Research Article

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On the continuity of functionals defined on partitions

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Abstract: We characterize the continuity of prototypical functionals acting on finite Caccioppoli partitions and prove that it is equivalent to convergence of the perimeter of the jump set.

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1 Introduction

In this short note we investigate the continuity of functionals defined on functions of bounded variation taking values in a finite set. More precisely, for an open set $\Omega \subset \mathbb{R}^d$ and $\mathcal{Z} = \{z_1, \dots, z_q\} \subset \mathbb{R}^N$ we consider functionals $F : \mathrm{BV}(\Omega, \mathcal{Z}) \to \mathbb{R}$ of the form

$$F(u) = \int_{S_u \cap \Omega} g(x, u^+, u^-, \nu_u) \, \mathrm{d}\mathcal{H}^{d-1}. \tag{1.1}$$

Here S_u denotes the discontinuity set of u, $\nu_u = \nu_u(x)$ is the corresponding normal vector at $x \in S_u$ and u^+, u^- are the traces of u on both sides of the discontinuity set. As it is usual in this framework, the functional is well-defined if we require the symmetry condition $g(x, a, b, \nu) = g(x, b, a, -\nu)$. Such functionals arise for example in the study of multiphase Cahn–Hilliard fluids [4] or the discrete to continuum analysis of spin systems with finitely many ground states [5]. A general treatment of these functionals from a variational point of view can be found in [1, 2]. In the recent paper [6] the authors proved a density result in the space $\mathrm{BV}(\Omega, \mathcal{Z})$ and established continuity of functionals of type (1.1) along the particular approximating sequence. Here we investigate general continuity properties. Assuming g to be bounded and continuous, we provide a precise characterization of the convergence under which all functionals of the form (1.1) are continuous.

We prove that functionals of the form (1.1) are continuous along sequences u_n such that $u_n \to u$ in $L^1(\Omega)$ and in addition $\mathcal{H}^{d-1}(S_{u_n} \cap \Omega) \to \mathcal{H}^{d-1}(S_u \cap \Omega)$. This is of course also a necessary condition when we take $g \equiv 1$. Even though the issue of the continuity of functionals as in (1.1) arises very naturally, to the best of our knowledge our result never appeared in the mathematical literature.

This short note is organized as follows: In Section 2 we give a short introduction to functions of bounded variation. In Section 3 we prove our main claim.

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2 Mathematical preliminaries

In this section we recall basic facts about functions of bounded variation that can be found in [3].

A function $u \in L^1(\Omega)$ is a function of bounded variation if there exists a finite vector-valued Radon measure μ on Ω such that for any $\varphi \in C_c^\infty(\Omega, \mathbb{R}^d)$ there holds

$$\int_\Omega u \operatorname{div} \varphi \, \mathrm{d}x = -\int_\Omega \langle \varphi, \mu \rangle.$$

In this case we write $u \in \mathrm{BV}(\Omega)$ and $Du = \mu$ is the distributional derivative of u. A function $u \in L^1(\Omega, \mathbb{R}^N)$ belongs to $\mathrm{BV}(\Omega, \mathbb{R}^N)$ if every component belongs to $\mathrm{BV}(\Omega)$. In this case Du denotes the matrix-valued Radon measure consisting of the distributional derivatives of each component.

The spaces $\mathrm{BV}_{\mathrm{loc}}(\Omega)$ and $\mathrm{BV}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ are defined as usual. The space $\mathrm{BV}(\Omega, \mathbb{R}^N)$ becomes a Banach space when endowed with the norm $\|u\|_{\mathrm{BV}(\Omega)} = \|u\|_{L^1(\Omega)} + |Du|(\Omega)$, where $|Du|$ denotes the total variation measure of Du. When Ω is a bounded Lipschitz domain, then $\mathrm{BV}(\Omega, \mathbb{R}^N)$ is compactly embedded in $L^1(\Omega, \mathbb{R}^N)$. We say that a sequence u_n converges weakly* in $\mathrm{BV}(\Omega, \mathbb{R}^N)$ to u if $u_n \to u$ in $L^1(\Omega, \mathbb{R}^N)$ and $Du_n \overset{*}{\rightharpoonup} Du$ in the sense of measures. We say that u_n converges strictly to u if $u_n \to u$ in $L^1(\Omega, \mathbb{R}^N)$ and $|Du_n|(\Omega) \to |Du|(\Omega)$. Note that strict convergence implies weak*-convergence and that for Ω with Lipschitz boundary norm-bounded sequences in $\mathrm{BV}(\Omega, \mathbb{R}^N)$ are compact with respect to weak*-convergence, but not necessarily with respect to strict convergence.

We say that a Lebesgue-measurable set $E \subset \mathbb{R}^d$ has finite perimeter in Ω if its characteristic function $\mathbb{1}_E$ belongs to $\mathrm{BV}(\Omega)$. We say it has locally finite perimeter in Ω if $\mathbb{1}_E \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$. Let Ω' be the largest open set such that E has locally finite perimeter in Ω'. The reduced boundary $\mathcal{F}E$ of E is defined as

$$\mathcal{F}E = \Big\{x \in \Omega' \cap \operatorname{supp}|D\mathbb{1}_E| : \nu(x) = \lim_{\rho\to 0} \frac{D\mathbb{1}_E(B_\rho(x))}{|D\mathbb{1}_E|(B_\rho(x))} \text{ exists and } |\nu(x)| = 1\Big\}$$

and it holds that $|D\mathbb{1}_E| = \mathcal{H}^{d-1} \llcorner \mathcal{F}E$. Moreover, ν can be interpreted as a measure theoretic inner normal vector (see also [3, Theorem 3.59]).

Now we state some fine properties of BV-functions. To this end, we need some definitions. A function $u \in L^1(\Omega, \mathbb{R}^N)$ is said to have an approximate limit at $x \in \Omega$ whenever there exists $z \in \mathbb{R}^N$ such that

$$\lim_{\rho\to 0} \frac{1}{\rho^d} \int_{B_\rho(x)} |u(y) - z| \, \mathrm{d}y = 0.$$

We let $S_u \subset \Omega$ be the set where u has no approximate limit. Now we introduce so-called approximate jump points. Given $x \in \Omega$ and $\nu \in S^{d-1}$ we set

$$\begin{cases} B_\rho^+(x,\nu) = \{y \in B_\rho(x) : \langle y - x, \nu \rangle > 0\}, \\ B_\rho^-(x,\nu) = \{y \in B_\rho(x) : \langle y - x, \nu \rangle < 0\}. \end{cases}$$

We say that $x \in \Omega$ is an approximate jump point of u if there exist $a \neq b \in \mathbb{R}^N$ and $\nu \in S^{d-1}$ such that

$$\lim_{\rho\to 0} \frac{1}{\rho^d} \int_{B_\rho^+(x,\nu)} |u(y) - a| \, \mathrm{d}y = \lim_{\rho\to 0} \frac{1}{\rho^d} \int_{B_\rho^-(x,\nu)} |u(y) - b| \, \mathrm{d}y = 0.$$

Note that the triplet (a, b, ν) is determined uniquely up to the change to $(b, a, -\nu)$. We denote it by

$$(u^+(x), u^-(x), \nu_u(x))$$

and let J_u be the set of approximate jump points of u. Then the triplet (u^+, u^-, ν_u) can be chosen as a Borel function on the Borel set J_u. If $u \in \mathrm{BV}(\Omega, \mathbb{R}^N)$, it can be shown that $\mathcal{H}^{d-1}(S_u \backslash J_u) = 0$. Denoting by ∇u the density of the absolutely continuous part of Du with respect to the Lebesgue measure, we can decompose the measure Du via

$$Du(B) = \int_B \nabla u \, \mathrm{d}x + \int_{J_u \cap B} (u^+(x) - u^-(x)) \otimes \nu_u(x) \, \mathrm{d}\mathcal{H}^{d-1} + D^c u(B),$$

where $D^c u$ is the so-called Cantor part.

From now on we assume that Ω is a bounded open set. Given a finite set $\mathcal{Z} = \{z_1, \dots, z_q\} \subset \mathbb{R}^N$ we define the space $BV(\Omega, \mathcal{Z})$ as the space of those functions $u \in BV(\Omega, \mathbb{R}^N)$ such that $u(x) \in \mathcal{Z}$ almost everywhere. As an immediate consequence of the coarea formula applied to each component, it follows that all level sets $E_i := \{u = z_i\}$ have finite perimeter in Ω. Moreover, the total variation of Du and the surface measure of S_u are given by

$$|Du| = \frac{1}{2}\sum_{i=1}^{q}\sum_{j\neq i}|z_i - z_j|\mathcal{H}^{d-1}(\mathcal{F}E_i \cap \mathcal{F}E_j \cap \Omega),$$

$$\mathcal{H}^{d-1}(S_u) = \frac{1}{2}\sum_{i=1}^{q}\mathcal{H}^{d-1}(\mathcal{F}E_i \cap \Omega).$$

3 Statement and proof of the main result

The following theorem is the main result of this short note. For our proof we use minimal liftings in BV as in [9] (see also [7]).

Theorem 3.1. *Let $g : \Omega \times \mathcal{Z}^2 \times S^{d-1} \to \mathbb{R}$ be bounded and continuous and let $u_n, u \in BV(\Omega, \mathcal{Z})$ be such that $u_n \to u$ in $L^1(\Omega)$ and $\mathcal{H}^{d-1}(S_{u_n} \cap \Omega) \to \mathcal{H}^{d-1}(S_u \cap \Omega)$. Then*

$$\lim_n \int_{S_{u_n}\cap\Omega} g(x, u_n^+, u_n^-, \nu_{u_n})\,d\mathcal{H}^{d-1} = \int_{S_u\cap\Omega} g(x, u^+, u^-, \nu_u)\,d\mathcal{H}^{d-1}.$$

Remark 1. Since we aim for a rather weak kind of convergence, it is convenient to require that g is bounded and continuous. While continuity in the trace variables is redundant since $\mathcal{Z}$ is a finite set, continuity in x and ν can be dropped if we aim for norm convergence in $BV(\Omega, \mathcal{Z})$. On the other hand, given a sequence $u_n \in BV(\Omega, \mathcal{Z})$ such that $u_n \to u$ in $L^1(\Omega)$ we cannot expect the energy to converge as well.

Proof of Theorem 3.1. Let us set $F(u) = \int_{S_u\cap\Omega} g(x, u^+, u^-, \nu_u)\,d\mathcal{H}^{d-1}$. We will just prove upper semicontinuity. The general result then follows applying upper semicontinuity to the functional $-F$. By our assumptions we can assume without loss of generality that $g \geq 0$. For an arbitrary $v \in BV(\Omega, \mathcal{Z})$ we define for $|Dv|$-almost every $x \in \Omega$ the vector measure λ_x via its action on functions $\varphi \in C_0(\mathbb{R}^N)$ by

$$\int_{\mathbb{R}^N} \varphi(y)\,d\lambda_x(y) = \frac{dDv}{d|Dv|}(x)\int_0^1 \varphi(\theta v^+(x) + (1-\theta)v^-(x))\,d\theta.$$

To reduce notation, we write $v^\theta = \theta v^+ + (1-\theta)v^-$. Since v^+, v^- are $|Dv|$-measurable, by using Fubini's theorem, one can show that for any $\varphi \in C_0(\Omega \times \mathbb{R}^N)$ the mapping

$$x \mapsto \int_{\mathbb{R}^N} \varphi(x, y)\,d\lambda_x(y)$$

is $|Dv|$-measurable and essentially bounded. Hence we can define the generalized product $\mu[v] = |Dv| \otimes \lambda_x$ again by its action on $C_0(\Omega \times \mathbb{R}^N)$ by setting

$$\int_{\Omega\times\mathbb{R}^N} \varphi(x, y)\,d\mu[v](x, y) = \int_\Omega\int_{\mathbb{R}^N} \varphi(x, y)\,d\lambda_x(y)\,d|Dv|(x);$$

see also [3, Definition 2.27]. We next claim that up to a negligible set it holds that

$$\frac{d\mu[v]}{d|\mu[v]|}(x, y) = \frac{dDv}{d|Dv|}(x). \tag{3.1}$$

Indeed, [3, Corollary 2.29] yields $|\mu[v]| = |Dv| \otimes |\lambda_x|$. As the defining formula for the generalized product extends to integrable functions, we infer that

$$\int_{\Omega\times\mathbb{R}^N} \varphi(x,y)\frac{\mathrm{d}Dv}{\mathrm{d}|Dv|}(x)\,\mathrm{d}|\mu[v]|(x,y) = \int_\Omega\int_{\mathbb{R}^N} \varphi(x,y)\frac{\mathrm{d}Dv}{\mathrm{d}|Dv|}(x)\,\mathrm{d}|\lambda_x|(y)\,\mathrm{d}|Dv|(x)$$
$$= \int_\Omega\int_{\mathbb{R}^N} \varphi(x,y)\,\mathrm{d}\lambda_x(y)\,\mathrm{d}|Dv|(x) = \int_{\Omega\times\mathbb{R}^N} \varphi(x,y)\,\mathrm{d}\mu[v](x,y),$$

where we have used that $\lambda_x = \frac{\mathrm{d}Dv}{\mathrm{d}|Dv|}(x)|\lambda_x|$. Hence (3.1) follows by the uniqueness of the polar decomposition of measures. Because of (3.1) and the generalized product structure of $|\mu[v]|$, by an approximation argument it holds that

$$\int_{\Omega\times\mathbb{R}^N} f\Big(x,y,\frac{\mathrm{d}\mu[v]}{\mathrm{d}|\mu[v]|}(x,y)\Big)\,\mathrm{d}|\mu[v]|(x,y) = \int_{\Omega\times\mathbb{R}^N} f\Big(x,y,\frac{\mathrm{d}Dv}{\mathrm{d}|Dv|}(x)\Big)\,\mathrm{d}|\mu[v]|(x,y)$$
$$= \int_\Omega\int_{\mathbb{R}^N} f\Big(x,y,\frac{\mathrm{d}Dv}{\mathrm{d}|Dv|}(x)\Big)\,\mathrm{d}|\lambda_x|(y)\,\mathrm{d}|Dv|(x)$$
$$= \int_\Omega\int_0^1 f\Big(x,v^\theta,\frac{\mathrm{d}Dv}{\mathrm{d}|Dv|}(x)\Big)\,\mathrm{d}\theta\,\mathrm{d}|Dv|(x) \tag{3.2}$$

for every nonnegative function $f \in C(\Omega\times\mathbb{R}^N\times S^{N\times d-1})$. In [7] it is proven that if $v_n \to v$ strictly in $\mathrm{BV}(\Omega,\mathbb{R}^N)$, then $\mu[v_n] \overset{*}{\rightharpoonup} \mu[v]$ and $|\mu[v_n]|(\Omega\times\mathbb{R}^N) \to |\mu[v]|(\Omega\times\mathbb{R}^N)$. The idea now is to apply the classical Reshetnyak continuity theorem (see for instance [8, 10]) with an appropriate f and a strictly converging sequence v_n. To this end, we transform the set $\mathcal{Z}$ so that averages of the jump functions $u^\pm$ encode the values of the traces and such that the convergence assumptions yield strict convergence. Recall that $q = \#\mathcal{Z}$. We define the mapping $T : \mathcal{Z} \to \mathbb{R}^q$ via $T(z_i) = e_i$, where e_i denotes the i-th unit vector. Next we construct the function f. Given $i < j$ we consider the set

$$L_{ij} = \Big\{\lambda T(z_i) + (1-\lambda)T(z_j) : \lambda \in \Big(\frac{1}{4},\frac{3}{4}\Big)\Big\}.$$

Observe that by construction of the set $T(\mathcal{Z})$ there holds $L_{ij}\cap L_{kl} = \emptyset$ whenever $\{i,j\} \neq \{k,l\}$. Given $\delta > 0$ we next choose a cut-off function $\theta_{ij}^\delta : [T(z_i),T(z_j)] \to [0,1]$ such that $\theta_{ij}^\delta = 1$ on L_{ij} and $\theta_{ij}^\delta(x) = 0$ if $\operatorname{dist}(x,L_{ij}) \geq \delta$. Set $f_\delta \in C(D\times\mathbb{R}^q\times S^{q\times d-1})$ as any continuous, nonnegative extension of the function

$$f_\delta(x,u,\xi) = \frac{\theta_{ij}^\delta(u)}{\sqrt{2}\mathcal{H}^1(L_{ij})} g\Big(x,z_i,z_j,\frac{\xi^T e_1}{|\xi^T e_1|}\Big)|\xi^T e_1| \quad \text{if } u \in [T(z_i),T(z_j)].$$

First observe that this is well-defined due to the ordering $i < j$ (also in the case $\xi^T e_1 = 0$ since g is bounded). Moreover, for δ small enough such an extension exists by the properties of the cut-off function. Now for any $T(u) \in \mathrm{BV}(\Omega, T(\mathcal{Z}))$, with a suitable orientation of the normal vector, for $|DT(u)|$-almost every $x \in \Omega$ it holds that

$$\frac{\mathrm{d}DT(u)}{\mathrm{d}|DT(u)|}(x) = \frac{1}{\sqrt{2}}\sum_{i<j}(T(z_i)-T(z_j))\otimes\nu_u(x)\mathbb{1}_{\mathcal{F}E_i\cap\mathcal{F}E_j}(x),$$
$$|DT(u)| = \sqrt{2}\sum_{i<j}\mathcal{H}^{d-1}\llcorner(\mathcal{F}E_i\cap\mathcal{F}E_j),$$

where $E_i = \{u = z_i\}$. Therefore we can rewrite

$$\int_\Omega\int_0^1 f_\delta\Big(x,T(u)^\theta,\frac{\mathrm{d}DT(u)}{\mathrm{d}|DT(u)|}(x)\Big)\,\mathrm{d}\theta\,\mathrm{d}|DT(u)|(x)$$
$$= \int_\Omega \sum_{i<j} g(x,z_i,z_j,\nu_u)\,\mathrm{d}\mathcal{H}^{d-1}\llcorner(\mathcal{F}E_i\cap\mathcal{F}E_j) + \mathcal{O}(\delta)\mathcal{H}^{d-1}(S_u\cap\Omega)$$
$$= F(u) + \mathcal{O}(\delta)\mathcal{H}^{d-1}(S_u\cap\Omega)$$

with a nonnegative error $\mathcal{O}(\delta)$.

If u_n, u are as in the claim, then $T(u_n) \to T(u)$ in $L^1(\Omega)$, and moreover

$$|DT(u_n)| = \sqrt{2}\mathcal{H}^{d-1}(S_{u_n} \cap \Omega) \to \sqrt{2}\mathcal{H}^{d-1}(S_u \cap \Omega) = |DT(u)|,$$

so that $T(u_n)$ converges strictly to $T(u)$. Hence we conclude from (3.2) and the classical Reshetnyak continuity theorem applied to the measures $\mu[T(u_n)]$, $\mu[T(u)]$ that

$$\begin{aligned}\limsup_n F(u_n) &\leq \lim_n \int_\Omega \int_0^1 f_\delta\Big(x, T(u_n)^\theta, \frac{\mathrm{d}DT(u_n)}{\mathrm{d}|DT(u_n)|}(x)\Big)\,\mathrm{d}\theta\,\mathrm{d}|DT(u_n)|(x) \\ &= \int_\Omega \int_0^1 f_\delta\Big(x, T(u)^\theta, \frac{\mathrm{d}DT(u)}{\mathrm{d}|DT(u)|}(x)\Big)\,\mathrm{d}\theta\,\mathrm{d}|DT(u)|(x) \\ &\leq F(u) + \mathcal{O}(\delta)\mathcal{H}^{d-1}(S_u \cap \Omega).\end{aligned}$$

The claim follows by the arbitrariness of δ. □

Remark 2. Taking $g(x, u^+, u^-, \nu) = |u^+ - u^-|$, Theorem 3.1 yields that $L^1(\Omega)$-convergence combined with the convergence of $\mathcal{H}^{d-1}(S_u)$ implies strict convergence in $\mathrm{BV}(\Omega, \mathbb{Z})$. The converse is false in general as can be seen by the following one-dimensional example: Given $n \in \mathbb{N}$ we set $u_n : (-1, 1) \to \{0, 1, 2\}$ as $u_n(x) = \mathbb{1}_{(1/n,2/n)} + 2\mathbb{1}_{(2/n,1)}$. Then u_n converges strictly to the function $u = 2\mathbb{1}_{(0,1)}$, while $\mathcal{H}^0(S_{u_n}) = 2$, but $\mathcal{H}^0(S_u) = 1$.

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