From Statistical Polymer Physics to Nonlinear Elasticity

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Abstract

A polymer-chain network is a collection of interconnected polymer-chains, made themselves of the repetition of a single pattern called a monomer. Our first main result establishes that, for a class of models for polymer-chain networks, the thermodynamic limit in the canonical ensemble yields a hyperelastic model in continuum mechanics. In particular, the discrete Helmholtz free energy of the network converges to the infimum of a continuum integral functional (of an energy density depending only on the local deformation gradient) and the discrete Gibbs measure converges (in the sense of a large deviation principle) to a measure supported on minimizers of the integral functional. Our second main result establishes the small temperature limit of the obtained continuum model (provided the discrete Hamiltonian is itself independent of the temperature), and shows that it coincides with the Γ-limit of the discrete Hamiltonian, thus showing that thermodynamic and small temperature limits commute. We eventually apply these general results to a standard model of polymer physics from which we derive nonlinear elasticity. We moreover show that taking the Γ-limit of the Hamiltonian is a good approximation of the thermodynamic limit at finite temperature in the regime of large number of monomers per polymer-chain (which turns out to play the role of an effective inverse temperature in the analysis).

Contents

1. Introduction and Statement of the Main Results

1.1. Polymer Physics and Nonlinear Elasticity

As opposed to (standard) fluids and to crystalline solids, the interesting properties of rubber stem from statistical physics rather than from quantum or molecular physics. As Rubinstein and Colby write in the chapter "Networks and gels" of their reference monograph [43] on polymer physics, *Such networks, with either chemical or strong physical bonds, are important soft solids. (…) The entropic nature of elasticity in rubbers is the origin of their remarkable mechanical properties.*

The aim of the present work is to rigorously relate the "remarkable mechanical properties" of rubber (best described by hyperelasticity at large deformation in continuum mechanics) to "the entropic nature of elasticity" (best described at the level of the statistical physics of polymer-chain networks). This question is a typical instance of the general program of deriving macroscopic models from microscopic descriptions in the vein of the sixth Hilbert problem.

Our contribution is twofold:

- On the one hand, we perform a rigorous thermodynamic limit of a general class of statistical physics models towards nonlinear elasticity, which raises interesting questions in mathematical analysis and probability.
- On the other hand, we present the state-of-the-art models of polymer-chain physics, that we rewrite in a form suitable for the analysis, emphasizing the modeling aspects, the physical aspects, and the relevant orders of magnitude involved (as needed in asymptotic analysis). This enables us to justify standard approaches in polymer physics in some relevant regimes, and to establish the validity of a two-temperature model introduced in [30].

This work rigorously derives nonlinear elasticity from a polymer physics model at finite temperature for the first time. It constitutes the first interaction between polymer physics and mathematical analysis at a level that allows to answer questions of interest to both communities. In particular, the use of mathematical analysis allows one to turn qualitative physical insight into quantitative statements, which is what this contribution is about. We expect further such interactions to develop.

This article is written in such a way that physics and mathematics can be read separately. In the rest of this introduction, we focus on the mathematical aspects: we introduce the notation, the mathematical (and statistical) description of the network, and state the main results of the paper on the thermodynamic limit of the Gibbs measure and the free energy, and discuss the structure of the proofs. Section 2 is dedicated to physical aspects, and gives a gentle introduction to the polymer physics of rubber-like materials. Not only does this allow us to motivate the class of models considered in the analysis part of this contribution, but it also allows us to apply these results to specific models of interest in polymer physics.

1.2. Discrete Free Energies and Gibbs Measures

We start with the definition of admissible Euclidean graphs, and first fix once and for all constants $R, r, C_0 > 0$, and dimensions d, n. For technical reasons we assume that $6R < C_0$. For more details on point processes, we refer to the monograph [39].

Definition 1.1. Let $\mathcal{P} \subset \mathbb{R}^d$ be a countable set.

(i) $\mathcal{P}$ is said to be in general position if there are no $k+1$ points contained in a common $k-1$-dimensional affine subspace ($1 \leqq k \leqq d$) and no $d+2$ points lie on the boundary of the same sphere.
(ii) A Delaunay tessellation $\mathbb{T} = \{T_i\}_{i\in\mathbb{N}}$ associated with $\mathcal{P}$ is a partition of $\mathbb{R}^d$ into d-simplices T_i whose vertices are in $\mathcal{P}$ and such that no point of $\mathcal{P}$ is contained inside the circumsphere of any simplex in $\mathbb{T}$.

Recall that Delaunay tessellations are dual to Voronoi tessellations (if the point set is in general position—otherwise one has to choose a specific representative, as one would do for $\mathbb{Z}^d$). This will only be used to define piecewise affine interpolations as needed for the volumetric part of the Hamiltonian—this definition does not restrict generality.

We now introduce our model for the reference configuration of a polymer network.

Definition 1.2. An extended Euclidean graph $G = (\mathcal{L}, E, S) \in (\mathbb{R}^d)^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}\times\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$ is a set of points $\mathcal{L} = \{x_i\}_{i\in\mathbb{N}} \subset (\mathbb{R}^d)^{\mathbb{N}}$, an associated connectivity graph $E \in \{0,1\}^{\mathbb{N}\times\mathbb{N}}$, and a subset of points $\mathcal{L}_1 := \cup_{i|S_i=1}\{x_i\}$. If $E_{ij} = 1$, we say that (x_i, x_j) is an edge of the graph, whereas if $S_i = 1$ we say that x_i is a "volumetric point" (this wording will be clear later). We call $\mathbb{B}$ the set of edges of $(\mathcal{L}, E, S)$, and $\mathbb{T}$ the Delaunay tessellation of $\mathbb{R}^d$ associated with $\cup_{i|S_i=1}\{x_i\}$.[1] We say that $G = (\mathcal{L}, E, S)$ is an admissible extended Euclidean graph (graph in short) if it satisfies

(i) $\mathrm{dist}(z, \mathcal{L}_1) \leqq R$ for all $z \in \mathbb{R}^d$;
(ii) $\mathrm{dist}(x, \mathcal{L} \setminus \{x\}) \geqq r$ for all $x \in \mathcal{L}$;
(iii) for all $x \in \mathcal{L}: \ \{y \in \mathcal{L} : \ (x,y) \in \mathbb{B}\} \subset B_{C_0}(x)$;
(iv) For all $x, y \in \mathcal{L}$ there exists a path $P(x,y)$ of edges of $\mathbb{B}$ connecting x to y with

$$P(x,y) \subset [x,y] + B_{C_0}(0),$$

where $[x,y] = \{x + t(y-x) : \ t \in [0,1]\}$;
(v) $\mathcal{L}_1$ is in general position.

We denote by $\mathcal{G}$ the set of graphs for which (i)–(v) hold.

[1] Uniqueness fails when points are not in general position, which we rule out by condition (v) of this definition.

In the above definition one can obviously choose $\mathcal{L}_1 = \mathcal{L}$. The wording "volumetric points" is chosen in reference to the volumetric part of the Hamiltonian which will be defined using points in $\mathcal{L}_1$—and not necessarily all the points of $\mathcal{L}$. This is related to the scale at which one wishes to impose incompressibility in the physical model, and we refer the reader to Paragraph 2.2.4 for details. In mathematical terms, considering $\mathcal{L}_1$ on top of $\mathcal{L}$ adds some degree of freedom to define the Hamiltonian—it does not yield any additional difficulty in the analysis, and should be seen as a requirement of physical modeling only. Note that the set of vertices $\mathcal{L}$ also satisfies (i). Point sets with the properties (i) and (ii) are sometimes called Delone sets. This class of point sets has already been used as a reference configuration for atomistic models in elasticity in [4]—albeit at zero temperature. Other assumptions have also been considered in the literature for different models. In [8], the authors define random perturbations of periodic lattices and study the limit of the ground state of the electronic cloud associated with atoms that are placed at the vertices of this perturbed lattice. In [9], the authors address a similar model (where electrons interact via a Coulomb potential) at finite temperature. The main difference between [8,9] and [4] and the present contribution (besides the assumptions on the graphs, which is not essential for our analysis) is the type of unknown. In [8,9] the authors characterize the density of electrons, whereas in [4] and in the present contribution we characterize the deformation of the lattice points. Note next that (ii) and (iii) imply that the degree of each vertex is bounded uniformly (this is one of the physical constants of the model). Assumption (iv) is technical and ensures a coercivity property (see Lemma 4.3). Assumption (v) is to avoid the non-uniqueness of Delaunay tessellations—this is not essential but convenient to simplify measurability issues.

Next we endow $\mathcal{G}$ with a probabilistic structure, and consider on $\mathcal{G} \subset (\mathbb{R}^d)^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}\times\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$ the σ-algebra Σ given by the trace σ-algebra of $\mathcal{B}_{(\mathbb{R}^d)^{\mathbb{N}}} \otimes \mathcal{B}_{\{0,1\}^{\mathbb{N}\times\mathbb{N}}} \otimes \mathcal{B}_{\{0,1\}^{\mathbb{N}}}$, where each factor denotes the Borel σ-algebra given by the product topology on the factors. We do not distinguish between $(\mathcal{L}, E, S)$ and $(\mathcal{L}, \mathbb{B}, \mathbb{T})$, which we will both denote by G. We then give ourselves a statistics on this set of Euclidean graphs described by a measure $\mathbb{E}$ on $(\mathcal{G}, \Sigma)$, and address the minimal assumptions on this distribution $\mathbb{E}$. They are related to the operation of the shift group $(\mathbb{Z}^d, +)$ on $\mathcal{G}$, that is, for any shift vector $z \in \mathbb{Z}^d$ and any Euclidean graph $G = (\mathcal{L}, \mathbb{B}, \mathbb{T})$, the shifted graph $G+z = (\mathcal{L}+z, \mathbb{B}+z, \mathbb{T}+z)$ is again a Euclidean graph. The first assumption is *stationarity*, which means that for any shift $z \in \mathbb{Z}^d$ the random Euclidean graphs G and $G+z$ have the same (joint) distribution. The second assumption is *ergodicity*, which means that any (integrable) random variable $F(G)$ (that is a measurable map of the random graph) that is shift invariant, in the sense that $F(G+z) = F(G)$ for all shift vectors $z \in \mathbb{Z}^d$ and almost-every Euclidean graph G is actually constant, that is $F = \mathbb{E}[F]$ for almost every Euclidean graph G. Throughout this paper we will tacitly assume stationarity and ergodicity. Our results (except Remark 2) remain valid under the mere assumption of stationarity, but all asymptotic quantities may still be random.

We are now in the position to introduce the Hamiltonian and the free energy at the microscopic level. Let $D \subset \mathbb{R}^d$ be an open bounded reference domain with Lipschitz boundary. Given a small parameter $0 < \varepsilon \ll 1$ and any $U \subseteq \mathbb{R}^d$ we set $U_\varepsilon = \frac{U}{\varepsilon}$, and use the short-hand notation $U_\varepsilon^{\mathcal{L}} := \mathcal{L} \cap U_\varepsilon$—with this choice the microscopic scale is set to 1 and the macroscopic scale to $\frac{1}{\varepsilon}$. We consider microscopic deformations $u : D_\varepsilon^{\mathcal{L}} \to \mathbb{R}^n$, whose internal energy takes the form

$$H_\varepsilon(D, u) = \sum_{\substack{(x,y)\in\mathbb{B} \\ x,y\in D_\varepsilon}} f\,(x - y, u(x) - u(y)) + H_{\mathrm{vol},\varepsilon}(D, u), \tag{1.1}$$

for some map f and a volumetric term that penalizes large changes of volume and change of "orientation" (if it is not identically zero we always consider the case $n = d$). In order to define such a term we need to introduce some further notation. Denote by $\mathcal{V}_1 = \{\mathcal{C}_1(x)\}_{x\in\mathcal{L}_1}$ the Voronoi tessellation of $\mathbb{R}^d$ with respect to the volumetric points $\mathcal{L}_1$, and recall that

$$\mathcal{C}_1(x) := \{z \in \mathbb{R}^d : \ |z - x| \leqq |z - y| \quad \forall y \in \mathcal{L}_1\}.$$

We define the interior Voronoi cells by

$$\mathcal{V}_{1,\varepsilon}(D) = \{\mathcal{C}_1(x) \in \mathcal{V}_1 : \ T \subset D_\varepsilon \text{ for all } T \in \mathbb{T} \text{ such that } T \cap \mathcal{C}_1(x) \neq \emptyset\}.$$

The intuition behind these cells is that we want to define $H_{\mathrm{vol},\varepsilon}(D, \cdot)$ using only the volumetric points inside the domain D_ε. Given $u : D_\varepsilon^{\mathcal{L}} \to \mathbb{R}^d$ we denote by $u_{\mathrm{aff}} : \bigcup_{T\subset D_\varepsilon} T \to \mathbb{R}^d$ the continuous and piecewise affine interpolation with respect to the triangulation $\mathbb{T}$ and the values of $\{u(x)\}_{x\in\mathcal{L}_1}$. With these quantities at hand the volumetric term takes the form

$$H_{\mathrm{vol},\varepsilon}(D, u) = \sum_{\mathcal{C}_1(x)\in\mathcal{V}_{1,\varepsilon}(D)} |\mathcal{C}_1(x)| W\Big(\det_{\mathcal{C}_1(x)}(\nabla u_{\mathrm{aff}})\Big) \tag{1.2}$$

for some map W and the short-hand notation $\det_{\mathcal{C}_1(x)}(\nabla u_{\mathrm{aff}}) := \fint_{\mathcal{C}_1(x)} \det(\nabla u_{\mathrm{aff}})\,\mathrm{d}z$. By definition of the interior Voronoi cells, this sum is well-defined and, since u_{aff} is piecewise affine, the integrals over the Voronoi cells can be rewritten as finite sums. The random character of this Hamiltonian H_ε is encoded by $\mathbb{B}$ and $\mathbb{T}$ (which is a more descriptive notation of the actual event—a random graph—than the standard "ω"). We make three assumptions on the discrete energy densities f and W. The first set of assumptions is used for the general results.

Hypothesis 1. The functions $f : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}_+$ and $W : \mathbb{R} \to \mathbb{R}_+$ are (jointly) measurable, nonnegative and there exist a constant $C > 0$ and an exponent $p > 1$ such that for all $z \in \mathbb{R}^d, \xi, \zeta \in \mathbb{R}^n, \lambda \in \mathbb{R}$ we have the (two-sided) p-growth condition

$$\frac{1}{C}|\xi|^p - C \leqq f(z, \xi) \leqq C(1 + |\xi|^p), \quad 0 \leqq W(\lambda) \leqq C(1 + |\lambda|^{\frac{p}{d}}). \tag{1.3}$$

Some of our results require a slightly stronger set of assumptions.

Hypothesis 2. The function $f : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}_+$ is jointly measurable, $W : \mathbb{R} \to \mathbb{R}_+$ is continuous, and there exist a constant $C > 0$ and an exponent $p > 1$ such that for all $z \in \mathbb{R}^d$, $\xi, \zeta \in \mathbb{R}^n$, $\lambda, \lambda' \in \mathbb{R}$ we have the (two-sided) p-growth condition

$$\frac{1}{C}|\xi|^p - C \leqq f(z,\xi) \leqq C(1+|\xi|^p), \quad 0 \leqq W(\lambda) \leqq C(1+|\lambda|^{\frac{p}{d}}), \tag{1.4}$$

and the local Lipschitz conditions

$$\begin{aligned} |f(z,\xi) - f(z,\zeta)| &\leqq C|\xi - \zeta|(1+|\xi|^{p-1} + |\zeta|^{p-1}), \\ |W(\lambda) - W(\lambda')| &\leqq C|\lambda - \lambda'|(1+|\lambda|^{\frac{p}{d}-1} + |\lambda'|^{\frac{p}{d}-1}). \end{aligned} \tag{1.5}$$

If $W \not\equiv 0$, then we assume in addition that $n = d$ and $p \geqq d$.

The third set of assumptions is similar to Hypothesis 2, but is tuned for our applications to polymer physics, and exploits the specific form of the model.

Hypothesis 3. The function $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ is jointly measurable, $W : \mathbb{R} \to \mathbb{R}_+$ is continuous, and there exist an exponent $p \geqq d$ and constants $0 < C_p, C_2, C_p', C_2', C, C'$ such that for all $z \in \mathbb{R}^d$, $\xi, \zeta \in \mathbb{R}^d$, $\lambda, \lambda' \in \mathbb{R}$ we have the (two-sided) p-growth condition

$$C_2|\xi|^2 + C_p|\xi|^p \leqq f(z,\xi) \leqq C_2'|\xi|^2 + C_p'|\xi|^p, \quad 0 \leqq W(\lambda) \leqq C(1+|\lambda|^{\frac{p}{d}}), \tag{1.6}$$

and the local Lipschitz conditions

$$\begin{aligned} |f(z,\xi) - f(z,\zeta)| &\leqq |\xi - \zeta|(C_2'(|\xi| + |\zeta|) + C_p'(|\xi|^{p-1} + |\zeta|^{p-1})), \\ |W(\lambda) - W(\lambda')| &\leqq C'|\lambda - \lambda'|(1+|\lambda|^{\frac{p}{d}-1} + |\lambda'|^{\frac{p}{d}-1}). \end{aligned} \tag{1.7}$$

Note that the second condition in (1.7) follows automatically from (1.6) if W is assumed to be convex.

Under the above Hypotheses the passage from discrete Hamiltonians to continuum energies is well-understood *at zero temperature* (e.g. by Γ-convergence in [3,4,19]; see also [18] for results on local minimizers). In this paper we are interested in the asymptotic behavior of the free energy (that is, *at positive temperature*) when we prescribe boundary conditions. To this end, given $\varphi \in \mathrm{Lip}(D, \mathbb{R}^n)$ we define the class of states associated with φ at scale $\frac{1}{\varepsilon}$ as

$$\mathcal{B}_\varepsilon(D,\varphi) = \{u : D_\varepsilon \cap \mathcal{L} \to \mathbb{R}^n,\ |u(x) - \tfrac{1}{\varepsilon}\varphi(\varepsilon x)| < 1 \text{ if } \mathrm{dist}(x, \partial D_\varepsilon) \leqq C_0\}. \tag{1.8}$$

We denote by $\mathcal{V} := \{\mathcal{C}(x)\}_{x\in\mathcal{L}}$ the Voronoi tessellation of $\mathbb{R}^d$ associated with $\mathcal{L}$ (note that this is not necessarily the dual tessellation of $\mathbb{T}$—the latter is given by $\mathcal{V}_1$ which could and *will* be coarser). We shall identify functions of $\mathcal{B}_\varepsilon(D,\varphi)$ with their piecewise constant extensions on the union of Voronoi cells $\mathcal{C}(x)$ for $x \in D_\varepsilon \cap \mathcal{L}$.

The partition function $Z^{\beta}_{\varepsilon,D,\varphi}$ at inverse temperature $\beta > 0$ with boundary condition $\varphi \in \mathrm{Lip}(D, \mathbb{R}^n)$ is defined as

$$Z^{\beta}_{\varepsilon,D,\varphi} := \int_{\mathcal{B}_\varepsilon(D,\varphi)} \exp(-\beta H_\varepsilon(D,u))\,\mathrm{d}u, \tag{1.9}$$

where the integration is understood in the sense of the product measure $\mathrm{d}u = \prod_{x_j \in D_\varepsilon \cap \mathcal{L}} \mathrm{d}u(x_j)$, whereas the Helmholtz free energy writes

$$\mathcal{E}^{\beta}_{\varepsilon}(D,\varphi) := -\frac{1}{\beta|D_\varepsilon|} \log\big(Z^{\beta}_{\varepsilon,D,\varphi}\big). \tag{1.10}$$

We conclude this section by the definition of the Gibbs measure. For all $\varepsilon > 0$ and $v \in L^p(D)$, we introduce the rescaled version $u := \Pi_{1/\varepsilon} v$ of v as

$$\Pi_{1/\varepsilon} v : D_\varepsilon \to \mathbb{R}^n,\, z \mapsto \tfrac{1}{\varepsilon} v(\varepsilon z).$$

We define the Gibbs measure $\mu^{\beta}_{\varepsilon,D,\varphi}$ at temperature β associated with the Hamiltonian $H_\varepsilon(D,\cdot)$ and the boundary condition φ as the probability measure on $L^p(D,\mathbb{R}^n)$ characterized by

$$L^p(D) \ni V \mapsto \mu^{\beta}_{\varepsilon,D,\varphi}(V) := \frac{1}{Z^{\beta}_{\varepsilon,D,\varphi}} \int_{\Pi_{1/\varepsilon} V \cap \mathcal{B}_\varepsilon(D,\varphi)} \exp(-\beta H_\varepsilon(D,u))\,\mathrm{d}u, \tag{1.11}$$

where we divided by the partition function to ensure that $\mu^{\beta}_{\varepsilon,D,\varphi}(L^p(D)) = 1$ (see Section 5 for a rigorous definition). The main aim of this article is to study the thermodynamic limit of $\mathcal{E}^{\beta}_{\varepsilon}(D,\varphi)$ and $\mu^{\beta}_{\varepsilon,D,\varphi}$, that is their asymptotic behavior as $\frac{1}{\varepsilon} \uparrow \infty$ (large-volume limit).

In all the results to come, and in the proofs, quantities of interest are random variables (or random measures or functionals). As such, they depend on the realization of the random graph. We do not make this dependence explicit in the notation, except when it is strictly necessary (in which case we put an additional argument, e.g. we write $H_\varepsilon(D,u,G)$ instead of $H_\varepsilon(D,u)$).

1.3. Thermodynamic Limit

The following analysis is essentially an extension to general random graphs of the inspiring results [34] by Kotecký and Luckhaus on the $\mathbb{Z}^d$ lattice. We start with the convergence of the Helmholtz free energy for linear boundary conditions $\overline{\varphi}_\Lambda : x \mapsto \Lambda x$ and the definition of the limiting (free) energy density of the continuum hyperelastic model.

Theorem 1.3. *Assume Hypothesis 1. Then for all $\beta > 0$ there exists a deterministic quasiconvex function $\overline{W}^{\beta} : \mathbb{R}^{n\times d} \to \mathbb{R}$ satisfying the two-sided p-growth condition*

$$\forall \Lambda \in \mathbb{R}^{n\times d}: \qquad \frac{1}{C}|\Lambda|^p - C \leqq \overline{W}^{\beta}(\Lambda) \leqq C(1+|\Lambda|^p),$$

and for all bounded Lipschitz domains $D \subset \mathbb{R}^d$, the Helmholtz free energy defined in (1.10) *satisfies almost surely*

$$\forall \Lambda \in \mathbb{R}^{n\times d}: \qquad \lim_{\varepsilon\downarrow 0} \mathcal{E}_\varepsilon^\beta(D, \overline{\varphi}_\Lambda) = \overline{W}^\beta(\Lambda).$$

Theorem 1.3 is only an existence result. It is essentially of no use for numerical purposes. Part of the rest of the present analysis is dedicated to the justification of approximations of $\overline{W}^\beta$ that are "computable" numerically. This concerns the small temperature limit (cf. Theorem 1.6 below) and the case of a quadratic Hamiltonian (cf. Section 6.4 below).

The extension of this result to general boundary conditions $\varphi \in \mathrm{Lip}(D, \mathbb{R}^n)$ is as follows, and implies the convergence of the Helmholtz free energy to the infimum of an energy functional associated with the free energy density $\overline{W}^\beta$—a continuum hyperelastic model.

Theorem 1.4. *Assume Hypothesis 1 and for all $\beta > 0$, let $\overline{W}^\beta$ be the well-defined energy density of Theorem 1.3. Then for all bounded Lipschitz domains $D \subset \mathbb{R}^d$ we have almost surely for all boundary conditions $\varphi \in \mathrm{Lip}(D, \mathbb{R}^n)$*

$$\lim_{\varepsilon\downarrow 0} \mathcal{E}_\varepsilon^\beta(D, \varphi) = \inf\left\{ \fint_D \overline{W}^\beta(\nabla u(x))dx \;:\; u \in \varphi + W_0^{1,p}(D)\right\},$$

where $\fint_D$ is a short-hand notation for $\frac{1}{|D|}\int_D$.

We conclude the study of the thermodynamic limit by establishing a large-deviation principle which ensures that the Gibbs measure concentrates as $\varepsilon \downarrow 0$ on states that minimize the energy functional associated with $\overline{W}^\beta$ in the set of continuum deformations that satisfy the boundary condition φ. For a general introduction to the subject we refer to [24].

Theorem 1.5. *Assume Hypothesis 1 and for all $\beta > 0$, let $\overline{W}^\beta$ be the well-defined energy density of Theorem 1.3. Then for all bounded Lipschitz domains $D \subset \mathbb{R}^d$, almost surely, and for all boundary conditions $\varphi \in \mathrm{Lip}(D, \mathbb{R}^n)$, the measure $\mu^\beta_{\varepsilon,D,\varphi}$ satisfies a strong large deviation principle with speed $(\beta|D_\varepsilon|)^{-1}$ and good rate functional $\mathcal{I}^\beta_{D,\varphi} : L^p(D, \mathbb{R}^n) \to [0, +\infty]$ finite only on $\varphi + W_0^{1,p}(D, \mathbb{R}^n)$ and characterized by*

$$\varphi + W_0^{1,p}(D, \mathbb{R}^n) \ni u \mapsto \mathcal{I}^\beta_{D,\varphi}(u) := \fint_D \overline{W}^\beta(\nabla u(x))\,\mathrm{d}x - \inf_{v\in\varphi+W_0^{1,p}(D,\mathbb{R}^n)} {}_D\, \overline{W}^\beta(\nabla v(x))\,\mathrm{d}x.$$

More precisely, for every open and closed sets $U \subset L^p(D, \mathbb{R}^n)$ and $V \subset L^p(D, \mathbb{R}^n)$, we have

$$\liminf_{\varepsilon\downarrow 0} \frac{1}{|D_\varepsilon|} \log(\mu^\beta_{\varepsilon,D,\varphi}(U)) \geqq -\inf_{u\in U} \mathcal{I}^\beta_{D,\varphi}(u),$$

$$\limsup_{\varepsilon\downarrow 0} \frac{1}{|D_\varepsilon|} \log(\mu^\beta_{\varepsilon,D,\varphi}(V)) \leqq -\inf_{u\in V} \mathcal{I}^\beta_{D,\varphi}(u).$$

Remark 1. In the scalar case in some regimes the strict convexity of $\overline{W}^{\beta}$ is known (see, e.g., [1,21,22,25]). Given such a result the large deviation principle immediately implies that the Gibbs measures converge to the Dirac measure supported on the unique minimizer of the rate functional. However, in our vectorial setting, in general we don't even expect convexity of $\overline{W}^{\beta}$; see also [34, Lemma 8].

Remark 2. From a continuum mechanics point of view the energy density $\overline{W}^{\beta}$ should be frame-indifferent, that is, $\overline{W}^{\beta}(R\Lambda) = \overline{W}^{\beta}(\Lambda)$ for all deformation gradients $\Lambda \in \mathbb{R}^{n\times d}$ and all rotations $R \in SO(n)$. Our model yields a frame-indifferent energy density whenever the discrete interactions are of the form $f(z,\xi) = \tilde{f}(z,|\xi|)$ for some function $\tilde{f} : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$. Indeed, in this case one can use the change of variables $\varphi \mapsto R\varphi$ to show that $\mathcal{E}^{\beta}_{\varepsilon}(D,\overline{\varphi}_{R\Lambda}) = \mathcal{E}^{\beta}_{\varepsilon}(D,\overline{\varphi}_{\Lambda})$, which implies frame-indifference of the limit. In case of rubber elasticity it is also customary to assume that $\overline{W}^{\beta}$ is isotropic, that is, $\overline{W}^{\beta}(\Lambda R) = \overline{W}^{\beta}(\Lambda)$ for all deformation gradients $\Lambda \in \mathbb{R}^{n\times d}$ and all rotations $R \in SO(d)$. In order for $\overline{W}^{\beta}$ to be isotropic it is enough to assume that the discrete interactions are of the form $f(z,\xi) = \widehat{f}(|z|,\xi)$ for some function $\widehat{f} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and that in addition the graph G is isotropic in law (that is, the random variable $RG := (R\mathcal{L}, R\mathbb{B}, R\mathbb{T})$ has the same (joint) distribution as G for all $R \in SO(d)$). Under this assumptions $\overline{W}^{\beta}$ is isotropic. The proof of this result, which proceeds as that of [4, Theorem 9] up to minor modifications, uses a change of variables in probability in the formula

$$\overline{W}^{\beta}(\Lambda R) = \lim_{\varepsilon\downarrow 0} \mathbb{E}[\mathcal{E}^{\beta}_{\varepsilon}(B_1(0),\overline{\varphi}_{R\Lambda})],$$

where $B_1(0)$ denotes the unit ball. The following two facts should be stressed. First, from a modeling point of view the isotropy of the polymer network is a natural assumption (otherwise no isotropy of the model should be expected). Second, the assumptions above do not involve the volumetric part (which plays a role in our models only in the case $n = d$) since it has the correct invariant structure by definition.

In the following section we complete the study of the thermodynamic limit by analyzing the behavior of $\overline{W}^{\beta}$ and $\mathcal{I}^{\beta}_{D,\varphi}$ when the temperature tends to zero, that is in the regime $\beta \uparrow \infty$.

1.4. Zero-Temperature Limit

A natural guess for the zero-temperature limit of the Helmholtz free energy $\overline{W}^{\beta}$ is the large-volume limit of the infimum of the Hamiltonian $H_{\varepsilon}(D,\cdot)$. The following result, which is new even for periodic lattices, establishes rigorously the Γ-convergence of the rate functional $\mathcal{I}^{\beta}_{D,\varphi}$ towards the Γ-limit of the discrete Hamiltonian studied in [4], and therefore indeed implies the commutation of the limits $\beta \uparrow +\infty$ and $\varepsilon \downarrow 0$. In a nutshell, this relates the large deviation principle to Γ-convergence of the Hamiltonian at vanishing temperature. Note that we require the stronger Hypothesis 2.

Theorem 1.6. *Assume Hypothesis 2 and for all $\beta > 0$, let $\overline{W}^{\beta}$ be the well-defined energy density of Theorem 1.3 and for all Lipschitz domains D and boundary conditions $\varphi \in \mathrm{Lip}(D, \mathbb{R}^n)$, let $\mathcal{I}^{\beta}_{D,\varphi} : L^p(D, \mathbb{R}^n) \to [0, +\infty]$ be the rate functional of Theorem 1.5. Then, as $\beta \uparrow +\infty$, $\mathcal{I}^{\beta}_{D,\varphi}$ almost-surely $\Gamma(L^p)$-converges towards the integral functional $\mathcal{I}^{\infty}_{D,\varphi} : L^p(D, \mathbb{R}^n) \to [0, +\infty]$ finite only on $\varphi + W^{1,p}_0(D, \mathbb{R}^n)$ and characterized by*

$$\varphi + W^{1,p}_0(D, \mathbb{R}^n) \ni u \mapsto \mathcal{I}^{\infty}_{D,\varphi}(u) := \fint_D \overline{W}^{\infty}(\nabla u(x))\, \mathrm{d}x - \inf_{v \in \varphi + W^{1,p}_0(D,\mathbb{R}^n)} \fint_D \overline{W}^{\infty}(\nabla v(x))\, \mathrm{d}x,$$

where $\overline{W}^{\infty}$ is an almost-surely well-defined quasiconvex energy density satisfying the two-sided growth condition

$$\forall \Lambda \in \mathbb{R}^{d\times n} : \qquad \frac{1}{C}|\Lambda|^p - C \leqq \overline{W}^{\infty}(\Lambda) \leqq C(|\Lambda|^p + 1), \tag{1.12}$$

and given for all $\Lambda \in \mathbb{R}^{n\times d}$ by

$$\overline{W}^{\infty}(\Lambda) := \lim_{\varepsilon \downarrow 0} \inf_{u \in \mathcal{B}_\varepsilon(D', \varphi_\Lambda)} \frac{1}{|D'_\varepsilon|} H_\varepsilon(D', u)$$

for any Lipschitz bounded domain $D' \subset \mathbb{R}^d$, where $\mathcal{B}_\varepsilon(D', \varphi_\Lambda)$ is defined in (1.8). In addition, for all $\Lambda \in \mathbb{R}^{n\times d}$,

$$|\overline{W}^{\infty}(\Lambda) - \overline{W}^{\beta}(\Lambda)| \leqq \frac{\log \beta}{\beta} C\big(1 + |\Lambda|^{p-1}\big). \tag{1.13}$$

Remark 3. Theorem 1.6 implies in particular that the minimizers of the rate functionals $\mathcal{I}^{\beta}_{D,\varphi}$ given by Theorem 1.5 at inverse temperature β converge weakly in $W^{1,p}(D, \mathbb{R}^n)$ to minimizers of $I^{\infty}_{D,\varphi}$. Moreover, due to equicoercivity of both functionals, from [12, Proposition 1.18] we infer that for every open and closed sets $U \subset L^p(D, \mathbb{R}^n)$ and $V \subset L^p(D, \mathbb{R}^n)$

(i) $\limsup_{\beta\uparrow+\infty} \inf_{u\in U} \mathcal{I}^{\beta}_{D,\varphi}(u) \leqq \inf_{u\in U} \mathcal{I}^{\infty}_{D,\varphi}(u)$;
(ii) $\liminf_{\beta\uparrow+\infty} \inf_{u\in V} \mathcal{I}^{\beta}_{D,\varphi}(u) \geqq \inf_{u\in V} \mathcal{I}^{\infty}_{D,\varphi}(u)$.

Those inequalities allow to pass to the limit $\beta \to +\infty$ in the inequalities of the large deviation principle. For quadratic functionals we shall prove a much stronger statement in Corollary 2, namely the limit free energy and the density of the Γ-limit differ only by a β-dependent constant (which does not affect minimization). This provides a rigorous justification of the so-called phantom model (for which the free energies of polymer-chains are assumed to be Gaussian), an elementary linear model of polymer physics (see e.g. [43, Section 7.2.2]).

Remark 4. If Hypothesis 2 is replaced by Hypothesis 3, the conclusion (1.13) can be strengthened to

$$|\overline{W}^{\infty}(\Lambda) - \overline{W}^{\beta}(\Lambda)| \leqq \frac{\log \beta}{\beta}\big(C_2''|\Lambda|^2 + C_p''|\Lambda|^p + d\big), \tag{1.14}$$

for some C_2'' and C_p'' depending on d, p, C, C', C_2, C_2', C_p, C_p'.

1.5. Relation to the Literature

This contribution belongs to the large body of literature that aims at deriving macroscopic models from microscopic descriptions of solids. To classify these works, one needs to distinguish between

- solids: crystals versus disordered solids;
- geometric description: Eulerian (in which case a point set is enough to describe the Hamiltonian) or Lagrangian (in which case, one needs a graph on top of a point set to describe the Hamiltonian);
- interactions: long-range interactions (typically via a two-body Lennard–Jones potential) or short-range;
- temperature: ground-states or Gibbs measures.

For crystalline solids (like metals), typical questions concern crystallization, the Cauchy–Born rule, phase transitions, etc. Despite many contributions and much progress, there is still yet no complete picture on how to pass from quantum mechanical descriptions to linear elasticity (and plasticity, etc.). The mathematical tools developed in this context are completely different from the tools used in the present contribution, which makes these works and the present work quite unrelated.

Disordered solids—like rubber—have received much less attention from the mathematical community than crystals (for related contributions from the physics and engineering literature, we refer the reader to Section 2). The twist for such models is that the microscopic description is not that of the atoms and of the quantum world, but the statistical physics of random graphs. At zero temperature this reduces the problem to taking a large-scale limit of a ground state of some graph. Such problems are reminiscent of the homogenization theory, and their study using the framework of Γ-convergence was pioneered by Braides and collaborators [14–17], etc. For periodic lattices, the first result treating Hamiltonians of the form considered here at zero temperature was obtained in [3], whereas the case of positive temperature was successfully analyzed by Kotecký and Luckhaus in [34], as we already mentioned. Such Hamiltonians are however not meant to describe crystalline solids at the atomic length scale, and are more reminiscent of Hamiltonians used in polymer physics, which is however the realm of disordered materials rather than crystals. In this context, the work [3] was extended to stochastic lattices in [4], whereas the extension of [34] is the aim of the present contribution.

1.6. Structure of the Paper and of the Proofs

Assume momentarily that the volumetric term $W \equiv 0$ vanishes. Our main two results are Theorems 1.5 and 1.6.

The general structure of the proof of Theorem 1.5 follows that of Kotecký and Luckhaus in [34], and adaptations are mainly technical (due to the randomness and the general structure of the graph). One of the most important quantities to study is the localized partition function defined for a Lipschitz domain $O \subset \mathbb{R}^d$ and for subsets of deformations $V \subset \{u : O_\varepsilon^{\mathcal{L}} \to \mathbb{R}^n\}$ by

$$Z^{\beta}_{\varepsilon,O}(V) := \int_V \exp(-\beta H_\varepsilon(O,u))\,\mathrm{d}u.$$

For $v \in L^p_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{R}^n)$ and $O \in \mathcal{A}^R(\mathbb{R}^d)$ we then define

$$\mathcal{F}^-_\kappa(O,v) = \liminf_{\varepsilon\downarrow 0} -\frac{1}{\beta|O_\varepsilon|}\log(Z^\beta_{\varepsilon,O}(\mathcal{N}_p(v,O,\varepsilon,\kappa))),$$
$$\mathcal{F}^+_\kappa(O,v) = \limsup_{\varepsilon\downarrow 0} -\frac{1}{\beta|O_\varepsilon|}\log(Z^\beta_{\varepsilon,O}(\mathcal{N}_p(v,O,\varepsilon,\kappa))),$$

where

$$\mathcal{N}_p(v,O,\varepsilon,\kappa) := \{u : O_\varepsilon^{\mathcal{L}} \to \mathbb{R}^n,\ \sum_{O_\varepsilon^{\mathcal{L}}} \varepsilon^d |v_\varepsilon(x) - \varepsilon u(x)|^p < \kappa^p |O|^{1+\frac{p}{d}}\}$$

is a rescaled discrete L^p-neighborhood of some discrete approximation of v (see Section 3.1 for details). Both quantities are decreasing in κ, so that we may consider their limits as $\kappa \downarrow 0$

$$\mathcal{F}^-(O,v) = \lim_{\kappa\to 0}\mathcal{F}^-_\kappa(O,v),$$
$$\mathcal{F}^+(O,v) = \lim_{\kappa\to 0}\mathcal{F}^+_\kappa(O,v).$$

In view of [24, Theorem 4.1.11] these quantities (if equal) are a natural candidate for the rate functional of a large deviation principle for the Gibbs measures—except that they do not take into account the boundary values. Incidentally, notice that if we would replace the integral in the localized partition function by the infimum of the Hamiltonian over the set V, the quantities $\mathcal{F}^-(O,v)$ and $\mathcal{F}^+(O,v)$ would coincide with the $\Gamma(L^p(D))$-liminf and $\Gamma(L^p(D))$-limsup, respectively, of the rescaled Hamiltonian $\tilde{H}_\varepsilon : L^p(D,\mathbb{R}^n) \to [0,+\infty]$ defined by

$$\tilde{H}_\varepsilon(v,O) = \begin{cases} \dfrac{1}{|O_\varepsilon|}\displaystyle\sum_{\substack{(x,y)\in\mathbb{B}\\ \varepsilon x,\varepsilon y\in O}} f\left(x-y, \dfrac{v(\varepsilon x)-v(\varepsilon y)}{\varepsilon}\right) & \text{if } v \in \mathcal{PC}_\varepsilon. \\ +\infty & \text{otherwise,}\end{cases} \tag{1.15}$$

where $\mathcal{PC}_\varepsilon$ denotes a suitable class of piecewise constant functions that can be identified with functions $v : \varepsilon\mathcal{L} \to \mathbb{R}^n$ (see Section 6.1 where we make this connection rigorous in the small temperature regime).

Let us now describe the main steps of the proof of the main results and the related flow of lemmas.

Section 3: Preliminary estimates. In this section we extend some auxiliary results of [34] to the setting of random graphs. More precisely,

- We first prove a discrete Poincaré inequality on bounded subsets of an admissible graph $(\mathcal{L}, E)$ for functions with zero boundary values (Lemma 3.1).
- In Lemmata 3.2 and 3.3 we obtain a control on the localized partition function $Z^{\beta}_{\varepsilon,O}(V)$ for sets V of deformations satisfying a discrete boundary condition. This estimate is particularly useful because it scales with the size of the reference set O. (The only difference with [34, Lemma 12] is that the restriction of the graph to $O^{\mathcal{L}}_{\varepsilon}$ might not be connected in the random setting.)
- Lemma 3.4 (similar to [34, Lemma 1]) shows that one can neglect deformations with large energy in the computation of the localized partition function. More precisely, this estimate will imply an exponential tightness on the sublevel sets of the Hamiltonian and allows to restrict most of the analysis to deformations with a uniformly bounded discrete p-Dirichlet energy.
- The final technical ingredient, Proposition 1 (similar to [34, Lemma 2]), is an interpolation inequality which is to large deviation principles what the so-called 'fundamental estimate' is to homogenization of integral functionals by Γ-convergence (cf. [13, Chapter 11], and Proposition 11.7 therein). In a nutshell this result ensures that one can compute the quantities $\mathcal{F}^{\pm}_{\kappa}(O, v)$ either with or without an imposed soft boundary condition as long as the (rescaled) boundary condition is L^p-close to the function v. For the reader's convenience we display in the appendix a proof of that technical result that we hope to be slightly more transparent than the original version presented in [34].

Section 4: Definition and properties of the Helmholtz free energy. This section is dedicated to the analysis of the Helmholtz free energy defined in (1.10) with linear boundary conditions. This is a necessary step to be able to treat more general boundary conditions by localizing the partition function via a suitable partition of the reference set D (that allows to treat deformations locally as affine functions). More precisely, defining the linear deformation of the boundary as the linear map $x \mapsto \Lambda x$:

- We first prove the almost sure (with respect to the randomness of the graph) existence of the limit of the Helmholtz free energy $\mathcal{E}^{\beta}_{\varepsilon}(D, \overline{\varphi}_{\Lambda})$ as $\varepsilon \to 0$, that this limit $\overline{W}^{\beta}(\Lambda)$ is deterministic, and that it does not depend on the reference set D (cf. Proposition 2). To this end, we replace the classical (deterministic) subadditivity arguments used in [34] by the subadditive ergodic theorem [2].
- In Lemma 4.1 we give several equivalent formulas for the limit $\overline{W}^{\beta}(\Lambda)$ of the Helmholtz free energy, that will be convenient in different steps of the proof. On the one hand, we show that we can restrict the class of deformations to any discrete L^p-ball centered at the linear map $x \mapsto \Lambda x$ rather than only imposing this deformation at the boundary. On the other hand, we prove that $\overline{W}^{\beta}(\Lambda) = \mathcal{F}^{+}(O, \overline{\varphi}_{\Lambda})$, which provides a formula that only takes into account deformations that are L^p-close to the linear deformation but this time without imposing boundary conditions. Again the main difference with [34, Lemma 3] is the use of the ergodic theorem.
- The first significant difference in this random setting comes with Proposition 3, which is related to null sets. Since we want to use the values $\overline{W}^{\beta}(\Lambda)$ to

reconstruct the limit of the free energy for general boundary conditions, we have to ensure that we do not take an uncountable union of null sets when applying the ergodic theorem to linear deformations Λ. This requires to establish some uniform stability of $\mathcal{E}_\varepsilon^\beta(D, \overline{\varphi}_\Lambda)$ with respect to Λ as $\varepsilon \to 0$.

- Lemma 4.2 establishes the p-growth of $\Lambda \mapsto \overline{W}^\beta(\Lambda)$ from above (which follows again from the subadditive ergodic theorem). Lemma 4.3 deals with the p-growth from below (and indeed quantifies the statement of [34, Lemma 3 (c)]). This is where the geometric assumptions (i)–(iv) of Definition 1.2 come into play, which is the price to pay to consider general graphs as we do here.
- Next we show that the functions $v \mapsto \mathcal{F}^\pm(O, v)$ are $L^p(O)$-lower semicontinuous (Lemma 4.4).
- Finally, in Theorem 4.5 we establish the identity

$$\mathcal{F}^-(D, v) = \mathcal{F}^+(D, v) = \fint_D \overline{W}^\beta(\nabla v)\,\mathrm{d}x, \tag{1.16}$$

which then implies the quasiconvexity of the map $\Lambda \mapsto \overline{W}^\beta(\Lambda)$ (that is Theorem 1.3) by the previously proven lower semicontinuity results. For the upper bound $\mathcal{F}^+(D, v) \leqq \fint \overline{W}^\beta(\nabla v)\,\mathrm{d}x$ we may restrict the analysis to piecewise affine function by a density argument using the continuity and p-growth conditions of the map $\Lambda \mapsto \overline{W}^\beta(\Lambda)$ as well as the lower semicontinuity of the LHS established above. At this point it is crucial to put additional soft boundary conditions on the boundary of each triangle on which the macroscopic deformation is affine to make the partition function almost superadditive. Here we use the alternative characterization of $\overline{W}^\beta(\Lambda)$ from Lemma 4.1. The lower bound $\mathcal{F}^-(D, v) \geqq \fint_D \overline{W}^\beta(\Lambda)\,\mathrm{d}x$ is achieved via blow-up which allows to treat v locally as an affine function. Although the basic idea is the same as in [34], the disorder of the graph introduces nontrivial additional boundary terms. Note that the equality (1.16) is reminiscent of a Γ-convergence result without boundary conditions.

Section 5: Proof of the large deviation principle. With the identity (1.16) at hand the large deviation principle for the Gibbs measure is rather standard (cf. [24, Theorem 4.1.11]) using that the interpolation estimate (Proposition 1) allows one to remove or impose boundary conditions without changing the value of the logarithm of the partition function too much. Again, note the similarity with Γ-convergence problems in terms of addition of boundary conditions once the Γ-limit is known and a fundamental estimate is available (e.g. [13, Proposition 11.7]).

- In Lemma 5.1 we show that rescaled sublevel sets of the Hamiltonian are compact in L^p (which relies on the discrete Poincaré inequality). Combined with Lemma 3.4 this yields the exponential tightness of the Gibbs measures (cf. Lemma 5.2) which allows to show the upper bound of the large deviation principle for compact sets.
- We conclude by proving Theorem 1.5 following the standard approach up to some minor modifications (our topological neighborhoods indeed depend on ε). As a corollary we deduce Theorem 1.4.

We then turn to Theorem 1.6.

Section 6: The small temperature limit. From the mathematical point of view, this section contains the main novelty of this contribution. It relates the rate functional of the large deviation principle to the Γ-limit of the rescaled Hamiltonians $\tilde{H}_\varepsilon$ defined in (1.15) in the small temperature regime. This makes a rigorous connection (for the models under consideration) between the variational and the statistical physics approaches (which should not come as a surprise neither from a physical point of view, nor from the mathematical side in view of the similarities between large deviation principles and Γ-convergence pointed out above in this context).

- In Theorems 6.1 and 6.2 we first briefly recall the variational results on the Hamiltonian proven in [4].
- in Lemma 6.3 we argue that the density of the Γ-limit can be calculated either with clamped boundary conditions or the softer ones considered for the Gibbs measures.
- In Lemmata 6.4 and 6.5 we prove upper and lower bounds for the difference $\overline{W}^{\beta}(\Lambda) - \overline{W}^{\infty}(\Lambda)$, where $\overline{W}^{\infty}(\Lambda)$ denotes the density of the Γ-limit of the rescaled Hamiltonians. From those bounds Theorem 1.6 easily follows.
- We conclude this section by considering the so-called phantom model which corresponds to a quadratic Hamiltonian. In Corollary 2 we prove that in this case the density of the Γ-limit differs from the limit free energy $\overline{W}^{\beta}(\Lambda)$ only by a temperature-dependent constant. Hence minimizing the rate functional or the Γ-limit yields the same optimal deformation.

Section 7: Volumetric effects In this last section we show how to incorporate the volumetric term in the analysis, which was not considered in [34] and is crucial for our applications to polymer physics. More precisely,

- in Lemma 7.1 we prove a local upper bound for the volumetric part in terms of finite differences on the graph while in Lemma 7.2 we show that also the volumetric term leads to a stationary Helmholtz free energy when we impose linear boundary conditions.
- Lemma 7.3 provides the global continuity estimate of the volumetric part of the Hamiltonian that was needed to obtain the estimates in the small temperature regime.

2. From Polymer Physics to Rubber Elasticity

2.1. Continuum Mechanics and Phenomenology

Rubber-like materials are the realm of continuum mechanics and constitute the paradigmatic example of hyperelastic materials at large deformations—that is, their energy density and stress tensor only depend locally on the gradient of deformation.

2.1.1. Kinematics and Hyperelasticity Consider a piece of material that occupies a domain D at rest, and which is deformed according to some map $u : D \to \mathbb{R}^3$

(in Lagrangian coordinates). The energy of the deformed configuration then takes the form

$$\mathcal{I}(D,u) := \int_D \overline{W}(\nabla u(x))\,\mathrm{d}x,$$

where $\overline{W} : \mathbb{R}^{3\times 3} \to [0,+\infty]$, $\Lambda \mapsto \overline{W}(\Lambda)$ is the energy density of the material (minimal at $\Lambda = \mathrm{Id}$), and is referred to as its constitutive law. The associated Piola stress tensor is given by $D_\Lambda \overline{W}(\Lambda)$. A crucial *physical* requirement on the map $\overline{W}$ is frame-indifference, that is, for all rotations $R \in SO_3(\mathbb{R})$ and deformation gradients $\Lambda \in \mathbb{R}^{3\times 3}$, $\overline{W}(R\Lambda) = \overline{W}(\Lambda)$. Rubber materials are also usually isotropic, which reads as follows on $\overline{W}$: For all rotations $R \in SO_3(\mathbb{R})$ and deformation gradients $\Lambda \in R^{3\times 3}$, $\overline{W}(\Lambda R) = \overline{W}(\Lambda)$. Finally, rubber materials are nearly-incompressible, which typically requires that $\overline{W}(\Lambda)$ gets large when $|\det \Lambda - 1| \gg 1$, and should not allow interpenetration of matter, which at least imposes that $\overline{W}(\Lambda) = +\infty$ if $\det \Lambda \leqq 0$. For a given deformation $\varphi : \partial D \to \mathbb{R}^3$ of the boundary, the piece of deformed material (that occupied D in the reference configuration) has now energy

$$\mathcal{E}(\varphi) := \inf \left\{ \int_D \overline{W}(\nabla u) \,|\, u : D \to \mathbb{R}^3,\, u|_{\partial D} \equiv \varphi \right\}, \tag{2.1}$$

and its deformation is given by the minimizer of this functional (if attained and unique). We refer to [20] for classical mathematical aspects of nonlinear elasticity. Standard mechanical experiments illustrate the complexity of the nonlinear response of these materials at large deformations—see Fig. 1 for the Treloar experiments in uniaxial traction [45].

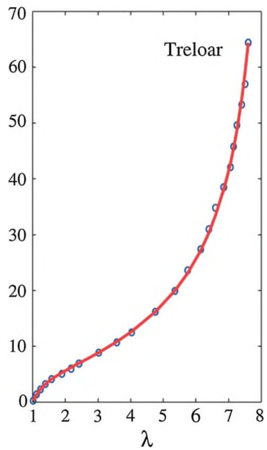
2.1.2. Phenomenological Constitutive Laws and Their Limitations The choice of the energy density $\overline{W}$ depends on the actual material considered. The energy densities used in practice in applied mechanics and in the rubber industry are phenomenological—see e.g. the survey [6] on constitutive laws for rubber. Deriving suitable constitutive laws in applied mechanics and engineering remains a hot topic in the field. A fair statement is that discrepancies between experiments and numerical simulations are due to the choice of the constitutive laws rather than the numerical accuracy achieved. Discrepancies are not only quantitative but also qualitative. An example of such a qualitative discrepancy is the so-called Rivlin effect. For uniaxial deformations

$$\Lambda_\lambda := \mathrm{diag}(\lambda, \lambda^{-1/2}, \lambda^{-1/2})$$

with $\lambda > 0$, consider the Mooney plot

$$\lambda \mapsto M(\lambda) = \frac{\sigma_{11}(\lambda) - \sigma_{22}(\lambda)}{2(\lambda^2 - 1/\lambda)},$$

where σ is the Cauchy stress tensor (that is, $\nabla \overline{W}$ written in the deformed configuration). A material displays the Rivlin effect if this map is concave around $\lambda = 1$. Rubber materials generically exhibit such a Rivlin effect, see for instance Figure 9 in [42]. However, for all of the constitutive laws listed in [6], the map M is convex



A sample of rubber is submitted to prescribed linear deformation at its boundary: $\varphi_\Lambda : x \mapsto \Lambda \cdot x$, for $\Lambda = \text{diag}(\lambda, \lambda^{-\frac{1}{2}}, \lambda^{-\frac{1}{2}})$ and $\lambda \in [1, 7.5]$.

Three regimes:

- linear response ($|\lambda - 1| \ll 1$),
- strain softening ($2 \leq \lambda \leq 4$),
- strain hardening ($\lambda \geq 6$).

Fig. 1. Treloar experiments in unaxial traction (engineering stress versus λ)

around $\lambda = 1$, which shows that something is missing in the phenomenological understanding of rubber in this regime. The common interpretation is that phenomenological models at the continuum level are missing physical insight, which is however only available at the scale of the polymer-chain network. This raises the question: How can one upscale polymer physics models to the continuum level in a *quantitative* way?

2.1.3. Towards Constitutive Laws Based on Polymer Physics Before we turn more thoroughly to polymer physics, let us quickly describe some specific approaches to upscaling. So far, all these approaches take as a "discrete model" a network of elastic springs (the elasticity of which is reminiscent of that of a polymer chain at a given temperature), and propose a way to upscale it. Some further assumptions are made to that end. In some works, an additional phenomenological assumption is made at the discrete level to be able to explicitly upscale the model (e.g. [5,33,38,44] to cite a few). In some other works [10,11], a numerical model is introduced based on finite elements, which imposes that the discrete network be a Delaunay tessellation, and does not give rise to an effective model (no limit is taken). From the *mathematical* point of view, none of these works are satisfactory: they either shift the phenomenological assumptions from the continuum scale to the discrete scale, or they remain at a discrete level (finite elements e.g.). The model which has least phenomenological assumptions is the two-temperature model introduced, analyzed, and numerically investigated in [4,30]. As in [10,11], it is based on a simplified polymer physics model. But as opposed to [10,11], the "thermodynamic limit" thereof is established (see below for details), and gives rise to a continuum model. Incidentally, the numerical simulations of the energy density associated with the two-temperature model display the desired Rivlin effect, cf. [30].

This state-of-the-art of constitutive laws for rubber-like materials constitutes the starting point of the series of works [4,26,30,31] in the field. We believe that rigorous upscaling methods can be of added value to the *quantitative* and *practical* modeling of rubber elasticity. In particular, although the two-temperature model of [4,30] yields promising results, it does not appear yet as a consistent approximation of a consensual polymer physics model. Our analysis (and in particular Theorem 1.6) establishes this result in the regime of large number of monomers per polymer-chain (which will play the role of an effective inverse temperature)—see details in Section 2.3. In order to draw the link between the analytical results of this paper and the derivation of actual rubber-like materials from actual polymer physics models, we need to give some background on polymer physics.

2.2. Polymer Physics

Rubber-like materials are also the realm of the statistical physics of polymer-chain networks and constitute the paradigmatic example of materials for which elastic properties are purely entropic—that is, they are only due to thermal fluctuations. The following constitutes a gentle introduction to polymer physics with the thermodynamic limit in mind.

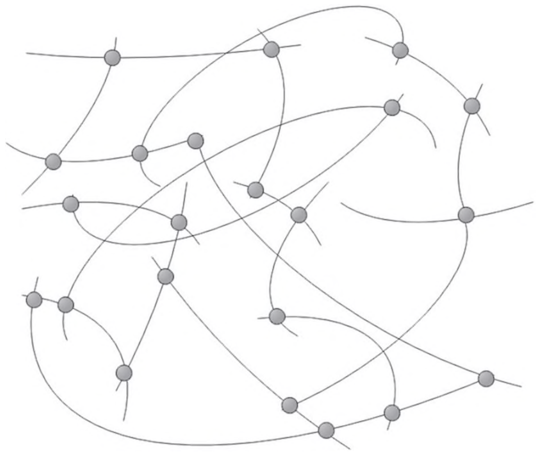


Fig. 2. Polymer-chain network (balls represent cross-links, lines represent polymer-chains)

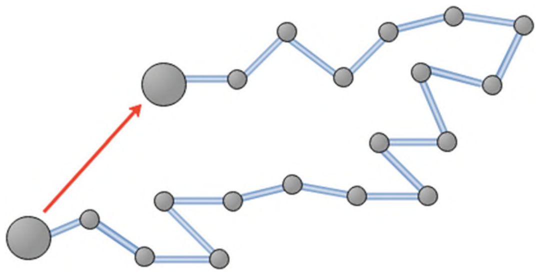


Fig. 3. Polymer-chain: end-to-end vector (red arrow) between two cross-links (large balls) and monomers (thick edges)

2.2.1. The Network We first start with some vocabulary. A network of cross-linked polymer-chains is a set of polymer-chains attached to each other through cross-links, cf. Fig. 2. Polymer-chains can be represented as the edges of a graph, and cross-links should be thought of as the vertices of a graph.

Each polymer-chain is itself a sequence of monomers. The edges of the graph only represent the so-called end-to-end vectors of the polymer-chains. In particular, the state of a polymer-chain is fully described by the end-to-end vector and the positions of all the physical (or chemical) bonds between monomers, see Fig. 3.

The rest of this section aims at

- introducing the statistical physics of such networks (in the canonical ensemble),
- presenting a coarse-grained version where the effect of the positions of the monomers are averaged out, and only the positions of the cross-links matter.

Before we turn to the kinematics of polymer-chain networks, let us comment on the notion of network. As customary in polymer physics, one considers a network to be given once and for all: the network is obtained through a chemical process of cross-linking, and the obtained cross-links are permanent (the energy needed to break a cross-link is much larger than mechanical forces). This cross-linking process gives rise to several statistical properties of such networks: their connectivity (how many chains are attached to the same cross-links), the typical distance between two neighboring cross-links, the typical length of an end-to-end vector, the typical number of monomers per polymer-chain. We will come to orders of magnitude later on. In this contribution we consider the network to be given. *In the mathematical description of a network, we need to have the freedom to choose the above four geometric properties* and we assume stationarity and ergodicity (which are mild assumptions). In particular, the network cannot be described as the Delaunay graph of a point set (the connectivity would be forced upon us), whence the larger class of graphs we consider here. We shall not discuss the cross-linking process further, and consider the probability measure describing the graph as a choice of the specific polymer-chain model under investigation—our assumptions are general enough for this.

Let us now be more specific and describe the network as a set of labels and relations between these labels. We call $\tilde{\mathfrak{N}}$ a finite network of cross-linked polymer-chains, described by a (finite) set $\tilde{\mathfrak{L}} \subset \mathbb{N}$ of cross-links i, a subset $\tilde{\mathfrak{L}}_b \subset \tilde{\mathfrak{L}}$ of boundary cross-links, a (finite) set $\tilde{\mathfrak{B}} \subset \tilde{\mathfrak{L}} \times \tilde{\mathfrak{L}}$ of undirected chains $c_{ij} = (i, j)$ (such that $\tilde{\mathfrak{L}}$ has one single connected component via $\tilde{\mathfrak{B}}$). Each chain c_{ij} is itself made of a sequence of $N_{ij} \in \mathbb{N}$ monomers, characterized by i, j and $\tilde{\mathfrak{M}}_{ij} = \{(i, j, 1), \ldots, (i, j, N_{ij} - 1)\}$. Up to this point, the above only describes a graph and quantities attached to it, that is, how labeled monomers, cross-links, and polymer-chains are organized together. The next nontrivial step is to define the kinematics of a network by specifying the position of the monomers, cross-links, and polymer-chains in the physical space $\mathbb{R}^3$. We have to specify the scale of the observer: we place ourselves at the physical scale of a monomer (the smallest constituent of matter considered in our description), which has from now on size unity.

2.2.2. The Kinematics and the Helmholtz Free Energy As usual in statistical physics, one may simplify a model by making the state space discrete—this makes the notion of partition function easier to grasp. This is what we do now by assuming that monomers can only be placed on the edges of the canonical graph $\mathbb{Z}^3$ (instead of anywhere in $\mathbb{R}^3$—this is only done for convenience in this presentation, and not in the rest of the paper). Here comes the kinematics of a network: a deformation $\tilde{u}$ of the polymer-chain network $\tilde{\mathfrak{N}}$ is a map $\tilde{\mathfrak{L}} \cup \bigcup_{(i,j)\in\tilde{\mathfrak{B}}} \tilde{\mathfrak{M}}_{ij} \to \mathbb{Z}^3$ with the following properties. The map $\tilde{u}$ is edge-injective (that is, two distinct monomers cannot occupy the same edge of $\mathbb{Z}^3$) and has unit increments in the sense that for all $(i, j) \in \tilde{\mathfrak{B}}$ and all $0 \leqq k \leqq N_{ij} - 1$ we have with the notation $\tilde{u}_{(i,j,0)} = \tilde{u}_i$ and $\tilde{u}_{(i,j,N_{ij})} = \tilde{u}_j$:

$$|\tilde{u}_{(i,j,k)} - \tilde{u}_{(i,j,k+1)}| = 1. \tag{2.2}$$

Given a boundary map $\tilde{\varphi}_b : \tilde{\mathfrak{L}}_b \to \mathbb{Z}^3$, we denote by $\Omega(\tilde{\mathfrak{N}}, \tilde{\varphi}_b)$ the cardinality of the set $\{\tilde{u} \text{ deformation } : \tilde{\mathfrak{L}} \cup \bigcup_{(i,j)\in\tilde{\mathfrak{B}}} \tilde{\mathfrak{M}}_{ij} \to \mathbb{Z}^3 \ \mid\ \tilde{u}|_{\tilde{\mathfrak{L}}_b} \equiv \tilde{\varphi}_b\}$. The Helmholtz free energy $\mathcal{E}^\beta(\tilde{\mathfrak{N}}, \tilde{\varphi}_b)$ of the network $\tilde{\mathfrak{N}}$ with boundary deformation $\tilde{\varphi}_b$ at inverse temperature β is then given by

$$\mathcal{E}^{\beta}(\tilde{\mathfrak{N}}, \tilde{\varphi}_b) := -\frac{1}{\beta} \log \Omega(\tilde{\mathfrak{N}}, \tilde{\varphi}_b), \tag{2.3}$$

with the understanding that $\mathcal{E}^{\beta}(\tilde{\mathfrak{N}}, \varphi_b) = +\infty$ if $\Omega(\tilde{\mathfrak{N}}, \tilde{\varphi}_b) = 0$. Before we make further restrictions on $\tilde{\mathfrak{L}}$ and $\tilde{\mathfrak{B}}$, observe that if the network $\tilde{\mathfrak{B}}$ is made of one single polymer-chain $(1, 2)$ with N monomers, then given $\tilde{\varphi}_b : \{1, 2\} \mapsto \{\tilde{u}(1), \tilde{u}(2)\} \in (\mathbb{Z}^3)^2$, $\Omega(\tilde{\mathfrak{N}}, \tilde{\varphi}_b)$ is explicit and obviously only depends on N and the length $|\tilde{u}(1) - \tilde{u}(2)|$. Indeed, for large N, (2.3) can be explicitly computed (and typically leads to (2.5), see below).

2.2.3. The Lagrangian Description and the Reference Configuration So far, we have defined the notion of network, its kinematics, and the Helmholtz free energy. In order to relate this description to continuum mechanics, it is convenient to have a Lagrangian description of the network, and thus a *reference configuration*. This is usually not presented in monographs on polymer physics [27,43,45]. Although this is elementary and pedestrian, we display this construction in detail since it is at the *very origin of the passage from the network description to the kinematics of continuum media*. A reference configuration for $\tilde{\mathfrak{N}}$ is a specific deformation, which we denote by x (see below for the physical interpretation). In particular, for all $(i, j) \in \tilde{\mathfrak{B}}$, x_i and x_j are the reference positions of the cross-links i and j, for all $0 \leqq k \leqq N_{ij} - 1$, (x_{ij}^k, x_{ij}^{k+1}) is the reference position (an edge) of the $(k+1)$-th monomer of the polymer-chain $c_{i,j}$ (and with $x_{ij}^0 = x_i$, $x_{ij}^{N_{ij}} = x_j$). We then now denote by $\mathfrak{N} = \{\tilde{\mathfrak{N}}, x\}$ the network and its reference configuration (which we still abusively call network), and let $\mathfrak{L}$, $\mathfrak{L}_b$, $\mathfrak{B}$ denote the sets of reference positions of cross-links, boundary cross-links, and end-to-end points of polymer-chains. This description allows us to view deformations $\tilde{u}$ of the graph as deformations u of the reference configuration x via the relation $u(x_{ij}^k) := \tilde{u}_{(i,j,k)}$. As above, for all $(x_i, x_j) \in \mathfrak{B}$, $\mu_{ij} : \{x_{ij}^0, \ldots, x_{ij}^{N_{ij}}\} \to \mathbb{Z}^3$ is an admissible deformation of the polymer-chain (x_i, x_j) if it is edge-injective and has unit increments in the sense of (2.2).

2.2.4. The Coarse-Grained Helmholtz Free Energy The next classical step in polymer physics is to relax to some extent the edge-injectivity condition between monomers from different chains. Given a deformation $u : \mathfrak{L} \to \mathbb{Z}^3$ and a polymer-chain $(x_i, x_j) \in \mathfrak{B}$, we denote by $\Omega_{ij}(u)$ the cardinality of the set of admissible deformations μ_{ij} such that $\mu_{ij}(x_i) = u(x_i)$ and $\mu_{ij}(x_j) = u(x_j)$. This accounts for local injectivity *within each chain*. Given a boundary map $\varphi_b : \mathfrak{L}_b \to \mathbb{Z}^3$, we define $\mathfrak{U}(\varphi_b) := \{u : \mathfrak{L} \to \mathbb{Z}^3 \mid u|_{\mathfrak{L}_b} \equiv \varphi_b\}$ the subset of deformations of cross-links that coincide with the boundary map φ_b on $\mathfrak{L}_b$. We finally replace the edge-injectivity assumption between monomers of different chains by some steric effect *between chains* (that is, the monomers of a polymer-chain, and thus the polymer-chain itself, occupy some volume in which the monomers of other chains, and thus the other polymer-chains themselves, are excluded), that can be accounted for by restricting admissible deformations to a suitable subset $\mathfrak{V}$ of $\{u : \mathfrak{L} \to \mathbb{Z}^3\}$ (which does not describe the positions of monomers any longer), and define $\widehat{\mathfrak{U}}(\varphi_b) := \mathfrak{U}(\varphi_b) \cap \mathfrak{V}$. This *assumption allows one to coarsen the model by factorizing the number of admissible deformations* in the form

$$\Omega(\mathfrak{N}, \varphi_b) \sim \sum_{u \in \widehat{\mathfrak{U}}(\varphi_b)} \prod_{(x_i, x_j) \in \mathfrak{B}} \Omega_{ij}(u).$$

Of course, this step is *not rigorous* in the sense that Ω does not exactly factorize. It is however considered to be a good approximation in polymer physics. This procedure has the effect to integrate out the positions of the monomers and to reduce the characterization of the model to $(\mathfrak{L}, \mathfrak{B})$ and the definition of the state space $\widehat{\mathfrak{U}}(\varphi_b)$ for any boundary deformation $\varphi_b : \mathfrak{L}_b \to \mathbb{Z}^3$. There is quite some flexibility and arbitrariness in this choice. Since these quantities only depend on distances (and/or angles) between cross-links, $\Omega(\mathfrak{N}, \varphi_b)$ does not depend on the frame of the Lagrangian description.

We now enrich the physics: On top of the non-interpenetrability of matter, polymer-chains feel the effect of a solvent which yields an internal energy that penalizes changes of volume with respect to the reference network, which we model in the form of $\hat{H}(u)$, an internal energy that only depends locally on u at a scale larger than that of a polymer-chain (in a frame-indifferent way). Again there is flexibility and arbitrariness in the choice of that scale (which is accounted for in Definition 1.1). The Helmholtz free energy of the network $\mathfrak{N}$ with boundary deformation φ_b at inverse temperature β is then given by the following modified version of (2.3)

$$\begin{aligned}\mathcal{E}^\beta(\mathfrak{N}, \varphi_b) &= -\frac{1}{\beta} \log \left(\sum_{u \in \widehat{\mathfrak{U}}(\varphi_b)} \left(\prod_{(x_i, x_j) \in \mathfrak{B}} \Omega_{ij}(u) \right) \exp(-\beta \hat{H}(u)) \right) \\ &= -\frac{1}{\beta} \log \left(\sum_{u \in \widehat{\mathfrak{U}}(\varphi_b)} \exp \left(-\beta \left(\hat{H}(u) + \sum_{(x_i, x_j) \in \mathfrak{B}} \frac{-1}{\beta} \log(\Omega_{ij}(u)) \right) \right) \right),\end{aligned}$$

which we rewrite as

$$\mathcal{E}^\beta(\mathfrak{N}, \varphi_b) := -\frac{1}{\beta} \log \Bigg(\sum_{u \in \mathfrak{U}(\varphi_b)} \exp \left(-\beta \left(H(u) + \sum_{(x_i, x_j) \in \mathfrak{B}} \frac{-1}{\beta} \log(\Omega_{ij}(u)) \right) \right) \Bigg), \tag{2.4}$$

by setting $H(u) = \hat{H}(u) + \tilde{H}(u)$ where $\tilde{H}(u) = +\infty$ if $u \notin \mathfrak{V}$ and $\tilde{H}(u) = 0$ if $u \in \mathfrak{V}$. The latter rewriting amounts to penalizing that a deformation $u : \mathfrak{L} \to \mathbb{Z}^3$ be admissible rather than restricting the set of states. The thermally fluctuating network with imposed boundary deformation φ_b has then free Helmholtz energy $\mathcal{E}^\beta(\mathfrak{N}, \varphi_b)$, and its configuration is described by a probability measure $\mu^\beta_{\varphi_b}$ on the set of admissible deformations defined as follows: for all $V \subset \mathfrak{U}(\varphi_b)$,

$$\mu^\beta_{\varphi_b}(V) := \frac{\sum_{u \in V} \exp\Big(-\beta \big(H(u) + \sum_{(x_i, x_j) \in \mathfrak{B}} \frac{-1}{\beta} \log(\Omega_{ij}(u)) \big) \Big)}{\sum_{u \in \mathfrak{U}(\varphi_b)} \exp\Big(-\beta \big(H(u) + \sum_{(x_i, x_j) \in \mathfrak{B}} \frac{-1}{\beta} \log(\Omega_{ij}(u)) \big) \Big)}.$$

The above consensual model of polymer physics is of the form studied in the analysis part of the present contribution. There is however a fundamental difference between (1.1) and (2.4): In the former the Hamiltonian has no dependence with respect to β, whereas in the latter the "effective" Hamiltonian of a polymer chain has a dependence on β and vanishes in the limit of small temperature (this is, again, not surprising since rubber elasticity is an entropic effect solely due to temperature). This is not an issue for the application of Theorem 1.5 since β is a parameter there. This is different for Theorem 1.6 since we do not wish to take a limit $1/\beta \to 0$ at which elasticity disappears. In this respect, let us quickly anticipate on Paragraph 2.3.4, and mention that the two-temperature model analyzed below introduces two temperatures, the inverse physical temperature β (which is arbitrary yet fixed) and an "**effective temperature**", which turns out to be small in the regime of large number of monomers per polymer-chain. In this context, taking the small temperature limit means taking the "effective temperature" small (while keeping the physical temperature unchanged), which is the way Theorem 1.6 will be used in Paragraph 2.3.4.

To conclude this paragraph, let us emphasize that the only phenomenological aspect of this model lies in the choice of H and Ω_{ij}, and in particular on the fact that they can be chosen *not depending on the positions of the monomers inside a chain*. Classical choices are as follows. The Hamiltonian H is often chosen to reflect incompressibility, albeit at a scale slightly larger than that of a polymer-chain (because chains can intertwin and extend to distances that are larger than the end-to-end distance). Evaluating Ω_{ij} reduces to counting the number of states of a polymer-chain given its end-to-end vector (that is, the number of sequences of monomers that lead from one end to the other) and given some rules (e.g. two monomers cannot overlap, the monomers must lie in some tube, and so on), which often leads to semi-explicit formulas, cf. the several choices discussed in [43, Chapter 3]. Establishing the regime of validity of such an assumption and specific forms for H and Ω_{ij} should definitely be investigated using molecular dynamics. Again, our analysis makes rather general assumptions on H and Ω_{ij}.

2.2.5. Towards Constitutive Laws Based on Polymer Physics Let us quickly revisit some classical models and some of the models of Paragraph 2.1.3. For simplicity, assume that the boundary condition is linear: $\varphi_b(x) = \Lambda \cdot x$ for some matrix Λ.

- Treloar (or affine) model: (2.4) is approximated by evaluating $H(u) + \sum_{(x_i,x_j)\in\mathfrak{B}} \frac{-1}{\beta} \log(\Omega_{ij}(u))$ at $u(x) = \Lambda \cdot x$;
- Arruda-Boyce model: (2.4) is approximated by replacing the network by a representative element made of 8 chains which spontaneously align with the principal directions of Λ, and use the affine deformation;
- Path-bases, non-affine micro-sphere (etc.) models: (2.4) is approximated by restricting the class of test-functions u in the average using some form of representative element;

- Two-temperature model: (2.4) is approximated by

$$\inf_u \left\{ H(u) + \sum_{(x_i,x_j)\in\mathfrak{B}} \frac{-1}{\beta} \log(\Omega_{ij}(u)) \right\},$$

which amounts to replacing the Helmholtz free energy by the ground state (whence the wording: cross-links are taken at zero temperature, but monomers—via Ω_{ij}—are taken at inverse temperature β).

Let us quickly interpret Treloar's experiments and the three regimes of Fig. 1 in terms of polymer physics. The linear regime essentially represents the fact that for φ_b close to the identity map, (2.4) is close to quadratic. The regimes of strain softening and strain hardening are related to the entropic term, the geometry of the network, and H. Let us give some intuition on the entropic term by considering a system of two cross-linked polymer-chains of possibly different length, for which the deformation of the boundary of the system (that is, the end points of the two polymer-chains except the cross-link) is fixed. For large boundary deformations, monomers tend to align so that there are less configurations available and the free energy of the system gets large and ultimately blows up. For moderately large boundary deformations, among the possible deformations of the cross-link, the one with the largest number of configurations is the linear interpolation of the deformation of the boundary only if the chains have the same length—otherwise it is advantageous to deform the longer chain more, which yields redistribution of strain and therefore leads to softening.

2.2.6. Orders of Magnitude We conclude with some orders of magnitude, and a discussion of the reference configuration. The reference configuration is obtained after cross-linking (that is, attach together) polymer-chains that were evolving freely in a solvent, cf. Fig. 4. First, as measured in physical experiments, the connectivity of such obtained polymer-chain networks is between 3 and 4 (depending on the polymer). Indeed, the cross-linking process takes place when a cross-linker meets several polymer-chains together: the probability that more than two polymer-chains are within range of the cross-linker is small. Second, the end-to-end vector of a polymer-chain in the reference configuration is a function of the number of monomers it has. Assume that the polymer-chain is a sequence of N monomers (recall that monomers have size unity in this discussion). Then the length of the end-to-end vector of this chain in the reference configuration is random itself and obeys some distribution which is peaked at $\sqrt{N}$. This is not surprising since one can think of a polymer-chain as a random walk after N steps (in which case the expectation of the distance to the origin is $\sqrt{N}$). In our analysis, although we have assumed that Ω_{ij} is a deterministic function of the network, our arguments can treat this additional randomness (provided it is chosen as an iid process on each chain). Finally, N ranges from 25 to 1000 in practical examples.

2.3. Application of the Main Results to Polymer Physics

In this second main part of this contribution, we apply our analysis to the physical model (2.4) of polymer physics recast in the form of (1.10), which allows

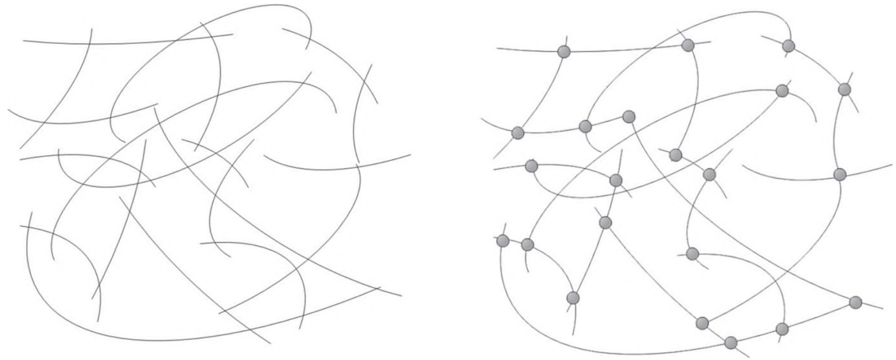


Fig. 4. Polymer-chains before and after cross-linking

us to justify the two-temperature model. In this section, we use physical parameters. In particular, monomers have length ℓ (a few nanometers at most).

2.3.1. The Precise Hamiltonian We need to make precise the form of the free energies of isolated polymer-chains in function of the number of monomers in the chain. The Kuhn and Grün formula (see e.g. [35] and [43, Section 3.4]) for the free energy of an isolated chain made of N monomers of size ℓ with end-to-end length L at temperature β is given by

$$f^{\beta}(L,N) := \frac{1}{\beta} N \left(\frac{L}{N\ell} \theta\left(\frac{L}{N\ell}\right) + \log \frac{\theta\left(\frac{L}{N\ell}\right)}{\sinh \theta\left(\frac{L}{N\ell}\right)} \right),$$

where θ is the inverse of the Langevin function $t \mapsto \coth t - \frac{1}{t}$. In particular, $L \mapsto f^{\beta}(L,N)$ is a non-negative convex increasing function in the variable L^2, that vanishes at $L=0$ and blows up as $L \uparrow N\ell$. This formula is based on a self-avoiding random bridge. For technical considerations (cf. discussion in [26] Paragraph 2.3.5), we replace this function by a function with p-growth from above and below, which yields our starting point

$$f^{\beta,(p)}(L,N) := \frac{N}{\beta} f^{(p)}\left(\frac{L}{N\ell}\right), \tag{2.5}$$

where $f^{(p)}$ is a suitable approximation of $t \mapsto f(t) := t\theta(t) + \log \frac{\theta(t)}{\sinh \theta(t)}$ (that remains convex and increasing). At order $p=10$, a Taylor-expansion (cf. [30]) simply yields

$$f^{\beta,(10)}(L,N) = \frac{N}{\beta}\left[\frac{3}{2}\left(\frac{L}{N\ell}\right)^2 + \frac{9}{20}\left(\frac{L}{N\ell}\right)^4 + \frac{9}{350}\left(\frac{L}{N\ell}\right)^6 + \frac{81}{7000}\left(\frac{L}{N\ell}\right)^8 + \frac{243}{673750}\left(\frac{L}{N\ell}\right)^{10}\right].$$

Consider now an ergodic random graph $G^\circ = (\mathcal{L}, \mathbb{B}, \mathbb{T})$, a fixed inverse temperature $\beta^\circ \geqq 1$, and fix p (say, $p=10$). Recall that we assume that the length of an edge $b \in \mathbb{B}$ of the random graph writes $\sqrt{N_b^\circ}\ell$, which we use to define the number N_b° of monomers in the polymer-chain b. We denote by $N^\circ := \mathbb{E}[N_b^\circ : b \in \mathbb{B}]$ the average number of monomers per polymer-chain in the graph. We then rewrite (2.5) in terms of the deformation ratio $\lambda = \frac{L}{\sqrt{N_b^\circ}\ell}$ as

$$f^{\beta^\circ}(L, N_b^\circ) := \frac{N_b^\circ}{\beta^\circ} f^{(p)}\left(\lambda \frac{1}{\sqrt{N_b^\circ}}\right),$$

and make the volumetric term more precise by considering for some $K>0$

$$W(\Lambda) := \frac{1}{K} W_{\mathrm{vol}}(\det \Lambda),$$

where $W_{\mathrm{vol}} : \mathbb{R} \to \mathbb{R}_+$ is a convex function that is minimal at $t=1$ and satisfies the growth condition

$$\forall t \geqq 0 : \quad W_{\mathrm{vol}}(t) \leqq 1 + t^{\frac{p}{d}}. \tag{2.6}$$

For all Lipschitz domains $D\varepsilon > 0$ and microscopic deformations $u : \mathcal{L} \cap D_\varepsilon \to \mathbb{R}^d$, the discrete Hamiltonian takes the form

$$H_\varepsilon^\circ(D,u) = \sum_{\substack{(x,y)\in\mathbb{B}\\ x,y\in D_\varepsilon}} \frac{N_{xy}^\circ}{\beta^\circ} f^{(p)}\left(\frac{|u(x)-u(y)|}{|x-y|}\frac{1}{\sqrt{N_{xy}^\circ}}\right) + \sum_{\mathcal{C}\in\mathcal{V}_{1,\varepsilon}(D)} |\mathcal{C}|\frac{1}{K} W_{\mathrm{vol}}\Big(\det_{\mathcal{C}}(\nabla u_{\mathrm{aff}})\Big),$$

which we rewrite in the equivalent form

$$H_\varepsilon^\circ(D,u) = \frac{N^\circ}{\beta^\circ}\tilde{H}_\varepsilon^\circ(D,u), \tag{2.7}$$

where $\tilde{H}_\varepsilon^\circ(D,u)$ is given by

$$\tilde{H}_\varepsilon^\circ(D,u) := \sum_{\substack{(x,y)\in\mathbb{B}\\ x,y\in D_\varepsilon}} f_{xy}^{\circ,(p)}\left(\frac{|u(x)-u(y)|}{|x-y|}\right) + \sum_{\mathcal{C}\in\mathcal{V}_{1,\varepsilon}(D)} |\mathcal{C}| W_{\mathrm{vol}}^\circ\Big(\det_{\mathcal{C}}(\nabla u_{\mathrm{aff}})\Big),$$

and for all $\lambda \geqq 0$, $t \in \mathbb{R}$,

$$f_{xy}^{\circ,(p)}(\lambda) := \frac{N_{xy}^\circ}{N^\circ} f^{(p)}\left(\lambda\frac{1}{\sqrt{N_{xy}^\circ}}\right), \qquad W_{\mathrm{vol}}^\circ(t) := \frac{\beta^\circ}{N^\circ}\frac{1}{K} W_{\mathrm{vol}}(t).$$

In terms of scaling, since volumetric and entropic terms compete, we choose $\frac{\beta^\circ}{K} \sim 1$, in which case (1.6) and (1.7) are valid for $p = 10$ with the constants

$$C \sim \frac{1}{N^\circ},\ C_2 \sim \frac{3}{2N^\circ},\ C_{10} \sim \frac{243}{673750\sqrt{N^\circ}^5}, \tag{2.8}$$

where $\sim$ means $\leqq c\times$ and $\geqq \frac{1}{c}\times$ for some constant c independent of N°. We are in the position to apply our general results. In what follows, β° and N° are fixed physical quantities, whereas β_1 and N_1 are dummy variables.

2.3.2. Thermodynamic Limit for $H_\varepsilon^\circ(D,u)$ By Theorems 1.3, 1.4, and 1.5, for all temperatures β_1 there exists a macroscopic energy density $\overline{W}_{N^\circ}^{\circ,\beta_1}$ associated with the Hamiltonian $H_\varepsilon^\circ(D,u)$ (recall that β° and N° are *fixed* parameters) via

$$\forall \Lambda \in \mathbb{R}^{d\times d}: \qquad \lim_{\varepsilon\downarrow 0} -\frac{1}{\beta_1|D_\varepsilon|}\log\int_{\mathcal{B}_\varepsilon(D,\varphi_\Lambda)} \exp(-\beta_1 H_\varepsilon^\circ(D,u))\,\mathrm{d}u = \overline{W}_{N^\circ}^{\circ,\beta_1}(\Lambda). \tag{2.9}$$

For the physical choice $\beta_1 = \beta^\circ$, this implies that the free energy of the discrete network of polymer-chains and the associated Gibbs measure are well-described at the thermodynamic limit (with given Dirichlet boundary data φ) by the infimum of the continuum energy functional

$$\varphi + W_0^{1,p}(D) \ni u \mapsto \mathcal{I}_{N^\circ}^{\circ,\beta_1}(u) := \fint_D \overline{W}_{N^\circ}^{\circ,\beta_1}(\nabla u(x))dx,$$

and by the Dirac mass at the set of minimizers. Next we argue that a direct application of Theorem 1.6 does not allow to justify the two-temperatures model which amounts to taking the limit $\beta_1 \uparrow \infty$ while keeping β° fixed. In this setting, Theorem 1.6 yields the existence of some energy density $\overline{W}_{N^\circ}^{\circ,\infty}$ such that

$$\forall \Lambda \in \mathbb{R}^{d\times d}: \qquad \lim_{\beta_1\uparrow\infty} \overline{W}_{N^\circ}^{\circ,\beta_1}(\Lambda) = \overline{W}_{N^\circ}^{\circ,\infty}(\Lambda).$$

However, the quantitative estimate (1.13) of Theorem 1.6, that takes the form

$$|\overline{W}_{N^\circ}^{\circ,\beta_1}(\Lambda) - \overline{W}_{N^\circ}^{\circ,\infty}(\Lambda)| \leqq \frac{\log \beta_1}{\beta_1}(d + \frac{1}{\beta^\circ}C(1+|\Lambda|^p)), \tag{2.10}$$

is not precise enough for $\beta_1 = \beta^\circ$ since $\overline{W}_{N^\circ}^{\circ,\infty}$ is itself of order $\frac{1}{\beta^\circ}C(1+|\Lambda|^p)$. The rest of this section aims at justifying the two-temperatures model in the regime $N^\circ \gg 1$ rather than $\beta^\circ \gg 1$.

2.3.3. Thermodynamic Limit for $\tilde{H}_\varepsilon^\circ(D,u)$ We denote by $\overline{W}^{\circ,N_1}$ the macroscopic free energy at temperature "N_1" (the number of monomers will indeed play the role of an inverse physical temperature in what follows) associated with the Hamiltonian $\tilde{H}_\varepsilon^\circ$ via Theorem 1.3, that is,

$$\forall \Lambda \in \mathbb{R}^{d\times d}: \qquad \lim_{\varepsilon\downarrow 0} -\frac{1}{N_1|D_\varepsilon|}\log \int_{\mathcal{B}_\varepsilon(D,\varphi_\Lambda)} \exp(-N_1\tilde{H}_\varepsilon^\circ(D,u))\,\mathrm{d}u = \overline{W}^{\circ,N_1}(\Lambda).$$

In view of (2.7) and (2.9), we have the identity

$$\overline{W}_{N^\circ}^{\circ,\beta_1}|_{\beta_1=\beta^\circ} = \frac{N^\circ}{\beta^\circ}\overline{W}^{\circ,N_1}|_{N_1=N_\circ}. \tag{2.11}$$

Whereas the $\overline{W}_{N^\circ}^{\circ,\beta_1}$ is well-suited to take the zero-temperature limit $\beta_1 \uparrow \infty$, $\overline{W}^{\circ,N_1}$ is well-suited to take the limit of large number of monomers per chain $N_1 \uparrow \infty$. By Theorem 1.6 (in form of (1.14)), there exists a macroscopic energy density $\overline{W}^{\circ,\infty}$ such that for all $N_1 \gg 1$

$$\forall \Lambda \in \mathbb{R}^{d\times d}: \quad |\overline{W}^{\circ,\infty}(\Lambda) - \overline{W}^{\circ,N_1}(\Lambda)| \leqq \frac{\log N_1}{N_1}\Big(C_2''|\Lambda|^2 + C_p''|\Lambda|^p + d\Big), \tag{2.12}$$

and so that the integral functional $u \mapsto \mathcal{I}^{\circ,N_1} := \fint_D \overline{W}^{\circ,N_1}(\nabla u(x))dx$ $\Gamma(L^p)$-converges towards $u \mapsto \mathcal{I}^{\circ,\infty}(u) := \fint_D \overline{W}^{\circ,\infty}(\nabla u(x))dx$ on $\varphi + W_0^{1,p}(D)$ as $N_1 \uparrow \infty$. Note that the lower and upper bounds in (1.12) are crude and could be largely improved if more precise assumptions are made on the random graph—in particular, we expect the coefficients of the terms of order $|\Lambda|^p$ to be comparable in both sides of the two-sided estimate, so that the RHS of (2.12) would indeed scale like $\frac{\log N_1}{N_1}$ times the order of magnitude of $\overline{W}^{\circ,N_1}(\Lambda)$.

2.3.4. Justification of the Two-Temperatures Model in the Regime $N^\circ \gg 1$ and the Effective Temperature The combination of (2.12) and (2.11) yields

$$\forall \Lambda \in \mathbb{R}^{d\times d}: \quad |\frac{N^\circ}{\beta^\circ}\overline{W}^{\circ,\infty}(\Lambda) - \overline{W}^{\circ,\beta^\circ}_{N^\circ}(\Lambda)| \leqq \left(\frac{\log N^\circ}{N^\circ}\right) \times \frac{N^\circ}{\beta^\circ}\Big(C_2''|\Lambda|^2 + C_p''|\Lambda|^p + d\Big). \tag{2.13}$$

In view of the parameters (2.8) and lower bounds for the Γ-limit, for deformations Λ such that $|\Lambda| \sim \sqrt{N^\circ}$ (that is, in the nonlinear regime), we have

$$\begin{aligned}\overline{W}^{\circ,\infty}(\Lambda) &\sim C_2|\Lambda|^2 + (C_p + C)|\Lambda|^p \\ &\sim C_2''|\Lambda|^2 + C_p''|\Lambda|^p\end{aligned} \quad \gtrsim 1 \quad \Longrightarrow \quad \frac{N^\circ}{\beta^\circ}\overline{W}^{\circ,\infty}(\Lambda) \gtrsim \frac{N^\circ}{\beta^\circ}\Big(C_2''|\Lambda|^2 + C_p''|\Lambda|^p + d\Big),$$

so that (2.13) shows that the relative error between $\overline{W}^{\circ,\beta^\circ}_{N^\circ}$ and its approximation $\frac{N^\circ}{\beta^\circ}\overline{W}^{\circ,\infty}$ is of order $\frac{\log N^\circ}{N^\circ} \ll 1$ in the regime $N^\circ \gg 1$ of large number of monomers per polymer-chain. Combined with the observation that the identity (2.7) also yields

$$\forall \Lambda \in \mathbb{R}^{d\times d}: \quad \frac{N^\circ}{\beta^\circ}\overline{W}^{\circ,\infty}(\Lambda) = \lim_{\beta_1\uparrow\infty} \overline{W}^{\circ,\beta_1}_{N^\circ}(\Lambda) = \overline{W}^{\circ,\infty}_{N^\circ}(\Lambda),$$

(2.13) takes the form

$$\begin{aligned}&\forall \Lambda \in \mathbb{R}^{d\times d}: |\overline{W}^{\circ,\infty}_{N^\circ}(\Lambda) - \overline{W}^{\circ,\beta^\circ}_{N^\circ}(\Lambda)| \leqq \left(\frac{\log N^\circ}{N^\circ}\right) \times \frac{N^\circ}{\beta^\circ}\Big(C_2''|\Lambda|^2 \\ &\quad + C_p''|\Lambda|^p + d\Big), \\ &|\Lambda| \sim \sqrt{N^\circ} \implies \overline{W}^{\circ,\infty}_{N^\circ}(\Lambda) \gtrsim \frac{N^\circ}{\beta^\circ}\Big(C_2''|\Lambda|^2 + C_p''|\Lambda|^p + d\Big),\end{aligned}$$

which improves on (2.10). The above applications of Theorems 1.3, 1.4, 1.5, and 1.6 therefore yield a rigorous justification of the two-temperatures model $\overline{W}^{\circ,\infty}_{N^\circ}$, which consists in assuming that the monomers of the polymer-chains fluctuate at inverse temperature β°, whereas cross-links are considered at zero temperature ($\beta_1 = +\infty$). In particular, one can interpret $\frac{\log N^\circ}{N^\circ}$ as the **effective temperature** of the cross-links. This sets on rigorous ground the approach introduced and analyzed in [4,30] to derive nonlinear elasticity from polymer physics.

2.3.5. Extensions and Comments We conclude this section on the derivation of rubber elasticity from statistical polymer physics with a list of possible extensions and open problems.

- The process of vulcanization of rubber generates metallic inclusions (zinc oxides) in the matrix phase, which modifies the elastic behavior of rubber-like materials at large deformation since the former are more rigid than the polymer-chains. This can be included in the discrete model as follows. Enrich the probability space by adding a state $Z \in \{0,1\}^{\mathbb{N}}$, and say that a vertex i is in the set of zinc oxides if $Z(i) = 1$. If an edge $b = (x_i, x_j) \in \mathbb{B}$ is such that $Z(i) + Z(j) \geqq 1$, then the free energy of the polymer-chain f_{ij} is multiplied by some large constant $K \gg 1$ (which encodes the larger rigidity of the zinc oxides). We may then perform the same analysis as above.

- We now comment on the two main analytical simplifications of this work, namely that f_{ij} and W have p-growth from above. We believe that at least parts of the results should survive if we let f_{ij} blow up at finite deformation. For the homogenization of multiple integrals, such a result was recently obtained by Duerinckx and the second author in [26]. Whereas Γ-convergence focuses on minimizers, large-deviation principles focus on neighborhoods of minimizers, so that one needs finer quantitative control for the latter. Since such quantitative control is already quite subtle in [26], the extension of our analysis to this setting might hold, but not without additional and substantial work. In contrast, the growth condition on W is crucial for our arguments to work. Relaxing this assumption constitutes the *major open problem of homogenization* of integral functionals with quasiconvex integrands. For first results in that direction (with small data) we refer to [41].
- Both in (2.1) and in (2.4), the setting is time-independent. Although this is correct at first approximation, rubber-like materials also display some visco-elastic effects in practice, which find their origin in a feature of the discrete network we have not touched upon: when a polymer-chain gets extended, it needs to "un-entangle", and there is some friction at that level. This could be included in the two-temperature model following the recent work [37] by Lequeux and collaborators.
- The model (2.4) also neglects one feature of polymer-chain networks: topological constraints. Indeed, chains can be prevented from extending too much because they cannot pass through other chains. Such topological constraints are not taken into account, and partly contribute to the strain hardening of Fig. 1. It is not yet clear to us how to enrich the model in that direction.
- Last there is yet no satisfactory explanation of the origin of the volumetric term H, which, according to polymer-physicists, comes from a smaller scale than that of the monomer. Not unrelated to this, one could hope to better understand cavitation phenomena at the scale of the polymer-chain network. This is not clear, even for polymer-physicists. However, let us emphasize that the growth condition on the Ω_{ij}'s (which are expected to blow up at finite deformation) rules out the classical results of cavitation à la Müller [40]. It would be worth investigating possible relations to the recent interpretation of cavitation as a healing process, cf. [29,36].

Next to the analysis side of this work, our results also raise interesting questions at the level of physical experiments and numerical simulations.

- As already mentioned, some explicit form for Ω_{ij} and H could be obtained by direct molecular simulations of a polymer-chain network of moderate size.
- For applications to polymer physics, one needs input on the polymer-chain network, such as the connectivity, the typical number of monomers per polymer-chains, the typical distance between cross-links, etc.

- The question of the experimental validation of the models is of particular interest. Next to standard mechanical experiments at the level of $\overline{W}$, one can also “validate” the model at the discrete level. Indeed, small-angle diffraction experiments give access to local deformation at the scale of polymer-chains [7]. These could be compared to the output of a direct numerical simulation of the two-temperature model. This can be done at the qualitative level—for instance regarding the so-called butterfly effect, and at the quantitative level (which would require a close collaboration with physicists). The butterfly effect for the two-temperature model is currently under investigation [31].

3. Notation and Preliminary Geometric Estimates

Let us fix some notation. Given a measurable set $B \subset \mathbb{R}^d$ we denote by $|B|$ its d-dimensional Lebesgue measure. The same notation is used to denote the cardinality of B whenever it is a finite set. More generally we denote by $\mathcal{H}^k(B)$ the k-dimensional Hausdorff measure of B. Given $x \in \mathbb{R}^d$ we let $|x|$ denote its Euclidean norm and we let $B_r(x)$ be the open ball with center x and radius r. Moreover, $Q(x, r) = x+(-r/2, r/2)^d$ denotes the open cube with center x and side length r. We set $\operatorname{dist}(x, B) = \inf_{y\in B} |x - y|$. Given an open set $U \subset \mathbb{R}^d$ we define $\mathcal{A}^R(U)$ to be the family of open, bounded subsets of U with Lipschitz boundary. We denote by $L^p(U, \mathbb{R}^n)$, $W^{1,p}(U, \mathbb{R}^n)$ the usual vector-valued Lebesgue and Sobolev spaces. We use the short-hand notation $L^p(U)$ or $W^{1,p}(U)$ when we refer to convergence in these spaces and no confusion about the co-domain is possible. In the proofs C denotes a generic constant (depending only on the dimension or other fixed parameters) that may change every time it appears.

3.1. Geometric Considerations

In this subsection we establish some geometric properties of admissible extended Euclidean graphs that will be useful throughout this article. Recall that given $G = (\mathcal{L}, E, S)$, we denote by $\mathcal{V} = \{\mathcal{C}(x)\}_{x\in\mathcal{L}}$ the Voronoi tessellation associated to the vertices $\mathcal{L}$. Note that if the vertices fulfill conditions (i) and (ii) of Definition 1.2, then the Voronoi cells satisfy $B_{\frac{r}{2}}(x) \subset \mathcal{C}(x) \subset B_R(x)$ for all $x \in \mathcal{L}$. In particular it holds that

$$\forall x \in \mathcal{L} : \quad \frac{1}{C} \leqq |\mathcal{C}(x)| \leqq C \tag{3.1}$$

and, for fixed $O \in \mathcal{A}^R(D)$ and ε small enough, we have the estimate

$$\frac{1}{C}|O|\varepsilon^{-d} \leqq |O_\varepsilon^{\mathcal{L}}| \leqq C|O|\varepsilon^{-d}. \tag{3.2}$$

In some geometric constructions we will also need a bound on the cardinality of sets of the form

$$\{x \in \mathcal{L} : \ \operatorname{dist}(x, \partial O_\varepsilon) \leqq C_0\}.$$

For x in this set the rescaled Voronoi cells $\varepsilon\mathcal{C}(x)$ are contained in the $(C_0 + R)\varepsilon$-tubular neighbourhood of ∂O. Since for Lipschitz boundaries, the Minkowski content agrees (up to a dimensional constant) with $\mathcal{H}^{d-1}(\partial O)$, we deduce that for ε small enough we have

$$|\{x \in \mathcal{L} : \ \mathrm{dist}(x, \partial O_\varepsilon) \leqq C_0\}| \leqq C\varepsilon^{1-d}\mathcal{H}^{d-1}(\partial O). \tag{3.3}$$

Similar estimates hold for finite unions or intersections of Lipschitz sets.

We shall identify functions $u : \mathcal{L} \to \mathbb{R}^n$ with their piecewise constant interpolations on the Voronoi tessellation $\mathcal{V}$ associated with $\mathcal{L}$. Conversely, given a function $u \in L^p_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^n)$ we define a (random) discrete approximation $u_\varepsilon : \mathcal{L} \to \mathbb{R}^n$ via

$$u_\varepsilon(x) := \frac{1}{|\varepsilon\mathcal{C}(x)|} \int_{\varepsilon\mathcal{C}(x)} u(z)\ \mathrm{d}z. \tag{3.4}$$

Remark 5. The rescaled (piecewise constant) functions $\tilde{u}_\varepsilon(\varepsilon x) := u_\varepsilon(x)$ converge to u in $L^p_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^n)$. Indeed, given any bounded set $B \subset \mathbb{R}^d$ we choose another bounded, open set $U \subset \mathbb{R}^d$ such that $B \subset\subset U$ and redefine $u \equiv 0$ on $\mathbb{R}^d \setminus U$. This does not affect the values of $\tilde{u}_\varepsilon$ and u on B, but now $u \in L^p(\mathbb{R}^d, \mathbb{R}^n)$. Assume that $n = 1$. From (3.1) and Lebesgue's differentiation Theorem we infer that $\tilde{u}_\varepsilon \to u$ almost everywhere in B. Moreover, again by definition (3.4), we have $|\tilde{u}_\varepsilon| \leqq C\mathcal{M}u$, where $\mathcal{M}u$ denotes the Hardy-Littlewood maximal function. Hence $\tilde{u}_\varepsilon \to u$ in $L^p(B, \mathbb{R}^d)$ by dominated convergence. The general case $n \geqq 1$ follows by treating each component separately.

For notational convenience we also define discrete ℓ^p norms as follows: for all $\varepsilon > 0$ and $u : \mathcal{L} \to \mathbb{R}^n$

$$\|u\|_{\ell^p_\varepsilon(O)} := \left(\sum_{x \in O^{\mathcal{L}}_\varepsilon} |u(x)|^p \right)^{\frac{1}{p}}, \qquad \|\nabla_{\mathbb{B}} u\|_{\ell^p_\varepsilon(O)} := \left(\sum_{\substack{(x,y)\in\mathbb{B} \\ x,y \in O^{\mathcal{L}}_\varepsilon}} |u(x) - u(y)|^p \right)^{\frac{1}{p}},$$

where $\nabla_{\mathbb{B}}$ denotes the gradient on the graph, which maps functions on vertices to functions on edges (and is convenient to estimate the Hamiltonian).

As we show now, admissible graphs enjoy discrete Poincaré-type inequalities with respect to these norms. Recall that for any set $O \subset \mathbb{R}^d$ and $\varphi \in \mathrm{Lip}(O, \mathbb{R}^n)$ we let

$$\mathcal{B}_\varepsilon(O, \varphi) = \{u : O_\varepsilon \cap \mathcal{L} \to \mathbb{R}^n, \ |u(x) - \tfrac{1}{\varepsilon}\varphi(\varepsilon x)| < 1 \text{ if } \mathrm{dist}(x, \partial D_\varepsilon) \leqq C_0\}.$$

Lemma 3.1. *Let $G \in \mathcal{G}$ and let $O \in \mathcal{A}^R(\mathbb{R}^d)$. Then there exists a constant $C = C_{O,p}$ such that for all ε small enough and all $u \in \mathcal{B}_\varepsilon(O, 0)$ we have*

$$\|u\|^p_{\ell^p_\varepsilon(O)} \leqq \frac{C}{\varepsilon^p} \left(\|\nabla_{\mathbb{B}} u\|^p_{\ell^p_\varepsilon(O)} + \varepsilon^{1-d} \right).$$

Proof of Lemma 3.1. We extend u setting $u(x) = 0$ for $x \in \mathcal{L}\backslash O_\varepsilon^{\mathcal{L}}$. Take any cube $Q \subset \mathbb{R}^d$ such that $O \subset\subset Q$. For $x \in O_\varepsilon^{\mathcal{L}}$, define the ray $R_x := \{x + te_1 : t \geqq 0\}$. Then there exists a smallest number $t_* > 0$ such that $x + t_* e_1 \in \mathcal{C}(z)$ for some $z \in \mathcal{L}\backslash O_\varepsilon^{\mathcal{L}}$. We let $z_x \in \mathcal{L}$ be (one of) such point(s). Then $z_x \in Q_\varepsilon$ for ε small enough and moreover $|x - z_x| \leqq \varepsilon^{-1}\mathrm{diam}\, O + 2R$. As G is admissible, there exists a path $P(x)$ connecting x and z_x such that $P(x) \subset [x, z_x] + B_{C_0}(0)$. By (3.1) the number of edges in such a path is bounded by $\#\{(x', x'') \in P(x)\} \leqq C\varepsilon^{-1}\mathrm{diam}\, O$ and moreover we may assume that $P(x) \subset Q_\varepsilon$ for ε small enough. Jensen's inequality then yields

$$
\begin{aligned}
|u(x)|^p = |u(x) - u(z_x)|^p &\leqq \left(\sum_{(x',x'')\in P(x)} |u(x') - u(x'')| \right)^p \\
&\leqq C \left(\frac{\mathrm{diam}\, O}{\varepsilon} \right)^{p-1} \sum_{(x',x'')\in P(x)} |u(x') - u(x'')|^p, \qquad (3.5)
\end{aligned}
$$

where we used that $u(z_x) = 0$. Next, for any edge $(x', x'') \in \mathbb{B}$ we set

$$
K_\varepsilon(x', x'') := \{x \in O_\varepsilon^{\mathcal{L}} : \ (x', x'') \in P(x)\}.
$$

We need to bound the cardinality of this set. If $x \in K_\varepsilon(x', x'')$, then there exists $\lambda \in [0, 1]$ such that the point $x_\lambda = x + \lambda(z_x - x)$ satisfies $|x_\lambda - x'| \leqq C_0$. Hence we infer

$$
\begin{aligned}
x &= x - x_\lambda + x' + (x_\lambda - x') \\
&= -\lambda(z_x - x) + x' + (x_\lambda - x') \in (-R_{-x'} + B_{R+C_0}(0)) \cap \varepsilon^{-1} O.
\end{aligned}
$$

By (3.1) we conclude that $\# R_\varepsilon(x', x'') \leqq C\varepsilon^{-1}\mathrm{diam}\, O$, so that summing (3.5) over $x \in O_\varepsilon^{\mathcal{L}}$ yields

$$
\|u\|^p_{\ell^p_\varepsilon(O)} \leqq C \left(\frac{\mathrm{diam}\, O}{\varepsilon} \right)^p \|\nabla_{\mathbb{B}} u\|^p_{\ell^p_\varepsilon(Q)}. \qquad (3.6)
$$

Due to the constant extension and the soft boundary conditions, for small ε the contributions on the large cube Q can be bounded via the estimate

$$
\begin{aligned}
\|\nabla_{\mathbb{B}} u\|^p_{\ell^p_\varepsilon(Q)} &\leqq \|\nabla_{\mathbb{B}} u\|^p_{\ell^p_\varepsilon(O)} + \sum_{\substack{(x,y)\in\mathbb{B} \\ [x,y]\cap\partial O_\varepsilon \neq \emptyset}} |u(x) - u(y)|^p \\
&\leqq \|\nabla_{\mathbb{B}} u\|^p_{\ell^p_\varepsilon(O)} + C\varepsilon^{1-d}\mathcal{H}^{d-1}(\partial O),
\end{aligned}
$$

where we used (3.3). Inserting this estimate in (3.6) concludes the proof. □

Remark 6. In the discrete setting there is also a trivial reverse Poincaré inequality. Indeed, as the degree of every vertex in $\mathcal{L}$ is equibounded due to (3.1), there exists $C = C_p$ such that for all $O \in \mathcal{A}^R(\mathbb{R}^d)$ and $u : \mathcal{L} \to \mathbb{R}^n$, $\|\nabla_{\mathbb{B}} u\|^p_{\ell^p_\varepsilon(O)} \leqq C\|u\|^p_{\ell^p_\varepsilon(O)}$.

Next we prove a technical Lemma which is the analogue of Lemma 12 in [34] for non-periodic graphs.

Lemma 3.2. *Let $G' = (\mathcal{L}', \mathbb{B}')$ be a finite connected subgraph of G and $\bar{x} \in \mathcal{L}'$. Then there exists a dimensional constant C_1 such that, for all $z \in \mathbb{R}^n$ and $\alpha, \gamma > 0$,*

$$\int_{(\mathbb{R}^n)^{\mathcal{L}'}} \mathbb{1}_{\{|u(\bar{x})-z|<\gamma\}} \exp\left(-\alpha \sum_{(x,y)\in\mathbb{B}'} |u(x)-u(y)|^p\right) \mathrm{d}u \leqq C_1 \gamma^n \left(\alpha^{-\frac{n}{p}} C_1\right)^{|\mathcal{L}'|-1}.$$

Proof of Lemma 3.2. As G' is connected, there exists a rooted spanning tree $T_{\bar{x}} = (\mathcal{L}', \mathbb{B}_{\bar{x}})$ with root $\bar{x}$ (note that here we exceptionally consider a directed graph). We now prove inductively that we can integrate out all the vertices except the root. Since $T_{\bar{x}}$ has less edges than G', it holds that

$$\sum_{(x,y)\in\mathbb{B}'} |u(x)-u(y)|^p \geqq \sum_{(x,y)\in\mathbb{B}_{\bar{x}}} |u(x)-u(y)|^p.$$

Consider any leaf $x_0 \in \mathcal{L}'$, that means x_0 has no outgoing edges and only one incoming edge $(x_1, x_0) \in \mathbb{B}_{\bar{x}}$. Then, by Fubini's Theorem and a change of variables, we deduce that

$$\begin{aligned}
&\int_{(\mathbb{R}^n)^{\mathcal{L}'}} \mathbb{1}_{\{|u(\bar{x})-z|<\gamma\}} \exp\left(-\alpha \sum_{(x,y)\in\mathbb{B}'} |u(x)-u(y)|^p\right) \mathrm{d}u \\
&\leqq \int_{(\mathbb{R}^n)^{\mathcal{L}'}} \mathbb{1}_{\{|u(\bar{x})-z|<\gamma\}} \exp\left(-\alpha \sum_{(x,y)\in\mathbb{B}_{\bar{x}}} |u(x)-u(y)|^p\right) \mathrm{d}u \\
&\leqq \int_{(\mathbb{R}^n)^{\mathcal{L}'\setminus x_0}} \mathbb{1}_{\{|u(\bar{x})-z|<\gamma\}} \exp\left(-\alpha \sum_{\substack{(x,y)\in\mathbb{B}_{\bar{x}} \\ (x,y)\neq(x_1,x_0)}} |u(x)\right. \\
&\qquad \left. -u(y)|^p\right) \int_{\mathbb{R}^n} \exp(-\alpha|u(x_1)-u(x_0)|^p)\,\mathrm{d}u(x_0)\,\mathrm{d}u \\
&= \left(\alpha^{-\frac{n}{p}} \int_{\mathbb{R}^n} \exp(-|\zeta|^p)\,\mathrm{d}\zeta\right) \int_{(\mathbb{R}^n)^{\mathcal{L}'\setminus x_0}} \mathbb{1}_{\{|u(\bar{x})-z|<\gamma\}} \exp\left(-\alpha \sum_{\substack{(x,y)\in\mathbb{B}_{\bar{x}} \\ (x,y)\neq(x_1,x_0)}} |u(x)\right. \\
&\qquad \left. -u(y)|^p\right) \mathrm{d}u.
\end{aligned}$$

The (directed) graph $(\mathcal{L}'\setminus\{x_0\}, \mathbb{B}_{\bar{x}}\setminus(x_1, x_0))$ is still a rooted tree for the set of edges $\mathcal{L}'\setminus\{x_0\}$ with root $\bar{x}$. By iteration we thus obtain the claim upon setting

$$C_1 = \max\left\{\int_{B_1(0)} \mathrm{d}\zeta, \int_{\mathbb{R}^n} \exp(-|\zeta|^p)\,\mathrm{d}\zeta\right\},$$

where the volume of the unit ball is the remaining term when we integrated out the contributions of all the edges in $\mathbb{B}_{\bar{x}}$. □

Remark 7. Given a set $U \in \mathcal{A}^R(\mathbb{R}^d)$, the graph $G_U = (U \cap \mathcal{L}, \{(x, y) \in \mathbb{B} : x, y \in U\})$ is in general not connected but can be decomposed into its connected components. If N_U denotes the number of such components, then it follows that

$$N_U \leqq \#\{x \in \mathcal{L} : \ \mathrm{dist}(x, \partial U) \leqq C_0\}.$$

Indeed, for any component $G_j = (V_j, \mathbb{B}_j)$ take $x \in V_j$ and $y \in \mathcal{L}\backslash U$. As G is connected we find a path in G connecting x and y. Starting at x, let y_j be the first vertex of the path such that $y_j \notin U$. Then its preceding vertex x_j satisfies $\mathrm{dist}(x_j, \partial U) \leqq C_0$ because G is admissible. By construction $x_j \in V_j$.

Combining Remark 7, Lemma 3.2, and Fubini's theorem, we immediately obtain the following bound for possibly disconnected subgraphs.

Lemma 3.3. *Let $\varepsilon > 0$. Given $O \in \mathcal{A}^R(\mathbb{R}^d)$, we define the graph $G_{O,\varepsilon} = (O_\varepsilon^{\mathcal{L}}, \{(x, y) \in \mathbb{B} : \ x, y \in O_\varepsilon^{\mathcal{L}}\})$. Consider a set V such that there exist $\gamma > 0$ and $\{z_x\}_{\{x \in \mathcal{L}:\, \mathrm{dist}(x, \partial O_\varepsilon) \leqq C_0\}} \subset \mathbb{R}^n$ with*

$$V \subset \{u : O_\varepsilon^{\mathcal{L}} \to \mathbb{R}^n : \ |u(x) - z_x| < \gamma \text{ for all } x \in \mathcal{L} \text{ such that } \mathrm{dist}(x, \partial O_\varepsilon) \leqq C_0\}.$$

Then there exists $C_1 > 0$ such that for all $\alpha > 0$

$$\int_V \exp(-\alpha \|\nabla_{\mathbb{B}} u\|^p_{\ell^p_\varepsilon(O)})\, \mathrm{d}u \leqq \Big(C_1 \gamma^n\Big)^{N_{O,\varepsilon}} \Big(\alpha^{-\frac{n}{p}} C_1\Big)^{|O_\varepsilon^{\mathcal{L}}| - N_{O,\varepsilon}},$$

where $N_{O,\varepsilon}$ denotes the number of connected components of the graph $G_{O,\varepsilon}$.

3.2. Estimates on the Partition Function

For the analysis, we need to introduce further functional spaces. Given $O \in \mathcal{A}^R(\mathbb{R}^d)$, $v \in L^p_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^n)$, $w : O_\varepsilon^{\mathcal{L}} \to \mathbb{R}^n$, and $\kappa, M > 0$, we define the following three sets:

$$\begin{aligned} \mathcal{N}_p(v, O, \varepsilon, \kappa) &:= \{u : O_\varepsilon^{\mathcal{L}} \to \mathbb{R}^n, \ \sum_{O_\varepsilon^{\mathcal{L}}} \varepsilon^d |v_\varepsilon(x) - \varepsilon u(x)|^p < \kappa^p |O|^{1+\frac{p}{d}}\}, \\ \mathcal{N}_\infty(w, O, \varepsilon) &:= \{u : O_\varepsilon^{\mathcal{L}} \to \mathbb{R}^n : \ \|w - u\|_\infty < 1\}, \\ \mathcal{S}_M(O, \varepsilon) &:= \{u : O_\varepsilon^{\mathcal{L}} \to \mathbb{R}^n : \ H_\varepsilon(O, u) \leqq M |O_\varepsilon^{\mathcal{L}}|\}. \end{aligned} \tag{3.7}$$

The first two sets define neighborhoods of φ_ε (defined via (3.4)) and v, respectively, in a suitable topology. The third set contains deformations of uniformly finite energy.

Next, we introduce a localized version of the partition function (1.9), and define for all sets $V \subset \{u : O_\varepsilon^{\mathcal{L}} \to \mathbb{R}^n\}$

$$Z^\beta_{\varepsilon,O}(V) := \int_V \exp(-\beta H_\varepsilon(O, u))\, \mathrm{d}u,$$

and two (β-dependent) quantities that play a major role in the analysis: For $v \in L^p_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^n)$ and $O \in \mathcal{A}^R(\mathbb{R}^d)$ we set

$$\mathcal{F}^-_\kappa(O, v) = \liminf_{\varepsilon \downarrow 0} -\frac{1}{\beta |O_\varepsilon|} \log(Z^\beta_{\varepsilon, O}(\mathcal{N}_p(v, O, \varepsilon, \kappa))),$$
$$\mathcal{F}^+_\kappa(O, v) = \limsup_{\varepsilon \downarrow 0} -\frac{1}{\beta |O_\varepsilon|} \log(Z^\beta_{\varepsilon, O}(\mathcal{N}_p(v, O, \varepsilon, \kappa))).$$

Since both quantities are decreasing in κ, we can consider their limits as $\kappa \downarrow 0$ and define

$$\mathcal{F}^-(O, v) = \lim_{\kappa \to 0} \mathcal{F}^-_\kappa(O, v) = \sup_{\kappa > 0} \mathcal{F}^-_\kappa(O, v),$$
$$\mathcal{F}^+(O, v) = \lim_{\kappa \to 0} \mathcal{F}^+_\kappa(O, v) = \sup_{\kappa > 0} \mathcal{F}^+_\kappa(O, v).$$

We conclude this section with two results. The first one rules out concentration on high energy configurations, and the second is an interpolation result, which will both be crucial to prove the exponential tightness at the origin of the large deviation principle for the Gibbs measure.

Lemma 3.4. *Assume Hypothesis 1 and let $G \in \mathcal{G}$. Fix $O \in \mathcal{A}^R(\mathbb{R}^d)$, $v \in L^p_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^n)$ and $\varphi \in \mathrm{Lip}(O, \mathbb{R}^n)$. Then there exists a constant $C_\beta > 0$ such that for all $\kappa > 0$, $\varepsilon = \varepsilon(\kappa) > 0$ small enough, all $\beta > 0$ and $M \geqq C_\beta$,*

$$Z^\beta_{\varepsilon, O}(\mathcal{N}_p(v, O, \varepsilon, \kappa) \backslash \mathcal{S}_M(O, \varepsilon)) \leqq \exp\left(-\frac{M}{2}\beta |O^{\mathcal{L}}_\varepsilon|\right) \exp(C_\beta |O^{\mathcal{L}}_\varepsilon|),$$
$$Z^\beta_{\varepsilon, O}(\mathcal{B}_\varepsilon(O, \varphi) \backslash \mathcal{S}_M(O, \varepsilon)) \leqq \exp\left(-\frac{M}{2}\beta |O^{\mathcal{L}}_\varepsilon|\right) \exp(C_\beta |O^{\mathcal{L}}_\varepsilon|).$$

The constant C_β can be chosen as

$$C_\beta = \begin{cases} C & \text{if } \beta \geqq \frac{1}{2}, \\ -C \log(\beta) & 0 < \beta < \frac{1}{2}. \end{cases}$$

Proof of Lemma 3.4. Note that by Hypothesis 1, for any $u \notin \mathcal{S}_M(O, \varepsilon)$ it holds that

$$H_\varepsilon(u, O) \geqq \frac{3M}{4} |O^{\mathcal{L}}_\varepsilon| + \frac{1}{4} H_\varepsilon(u, O) \geqq \frac{3M}{4} |O^{\mathcal{L}}_\varepsilon| + \frac{1}{4C} \|\nabla_{\mathbb{B}} u\|^p_{\ell^p_\varepsilon(O)} - \frac{C}{4} |O^{\mathcal{L}}_\varepsilon|.$$

Hence we obtain that

$$\begin{aligned} & Z^\beta_{\varepsilon, O}(\mathcal{N}_p(v, O, \varepsilon, \kappa) \backslash \mathcal{S}_M(O, \varepsilon)) \\ & \quad \leqq \exp\left(-\frac{M}{2}\beta |O^{\mathcal{L}}_\varepsilon|\right) \int_{\mathcal{N}_p(v, O, \varepsilon, \kappa)} \exp\left(-\beta \frac{1}{C} \|\nabla_{\mathbb{B}} u\|^p_{\ell^p_\varepsilon(O)}\right) \mathrm{d}u, \end{aligned}$$

up to redefining C. In order to bound the last integral, first note that for every $u \in \mathcal{N}_p(v, O, \varepsilon, \kappa)$ the definition (3.7) implies that for all $x \in O^{\mathcal{L}}_\varepsilon$ we have

$$|u(x) - \varepsilon^{-1} v_\varepsilon(x)| < \kappa (|O| \varepsilon^{-d})^{\frac{1}{p} + \frac{1}{d}}.$$

Therefore we may apply Lemma 3.3 with the family $z_x = \varepsilon^{-1} v_\varepsilon(x)$ and obtain the estimate

$$\int_{\mathcal{N}_p(v,O,\varepsilon,\kappa)} \exp(-\beta \frac{1}{C} \|\nabla_{\mathbb{B}} u\|^p_{\ell^p_\varepsilon(O)})\,\mathrm{d}u \leqq \left(C\kappa^n \left(|O|\varepsilon^{-d}\right)^{\frac{n}{p}+\frac{n}{p}}\right)^{N_{O,\varepsilon}} \left(C\beta^{-\frac{n}{p}}\right)^{|O^{\mathcal{L}}_\varepsilon|-N_{O,\varepsilon}}, \tag{3.8}$$

where the graph $G_{O,\varepsilon}$ is defined as in Lemma 3.3. By Remark 7 and (3.3), taking ε small enough (depending on O) the number of connected components of $G_{O,\varepsilon}$ can be bounded via

$$N_{O,\varepsilon} \leqq C\mathcal{H}^{d-1}(\partial O)\varepsilon^{1-d}.$$

Set C_β as in the statement. Up to further decreasing $\varepsilon = \varepsilon(O,\kappa)$, we deduce from (3.8) the estimate

$$\int_{\mathcal{N}_p(v,O,\varepsilon,\kappa)} \exp\left(-\frac{1}{C} \|\nabla_{\mathbb{B}} u\|^p_{\ell^p_\varepsilon(O)}\right)\mathrm{d}u \leqq \exp(C_\beta |O^{\mathcal{L}}_\varepsilon|).$$

This proves the first estimate. The second one is easier as we have a better control for Lemma 3.3 using the boundary conditions. We leave the details to the reader. □

Remark 8. Observe that in Lemma 3.4 the condition on ε is independent of M, so that the estimate holds uniformly with respect to $M \geqq C$.

The last result we state in this section is one of the main tools in [34] to prove large deviation principles for the Gibbs measures associated with elastic energies on periodic lattices. It is an interpolation inequality that allows to impose additional boundary conditions. We extend the validity of this inequality to admissible graphs. Although there are only minor changes in the argument, we display the proof with our notation in the appendix. Since it is a technical tool, we don't quantify the dependence on β here. However, we stress that we have to keep track of how the estimate depends on the set O after letting $\varepsilon \downarrow 0$ (see Remark 14 in the appendix).

Proposition 1. *Assume Hypothesis 1 and let $G \in \mathcal{G}$. Fix $O \in \mathcal{A}^R(D)$ and $\beta > 0$. Let $v \in L^p_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^n)$. For $\delta > 0$ we set $O^\delta = \{x \in O : \operatorname{dist}(x, \partial O) < 2\delta\}$. Then for all $\delta > 0$ small enough, $N \in \mathbb{N}$ and $\kappa > 0$ there exists $\varepsilon_0 > 0$ and $C = C_\beta < +\infty$ such that for all $0 < \varepsilon < \varepsilon_0$ and all $\varphi \in \mathcal{N}_p(v, O, \varepsilon, \kappa)$ we have*

$$\left(Z^\beta_{\varepsilon,O}(\mathcal{N}_p(v,O,\varepsilon,\kappa))\right)^{\frac{N-C}{N}} \leqq 2N Z^\beta_{\varepsilon,O}(\mathcal{N}_p(v,O,\varepsilon,3\kappa)\cap\mathcal{B}_\varepsilon(O,\varphi)) \times \exp\left(C\Big(|(O^\delta)_\varepsilon| + \left(\frac{(N\kappa|O|^{\frac{1}{d}})^p}{\delta^p} + \frac{1}{N}\right)|O^{\mathcal{L}}_\varepsilon| + H_\varepsilon(O^\delta,\varphi)\Big)\right).$$

4. Thermodynamic Limit of the Free Energy: Proof of Theorem 1.3

As made clear in the statements of the main result, linear boundary conditions are the basic ingredients to define the continuum free energy density. Let $\Lambda \in \mathbb{R}^{n\times d}$, and D be a Lipschitz subset of $\mathbb{R}^d$. We denote by $\varphi_\Lambda : \mathcal{L} \to \mathbb{R}^n$ the function defined by $\varphi_\Lambda(x) = \Lambda x$ and by $\overline{\varphi}_\Lambda : \mathbb{R}^d \to \mathbb{R}^n$ its continuum version $x \mapsto \Lambda x$ (this distinction will be needed when we identify φ_Λ with its piecewise constant interpolation on the Voronoi tessellation; see below). In this section we better characterize the asymptotic behavior of the functionals $\mathcal{E}_\varepsilon^\beta(D, \overline{\varphi}_\Lambda)$. We first show that for stationary graphs there exists a limit of the free energy when $\varepsilon \to 0$ and, following the approach of [34], we give some useful equivalent characterizations. Then we show that the limit inherits the p-growth conditions of Hypothesis 1. Finally, we prove its quasiconvexity and conclude with Theorem 1.3.

4.1. Existence of $\overline{W}^\beta$ and Equivalent Definitions

We shall prove the almost sure existence of the limit $\lim_{\varepsilon\downarrow 0} \mathcal{E}_\varepsilon(D, \overline{\varphi}_\Lambda)$ using the subadditive ergodic theorem, cf. [2, Theorem 2.7]. We set $\mathcal{I} = \{[a, b) : a, b \in \mathbb{R}^d, a \neq b\}$, where $[a, b) := \{x \in \mathbb{R}^d : \ a_i \leqq x_i < b_i \ \forall i\}$.

Proposition 2. *Assume Hypothesis 1. Fix $\Lambda \in \mathbb{R}^{n\times d}$. Then there exists a deterministic constant $\overline{W}^\beta(\Lambda)$ such that for all Lipschitz domains D we have almost surely*

$$\overline{W}^\beta(\Lambda) = \lim_{\varepsilon\downarrow 0} \mathcal{E}_\varepsilon^\beta(D, \overline{\varphi}_\Lambda).$$

Remark 9. In the above statement the exceptional set may depend on Λ (and β). Later on we shall prove that $\overline{W}$ is continuous, which implies that the set can be taken independent of Λ (and β).

Proof of Proposition 2. We drop the superscript β, and start with defining a suitable stochastic process (that is, a measurable function on the set of graphs $\mathcal{G}$). Given $I \in \mathcal{I}$, set

$$\sigma(I) := -\log\left(\int_{\mathcal{B}_1(I,\varphi_\Lambda)} \exp\big(-H_1(I, u)\big)\,\mathrm{d}u\right) + C_\Lambda\,\mathcal{H}^{d-1}(\partial I), \qquad (4.1)$$

where C_Λ will be chosen later to make the process subadditive. In order to apply the subadditive ergodic Theorem it is enough to prove:

(a) that $|\sigma(I, G)|$ is bounded uniformly with respect to G,
(b) that $G \mapsto \sigma(I, G)$ is a stationary process,
(c) that $I \mapsto \sigma(I, G)$ is subadditive.

We split the rest of the proof into four steps, prove (a), (b), and (c) separately, and then conclude.
Step 1. Proof of (a).

In order to show that $\sigma(I,\cdot)$ is integrable, we use Hypothesis 1, (3.1), and Remark 6 in the form

$$H_1(I,u) \leqq C\|\nabla_{\mathbb{B}} u\|^p_{\ell^p_1(I)} + C|I| \leqq C\|\nabla_{\mathbb{B}}(u-\varphi_\Lambda)\|^p_{\ell^p_1(I)} + C\|\nabla_{\mathbb{B}}\varphi_\Lambda\|^p_{\ell^p_1(I)} + C|I|$$
$$\leqq C\|u-\varphi_\Lambda\|^p_{\ell^p_1(I)} + C(|\Lambda|^p+1)|I|.$$

Since $\mathcal{B}_1(I,\varphi_\Lambda)-\varphi_\Lambda = \mathcal{B}_1(I,0)$, we obtain by a change of variables, monotonicity, Fubini's theorem, and (3.1) again,

$$\begin{aligned}\sigma(I) \leqq\ & C(|\Lambda|^p+1)|I| - \log\left(\int_{\mathcal{B}_1(I,0)} \exp\big(-C\|u\|^p_{\ell^p_1(I)}\big)\,\mathrm{d}u\right) + C_\Lambda \mathcal{H}^{d-1}(\partial I)\\ \leqq\ & C(|\Lambda|^p+1)|I| - \#\{x\in\mathcal{L}\cap I:\ \operatorname{dist}(x,\partial I)\\ & > C_0\}\log\left(\int_{\mathbb{R}^n}\exp(-C|\zeta|^p)\,\mathrm{d}\zeta\right)\\ & - \#\{x\in\mathcal{L}\cap I:\ \operatorname{dist}(x,\partial I)\leqq C_0\}\log\left(\int_{B_1(0)}\exp(-C|\zeta|^p)\,\mathrm{d}\zeta\right)\\ & + C_\Lambda\mathcal{H}^{d-1}(\partial I)\\ \leqq\ & C(|\Lambda|^p+1)|I| + C|I|\big|\log C\big| + C\mathcal{H}^{d-1}(\partial I)\big|\log C\big| + C_\Lambda\mathcal{H}^{d-1}(\partial I).\end{aligned} \tag{4.2}$$

From Hypothesis 1 and Lemma 3.3 applied with $V=\mathcal{B}_1(I,\varphi_\Lambda)$, $z_x=\Lambda x$, $\gamma=1$ and $\alpha=\frac{1}{C}$, we also deduce that

$$Z_{1,I,\varphi_\Lambda} \leqq \int_{\mathcal{B}_1(I,\varphi_\Lambda)} \exp\left(-\frac{1}{C}\|\nabla_{\mathbb{B}}u\|^p_{\ell^p_1(I)} + C|I|\right)\mathrm{d}u \leqq \exp(C|I|)C^{|\mathcal{L}\cap I|} \leqq C^{|I|}.$$

Taking minus the logarithm we obtain that

$$\sigma(I) \geqq -|I|\log C. \tag{4.3}$$

The desired estimate (a) follows from the combination of (4.2) and (4.3).
Step 2. Proof of (b).
Let $z\in\mathbb{Z}^d$. By definition of a graph and of $\mathcal{B}_1$, we have the equivalence

$$u\in\mathcal{B}_1(I,\varphi_\Lambda,G+z) \iff w(\cdot)=u(\cdot+z)-\Lambda z\in\mathcal{B}_1(I-z,\varphi_\Lambda,G)$$

and $H_1(I,u,G+z)=H_1(I-z,w,G)$. This implies stationarity of σ in the form $\sigma(I,z+G)=\sigma(I-z,G)$.
Step 3. Proof of (c).
In order to prove subadditivity, let $I\in\mathcal{I}$ and consider a finite partition $I=\bigcup_i I_i$ with $I_i\in\mathcal{I}$. By definition,

$$\mathcal{B}_1(I,\varphi_\Lambda)\supset\{u:\mathcal{L}\cap I\to\mathbb{R}^n:\ u_{|\mathcal{L}\cap I_i}\in\mathcal{B}_1(I_i,\varphi_\Lambda)\} = \prod_i \mathcal{B}_1(I_i,\varphi_\Lambda). \tag{4.4}$$

Moreover, for any $u \in \prod_i \mathcal{B}_1(I_i, \varphi_\Lambda)$ the monotonicity of $H_1(\cdot, u)$ with respect to set inclusion yields the (almost) subadditivity estimate

$$H_1(I, u) \leqq \sum_i H_1(I_i, u_{|\mathcal{L}\cap I_i}) + \sum_i \sum_{\substack{(x,y)\in\mathbb{B} \\ [x,y]\cap\partial I_i\setminus\partial I\neq\emptyset}} C(|\Lambda|^p + 1)$$
$$\leqq \sum_i H_1(I_i, u_{|\mathcal{L}\cap I_i}) + C(|\Lambda|^p + 1)\sum_i \mathcal{H}^{d-1}(\partial I_i\setminus\partial I). \qquad (4.5)$$

From (4.4) and (4.5) we conclude by Fubini's Theorem that for $C_\Lambda \geqq C(|\Lambda|^p + 1)$ we have

$$\sigma(I) \leqq \sum_i \big(\sigma(I_i) - C_\Lambda \mathcal{H}^{d-1}(\partial I_i)\big) + C(|\Lambda|^p + 1)\sum_i \mathcal{H}^{d-1}(\partial I_i\setminus\partial I)$$
$$+ C_\Lambda \mathcal{H}^{d-1}(\partial I)$$
$$\leqq \sum_i \sigma(I_i) + \big(C(|\Lambda|^p + 1) - C_\Lambda\big)\sum_i \mathcal{H}^{d-1}(\partial I_i\setminus\partial I) \leqq \sum_i \sigma(I_i),$$

that is, the desired subadditivity.

Step 4. Conclusion.

By the subadditive ergodic Theorem (combined with an elementary approximation argument to pass from integer rectangles to general rectangles and Lipschitz domains, see for instance Step 4 of the proof of Theorem 3.1 in [32]), we obtain the existence of the deterministic field $\overline{W}(\Lambda)$ satisfying almost surely for all Lipschitz domains D

$$\overline{W}(\Lambda) = \lim_{t\uparrow\infty} \frac{1}{t^d}\mathcal{E}_1(tD, \varphi_\Lambda).$$

□

Following [34] we next prove two equivalent characterizations of $\overline{W}^\beta$.

Lemma 4.1. *Assume Hypothesis 1. Fix $\Lambda \in \mathbb{R}^{n\times d}$. Then $\overline{W}^\beta(\Lambda)$ defined in Proposition 2 almost surely satisfies: For all $\kappa > 0$ and all $O \in \mathcal{A}^R(\mathbb{R}^d)$,*

$$\overline{W}^\beta(\Lambda) = \lim_{\varepsilon\downarrow 0} -\frac{1}{|O_\varepsilon|}\log\Big(Z^\beta_{\varepsilon,O}(\mathcal{N}_p(\overline{\varphi}_\Lambda, O, \varepsilon, \kappa)\cap\mathcal{B}_\varepsilon(O, \varphi_\Lambda))\Big) \qquad (\mathrm{I})$$

$$= \mathcal{F}^+(O, \overline{\varphi}_\Lambda). \qquad (\mathrm{II})$$

Proof of Lemma 4.1. We split the proof into 4 steps. Again we drop the superscript β.

Step 1. Existence of the limit.

We first prove that the right hand side of (I) is well-defined. Again we use the subadditive ergodic theorem. To this end, note that due to (3.1) and the definition of the sets $\mathcal{N}_p$ in (3.7) there exists a deterministic length of the form $l_\kappa = \lceil C_1(|\Lambda| + 1)/\kappa\rceil \in \mathbb{N}$ such that, for any $I \in \mathcal{I}$,

$$\varphi_\Lambda \in \mathcal{N}_p\Big(\overline{\varphi}_\Lambda, l_\kappa I, 1, \frac{\kappa}{2}\Big)\cap\mathcal{B}_1(l_\kappa I, \varphi_\Lambda). \qquad (4.6)$$

We fix such C_1 from now on and define the stochastic process $\sigma_\kappa : \mathcal{I} \to L^1(\mathcal{G})$ by

$$\sigma_\kappa(I) = -\log\big(Z_{1,I}(\mathcal{N}_p(\overline{\varphi}_\Lambda, l_\kappa I, 1, \kappa) \cap \mathcal{B}_1(l_\kappa I, \varphi_\Lambda))\big) + C_{\sigma_\kappa}\mathcal{H}^{d-1}(\partial(l_\kappa I)), \tag{4.7}$$

where C_{σ_κ} will be chosen later to obtain subadditivity. To show integrability, we first note that $\sigma_\kappa(I) \geqq \sigma(l_\kappa I) - (C_\Lambda - C_{\sigma_\kappa})l_\kappa\mathcal{H}^{d-1}(\partial I)$, where σ is the process defined in the proof of Proposition 2. In order to prove an upper bound, observe that there exists a constant $c > 0$ such that

$$\begin{aligned}&\varphi_\Lambda + \big\{u : l_\kappa I \cap \mathcal{L} \to \mathbb{R}^n : \ |u(x)| \\ &\quad \leqq \min\Big\{1, c(|\Lambda|+1)|I|^{\frac{1}{d}}\Big\}\Big\} \subset \mathcal{N}_p(\overline{\varphi}_\Lambda, l_\kappa I, 1, \kappa) \cap \mathcal{B}_1(l_\kappa I, \varphi_\Lambda).\end{aligned}$$

Indeed, the set on the left hand side clearly satisfies the boundary conditions and is thus contained in $\mathcal{B}_1(l_\kappa I, \varphi_\Lambda)$. The remaining inclusion follows by the triangle inequality since (3.1) and (4.6) yield

$$\begin{aligned}\|(\varphi_\Lambda)_1 - \varphi_\Lambda - u\|_{\ell_1^p(l_\kappa I)} &< \frac{\kappa}{2}|l_\kappa I|^{\frac{1}{p}+\frac{1}{d}} + C|l_\kappa I|^{\frac{1}{p}}\|u\|_\infty \leqq \frac{\kappa}{2}|l_\kappa I|^{\frac{1}{p}+\frac{1}{d}} \\ &\quad + C|l_\kappa I|^{\frac{1}{p}} c(|\Lambda|+1)|I|^{\frac{1}{d}} \\ &\leqq \frac{\kappa}{2}|l_\kappa I|^{\frac{1}{p}+\frac{1}{d}} + C|l_\kappa I|^{\frac{1}{p}}\frac{c}{C_1}\kappa|l_\kappa I|^{\frac{1}{d}} \leqq \kappa|l_\kappa I|^{\frac{1}{p}+\frac{1}{d}},\end{aligned}$$

provided that $c \leqq \frac{C_1}{2C}$. Having in mind the established set-inclusion, the argument for (4.2) also yields a deterministic upper bound, the proof of which we omit.

Concerning stationarity, we recall that the interpolation in (3.4) is random as it depends on the Voronoi cells. By stationarity of G, which is inherited by the Voronoi tessellation, for every $z \in \mathbb{Z}^d$ we have $(\varphi_\Lambda)_1(x+z, G+z) = (\varphi_\Lambda)_1(x, G) + \Lambda z$. Hence, with a slight abuse of notation,

$$\begin{aligned}&\big(\mathcal{N}_p(\overline{\varphi}_\Lambda, l_\kappa(I-z), 1, \kappa, G) \cap \mathcal{B}_1(l_\kappa(I-z), \varphi_\Lambda, G)\big) + l_\kappa\Lambda z \\ &\quad = \mathcal{N}_p(\overline{\varphi}_\Lambda, l_\kappa I, 1, \kappa, G + l_\kappa z) \cap \mathcal{B}_1(l_\kappa I, \varphi_\Lambda, G + l_\kappa z).\end{aligned}$$

where we used that $l_\kappa \in \mathbb{N}$. By a change of variables we obtain the $l_\kappa\mathbb{Z}^d$-stationarity condition

$$\sigma_\kappa(I - l_\kappa z, G) = \sigma_\kappa(I, G + l_\kappa z).$$

From now on the proof is very similar to the one of Proposition 2. Just note that for proving subadditivity, given a partition $I = \bigcup_i I_i$ it holds that

$$\mathcal{N}_p(\overline{\varphi}_\Lambda, l_\kappa I, 1, \kappa) \cap \mathcal{B}_1(l_\kappa I, \varphi_\Lambda) \supset \prod_i \mathcal{N}_p(\overline{\varphi}_\Lambda, l_\kappa I_i, 1, \kappa) \cap \mathcal{B}_1(l_\kappa I_i, \varphi_\Lambda). \tag{4.8}$$

This is clear for the boundary conditions, while for the discrete neighbourhoods it follows from the inequality $\sum_j |I_j|^{1+\frac{p}{d}} \leqq |I|^{1+\frac{p}{d}}$, which is due to the fact that the discrete ℓ^p-norms are maximal for $p = 1$. With (4.8) at hand and choosing a suitable C_{σ_κ}, we conclude, as we did for Proposition 2, that with probability one the following limit exists

$$W_\kappa(\Lambda) = \lim_{\varepsilon\to 0} -\frac{1}{|O_\varepsilon|}\log\left(Z_{\varepsilon,O}(\mathcal{N}_p(\overline{\varphi}_\Lambda, O, \varepsilon, \kappa)\cap\mathcal{B}_\varepsilon(O,\varphi_\Lambda))\right),$$

and is independent of the Lipschitz domain O (note that (3.4) and the definition of the sets $\mathcal{N}_p(\overline{\varphi}_\Lambda, O, \varepsilon, \kappa)$ in (3.7) are compatible with rescaling in the sense that $(\overline{\varphi}_\Lambda)_\varepsilon = \varepsilon(\overline{\varphi}_\Lambda)_1$ and therefore $u \in \mathcal{N}_p(\overline{\varphi}_\Lambda, O, \varepsilon, \kappa)\cap\mathcal{B}_\varepsilon(O,\varphi_\Lambda)$ if and only if $u \in \mathcal{N}_p(\overline{\varphi}_\Lambda, O/\varepsilon, 1, \kappa)\cap\mathcal{B}_1(O/\varepsilon,\varphi_\Lambda))$.

Step 2. Independence with respect to κ.

Let us prove that the limit is independent of κ which turns out to be useful for proving (I) in the next step. We may assume that the limit of Step 1 exists almost surely for all positive $\kappa \in \mathbb{Q}$. As in [34] we compute the energy on a half-open cube $O = [0,1)^d$ which we subdivide again into 2^d smaller half-open cubes $\{O_i\}_{i=1}^{2^d}$ of equal size. In this case we can improve the rescaled version (4.8) in the sense that

$$\mathcal{N}_p(\overline{\varphi}_\Lambda, O, \varepsilon, \frac{\kappa}{2})\cap\mathcal{B}_\varepsilon(O,\varphi_\Lambda) \supset \prod_i \mathcal{N}_p(\overline{\varphi}_\Lambda, O_i, \varepsilon, \kappa)\cap\mathcal{B}_\varepsilon(O_i,\varphi_\Lambda).$$

Indeed, the boundary conditions on the large cube hold by the boundary conditions on the smaller cubes and, for each function u belonging to the right hand side set, the discrete norms in the definition (3.7) can be estimated via

$$\begin{aligned}\sum_{x\in O_\varepsilon^{\mathcal{L}}} \varepsilon^d|(\overline{\varphi}_\Lambda)_\varepsilon(x) - \varepsilon u(x)|^p &\leqq 2^d\kappa^p|O_1|^{1+\frac{p}{d}}\\ &= \left(\frac{\kappa}{2}\right)^p (2^d)^{1+\frac{p}{d}}|O_1|^{1+\frac{p}{d}} = \left(\frac{\kappa}{2}\right)^p |O|^{1+\frac{p}{d}}.\end{aligned}$$

A rescaled version of the almost subadditivity estimate (4.5) and Fubini's Theorem then yield

$$\begin{aligned}\sum_{i=1}^{2^d}\Big(&C(|\Lambda|^p+1)\mathcal{H}^{d-1}(\partial O_i)\varepsilon\\ &-\frac{1}{|(O_i)_\varepsilon|}\log\left(Z_{\varepsilon,O_i}(\mathcal{N}_p(\overline{\varphi}_\Lambda, O_i, \varepsilon, \kappa)\cap\mathcal{B}_\varepsilon(O_i,\varphi_\Lambda))\right)\frac{|(O_i)_\varepsilon|}{|O_\varepsilon|}\Big)\\ &\geqq -\frac{1}{|O_\varepsilon|}\log\left(Z_{\varepsilon,O}(\mathcal{N}_p(\overline{\varphi}_\Lambda, O, \varepsilon, \frac{\kappa}{2})\cap\mathcal{B}_\varepsilon(O,\varphi_\Lambda))\right).\end{aligned}$$

Passing to the limit when $\varepsilon \to 0$ we obtain by definition

$$W_\kappa(\Lambda) \geqq W_{\kappa/2}(\Lambda),$$

where used that the limit is indeed also given by half-open cubes because the contributions at the boundary are negligible. Since the reverse inequality is obvious this proves that the two terms actually agree. In particular, by a sandwich argument we deduce that for a set of full probability the limit exists for all $\kappa > 0$ and is independent of κ.

Step 3. Proof of (I).
We argue by letting $\kappa \to +\infty$. Since we shall use this equality to show that the exceptional set can be taken independent of Λ, we only use deterministic arguments.

On the one hand, by Hypothesis 1, for any $u \in \mathcal{B}_\varepsilon(O, \varphi_\Lambda)$ and ε small enough we have by the discrete Poincaré inequality of Lemma 3.1

$$\begin{aligned} H_\varepsilon(u, O) &\geqq \frac{1}{C}\|\nabla_{\mathbb{B}} u\|^p_{\ell^p_\varepsilon(O)} - C|O^{\mathcal{L}}_\varepsilon| \geqq \frac{1}{C}\|\nabla_{\mathbb{B}}(u - \varphi_\Lambda)\|^p_{\ell^p_\varepsilon(O)} - C(|\Lambda|^p + 1)|O^{\mathcal{L}}_\varepsilon| \\ &\geqq \frac{1}{C}\|\varepsilon u - \varepsilon\varphi_\Lambda\|^p_{\ell^p_\varepsilon(O)} - C(|\Lambda|^p + 1)|O^{\mathcal{L}}_\varepsilon|. \end{aligned}$$

On the other hand, if $u \notin \mathcal{N}_p(\overline{\varphi}_\Lambda, O, \varepsilon, \kappa)$ then from the definition in (3.4) and (3.1) we infer that

$$\begin{aligned} \kappa^p|O|^{1+\frac{p}{d}} \leqq C\sum_{x\in O^{\mathcal{L}}_\varepsilon} \varepsilon^d\left(|\varepsilon u(x) - \varepsilon\Lambda x|^p + |\Lambda|^p\varepsilon^p\right) \leqq C\varepsilon^d\|\varepsilon u - \varepsilon\varphi_\Lambda\|^p_{\ell^p_\varepsilon(O)} \\ +C|\Lambda|^p\varepsilon^{p+d}|O^{\mathcal{L}}_\varepsilon|. \end{aligned}$$

Combining these inequalities, we infer that, given $M > 0$ there exists $\kappa_0 > 0$ such that for all $\kappa > \kappa_0$ it holds that $\mathcal{B}_\varepsilon(O, \varphi_\Lambda)\backslash\mathcal{N}_p(\overline{\varphi}_\Lambda, O, \varepsilon, \kappa) \subset \mathcal{B}_\varepsilon(O, \varphi_\Lambda)\backslash \mathcal{S}_M(O, \varepsilon)$. Now we choose

$$M > 2\left(\overline{C} + \log(2)|O^{\mathcal{L}}_\varepsilon|^{-1} + \frac{1}{|O^{\mathcal{L}}_\varepsilon|}\log\left(Z_{\varepsilon,O}(\mathcal{B}_\varepsilon(O, \varphi_\Lambda))\right)\right), \tag{4.9}$$

where $\overline{C}$ denotes the constant from Lemma 3.4. Note that due to (4.3), such M can be chosen independent of $0 < \varepsilon \leqq \varepsilon_0$ for some $\varepsilon_0 = \varepsilon_0(O)$ which depends not on the graph G. The second estimate of Lemma 3.4 yields for κ large enough and this choice of M

$$\begin{aligned} Z_{\varepsilon,O}(\mathcal{B}_\varepsilon(O, \varphi_\Lambda)\backslash\mathcal{N}_p(\overline{\varphi}_\Lambda, O, \varepsilon, \kappa)) &\leqq Z_{\varepsilon,O}(\mathcal{B}_\varepsilon(O, \varphi_\Lambda)\backslash\mathcal{S}_M(O, \varepsilon)) \\ &\leqq \exp\left(-\frac{M}{2}|O^{\mathcal{L}}_\varepsilon|\right)\exp(\overline{C}|O^{\mathcal{L}}_\varepsilon|) \\ &\leqq \frac{1}{2}Z_{\varepsilon,O}(\mathcal{B}_\varepsilon(O, \varphi_\Lambda)) \end{aligned}$$

or equivalently

$$Z_{\varepsilon,O}(\mathcal{B}_\varepsilon(O, \varphi_\Lambda)) \geqq Z_{\varepsilon,O}(\mathcal{B}_\varepsilon(O, \varphi_\Lambda) \cap \mathcal{N}_p(\overline{\varphi}_\Lambda, O, \varepsilon, \kappa)) \geqq \frac{1}{2}Z_{\varepsilon,O}(\mathcal{B}_\varepsilon(O, \varphi_\Lambda)).$$

Taking logarithms and dividing by $-|O_\varepsilon|$ we obtain the claim letting $\varepsilon \to 0$ and using Step 2.

Step 4. Proof of (II).
We apply the interpolation inequality in Proposition 1 with $\varphi = \overline{\varphi}_\Lambda$ and $\phi = \varphi_\Lambda$. Note that for ε small enough it holds that $\varphi_\Lambda \in \mathcal{N}_p(\overline{\varphi}_\Lambda, O, \varepsilon, \kappa)$. Taking logarithms

in the interpolation inequality and dividing by $-|O_\varepsilon|$, we obtain by Steps 2 and 3, and Hypothesis 1

$$\frac{N-C}{N}\mathcal{F}_\kappa^+(O,\overline{\varphi}_\Lambda) \geqq \overline{W}(\Lambda) - C\left(\left(\frac{(N\kappa|O|^{\frac{1}{d}})^p}{\delta^p}+\frac{1}{N}\right)+(|\Lambda|^p+1)\frac{|O^\delta|}{|O|}\right).$$

The claim now follows taking first the limit $\kappa \to 0$ and then $N \to +\infty$ and $\delta \to 0$. On the other hand the reverse inequality follows by Step 3 and a monotonicity argument based on set inclusion. □

Remark 10. The estimates of Step 3 in the proof of Lemma 4.1 show that the limit defining $\overline{W}^\beta(\Lambda)$ exists whenever G is admissible and the limit defining $W_\kappa(\Lambda)$ exists.

Using the observation of the previous remark, we now show that the exceptional set where convergence of the free energy may fail can be taken independent of the macroscopic boundary condition Λ. As a byproduct we obtain the continuity of the maps $\Lambda \mapsto \overline{W}^\beta(\Lambda)$.

Proposition 3. *Assume Hypothesis 1. Then for almost all $G \in \mathcal{G}$, all $\Lambda \in \mathbb{R}^{n\times d}$, all $\beta > 0$ and all bounded Lipschitz domains $D \subset \mathbb{R}^d$ there exists the limit*

$$\overline{W}^\beta(\Lambda) = \lim_{\varepsilon\to 0}\mathcal{E}_\varepsilon^\beta(D,\overline{\varphi}_\Lambda)$$

and the further statements of Proposition 2 and the equivalent characterizations of Lemma 4.1 remain true. In addition the map $\Lambda \mapsto \overline{W}^\beta(\Lambda)$ is continuous.

Proof of Proposition 3. In view of Remark 10, the first claim proven once we show that there exists a common set $\mathcal{G}' \subset \mathcal{G}$ of full probability such that the limit

$$W_\kappa(\Lambda) = \lim_{\varepsilon\to 0} -\frac{1}{|O_\varepsilon|}\log\left(Z_{\varepsilon,O}(\mathcal{N}_p(\overline{\varphi}_\Lambda, O, \varepsilon, \kappa)\cap\mathcal{B}_\varepsilon(O,\varphi_\Lambda))\right) \tag{4.10}$$

exists for all $G \in \mathcal{G}'$, all $\kappa > 0$, all $\Lambda \in \mathbb{R}^{n\times d}$ and all $O \in \mathcal{A}^R(\mathbb{R}^d)$. By Step 3 of the proof of Lemma 4.1 we know that for any $\Lambda \in \mathbb{Q}^{n\times d}$ we find a set $\mathcal{G}_\Lambda \subset \mathcal{G}$ of full probability such that the limit in (4.10) exists for all $\kappa > 0$, all $G \in \mathcal{G}_\Lambda$ and all $O \in \mathcal{A}^R(\mathbb{R}^d)$. Let us set $\mathcal{G}' = \bigcap_{\Lambda\in\mathbb{Q}^{n\times d}}\mathcal{G}_\Lambda$. We fix $G \in \mathcal{G}'$ and set

$$\overline{W}_\kappa(\Lambda, O) = \limsup_{\varepsilon\to 0} -\frac{1}{|O_\varepsilon|}\log\left(Z_{\varepsilon,O}(\mathcal{N}_p(\overline{\varphi}_\Lambda, O, \varepsilon, \kappa)\cap\mathcal{B}_\varepsilon(O,\varphi_\Lambda))\right),$$
$$\underline{W}_\kappa(\Lambda, O) = \liminf_{\varepsilon\to 0} -\frac{1}{|O_\varepsilon|}\log\left(Z_{\varepsilon,O}(\mathcal{N}_p(\overline{\varphi}_\Lambda, O, \varepsilon, \kappa)\cap\mathcal{B}_\varepsilon(O,\varphi_\Lambda))\right).$$

We first argue by rational approximation that these two terms actually agree. To this end fix $\Lambda, \Lambda' \in \mathbb{R}^{n\times d}$ and $O \in \mathcal{A}^R(\mathbb{R}^d)$. For $\delta > 0$ we define the set $O^\delta = \{x \in O :\ \operatorname{dist}(x, \partial O) > \delta\}$. Taking δ small enough, we may assume that $O_{2\delta} \in \mathcal{A}^R(D)$

(see, for instance, [32, Lemma 2.2]). We now define suitable interpolations. Let $\theta_\delta : O \to [0,1]$ be the Lipschitz-continuous cut-off function

$$\theta_\delta(z) = \min\left\{\max\left\{\frac{1}{\delta}\mathrm{dist}(z,\partial O) - 1, 0\right\}, 1\right\}.$$

By construction we have $\theta_\delta \equiv 1$ on $\overline{O_{2\delta}}$, $\theta_\delta \equiv 0$ on $O\backslash O^\delta$ and the Lipschitz constant is bounded by $\mathrm{Lip}(\theta_\delta) \leqq \frac{1}{\delta}$. Given $\varphi : O_\varepsilon^{\mathcal{L}} \to \mathbb{R}^n$ and $\psi : (O\backslash O_{2\delta})_\varepsilon \to \mathbb{R}^n$ we define the interpolation $T_{\varepsilon,\delta}(\varphi,\psi) : O_\varepsilon^{\mathcal{L}} \to \mathbb{R}^n$ setting

$$T_{\varepsilon,\delta}(\varphi,\psi)(x) = \theta_\delta(\varepsilon x)\varphi(x) + (1-\theta_\delta(\varepsilon x))\psi(x).$$

Assume in addition that $\varphi \in \mathcal{B}_\varepsilon(O_{2\delta},\varphi_\Lambda)\times\mathcal{N}_\infty(\varphi_\Lambda, O\backslash O_{2\delta},\varepsilon)$ and $\psi \in \mathcal{N}_\infty(\varphi_{\Lambda'}, O\backslash O_{2\delta},\varepsilon)$. Then Hypothesis 1 implies that

$$H_\varepsilon(O, T_{\varepsilon,\delta}(\varphi,\psi)) \leqq H_\varepsilon(O_{2\delta},\varphi) + C\sum_{\substack{(x,y)\in\mathbb{B}\\ \varepsilon x\in O\backslash O_{2\delta},\varepsilon y\in O}} (|T_{\varepsilon,\delta}(\varphi,\psi)(x) - T_{\varepsilon,\delta}(\varphi,\psi)(y)|^p + 1) \tag{4.11}$$

To bound the last term we use the algebraic formula

$$T_{\varepsilon,\delta}(\varphi,\psi)(x) - T_{\varepsilon,\delta}(\varphi,\psi)(y) = \theta_\delta(\varepsilon y)(\varphi(x)-\varphi(y)) + (1-\theta_\delta(\varepsilon y))(\psi(x)-\psi(y)) + (\theta_\delta(\varepsilon x)-\theta_\delta(\varepsilon y))(\varphi(x)-\psi(x)).$$

For all $(x,y)\in E$ such that $\varepsilon x \in O\backslash O_{2\delta}$ and $\varepsilon y \in O$, the boundary values of φ and the L^∞-restrictions on φ,ψ combined with the bound on $\mathrm{Lip}(\theta_\delta)$ yield

$$|T_{\varepsilon,\delta}(\varphi,\psi)(x) - T_{\varepsilon,\delta}(\varphi,\psi)(y)| \leqq C(1+|\Lambda|+|\Lambda'|) + \frac{C\varepsilon}{\delta}|\Lambda-\Lambda'||x| \leqq C(1+|\Lambda|+|\Lambda'|) + \frac{C}{\delta}|\Lambda-\Lambda'|,$$

where we used that $\varepsilon x \in O$ has equibounded norm for fixed O. Taking the p^{th} power in this inequality, we can further estimate (4.11) by

$$H_\varepsilon(O, T_{\varepsilon,\delta}(\varphi,\psi)) \leqq H_\varepsilon(O_{2\delta},\varphi) + \Big(C(1+|\Lambda|+|\Lambda'|)^p + \frac{C}{\delta^p}|\Lambda-\Lambda'|^p\Big)|(O\backslash O_{2\delta})_\varepsilon^{\mathcal{L}}|. \tag{4.12}$$

To reduce notation, we set $\sigma_\delta(\Lambda,\Lambda') := (1+|\Lambda|+|\Lambda'|)^p + \frac{1}{\delta^p}|\Lambda-\Lambda'|^p$ and define the set

$$\mathcal{S} = \Big(\mathcal{N}_p(\overline{\varphi}_\Lambda, O_{2\delta},\varepsilon,\kappa)\cap\mathcal{B}_\varepsilon(O_{2\delta},\varphi_\Lambda)\Big)\times\mathcal{N}_\infty(\varphi_\Lambda, O\backslash O_{2\delta},\varepsilon).$$

Using (4.12) and the fact that $|\mathcal{N}_\infty(\varphi_\Lambda, O\backslash O_{2\delta},\varepsilon)| \times |\mathcal{N}_\infty(\varphi_{\Lambda'}, O\backslash O_{2\delta},\varepsilon)| \geqq \exp(-C|(O\backslash O_{2\delta})_\varepsilon^{\mathcal{L}}|)$, we deduce from Fubini's Theorem that

$$Z_{\varepsilon,O_{2\delta}}(\mathcal{N}_p(\overline{\varphi}_\Lambda, O_{2\delta},\varepsilon,\kappa)\cap\mathcal{B}_\varepsilon(O_{2\delta},\varphi_\Lambda))\exp\big(-C\sigma_\delta(\Lambda,\Lambda')|(O\backslash O_{2\delta})_\varepsilon^{\mathcal{L}}|\big)$$

$$\leqq \int_{\mathcal{S}\times\mathcal{N}_\infty(\varphi_{\Lambda'},O\setminus O_{2\delta},\varepsilon)} \exp(-H_\varepsilon(O,T_{\varepsilon,\delta}(\varphi,\psi)))\,\mathrm{d}\varphi\,\mathrm{d}\psi. \tag{4.13}$$

In order to provide an upper bound for the integral on the right hand side, we change the variables. A computation yields for any $(\varphi,\psi)\in\mathcal{S}\times\mathcal{N}_\infty(\varphi_{\Lambda'},O\setminus O_{2\delta},\varepsilon)$ the estimate

$$\begin{aligned}&\varepsilon^{\frac{d}{p}}\|(\overline{\varphi}_{\Lambda'})_\varepsilon-\varepsilon T_{\varepsilon,\delta}(\varphi,\psi)\|_{\ell^p_\varepsilon(O)}\\&\quad\leqq\kappa|O_{2\delta}|^{\frac{1}{d}+\frac{1}{p}}+C|\Lambda-\Lambda'||O|^{\frac{1}{p}}+C\varepsilon(1+|\Lambda|+|\Lambda'|)|(O\setminus O_{2\delta})_\varepsilon|^{\frac{1}{p}}\varepsilon^{\frac{d}{p}}.\end{aligned}$$

Hence we find $\eta_\kappa=\eta_\kappa(O)>0$ such that for all ε small enough we have the implication

$$|\Lambda-\Lambda'|<\eta_\kappa \qquad\Rightarrow\qquad T_{\varepsilon,\delta}(\varphi,\psi)\in\mathcal{N}_p(\overline{\varphi}_{\Lambda'},O,\varepsilon,3\kappa)\cap\mathcal{B}_\varepsilon(O,\varphi_{\Lambda'}) \tag{4.14}$$

for all $(\varphi,\psi)\in\mathcal{S}\times\mathcal{N}_\infty(\varphi_{\Lambda'},O\setminus O_{2\delta},\varepsilon)$. In particular this implication is independent of δ. Introducing the function $b:(O\setminus O_{2\delta})^{\mathcal{L}}_\varepsilon\to\mathbb{R}^n$ defined by

$$b(x)=\begin{cases}\Lambda'x & \text{if } \theta_\delta(\varepsilon x)\geqq\frac{1}{2},\\ \Lambda x & \text{if } \theta_\delta(\varepsilon x)<\frac{1}{2},\end{cases}$$

for ε small enough and $|\Lambda-\Lambda'|<\eta_\kappa$ we can define the linear mapping $\Phi_{\varepsilon,\delta}:\mathcal{S}\times\mathcal{N}_\infty(\varphi_{\Lambda'},O\setminus O_{2\delta},\varepsilon)\to\mathcal{N}_p(\overline{\varphi}_{\Lambda'},O,\varepsilon,3\kappa)\times\mathcal{N}_\infty(b,O\setminus O_{2\delta},\varepsilon)$ by setting (with a slight abuse of notation)

$$\Phi_{\varepsilon,\delta}(\varphi,\psi)(x)=\begin{cases}(T_{\varepsilon,\delta}(\varphi,\psi)(x),\psi(x)) & \text{if } \theta_\delta(\varepsilon x)\geqq\frac{1}{2},\\ (T_{\varepsilon,\delta}(\varphi,\psi)(x),\varphi(x)) & \text{if } \theta_\delta(\varepsilon x)<\frac{1}{2}.\end{cases}$$

Note that $\Phi_{\varepsilon,\delta}$ is well-defined due to (4.14) and bijective onto its range $\mathcal{R}(\Phi_{\varepsilon,\delta})$. In order to calculate the Jacobian, it is convenient to number the points $x\in O^{\mathcal{L}}_\varepsilon$ and view the state space as large vectors by putting as first component the value $\varphi(x(1))$ and as second either $\psi(x(1))$ if $x(1)\in O\setminus O_{2\delta}$ or $\varphi(x(2))$ otherwise. Continuing this procedure the matrix representation of $\Phi_{\varepsilon,\delta}$ has non-zero entries only in 2×2-matrices around the diagonal. Thus the determinant splits into products and we obtain

$$\begin{aligned}|\det(D\Phi_{\varepsilon,\delta}(\varphi,\psi))|^{-1}&=\left(\prod_{x:\theta_\delta(\varepsilon x)\geqq\frac{1}{2}}|\theta_\delta(\varepsilon x)|^n\prod_{x:\theta_\delta(\varepsilon x)<\frac{1}{2}}|1-\theta_\delta(\varepsilon x)|^n\right)^{-1}\\&\leqq\exp(C|(O\setminus O_{2\delta})^{\mathcal{L}}_\varepsilon|).\end{aligned}$$

Via the change of variables $(g,h)=\Phi_{\varepsilon,\delta}(\varphi,\psi)$ and (4.14) we can estimate the right hand side integral in (4.13) by

$$\int_{\mathcal{S}\times\mathcal{N}_\infty(\varphi_{\Lambda'},O\setminus O_{2\delta},\varepsilon)}\exp(-H_\varepsilon(O,T_{\varepsilon,\delta}(\varphi,\psi)))\,\mathrm{d}\varphi\,\mathrm{d}\psi$$

$$\leqq Z_{\varepsilon,O}(\mathcal{N}_p(\overline{\varphi}_{\Lambda'}, O, \varepsilon, 3\kappa) \cap \mathcal{B}_\varepsilon(O, \varphi_{\Lambda'})) \\ \times |\mathcal{N}_\infty(b, O\backslash O_{2\delta}, \varepsilon)| \exp(C|(O\backslash O_{2\delta})^{\mathcal{L}}_\varepsilon|). \tag{4.15}$$

Combining (4.13) and (4.15) we conclude the estimate

$$Z_{\varepsilon,O_{2\delta}}(\mathcal{N}_p(\overline{\varphi}_\Lambda, O_{2\delta}, \varepsilon, \kappa) \cap \mathcal{B}_\varepsilon(O_{2\delta}, \varphi_\Lambda)) \\ \leqq Z_{\varepsilon,O}(\mathcal{N}_p(\Lambda' x, O, \varepsilon, 3\kappa) \cap \mathcal{B}_\varepsilon(O, \varphi_{\Lambda'})) \\ \times \exp\left(C\sigma_\delta(\Lambda, \Lambda')|(O\backslash O_{2\delta})^{\mathcal{L}}_\varepsilon|\right).$$

Taking logarithms, dividing by $-|O_\varepsilon|$ and taking the limes superior on both sides yields

$$\frac{|O_{2\delta}|}{|O|}\overline{W}_\kappa(\Lambda, O_{2\delta}) \geqq \overline{W}_{3\kappa}(\Lambda', O) - C\sigma_\delta(\Lambda, \Lambda')\frac{|O\backslash O_{2\delta}|}{|O|}. \tag{4.16}$$

Replacing $O_{2\delta}$ and O by the sets O and $O^{2\delta}$ respectively, where $O^{2\delta} = \{x \in \mathbb{R}^d : \mathrm{dist}(x, O) < 2\delta\}$, and switching the roles of Λ and Λ' we can prove in exactly the same way the estimate

$$\underline{W}_{3\kappa}(\Lambda', O) \geqq \frac{|O^{2\delta}|}{|O|}\underline{W}_{9\kappa}(\Lambda, O^{2\delta}) - C\sigma_\delta(\Lambda, \Lambda')\frac{|O^{2\delta}\backslash O|}{|O|}. \tag{4.17}$$

Further we may assume that $O^{2\delta} \in \mathcal{A}^R(\mathbb{R}^d)$. Choosing $\Lambda = \Lambda_j \in \mathbb{Q}^{n\times d}$ such that $\Lambda_j \to \Lambda'$, the two inequalities (4.16) and (4.17) yield

$$0 \leqq \overline{W}_{3\kappa}(\Lambda', O) - \underline{W}_{3\kappa}(\Lambda', O) \leqq \frac{|O^{2\delta}\backslash O_{2\delta}|}{|O|}\left(|\overline{W}(\Lambda_j)| + C\sigma_\delta(\Lambda_j, \Lambda')\right).$$

Letting first $j \to +\infty$ and then $\delta \to 0$ the right hand side vanishes since $\overline{W}(\Lambda)$ is locally bounded (see the estimates (4.2) and (4.3)). Hence the limit in (4.10) indeed exists.

We now show that it is independent of O and κ. Choosing Λ_j as above, we infer again from (4.16) that

$$W_{3\kappa}(\Lambda', O) \leqq \liminf_j \overline{W}(\Lambda_j) + \left(1 - \frac{|O_{2\delta}|}{|O|}\right)\sup_j |\overline{W}(\Lambda_j)| \\ + C\frac{|O\backslash O_{2\delta}|}{|O|}(1 + 2|\Lambda'|^p).$$

Letting $\delta \to 0$ we obtain $W_{3\kappa}(\Lambda', O) \leqq \liminf_j \overline{W}(\Lambda_j)$. On the other hand, (4.17) and a similar reasoning yield $W_{3\kappa}(\Lambda', O) \geqq \limsup_j \overline{W}(\Lambda_j)$, so that

$$W_{3\kappa}(\Lambda', O) = \lim_j \overline{W}(\Lambda_j)$$

is independent of κ and O. Repeating the deterministic argument from Step 3 of the proof of Lemma 4.1 one can show that $W_{3\kappa}(\Lambda', O) = \overline{W}(\Lambda')$ for all $\kappa > 0$. Thus continuity can be proven using again (4.16) and (4.17) since there is no κ-dependence any more.

Finally, as the only random construction in the proof of Lemma 4.1 was the existence of the limits in (4.10) it is clear that the characterizations still hold true and, by continuity, so do the additional properties stated in Proposition 2. □

Remark 11. As we intend to vary the temperature in Section 6, let us observe that the set $\mathcal{G}' \subset \mathcal{G}$ of full probability given by Proposition 3 can be chosen also independent of $\beta > 0$. Indeed, first we choose a set of full probability such that Proposition 3 holds for all rational $\beta \geqq 1$. Then for given $\beta \geqq 1$ we take a rational sequence $\beta_j > \beta$ such that $\beta_j \downarrow \beta$. The remaining argument relies on Remark 10: Fix $\kappa > 0$ and a set $O \in \mathcal{A}^R(\mathbb{R}^d)$. Then by monotonicity and Lemma 3.4 with a suitable $M = M(\beta, \Lambda)$ (see e.g. (4.9)), we have for all ε small enough the inequality

$$
\begin{aligned}
&Z^{\beta_j}_{\varepsilon,O}(\mathcal{N}_p(\overline{\varphi}_\Lambda, O, \varepsilon, \kappa) \cap \mathcal{B}_\varepsilon(O, \varphi_\Lambda)) \\
&\quad \leqq Z^{\beta}_{\varepsilon,O}(\mathcal{N}_p(\overline{\varphi}_\Lambda, O, \varepsilon, \kappa) \cap \mathcal{B}_\varepsilon(O, \varphi_\Lambda)) \\
&\quad \leqq Z^{\beta}_{\varepsilon,O}(\mathcal{B}_\varepsilon(O, \varphi_\Lambda)) \leqq 2Z^{\beta}_{\varepsilon,O}(\mathcal{B}_\varepsilon(O, \varphi_\Lambda) \cap \mathcal{S}_M(O, \varepsilon)) \\
&\quad \leqq 2Z^{\beta_j}_{\varepsilon,Q}(\mathcal{B}_\varepsilon(O, \varphi_\Lambda) \cap \mathcal{S}_M(O, \varepsilon)) \times \exp(M(\beta_j - \beta)|O^{\mathcal{L}}_\varepsilon|) \\
&\quad \leqq 2Z^{\beta_j}_{\varepsilon,O}(\mathcal{B}_\varepsilon(O, \varphi_\Lambda)) \times \exp(M(\beta_j - \beta)|O^{\mathcal{L}}_\varepsilon|).
\end{aligned}
$$

Taking the logarithm and dividing $-\beta|O_\varepsilon|$, we obtain by Lemma 4.1 and Proposition 3 that the limit corresponding to β exists and is independent of κ and O. Moreover, as a by-product, we proved a continuous dependence on β.

4.2. p-Growth From Above and Below

For the limit free energy $\overline{W}(\Lambda)$ we now prove suitable two-sided growth estimates. Here we keep track of the dependence on the inverse temperature β.

Lemma 4.2. *Assume Hypothesis 1. Let $\overline{W}^\beta$ be given by Proposition 2. Then there exists a constant $C > 0$ such that for all $\Lambda \in \mathbb{R}^{n\times d}$ and all $\beta > 0$*

$$
\overline{W}^\beta(\Lambda) \leqq C|\Lambda|^p + C\left(1 + \frac{1 + |\log(\beta)|}{\beta}\right).
$$

Proof of Lemma 4.2. This estimate as an immediate consequence of the bound (4.2) taking into account that there is a prefactor β in the exponential functions. □

We now turn to the lower bound. Here we use the full assumptions on the graph in Definition 1.2.

Lemma 4.3. *Assume Hypothesis 1 and let $G \in \mathcal{G}$. Let $v \in L^p_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^n)$. Then $\mathcal{F}^-(O, v) < +\infty$ only if $v \in W^{1,p}(O, \mathbb{R}^n)$. In this case there exists a constant $c > 0$ such that*

$$
\mathcal{F}^-(O, v) \geqq \frac{c}{|O|}\int_O |\nabla v(z)|^p \, \mathrm{d}z - \frac{1}{c}\left(1 + \frac{1 + \log(\beta)|}{\beta}\right).
$$

In particular

$$
\overline{W}^\beta(\Lambda) \geqq c|\Lambda|^p - \frac{1}{c}\left(1 + \frac{1 + |\log(\beta)|}{\beta}\right).
$$

Proof of Lemma 4.3. First observe that the lower bound on $\overline{W}(\Lambda)$ follows by the first estimate and Lemma 4.1. To prove the first estimate, we split the free energy in a purely variational part and an integral part over a translated neighborhood. Given ε, κ fixed, we first choose $u_{\varepsilon,\kappa} \in \mathcal{N}_p(v, O, \varepsilon, \kappa)$ such that

$$\|\nabla_{\mathbb{B}} u_{\varepsilon,\kappa}\|^p_{\ell^p_\varepsilon(O)} \leqq \inf_{u \in \mathcal{N}_p(v,O,\varepsilon,\kappa)} \|\nabla_{\mathbb{B}} u\|^p_{\ell^p_\varepsilon(O)} + 1.$$

By convexity, for every $u \in \mathcal{N}_p(\varphi, O, \varepsilon, \kappa)$ we have $\frac{1}{2}u_{\varepsilon,\kappa} + \frac{1}{2}u \in \mathcal{N}_p(v, O, \varepsilon, \kappa)$. Hence by Lemma A.1 there exists $C_p < 2^p$ such that

$$\begin{aligned}\|\nabla_{\mathbb{B}} u_{\varepsilon,\kappa}\|^p_{\ell^p_\varepsilon(O)} - 1 &\leqq \|\frac{1}{2}\nabla_{\mathbb{B}} u_{\varepsilon,\kappa} + \frac{1}{2}\nabla_{\mathbb{B}} u\|^p_{\ell^p_\varepsilon(O)} \\ &\leqq \frac{C_p}{2^p}\|\nabla_{\mathbb{B}} u_{\varepsilon,\kappa}\|^p_{\ell^p_\varepsilon(O)} + \frac{C_p}{2^p}\|\nabla_{\mathbb{B}} u\|^p_{\ell^p_\varepsilon(O)} \\ &\quad - \frac{1}{2^p}\|\nabla_{\mathbb{B}}(u_{\varepsilon,\kappa} - u)\|^p_{\ell^p_\varepsilon(O)}.\end{aligned}$$

Subtracting the first and the last term on the right hand side, we infer that

$$\left(1 - \frac{C_p}{2^p}\right)\|\nabla_{\mathbb{B}} u_{\varepsilon,\kappa}\|^p_{\ell^p_\varepsilon(O)} + \frac{1}{2^p}\|\nabla_{\mathbb{B}}(u_{\varepsilon,\kappa} - u)\|^p_{\ell^p_\varepsilon(O)} - 1 \leqq \frac{C_p}{2^p}\|\nabla_{\mathbb{B}} u\|^p_{\ell^p_\varepsilon(O)}.$$

As $\frac{C_p}{2^p} < 1$, this estimate combined with Hypothesis 1 yields

$$\begin{aligned}H_\varepsilon(O, u) &\geqq \frac{1}{C}\|\nabla_{\mathbb{B}} u\|^p_{\ell^p_\varepsilon(O)} - C|O^{\mathcal{L}}_\varepsilon| \geqq \frac{1}{C}\|\nabla_{\mathbb{B}} u_{\varepsilon,\kappa}\|^p_{\ell^p_\varepsilon(O)} - C|O^{\mathcal{L}}_\varepsilon| \\ &\quad + \frac{1}{C}\|\nabla_{\mathbb{B}}(u_{\varepsilon,\kappa} - u)\|^p_{\ell^p_\varepsilon(O)}.\end{aligned}$$

As the function $u_{\varepsilon,\kappa} - u$ belongs to $\mathcal{N}_p(0, O, \varepsilon, 2\kappa)$, after a change of variables we obtain

$$\begin{aligned}-\frac{1}{\beta|O_\varepsilon|}\log(Z^\beta_{O,\varepsilon}(\mathcal{N}_p(v, O, \varepsilon, \kappa))) &\geqq \frac{1}{C|O_\varepsilon|}\|\nabla_{\mathbb{B}} u_{\varepsilon,\kappa}\|^p_{\ell^p_\varepsilon(O)} - C \\ &\quad - \frac{1}{\beta|O_\varepsilon|}\log\left(\int_{\mathcal{N}_p(0,O,\varepsilon,2\kappa)} \exp\left(-\frac{\beta}{C}\|\nabla_{\mathbb{B}} u\|^p_{\ell^p_\varepsilon(O)}\right) \mathrm{d}u\right)\end{aligned} \tag{4.18}$$

Step 1. Estimate of the first right hand side term of (4.18).
Let us start with estimating $|O_\varepsilon|^{-1}\|\nabla_{\mathbb{B}} u_{\varepsilon,\kappa}\|^p_{\ell^p_\varepsilon(O)}$. Here we follow [4] and use a difference quotient estimate. To this end, let $O' \subset\subset O$ and fix $h \in \mathbb{R}^d$ with $2|h| \leqq \mathrm{dist}(O', \partial O)$. For any $y \in \mathcal{L}$ we set $U^h_\varepsilon(y) = \{x \in \mathcal{L} : \mathcal{C}(x) \cap (\mathcal{C}(y) + \frac{h}{\varepsilon}) \neq \emptyset\}$. Then

$$\begin{aligned}&\int_{O'/\varepsilon} |\varepsilon u_{\varepsilon,\kappa}(z + h/\varepsilon) - \varepsilon u_{\varepsilon,\kappa}(z)|^p \,\mathrm{d}z \\ &\leqq \sum_{\substack{y \in \mathcal{L} \\ \mathcal{C}(y) \cap \frac{O'}{\varepsilon} \neq \emptyset}} \int_{\mathcal{C}(y)} \varepsilon^p |u_{\varepsilon,\kappa}(z + h/\varepsilon) - u_{\varepsilon,\kappa}(y)|^p \,\mathrm{d}z\end{aligned}$$

$$= \sum_{\substack{y\in\mathcal{L}\\ \mathcal{C}(y)\cap\frac{O'}{\varepsilon}\neq\emptyset}} \int_{\mathcal{C}(y)+\frac{h}{\varepsilon}} \varepsilon^p |u_{\varepsilon,\kappa}(z)-u_{\varepsilon,\kappa}(y)|^p\,\mathrm{d}z$$

$$\leqq \sum_{\substack{y\in\mathcal{L}\\ \mathcal{C}(y)\cap\frac{O'}{\varepsilon}\neq\emptyset}} \sum_{x\in U_\varepsilon^h(y)} \int_{\mathcal{C}(x)} \varepsilon^p |u_{\varepsilon,\kappa}(x)-u_{\varepsilon,\kappa}(y)|^p\,\mathrm{d}z. \tag{4.19}$$

We next derive a pointwise estimate of $\varepsilon^p|u_{\varepsilon,\kappa}(x)-u_{\varepsilon,\kappa}(y)|^p$ for $x\in U_\varepsilon^h(y)$. Let $P(x,y)$ be a path connecting x,y satisfying the properties of Definition 1.2 (iv). From (3.1) we deduce that $|x-y|\leqq|h|\varepsilon^{-1}+2R$ and thus $\#P(x,y)\leqq C(|h|\varepsilon^{-1}+1)$. Using Jensen's inequality we obtain

$$\begin{aligned}\varepsilon^p|u_{\varepsilon,\kappa}(x)-u_{\varepsilon,\kappa}(y)|^p &\leqq \varepsilon^p\,(\#P(x,y))^{p-1}\sum_{(x',x'')\in P(x,y)}|u_{\varepsilon,\kappa}(x')-u_{\varepsilon,\kappa}(x'')|^p\\ &\leqq (C\varepsilon|h|^{p-1}+C\varepsilon^p)\sum_{(x',x'')\in P(x,y)}|u_{\varepsilon,\kappa}(x')-u_{\varepsilon,\kappa}(x'')|^p.\end{aligned} \tag{4.20}$$

Moreover note that $\mathcal{C}(y)\cap O'_\varepsilon\neq\emptyset$ and $x\in U_\varepsilon^h(y)$ imply that $x',x''\in O_\varepsilon^{\mathcal{L}}$ for ε small enough. Indeed, applying the triangle inequality several times one can show that for any $v\in[x',x'']$ one has

$$\mathrm{dist}\left(v,\frac{O'}{\varepsilon}\right)\leqq\frac{|h|}{\varepsilon}+(2C_0+3R),$$

where C_0 is given by Definition 1.2. Conversely, given any $(x',x'')\in\mathbb{B}$, we define the sets

$$K_\varepsilon^h(x',x''):=\{y\in\mathcal{L}:\ \exists x\in U_\varepsilon^h(y)\text{ such that }(x',x'')\in P(x,y)\}.$$

As G is admissible, for any $(x',x'')\in P(x,y)$ there exists $\lambda\in[0,1]$ such that $z=y+\lambda(x-y)$ satisfies $|z-x'|\leqq C_0$. Hence for any $y\in K_\varepsilon^h(x',x'')$ we obtain

$$\begin{aligned}y=y-z+x'+(z-x')&=-\lambda\frac{h}{\varepsilon}+x'+\lambda\left(\frac{h}{\varepsilon}-(x-y)\right)\\ &+(z-x')\in\left[-\frac{h}{\varepsilon},0\right]+x'+B_{2R+C_0}(0),\end{aligned}$$

where we have used that $|x-y-\frac{h}{\varepsilon}|\leqq 2R$ for any $x\in U_\varepsilon^h(y)$. Using again (3.1) we conclude that $\#K_\varepsilon^h(x',x'')\leqq C(|h|\varepsilon^{-1}+1)$. Furthermore the set $U_\varepsilon^h(y)$ has equibounded cardinality, so that the inequalities (4.19), (4.20) and the uniform bound on the measure of the Voronoi cells imply

$$\int_{O'/\varepsilon}|\varepsilon u_{\varepsilon,\kappa}(z+h/\varepsilon)-\varepsilon u_{\varepsilon,\kappa}(z)|^p\,\mathrm{d}z$$

$$\leqq C(|h|^p + |h|\varepsilon^{p-1} + \varepsilon|h|^{p-1} + \varepsilon^p) \sum_{\substack{(x,y)\in\mathbb{B} \\ x,y\in O_\varepsilon^{\mathcal{L}}}} |u_{\varepsilon,\kappa}(x) - u_{\varepsilon,\kappa}(y)|^p$$
$$= C(|h|^p + |h|\varepsilon^{p-1} + \varepsilon|h|^{p-1} + \varepsilon^p)\|\nabla_{\mathbb{B}} u_{\varepsilon,\kappa}\|^p_{\ell^p_\varepsilon(O)}. \tag{4.21}$$

As $u_{\varepsilon,\kappa} \in \mathcal{N}_p(v, O, \varepsilon, \kappa)$, the function $v_{\varepsilon,\kappa} : O \to \mathbb{R}^n$ defined by $v_{\varepsilon,\kappa}(x) := \varepsilon u_{\varepsilon,\kappa}(x/\varepsilon)$ satisfies

$$\int_{O'} |v_{\varepsilon,\kappa}(z) - v_\varepsilon(z/\varepsilon)|^p \,\mathrm{d}z \leqq C\varepsilon^d \|\varepsilon u_{\varepsilon,\kappa} - v_\varepsilon\|^p_{\ell^p_\varepsilon(O)} \leqq C\kappa^p|O|^{1+\frac{p}{d}}.$$

In particular, by Remark 5 it is bounded in $L^p(O')$ and thus there exists a subsequence (not relabeled) such that $v_{\varepsilon,\kappa} \rightharpoonup v_\kappa$ in $L^p(O')$. Moreover, by Remark 5 and lower semicontinuity of the L^p-norm it holds that $\|v_\kappa - v\|_{L^p(O')} \leqq C\kappa|O|^{\frac{1}{p}+\frac{1}{d}}$. By a change of variables in the left hand side of (4.21) we further obtain that

$$\int_{O'} |v_{\varepsilon,\kappa}(z+h) - v_{\varepsilon,\kappa}(z)|^p \,\mathrm{d}z \leqq C(|h|^p\varepsilon^d + |h|\varepsilon^{p+d-1}$$
$$+\varepsilon^{d+1}|h|^{p-1} + \varepsilon^{p+d})\|\nabla_{\mathbb{B}} u_{\varepsilon,\kappa}\|^p_{\ell^p_\varepsilon(O)}. \tag{4.22}$$

Applying weak lower semicontinuity in the above estimate we deduce

$$\liminf_{\kappa\to 0}\liminf_{\varepsilon\to 0} \frac{1}{|O_\varepsilon|}\|\nabla_{\mathbb{B}} u_{\varepsilon,\kappa}\|^p_{\ell^p_\varepsilon(A)} \geqq \frac{1}{C|O|}\int_{O'} \left|\frac{v(z+h) - v(z)}{|h|}\right|^p \mathrm{d}z. \tag{4.23}$$

Before we conclude Sobolev-regularity of v, we have to ensure that the third right hand side term in (4.18) remains finite.
Step 2. Control of the third right hand side term of (4.18).
We want to apply Lemma 3.3. To this end, we observe that

$$|u(x)| \leqq 2\kappa\left(\frac{|O|}{\varepsilon^d}\right)^{\frac{1}{p}+\frac{1}{d}}$$

for any $u \in \mathcal{N}_p(0, O, \varepsilon, 2\kappa)$ and all $x \in O_\varepsilon^{\mathcal{L}}$. Hence, setting $\gamma = 2\kappa\left(\frac{|O|}{\varepsilon^d}\right)^{\frac{1}{p}+\frac{1}{d}}$, $z_x = 0$ and $\alpha = \frac{\beta}{C}$, Lemma 3.3 yields

$$\log\left(\int_{\mathcal{N}_p(0,O,\varepsilon,2\kappa)} \exp(-\frac{\beta}{C}\|\nabla_{\mathbb{B}} u\|^p_{\ell^p_\varepsilon(O)})\,\mathrm{d}u\right) \leqq N_{O,\varepsilon}\log\left(C(2\kappa)^n\left(\frac{|O|}{\varepsilon^d}\right)^{\frac{n}{p}+\frac{n}{d}}\right)$$
$$+ C\big(|O_\varepsilon^{\mathcal{L}}| - N_{O,\varepsilon}\big)(1 + |\log(\beta)|), \tag{4.24}$$

where $N_{O,\varepsilon}$ denotes the number of connected components of the graph $G_{O,\varepsilon}$. Since O has Lipschitz boundary, by Remark 7 and (3.3) it holds that

$$N_{O,\varepsilon} \leqq C\varepsilon^{1-d}\mathcal{H}^{d-1}(\partial O)$$

for ε small enough. Dividing (4.24) by $-\beta|O_\varepsilon|$ and letting $\varepsilon \to 0$ we find that

$$\begin{aligned}&\liminf_{\varepsilon\to 0} -\frac{1}{\beta|O_\varepsilon|}\log\left(\int_{\mathcal{N}_p(0,O,\varepsilon,2\kappa)}\exp\left(-\frac{1}{C}\|\nabla_{\mathbb{B}}u\|^p_{\ell^p_\varepsilon(O)}\right)\mathrm{d}u\right)\\&\geqq -C\left(\frac{1+|\log(\beta)|}{\beta}\right).\end{aligned}$$

From (4.18), (4.23) and the previous inequality we finally obtain

$$\mathcal{F}^-(O,v) \geqq \frac{1}{C|O|}\int_{O'}\left|\frac{v(z+h)-v(z)}{|h|}\right|^p \mathrm{d}z - C\left(1+\frac{1+|\log(\beta)|}{\beta}\right)$$

for every $h \in \mathbb{R}^d$ such that $2|h| \leqq \mathrm{dist}(O', \partial O)$. Using the difference quotient characterization of $W^{1,p}$-spaces we conclude that $v \in W^{1,p}(O,\mathbb{R}^n)$ and letting $|h| \to 0$ yields by the arbitrariness of O' that

$$\mathcal{F}^-(O,v) \geqq \frac{1}{C|O|}\int_O |\nabla v|^p \,\mathrm{d}z - C\left(1+\frac{1+\log(\beta)|}{\beta}\right).$$

□

4.3. Quasiconvexity of the Limit Free Energy

For the proof of Theorem 1.3 we next establish a lower semicontinuity result that we use to show the quasiconvexity of the free energy by soft arguments.

Lemma 4.4. *Let $G \in \mathcal{G}$ and let $O \in \mathcal{A}^R(\mathbb{R}^d)$. If $v, \hat{v} \in L^p_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^n)$ are such that $v = \hat{v}$ almost everywhere on O, then $\mathcal{F}^\pm(O,v) = \mathcal{F}^\pm(O,\hat{v})$. Hence the maps $L^p(O,\mathbb{R}^n) \ni v \mapsto \mathcal{F}^\pm(O,v)$ are well-defined. Moreover both are lower semicontinuous with respect to the strong $L^p(O,\mathbb{R}^n)$-topology.*

Proof of Lemma 4.4. Let $v_j, v \in L^p_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{R}^n)$ such that $v_j \to v$ in $L^p(O,\mathbb{R}^n)$. Both claims follow if we establish the lower semicontinuity along such sequences. Given $u \in \mathcal{N}_p(v_j, O, \varepsilon, \kappa)$, by (3.1) we have

$$\begin{aligned}\varepsilon^{\frac{d}{p}}\|v_\varepsilon - \varepsilon u\|_{\ell^p_\varepsilon(O)} &\leqq \varepsilon^{\frac{d}{p}}\|v_\varepsilon - (v_j)_\varepsilon\|_{\ell^p_\varepsilon(O)} + \varepsilon^{\frac{d}{p}}\|(v_j)_\varepsilon - \varepsilon u\|_{\ell^p_\varepsilon(O)}\\&\leqq C\left(\sum_{x\in O^{\mathcal{L}}_\varepsilon}\int_{\varepsilon\mathcal{C}(x)}|v(z)-v_j(z)|^p\,\mathrm{d}z\right)^{\frac{1}{p}} + \kappa|O|^{\frac{1}{p}+\frac{1}{d}}\\&\leqq C\left(\|u-u_j\|^p_{L^p(O)} + \int_{\partial O + B_{R\varepsilon}(0)}|v(z)-v_j(z)|^p\,\mathrm{d}z\right)^{\frac{1}{p}}\\&\quad + \kappa|O|^{\frac{1}{p}+\frac{1}{d}}.\end{aligned}$$

Since $O \in \mathcal{A}^R(D)$ we deduce that for all $j = j(\kappa)$ large enough there exists $\varepsilon_0 = \varepsilon_0(j)$ such that for all $\varepsilon < \varepsilon_0$ we have $\mathcal{N}_p(v_j, O, \varepsilon, \kappa) \subset \mathcal{N}_p(v, O, \varepsilon, 2\kappa)$. For every fixed κ_0 and $j = j(\kappa_0)$ large enough this yields

$$\mathcal{F}^+(O, v_j) = \sup_{\kappa>0} \mathcal{F}^+_\kappa(O, v_j) \geqq \mathcal{F}^+_{\kappa_0}(O, v_j) \geqq \mathcal{F}^+_{2\kappa_0}(O, v).$$

Letting first $j \to +\infty$ and then $\kappa_0 \to 0$ we conclude that

$$\liminf_j \mathcal{F}^+(O, v_j) \geqq \lim_{\kappa_0 \to 0} \mathcal{F}^+_{2\kappa_0}(O, v) = \mathcal{F}^+(O, v).$$

The proof for $\mathcal{F}^-(O, v)$ is similar. □

We now state an important intermediate result that will imply by more or less standard arguments a large deviation principle for large volume Gibbs measures under clamped boundary conditions. Due to that reason, we postpone its proof to the end of Section 5 on the large deviation principle.

Theorem 4.5. *Assume Hypothesis 1. Then for a set of full probability and for any $v \in W^{1,p}(D, \mathbb{R}^n)$ it holds that*

$$\mathcal{F}^-(D, v) = \mathcal{F}^+(D, v) = \frac{1}{|D|} \int_D \overline{W}^\beta(\nabla v)\, \mathrm{d}x,$$

where $\overline{W}$ is given by Proposition 2.

Proof of Theorem 1.3. The almost sure existence of the limit of the free energy for all $\Lambda \in \mathbb{R}^{n\times d}$, all $\beta > 0$ and all bounded Lipschitz domains D follows from Proposition 3 and Remark 11. The claimed p-growth conditions have been proven in the Lemmata 4.2 and 4.3. Quasiconvexity is a standard result on necessary conditions for weak lower semicontinuity of integral functionals on $W^{1,p}(D, \mathbb{R}^n)$, provided the integrand is continuous (as proven in Proposition 3) and satisfies the proven p-growth. Thus quasiconvexity of the map $\Lambda \mapsto \overline{W}^\beta(\Lambda)$ is a consequence of Theorem 4.5, Lemma 4.4 and the Sobolev embedding theorem. Finally the claim on the ergodic case is contained in Proposition 2. □

5. Large Deviation Principle for the Gibbs Measures: Proof of Theorems 1.4 and 1.5

We now turn our attention to the announced large deviation principle for Gibbs measures associated with the discrete Hamiltonian H_ε. As a by-product we shall prove Theorem 1.4.

5.1. Notation for Gibbs Measures and Exponential Tightness

In order to avoid technical issues when discretizing the gradient of a Sobolev function on a vanishing set, we restrict our analysis to boundary data $\varphi \in \mathrm{Lip}(\mathbb{R}^d, \mathbb{R}^n)$. Moreover, in order to identify the discrete variables with a function defined on the continuum we proceed as follows: Recall that given any $v : D \to \mathbb{R}^n$, we have set $u := \Pi_{1/\varepsilon} v : D_\varepsilon \to \mathbb{R}^n$ as

$$\Pi_{1/\varepsilon} v(z) = \frac{1}{\varepsilon} v(\varepsilon z).$$

Given such v, with a slight abuse of notation we write $u = \Pi_{1/\varepsilon} v \in \mathcal{B}_\varepsilon(D, \varphi)$ if and only if the following conditions are met:

(i) $\big(\Pi_{1/\varepsilon} v\big)_{|\mathcal{C}(x)\cap D_\varepsilon}$ is constant for all $x \in \mathcal{L}$;
(ii) $\big(\Pi_{1/\varepsilon} v\big)_{|D_\varepsilon^{\mathcal{L}}} \in \mathcal{B}_\varepsilon(D, \varphi)$ in the usual sense;
(iii) $\big(\Pi_{1/\varepsilon} v\big)_{|\mathcal{C}(x)\cap D_\varepsilon} = \big(\Pi_{1/\varepsilon}\varphi\big)(x)$ whenever $x \in \mathcal{L} \setminus D_\varepsilon$

Then the Gibbs measure $\mu^\beta_{\varepsilon,D,g}$ with respect to the Hamiltonian H_ε and boundary value φ is the probability measure on $L^p(D, \mathbb{R}^n)$ given by the formula (1.11), that is,

$$\mu^\beta_{\varepsilon,D,\varphi}(V) = \frac{1}{Z^\beta_{\varepsilon,D,\varphi}} \int_{\Pi_{1/\varepsilon} V \cap \mathcal{B}_\varepsilon(D,\varphi)} \exp(-\beta H_\varepsilon(D, u))\, \mathrm{d}u,$$

where the partition function $Z^\beta_{\varepsilon,D,\varphi}$ is the normalizing factor that ensure that $\mu^\beta_{\varepsilon,D,\varphi}(L^p(D, \mathbb{R}^n)) = 1$. With what we have proved so far, we are now in a position to state and prove a large deviation principle for these Gibbs measures in the many-particle limit. As usual, we first have to establish an exponential tightness estimate. This will be achieved in the two lemmata below.

Lemma 5.1. *Assume Hypothesis 1 and let $G \in \mathcal{G}$. Fix $O \in \mathcal{A}^R(\mathbb{R}^d)$ and $\varphi \in \mathrm{Lip}(\mathbb{R}^d, \mathbb{R}^n)$. If $u^\varepsilon \in \mathcal{B}_\varepsilon(O, \varphi) \cap \mathcal{S}_M(O, \varepsilon)$, then there exists a subsequence u^{ε_j} and $v \in \varphi + W^{1,p}_0(O, \mathbb{R}^n)$ such that $\Pi_{\varepsilon_j} u^{\varepsilon_j} := \varepsilon_j u^{\varepsilon_j}(\varepsilon_j^{-1}\cdot) \to v$ in $L^p(O, \mathbb{R}^n)$.*

Proof of Lemma 5.1. We just sketch the argument. First extend u^ε to the whole vertex set $\mathcal{L}$ setting $u^\varepsilon(x) = \big(\Pi_{1/\varepsilon}\varphi\big)(x)$ whenever $x \in \mathcal{L}\setminus O_\varepsilon^{\mathcal{L}}$. Now take $O_1, O_2 \in \mathcal{A}^R(\mathbb{R}^d)$ such that $O \subset\subset O_1 \subset\subset O_2$. To reduce notation, we introduce $v^\varepsilon : O_2 \to \mathbb{R}^n$ as $v^\varepsilon = \Pi_\varepsilon u^\varepsilon$. Since $u^\varepsilon \in \mathcal{B}_\varepsilon(O, \varphi) \cap \mathcal{S}_M(O, \varepsilon)$ and φ is Lipschitz, one can show that

$$\sup_{\varepsilon>0} |(O_2)_\varepsilon|^{-1} H_\varepsilon(u^\varepsilon, O_2) < +\infty.$$

Using the same construction as for the proof of (4.22) we obtain that, for $h \in \mathbb{R}^d$ such that $2|h| \leqq \mathrm{dist}(O_1, \partial O_2)$, it holds that

$$\int_{O_1} |v^\varepsilon(z+h) - v^\varepsilon(z)|^p\, \mathrm{d}z \leqq C(|h|^p + |h|\varepsilon^{p-1} + \varepsilon|h|^{p-1} + \varepsilon^p). \quad (5.1)$$

According to [32, Lemma 4.6], strong $L^p(O_1)$-compactness follows if we prove that v^ε is bounded in $L^p(O_2)$. This can be achieved combining the energy bound with the growth assumptions from Hypothesis 1 and the properly scaled discrete Poincaré inequality stated in Lemma 3.1. The regularity of any limit function v follows again by the difference quotient characterization of $W^{1,p}(O_1, \mathbb{R}^n)$ and (5.1). Since φ is Lipschitz, it can be shown that $v = \varphi$ on $O_1 \backslash O$ and therefore v has trace φ on ∂O. $\square$

Lemma 5.2. *Assume Hypothesis 1 and let $G \in \mathcal{G}$. Then, for each $N \in \mathbb{N}$ there exists a compact set $K_N \subset L^p(D, \mathbb{R}^n)$ such that*

$$\limsup_{\varepsilon \to 0} \frac{1}{\beta |D_\varepsilon|} \log \left(\mu^\beta_{\varepsilon, D, \varphi}(L^p(D, \mathbb{R}^n) \backslash K_N) \right) \leqq -N.$$

Proof of Lemma 5.2. For a given number $M > 0$ we define the set $K_M \subset L^p(D, \mathbb{R}^n)$ by

$$K_M := \bigcup_{0<\varepsilon<1} \left\{ v : D \to \mathbb{R}^n : \ \Pi_{1/\varepsilon} v \in \mathcal{B}_\varepsilon(D, \varphi), \ H_\varepsilon(D, \Pi_{1/\varepsilon} v) \leqq M |D_\varepsilon| \right\},$$

where we identify again discrete functions with piecewise constant function on Voronoi cells. We argue that the set K_M is precompact in $L^p(D, \mathbb{R}^n)$. To this end consider a sequence $\{v_j\} \subset K_M$. Then for each j we find ε_j such that v_j is defined on the nodes of $\varepsilon_j \mathcal{L}$. First let us extend the functions to all of $\varepsilon_j \mathcal{L}$ setting $v_j(\varepsilon_j x) = \varphi(\varepsilon_j x)$ for $x \in \mathcal{L} \backslash D_{\varepsilon_j}$. We distinguish two cases: If $\liminf_j \varepsilon_j > 0$, then we can use the boundary conditions and the energy bound to prove that v_j contains a converging subsequence since it can be identified with an equibounded sequence in a finite dimensional space. Here we use again the fact that each connected component of $G_{D,\varepsilon}$ contains a vertex with active boundary conditions. Next we treat the case when $\liminf_j \varepsilon_j = 0$. In that case we can apply Lemma 5.1 to conclude that K_M is precompact for every M.

For the claimed estimate we have to control the contribution from the partition function. Using the upper bound from Hypothesis 1, Remark 6, (3.2) and a change of variables, for ε small enough we obtain

$$\begin{aligned}
&\frac{-1}{\beta |D_\varepsilon|} \log \left(Z^\beta_{\varepsilon, D, \varphi} \right) \\
&\leqq \frac{-1}{\beta |D_\varepsilon|} \log \left(\int_{\mathcal{B}_\varepsilon(D,\varphi)} \exp\left(-C\beta (\|\nabla_{\mathbb{B}} u\|^p_{\ell^p_\varepsilon(D)} + |D^{\mathcal{L}}_\varepsilon|) \right) \mathrm{d}u \right) \\
&\leqq \frac{-1}{\beta |D_\varepsilon|} \log \left(\int_{\mathcal{B}_\varepsilon(D,\varphi)} \exp\left(-C\beta (\|\nabla_{\mathbb{B}} (u - \Pi_{1/\varepsilon}\varphi)\|^p_{\ell^p_\varepsilon(D)} \right.\right. \\
&\quad \left.\left. + (\|\nabla \varphi\|^p_\infty + 1) |D^{\mathcal{L}}_\varepsilon|) \right) \mathrm{d}u \right) \\
&\leqq C(\|\nabla\varphi\|^p_\infty + 1) - \frac{1}{\beta |D_\varepsilon|} \log \left(\int_{\mathcal{B}_\varepsilon(D,0)} \exp(-C\beta \|u\|^p_{\ell^p_\varepsilon(D)}) \, \mathrm{d}u \right) \\
&\leqq C(\|\nabla\varphi\|^p_\infty + 1) + \frac{C}{\beta} \left| \log \left(\int_{B_1(0)} \exp(-C\beta |\zeta|^p) \, \mathrm{d}\zeta \right) \right|.
\end{aligned} \tag{5.2}$$

Combining this bound with Lemma 3.4 we obtain the claim choosing K_M with $M = M(N, \beta)$ large enough and taking the L^p-closure of this set. $\square$

5.2. Proof of the Large Deviation Principle

Having established the exponential tightness we can now prove a strong large deviation principle for the large volume Gibbs measures as stated in the main results. Theorem 1.4 will then be a straightforward consequence of the proof.

Proof of Theorem 1.5. Observe first that the term $-\frac{1}{\beta|D_\varepsilon|}\log(Z^\beta_{\varepsilon,D,\varphi})$ is bounded from above as shown in (5.2). A corresponding lower bound can be achieved using the lower bound of Hypothesis 1 and Lemma 3.3. Hence we may assume that, passing to a subsequence (not relabeled), it holds that

$$\lim_{\varepsilon\to 0} -\frac{1}{\beta|D_\varepsilon|}\log(Z^\beta_{\varepsilon,D,\varphi}) = c_{\varphi,\beta}$$

for some constant $c_{\varphi,\beta} \in \mathbb{R}$. To reduce notation, we define the functional $I_g : L^p(D,\mathbb{R}^n) \to (-\infty, +\infty]$ via

$$I^\beta_{D,\varphi}(v) = \begin{cases} \frac{1}{|D|}\int_D \overline{W}^\beta(\nabla v)\,\mathrm{d}x & \text{if } v \in \varphi + W^{1,p}_0(D,\mathbb{R}^n), \\ +\infty & \text{otherwise.} \end{cases}$$

Note that by the upper and lower bounds established in Lemmas 4.2 and Lemma 4.3, respectively, as well as the quasiconvexity proven in Theorem 4.5, we know that $I^\beta_{D,\varphi}$ is lower-semicontinuous with respect to strong $L^p(D,\mathbb{R}^n)$-convergence.

Step 1. Proof of the lower bound on open sets.
We start the proof with the case of an open set $U \subset L^p(D,\mathbb{R}^n)$. If $U \cap (\varphi + W^{1,p}_0(D,\mathbb{R}^n)) = \emptyset$, then there is nothing to prove. Therefore consider $v \in U \cap (\varphi + W^{1,p}_0(D,\mathbb{R}^n))$. Since U is open, given $\eta > 0$ we can find $v^\eta \in U \cap (\varphi + C^\infty_c(D,\mathbb{R}^n))$ such that $\|v^\eta - v\|_{W^{1,p}(D)} < \eta$. We claim that, for fixed $\eta > 0$, there exist $\kappa_0, \varepsilon_0 > 0$ such that for all $\kappa < \kappa_0$ and $\varepsilon < \varepsilon_0$ it holds that

$$\Pi_\varepsilon\Big(\mathcal{N}_p(v^\eta, D, \varepsilon, 3\kappa) \cap \mathcal{B}_\varepsilon(D,\varphi)\Big) \subset U. \tag{5.3}$$

Indeed, recalling the definition of $\tilde{v}^\eta_\varepsilon$ in Remark 5, for every $u \in \mathcal{N}_p(v^\eta, D, \varepsilon, 3\kappa) \cap \mathcal{B}_\varepsilon(D,\varphi)$ we have that

$$\begin{aligned}
&\|v^\eta - \Pi_\varepsilon u\|_{L^p(D)} \\
&\quad \leqq \|v^\eta - \tilde{v}^\eta_\varepsilon\|_{L^p(D)} + C\left(\sum_{x\in D_\varepsilon} \varepsilon^d |v^\eta_\varepsilon(x) - \varepsilon u(x)|^p \right.\\
&\qquad \left. + \sum_{\varepsilon\mathcal{C}(x)\cap\partial D\neq\emptyset} \varepsilon^d |\tilde{v}^\eta_\varepsilon(\varepsilon x) - \varphi(\varepsilon x)|^p\right)^{\frac{1}{p}}
\end{aligned}$$

$$\leqq \|v^\eta - \tilde{v}^\eta_\varepsilon\|_{L^p(D)} + C\left((3\kappa)^p|D|^{1+\frac{p}{d}}\right.$$
$$\left.+\sum_{\varepsilon\mathcal{C}(x)\cap\partial D\neq\emptyset}\varepsilon^d|\tilde{v}^\eta_\varepsilon(\varepsilon x)-\varphi(\varepsilon x)|^p\right)^{\frac{1}{p}}. \tag{5.4}$$

The ε-dependent terms vanish by Remark 5 combined with an equiintegrability argument for the last sum. Hence (5.3) holds provided we choose ε_0, κ_0 small enough. Then from the definition of the Gibbs measure we infer for $\kappa < \kappa_0$ that

$$\liminf_{\varepsilon\to 0}\frac{\log(\mu^\beta_{\varepsilon,D,\varphi}(U))}{\beta|D_\varepsilon|}$$
$$\geqq \liminf_{\varepsilon\to 0}\frac{1}{\beta|D_\varepsilon|}\log\left(Z^\beta_{\varepsilon,D}(\mathcal{N}_p(v^\eta,D,\varepsilon,3\kappa)\cap\mathcal{B}_\varepsilon(D,\varphi)\right)+c_{\varphi,\beta}.$$

Applying Proposition 1, we deduce that for any $\delta > 0$ small enough and any $N \in \mathbb{N}$ it holds that

$$\liminf_{\varepsilon\to 0}\frac{\log(\mu^\beta_{\varepsilon,D,\varphi}(U))}{\beta|D_\varepsilon|} \geqq -\frac{N-C}{N}\mathcal{F}^+_\kappa(D,v^\eta) - C(\|\nabla\varphi\|^p_\infty+1)\frac{|D_\delta|}{|D|}$$
$$-\left(\frac{N\kappa|D|^{\frac{1}{d}}}{\delta}\right)^p - \frac{C}{N} + c_{\varphi,\beta}.$$

Letting first $\kappa \to 0$ and then $N \to +\infty$ as well as $\delta \to 0$, from Theorem 4.5 we infer

$$\liminf_{\varepsilon\to 0}\frac{\log(\mu^\beta_{\varepsilon,D,\varphi}(U))}{\beta|D_\varepsilon|} \geqq -\frac{1}{|D|}\int_D \overline{W}^\beta(\nabla v^\eta)\,\mathrm{d}x + c_{\varphi,\beta}.$$

As $\eta > 0$ was arbitrary, the continuity of $\Lambda \mapsto \overline{W}^\beta(\Lambda)$ and its growth condition allow to pass from v^η to v and since $v \in U \cap (\varphi + W^{1,p}_0(D,\mathbb{R}^n))$ was arbitrary too, we conclude the lower bound

$$\liminf_{\varepsilon\to 0}\frac{\log(\mu^\beta_{\varepsilon,D,\varphi}(U))}{\beta|D_\varepsilon|} \geqq -\inf_{v\in U} I^\beta_{D,\varphi}(v) + c_{\varphi,\beta}.$$

Step 2. Proof of the upper bound on closed sets.
In order to prove an upper bound, we first recall that due to the exponential tightness established in Lemma 5.2, it suffices to consider the case when V is compact (see for example Lemma 1.2.18 in [24]). Then, for $\delta > 0$ we define the truncated energy via

$$\mathcal{F}_\delta(D,v) = \min\left\{\mathcal{F}^-(D,v)-\delta, \frac{1}{\delta}\right\}.$$

Note that by definition of $\mathcal{F}^-(D, v)$, for every $v \in V$ there exists $\kappa > 0$ such that

$$-\limsup_{\varepsilon\to 0} \frac{1}{\beta|D_\varepsilon|} \log(Z^\beta_{\varepsilon,D}(\mathcal{N}_p(v, D, \varepsilon, \kappa)) \geqq \mathcal{F}_\delta(D, v). \tag{5.5}$$

Let us fix C_1 such that $|\mathcal{C}(x)|^{\frac{1}{p}} \leqq C_1$ for all $x \in \mathcal{L}$. By lower semicontinuity of the functional $I^\beta_{D,\varphi}$, up to reducing κ we may assume that

$$I^\beta_{D,\varphi}(v) \leqq I^\beta_{D,\varphi}(w) + 1 \tag{5.6}$$

for all $w \in L^p(D, \mathbb{R}^n)$ such that $\|v - w\|_{L^p(D,\mathbb{R}^n)} \leqq C_1\kappa|D|^{\frac{1}{p}+\frac{1}{d}}$. As we show now, for a suitable $0 < \kappa' < \kappa$ and all ε small enough, we have the inclusion

$$B_{\kappa'}(v) \cap \Pi_\varepsilon(\mathcal{B}_\varepsilon(D, \varphi)) \subset \Pi_\varepsilon\Big(\mathcal{N}_p(v, D, \varepsilon, \kappa) \cap \mathcal{B}_\varepsilon(D, \varphi)\Big), \tag{5.7}$$

where here we denote by $B_{\kappa'}(v)$ the $L^p(D, \mathbb{R}^n)$-ball centered at u with radius κ'. Indeed, from (3.1) we deduce that any $u \in \mathcal{B}_\varepsilon(D, \varphi)$ with $\Pi_\varepsilon u \in B_{\kappa'}(v)$ satisfies

$$\begin{aligned}
&\sum_{x\in D^{\mathcal{L}}_\varepsilon} \varepsilon^d |v_\varepsilon(x) - \varepsilon u(x)|^p \\
&\quad\leqq C\|\tilde{v}_\varepsilon - \Pi_\varepsilon u\|^p_{L^p(D)} + C \sum_{\varepsilon\mathcal{C}(x)\cap\partial D\neq\emptyset} \varepsilon^d \left(|v_\varepsilon(x) - \varphi(\varepsilon x)|^p + \varepsilon^p\right) \\
&\quad\leqq C\|\tilde{v}_\varepsilon - v\|^p_{L^p(D)} + C\kappa' + \sum_{\varepsilon\mathcal{C}(x)\cap\partial D\neq\emptyset} \varepsilon^d \left(|v_\varepsilon(x) - \varphi(\varepsilon x)|^p + \varepsilon^p\right)
\end{aligned}$$

and again the ε-dependent terms converge to zero by Remark 5 and an equiintegrability argument for the sum in the second line. Since V is compact, we can find a finite covering by the open balls $B_{\kappa'}(u)$, that is there exist $v_1, \ldots, v_m$ such that $V \subset \bigcup_{i=1}^m B_{\kappa'_i}(v_i)$. Together with the the inclusion (5.7) this covering implies

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \frac{\log(\mu^\beta_{\varepsilon,D,\varphi}(V))}{\beta|D_\varepsilon|} \\
&\quad\leqq \limsup_{\varepsilon\to 0} \frac{1}{\beta|D_\varepsilon|} \log\left(\sum_{i=1}^m \mu^\beta_{\varepsilon,D,\varphi}(B_{\kappa'_i}(v_i))\right) \\
&\quad\leqq \max_i \limsup_{\varepsilon\to 0} \frac{1}{\beta|D_\varepsilon|}\Big(\log(Z^\beta_{\varepsilon,D}(\mathcal{N}_p(v_i, D, \varepsilon, \kappa_i) \cap \mathcal{B}_\varepsilon(D, \varphi))\Big) + c_{\varphi,\beta}
\end{aligned} \tag{5.8}$$

and therefore it remains to bound the term for a fixed v_{i_0}. First note that if

$$\limsup_{\varepsilon\to 0} \frac{1}{\beta|D_\varepsilon|}\Big(\log(Z^\beta_{\varepsilon,D}(\mathcal{N}_p(v_{i_0}, D, \varepsilon, \kappa_{i_0}) \cap \mathcal{B}_\varepsilon(D, \varphi))\Big) = -\infty,$$

then there is nothing left to prove. Otherwise, Lemma 3.4 implies that, for a suitable large M, it holds that

$$\mathcal{N}_p(v_{i_0}, D, \varepsilon, \kappa_{i_0}) \cap \mathcal{B}_\varepsilon(D, \varphi) \cap \mathcal{S}_M(D, \varepsilon) \neq \emptyset$$

along some infinitesimal sequence $\varepsilon \to 0$ (not relabeled). Applying Lemma 5.1 to this element, we deduce that there exists $v \in \varphi + W_0^{1,p}(D, \mathbb{R}^n)$ such that, similar to estimate (5.4), it holds that

$$\|v_{i_0} - \varphi\|_{L^p(D)} \leqq C_1 \kappa_{i_0} |D|^{\frac{1}{p}+\frac{1}{d}}$$

Together with (5.6) this implies that $v_{i_0} \in \varphi + W_0^{1,p}(D, \mathbb{R}^n)$, too. Therefore, using also (5.5), we can further estimate (5.8) by

$$\begin{aligned}
\limsup_{\varepsilon \to 0} \frac{\log(\mu_{\varepsilon,g}(A))}{\beta |D_\varepsilon|} &\leqq \limsup_{\varepsilon \to 0} \frac{1}{\beta |D_\varepsilon|} \log\left(Z^\beta_{\varepsilon,D}(\mathcal{N}_p(v_{i_0}, D, \varepsilon, \kappa_{i_0}))\right) + c_{\varphi,\beta} \\
&\leqq -\mathcal{F}_\delta(D, v_{i_0}) + c_{\varphi,\beta} \\
&\leqq - \inf_{v \in V \cap g + W_0^{1,p}(D,\mathbb{R}^n)} \mathcal{F}_\delta(D, v) + c_{\varphi,\beta}.
\end{aligned}$$

Letting $\delta \to 0$, by monotonicity and Theorem 4.5 we obtain the estimate

$$\limsup_{\varepsilon \to 0} \frac{1}{\beta |D_\varepsilon|} \log(\mu^\beta_{\varepsilon,D,\varphi}(V)) \leqq - \inf_{v \in V} I^\beta_{D,\varphi}(u) + c_{\varphi,\beta}.$$

Step 3. Identification of $c_{\varphi,\beta}$ and conclusion.
It remains to show that $c_{\varphi,\beta}$ does not depend on the subsequence. Testing the open and closed set $L^p(D, \mathbb{R}^n)$ it immediately follows that $c_{\varphi,\beta} = \inf_{v \in L^p(D,\mathbb{R}^n)} I^\beta_{D,\varphi}(v)$ and this proves the large deviation principle with rate functional $\mathcal{I}^\beta_{D,\varphi}$ as claimed in Theorem 1.5. □

Proof of Theorem 1.4. Observe that, by the definitions in (1.9) and (1.10), in Step 3 above we also proved the claim on the Helmholtz free energy with boundary condition φ. □

From the large deviation principle we obtain the following qualitative behavior of the Gibbs measures.

Corollary 1. *Let $\varepsilon_j \to 0$. Under the assumptions of Theorem 1.5, for a set of full probability the sequence of measures $\mu^\beta_{\varepsilon_j D,\varphi}$ is compact with respect to weak*-convergence and each cluster point as $\varepsilon_j \to 0$ is a probability measure whose support is contained in the set of minimizers of the rate functional $\mathcal{I}^\beta_{D,\varphi}$.*

5.3. Asymptotic Analysis of the Localized Partition Function

At the end of this section we now give the technical proof of Theorem 4.5, which was used in the proof of the large deviation principle.

Proof of Theorem 4.5. Let $G \in \mathcal{G}'$ with $\mathcal{G}'$ the set of full probability implicitly given by Proposition 3. The argument consists of two steps.
Step 1. Proof of the upper bound.

We show that

$$\mathcal{F}^+(D, v) \leqq \frac{1}{|D|} \int_D \overline{W}^\beta(\nabla v)\, \mathrm{d}x. \tag{5.9}$$

By the lower semicontinuity of $v \mapsto \mathcal{F}^+(D, v)$ established in Lemma 4.4 and the p-growth conditions and continuity of $\overline{W}^\beta(\Lambda)$ (cf. the Lemmata 4.2 and 4.3 and Proposition 3) it is enough to prove the estimate for continuous piecewise affine functions. More precisely, we consider a locally finite triangulation $\mathcal{T} = \{T\}$ of $\mathbb{R}^d$ and a Lipschitz function $v \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^n)$ such that for all $T \in \mathcal{T}$ there exists $\Lambda_T \in \mathbb{R}^{n\times d}$ and $b_T \in \mathbb{R}^n$ with $v_{|T}(y) = \Lambda_T y + b_T$. To simplify notation we introduce the almost lower-dimensional set

$$S = \bigcup_{T\in\mathcal{T}} (\partial T \cap D) \cup \bigcup_{T\cap\partial D\neq\emptyset} (T \cap D).$$

A direct computation shows that, for fixed $\kappa > 0$ we find ε_0 such that for all $\varepsilon < \varepsilon_0$ we have the inclusion

$$\prod_{T\subset D} \left(\mathcal{N}_p\left(v, T, \varepsilon, \frac{\kappa}{2}\right) \cap \mathcal{B}_\varepsilon(T, \varphi_{\Lambda_T} + b_T)\right) \times \mathcal{N}_\infty(\varepsilon^{-1}v_\varepsilon, S, \varepsilon) \subset \mathcal{N}_p(v, D, \varepsilon, \kappa). \tag{5.10}$$

We aim to establish a kind of subadditivity estimate. To this end, observe that when u belongs to the set on the left hand side and $x, x' \in D_\varepsilon^{\mathcal{L}}$ are such that $|x - x'| \leqq C_0$, we have the following bounds:

(i) If $x \in T_\varepsilon$ and $x' \in T'_\varepsilon$ for $T \neq T'$, then

$$|u(x) - u(x')| \leqq 2 + \frac{1}{\varepsilon}|v(\varepsilon x) - v(\varepsilon x')| \leqq 2 + C_0\|\nabla v\|_\infty.$$

(ii) If $x \in T_\varepsilon$ and $x' \in S_\varepsilon$, then by definition of v_ε in (3.4) and (3.1)

$$\begin{aligned}|u(x) - u(x')| &\leqq 2 + \frac{1}{\varepsilon}|v(\varepsilon x) - v_\varepsilon(x')| \leqq 2 + R\|\nabla u\|_\infty + \frac{1}{\varepsilon}|v(\varepsilon x) - v(\varepsilon x')|\\ &\leqq 2 + (R + C_0)\|\nabla v\|_\infty.\end{aligned}$$

(iii) If $x, x' \in S_\varepsilon$, then by same reasoning as for (ii)

$$|u(x) - u(x')| \leqq 2 + \frac{1}{\varepsilon}|v_\varepsilon(x) - v_\varepsilon(x')| \leqq 2 + (2R + C_0)\|\nabla v\|_\infty.$$

By Hypothesis 1, the above bounds, (3.2) and (3.3) there exists a constant C_v depending only on $\|\nabla v\|_\infty^p$ such that, for ε sufficiently small,

$$H_\varepsilon(D, u) \leqq \sum_{T\subset D} H_\varepsilon(T, u) + C_v \left(\varepsilon^{1-d} \sum_{T\cap D\neq\emptyset} \mathcal{H}^{d-1}(\partial T) + \varepsilon^{-d} \sum_{T\cap\partial D\neq\emptyset} |T \cap D|\right).$$

Upon taking the inverse exponential, integrating the above inequality over the left hand side set in (5.10) and applying Fubini's Theorem yields the estimate

$$Z^\beta_{\varepsilon,D}(\mathcal{N}_p(v, D, \varepsilon, \kappa))$$

$$\geqq \prod_{T\subset D} Z^{\beta}_{\varepsilon,T}(\mathcal{N}_p(v,T,\varepsilon,\frac{\kappa}{2})\cap\mathcal{B}_\varepsilon(T,\varphi_{\Lambda_T}+b_T))$$
$$\times\exp\Bigg(-C_v\beta\varepsilon^{1-d}\sum_{T\cap D\neq\emptyset}\mathcal{H}^{d-1}(\partial T)-C_v\beta\varepsilon^{-d}\sum_{T\cap\partial D\neq\emptyset}|T\cap D|\Bigg),$$

where we incorporated the measure of $\mathcal{N}_\infty(\varepsilon^{-1}v_\varepsilon, S, \varepsilon)$ in the exponential term in the last line, possibly increasing the value of C_v by a multiplicative factor. On each simplex $T \subset D$ we use the translation invariance of the Hamiltonian to get rid of the constant b_T and obtain

$$Z^{\beta}_{\varepsilon,D}(\mathcal{N}_p(v,D,\varepsilon,\kappa))$$
$$\geqq \prod_{T\subset D} Z^{\beta}_{\varepsilon,T}(\mathcal{N}_p(\overline{\varphi}_{\Lambda_T},T,\varepsilon,\frac{\kappa}{2})\cap\mathcal{B}_\varepsilon(T,\varphi_{\Lambda_T}))$$
$$\times\exp\left(-C_v\beta\varepsilon^{1-d}\sum_{T\cap D\neq\emptyset}\mathcal{H}^{d-1}(\partial T)-C_v\beta\varepsilon^{-d}\sum_{T\cap\partial D\neq\emptyset}|T\cap D|\right),$$

Taking logarithms and dividing by $-\beta|D_\varepsilon|$, when $\varepsilon\to 0$ we infer from Lemma 4.1 and Proposition 3 that

$$\mathcal{F}^+_\kappa(D,v)\leqq\sum_{D\subset T}\frac{|T|}{|D|}\overline{W}^{\beta}(\nabla v_{|T})+C_v\sum_{T\cap\partial D\neq\emptyset}\frac{|T\cap D|}{|D|}$$
$$\leqq\frac{1}{|D|}\int_D\overline{W}^{\beta}(\nabla v)\,\mathrm{d}x+\sum_{T\cap\partial D\neq\emptyset}\Big(C_u-\overline{W}^{\beta}(\Lambda_T)\Big)\frac{|T\cap D|}{|D|}.$$

Now keeping v fixed, we let first $\kappa\to 0$ and then we refine the triangulation $\mathcal{T}$ and by the regularity of ∂D the last sum can be made arbitrarily small. This proves (5.9).

Step 2. Proof of the lower bound.
We now turn to the argument for the inequality

$$\mathcal{F}^-(D,v)\geqq\frac{1}{|D|}\int_D\overline{W}^{\beta}(\nabla v)\,\mathrm{d}x.$$

Let us assume without loss of generality that $\overline{W}^{\beta}(\Lambda)\geqq 0$. Due to Lemma 4.3 this can be achieved by adding a large constant to the discrete energy density f. This perturbation yields a (random) additive constant on both sides due to the superadditive version of the ergodic theorem. We want to apply the blow-up Lemma proven in [34] that allows to treat v locally as an affine function. To this end, we need some notation. First we extend the target function $v\in W^{1,p}(D,\mathbb{R}^n)$ (without relabeling) to a function $v\in W^{1,p}(\mathbb{R}^d,\mathbb{R}^n)$ with compact support. Next, given $\rho>0$ and $\xi\in\mathbb{R}^d$ we define the periodic lattice $\mathcal{L}_{\rho,\xi}=\xi+\rho\mathbb{Z}^d$. Note that for fixed ρ, for almost all $\xi\in\mathbb{R}^d$ the set $\mathcal{L}_{\rho,\xi}$ consists of Lebesgue points of ∇v. Hence we can define for such ξ a (not necessarily continuous) piecewise affine approximation as

$$L_{\rho,\xi}v(y)=\nabla v(z)(y-z)+\frac{1}{\rho^d}\int_{Q(z,\rho)}v(x)\,\mathrm{d}x\qquad\text{if } y\in Q(z,\rho),\ z\in\mathcal{L}_{\rho,\xi}.$$

The main object will be the local difference between the linearization and the function itself defined for $z \in \mathcal{L}_{\rho,\xi}$ as

$$\phi^z_{\rho,\xi}(y) = \frac{1}{\rho}\big(v(z+\rho y) - L_{\rho,\xi} v(z+\rho y)\big) \qquad \text{if } y \in Q(0,1).$$

Note that due the bounds in Lemma 4.2, for any $D' \subset\subset D$ the function $x \mapsto \overline{W}^\beta(\nabla v(x))\mathbb{1}_{D'}(x)$ is integrable on $\mathbb{R}^d$. Hence we are in the position to use the blow-up Lemma by Kotecký and Luckhaus (see [34, Corollary 1]). Note that we apply it also for the function itself which follows simply by the Poincaré inequality since the function $\phi^z_{\rho,\xi}$ has mean value zero on $Q(0,1)$. It states that for each $\eta > 0$ we find $\rho_0 > 0$ such that for each $\rho < \rho_0$ there exists $\xi \in Q(0,\rho)$ with

$$\begin{aligned} &\sum_{z\in\mathcal{L}_{\rho,\xi}} \rho^d \left(\int_{Q(0,1)} |\nabla \phi^z_{\rho,\xi}(y)|^p \,\mathrm{d}y + \int_{Q(0,1)} |\phi^z_{\rho,\xi}(y)|^p \,\mathrm{d}y \right) < \eta \\ &\sum_{z\in\mathcal{L}_{\rho,\xi}} \rho^d \overline{W}^\beta(\nabla v(z))\mathbb{1}_{D'}(z) > \int_{D'} \overline{W}(\nabla v)\,\mathrm{d}x - \eta. \end{aligned} \tag{5.11}$$

Next, if $u \in \mathcal{N}_p(v, D, \varepsilon, \kappa)$, then for all cubes $Q(z,\rho) \subset D$ and $\varepsilon = \varepsilon(\rho)$ small enough the triangle inequality, the definition (3.4) and a change of variables imply

$$\begin{aligned} &\left(\sum_{x\in Q(z,\rho)^{\mathcal{L}}_\varepsilon} \varepsilon^d |\varepsilon u(x) - (L_{\rho,\xi} v)_\varepsilon(x)|^p \right)^{\frac{1}{p}} \\ &\leqq \kappa |D|^{\frac{1}{p}+\frac{1}{d}} + \left(\sum_{x\in Q(z,\rho)^{\mathcal{L}}_\varepsilon} \varepsilon^d |v_\varepsilon(x) - (L_{\rho,\xi} v)_\varepsilon(x)|^p \right)^{\frac{1}{p}} \\ &\leqq \kappa |D|^{\frac{1}{p}+\frac{1}{d}} + C\rho^{1+\frac{d}{p}} \left(\sum_{\substack{z'\in\mathcal{L}_{\rho,\xi} \\ |z-z'|\leqq\rho}} \int_{Q(0,1)} |\phi^{z'}_{\rho,\xi}(y)|^p \,\mathrm{d}y \right)^{\frac{1}{p}} \\ &= \kappa |D|^{\frac{1}{p}+\frac{1}{d}} + C|Q(z,\rho)|^{\frac{1}{p}+\frac{1}{d}} \left(\sum_{\substack{z'\in\mathcal{L}_{\rho,\xi} \\ |z-z'|\leqq\rho}} \int_{Q(0,1)} |\phi^{z'}_{\rho,\xi}(y)|^p \,\mathrm{d}y \right)^{\frac{1}{p}}. \end{aligned}$$

On setting $S_\rho = S^1_\rho \cup S^2_\rho$, where $S^1_\rho = \bigcup_{z\in\mathcal{L}_{\rho,\xi}} (\partial Q(z,\rho) \cap D)$ and $S^2_\rho = \bigcup_{Q(z,\rho)\cap\partial D\neq\emptyset} (Q(z,\rho) \cap D)$ as well as

$$\kappa_z = \max\left\{ 2C \left(\sum_{\substack{z'\in\mathcal{L}_{\rho,\xi} \\ |z-z'|\leqq\rho}} \int_{Q(0,1)} |\phi^{z'}_{\rho,\xi}(y)|^p \,\mathrm{d}y \right)^{\frac{1}{p}}, \rho \right\},$$

we obtain for $\kappa = \kappa(\rho)$ small enough the set inclusion

$$\mathcal{N}_p(v, D, \varepsilon, \kappa) \subset \prod_{\substack{z \in \mathcal{L}_{\rho,\xi} \\ Q(z,\rho) \subset D}} \mathcal{N}_p(L_{\rho,\xi} v, Q(z, \rho), \varepsilon, \kappa_z) \times \prod_{i=1}^{2} S_\varepsilon^i(\kappa, \rho), \tag{5.12}$$

where for $i = 1, 2$ we define the sets

$$S_\varepsilon^i(\kappa, \rho) := \left\{ u : (S_\rho^i)_\varepsilon^{\mathcal{L}} \to \mathbb{R}^n : \ \|u - \varepsilon^{-1} v_\varepsilon\|_\infty \leqq \kappa |\varepsilon^{-1} D|^{\frac{1}{p}+\frac{1}{d}} \right\}.$$

In order to control the integration over these two sets, we note that for ε small enough

$$\begin{aligned} \log \left(\int_{S_\varepsilon^1(\kappa,\rho)} \mathrm{d}u \right) &\leqq C \log \left(C\kappa^n \varepsilon^{-\frac{nd}{p}-n} \right) \sum_{Q(z,\rho) \cap D \neq \emptyset} \rho^{d-1} \varepsilon^{1-d} \\ &\leqq C \log \left(C\kappa^n \varepsilon^{-\frac{nd}{p}-n} \right) \rho^{-1} \varepsilon^{1-d}. \end{aligned} \tag{5.13}$$

To treat the contributions from the points in $(S_\rho^2)_\varepsilon$ we have to use once again Lemma 3.3. To this end we observe that

$$\partial S_\rho^2 \subset \partial D \cup \bigcup_{Q(z,\rho) \cap \partial D \neq \emptyset} (\partial Q(z, \rho) \cap D).$$

As the union on the right hand side is finite, we can argue as for Remark 7 and (3.3) to show that, for ε small enough, the number of connected components $N_{\varepsilon,\rho}$ of the graph $G_{S_\rho^2,\varepsilon}$ can be bounded by

$$N_{\varepsilon,\rho} \leqq C\varepsilon^{1-d} \left(\mathcal{H}^{d-1}(\partial D) + \rho^{-1} \right). \tag{5.14}$$

Due to Hypothesis 1 and Lemma 3.3 we deduce the bound

$$\begin{aligned} &\log \left(\int_{S_\varepsilon^2(\kappa,\rho)} \exp(-\beta H_\varepsilon(S_\rho^2, u) \, \mathrm{d}u) \right) \\ &\leqq \log \left(\int_{S_\varepsilon^2(\kappa,\rho)} \exp \left(-\frac{\beta}{C} \|\nabla_{\mathbb{B}} u\|_{\ell_\varepsilon^p(S_\rho^2)}^p \right) \mathrm{d}u \right) + C\beta |(S_\rho^2)_\varepsilon^{\mathcal{L}}| \\ &\leqq \log \left(C\kappa^n \varepsilon^{-\frac{nd}{p}-n} \right) N_{\varepsilon,\rho} + C(1 + |\log(\beta)| + \beta) |(S_\rho^2)_\varepsilon^{\mathcal{L}}| \end{aligned} \tag{5.15}$$

Together with the inequality $H_\varepsilon(D, u) \geqq \sum_{Q(z,\rho) \subset D} H_\varepsilon(Q(z, \rho), u) + H_\varepsilon(S_\rho^2, u)$, the inclusion in (5.12) and Fubini's Theorem imply

$$Z_{\varepsilon,D}^\beta(\mathcal{N}_p(v, D, \varepsilon, \kappa)) \leqq \prod_{\substack{z \in \mathcal{L}_{\rho,\xi} \\ Q(z,\rho) \subset D}} Z_{\varepsilon,Q(z,\rho)}^\beta(\mathcal{N}_p(L_{\rho,\xi} v, Q(z, \rho), \varepsilon, \kappa_z))$$

$$\times \int_{S^1_\varepsilon(\kappa)} \mathrm{d}u_1 \int_{S^2_\varepsilon(\kappa,\rho)} \exp(-H_\varepsilon(S^2_\rho, u_2)\,\mathrm{d}u_2.$$

Taking logarithms and dividing by $-\beta|D_\varepsilon|$ we infer from (5.13), (5.14) combined with (5.15) that

$$\mathcal{F}^-(D,v) \geqq \sum_{\substack{z\in\mathcal{L}_{\rho,\xi}\\ Q(z,\rho)\subset D}} \frac{\rho^d}{|D|}\mathcal{F}^-_{\kappa_z}(Q(z,\rho),\overline{\varphi}_{\nabla v(z)}) \\ -C\left(\frac{1+|\log(\beta)|+\beta}{\beta}\right)\frac{|\partial D+Q(0,2\rho)|}{|D|},$$

where we also used that the energy is invariant under constant shifts so that we can pass from the affine approximation to the linear one. Since we assume that $\overline{W}(\Lambda)\geqq 0$, using Remark 14 and (5.11) we infer that for arbitrary $N\in\mathbb{N}$ and δ, ρ sufficiently small

$$\begin{aligned}
&\mathcal{F}^-(D,v)\\
&\quad\geqq \sum_{\substack{z\in\mathcal{L}_{\rho,\xi}\\ Q(z,\rho)\subset D}} \frac{\rho^d}{|D|}\left(\overline{W}^\beta(\nabla v(z)) - C\left((1+|\nabla v(z)|^p)\delta + \frac{(N\kappa_z)^p}{\delta^p}+\frac{1}{N}\right)\right)\\
&\qquad - C_\beta\frac{|\partial D+Q(0,2\rho)|}{|D|}\\
&\quad\geqq \frac{1}{|D|}\int_{D'}\overline{W}^\beta(\nabla v)\,\mathrm{d}x - \frac{\eta}{|D|}\\
&\qquad - \sum_{\substack{z\in\mathcal{L}_{\rho,\xi}\\ Q(z,\rho)\subset D}} C\frac{\rho^d}{|D|}\left((1+|\nabla u(z)|^p)\delta+\frac{(N\kappa_z)^p}{\delta^p}+\frac{1}{N}\right)\\
&\qquad - C_\beta\frac{|\partial D+Q(0,2\rho)|}{|D|}.
\end{aligned}\tag{5.16}$$

Using again (5.11) we can bound the sum of the gradients. Indeed, by a change of variables it holds that

$$\begin{aligned}
\sum_{\substack{z\in\mathcal{L}_{\rho,\xi}\\ Q(z,\rho)\subset D}} \rho^d|\nabla v(z)|^p &\leqq C \sum_{\substack{z\in\mathcal{L}_{\rho,\xi}\\ Q(z,\rho)\subset D}} \int_{Q(z,\rho)} |\nabla v(y)-\nabla v(z)|^p + |\nabla v(y)|^p\,\mathrm{d}y\\
&\leqq C\sum_{z\in\mathcal{L}_{\rho,\xi}}\rho^d\int_{Q(0,1)}|\nabla\phi^z_{\rho,\xi}(y)|^p\,\mathrm{d}y + C\|\nabla v\|^p_{L^p(D)}\\
&\leqq C(\eta+\|\nabla v\|^p_{L^p(D)}).
\end{aligned}\tag{5.17}$$

To control the sum over κ_z^p, note that by (5.11) and the definition of κ_z we have

$$\sum_{\substack{z\in\mathcal{L}_{\rho,z}\\ Q(z,\rho)\subset D}} \frac{\rho^d}{|D|}\kappa_z^p \leqq C\rho^p + \frac{C}{|D|}\sum_{z\in\mathcal{L}_{\rho,\xi}}\rho^d\int_{Q(0,1)}|\phi^z_{\rho,\xi}(y)|^p\,\mathrm{d}y \leqq C(\rho^p+\frac{\eta}{|D|}).$$

(5.18)

Putting together (5.16), (5.17) and (5.18) we obtain

$$\begin{aligned}\mathcal{F}^-(D,v) \geqq \frac{1}{|D|}\int_{D'} \overline{W}^{\beta}(\nabla v)\,\mathrm{d}x \\ - \frac{C}{|D|}\left(\left(|D| + \eta + \|\nabla v\|^p_{L^p(D)}\right)\delta + \frac{N^p}{\delta^p}(\rho^p|D| + \eta) + \frac{|D|}{N}\right) \\ - C_\beta \frac{|\partial D + Q(0,\rho)|}{|D|}.\end{aligned}$$

The last inequality concludes the proof after letting first $\rho \to 0$, then $\eta \to 0$ followed by $N \to +\infty$ and $\delta \to 0$ and finally using the arbitrariness of $D' \subset\subset D$ (recall the integrability of $\overline{W}^{\beta}(\nabla v)$). □

6. Zero Temperature Limit of the Elastic Free Energy: Proof of Theorem 1.6

In this section we investigate the asymptotic behavior of the rate functional from the large deviation principle when the temperature vanishes, or equivalently when $\beta \to +\infty$. To this end, we bound from above and below the entropic part whenever we consider the energy difference between a general configuration of the system and the ground state when we prescribe linear boundary conditions. We shall prove that, under the standard p-growth conditions (1.3) and an additional local Lipschitz property (see Hypothesis 2) we indeed recover the density of the Γ-limit of the rescaled versions of the Hamiltonians $H_\varepsilon(D, v)$. Since Γ-convergence focuses on the convergence of global minimizers of the Hamiltonian $H_\varepsilon(D, \cdot)$ (for a general reference on the subject we refer to the standard literature [12,23]), our result shows that at low temperatures entropic effects can be neglected and energy minimization is indeed meaningful also from a statistical physics point of view.

6.1. Variational Results Neglecting Temperature

For completeness we briefly recall Γ-convergence results at zero temperature. First we rescale the Hamiltonian and its domain as for the definition of the Gibbs measure. Given $\varepsilon > 0$ and a function $u : \mathcal{L} \to \mathbb{R}^n$ we define the function $v : \varepsilon\mathcal{L} \to \mathbb{R}^n$ setting $v(\varepsilon x) = \varepsilon u(x)$. As usual this function can be identified with a function that is constant on the scaled Voronoi cells, so that it belongs to the class

$$\mathcal{PC}_\varepsilon := \{v : \mathbb{R}^d \to \mathbb{R}^n : \ u_{|\varepsilon \mathcal{C}(x)} \text{ is constant for all } x \in \mathcal{L}\}.$$

We may embed $\mathcal{PC}_\varepsilon \subset L^p(D, \mathbb{R}^n)$. Then, for every $O \in \mathcal{A}^R(D)$, we introduce the rescaled Hamiltonian $\tilde{H}_\varepsilon(O, \cdot) : L^p(D, \mathbb{R}^n) \to [0, +\infty]$ setting

$$\tilde{H}_\varepsilon(v, O) = \begin{cases} \dfrac{1}{|O_\varepsilon|} \displaystyle\sum_{\substack{(x,y)\in\mathbb{B} \\ \varepsilon x, \varepsilon y \in O}} f\left(x - y, \frac{v(\varepsilon x) - v(\varepsilon y)}{\varepsilon}\right) & \text{if } v \in \mathcal{PC}_\varepsilon. \\ +\infty & \text{otherwise} \end{cases}$$

We then define the set of clamped displacements

$$\mathcal{BC}_\varepsilon(O,\varphi_\Lambda) = \{u : O_\varepsilon^{\mathcal{L}} \to \mathbb{R}^n :\ u(x) = \Lambda x \text{ if } \operatorname{dist}(x,\partial O_\varepsilon) \leqq C_0\}.$$

Note that in contrast to the soft boundary conditions defining the set $\mathcal{B}_\varepsilon(O,\varphi_{\Lambda'})$ here the boundary conditions are exactly satisfied. The density of the Γ-limit is then given by the formula

$$\overline{W}^\infty(\Lambda) = \lim_{\varepsilon\to 0} \frac{1}{|Q_\varepsilon|} \inf\{H_\varepsilon(u,Q) :\ u \in \mathcal{BC}_\varepsilon(Q,\varphi_\Lambda)\},$$

where $Q = (-\frac{1}{2},\frac{1}{2})^d$. The existence of this limit is a consequence of the subadditive ergodic Theorem, as for Proposition 2. By [4, Theorems 2 & 3] we have the following Γ-convergence result:

Theorem 6.1. *Assume* (1.3) *and let G be an admissible, stationary random Euclidean graph. Assume in addition that f is continuous in the second variable. Then almost surely the functionals $\tilde{H}_\varepsilon$ Γ-converge with respect to the $L^p(D,\mathbb{R}^n)$-topology to the functional $\overline{H} : L^p(D,\mathbb{R}^n) \to [0,+\infty]$ finite only on $W^{1,p}(D,\mathbb{R}^n)$ and characterized by*

$$\overline{H}(v) = \frac{1}{|D|}\int_D \overline{W}^\infty(\nabla v(x))\,\mathrm{d}x.$$

Moreover, for any $O \in \mathcal{A}^R(D)$ and any $v \in W^{1,p}(D,\mathbb{R}^n)$ we have the local version

$$\Gamma\text{-}\lim_{\varepsilon\to 0} \tilde{H}_\varepsilon(O,v) = \frac{1}{|O|}\int_O \overline{W}^\infty(\nabla v(x))\,\mathrm{d}x.$$

The map $\Lambda \mapsto \overline{W}^\infty(\Lambda)$ is quasiconvex and satisfies the p-growth condition

$$\frac{1}{C}|\Lambda|^p - C \leqq \overline{W}^\infty(\Lambda) \leqq C(|\Lambda|^p + 1).$$

In order to also incorporate Dirichlet boundary conditions $\varphi \in \operatorname{Lip}(\mathbb{R}^d,\mathbb{R}^n)$, we introduce the class

$$\mathcal{PC}_{\varepsilon,\varphi} = \{v \in \mathcal{PC}_\varepsilon :\ v(\varepsilon x) = \varphi(\varepsilon x) \text{ for all } x \in \mathcal{L} \text{ such that } \operatorname{dist}(\varepsilon x,\partial D) \leqq C_0\varepsilon\}.$$

We restrict the domain of the discrete Hamiltonian $\tilde{H}_\varepsilon$ to $\mathcal{PC}_{\varepsilon,\varphi}$ setting $\tilde{H}_{\varepsilon,\varphi} : L^p(D,\mathbb{R}^n) \to [0,+\infty]$ as

$$\tilde{H}_{\varepsilon,\varphi}(v) = \begin{cases} \tilde{H}_\varepsilon(v) & \text{if } v \in \mathcal{PC}_{\varepsilon,\varphi}, \\ +\infty & \text{otherwise.} \end{cases}$$

Then [4, Theorem 4] yields the following Γ-convergence result under Dirichlet boundary conditions.

Theorem 6.2. *Under the assumptions of Theorem 6.1, the functionals $\tilde{H}_{\varepsilon,\varphi}$ Γ-converge with respect to the $L^p(D,\mathbb{R}^n)$-topology to the functional $\overline{H}_\varphi : L^p(D,\mathbb{R}^n) \to [0,+\infty]$ finite only for $\varphi + W_0^{1,p}(D,\mathbb{R}^n)$ and characterized by*

$$\overline{H}_\varphi(v) = \frac{1}{|D|}\int_D \overline{W}^\infty(\nabla v(x))\,\mathrm{d}x.$$

Remark 12. From Lemma 5.1 and the fundamental property of Γ-convergence we deduce in particular the convergence of (almost-)minimizers to minimizers of the limit energy. Moreover it follows that

$$\lim_{\varepsilon\to 0}\Big(\inf_{v\in L^p(D,\mathbb{R}^n)} \tilde{H}_{\varepsilon,\varphi}(v)\Big) = \min_{v\in L^p(D,\mathbb{R}^n)} \overline{H}_\varphi(v). \tag{6.1}$$

Before comparing the Γ-limit and the limit free energy we prove that one can replace the clamped boundary conditions by the soft version considered for the free energies and obtains the same limit. Note that in what follows both definitions will be used.

Lemma 6.3. *Assume Hypothesis 1. Fix $\Lambda \in \mathbb{R}^{n\times d}$. Then almost surely it holds that*

$$\overline{W}^\infty(\Lambda) = \lim_{\varepsilon\to 0}\frac{1}{|Q_\varepsilon|}\inf\{H_\varepsilon(u,Q):\ u\in\mathcal{B}_\varepsilon(Q,\varphi_\Lambda)\}.$$

Proof of Lemma 6.3. The result is a special case of Γ-convergence. Indeed, consider the auxiliary functional $H_{\varepsilon,\Lambda} : L^p(Q,\mathbb{R}^n) \to [0,+\infty]$ defined by

$$H_{\varepsilon,\Lambda}(v) = \begin{cases} \tilde{H}_\varepsilon(Q,v) & \text{if } v\in\mathcal{PC}_\varepsilon \text{ and } \Pi_{1/\varepsilon}v\in\mathcal{B}_\varepsilon(Q,\varphi_\Lambda),\\ +\infty & \text{otherwise.}\end{cases}$$

Due to Lemma 5.1 we know that the Γ-limit of $H_{\varepsilon,\Lambda}$ can be finite only for $v \in \overline{\varphi}_\Lambda + W_0^{1,p}(Q,\mathbb{R}^n)$. From Theorems 6.1 and 6.2 applied with $D = Q$ and $\varphi = \overline{\varphi}_\Lambda$, for any such v we deduce from monotonicity that

$$\begin{aligned}\int_Q \overline{W}^\infty(\nabla v(x))\,\mathrm{d}x &\leqq \Gamma\text{-}\liminf_{\varepsilon\to 0} H_{\varepsilon,\Lambda}(v) \leqq \Gamma\text{-}\limsup_{\varepsilon\to 0} H_{\varepsilon,\Lambda}(u)\\ &\leqq \Gamma\text{-}\limsup_{\varepsilon\to 0}\tilde{H}_{\varepsilon,\overline{\varphi}_\Lambda}(v) \leqq \int_Q \overline{W}^\infty(\nabla v(x))\,\mathrm{d}x.\end{aligned}$$

Using (6.1), which holds by the same arguments for the functionals $H_{\varepsilon,\Lambda}$, the result follows by quasiconvexity of $\Lambda \mapsto \overline{W}^\infty(\Lambda)$ and a rescaling since $\frac{1}{|Q_\varepsilon|}H_\varepsilon(Q,\Pi_{1/\varepsilon}(\cdot)) = \tilde{H}_\varepsilon(Q,\cdot)$. □

6.2. Quantitative Comparison Between $\overline{W}^{\beta}$ and $\overline{W}^{\infty}$

As announced earlier, in this section we replace Hypothesis 1 by the stronger assumptions of Hypothesis 2. In the following two lemmata we establish a quantitative estimate between $\overline{W}^{\beta}(\Lambda)$ and $\overline{W}^{\infty}(\Lambda)$ in the regime $\beta \gtrsim 1$. Note that the additional properties of Hypothesis 2 are only needed for Lemma 6.4 below.

Lemma 6.4. *Assume Hypothesis 2. Then for every $\Lambda \in \mathbb{R}^{n\times d}$ there exists a constant $0 < C_\Lambda \leqq C(1+|\Lambda|^{p-1})$ such that, for all $\beta \geqq \exp(1)$,*

$$\overline{W}^{\beta}(\Lambda) - \overline{W}^{\infty}(\Lambda) \leqq C_\Lambda \frac{\log(\beta)}{\beta}.$$

Proof of Lemma 6.4. Due to Proposition 2 we can consider the free energy on the unit cube Q. First note that due to the assumptions, the discrete energy $H_\varepsilon(Q,\cdot)$ is equicoercive and continuous on the closed set $\mathcal{BC}_\varepsilon(Q,\varphi_\Lambda)$. Hence the minimum is attained and we denote by $\hat{u}_\varepsilon$ a minimizer. By testing the function φ_Λ and using the p-growth conditions of Hypothesis 2, we obtain the a priori bound

$$\|\nabla_{\mathbb{B}}\hat{u}_\varepsilon\|^p_{\ell^p_\varepsilon(Q)} \leqq C H_\varepsilon(\hat{u}_\varepsilon) + C|Q_\varepsilon^{\mathcal{L}}| \leqq C(1+|\Lambda|^p)|Q_\varepsilon^{\mathcal{L}}|. \tag{6.2}$$

According to the continuity property in Hypothesis 2, for any $u \in \mathcal{B}_\varepsilon(Q,\varphi_\Lambda)$ we have the estimate

$$\begin{aligned}
&H_\varepsilon(Q,u) - H_\varepsilon(Q,\hat{u}_\varepsilon)\\
&\quad\leqq \sum_{\substack{(x,y)\in\mathbb{B}\\ x,y\in Q_\varepsilon}} |f(x-y,u(x)-u(y)) - f(x-y,\hat{u}_\varepsilon(x)-\hat{u}_\varepsilon(y))|\\
&\quad\leqq C\sum_{\substack{(x,y)\in\mathbb{B}\\ x,y\in Q_\varepsilon}} (1+|u(x)-u(y)|^{p-1} + |\hat{u}_\varepsilon(x)-\hat{u}_\varepsilon(y)|^{p-1})|(u-\hat{u}_\varepsilon)(x)\\
&\qquad - (u-\hat{u}_\varepsilon)(y)|\\
&\quad\leqq C\big(|Q_\varepsilon^{\mathcal{L}}|^{\frac{p-1}{p}} + \|\nabla_{\mathbb{B}}(u-\hat{u}_\varepsilon)\|^{p-1}_{\ell^p_\varepsilon(Q)} + \|\nabla_{\mathbb{B}}\hat{u}_\varepsilon\|^{p-1}_{\ell^p_\varepsilon(Q)}\big)\|\nabla_{\mathbb{B}}(u-\hat{u}_\varepsilon)\|_{\ell^p_\varepsilon(Q)}\\
&\quad\leqq C|Q_\varepsilon^{\mathcal{L}}|^{\frac{p-1}{p}}(1+|\Lambda|^{p-1})\|\nabla_{\mathbb{B}}(u-\hat{u}_\varepsilon)\|_{\ell^p_\varepsilon(Q)} + C\|\nabla_{\mathbb{B}}(u-\hat{u}_\varepsilon)\|^p_{\ell^p_\varepsilon(Q)}\\
&\quad\leqq C|Q_\varepsilon^{\mathcal{L}}|^{\frac{p-1}{p}}(1+|\Lambda|^{p-1})\|u-\hat{u}_\varepsilon\|_{\ell^p_\varepsilon(Q)} + C\|u-\hat{u}_\varepsilon\|^p_{\ell^p_\varepsilon(Q)},
\end{aligned}$$

where we have used Hölder's inequality, (6.2) and Remark 6. From the change of variables $u \mapsto u - \hat{u}_\varepsilon$, which maps one-to-one from $\mathcal{B}_\varepsilon(Q,\varphi_\Lambda)$ to $\mathcal{B}_\varepsilon(Q,0)$, we infer that the discrete error can be bounded by

$$\begin{aligned}
e_{\varepsilon,\beta} :&= \mathcal{E}^{\beta}_\varepsilon(Q,\overline{\varphi}_\Lambda) - \frac{1}{|Q_\varepsilon|}H_\varepsilon(\hat{u}_\varepsilon)\\
&= -\frac{1}{\beta|Q_\varepsilon|}\log\left(\int_{\mathcal{B}_\varepsilon(Q,\varphi_\Lambda)} \exp\big(-\beta(H_\varepsilon(Q,u)-H_\varepsilon(Q,\hat{u}_\varepsilon))\big)\,\mathrm{d}u\right)
\end{aligned}$$

$$\leqq -\frac{1}{\beta|Q_\varepsilon|}\log\left(\int_{\mathcal{B}_\varepsilon(Q,0)}\exp\left(-\beta C(|Q_\varepsilon^{\mathcal{L}}|^{\frac{p-1}{p}}(1+|\Lambda|^{p-1})\|u\|_{\ell_\varepsilon^p(Q)}\right.\right.$$

$$\left.\left.+\|u\|^p_{\ell^p_\varepsilon(Q)})\right)\,\mathrm{d}u\right). \tag{6.3}$$

In this last integral we aim to get rid of the boundary conditions. To this end, let us write the domain of integration as a product and implicitly define the numbers $d_{\varepsilon,i}$ and $d_{\varepsilon,b}$ via

$$\mathcal{B}_\varepsilon(Q,0) = \left(\prod_{\substack{x\in Q_\varepsilon^{\mathcal{L}}\\ \mathrm{dist}(x,\partial Q_\varepsilon)>C_0}}\mathbb{R}^n\right)\times\left(\prod_{\substack{x\in Q_\varepsilon^{\mathcal{L}}\\ \mathrm{dist}(x,\partial Q_\varepsilon)\leqq C_0}}B_1(0)\right)$$
$$= (\mathbb{R}^n)^{d_{\varepsilon,\mathrm{int}}}\times(B_1(0))^{d_{\varepsilon,\mathrm{bd}}}.$$

With a slight abuse of notation, we write any $u\in\mathcal{B}_\varepsilon(Q,0)$ as a sum via $u=u_1+u_2$, where $u_1\in\mathcal{BC}_\varepsilon(Q,0)$ and $|u_2(x)|\leqq 1$ with support contained in $\{x\in Q_\varepsilon : \mathrm{dist}(x,\partial Q_\varepsilon)\leqq C_0\}$. Interpreting a deformation as a large vector $u\in\mathbb{R}^{n|Q_\varepsilon^{\mathcal{L}}|}$ we denote its standard p-norm by $|u|_p$. As on $\mathbb{R}^n$ all norms are equivalent, it holds that $\|u\|_{\ell^p_\varepsilon(Q)}\leqq C|u|_p$. By the triangle inequality and the structure of u_2 we get

$$|Q_\varepsilon^{\mathcal{L}}|^{\frac{p-1}{p}}(1+|\Lambda|^{p-1})\|u\|_{\ell^p_\varepsilon(Q)}+\|u\|^p_{\ell^p_\varepsilon(Q)}$$
$$\leqq C\left(|Q_\varepsilon^{\mathcal{L}}|^{\frac{p-1}{p}}(1+|\Lambda|^{p-1})\left(|u_1+u_2|_p\right)+(|u_1+u_2|_p^p)\right)$$
$$\leqq C\Big(|Q_\varepsilon^{\mathcal{L}}|^{\frac{p-1}{p}}(1+|\Lambda|^{p-1})(|u_1|_p+(d_{\varepsilon,\mathrm{bd}})^{\frac{1}{p}})+|u_1|_p^p+d_{\varepsilon,\mathrm{bd}}\Big).$$

Using Fubini's Theorem we can factorize the integral and therefore (6.3) yields

$$e_{\varepsilon,\beta}\leqq -\frac{1}{\beta|Q_\varepsilon|}\log\left(\int_{\mathbb{R}^{nd_{\varepsilon,\mathrm{int}}}}\exp\big(-\beta C(|Q_\varepsilon^{\mathcal{L}}|^{\frac{p-1}{p}}(1+|\Lambda|^{p-1})|u_1|_p+|u_1|_p^p)\big)\mathrm{d}u_1\right)$$
$$-\frac{1}{\beta|Q_\varepsilon|}\log\left(|B_1(0)|^{d_{\varepsilon,\mathrm{bd}}}\exp\big(-\beta C((1+|\Lambda|^{p-1})|Q_\varepsilon^{\mathcal{L}}|^{\frac{p-1}{p}}d_{\varepsilon,\mathrm{bd}}^{\frac{1}{p}}+d_{\varepsilon,\mathrm{bd}})\big)\right)$$
$$=: e^{\mathrm{int}}_{\varepsilon,\beta}+e^{\mathrm{bd}}_{\varepsilon,\beta}.$$

We first argue that $e^{\mathrm{bd}}_{\varepsilon,\beta}$ vanishes when $\varepsilon\to 0$. Indeed, as $d_{\varepsilon,\mathrm{bd}}\leqq C\varepsilon^{1-d}$ by (3.3) and the Lipschitz regularity of ∂Q, for ε small enough it holds that

$$|e^{\mathrm{bd}}_{\varepsilon,\beta}|\leqq (C\beta^{-1}+C)\varepsilon+C(1+|\Lambda|^{p-1})\varepsilon^{\frac{1}{p}}. \tag{6.4}$$

To treat the contribution of $e^{\mathrm{int}}_{\varepsilon,\beta}$, we make use of the coarea formula. Therefore we consider the Lipschitz-continuous function $f_\varepsilon:\mathbb{R}^{nd_{\varepsilon,\mathrm{int}}}\to[0,+\infty)$ defined by $y\mapsto f_\varepsilon(y)=|Q_\varepsilon^{\mathcal{L}}|^{-\frac{1}{p}}|y|_p$. For $y\neq 0$ it is differentiable and, since f_ε is

$|Q_\varepsilon^{\mathcal{L}}|^{-\frac{1}{p}}$-Lipschitz with respect to the p-norm, for ε small enough we have the rough estimate

$$
\begin{aligned}
|\nabla f_\varepsilon(y)|_2 &= \sup_{|x|_2=1} \langle \nabla f_\varepsilon(y), x\rangle = \sup_{|x|_2=1} \lim_{t\to 0} \frac{f_\varepsilon(y+tx) - f_\varepsilon(y)}{t} \\
&\leqq \sup_{|x|_2=1} |Q_\varepsilon^{\mathcal{L}}|^{-\frac{1}{p}} |x|_p \leqq |Q_\varepsilon^{\mathcal{L}}|^{-\frac{1}{p}} \max\{1, (nd_{\varepsilon,\mathrm{int}})^{\frac{1}{p}-\frac{1}{2}}\} \leqq 1. \qquad (6.5)
\end{aligned}
$$

Using (6.5), we deduce from the coarea formula that for arbitrary $t_* > 0$

$$
\begin{aligned}
e_{\varepsilon,\beta}^{\mathrm{int}} &\leqq -\frac{1}{\beta|Q_\varepsilon|} \log\bigg(\int_{\mathbb{R}^{nd_{\varepsilon,\mathrm{int}}}} |\nabla f_\varepsilon(u_1)|_2 \exp\big(-\beta C(|Q_\varepsilon^{\mathcal{L}}|(1+|\Lambda|^{p-1}) f_\varepsilon(u_1) \\
&\quad + |Q_\varepsilon^{\mathcal{L}}| f_\varepsilon(u_1)^p)\big)\, \mathrm{d}u_1 \bigg) \\
&= -\frac{1}{\beta|Q_\varepsilon|} \log\bigg(\int_0^\infty \mathcal{H}^{nd_{\varepsilon,\mathrm{int}}-1}(\{f_\varepsilon = t\}) \exp\big(-\beta C |Q_\varepsilon^{\mathcal{L}}|((1+|\Lambda|^{p-1})t \\
&\quad + t^p)\big)\, \mathrm{d}t \bigg) \\
&\leqq -\frac{1}{\beta|Q_\varepsilon|} \log\bigg(\int_0^{t_*} \mathcal{H}^{nd_{\varepsilon,\mathrm{int}}-1}(\{|y|_p = |Q_\varepsilon^{\mathcal{L}}|^{\frac{1}{p}} t\}) \exp\big(-\beta C |Q_\varepsilon^{\mathcal{L}}|((1 \\
&\quad + |\Lambda|^{p-1})t + t^p)\big)\, \mathrm{d}t \bigg), \qquad (6.6)
\end{aligned}
$$

We next bound from below the surface measure inside the integral. To this end, we make the restriction $t_* \leqq 1$. By Lemma A.2 and the scaling properties of the Hausdorff measure, for some small constant $c = c(n, p)$ we have the lower bound

$$
\begin{aligned}
\mathcal{H}^{nd_{\varepsilon,\mathrm{int}}-1}\left(\left\{ |y|_p = |Q_\varepsilon^{\mathcal{L}}|^{\frac{1}{p}} t \right\} \right) &\geqq (|Q_\varepsilon^{\mathcal{L}}|^{\frac{1}{p}} t)^{nd_{\varepsilon,\mathrm{int}}-1} \left(\frac{c_p}{nd_{\varepsilon,\mathrm{int}}} \right)^{\frac{nd_{\varepsilon,\mathrm{int}}}{p}} \\
&\geqq |Q_\varepsilon^{\mathcal{L}}|^{-\frac{1}{p}} (ct)^{nd_{\varepsilon,\mathrm{int}}} \left(\frac{|Q_\varepsilon^{\mathcal{L}}|}{d_{\varepsilon,\mathrm{int}}} \right)^{\frac{nd_{\varepsilon,\mathrm{int}}}{p}} \\
&\geqq |Q_\varepsilon^{\mathcal{L}}|^{-\frac{1}{p}} (ct)^{n|Q_\varepsilon^{\mathcal{L}}|},
\end{aligned}
$$

where we used that $t \leqq 1$. Plugging this bound into (6.6), for any $t_* \leqq 1$ we can further estimate

$$
\begin{aligned}
e_{\varepsilon,\beta}^{\mathrm{int}} &\leqq -\frac{1}{\beta|Q_\varepsilon|} \log\left(\int_0^{t_*} \exp\Big(|Q_\varepsilon^{\mathcal{L}}| \big(n \log(ct) - \beta C((1+|\Lambda|^{p-1})t + t^p) \big) \Big)\, \mathrm{d}t \right) \\
&\quad + \frac{1}{p\beta|Q_\varepsilon|} \log(|Q_\varepsilon^{\mathcal{L}}|). \qquad (6.7)
\end{aligned}
$$

We now choose an appropriate $t_* \leqq 1$. More precisely, we try to find t_* and $\overline{C} = \overline{C}(\Lambda, n, p)$ such that for all $t \leqq t_*$

$$
n \log(ct) - \beta C((1+|\Lambda|^{p-1})t + t^p) \geqq \overline{C} \log(t).
$$

To this end, first observe that the function $t \mapsto (n-\overline{C})\log(ct)-\beta C((1+|\Lambda|^{p-1})t+t^p)$ is decreasing whenever $\overline{C} \geqq n$. Moreover, if we set $\overline{C} = n + C(2+|\Lambda|^{p-1})$ and $t_* = \frac{1}{\beta}$, then for $\beta > \exp(1)$ and $c \leqq 1$ we have

$$\begin{aligned}(n-\overline{C})\log(ct_*) - \beta C((1+|\Lambda|^{p-1})t_* + t_*^p)\\ \geqq (n-\overline{C})(\log(t_*) - \beta C(2+|\Lambda|^{p-1})t_*\\ \geqq C(\log(\beta)-1)(2+|\Lambda|^{p-1}) > 0.\end{aligned}$$

Thus with our choice of t_* and $\overline{C}$ we infer from (6.7) that

$$\begin{aligned} e^{\mathrm{int}}_{\varepsilon,\beta} &\leqq -\frac{1}{\beta|Q_\varepsilon|}\log\left(\int_0^{t_*} t^{\overline{C}|Q_\varepsilon^{\mathcal{L}}|}\,\mathrm{d}t\right) + \frac{1}{p\beta|Q_\varepsilon|}\log(|Q_\varepsilon^{\mathcal{L}}|)\\ &= -\frac{1}{\beta|Q_\varepsilon|}\log\left(\frac{t_*^{\overline{C}|Q_\varepsilon^{\mathcal{L}}|+1}}{\overline{C}|Q_\varepsilon^{\mathcal{L}}|+1}\right) + \frac{1}{p\beta|Q_\varepsilon|}\log(|Q_\varepsilon|)\end{aligned}$$

and we can conclude from (6.4) and (3.1) that

$$\begin{aligned}\overline{W}(\Lambda,\beta) - \overline{W}^{\infty}(\Lambda)) = \lim_{\varepsilon\to 0} e_{\varepsilon,\beta} &\leqq \limsup_{\varepsilon\to 0}\frac{\overline{C}|Q_\varepsilon^{\mathcal{L}}|+1}{\beta|Q_\varepsilon|}|\log(t_*)|\\ &\leqq \overline{C}\left(\frac{2R}{r}\right)^d\frac{\log(\beta)}{\beta}.\end{aligned}$$

This proves the claim by our definition of $\overline{C}$. □

Lemma 6.5. *Assume Hypothesis 1. Then for every $\Lambda \in \mathbb{R}^{n\times d}$ there exists a constant $0 < C_\Lambda \leqq C(1+\log(1+|\Lambda|))$ such that, for all $\beta \geqq 1$,*

$$\overline{W}^{\beta}(\Lambda) - \overline{W}^{\infty}(\Lambda) \geqq -\frac{C_\Lambda}{\beta}.$$

Proof of Lemma 6.5. Again we compute the energy densities with respect to the unit cube Q. Having in mind Lemma 6.3, we let $\tilde{u}_\varepsilon$ be a minimizer of the Hamiltonian H_ε on the set $\mathcal{B}_\varepsilon(Q,\varphi_\Lambda)$. Note that we assume without loss of generality that a minimizer exists, otherwise we could take an almost minimizer with an energy close to the infimum at a rate that vanishes much faster than ε^d. For any $u \in \mathcal{B}_\varepsilon(Q,\varphi_\Lambda)$, by the p-growth condition in Hypothesis 2 and (6.2), we have the inequality

$$\begin{aligned}H_\varepsilon(Q,u) - H_\varepsilon(Q,\tilde{u}_\varepsilon) &\geqq \frac{1}{C}\|\nabla_{\mathbb{B}}u\|^p_{\ell^p_\varepsilon(Q)} - C(1+|\Lambda|^p)|Q_\varepsilon^{\mathcal{L}}|\\ &\geqq \frac{1}{C}\|\nabla_{\mathbb{B}}(u-\varphi_\Lambda)\|^p_{\ell^p_\varepsilon(Q)} - C(1+|\Lambda|^p)|Q_\varepsilon^{\mathcal{L}}|.\end{aligned}$$

While this estimate turns out to be useful for deformations with large energy, we also need a suitable lower bound for deformations with small energy. To this end we observe that by minimality $H_\varepsilon(Q,u) - H_\varepsilon(Q,\tilde{u}_\varepsilon) \geqq 0$, so that we can write

$$H_\varepsilon(Q,u) - H_\varepsilon(Q,\tilde{u}_\varepsilon) \geqq \max\left\{0, \frac{1}{C}\|\nabla_{\mathbb{B}}(u-\varphi_\Lambda)\|^p_{\ell^p_\varepsilon(Q)} - C(1+|\Lambda|^p)|Q_\varepsilon^{\mathcal{L}}|\right\}.$$

(6.8)

This inequality motivates the partition $\mathcal{B}_\varepsilon(Q,0) = B_{1,\varepsilon} \cup B_{2,\varepsilon}$, where

$$B_{1,\varepsilon} := \Big\{\varphi \in \mathcal{B}_\varepsilon(Q,0): \ \|\nabla_{\mathbb{B}} u\|^p_{\ell^p_\varepsilon(Q)} \leqq 2C^2(1+|\Lambda|^p)|Q_\varepsilon^{\mathcal{L}}|\Big\},$$
$$B_{2,\varepsilon} := \mathcal{B}_\varepsilon(Q,0) \setminus B_{1,\varepsilon}.$$

With the change of variables $u \mapsto u - \varphi_\Lambda$ and (6.8) we then obtain

$$\begin{aligned} e^1_{\varepsilon,\beta} :&= \mathcal{E}^\beta_\varepsilon(\overline{\varphi}_\Lambda) - \frac{1}{|Q_\varepsilon|} H_\varepsilon(Q,\tilde{u}_\varepsilon) \\ &= -\frac{1}{\beta|Q_\varepsilon|}\log\Big(\int_{\mathcal{B}_\varepsilon(Q,\varphi_\Lambda)} \exp\big(-\beta(H_\varepsilon(Q,u) - H_\varepsilon(Q,\tilde{u}_\varepsilon))\big)\,\mathrm{d}u\Big) \\ &\geqq -\frac{1}{\beta|Q_\varepsilon|}\log\Big(|\mathcal{B}_{1,\varepsilon}| + \int_{\mathcal{B}_\varepsilon(Q,0)} \exp\big(-\frac{\beta}{2C}\|\nabla_{\mathbb{B}} u\|^p_{\ell^p_\varepsilon(Q)}\big)\,\mathrm{d}u\Big). \qquad (6.9) \end{aligned}$$

We treat the two terms inside the logarithm separately. Let us start with the integral. Using Lemma 3.3, for ε small enough (independent of β) and $\beta \geqq 1$, we obtain the bound

$$\begin{aligned} \int_{\mathcal{B}_\varepsilon(Q,0)} \exp\big(-\frac{\beta}{2C}\|\nabla_{\mathbb{B}} u\|^p_{\ell^p_\varepsilon(Q)}\big)\,\mathrm{d}u &\leqq C^{N_{Q,\varepsilon}}\left(\left(\frac{\beta}{2C}\right)^{-\frac{n}{p}} C\right)^{|Q_\varepsilon^{\mathcal{L}}| - N_{Q,\varepsilon}} \\ &\leqq \Big(C(p,n)\Big)^{n|Q_\varepsilon^{\mathcal{L}}|}. \qquad (6.10) \end{aligned}$$

In order to provide a bound for the measure of $\mathcal{B}_{1,\varepsilon}$ we first enlarge the set and then perform a suitable change of variables. To this end we number the vertices by the following algorithm: For every connected component $G_j = (V_j, \mathbb{B}_j)$ of the graph G_{Q_ε} we choose a minimal spanning tree $ST_j = (V_j, \mathbb{B}'_j)$ and a vertex where the boundary conditions are active (see Remark 7 for the existence of such a vertex). To this vertex we assign the number $k_j := \big(\sum_{i<j}|V_i|\big) + 1$. Then we start any path in the spanning tree and number the vertices consecutively until we cannot go on. If we have numbered all vertices of the connected component we go to the next one, otherwise we continue at the first (with respect to the numbering) vertex with multiple path possibilities and continue with the same procedure. Since we consider a minimal spanning tree, every vertex gets assigned a unique number. Moreover, for each vertex number $l \backslash \{k_j\}$, we find a number $l' < l$ such that $(x_{l'}, x_l) \in \mathbb{B}'_j$. Then we have the following set inclusion:

$$\mathcal{B}_{1,\varepsilon} \subset \left\{ u \in (\mathbb{R}^n)^{|Q_\varepsilon^{\mathcal{L}}|} : |u_{k_j}| < 1 \text{ for all } j \text{ and } \sum_j \sum_{l=k_j+1}^{k_{j+1}-1} |u_l - u_{l'}|^p \leqq 2C^2(1+|\Lambda|^p)|Q_\varepsilon^{\mathcal{L}}| \right\} =: \mathcal{U}_{1,\varepsilon}$$

We now define a linear transformation on $\mathcal{U}_{\varepsilon,1}$ setting $T : \mathcal{U}_{1,\varepsilon} \to (\mathbb{R}^n)^{|Q_\varepsilon^{\mathcal{L}}|}$ as

$$(T\varphi)_l = \begin{cases} \varphi_l & \text{if } l = k_j \text{ for some } j, \\ \varphi_l - \varphi_{l'} & \text{otherwise.} \end{cases}$$

Note that the mapping $l \mapsto l'$ is independent of φ, so that T is indeed linear. Moreover, it is straightforward to check that T is injective and thus a diffeomorphism onto its image allowing to perform a change of variables. Observe that its derivative DT admits a lower triangle matrix representation since the l^{th} component of $T\varphi$ depends only on entries with smaller index. On the diagonal we have all entries equal to 1. Hence it holds that $\det(DT) = 1$. By a change of variables we conclude that

$$|\mathcal{B}_{\varepsilon,1}| \leqq |\mathcal{U}_{\varepsilon,1}| = |T(\mathcal{U}_{\varepsilon,1})|,$$

and by construction it holds that

$$\begin{aligned} T(\mathcal{U}_{\varepsilon,1}) &= \left(\prod_{j=1}^{N_{Q,\varepsilon}} B_1(0)\right) \times \left\{\varphi \in (\mathbb{R}^n)^{|Q_\varepsilon| - N_{Q,\varepsilon}} : \|\varphi\|_p^p \leqq 2C^2(1 + |\Lambda|^p)|Q_\varepsilon^{\mathcal{L}}|\right\} \\ &\subset \left\{u \in \mathbb{R}^{n|Q_\varepsilon^{\mathcal{L}}|} : |u|_p \leqq C'(1 + |\Lambda|^p)^{\frac{1}{p}} |Q_\varepsilon^{\mathcal{L}}|^{\frac{1}{p}}\right\}, \end{aligned}$$

where the larger constant C' contains a factor derived from the equivalence of norms on $\mathbb{R}^n$. The last set is a high-dimensional ball with respect to the corresponding ℓ^p-norm, for which there exist exact formulas for the volume. Denoting (just in this proof) by Γ Euler's Gamma-function we deduce that, for ε small enough,

$$\begin{aligned} |\mathcal{B}_{\varepsilon,1}| &\leqq \frac{\left(C(p)(1 + |\Lambda|^p)|Q_\varepsilon^{\mathcal{L}}|\right)^{\frac{n|Q_\varepsilon^{\mathcal{L}}|}{p}}}{\Gamma\left(\frac{n|Q_\varepsilon^{\mathcal{L}}|}{p} + 1\right)} \\ &\leqq \left(C(p,n)(1 + |\Lambda|)\right)^{n|Q_\varepsilon^{\mathcal{L}}|}, \end{aligned} \tag{6.11}$$

where we used the lower bound $\Gamma(z + 1) \geqq (z/e)^z$ for all $z \geqq 1$. Combining (6.9), (6.10) and (6.11) we infer that, for ε small enough (but independent of β) and $\beta \geqq 1$,

$$\begin{aligned} e_{\varepsilon,\beta}^1 &\geqq -\frac{1}{\beta|Q_\varepsilon|} \log\left((C(p,n)(1 + |\Lambda|))^{n|Q_\varepsilon^{\mathcal{L}}|}\right) \\ &= -C(p,n)\left(\frac{2R}{r}\right)^d \frac{(1 + \log(1 + |\Lambda|)}{\beta}. \end{aligned}$$

Thanks due Lemma 6.3 and the definition of $e_{\varepsilon,\beta}^1$ the claim now follows after letting $\varepsilon \to 0$. $\square$

6.3. Γ-Convergence of the LDP Rate Functionals

The estimates proved in Lemmata 6.4 and 6.5 lead to Theorem 1.6 that also relates the support of the limits of Gibbs measures (see Corollary 1) to the minimizers of the Γ-limit at small temperatures.

Proof of Theorem 1.6. We let $G \in \mathcal{G}'$, where $\mathcal{G}'$ is given by Proposition 3 (see also Remark 11). Fix an arbitrary sequence $\beta_j \to +\infty$. For the moment we consider the functionals $F_j, F : L^p(D, \mathbb{R}^n) \to \mathbb{R} \cup \{+\infty\}$ finite only on $\varphi + W_0^{1,p}(D, \mathbb{R}^n)$ and characterized by

$$F_j(v) = \fint_D \overline{W}^{\beta_j}(\nabla v)\,\mathrm{d}x, \qquad F(v) = \fint_D \overline{W}^{\infty}(\nabla v)\,\mathrm{d}x.$$

By Lemmata 6.4 and 6.5 we have that $F_j \to F$ pointwise when $j \to +\infty$. Hence for all $v \in L^p(D, \mathbb{R}^n)$

$$\Gamma\text{-}\limsup_{j \to +\infty} F_j(v) \leqq F(v).$$

In order to prove the lim inf-inequality, consider $v \in L^p(D, \mathbb{R}^n)$ and a sequence $(v_j) \subset L^p(D, \mathbb{R}^n)$ such that $v_j \to v$ in $L^p(D, \mathbb{R}^n)$ and $\sup_j F_j(u_j) < +\infty$. Since Lemma 6.5 yields

$$\overline{W}^{\beta_j}(\Lambda) - W^{\infty}_{\text{hom}}(\Lambda) \geqq -\frac{C}{\beta}(1 + \log(1 + |\Lambda|),$$

where the constant C is independent of Λ and for j large enough it holds that $\overline{W}^{\beta_j}(\Lambda) \geqq \frac{1}{C}|\Lambda|^p - C$, we infer that $v \in \varphi + W_0^{1,p}(D, \mathbb{R}^n)$ and

$$\liminf_{j \to +\infty} F_j(v_j) \geqq \liminf_{j \to +\infty} F(v_j) \geqq F(v),$$

where we used that F is lower semicontinuous due to the quasiconvexity of the map $\Lambda \mapsto \overline{W}^{\infty}(\Lambda)$. is quasiconvex, the lower bound follows from weak lower semicontinuity. Thus F_j Γ-converges to F with respect to the $L^p(D < \mathbb{R}^n)$-topology. Since the Γ-convergence implies the convergence of the infimum values $\lim_j \inf_v F_j(v) = \inf_v F(v)$, Theorem 1.6 is proven. □

6.4. The Phantom Model

The convergence result proved in this section can be made much more precise when the discrete Hamiltonian is quadratic, that is,

$$H_\varepsilon(O, u) = \sum_{\substack{(x,y) \in \mathbb{B} \\ x,y \in O_\varepsilon}} \langle u(x) - u(y), A(x-y)(u(x) - u(y))\rangle \tag{6.12}$$

with a function $A : \mathbb{R}^d \to \mathbb{M}^{d \times d}_{\text{sym}}$ uniformly positive definite and bounded on $B_{C_0}(0)$, where C_0 is the maximal range of interactions given by Definition 1.2. The phantom model (see e.g. [43, Section 7.2.2]), which is an approximation of rubber elasticity of polymer-chain networks at small deformation and finite temperature, indeed corresponds to the homogenization of this energy density (using a self-consistent approach rather than the usual cell-formula). As the following shows, the limit free energy agrees with the density of the Γ-limit up to an additive constant which depends on β but not on Λ. In particular, this justifies the use of the self-consistent approach in [43, Section 7.2.2] even at finite temperature.

Corollary 2. *Assume that H_ε is given by (6.12). Then it holds that*

$$\overline{W}^\beta(\Lambda) = \overline{W}^\infty(\Lambda) + \overline{W}^\beta(0).$$

In particular the function $\Lambda \mapsto \overline{W}^\beta(\Lambda)$ is uniformly convex and quadratic.

Proof of Corollary 2. We consider the free energy on the unit cube Q and first find a unique minimizer of $u \mapsto H_\varepsilon(Q, u)$ on the set $\mathcal{BC}_\varepsilon(Q, \varphi_\Lambda)$, that we denote by $\hat{u}_\varepsilon$. We extend it to $\mathcal{L}$ setting $\hat{u}_\varepsilon(x) = \varphi_\Lambda(x)$ for all $x \in \mathcal{L}\backslash Q_\varepsilon$. Given $u \in \mathcal{B}_\varepsilon(Q, \varphi_\Lambda)$, we decompose it as $u = u_1 + u_2$ with $u_1 \in \mathcal{BC}_\varepsilon(Q, \varphi_\Lambda)$ and $u_2 : Q_\varepsilon \to \mathbb{R}^n$ satisfying $|u_2(x)| \leqq 1$ for all $x \in \mathcal{L}$ and $u_2(x) = 0$ for all $x \in \mathcal{L}$ such that $\operatorname{dist}(x, \partial Q_\varepsilon) > C_0$. By the quadratic structure we have

$$\begin{aligned} H_\varepsilon(Q, u) - H_\varepsilon(Q, \hat{u}_\varepsilon) &= H_\varepsilon(Q, u - \hat{u}_\varepsilon) \\ &\quad + 2 \sum_{\substack{(x,y)\in\mathbb{B} \\ x,y\in Q_\varepsilon}} \Big\langle (u - \hat{u}_\varepsilon)(x) - (u - \hat{u}_\varepsilon)(y), A(x-y)(\hat{u}_\varepsilon(x) - \hat{u}_\varepsilon(y)) \Big\rangle \\ &= H_\varepsilon(Q, u - \hat{u}_\varepsilon) + 2 \sum_{\substack{(x,y)\in\mathbb{B} \\ x,y\in Q_\varepsilon}} \Big\langle u_2(x) - u_2(y), A(x-y)(\hat{u}_\varepsilon(x) - \hat{u}_\varepsilon(y)) \Big\rangle, \end{aligned} \tag{6.13}$$

where we used pointwise symmetry of $A(z)$, the weak Euler-Lagrange equation satisfied by $\hat{u}_\varepsilon$ and that $u_1 - \hat{u}_\varepsilon \in \mathcal{BC}_\varepsilon(Q, 0)$ is an admissible test function for this equation. In order to bound the last term, we first introduce the set $\partial_\varepsilon Q = \{z \in Q_\varepsilon : \operatorname{dist}(z, \partial Q_\varepsilon) \leqq 2C_0\}$. Then by the properties of u_2 and boundedness of $A(z)$ we have

$$\begin{aligned} &\Bigg| 2 \sum_{\substack{(x,y)\in\mathbb{B} \\ x,y\in Q_\varepsilon}} \Big\langle u_2(x) - u_2(y), A(x-y)(\hat{u}_\varepsilon(x) - \hat{u}_\varepsilon(y)) \Big\rangle \Bigg| \\ &\leqq C \sum_{\substack{(x,y)\in\mathbb{B} \\ x,y\in\partial_\varepsilon Q}} c_0 |\hat{u}_\varepsilon(x) - \hat{u}_\varepsilon(y)|, \end{aligned}$$

where c_0 is a lower bound for the smallest eigenvalue of $A(z)$ for all $z \in B_{C_0}(0)$. Note that the right hand side does not depend on u any more. Since the change of variables $u \mapsto u - \hat{u}_\varepsilon$ maps $\mathcal{B}_\varepsilon(Q, \varphi_\Lambda)$ one-to-one to $\mathcal{B}_\varepsilon(Q, 0)$, we conclude by the very definition of the terms $\overline{W}^\beta(\Lambda)$ and $\overline{W}^\infty(\Lambda)$ and equation (6.13) that

$$|\overline{W}^\beta(\Lambda) - \overline{W}^\infty(\Lambda) - \overline{W}^\beta(0)| \leqq C \limsup_{\varepsilon\to 0} \frac{1}{|Q_\varepsilon|} \sum_{\substack{(x,y)\in\mathbb{B} \\ x,y\in\partial_\varepsilon Q}} c_0 |\hat{u}_\varepsilon(x) - \hat{u}_\varepsilon(y)|.$$

It remains to prove that the right hand side is zero. To this end, we note that due to Lemma 5.1 we can assume that $v_\varepsilon \in \mathcal{PC}_\varepsilon$ defined by $v_\varepsilon(\varepsilon x) = \varepsilon \hat{u}_\varepsilon(x)$ converges in $L^p(D, \mathbb{R}^n)$ to some function $v \in \overline{\varphi}_\Lambda + W_0^{1,p}(Q, \mathbb{R}^n)$ (actually one can prove that $v = \overline{\varphi}_\Lambda$, but this is not needed here). Given $\delta > 0$, Jensen's inequality and (quasi)convexity of $\Lambda \mapsto \overline{W}^\infty(\Lambda)$ imply

$$\begin{aligned}
&\limsup_{\varepsilon \to 0} \left(\frac{1}{|Q_\varepsilon|} \sum_{\substack{(x,y)\in\mathbb{B} \\ x,y\in\partial_\varepsilon Q}} c_0 |\hat{u}_\varepsilon(x) - \hat{u}_\varepsilon(y)| \right)^2 \\
&\ll \limsup_{\varepsilon \to 0} \frac{1}{|Q_\varepsilon|} \sum_{\substack{(x,y)\in E \\ x,y\in\partial^\varepsilon Q}} c_0 |\hat{u}_\varepsilon(x) - \hat{u}_\varepsilon(y)|^2 \\
&\leqq \overline{W}^\infty(\Lambda) - \frac{|(1-\delta)Q|}{|Q|} \liminf_{\varepsilon \to 0} \tilde{H}_\varepsilon((1-\delta)Q, u_\varepsilon) \leqq \int_{Q\setminus(1-\delta)Q} \overline{W}^\infty(\nabla v(x))\, \mathrm{d}x,
\end{aligned}$$

where for the last estimate we also used the local Γ-convergence result on $(1-\delta)Q$ stated in Theorem 6.1. Letting $\delta \to 0$ we deduce the claim since the right hand side vanishes. The quadratic structure follows by the general theory of Γ-convergence of quadratic functionals (see [23, Theorem 11.10]) while the uniform coercivity of A is conserved in the limit, too. This proves uniform convexity of $\Lambda \mapsto \overline{W}^\beta(\Lambda)$. □

7. Penalizing Volume Changes

Having in mind the model presented in the introduction, we now explain how to include the volumetric term defined in (1.2) in our previous analysis. We assume throughout this whole section that $n = d$ and $p \geqq d$. For the notation we also refer to Section 1.2.

In order to estimate the volumetric part of the Hamiltonian, it is convenient to rewrite the integral as sums over d-simplices, that is,

$$H_{\mathrm{vol},\varepsilon}(O, u) = \sum_{\mathcal{C}_1(x)\in\mathcal{V}_{1,\varepsilon}(O)} |\mathcal{C}_1(x)| W \left(\sum_{T\cap\mathcal{C}_1(x)\neq\emptyset} \frac{\det(\nabla u_{\mathrm{aff}|T}) |T \cap \mathcal{C}_1(x))|}{|\mathcal{C}_1(x)|} \right). \tag{7.1}$$

The Hamiltonian has a different structure than in the previous sections since it depends on multi-body interactions through the volumetric term. However, we emphasize that for our analysis in the previous sections, the precise structure was needed only for proving stationarity. The reader might remember that elsewhere we just used bounds from above and below and a certain locality given by the finite range of interactions (see also Remark 13). As we will prove in the lemmata below, the bounds from Hypothesis 1 lead to similar local bounds for the Hamiltonian and stationarity of the corresponding stochastic processes is preserved, too. To extend the validity of the results of Section 6, it then suffices to reprove a global continuity estimate. These technical details then allow to include the multi-body volumetric term in the proofs of the previous sections and in that sense complete the proofs of Theorems 1.3, 1.4, 1.5 and 1.6. We leave the details to the reader.

Lemma 7.1. *Assume that W satisfies Hypothesis 1 and let $G \in \mathcal{G}$. Then there exists a constant $C > 0$ such that, for every $u : \mathcal{L} \to \mathbb{R}^d$ and all $x \in \mathcal{L}_1$, it holds that*

$$W\left(\fint_{\mathcal{C}_1(x)} \det(\nabla u_{\mathrm{aff}})\,\mathrm{d}z\right) \leqq C\left(1 + \sum_{T\cap\mathcal{C}_1(x)\neq\emptyset}\ \sum_{y,y'\in\mathcal{L}_1\cap T} |u(y)-u(y')|^p\right).$$

Remark 13. Points y, y' appearing in the above upper bound satisfy $|x-y|, |x-y'| \leqq 3R$. Hence the condition $6R < C_0$ allows to establish almost subadditivity estimates using boundary values by the definition of the interior Voronoi cells $\mathcal{V}_{1,\varepsilon}(O)$.

Proof of Lemma 7.1. We rewrite the left hand side term similar to (7.1). Since $\mathcal{L}_1 \subset \mathcal{L}$, the volume of the Voronoi cells is uniformly bounded from below. Hence from the upper bound in Hypothesis 1 and the area formula we deduce

$$\begin{aligned} W\left(\fint_{\mathcal{C}_1(x)} \det(\nabla u_{\mathrm{aff}})\,\mathrm{d}z\right) &\leqq C + C\left(\sum_{T\cap\mathcal{C}_1(x)\neq\emptyset} |\det(\nabla u_{\mathrm{aff}_{|\mathrm{T}}}||T\cap\mathcal{C}_1(x))|\right)^{\frac{p}{d}} \\ &\leqq C + C\left(\sum_{T\cap\mathcal{C}_1(x)\neq\emptyset} |u_{\mathrm{aff}}(T)|\right)^{\frac{p}{d}}. \end{aligned} \tag{7.2}$$

We claim that for each $T \in \mathbb{T}$ with $|T \cap \mathcal{C}_1(x)| > 0$ it holds that

$$|u_{\mathrm{aff}}(T)| \leqq C\left(\sum_{y,y'\in\mathcal{L}_1\cap T} |u(y)-u(y')|\right)^d. \tag{7.3}$$

Indeed, if $\det(\nabla u_{\mathrm{aff}_{|\mathrm{T}}}) = 0$, then there is nothing to prove. Otherwise, the set $u_{\mathrm{aff}}(T)$ is again a d-simplex with vertices $\{u(y)\}_{y\in\mathcal{L}_1\cap T}$. By convexity its diameter can be bounded by

$$\mathrm{diam}(u_{\mathrm{aff}}(T)) \leqq \sum_{y,y'\in\mathcal{L}_1\cap T} |u(y)-u(y')|,$$

so that the bound $|u_{\mathrm{aff}}(T)| \leqq \mathrm{diam}(u_{\mathrm{aff}}(T))^d$ implies (7.3). Combining (7.2) and (7.3) we conclude that

$$W\left(\fint_{\mathcal{C}_1(x)} \det(\nabla u_{\mathrm{aff}})\,\mathrm{d}z\right) \leqq C + C\left(\sum_{T\cap\mathcal{C}_1(x)\neq\emptyset}\ \sum_{y,y'\in\mathcal{L}_1\cap T} |u(y)-u(y')|\right)^p$$

and the statement follows by Jensen's inequality since the number of terms in the above sum is equibounded with respect to $x \in \mathcal{L}_1$ and $G \in \mathcal{G}$. □

We continue our series of lemmas with the proof of stationarity as used when applying the ergodic theorem.

Lemma 7.2. *Let $u : \mathcal{L} \to \mathbb{R}^d$ and let $G \in \mathcal{G}$. Then, for all $z \in \mathbb{Z}^d$, all $I \in \mathcal{I}$ and every $\alpha \in \mathbb{R}^n$, it holds that*

$$H_{\mathrm{vol},1}(u(\cdot + z) + \alpha, I, G - z) = H_{\mathrm{vol},1}(u, I + z, G).$$

Proof of Lemma 7.2. Since we assume that $\mathcal{L}_1$ is stationary and in general position, the Delaunay tessellation $\mathbb{T}$ of $\mathbb{R}^d$ with respect to $\mathcal{L}_1$ is unique and hence also stationary. The claim then follows by the stationarity of $\mathcal{L}_1$ and $\mathbb{T}$ combined with the linearity of piecewise affine interpolations, a discrete change of variables and translation invariance of the Lebesgue measure. □

The last point left in order to repeat the analysis of the previous sections for the volumetric term concerns the quantitative continuity in order to prove the convergence in the zero temperature limit.

Before we prove the latter, we introduce some further notation. Define the set of neighbours for the volumetric points $\mathcal{L}_1$ by

$$\mathcal{N}_1 := \{(x, y) \in \mathcal{L}_1 \times \mathcal{L}_1 : \ \dim(\mathcal{C}_1(x) \cap \mathcal{C}_1(y)) = d - 1\}.$$

Since we assume $\mathcal{L}_1$ to be in general position, two points $x, y \in \mathcal{L}_1$ belong to the same simplex $T \in \mathbb{T}$ if and only if they are neighbours. Given $u : \mathcal{L} \to \mathbb{R}^d$ we set

$$\|\nabla_{\mathcal{N}} u\|_{\ell^p_\varepsilon(D)} = \left(\sum_{\substack{(x,y)\in\mathcal{N}_1 \\ \varepsilon x, \varepsilon y \in D}} |u(x) - u(y)|^p \right)^{\frac{1}{p}}.$$

Then the continuity estimate reads as follows:

Lemma 7.3. *Assume that W satisfies Hypothesis 2 and let $G \in \mathcal{G}$. Then, for any $u, \zeta : \mathcal{L} \to \mathbb{R}^d$ and any bounded Lipschitz domain $D \subset \mathbb{R}^d$, we have the global continuity estimate*

$$\begin{aligned}
&|H_{\mathrm{vol},\varepsilon}(D, u) - H_{\mathrm{vol},\varepsilon}(D, \zeta)| \\
&\quad \leqq C\Big(|D_\varepsilon^{\mathcal{L}}|^{\frac{p-1}{p}} + \|\nabla_{\mathcal{N}} u\|^{p-1}_{\ell^p_\varepsilon(D)} + \|\nabla_{\mathcal{N}} \zeta\|^{p-1}_{\ell^p_\varepsilon(D)}\Big) \|\nabla_{\mathcal{N}}(u - \zeta)\|_{\ell^p_\varepsilon(D)}.
\end{aligned}$$

Proof of Lemma 7.3. Fix $x \in \mathcal{L}_1$. Applying (1.5) we infer from the area formula and Jensen's inequality (recall that $p \geqq d$) that

$$\begin{aligned}
&\left| W\Big(\fint_{\mathcal{C}_1(x)} \det(\nabla u_{\mathrm{aff}})\,\mathrm{d}z\Big) - W\Big(\fint_{\mathcal{C}_1(x)} \det(\nabla \zeta_{\mathrm{aff}})\,\mathrm{d}z\Big) \right| \\
&\quad \leqq C\Big(1 + \sum_{T \cap \mathcal{C}_1(x) \neq \emptyset} |u_{\mathrm{aff}}(T \cap \mathcal{C}_1(x))|^{\frac{p}{d}-1} + |\zeta_{\mathrm{aff}}(T \cap \mathcal{C}_1(x))|^{\frac{p}{d}-1}\Big) \\
&\qquad \times \fint_{\mathcal{C}_1(x)} |\det(\nabla u_{\mathrm{aff}}) - \det(\nabla \zeta_{\mathrm{aff}})|\,\mathrm{d}z
\end{aligned}$$

$$\leqq C\Big(1+\sum_{T\cap\mathcal{C}_1(x)\neq\emptyset}|u_{\mathrm{aff}}(T)|^{\frac{p}{d}-1}+|\zeta_{\mathrm{aff}}(T)|^{\frac{p}{d}-1}\Big)$$
$$\times\sum_{T\cap\mathcal{C}_1(x)\neq\emptyset}|\det(\nabla u_{\mathrm{aff}_{|\mathrm{T}}})-\det(\nabla\zeta_{\mathrm{aff}_{|\mathrm{T}}})||T|.$$

Taking into account (7.3), we can again use Jensen's inequality to further estimate

$$\begin{aligned}
&\Big|W\Big(⨍_{\mathcal{C}_1(x)}\det(\nabla u_{\mathrm{aff}})\,\mathrm{d}z\Big)-W\Big(⨍_{\mathcal{C}_1(x)}\det(\nabla\zeta_{\mathrm{aff}})\,\mathrm{d}z\Big)\Big|\\
&\quad\leqq C\Big(1+\sum_{T\cap\mathcal{C}_1(x)\neq\emptyset}\ \sum_{y,y'\in\mathcal{L}_1\cap T}\Big(|u(y)-u(y')|^{p-d}+|\zeta(y)-\zeta(y')|^{p-d}\Big)\Big)\\
&\quad\times\sum_{T\cap\mathcal{C}_1(x)\neq\emptyset}|\det(\nabla u_{\mathrm{aff}_{|\mathrm{T}}})-\det(\nabla\zeta_{\mathrm{aff}_{|\mathrm{T}}})||T|. \qquad (7.4)
\end{aligned}$$

We bound the difference in each term of the last sum. Write $T=\mathrm{co}(x_0,\dots,x_d)$. Then by the volume formula for simplices

$$\begin{aligned}
\det(\nabla u_{\mathrm{aff}_{|\mathrm{T}}})|T| &= \frac{1}{d!}\det(\nabla u_{\mathrm{aff}_{|\mathrm{T}}})\det\big(x_1-x_0|\dots|x_d-x_0\big)\\
&=\frac{1}{d!}\det\big(u(x_1)-u(x_0)|\dots|u(x_d)-u(x_0)\big).
\end{aligned}$$

The same formula holds with ζ in place of u. From the standard continuity estimate for determinants, we deduce that

$$\begin{aligned}
|\det(\nabla u_{\mathrm{aff}_{|\mathrm{T}}})-\det(\nabla\zeta_{\mathrm{aff}_{|\mathrm{T}}})||T| &\leqq C\max_i\Big(|u(x_i)-u(x_0)|+|\zeta(x_i)-\zeta(x_0)|\Big)^{d-1}\\
&\quad\times\sum_{y,y'\in\mathcal{L}_1\cap T}|u(y)-u(y')-\zeta(y)+\zeta(y')|
\end{aligned}$$

Recall that $d\leqq p$. Hence inserting the above estimate into (7.4) it follows that

$$\begin{aligned}
&\left|W_1\left(\sum_{T\cap\mathcal{C}_1(x)\neq\emptyset}\frac{|u_{\mathrm{aff}}(T\cap\mathcal{C}_1(x))|}{|\mathcal{C}_1(x)|}\right)-W_1\left(\sum_{T\cap\mathcal{C}_1(x)\neq\emptyset}\frac{|\zeta_{\mathrm{aff}}(T\cap\mathcal{C}_1(x))|}{|\mathcal{C}_1(x)|}\right)\right|\\
&\quad\leqq C\Big(1+\sum_{T\cap\mathcal{C}_1(x)\neq\emptyset}\ \sum_{y,y'\in\mathcal{L}_1\cap T}\Big(|u(y)-u(y')|^{p-1}+|\zeta(y)-\zeta(y')|^{p-1}\Big)\Big)\\
&\quad\times\sum_{T\cap\mathcal{C}_1(x)\neq\emptyset}\ \sum_{y,y'\in\mathcal{L}_1\cap T}|u(y)-u(y')-\zeta(y)+\zeta(y')|. \qquad (7.5)
\end{aligned}$$

Note that each $T\in\mathbb{T}$ can intersect only a uniformly bounded number of Voronoi cells $\mathcal{C}_1$. Hence, summing (7.5) over all $\mathcal{C}_1(x)\in\mathcal{V}_{1,\varepsilon}(D)$ and using Hölder's inequality for the products yields

$$
\begin{aligned}
&|H_{\mathrm{vol},\varepsilon}(D,u) - H_{\mathrm{vol},\varepsilon}(D,\zeta)| \\
&\leqq C\Bigg(|D_\varepsilon^{\mathcal{L}}|^{\frac{p-1}{p}} + \Big(\sum_{T\subset D_\varepsilon}\sum_{y,y'\in\mathcal{L}_1\cap T}|u(y)-u(y')|^p\Big)^{\frac{p-1}{p}} \\
&\quad + \Big(\sum_{T\subset D_\varepsilon}\sum_{y,y'\in\mathcal{L}_1\cap T}|\zeta(y)-\zeta(y')|^p\Big)^{\frac{p-1}{p}}\Bigg) \\
&\quad \times \Big(\sum_{T\subset D_\varepsilon}\sum_{y,y'\in\mathcal{L}_1\cap T}|u(y)-u(y')-\zeta(y)+\zeta(y')|^p\Big)^{\frac{1}{p}} \\
&\leqq C\Big(|D_\varepsilon^{\mathcal{L}}|^{\frac{p-1}{p}} + \|\nabla_{\mathcal{N}}u\|^{p-1}_{\ell^p_\varepsilon(D)} + \|\nabla_{\mathcal{N}}\zeta\|^{p-1}_{\ell^p_\varepsilon(D)}\Big)\|\nabla_{\mathcal{N}}(u-\zeta)\|_{\ell^p_\varepsilon(D)},
\end{aligned}
$$

where we used in the last inequality that each element in $\mathbb{T}$ has as vertices only nearest neighbours and that a vertex can belong to only a uniformly bounded number of different cells. The last estimate yields the claim. □

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Appendix A

In this appendix we collect and prove some of the results we used in the paper. We start with the technical proof of the interpolation inequality.

Proof of Proposition 1. We set $\beta = 1$ to reduce the notation. Given $\delta > 0$ and $N \in \mathbb{N}$, for $i \in \{1, \dots, N+1\}$ we introduce the open sets

$$
O_i = \left\{x \in O : \ \mathrm{dist}(x, \partial O) > (i+1)\frac{\delta}{2N}\right\}.
$$

Then the stripes $S_i := O_{i-1}\backslash\overline{O_{i+2}}$ fulfill $S_i \cap S_j = \emptyset$ whenever $|i-j| > 2$. Thus for every $u : O_\varepsilon^{\mathcal{L}} \to \mathbb{R}^n$ we obtain by averaging

$$
\frac{1}{N}\sum_{i=1}^{N} H_\varepsilon(S_i, u) \leqq \frac{3}{N} H_\varepsilon(O, u),
$$

so that we can decompose the set $\mathcal{N}_p(v, O, \varepsilon, \kappa) = \bigcup_{i=1}^N \mathcal{P}_{i,\varepsilon}$ (we omit the dependence on O and κ), where

$$\mathcal{P}_{i,\varepsilon} = \left\{ u \in \mathcal{N}_p(v, O, \varepsilon, \kappa) : \; H_\varepsilon(S_i, u) \leqq \frac{3}{N} H_\varepsilon(O, u) \right\}.$$

Let $\theta_i : O \to [0, 1]$ be the Lipschitz-continuous cut-off function defined by

$$\theta_i(z) = \min \left\{ \max \left\{ \frac{2N}{\delta} \mathrm{dist}(z, \partial O) - (i+1), 0 \right\}, 1 \right\},$$

so that $\theta_i \equiv 1$ on $\overline{O_{i+1}}$, $\theta_i \equiv 0$ on $O \backslash O_i$ and its Lipschitz constant can be bounded by $\mathrm{Lip}(\theta_i) \leqq \frac{2N}{\delta}$. We then define an interpolation between functions $u, \psi : O_\varepsilon^{\mathcal{L}} \to \mathbb{R}^n$ as

$$T_{i,\varepsilon}(u, \psi)(x) = \theta_i(\varepsilon x) u(x) + (1 - \theta_i(\varepsilon x)) \psi(x).$$

Observe that if $u \in \mathcal{P}_{i,\varepsilon}$ as well as $\varphi \in \mathcal{N}_p(v, O, \varepsilon, \kappa)$ and $\psi \in \mathcal{N}_\infty(\varphi, O \backslash \overline{O_{i+1}}, \varepsilon)$, by the Minkowski inequality we have

$$\begin{aligned} \varepsilon^{\frac{d}{p}} \| v_\varepsilon - \varepsilon T_{i,\varepsilon}(u, \psi) \|_{\ell^p_\varepsilon(O)} &\leqq 2\kappa |O|^{\frac{1}{p} + \frac{1}{d}} + \varepsilon^{\frac{d}{p}} \| \varepsilon \psi - \varepsilon \varphi \|_{\ell^p_\varepsilon(O \backslash \overline{O_{i+1}})} \\ &\leqq 2\kappa |O|^{\frac{1}{p} + \frac{1}{d}} + C\varepsilon |O|^{\frac{1}{p}}, \end{aligned}$$

so that $T_{i,\varepsilon}(u, \psi) \in \mathcal{N}_p(v, O, \varepsilon, 3\kappa)$ for ε small enough.

For technical reasons the interpolations will not suffice to prove the estimates. For every i let us choose $t_i \in [\frac{1}{4}, \frac{3}{4}]$ such that, setting $S_i^t = \{x \in O : \; \theta_i(x) = t\}$, the coarea formula implies

$$\frac{1}{2} \mathcal{H}^{d-1}(S_i^{t_i}) \leqq \int_{\frac{1}{4}}^{\frac{3}{4}} \mathcal{H}^{d-1}(S_i^t) \, \mathrm{d}t \leqq \int_0^1 \mathcal{H}^{d-1}(S_i^t) \, \mathrm{d}t = \int_O |\nabla \theta_i| \leqq \frac{2N}{\delta} |O_i \backslash O_{i+1}|. \tag{A.1}$$

We set $S_i^* = \{x \in O : \theta_i(x) < t_i\}$. Note that for δ small enough (depending only on O), we have $S_i^* \in \mathcal{A}^R(D)$ (see for instance [32, Lemma 2.2]). Let us introduce the product set

$$\mathcal{U}_\varepsilon^i(M) := \big(\mathcal{P}_{i,\varepsilon} \cap \mathcal{S}_M(O, \varepsilon) \big) \times \mathcal{N}_\infty(\varphi, O \backslash \overline{O_{i+1}}, \varepsilon),$$

as well as the integral

$$e_\varepsilon^i(M) := \left(\int_{\mathcal{U}_\varepsilon^i(M)} \exp\Big(- H_\varepsilon(O, T_{i,\varepsilon}(u, \psi)) - c_0 \| \nabla_{\mathbb{B}} u \|^p_{\ell^p_\varepsilon(S_i^*)} \Big) \, \mathrm{d}u \, \mathrm{d}\psi \right),$$

where $c_0 > 0$ is a small constant such that $c_0 |\xi|^p \leqq f(\cdot, \xi) + c_0^{-1}$ (cf. Hypothesis 1). This integral quantity will be the main ingredient to prove the interpolation inequality. We split the remaining argument into several steps.

Step 1. Energy bounds for the interpolation.

To bound the energy of $T_{i,\varepsilon}(u, \psi)$, we use the pointwise inequality

$$\begin{aligned} |\psi(x) - \psi(y)|^p &\leqq C |(\psi - \varphi)(x) - (\psi - \varphi)(y)|^p + C |\varphi(x) - \varphi(y)|^p \\ &\leqq C + C |\varphi(x) - \varphi(y)|^p, \end{aligned}$$

which is valid for all $x, y \in (O \backslash \overline{O_{i+1}})_\varepsilon$. Combined with the two-sided growth condition in Hypothesis 1 we infer that

$$H_\varepsilon(O, T_{i,\varepsilon}(u,\psi)) \leqq H_\varepsilon(O_{i+1}, u) + H_\varepsilon(O\setminus\overline{O_i}, \psi) + H_\varepsilon(S_i, T_{i,\varepsilon}(u,\psi))$$
$$\leqq H_\varepsilon(O_{i+1}, u) + CH_\varepsilon(O^\delta, \varphi) + C|(O^\delta)^{\mathcal{L}}_\varepsilon| + H_\varepsilon(S_i, T_{i,\varepsilon}(u,\psi)), \quad \text{(A.2)}$$

where O^δ is defined in the statement of Proposition 1. In order to estimate the last term on the right hand side we use the formula

$$\begin{aligned} T_{i,\varepsilon}(u,\psi)(x) - T_{i,\varepsilon}(u,\psi)(y) &= \big(\theta_i(\varepsilon x) - \theta_i(\varepsilon y)\big)\big(u(x) - \psi(x)\big) \\ &\quad + \theta_i(\varepsilon y)\big(u(x) - u(y)\big) \\ &\quad + (1 - \theta_i(\varepsilon y))\big(\psi(x) - \psi(y)\big) \end{aligned}$$

and the bound on the Lipschitz constant of θ_i to estimate the energy on the interpolation stripe via

$$\begin{aligned} &H_\varepsilon(S_i, T_{i,\varepsilon}(u,\psi)) \\ &\quad \leqq C\|\nabla_{\mathbb{B}} T_{i,\varepsilon}(u,\psi)\|^p_{\ell^p_\varepsilon(S_i)} + C|(S_i)^{\mathcal{L}}_\varepsilon| \\ &\quad \leqq C\left(\|\nabla_{\mathbb{B}} u\|^p_{\ell^p_\varepsilon(S_i)} + \|\nabla_{\mathbb{B}}\psi\|^p_{\ell^p_\varepsilon(S_i)} + \frac{(N\varepsilon)^p}{\delta^p}\|u-\psi\|^p_{\ell^p_\varepsilon(S_i)} + |(S_i)^{\mathcal{L}}_\varepsilon|\right) \\ &\quad \leqq \frac{C}{N} H_\varepsilon(O,u) + CH_\varepsilon(O^\delta,\varphi) + C|(O^\delta)^{\mathcal{L}}_\varepsilon| + \frac{CN^p}{\delta^p}\kappa^p|O|^{1+\frac{p}{d}}\varepsilon^{-d}, \end{aligned} \quad \text{(A.3)}$$

where we have used again that the degree of each vertex is equibounded and that, after suitable extension, $\psi \in \mathcal{N}_p(v, O, \varepsilon, 2\kappa)$ for ε small enough. Combining (A.2) and (A.3) we infer that

$$\begin{aligned} H_\varepsilon(O, T_{i,\varepsilon}(u,\psi)) \leqq H_\varepsilon(O_{i+1}, u) + \frac{C}{N} H_\varepsilon(O,u) + CH_\varepsilon(O^\delta,\varphi) + C|(O^\delta)^{\mathcal{L}}_\varepsilon| \\ + \frac{CN^p}{\delta^p}\kappa^p|O|^{\frac{p}{d}}|O^{\mathcal{L}}_\varepsilon|. \end{aligned} \quad \text{(A.4)}$$

Step 2. Lower bound for $e^i_\varepsilon(M)$.
In order to prove a lower bound for the integral, first note that due to Hypothesis 1 and the definition of S_i^*

$$c_0\|\nabla_{\mathbb{B}} u\|^p_{\ell^p_\varepsilon(S_i^*)} \leqq H_\varepsilon(O\setminus O_{i+1}, u) + C|(O^\delta)^{\mathcal{L}}_\varepsilon|,$$

so that, up to increasing C, we can add this inequality to (A.4) and obtain the estimate

$$\begin{aligned} H_\varepsilon(O, T_{i,\varepsilon}(u,\psi)) + c_0\|\nabla_{\mathbb{B}} u\|^p_{\ell^p_\varepsilon(S_i^*)} \leqq \left(1 + \frac{C}{N}\right) H_\varepsilon(O,u) + CH_\varepsilon(O^\delta,\varphi) \\ + C|(O^\delta)^{\mathcal{L}}_\varepsilon| + \frac{CN^p}{\delta^p}\kappa^p|O|^{\frac{p}{d}}|O^{\mathcal{L}}_\varepsilon| \end{aligned}$$

Rearranging the terms we obtain by Fubini's Theorem that

$$e^i_\varepsilon(M) \geqq \exp\Bigg(-C\Big(|(O^\delta)^{\mathcal{L}}_\varepsilon| + \frac{(N\kappa|O|^{\frac{1}{d}})^p}{\delta^p}|O^{\mathcal{L}}_\varepsilon| + \frac{M}{N}|O^{\mathcal{L}}_\varepsilon|$$

$$
\begin{aligned}
&\quad + H_\varepsilon(O^\delta,\varphi)\Big)\Big) \int_{\mathcal{N}_\infty(\varphi, O\setminus\overline{O_{i+1}},\varepsilon)} \mathrm{d}\psi \\
&\quad \times \int_{\mathcal{P}_{i,\varepsilon}\cap\mathcal{S}_M(O,\varepsilon)} \exp(-H_\varepsilon(O,u))\,\mathrm{d}u \\
&\geqq \exp\left(-C\Big(|(O^\delta)^{\mathcal{L}}_\varepsilon| + \frac{(N\kappa|O|^{\frac{1}{d}})^p}{\delta^p}|O^{\mathcal{L}}_\varepsilon| + \frac{M}{N}|O^{\mathcal{L}}_\varepsilon| + H_\varepsilon(O^\delta,\varphi)\Big)\right) \\
&\quad \times Z_{\varepsilon,O}(\mathcal{P}_{i,\varepsilon}\cap\mathcal{S}_M(O,\varepsilon)) \qquad (A.5)
\end{aligned}
$$

where we used that the measure of $\mathcal{N}_\infty(\varphi, O\setminus\overline{O_{i+1}}, \varepsilon)$ can be bounded from below by $\exp(-C|(O^\delta)^{\mathcal{L}}_\varepsilon|)$.

Step 3. Upper bound for $e^i_\varepsilon(M)$ and conclusion.

To estimate $e^i_\varepsilon(M)$ from above, similar to [34] we perform a suitable change of variables. Define $\Phi_{i,\varepsilon} : \mathcal{N}_p(v, O, \varepsilon, \kappa) \times \mathcal{N}_\infty(\varphi, O\setminus\overline{O_{i+1}}, \varepsilon) \to \mathcal{N}_p(u, O, \varepsilon, 3\kappa) \times \mathcal{N}_p(u, O\setminus\overline{O_{i+1}}, \varepsilon, 3\kappa)$ by

$$
\Phi_{i,\varepsilon}(u,\psi)(x) = \begin{cases} (T_{i,\varepsilon}(u,\psi)(x), \psi(x)) & \text{if } \theta_i(\varepsilon x) \geqq t_i, \\ (T_{i,\varepsilon}(u,\psi)(x), u(x)) & \text{if } \theta_i(\varepsilon x) < t_i. \end{cases}
$$

Note that for ε small enough $\Phi_{i,\varepsilon}$ is well-defined and bijective onto its range $\mathcal{R}(\Phi_{i,\varepsilon})$. For the idea how to calculate the Jacobian, we refer to the proof of Proposition 3. As $t_i \in [\frac{1}{4}, \frac{3}{4}]$, it holds that

$$
\begin{aligned}
|\det(D\Phi_{i,\varepsilon}(u,\psi))|^{-1} &= \left(\prod_{x:\theta_i(\varepsilon x)\geqq t_i} |\theta_i(\varepsilon x)|^n \prod_{x:\theta_i(\varepsilon x)<t_i} |1-\theta_i(\varepsilon x)|^n\right)^{-1} \\
&\leqq \exp(C|(O^\delta)^{\mathcal{L}}_\varepsilon|).
\end{aligned}
$$

Setting $(g,h) = \Phi_{i,\varepsilon}(u,\psi)$, by construction of the interpolation we have

$$
\begin{aligned}
&g \in \mathcal{N}_p(u, O, \varepsilon, 3\kappa) \cap \mathcal{B}_\varepsilon(O,\varphi), \\
&h = (h_1, h_2) \in \mathcal{N}_\infty(\varphi, O\setminus(\overline{O_{i+1}} \cup S_i^*), \varepsilon) \\
&\quad \times \underbrace{\{h : (S_i^*)_\varepsilon \to \mathbb{R}^n :\ \|h - \varepsilon^{-1}v_\varepsilon\|_\infty \leqq C\kappa|O_\varepsilon|^{\frac{1}{p}+\frac{1}{d}}\}}_{=:R_{i,\varepsilon}}.
\end{aligned}
$$

As the measure of the set $\mathcal{N}_\infty(\varphi, O\setminus(\overline{O_{i+1}} \cup S_i^*), \varepsilon)$ can be bounded by $\exp(C|(O^\delta)^{\mathcal{L}}_\varepsilon|)$, the above change of variables and Fubini's Theorem imply

$$
\begin{aligned}
e^i_\varepsilon(M) &\leqq \exp(C|(O^\delta)^{\mathcal{L}}_\varepsilon|) \int_{\mathcal{R}(\Phi_{i,\varepsilon})} \exp\Big(-H_\varepsilon(g,O) - c_0\|\nabla_{\mathbb{B}} h\|^p_{\ell^p_\varepsilon(S_i^*)}\Big)\,\mathrm{d}g\,\mathrm{d}h \\
&\leqq \exp(C|(O^\delta)^{\mathcal{L}}_\varepsilon|) \int_{\mathcal{N}_\infty(\varphi, O\setminus(\overline{O_{i+1}}\cup S_i^*),\varepsilon)} \mathrm{d}h_1 \int_{R_{i,\varepsilon}} \exp(-c_0\|\nabla_{\mathbb{B}} h_2\|^p_{\ell^p_\varepsilon(S_i^*)})\,\mathrm{d}h_2 \\
&\quad \times Z(\mathcal{N}_p(u, O, \varepsilon, 3\kappa) \cap \mathcal{B}_\varepsilon(O,\varphi)) \\
&\leqq \exp(C|(O^\delta)^{\mathcal{L}}_\varepsilon|) \int_{R_{i,\varepsilon}} \exp(-c_0\|\nabla_{\mathbb{B}} h_2\|^p_{\ell^p_\varepsilon(S_i^t)})\,\mathrm{d}h_2
\end{aligned}
$$

$$\times Z_{\varepsilon,O}(\mathcal{N}_p(u, O, \varepsilon, 3\kappa) \cap \mathcal{B}_\varepsilon(O, \varphi)). \tag{A.6}$$

In order to bound the integral on the right hand side, we apply Lemma 3.3 to the graph $G_{S_i^*,\varepsilon}$ with $\alpha = c_0$ and $\gamma = C\kappa|O_\varepsilon|^{\frac{1}{p}+\frac{1}{d}}$ and infer

$$\int_{R_{i,\varepsilon}} \exp(-c_0\|\nabla_{\mathbb{B}} h_2\|^p_{\ell^p_\varepsilon(S_i^*)})\,\mathrm{d}h_2 \leqq \left(C\kappa^n|O_\varepsilon|^{\frac{n}{p}+\frac{n}{d}}\right)^{N_{i,\varepsilon}} C^{|(S_i^*)^{\mathcal{L}}_\varepsilon|-N_{i,\varepsilon}},$$

where we denoted by $N_{i,\varepsilon}$ the number of connected components of the graph $G_{S_i^*,\varepsilon}$. For ε small enough (possibly depending on N, δ), by Remark 7, (3.3) and the fact that $S_i^* \in \mathcal{A}^R(D)$ we can bound the number of components via

$$N_{i,\varepsilon} \leqq \#\{x \in O^{\mathcal{L}}_\varepsilon :\ \mathrm{dist}(x, \partial(S_i^*)_\varepsilon) \leqq C_0\} \leqq C\varepsilon^{1-d}(\mathcal{H}^{d-1}(S_i^{t_i}) + \mathcal{H}^{d-1}(\partial O)).$$

In particular, for N, δ and $\kappa > 0$ fixed, due to (A.1) there exists ε_0 such that for all $\varepsilon < \varepsilon_0$

$$\int_{R_{i,\varepsilon}} \exp(-c_0\|\nabla_{\mathbb{B}} h_2\|^p_{\ell^p_\varepsilon(S_i^*)})\,\mathrm{d}h_2 \leqq \exp(C|(O^\delta)^{\mathcal{L}}_\varepsilon|).$$

Plugging this bound into (A.6) and comparing with (A.5) yields

$$Z_{\varepsilon,O}(\mathcal{P}^i_\varepsilon \cap \mathcal{S}_M(O,\varepsilon)) \leqq Z_{\varepsilon,O}(\mathcal{N}_p(v, O, \varepsilon, 3\kappa) \cap \mathcal{B}_\varepsilon(O, \varphi))$$
$$\times \exp\left(C\left(|(O^\delta)^{\mathcal{L}}_\varepsilon| + \frac{(N\kappa|O|^{\frac{1}{d}})^p}{\delta^p}|O^{\mathcal{L}}_\varepsilon| + \frac{M}{N}|O^{\mathcal{L}}_\varepsilon| + H_\varepsilon(O^\delta, \varphi)\right)\right)$$

Summing this inequality over i, by the definition of the sets $\mathcal{P}_{i,\varepsilon}$ we infer that

$$Z_{\varepsilon,O}(\mathcal{N}_p(v, O, \varepsilon, \kappa) \cap \mathcal{S}_M(O,\varepsilon)) \leqq \sum_{i=1}^N Z_{\varepsilon,O}(\mathcal{P}^i_\varepsilon \cap \mathcal{S}_M(O,\varepsilon))$$
$$\leqq Z_{\varepsilon,O}(\mathcal{N}_p(v, O, \varepsilon, 3\kappa) \cap \mathcal{B}_\varepsilon(O, \varphi))$$
$$\times N\exp\left(C\left(|(O^\delta)^{\mathcal{L}}_\varepsilon| + \frac{(N\kappa|O|^{\frac{1}{d}})^p}{\delta^p}|O^{\mathcal{L}}_\varepsilon| + \frac{M}{N}|O^{\mathcal{L}}_\varepsilon| + H_\varepsilon(O^\delta, \varphi)\right)\right) \tag{A.7}$$

Now, choosing

$$M = 2\left(\frac{1}{|O^{\mathcal{L}}_\varepsilon|}\log\left(Z_{\varepsilon,O}(\mathcal{N}_p(v, O, \varepsilon, \kappa))\right) + \overline{C} + \frac{\log(2)}{|O^{\mathcal{L}}_\varepsilon|}\right),$$

where $\overline{C}$ is the constant of Lemma 3.4, we obtain by the same Lemma and Remark 8 that, for any $\kappa > 0$ fixed and all ε small enough,

$$Z_{\varepsilon,O}(\mathcal{N}_p(v, O, \varepsilon, \kappa)\backslash\mathcal{S}_M(O,\varepsilon)) \leqq \frac{1}{2}Z_{\varepsilon,O}(\mathcal{N}_p(v, O, \varepsilon, \kappa)).$$

Thus (A.7) and the definition of M yield the final estimate

$$Z_{\varepsilon,O}(\mathcal{N}_p(v, O, \varepsilon, \kappa))$$

$$\leqq 2Z_{\varepsilon,O}(\mathcal{N}_p(v, O, \varepsilon, \kappa) \cap \mathcal{S}_M(O, \varepsilon))$$
$$\leqq Z_{\varepsilon,O}(\mathcal{N}_p(v, O, \varepsilon, 3\kappa) \cap \mathcal{B}_\varepsilon(O, \varphi))\, Z_{\varepsilon,O}(\mathcal{N}_p(v, O, \varepsilon, \kappa))^{\frac{C}{N}}$$
$$\times\, 2N \exp\Big(C\big(|(O^\delta)^{\mathcal{L}}_\varepsilon| + \big(\frac{(N\kappa|O|^{\frac{1}{d}})^p}{\delta^p} + \frac{C}{N}\big)|O^{\mathcal{L}}_\varepsilon| + H_\varepsilon(O^\delta, \varphi)\big)\Big).$$

□

Remark 14. Note that the restriction on δ in the interpolation inequality comes only from the requirement that tubular neighbourhoods of the boundary have again Lipschitz boundary. In particular, if δ satisfies the condition for a set $O \subset \mathbb{R}^d$, then $\delta' = \delta\rho$ satisfies the condition for all sets of the form $O' = z + \rho O$. Applying this fact to the family of cubes $Q(z, \rho)$ with $z \in D$ and $\rho > 0$, we obtain that there exists $\delta_0 > 0$ such that for all $\delta < \delta_0$, all $N \in \mathbb{N}$ and all $\kappa > 0$ it holds that

$$\begin{aligned}\frac{N-C}{N}\mathcal{F}^-_\kappa(Q(z,\rho), \overline{\varphi}_\Lambda) &\geqq \overline{W}(\Lambda) - C(1 + |\Lambda|^p)\frac{|Q(z,\rho)^{\delta\rho}|}{|Q(z,\rho)|}\\ &\quad - C\left(\frac{(N\kappa|O(z,\rho)|^{\frac{1}{d}})^p}{(\delta\rho)^p} + \frac{1}{N}\right)\\ &\geqq \overline{W}(\Lambda) - C\left((1+|\Lambda|^p)\delta + \frac{(N\kappa)^p}{\delta^p} + \frac{1}{N}\right)\end{aligned}$$

$\mathbb{P}$-almost surely. Here we used Lemma 4.1, Proposition 3 and (3.1) in order to pass to the limit as $\varepsilon \to 0$ in the interpolation inequality almost surely. Note that the last bound is independent of ρ and z.

Lemma A.1. *Let $p \in (1, +\infty)$. For all $u, v \in \mathbb{R}^n$ it holds that*

$$|u - v|^p + |u + v|^p \leqq \max\{2^{p-1}, 2\}(|u|^p + |v|^p).$$

Proof of Lemma A.1. For $p \geqq 2$ the claimed estimate follows from Clarkson's inequality. If $p < 2$, then $(x_1^p + x_2^p)^{\frac{1}{p}} \geqq (x_1^2 + x_2^2)^{\frac{1}{2}}$ for all $x_1, x_2 \geqq 0$. Moreover, with elementary analysis one can show that $(x_1^p + x_2^p)^{\frac{1}{p}} \leqq 2^{\frac{1}{p}-\frac{1}{2}}(x_1^2 + x_2^2)^{\frac{1}{2}}$. Applying these two inequalities first with $x_1 = |u - v|$ and $x_2 = |u + v|$ and then with $x_1 = |u|$ and $x_2 = |v|$ we obtain

$$\begin{aligned}(|u - v|^p + |u + v|^p)^{\frac{1}{p}} &\leqq 2^{\frac{1}{p}-\frac{1}{2}}(|u - v|^2 + |u + v|^2)^{\frac{1}{2}} = 2^{\frac{1}{p}}(|u|^2 + |v|^2)^{\frac{1}{2}}\\ &\leqq 2^{\frac{1}{p}}(|u|^p + |v|^p)^{\frac{1}{p}}.\end{aligned}$$

□

Lemma A.2. *Let $p \in (1, +\infty)$. Then there exists a constant c_p such that the Hausdorff measure of the sphere $S_p^{n-1} = \{y \in \mathbb{R}^n : |y|_p = 1\}$ fulfills*

$$\mathcal{H}^{n-1}(S_p^{n-1}) \geqq \left(\frac{c_p}{n}\right)^{\frac{n}{p}}.$$

Proof of Lemma A.2. Note that S_p^{n-1} is a compact smooth $(n-1)$-dimensional manifold. Hence we can characterize its Hausdorff measure by its Minkowski content. To be more precise, it holds that

$$\mathcal{H}^{n-1}(S_p^{n-1}) = \lim_{\varepsilon\to 0} \frac{\mathcal{H}^n(S_p^{n-1} + B_\varepsilon(0))}{2\varepsilon}, \tag{A.8}$$

where the factor 2 comes from the Lebesgue measure of the 1D unit ball $[-1, 1]$. Note however that $B_\varepsilon(0)$ is a ball with respect to the Euclidean metric on $\mathbb{R}^n$. We now give a lower bound for the nominator on the right hand side of (A.8). To this end, set $c_{n,p} = \max\{1, n^{\frac{1}{2}-\frac{1}{p}}\}$. Then, for $y \neq 0$, we have

$$\left| y - \frac{y}{|y|_p} \right|_2 \leqq ||y|_p - 1| \frac{|y|_2}{|y|_p} \leqq ||y|_p - 1| c_{n,p},$$

where we used that by definition $|y|_2 \leqq c_{n,p}|y|_p$ for all $y \in \mathbb{R}^n$. We conclude that

$$\{y \in \mathbb{R}^n :\ 1 - c_{n,p}^{-1}\varepsilon < |y|_p < 1 + c_{n,p}^{-1}\varepsilon\} \subset S_p^{n-1} + B_\varepsilon(0).$$

Hence we deduce from (A.8) and the well-know formula for the volume of p-norm balls that

$$\begin{aligned}
\mathcal{H}^{n-1}(S_p^{n-1}) &\geqq \liminf_{\varepsilon\to 0} \frac{\mathcal{H}^n(\{|y|_p < 1 + c_{n,p}^{-1}\varepsilon\}) - \mathcal{H}^n(\{|y|_p < 1 - c_{n,p}^{-1}\varepsilon\})}{2\varepsilon} \\
&= \frac{(2\Gamma(\frac{1}{p}+1))^n}{\Gamma(\frac{n}{p}+1)} \lim_{\varepsilon\to 0} \frac{(1 + c_{n,p}^{-1}\varepsilon)^n - (1 - c_{n,p}^{-1}\varepsilon)^n}{2\varepsilon} \\
&= \frac{(2\Gamma(\frac{1}{p}+1))^n}{\Gamma(\frac{n}{p}+1)} n c_{n,p}^{-1} \geqq \frac{(2\Gamma(\frac{1}{p}+1))^n}{\Gamma(\frac{n}{p}+1)} n^{\frac{1}{2}}.
\end{aligned}$$

We conclude the proof using Stirling's formula in the form of the upper bound

$$\Gamma\left(\frac{n}{p}+1\right) \leqq \left(\frac{2\pi n}{p}\right)^{\frac{1}{2}} \left(\frac{n}{pe}\right)^{\frac{n}{p}} \exp(p/12).$$

□

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