelberg, 2013.
[43] A. Mielke, T. Roubíček, Thermoviscoelasticity in Kelvin-Voigt rheology at large strains, Arch. Ration. Mech. Anal. 238 (2020) 1–45, https://doi.org/10.1007/s00205-020-01537-z.
[44] W. Pompe, Korn's first inequality with variable coefficients and its generalization, Comment. Math. Univ. Carol. 44 (2003) 57–70.
[45] G.P. Galdi, An Introduction to the Mathematical Theory of the Navier-Stokes Equations, Springer, 2011.
[46] G. Allaire, Homogenization and two-scale convergence, SIAM J. Math. Anal. 23 (1992) 1482–1518, https://doi.org/10.1137/0523084.
[47] G. Nguetseng, A general convergence result for a functional related to the theory of homogenization, SIAM J. Math. Anal. 20 (1989) 608–623, https://doi.org/10.1137/0520043.
[48] D. Lukkassen, G. Nguetseng, P. Wall, Two-scale convergence, Int. J. Pure Appl. Math. 2 (2002) 35–86.
[49] D. Cioranescu, A. Damlamian, G. Griso, Periodic unfolding and homogenization, C. R. Math. Acad. Sci. Paris 335 (2002) 99–104, https://doi.org/10.1016/S1631-073X(02)02429-9.
[50] D. Cioranescu, A. Damlamian, G. Griso, The periodic unfolding method in homogenization, SIAM J. Math. Anal. 40 (2008) 1585–1620, https://doi.org/10.1137/080713148.
[51] D. Cioranescu, A. Damlamian, G. Griso, The periodic unfolding method, in: Theory and Applications to Partial Differential Problems, in: Series in Contemporary Mathematics, vol. 3, Springer, Singapore, 2018.