Light scattering in an ensemble of complex particles: a random-matrix approach

K. Ziegler[a,*], L. Kolokolova[b]

[a] *Institut für Physik, Universität Augsburg, Augsburg, Germany*
[b] *Department of Astronomy, University of Florida Gainesville, FL 32611, USA*

1. Introduction

This work consists of a discussion of standard concepts in light scattering like the T-matrix approach. Although there are many reviews on this subject we include this in order to establish our notation. In the main part of the paper we propose a random-matrix approach to describe scattering by an array of randomly shaped particles.

1.1. Maxwell theory

The electromagnetic field is described by the Maxwell equations

$$\nabla \times \mathbf{B} - \frac{\varepsilon}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{j},$$

* Corresponding author. Tel.: +49-821-598-3244; fax: +49-821-598-3262.
E-mail address: ziegler@physik.uni-augsburg.de (K. Ziegler).

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0,$$

where $\mathbf{j}$ is the current density that creates the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$. $\mathbf{j}$ can be split in the contribution from the light source $\mathbf{j}_0$ and the currents in the scattering medium which are related to the electric field by Ohm's law $\mathbf{j}' = \sigma\mathbf{E}$

$$\mathbf{j} = \mathbf{j}_0 + \sigma\mathbf{E}$$

with the conductivity σ of the scattering medium, $\varepsilon \geqslant 1$ is the relative dielectric coefficient, and μ_0 is the permeability of the medium. We consider a monochromatic field for light with frequency ω

$$\mathbf{E}(t) = \mathbf{E}\mathrm{e}^{\mathrm{i}\omega t}, \quad \mathbf{B}(t) = \mathbf{B}\mathrm{e}^{\mathrm{i}\omega t}, \quad \mathbf{j}_0(t) = \mathbf{j}_0\mathrm{e}^{\mathrm{i}\omega t}.$$

Then the electric field satisfies the equation

$$M\mathbf{E} \equiv \nabla^2\mathbf{E} - \nabla(\nabla \cdot \mathbf{E}) + \left(\varepsilon\frac{\omega^2}{c^2} - \mathrm{i}\omega\mu_0\sigma\right)\mathbf{E} = \mu_0\mathrm{i}\omega\mathbf{j}_0 \equiv \mathbf{J}_0 \tag{1}$$

with the Maxwellian M . It should be noticed that the term with $\nabla \cdot \mathbf{E}$ vanishes in the absence of a charge density. In an inhomogeneous space, however, there can be a charge density, corresponding with a spatially varying dielectric coefficient ε in Eq. (1).

The structure of the Maxwellian (1) in vacuum (i.e. $\varepsilon = 1$ and $\sigma = 0$) is represented by an expansion as

$$M_0 = \sum_{j=1}^{3}(\omega^2/c^2 + \nabla^2 - \nabla_j^2)\gamma_{jj} - \nabla_1\nabla_2\gamma_{12} - \nabla_1\nabla_3\gamma_{13} - \nabla_2\nabla_3\gamma_{23} \tag{2}$$

with 3×3 matrices

$$\gamma_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \gamma_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \gamma_{33} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\gamma_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \gamma_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \gamma_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{3}$$

The algebra of these matrices is given for $1 \leqslant i \leqslant j \leqslant 3$ and $1 \leqslant k \leqslant l \leqslant 3$ by the anti-commutator relation

$$\gamma_{ij}\gamma_{kl} + \gamma_{kl}\gamma_{ij} = \delta_{ik}(1 - \delta_{jl})(\gamma_{jl} + \gamma_{lj}) + \delta_{jl}(1 - \delta_{ik})(\gamma_{ik} + \gamma_{ki}) + 2\delta_{ik}\delta_{jl}(\gamma_{ii} + \gamma_{jj}), \tag{4}$$

where $\gamma_{jl} = 0$ for $j > l$. This algebra represents a fundamental structure of the light scattering theory. It allows us to expand not only M but also other quantities in this basis.

For a given current $\mathbf{J}_0$, the source of light in the system under consideration, the electric field can be obtained by the inversion of the Maxwellian

$$\mathbf{E}(\mathbf{r}) = \int M^{-1}(\mathbf{r}, \mathbf{r}')\mathbf{J}_0(\mathbf{r}')\,\mathrm{d}^3\mathbf{r}', \tag{5}$$

where $M^{-1}(\mathbf{r}, \mathbf{r}')$ is a function that gives the Dirac deltafunction for the positions in space (called sites subsequently)

$$MM^{-1}(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r}, \mathbf{r}').$$

In the following we shall use the compact notation of Eq. (5)

$$\mathbf{E} = M^{-1}\mathbf{J}_0$$

which has implicit integrations (convolutions) between operators (depending on two sites) and fields (depending on one site).

A (random) medium is described by a shift of the vacuum Maxwellian due to a space-dependent term

$$\Delta(\mathbf{r}) = \frac{\omega^2}{c^2}[\varepsilon(\mathbf{r}) - 1] - \mathrm{i}\omega\mu_0\sigma(\mathbf{r}) \tag{6}$$

as $M = M_0 + \sum_{\mathbf{r}\in\text{scatterers}}\Delta(\mathbf{r})$. $\Delta(\mathbf{r})$ contains all the information about the spatial distribution of optical properties of the medium that can, for example, consist of a number of individual particles. The treatment of the random fluctuations depends on their distribution. In practice, only spatially uncorrelated fluctuations are tractable by most methods. The common example for distributions is based on the conditions

$$\langle\Delta(\mathbf{r})\rangle = \Delta_0, \quad \langle\Delta(\mathbf{r})\Delta(\mathbf{r}')\rangle = \Delta_0^2 + g\delta(\mathbf{r}, \mathbf{r}'),$$

where $g \geqslant 0$ controls the strength of the random fluctuations of $\Delta(\mathbf{r})$. However, the restriction to randomly independent point scatterers is not realistic. This is often reflected by a singular behavior of the theory that has to be cured by additional regularizations [1]. This problem shall be briefly discussed in Section 4.2 and alternative models for avoiding it shall be proposed.

2. Green's function and the T matrix

Light scattering is conveniently described within a formalism using Green's functions and the T matrix. Here we briefly summarize the main ideas.

The propagation of a light amplitude, induced by a current source $\mathbf{J}_0$ at site $\mathbf{r}_0$

$$\mathbf{J}_0(\mathbf{r}) = \mathbf{J}_0\delta(\mathbf{r} - \mathbf{r}_0)$$

is given by

$$\mathbf{E}(\mathbf{r}) = M^{-1}(\mathbf{r}, \mathbf{r}_0)\mathbf{J}(\mathbf{r}_0) \equiv G(\mathbf{r}, \mathbf{r}_0)\mathbf{J}(\mathbf{r}_0),$$

where $G = M^{-1}$ is the Green's function. From this we can evaluate the intensity of light for an observer at $\mathbf{r}_1$ as

$$I(\mathbf{r}_1) = |\mathbf{E}(\mathbf{r}_1)|^2 = |G(\mathbf{r}_1, \mathbf{r}_0)\mathbf{J}(\mathbf{r}_0)|^2.$$

Similar expressions are available for other components of the Stokes vector, expressed by the components of the electric field amplitude [2].

Solutions for the propagation in a homogeneous medium

$$G_0 = M_0^{-1}$$

are known [3], whereas the Green's functions of the scattering problem

$$G = (M_0 + \Delta)^{-1}$$

have known solutions only in a few cases. An example is a spherical particle [2]. More general situations can be studied by perturbation theory and approximative methods.

The Green's function can be rewritten, using a Dyson equation (see Appendix A), in the form of

$$G = G_0 + G_0(\Delta G \Delta - \Delta)G_0. \tag{7}$$

This expression allows us to separate the propagation of the light from the source $\mathbf{J}_0$ directly to the observer and indirectly via the scattering region to the observer, since the electric field at the site of the observer reads

$$\mathbf{E} = (G_0 + G_0 T G_0)\mathbf{J}_0$$

with $T = \Delta G \Delta - \Delta$. The light going directly from the source to the observer is $\mathbf{E}_0 = G_0\mathbf{J}_0$, whereas the scattered light is given by the T matrix as

$$\mathbf{E}' = G_0 T G_0 \mathbf{J}_0.$$

Thus only the second term on the right-hand side of Eq. (7) is of interest in the scattering process, i.e. $G - G_0$. The T matrix for given sites $\mathbf{r}$ and $\mathbf{r}'$

$$T(\mathbf{r}, \mathbf{r}') = -\Delta(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}') + \Delta(\mathbf{r})G(\mathbf{r}, \mathbf{r}')\Delta(\mathbf{r}') \tag{8}$$

is the relevant quantity for the scattering process. It vanishes if its arguments $\mathbf{r}$ or $\mathbf{r}'$ are outside the scattering region. It should be noticed that the T matrix is often used in a special representation using, for instance, spherical waves as introduced by Waterman [4].

We consider now a situation which is typical for remote sensing where the source of light, the scatterers, and the observer are very far apart of each other (cf. Fig. 1). Then the Green's function G_0 enters the T matrix only through its asymptotic value at large distances $r = |\mathbf{r}|$ [1]

$$G_0(\mathbf{r}, \mathbf{0}) \sim \frac{\mathrm{e}^{\mathrm{i}kr}}{4\pi r^3} \begin{pmatrix} r^2 - x_1^2 & -x_1x_2 & -x_1x_3 \\ -x_2x_1 & r^2 - x_2^2 & -x_2x_3 \\ -x_3x_1 & -x_3x_2 & r^2 - x_3^2 \end{pmatrix} \quad (k = \omega/c), \tag{9}$$

where x_1, x_2, x_3 are the three cartesian components of $\mathbf{r}$. For the scattered light amplitude $\mathbf{E}'$, induced by a local current at $\mathbf{r}_0$, we have the expression

$$\begin{aligned} \mathbf{E}'(\mathbf{r}_1) &= \int\int G_0(\mathbf{r}_1, \mathbf{r})T(\mathbf{r}, \mathbf{r}')G_0(\mathbf{r}', \mathbf{r}_0)\,\mathrm{d}^3\mathbf{r}\,\mathrm{d}^3\mathbf{r}'\mathbf{J}_0 \\ &\sim \frac{\mathrm{e}^{\mathrm{i}k(r_1+r_0)}}{16\pi^2 r_0 r_1} \begin{pmatrix} \sin^2\Theta & -\cos\Theta\sin\Theta & 0 \\ -\cos\Theta\sin\Theta & \cos^2\Theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} \begin{pmatrix} 0 \\ J_2 \\ J_3 \end{pmatrix} \end{aligned} \tag{10}$$

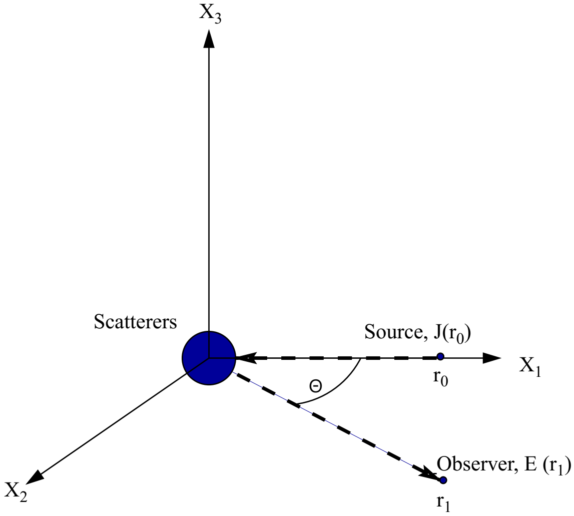

Fig. 1. Geometry of the light scattering. The observer is in the x_1–x_2 plane.

for r_0, r_1 very large in comparison with the size of the scattering region. Θ is the angle between $\mathbf{r}_0$ and $\mathbf{r}_1$ and t_{ij} are elements of the 3×3 matrix

$$\tilde{T}(\mathbf{r}_0/r_0, \mathbf{r}_1/r_1) = \int\int \mathrm{e}^{\mathrm{i}k\mathbf{r}_1\cdot\mathbf{r}/r_1} T(\mathbf{r}, \mathbf{r}')\mathrm{e}^{\mathrm{i}k\mathbf{r}_0\cdot\mathbf{r}\prime/r_0}\, \mathrm{d}^3\mathbf{r}\, \mathrm{d}^3\mathbf{r}'.$$

$\tilde{T}$ is the unitarily transformed T matrix T. More specifically, it is a Fourier transformation from the coordinates $\mathbf{r}$ and $\mathbf{r}'$ to the wave vectors $\mathbf{k} = k\mathbf{r}_1/r_1$ and $\mathbf{k}' = -k\mathbf{r}_0/r_0$, respectively. Therefore, $\tilde{T}$ can also be written in our operator notation as

$$\tilde{T} = U_{\mathbf{k}} T U^{\dagger}_{\mathbf{k}'}, \tag{11}$$

where $U_{\mathbf{k}}$ represents the Fourier transformation with wave vector $\mathbf{k}$. Using unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ (i.e., we use a fixed coordinate system which does not move with the phase angle Θ), and choosing $\mathbf{r}_0$ along the x_1-axis, $\mathbf{r}_1$ in the x_1–x_2 plane

$$\mathbf{r}_0 = r_0\mathbf{e}_1, \quad \mathbf{r}_1 = r_1 \cos\Theta\mathbf{e}_1 + r_1 \sin\Theta\mathbf{e}_2,$$

expression (11) reads

$$\tilde{T}(k, \Theta) = \int\int \mathrm{e}^{\mathrm{i}k(\mathbf{r}\cdot\mathbf{e}_1 \cos\Theta + \mathbf{r}\cdot\mathbf{e}_2 \sin\Theta)} T(\mathbf{r}, \mathbf{r}')\mathrm{e}^{\mathrm{i}k\mathbf{r}\prime\cdot\mathbf{e}_1}\, \mathrm{d}^3\mathbf{r}\, \mathrm{d}^3\mathbf{r}'. \tag{12}$$

The T matrix $\tilde{T}$ appears as a 3×3 matrix. However, the asymptotics in Eq. (10) implies that it is effectively only a 2×2 matrix, since the incident field at the scattering region has only two components perpendicular to $\mathbf{r}_0$ and the scattered field at the observer has only two components perpendicular to $\mathbf{r}_1$.

Using Eq. (10) the elements of matrix $\tilde{T}$ can be used to evaluate the intensity for $J_2 = J_3 \equiv J$

$$|E_3/J|^2 + |E_2/J|^2 + |E_1/J|^2 = |t_{33} + t_{32}|^2 + |(t_{12} + t_{13}) \sin\Theta + (t_{22} + t_{23}) \cos\Theta|^2 \tag{13}$$

and the polarization

$$\frac{|E_3/J|^2 - |E_2/J|^2 - |E_1/J|^2}{|E_3/J|^2 + |E_2/J|^2 + |E_1/J|^2} = \frac{|t_{33} + t_{32}|^2 - |(t_{12} + t_{13}) \sin\Theta + (t_{22} + t_{23}) \cos\Theta|^2}{|t_{33}|^2 + |(t_{12} + t_{13}) \sin\Theta + (t_{22} + t_{23}) \cos\Theta|^2}. \tag{14}$$

One possible strategy for treating T is to expand it in terms of a well-defined set of (orthogonal) eigenfunctions of M_0 [4,5]

$$T(\mathbf{r}, \mathbf{r}') = \sum_{\alpha,\alpha'} t_{\alpha\alpha'} f_\alpha(\mathbf{r}) f_{\alpha'}(\mathbf{r}')$$

with the expansion coefficients

$$t_{\alpha\alpha'} = \int f_\alpha(\mathbf{r}) T(\mathbf{r}, \mathbf{r}') f_{\alpha'}(\mathbf{r}')\, \mathrm{d}^3\mathbf{r}\, \mathrm{d}^3\mathbf{r}'.$$

Depending on the boundary conditions, this can be performed, for instance, with spherical wave functions. In the case of random scatterers, the coefficients of the expansion $t_{\alpha\alpha'}$ are random. To obtain generic properties of physical quantities, e.g. for the intensity or the polarization, we have to average them with respect to a distribution of their fundamental parameters like the refractive index or the particle size. In this paper we propose an alternative approach, where

$$\Delta^{1/2} M_0^{-1} \Delta^{1/2}$$

of the scattering problem is expressed in terms of a random set of scatterers, using special random matrices. This approach is based on the idea that (i) it is impossible to know the exact distribution of the scatterers in a real system and (ii) the generic properties of the physical quantities do not depend on the specific choice of the distribution, as long as some fundamental properties are included. In other words, in a specific measurement of the intensity or polarization of light our data scatter in some range, indicating a statistics which may or may not reveal information about the observed object as well as the instrument. In a first approximation we are only interested in the average values, according to the distributed data. This means that we fit the observed data with an interpolating curve.

3. Discussion of the scattering processes

Starting from T matrix we must separate the scattering region (where $\Delta(\mathbf{r}) \neq 0$) from the vacuum (where $\Delta(\mathbf{r}) = 0$). This can be done by introducing a projector P which projects the three-dimensional space on the areas or points with $\Delta(\mathbf{r}) \neq 0$. With this projector T matrix reads

$$T = -\Delta + \Delta G \Delta = -\Delta + \Delta P (M_0 + \Delta)^{-1} P \Delta. \tag{15}$$

Now for the term $P(M_0 + \Delta)^{-1} P$ we can use the identity [6]

$$P(M_0 + \Delta)^{-1} P = ((P M_0^{-1} P)^{-1} + \Delta)^{-1}. \tag{16}$$

to write with $G_P = \Delta^{1/2} M_0^{-1} \Delta^{1/2}$

$$\begin{aligned} T &= \Delta^{1/2}[(\Delta^{-1/2}(P M_0^{-1} P)^{-1} \Delta^{-1/2} + \mathbf{1})^{-1} - \mathbf{1}]\Delta^{1/2} \\ &= -\Delta^{1/2}(G_P + \mathbf{1})^{-1}\Delta^{1/2}. \end{aligned} \tag{17}$$

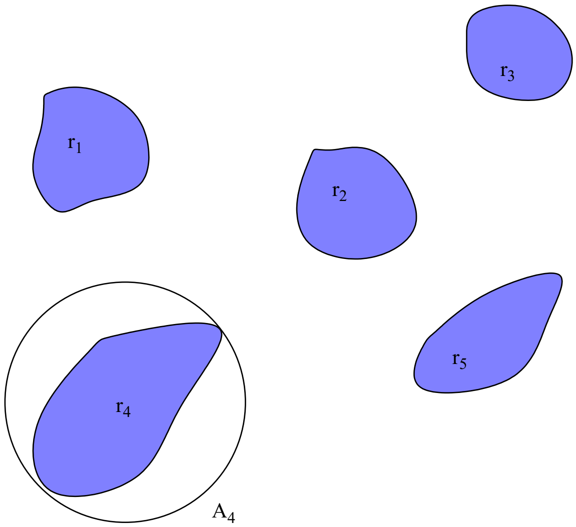

Fig. 2. Scattering region consisting of the scatterers with centers $\mathbf{r}_1$, …, $\mathbf{r}_5$. The scatterer at $\mathbf{r}_k$ is inside a sphere of volume A_k.

This result indicates that the central quantity of the T matrix, in contrast to the Green's function, is not the Maxwellian but G_P. Since M_0^{-1}, the Green's function of the vacuum, is known for different boundary conditions [3,1], the remaining problem is to evaluate $(G_P + \mathbf{1})^{-1}$ in Eq. (17). Our aim is to motivate that G_P can be replaced by an appropriate set of random matrices. The proper choice requires a discussion of the specifics of the scatterers. This is given subsequently.

4. Matrix representation

Scattering regions are typically composed of an ensemble of scattering objects. An example is shown in Fig. 2. Then $\Delta(\mathbf{r})$ is characterized by the centers of the scatterers $\mathbf{r}_1, \ldots, \mathbf{r}_n$, such that it can be expanded around these centers as

$$\Delta^{1/2}(\mathbf{r}) = \sum_{j=1}^{n} \theta_j(\mathbf{r}), \tag{18}$$

where $\theta_j(\mathbf{r})$ vanishes at the boundary of the scatterers. These are random quantities since $\Delta(\mathbf{r})$ is random in a random medium. In the general case of extended scatterers, the functions $\theta_j(\mathbf{r})$ can be expanded in a set of orthogonal functions $f_\alpha(\mathbf{r})$ as

$$\theta_j(\mathbf{r}) = \sum_{\alpha=1}^{N} \theta_{j\alpha} f_\alpha(\mathbf{r} - \mathbf{r}_j),$$

where we assume that $f_\alpha(\mathbf{r})$ decays with r on a sphere of radius R, the maximal sphere A_k of all scatterers. N is the number of the basis functions which is infinite. However, we can truncate the expansion at finite N if the coefficients $\theta_{j\alpha}$ are small for $\alpha > N$. Whether this is possible depends on the proper choice of the functions f_α. For instance, in the case of a point-like scatterer (see Section 4.2) $N = 1$ and f_1 is the Dirac deltafunction. Using spherical wavefunctions for f_α, we have $N = 1$ for a sphere, $N = 2$ for a spheroid and an increasing value of N for an increasing complexity of the scatterer.

The orthogonality of the functions f_α

$$\int f_\alpha(\mathbf{r} - \mathbf{r}_j) f_{\alpha'}(\mathbf{r} - \mathbf{r}_{j'})\, \mathrm{d}^3\mathbf{r} = \delta_{jj'}\delta_{\alpha\alpha'}$$

means that the integral

$$F_{j\alpha,j'\alpha'} = \int f_\alpha(\mathbf{r} - \mathbf{r}_j) F(\mathbf{r}, \mathbf{r}') f_{\alpha'}(\mathbf{r}' - \mathbf{r}_{j'})\, \mathrm{d}^3\mathbf{r}\, \mathrm{d}^3\mathbf{r}'$$

is an orthogonal transformation of F. We can apply this orthogonal transformation to $G_P + \mathbf{1}$ to obtain

$$\int f_\alpha(\mathbf{r} - \mathbf{r}_j)(G_P + \mathbf{1})(\mathbf{r}, \mathbf{r}') f_{\alpha'}(\mathbf{r}' - \mathbf{r}_{j'})\, \mathrm{d}^3\mathbf{r}\, \mathrm{d}^3\mathbf{r}' = G_{P;j\alpha,j'\alpha'} + \delta_{j,j'}\delta_{\alpha,\alpha'}. \tag{19}$$

The T matrix then reads

$$\tilde{T} = -\sum_{j,j'=1}^{n} \sum_{\alpha,\alpha'=1}^{N} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}_j - \mathrm{i}\mathbf{k}'\cdot\mathbf{r}_{j'}} \theta_{j\alpha}(G_P + \mathbf{1})^{-1}_{j\alpha,j'\alpha'}\theta_{j'\alpha'}. \tag{20}$$

$\tilde{T}$ can be studied for special types of scatterers. Subsequently we will consider weak scatterers, point-like scatterers and an array of randomly shaped particles.

4.1. Weak scatterers

If we assume that $|\Delta| \ll 1$ the T matrix in Eq. (15) can be expanded in powers of Δ. For the full Green's function

$$G = G_0(\mathbf{1} + \Delta G_0)^{-1}$$

we obtain in powers of Δ

$$(M_0 + \Delta)^{-1} = G_0 - G_0\Delta G_0 + \cdots + (-1)^n G_0(\Delta G_0)^n + \cdots.$$

The truncation after the linear term leads to

$$T \approx -\Delta + \Delta G_0 \Delta = -\Delta^{1/2}(\mathbf{1} - G_P)\Delta^{1/2}. \tag{21}$$

Partial summations for higher powers of G_P can also be carried out for this expansion. An example is the evaluation of the intensity or polarization, averaged over a random distribution of Δ. In this case we can choose the ladder and maximally crossed diagrams of Langer and Neal [7]. This was discussed for the case of light scattering in several papers [8–10].

4.2. Point-like scatterers

A particular case of our expansion (18) is that of n point scatterers:

$$\Delta^{1/2}(\mathbf{r}) = \sum_{j=1}^{n} \theta_j \delta(\mathbf{r} - \mathbf{r}_j).$$

Here it is not necessary to expand the Dirac deltafunction with orthogonal functions. Therefore, the T matrix is

$$\tilde{T} = -\sum_{j,j'} \mathrm{e}^{\mathbf{i}\mathbf{k}\cdot\mathbf{r}_j - \mathbf{i}\mathbf{k}'\cdot\mathbf{r}_{j'}} \theta_j (G_P + \mathbf{1})^{-1}_{j,j'} \theta_{j'},$$

where the $n \times n$ matrix $(G_{P;j,j'})$ has elements

$$G_{P;jj'} = \int \delta(\mathbf{r} - \mathbf{r}_j) G_P(\mathbf{r}, \mathbf{r}') \delta(\mathbf{r}' - \mathbf{r}_{j'})\, \mathrm{d}^3\mathbf{r}\, \mathrm{d}^3\mathbf{r}' = G_P(\mathbf{r}_j, \mathbf{r}_{j'}).$$

With the definition of G_P this we can also write as

$$G_P(\mathbf{r}_j, \mathbf{r}_{j'}) = \Delta^{1/2}(\mathbf{r}_j) G_0(\mathbf{r}_j, \mathbf{r}_{j'}) \Delta^{1/2}(\mathbf{r}_{j'}). \tag{22}$$

In this representation our calculation of the T matrix reduces to the inversion of an $n \times n$ matrix for n point scatterers:

$$\tilde{T} = -\sum_{j,j'=1}^{n} \mathrm{e}^{\mathbf{i}\mathbf{k}\cdot\mathbf{r}_j - \mathbf{i}\mathbf{k}'\cdot\mathbf{r}_{j'}} \theta_j (G_P + \mathbf{1})^{-1}_{jj'} \theta_{j'}. \tag{23}$$

However, a problem is that the Green's function $G_0(\mathbf{r}, \mathbf{r})$ is singular. It reflects the fact that the point scatterers are not realistic due to the sharp Dirac deltafunction. This can be solved by replacing the Dirac deltafunction by a smooth function. A standard way to do that is to introduce a cut-off for short lengths. This was discussed in great detail by de Vries et al. [1] and leads to a regularized Green's function $\tilde{G}_0$. It was shown that under some conditions regularized point-like particles represent small spherical particles. The parameters of the spheres (i.e., radius and their refractive index) are related to two regularization parameters Λ_L and Λ_T. The regularized Green's function $\tilde{G}_0$ that replaces G_0 then reads [1]

$$\tilde{G}_0(\mathbf{r}, \mathbf{0}) = \tilde{G}_0^T(\mathbf{r}, \mathbf{0}) + \tilde{G}_0^L(\mathbf{r}, \mathbf{0})$$

with

$$\tilde{G}_0^T(\mathbf{r}, \mathbf{0}) = -\frac{r^2\gamma_0 - 3\rho}{4\pi k^2 r^5} - \left\{ \frac{\mathrm{e}^{ikr}}{4\pi r}[P(\mathrm{i}kr)\gamma_0 + Q(\mathrm{i}kr)\frac{\rho}{r^2}] - \frac{\mathrm{e}^{-\Lambda_T r}}{4\pi r}[P(-\Lambda_T r)\gamma_0 + Q(-\Lambda_T r)\frac{\rho}{r^2}] \right\} \frac{\Lambda_T^2}{\Lambda_T^2 + k^2}$$

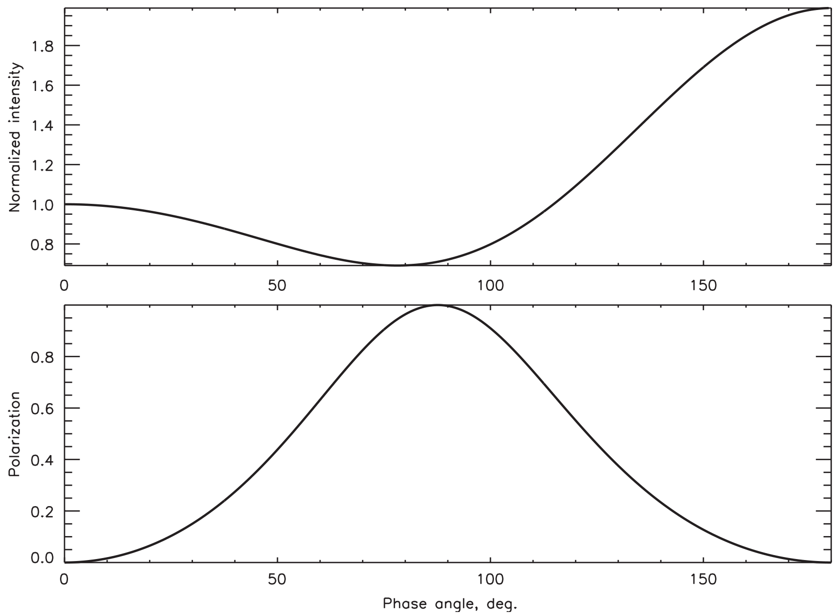

Fig. 3. Scattering on a cubic array of 125 point-like particles. The size parameter of each particle is 0.1. The center of the array is located in the origin of the coordinates and the edges are parallel to the coordinate axes.

and

$$\tilde{G}_0^L(\mathbf{r},\mathbf{0}) = \frac{r^2\gamma_0 - 3\rho}{4\pi k^2 r^5}\{1 - \mathrm{e}^{-\Lambda_L r}[\cos(\Lambda_L r) + \Lambda_L r(\cos(\Lambda_L r) + \sin(\Lambda_L r)]\} + \Lambda_L^2 \mathrm{e}^{-\Lambda_L r}\sin(\Lambda_L r)\frac{\rho}{2\pi k^2 r^3}.$$

γ_0 is the 3×3 unit tensor $(=\sum_{i=1}^{3}\gamma_{ii})$ and

$$\rho = \sum_{1\leqslant i\leqslant j\leqslant 3} x_i x_j \gamma_{ij} = \begin{pmatrix} x_1x_1 & x_1x_2 & x_1x_3 \\ x_2x_1 & x_2x_2 & x_2x_3 \\ x_3x_1 & x_3x_2 & x_3x_3 \end{pmatrix}.$$

The functions P and Q are

$$P(z) = 1 - \frac{1}{z} + \frac{1}{z^2}, \quad Q(z) = -1 + \frac{3}{z} - \frac{3}{z^2}.$$

In the case of n point scatterers this provides with Eq. (23) a well-defined $n \times n$ matrix, since for small values of r we get

$$\tilde{G}_0(\mathbf{r},\mathbf{0}) \sim \frac{1}{6\pi}\left[\frac{\Lambda_L^3}{k^2} - \frac{\Lambda_T^2}{\Lambda_T^2 + k^2}(\Lambda_T + \mathrm{i}k)\right]\gamma_0 - \frac{1}{32\pi k^2}(4\Lambda_L^4 + k^2\Lambda_T^2)\frac{\rho}{r}. \tag{24}$$

We have studied an array of 125 point-like scatterers with random coefficients θ_j. The result for the intensity and the polarization is shown in Fig. 3.

4.3. Randomly shaped scatterers

In the case of scatterers with random shape it is difficult and time consuming to perform the calculations for the T matrix without additional approximations, even if the number of scatterers is small. This problem is well known in many areas like nuclear [11,12] or mesoscopic physics [13,14]. The reason is that particles are characterized by many parameters, a very common situation in realistic physical systems. For instance, for scattering of small particles (e.g. protons or neutrons) on heavy atomic nuclei, the nucleons inside the nuclei act like randomly distributed scatterers. This situation was discussed by Wigner [11], using a random-matrix (R-matrix) ensemble. This is based on the idea that the realistic Hamiltonian H of the atomic nucleous is so complex that it has to be described by a statical approach. What is known from fundamental physics, however, is the fact that this Hamiltonian has to be symmetric [12]. Moreover, H can only be determined up to an orthogonal transformation, i.e. by its eigenvalues. The central and surprisingly simple idea is to replace the original Hamiltonian H by a specific distribution of matrices. In the case of a symmetric Hamiltonian H this leads to

$$H \to \begin{pmatrix} h_{11} & \dots & h_{1N} \\ \vdots & \ddots & \vdots \\ h_{N1} & \dots & h_{NN} \end{pmatrix} \quad (h_{ij} = h_{ji}),$$

where $\{h_{ij}\}$ $(1 \leqslant i \leqslant j \leqslant N)$ are independent random numbers:

$$\langle h_{ij} \rangle = 0, \quad \langle h_{ij} h_{kl} \rangle = \frac{g}{N}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

g is a parameter that controls the random fluctuations of the medium. It has to be chosen empirically.

In the case of the Maxwell theory the Maxwellian must be considered instead of the Hamiltonian H. Then the dyadic structure of Eq. (2), related to the algebra of the γ matrices, must be included in the construction of the corresponding random-matrix theory.

5. R-matrix approach to light scattering

The Green's function G_0 on a compact region is symmetric, a consequence of the fact that M_0 is symmetric. Therefore, it can be represented by the complete set of symmetric 3×3 matrices of Eq. (3) as

$$G_0 = \sum_{1 \leqslant i \leqslant j \leqslant 3} G_{0;ij} \gamma_{ij}. \tag{25}$$

Comparing this result, for instance, with the asymptotic behavior of $G_0(\mathbf{r}, \mathbf{0})$ in Eq. (9), we obtain

$$G_{0;ij}(\mathbf{r}, \mathbf{0}) \sim \frac{\cos(kr)}{4\pi r^3} \begin{cases} r^2 - x_i^2, & i = j, \\ -x_i x_j, & i < j. \end{cases}$$

The expansion of Eq. (25) implies for G_P

$$G_P = \sum_{i \leqslant j} \Delta^{1/2} G_{0;ij} \Delta^{1/2} \gamma_{ij}$$

which can be used to calculate the T matrix of Eq. (20) and finally the intensity and the polarization in Eqs. (13) and (14), respectively.

The orthogonal transformation of G_P of Eq. (19) creates from $\Delta^{1/2}G_{0;ij}\Delta^{1/2}$ an $nN \times nN$ matrix. The properties of this matrix are determined by Δ of Eq. (6) that describes the spatial distribution of optical properties. To study a generic situation for scattering in a random medium or a randomly distributed ensemble of scatterers we can consider the elements of a symmetric matrix

$$\Gamma_{ij;k\alpha,k'\alpha'} = \int\int f_\alpha(\mathbf{r}-\mathbf{r}_k)\Delta^{1/2}(\mathbf{r})G_{0;ij}(\mathbf{r},\mathbf{r}')\Delta^{1/2}(\mathbf{r}')f_{\alpha'}(\mathbf{r}'-\mathbf{r}_{k'})\,\mathrm{d}^3\mathbf{r}\,\mathrm{d}^3\mathbf{r}'$$

for $1 \leqslant i \leqslant j \leqslant 3$, $1 \leqslant k \leqslant k' \leqslant n$ and $1 \leqslant \alpha \leqslant \alpha' \leqslant N$. The contribution of a single scatterer (i.e., for $k' = k$) requires only the integration of the area of the scatterer A_k around $\mathbf{r}_k$. If we assume that this area is small, we can use the approximation of $G_0(\mathbf{r},\mathbf{r}')$ given in Eq. 0 (24):

$$G_{0;ij}(\mathbf{r},\mathbf{r}') \approx a\delta_{ij} + b\frac{(x_i - x_i')(x_j - x_j')}{|\mathbf{r}-\mathbf{r}'|}.$$

This allows us to write

$$\begin{aligned}\Gamma_{ij;k\alpha,k\alpha'} &= \int_{A_k}\int_{A_k} f_\alpha(\mathbf{r}-\mathbf{r}_k)\Delta^{1/2}(\mathbf{r})G_{0;ij}(\mathbf{r},\mathbf{r}')\Delta^{1/2}(\mathbf{r}')f_{\alpha'}(\mathbf{r}'-\mathbf{r}_k)\,\mathrm{d}^3\mathbf{r}\,\mathrm{d}^3\mathbf{r}'\\ &\approx A\delta_{\alpha\alpha'}\delta_{ij} + \Gamma'_{ij;k\alpha,k\alpha'}\end{aligned}$$

and for $k' \neq k$

$$\begin{aligned}\Gamma_{ij;k\alpha,k'\alpha'} &= \int_{A_k}\int_{A_{k'}} f_\alpha(\mathbf{r}-\mathbf{r}_k)\Delta^{1/2}(\mathbf{r})G_{0;ij}(\mathbf{r},\mathbf{r}')\Delta^{1/2}(\mathbf{r}')f_{\alpha'}(\mathbf{r}'-\mathbf{r}_{k'})\,\mathrm{d}^3\mathbf{r}\,\mathrm{d}^3\mathbf{r}'\\ &\approx \Gamma'_{ij;k\alpha,k'\alpha'}.\end{aligned}$$

The constant A describes the averaged properties of the scatterers and $\Gamma'_{ij;k\alpha,k'\alpha'}$ their random fluctuations. Using the assumption that the expectation values $\langle\Delta(\mathbf{r})\rangle$ and $\langle\Delta(\mathbf{r})\Delta(\mathbf{r}')\rangle$ are constant on A_k, we can calculate expectation values of $\Gamma_{ij;k\alpha,k'\alpha'}$. Choosing for the latter independent Gaussian random numbers, this implies the expectation values

$$\begin{aligned}&\langle\Gamma'_{ij;k\alpha,k'\alpha'}\rangle = 0,\\ &\langle\Gamma'_{ij;k\alpha,k'\alpha'}\Gamma'_{ij;l\beta,l'\beta'}\rangle = \frac{g_{kk'}}{N}(\delta_{kl}\delta_{k'l'}\delta_{\alpha\beta}\delta_{\alpha'\beta'} + \delta_{kl'}\delta_{k'l}\delta_{\alpha\beta'}\delta_{\alpha'\beta})\end{aligned} \tag{26}$$

and with the symmetry constraint

$$\Gamma'_{ij;k\alpha,k'\alpha'} = \Gamma'_{ij;k'\alpha',k\alpha} \quad (1 \leqslant i \leqslant j \leqslant 3).$$

Here we assumed that the fluctuations, characterized by the parameter $g_{kk'}$, are independent of the indices i, j. This simplification can be abandoned to describe, for instance, anisotropic scattering media.

From the T matrix of Eq. (20) we then obtain the expression

$$\tilde{T} = -\sum_{l,l'=1}^{n}\sum_{\alpha,\alpha'=1}^{N} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}_l - \mathrm{i}\mathbf{k}'\cdot\mathbf{r}_{l'}}\theta_{l\alpha}\left(\mathbf{1} + \sum_{i\leqslant j}\Gamma_{ij}\gamma_{ij}\right)^{-1}_{l\alpha,l'\alpha'}\theta_{l'\alpha'}. \tag{27}$$

This expression can be treated in the asymptotic regime of large values of N, leading to a perturbation theory in terms of $1/N$.

5.1. Perturbation theory: $1/N$ *expansion*

In order to obtain the averaged intensity and the averaged polarization in Eqs. (13) and (14) we must evaluate the average of products of elements of the T matrix. The averaging procedure can be performed easily if we expand the inverse matrix in $\tilde{T}$

$$\left(\mathbf{1}+\sum_{i\leqslant j}\Gamma_{ij}\gamma_{ij}\right)^{-1}=\sum_{l\geqslant 0}(-1)^l\left(\sum_{i\leqslant j}\Gamma_{ij}\gamma_{ij}\right)^l$$

and use the properties of the distribution given in Eq. (26). Then our calculation reduces to averaging products of matrix elements of Γ_{ij}. The expectation value of a multiple product of matrix elements can be decomposed into products of averaged pairs of matrix elements because higher moments of the Gaussian distribution can be related to products of second moments. As a first example, we discuss the average T matrix. A typical term is

$$\langle \Gamma_{ij;k\alpha,k_1\alpha_1}\Gamma_{ij;k_1\alpha_1,k_2\alpha_2}\cdots\Gamma_{ij;k_{l-1}\alpha_{l-1},k'\alpha'}\rangle, \tag{28}$$

where the intermediate indices α_1, ..., α_{l-1} are summed from 1 to N, and the labels of the intermediate scattering centers $k_1, k_2, ..., k_{l-1}$ are summed from 1 to n. According to properties (26), the matrix elements $\Gamma_{ij;k_l\alpha_l,k_{l+1}\alpha_{l+1}}$ must appear in (28) as an even power. This can be depicted by a diagrammatic representation of the perturbation expansion: the two sites $k_l\alpha_l$ and $k_{l+1}\alpha_{l+1}$ are connected by a matrix element $\Gamma_{ij;k_l\alpha_l,k_{l+1}\alpha_{l+1}}$. Then the string of matrix elements in Eq. (28) is represented by a line, connecting $k\alpha$ with $k'\alpha'$. Averaging gives a nonzero expression, for instance, if we fold the string in the middle such that every pair of sites coincide. This requires a string of length l, where l is an even number. The same construction must be applied for products of T-matrix elements, as shown in Fig. 4. It is crucial in all cases that there is a summation over the indices $\alpha_j=1$, ..., N, providing a factor N because the terms of the sum are degenerate. Moreover, there is a factor g/N from each pair of matrix elements after averaging, due to Eq. (26). As a result, the expansion can be organized in powers of $1/N$, in contrast to the weak-scattering expansion in powers of g, starting with the leading order 1. The leading order of the intensity and the polarization is related to the ladder and maximally crossed diagrams [7].

For practical purposes this diagrammatic approach is too involved and not very efficient though. A more transparent and efficient approach is the representation of the T matrix in a supersymmetric functional-integral formalism [14]. In this representation the $N\to\infty$ limit is a saddle point of an integral. This shall be discussed in a separate paper.

6. Discussion of the results

For a qualitative understanding of the T matrix of Eq. (27) we consider its expansion in terms of the γ matrices:

$$\tilde{T}=\sum_{1\leqslant i\leqslant j\leqslant 3}t_{ij}\gamma_{ij}.$$

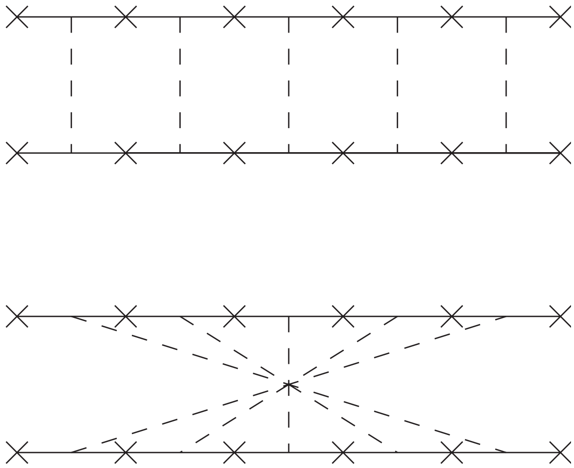

Fig. 4. Ladder and maximally crossed diagrams. Crosses indicate sites $k\alpha$, full lines between crosses indicate matrix elements $\Gamma_{ij,k\alpha,k'\alpha'}$ and dashed lines connect pairs of $\Gamma_{ij,k\alpha,k'\alpha'}$ (in the ladder) and pairs of $\Gamma_{ij,k\alpha,k'\alpha'}$ and $\Gamma_{ij,k'\alpha',k\alpha}$ (in the maximally crossed diagram). All these diagrams contribute in order $1/N$.

It is important to notice that the random variables t_{ij} are *correlated*, in contrast to the *independent* Gaussian random matrices Γ'_{ij} of Eq. (27). This can be seen, for instance, in a perturbation theory for weak scatterers, where we expand the inverse matrix in $\tilde{T}$ (cf. Section 5.1). The correlations are crucial for the properties of the scattering process. We discuss this in the following for a small single scatterer (Rayleigh particle) located at $\mathbf{r}=\mathbf{0}$. The fact that the scatterer is small allows us to approximate the unitary transformation $U_{\mathbf{k}}$ by the identity:

$$e^{i\mathbf{k}\cdot\mathbf{r}} \approx 1.$$

Consequently, we can use $\tilde{T} = U_{\mathbf{k}} T U^{\dagger}_{\mathbf{k}'} \approx T$. The dependence on the phase angle Θ then is only due to the trigonometric coefficients in the intensity and polarization of Eqs. (13), (14). The elements of T are independent of Θ and of the form

$$t_{ij} = a\delta_{ij} + \tau_{ij}$$

with a constant a and random variables τ_{ij} with $\langle \tau_{ij} \rangle = 0$.

If we assume uncorrelated t_{ij} with $\langle \tau_{ij}\tau^*_{i'j'} \rangle = g_{ij}\delta_{ii'}\delta_{jj'}$, for instance by truncating the perturbation theory at low-order terms, we get simple expressions for the average intensity

$$|a|^2(1+\cos^2\Theta) + g_{23} + g_{33} + (g_{12}+g_{13})\sin^2\Theta + (g_{22}+g_{23})\cos^2\Theta$$

and for the average polarization

$$\frac{|a|^2\sin^2\Theta + g_{23} + g_{33} - (g_{12}+g_{13})\sin^2\Theta - (g_{22}+g_{23})\cos^2\Theta}{|a|^2(1+\cos^2\Theta) + g_{23} + g_{33} + (g_{12}+g_{13})\sin^2\Theta + (g_{22}+g_{23})\cos^2\Theta}.$$

These two quantities are symmetric with respect to the phase angle $\Theta = \pi/2$. In the case of diagonal scattering (i.e. $g_{ij} = 0$ for $i \neq j$ and $g_{ii} = g$) we get for the polarization the result of a point

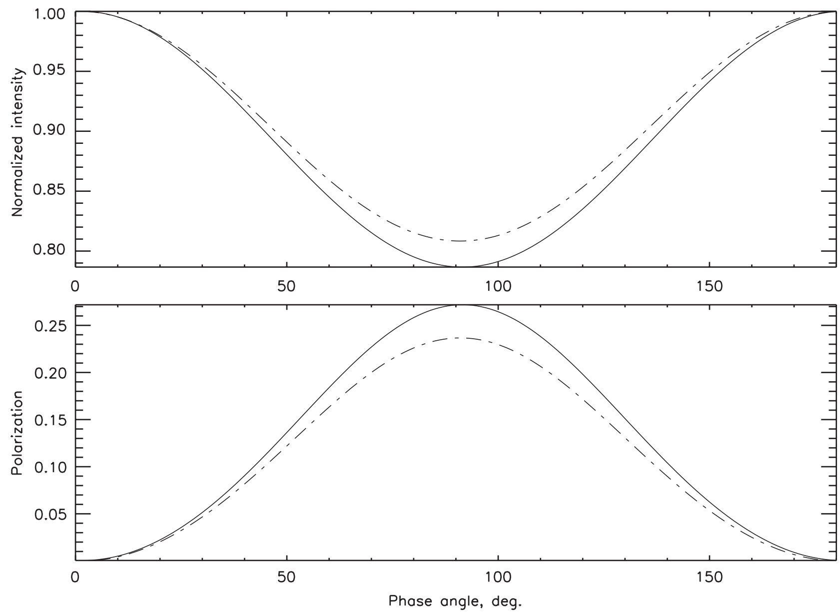

Fig. 5. Intensity and polarization of a single random particle ($n = N = 1$) for $A = 0.6$ (full curves) and $A = 1.1$ (dashed curves). The matrix elements are uniformally distributed on the interval $[-0.5, 0.5]$.

scatterer

$$\frac{\sin^2 \Theta}{1 + \cos^2 \Theta}.$$

With off-diagonal scattering $g_{ij} = g'$ $(i \neq j)$ the polarization is

$$\frac{(|a|^2 + g - g') \sin^2 \Theta}{(|a|^2 + g + g')(1 + \cos^2 \Theta) + 2g' \sin^2 \Theta}.$$

For isotropic scattering (i.e. $g' = g$) the average polarization reduces to

$$\frac{\sin^2 \Theta}{4g/|a|^2 + 1 + \cos^2 \Theta}.$$

Thus the off-diagonal scattering suppresses the polarization due to the extra term $4g/|a|^2$ in the denominator, in comparison with the result of a point scatterer.

Starting with the T matrix of Eq. (27), an asymmetric behavior is found for individual realizations of the random distribution but the average over all realizations leads again to a symmetric behavior with respect to $\Theta = \pi/2$. This is shown for a single random scatterer (i.e. $n = 1$) with $N = 1$ in Fig. 5. It indicates that there are no correlations between the matrix elements t_{ij} to create an asymmetric term

$$\langle (t_{12} + t_{13})(t^*_{22} + t^*_{23}) \rangle \cos \Theta \sin \Theta$$

in the intensity or in the polarization. Asymmetric contributions to small scatterers can be obtained from the T matrix in Eq. (12) by expanding the exponentials of the Fourier transformation in powers of $kr\ \cos\ \Theta$ and $kr\ \sin\ \Theta$.

7. Conclusions

To describe the scattering of light in a random medium we introduced a random-matrix model. The construction is based on a scattering medium that consists of small randomly shaped particles (grains). Their optical properties are defined by the quantity $\Delta(\mathbf{r})$ in Eq. (6) which is zero in vacuum but non-zero in the scatterers, where it depends on the dielectric constant. Our random-matrix approach relies on similar approaches in quantum theory. However, the vectorial structure of light, in contrast to the scalar structure of the Schrödinger wave function in quantum theory, requires some additional considerations. In our model each scattering particle is represented by an $3N \times 3N$ random matrix, and the scattering between each pair of different particles by another $3N \times 3N$ random matrix. The number N refers to N internal degrees of freedom which characterize the structure of the particle. For point-like spherical particles there is $N = 1$. Then a system of n particles is represented by an $3nN \times 3nN$ random matrix. From this we derived the effective T matrix and the corresponding expressions for the intensity and the polarization. In our approach two parameters (strength of fluctuations g of the random medium and $1/N$) appear which allow us to apply two types of perturbation theory. The approach can be used to describe the electromagnetic field, in particular, its intensity and polarization, caused by the scattering by an arbitrary ensemble of random particles. It can be used to study the characteristics of natural or man-made dusts and aerosols.

Acknowledgements

We are grateful to Pavel Litvinov for useful discussions.

Appendix A. Dyson equation and T matrix

The following calculations can be performed directly but it is more instructive to use perturbation theory. We expand the full Green's function in powers of Δ

$$G=(M_0+\Delta)^{-1} = M_0^{-1}\sum_{l\geqslant 0}(-\Delta M_0^{-1})^l$$
$$= M_0^{-1} - M_0^{-1}\Delta M_0^{-1} + M_0^{-1}\sum_{l\geqslant 2}(-\Delta M_0^{-1})^l.$$

Since

$$\sum_{l\geqslant 2}(-\Delta M_0^{-1})^l = \Delta M_0^{-1}\sum_{l\geqslant 0}(-\Delta M_0^{-1})^l\Delta M_0^{-1} = \Delta G\Delta M_0^{-1},$$

we can write

$$G = M_0^{-1} - M_0^{-1}\Delta M_0^{-1} + M_0^{-1}\Delta G \Delta M_0^{-1} \equiv M_0^{-1} + M_0^{-1} T M_0^{-1}.$$

This result can be considered as a Dyson equation of the Green's function G and a definition of the T matrix T.

References

[1] de Vries P, van Coevorden DV. Lagendijk. Point scatterers for classical waves. Rev Mod Phys 1998;70:447–66.

[2] van de Hulst HC. Light scattering by small particles. New York: Wiley; 1957 .

[3] Morse PM, Feshbach H. Methods of theoretical physics. New York: McGraw-Hill; 1953 .

[4] Waterman PC. Symmetry, unitarity, and geometry in electromagnetic scattering. Phys Rev D 1971;3:825–39.

[5] Mishchenko MI, Travis LD, Mackowski DW. T-matrix computations of light scattering by nonspherical particles: a review. JQSRT 1996;55:535–75.

[6] Ziegler K. Localization of electromagnetic waves in random media. JQSRT 2003;79–80:1189–98.

[7] Langer JS, Neal T. Breakdown of the Concentration Expansion for the Impurity Resistivity of Metals. Phys Rev Lett 1966;16:984–6.

[8] Stephen MJ, Cwilich G. Rayleigh scattering and weak localization: effects of polarization. Phys Rev B 1986;34:7564–72.

[9] Mishchenko MI. On the nature of the polarization opposition effect exhibited by Saturns's rings. Astrophys J 1993;411: 351–61.

[10] Ozrin VD. Exact solution for coherent backscattering of polarized light from a random medium of Rayleigh scatterers. Waves Random Media 1992;2:141–64.

[11] Wigner EP. On the distribution of the roots of certain symmetric matrices. Ann of Math 1958;67:325–7.

[12] Mehta ML. Random matrices. New York: Academic Press; 1967 .

[13] Marcus CM. et al. Quantum chaos in open versus closed quantum dots: signatures of interacting particles. Chaos, Solitons and fractals 1997;8:1261–79.

[14] Ziegler K. Quantum Hall transition in an array of quantum dots. Phys Rev B 1997;55:10602–6.