Convergence of equilibria of thin elastic plates in a discrete model - The von Kármán case

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Chapter 1

Introduction

A fundamentel problem in mathematical elasticity is to study the limiting behavior $h \to 0$ of elastic energies

$$\mathcal{E}^h(y) = \int_{\Omega_h} W(\nabla y)\ dx \tag{1.1}$$

of a deformation

$$y : \Omega_h \to \mathbb{R}^3,$$

which depend on a small parameter $h > 0$.

Prototypical examples are plates and rods. In the case of plates we have $\Omega_h = S \times \left(-\frac{h}{2}, \frac{h}{2}\right)$ for a bounded 2-dimensional domain S as for rods we have $\Omega_h = (0, L) \times hS$, where $L > 0$ is the length of the rod and S is the 2-dimensional cross section. For plates the height is assumed to be very small compared to the area of the mid-surface, in contrast for rods the area of the cross section should be small compared to the length.

When trying to predict the behavior of a material, of which one parameter is very small compared to the others, under a given load, we expect that the resulting deformations can be essentially described by a lower dimensional model:

- Deformations of thin plates should be understood by deformations of its midplane.
- Rods should be described by a 1D-model.

It is desirable to get a *mathematical rigorous* derivation of these *effective models*. Of course we cannot expect that an effective model is completely accurate. Still we do expect some advantages compared to the real world model:

- One trivial reason for the use of effective models is the impossibility to capture the whole complexity of reality.
- From a mathematical point of view it might be easier to show existence and uniqueness of solutions tho the corresponding differential equations.
- From a numerical point of view less complexity in the model should reduce complexity of calculations.

On an abstract level we can view at the problem as follows: Let $(P_h)_{h>0}$ be a family of problemes (where $P_{\tilde{h}}$ is the real problem for a small number $\tilde{h} > 0$) and P_0 be an effective problem. We give two possible ways (there are of course others) to connect P_0 with the problems P_h.

(i) Assume that there is a (sufficiently regular) solution to the approximating problem P_0 and show that there is a solution of problem P_h for small $h > 0$. Usually this is done with help of the implicit function theorem. We mention [Mon03] for von-Kármán-plates. A similar ansatz is used in [BS16] not for dimension reduction, but for a discrete-to-continuum problem.

(ii) Assume that the problems $(P_h)_{h>0}$ have solutions and show that the corresponding solutions converge to a solution of the limiting problem P_0. This is usually done by Γ-convergence, a type of convergence of functionals that, under certain additional conditions, ensures convergence of minimizers to a minimizer of the limiting functional. Good references for Γ-convergence are for example [Mas93] and [Bra02].

In many cases the first ansatz provides estimates of the error between the solutions of P_h and P_0 in a suitable norm. This is not the case for the second since Γ-convergence does not imply any convergence rate. In this thesis we will not follow the first approach. We give an overview of the Γ-convergence results with the functionals (1.1). In order to get interesting results the task is to compute the Γ-limit of the rescaled functionals $\frac{\mathcal{E}^h}{h^\beta}$ as $h \to 0$. Different values of β lead to different limiting theories. For plates among others we have:

(i) The case $\beta = 1$ has been treated in [DR95] and leads to membrane theory, which corresponds to a stretching of the midplane. This theory can be applied for example for stretching a thin piece of rubber.

(ii) For $\beta = 3$ the Γ-limit leads to Kirchhoff-theory, which is a bending theory where the midplane remains unstrained. This has been derived in [FJM02]. A key ingredient in this work is the famous geometric rigidity result: There is a constant $C = C(U)$ only dependent on the domain U such that for all $v \in W^{1,2}(U)$ there is a rotation $R \in SO(n)$ such that $\|\nabla v - R\|_{L^2(U)} \leq C \left\|\operatorname{dist}^2(\nabla v, SO(n))\right\|_{L^2(U)}$. An example where Kirchhoff-plate-theory applies is the bending of sheets of paper.

(iii) The case $\beta = 5$ leads to von-Kármán theory derived in [FJM06]. It is used for very small deformations, where the in-plane displacement is much smaller compared to the out-of-plane displacement. The latter one should be comparable to the height of the plate.

In all of these examples Γ-convergence is used for *dimension reduction*. A nice introduction into lower dimensional theories is given in [Mül17]. Dimension reduction however is not the only application of Γ-convergence in the field of mathematical elasticity theory. We want to mention the derivation of linearized elasticity from finite elasticity of Dal Maso, Negri and Percivale in [MNP02]. Γ-convergence is also an important tool when it comes to pass from discrete to continuum models. For a general overview of discrete to continuum approaches not limited to Γ-convergence we refer to [BBL07]. In [AC04] nonlinear elasticity functionals are derived from pair interaction models. This work has been extended to a wider class of interaction potentials in [BS13]. Another interesting discrete-to-continuum result is [Sch09], which can be seen as a discrete-to-continuum analogue to [MNP02].

The investigation of discrete interaction models has further advantages compared to purely continuum models. The latter might not fit for extremely thin structures, i.e. of films consisting only of a few layers of atoms. The discrete models we look at for thin films have two different scales, the interatomic distance ε and the thickness parameter h. In very thin (we will write *ultrathin*) films the two parameters are roughly the same order of magnitude, i.e. $\varepsilon \approx h$. In thicker (we will write *thin*) structures it holds that $\varepsilon \ll h$. Depending on the regime limiting models might be different. This can be seen in the work of [Sch06] where Kirchhoff plate theory is derived from an atomistic model. Additional surface terms, which can be neglected in the thin case, occur in the ultrathin model. This work included a discrete rigidity result which is an analogue to the one in [FJM02]. Different models for thin or ultrathin films can also be seen in [BS22] where von-Kármán plate theory was derived from an atomistic model. Such models for ultrathin structures could not be derived from purely continuum theory so far.

Let us mention that results for ultrathin films are not only available for plates, see for example [SZ23], where the authors derived a bending-torsion theory for thin and ultrathin rods.

From the view of a mathematician it is a natural question if we can tell something about stationary points that are not minimizers of the respective functionals. Γ-convergence only tells us something about the behavior of almost minimizing sequences. For integral functionals $T_h : X \to \mathbb{R},\ h > 0$, defined on a function space X we do not get any information about stationary points x_h, which are not absolute minimizers. For example x_h could be a local minimizer of T_h or a saddle point. But such points are also solutions of the corresponding Euler-Lagrange-equations of T_h and therefore it is clearly interesting to look at them as well. We mention [MP08], [MMR06] and [MM08] as examples.

Another reason why one might want to look beyond absolute minimizers is because Γ-convergence does not really fit for time-dependent problems, although there are recent works in that direction, c.f. [Mie23]. Understanding the *convergence of equilibria* for a static problem can help to prove a similar result for a corresponding time-dependent problem. In [AMM09] the results for von-Kármán-plates ([MP08]) have been extended to the time-dependent setting.

The goal of this thesis is to prove similar results as in [MP08] and [AMM09] starting from a discrete interaction model. In the static case we assume that we have atomic solutions that satisfy a force-balance of a particle system and show that these solutions converge to solutions of the von Kármán equations. In the dynamic case we consider solutions of the equations given by Newton's second law of motion and show that they converge to a solution of the time-dependent von Kármán equations. Especially in the time dependent settings with ultrathin films there are not many results yet to the authors best knowledge.

Working with discrete objects leads to many difficulties which are not present in the continuum setting. One question is how to relate continuum deformations (which appear in the limiting model) with displacements of atoms in a crystal lattice. One usually applies the Cauchy-Born hypothesis. Roughly speaking it says that each atom in a lattice follows the same affine deformation given at the boundary of the lattice. More precisely we can formulate it as follows: If the boundary atoms x of crystal lattice Λ is subject to an affine deformation $y(x) = Ax$, then the overall minimizer of the interaction energy of Λ is given by y. For validity of the Cauchy-Born hypothesis we refer to [FT02] [CDKM06]. Also we want to mention [Eri08] for a more detailed analysis.

Another question one faces when dealing with discrete objects is the choice of an appropriate interaction potential. Of course it is desirable that the analysis applies to widely used potentials in physics such as Lennard-Jones-potentials. In our case we had to choose more basic interaction potentials due to problems with growth conditions. Also the range of atomic interaction has to be considered. A longer interaction-range results in a more complex model, especially in the ultrathin setting.

Finally passing from discrete objects to continuum objects means to pass from sums to integral expressions. Hence it is necessary to choose suitable interpolations for the discrete mappings. We discuss two different interpolation schemes and show that they are more or less equivalent.

Chapter 2

The atomistic model

In this chapter we introduce the basic domains we are working with. We introduce discrete deformations and their gradients. Further we present the two interpolations schemes used in this thesis to pass from atomistic to continuum objects.

Then we define the energy of a discrete deformation as well as some quadratic forms which appear in the limiting model. Finally we prove some basic properties of the previously defined objects. For notation used in this chapter and the following thesis we refer to Appendix B.

2.1. Domain and atomistic deformations

Let $S \subset \mathbb{R}^2$ be an open, bounded and connected Lipschitz domain. Let ε_n, h_n such that $\varepsilon_n, h_n \to 0$, where ε_n corresponds to the interatomic distance and h_n to the height. By $\nu_n \in \mathbb{N}$ we denote the number of layers in the x_3-direction, i.e. we have

$$h_n = (\nu_n - 1)\varepsilon_n.$$

Let $\Omega_n = S \times (0, h_n)$ and $\Lambda_n = \overline{\Omega_n} \cap \varepsilon_n \mathbb{Z}^3$. Let $z^1, \cdots, z^8$ be the corners of the unit cube centered at 0 and let

$$Z = \left(z^1, \cdots, z^8\right) = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

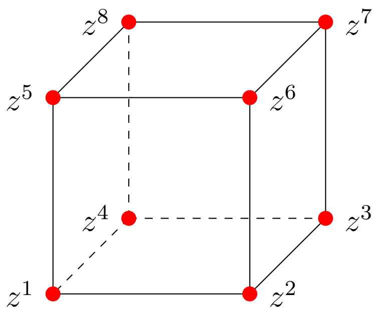

Figure 2.1: The unit cube Z.

By Λ_n' we denote the set of midpoints of lattice cells $x + [-\frac{\varepsilon_n}{2}, \frac{\varepsilon_n}{2}]^3$ contained in $\mathbb{R}^2 \times [0, h_n]$ for which at least one corner lies in Λ_n, i.e.

$$\Lambda_n' = \left(\bigcup_{x \in \Lambda_n} \left(x + \varepsilon_n \left\{ z^1, \cdots, z^8 \right\} \right) \right) \cap \left(\mathbb{R}^2 \times (0, h_n) \right).$$

The grid Λ_n' is also called the *dual grid* of Λ_n.

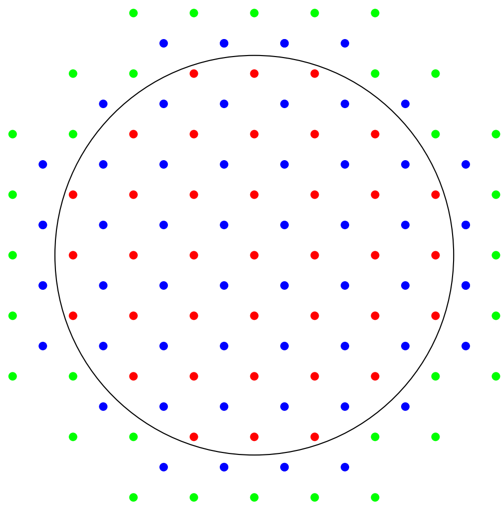

Figure 2.2: The red points correspond to Λ_n, the blue points are the midpoints of the cells. The green points are points to where deformations are extended.

For $x \in \Lambda_n'$ we set

$$Q_n(x) = x + \left(-\frac{\varepsilon_n}{2}, \frac{\varepsilon_n}{2} \right)^3.$$

We set $S_n = \{x \in S \colon \mathrm{dist}(x, \partial S) > \sqrt{2}\varepsilon_n\}$ and call $Q_n(x)$, $x \in \Lambda_n'$ an inner cell if $Q_n(x) \cap (S_n \times \mathbb{R}) \neq \emptyset$. In this case we write $x \in \Lambda_n'^{\circ}$. The corners of these cells are called interior atom positions $\Lambda_n^{\circ} = \Lambda_n'^{\circ} + \varepsilon_n \{z^1, \cdots, z^8\}$. We call $Q_n(x)$ a (lateral) *boundary cell* if

$$x \in \partial \Lambda_n' := \Lambda_n' \setminus \Lambda_n'^{\circ}.$$

Further we set

$$\bar{\Lambda}_n = \Lambda_n' + \varepsilon_n \{z^1, \cdots, z^8\}.$$

An atomistic deformation w is a mapping

$$w \colon \Lambda_n \to \mathbb{R}^3.$$

Later we will extend deformations to mappings $w \colon \bar{\Lambda}_n \to \mathbb{R}^3$. In the following atomistic deformations are also denoted by discrete deformations or lattice deformations.

2.1.1. The discrete gradient

For an (extended) atomistic deformation we define the associated discrete gradient

$$\bar{\nabla} w(x) = \left(\bar{\partial}_1 w(x), \cdots, \bar{\partial}_8 w(x) \right) \in \mathbb{R}^{3 \times 8}, \quad x \in \Lambda_n', \tag{2.1}$$

where

$$\bar{\partial}_i w(x) = \frac{1}{\varepsilon_n}\left(w(x+\varepsilon_n z^i) - \frac{1}{8}\sum_{j=1}^{8} w(x+\varepsilon_n z^j)\right). \tag{2.2}$$

In the following we will abbreviate

$$\langle w \rangle = \frac{1}{8}\sum_{j=1}^{8} w(x+\varepsilon_n z^j).$$

The discrete gradient can be viewed as a linear operator

$$\bar{\nabla}\colon \{\varphi\colon \bar{\Lambda}_n \to \mathbb{R}^3\} \to \{\phi : \Lambda_n' \to \mathbb{R}^{3\times 8}\},$$

$$\left[\left(\bar{\nabla}\varphi\right)(x)\right]_{ij} = \frac{1}{\varepsilon_n}\left(\varphi_i(x+\varepsilon_n z^j) - \frac{1}{8}\sum_{l=1}^{8}\varphi_i(x+\varepsilon_n z^l)\right).$$

If we consider the respective inner products on $l^2(\bar{\Lambda}_n;\mathbb{R}^3)$ and $l^2(\Lambda_n';\mathbb{R}^{3\times 8})$,

$$(f,g)_{l^2(\bar{\Lambda}_n;\mathbb{R}^3)} = \sum_{x\in\bar{\Lambda}_n} f(x)\cdot g(x),$$

$$(F,G)_{l^2(\Lambda_n';\mathbb{R}^{3\times 8})} = \sum_{\omega\in\Lambda_n'} F(\omega)\colon G(\omega),$$

it admits an adjoint $\bar{\nabla}^*\colon \{\phi : \Lambda_n' \to \mathbb{R}^{3\times 8}\} \to \{\varphi\colon \bar{\Lambda}_n \to \mathbb{R}^3\}$ given by the identity

$$\sum_{\omega\in\Lambda_n'} F(\omega)\colon \left(\bar{\nabla}\varphi\right)(\omega) = \sum_{x\in\bar{\Lambda}_n}\left(\bar{\nabla}^* F\right)(x)\cdot\varphi(x), \tag{2.3}$$

where $\varphi : \bar{\Lambda}_n \to \mathbb{R}^3$ and $F : \Lambda_n' \to \mathbb{R}^{3\times 8}$.

In some situations the discrete gradient $\bar{\nabla}y$ is difficult to handle because it lacks to have a product rule. Whenever needed this problem is circumvented by considering instead $\bar{D}y = (\bar{D}_1 y, \cdots, \bar{D}_8)$ with

$$\bar{D}_i y(x) = \frac{1}{\varepsilon_n}\left[y\left(\hat{x}+\varepsilon_n(a^i)\right) - y\left(\hat{x}\right)\right].$$

This is defined for $x \in \Lambda_n'$ with

$$\hat{x} = \left(\varepsilon_n\left\lfloor\frac{x_1}{\varepsilon_n}\right\rfloor, \varepsilon_n\left\lfloor\frac{x_2}{\varepsilon_n}\right\rfloor, \varepsilon_n\left\lfloor\frac{x_3}{\varepsilon_n}\right\rfloor\right),$$

such that $Q_n(x) = \hat{x} + (0,\varepsilon_n)^3$ and $A = (a^1,\cdots,a^8)$ defined by

$$A = Z + \frac{1}{2}(1,1,1)^T \otimes (1,1,1,1,1,1,1,1).$$

The a^j are the corners of the unit cube centered at $(\frac{1}{2},\frac{1}{2},\frac{1}{2})^T$. The following easy calculation shows the claimed product rule: For deformations y and w we have

$$\bar{D}_i\left(y(\hat{x})w(\hat{x})\right) = \frac{1}{\varepsilon_n}\left[y\left(\hat{x}+\varepsilon_n a^i\right) w\left(\hat{x}+\varepsilon_n a^i\right) - y(\hat{x})w(\hat{x})\right]$$

$$= \frac{1}{\varepsilon_n}\left[y\left(\hat{x}+\varepsilon_n a^i\right)w\left(\hat{x}+\varepsilon_n a^i\right) - y\left(\hat{x}\right)w\left(\hat{x}+\varepsilon_n a^i\right)\right]$$
$$+\frac{1}{\varepsilon_n}\left[y\left(\hat{x}\right)w\left(\hat{x}+\varepsilon_n a^i\right) - y(\hat{x})w(\hat{x})\right]$$
$$= \bar{D}_i y(x) w\left(\hat{x}+\varepsilon_n a^i\right) + y(\hat{x})\bar{D}_i w(\hat{x}).$$

The relation between those discrete gradients is given by

$$\bar{\partial}_i y(x) = \bar{D}_i y(\hat{x}) - \frac{1}{8}\sum_{j=1}^{8} \bar{D}_j y(\hat{x}) \tag{2.4}$$

and

$$\bar{D}_i y(\hat{x}) = \bar{\partial}_i y(x) - \bar{\partial}_1 y(x). \tag{2.5}$$

These equalities follow immediately from $\hat{x} = x + \varepsilon_n z^1$, $z^k = a^k + z^1$ and $y(x+\varepsilon_n z^k) = y(\hat{x}+\varepsilon a^k)$.

2.1.2. Extension

Later when we deal with interpolations of atomistic deformations $w : \Lambda_n \to \mathbb{R}^3$ it is necessary to define w on the corners of each cell $Q_n(x)$ with $x \in \Lambda_n'$. However there are cells $Q_n(x)$ with corners not contained in Λ_n. Thus we use an extension procedure from [Sch09] to obtain a deformation $w' : \bar{\Lambda}_n \to \mathbb{R}^3$. For details of this procedure we refer to Section 3.1 in [Sch09]. We want to mention the following lemma which can also be found in [BS22]. It states that rigidity and displacements of the boundary cells can be controlled by inner cells.

Lemma 2.1.1. *There are constants $c, C > 0$ such that for any $w : \Lambda_n \to \mathbb{R}^3$ and $R^* \in SO(3)$*

$$\sum_{x\in\partial\Lambda_n'} \left|\bar{\nabla} w'(x) - R^* Z\right|^2 \le c \sum_{x\in{\Lambda_n'}^\circ} \left|\bar{\nabla} w'(x) - R^* Z\right|^2$$

as well as

$$\sum_{x\in\partial\Lambda_n'} \operatorname{dist}^2(\bar{\nabla} w'(x), SO(3)Z) \le C \sum_{x\in{\Lambda_n'}^\circ} \operatorname{dist}^2(\bar{\nabla} w'(x), SO(3)Z).$$

2.1.3. Rescaling

It is convenient to work on a fixed domain. We achieve this by rescaling the reference domains Ω_n to $\Omega = H_n^{-1}\Omega_n = S \times (0,1)$ with

$$H_n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & h_n \end{pmatrix}.$$

Let $\tilde{\Lambda}_n = H_n^{-1}\Lambda_n$ and, for the extended maps, $\tilde{\bar{\Lambda}}_n = H_n^{-1}\bar{\Lambda}_n$. A deformation $w : \Lambda_n \to \mathbb{R}^3$ can be identified with a deformation $y : \tilde{\Lambda}_n \to \mathbb{R}^3$ via $y(x) = w(H_n x)$. The rescaled discrete gradients are given by

$$\bar{\partial}_i^n y(x) = y\left(x' + \varepsilon_n \left(z^i\right)', x_3 + \frac{\varepsilon_n}{h_n} z_3^i\right) - \frac{1}{8}\sum_{j=1}^{8} y\left(x' + \varepsilon_n \left(z^j\right)', x_3 + \frac{\varepsilon_n}{h_n} z_3^j\right) \tag{2.6}$$

and

$$\bar{D}_i^n y(x) = \frac{1}{\varepsilon_n}\left[y\left(\hat{x} + \varepsilon_n \begin{pmatrix} (a^i)' \\ h_n^{-1} a_3^i \end{pmatrix}\right) - y\left(\hat{x}\right)\right]. \tag{2.7}$$

2.1.4. Interpolation

To pass from discrete to continuum objects we need to interpolate in a suitable way. We shortly discuss the two interpolation schemes given in [BS22]. Let $w : \bar{\Lambda}_n \to \mathbb{R}^3$ be a lattice deformation. Let

$$\Omega_n^{\text{in}} := \left(\bigcup_{x \in \Lambda_n'^{\circ}} \overline{Q_n(x)}\right)^{\circ}$$
$$\Omega_n^{\text{out}} := \left(\bigcup_{x \in \Lambda_n'} \overline{Q_n(x)}\right)^{\circ}.$$

In order to extend w to $Q_n(x)$ we set

$$\tilde{w}(x) = \frac{1}{8}\sum_{j=1}^{8} w(x + \varepsilon_n z^i) \quad \text{for } x \in \Lambda_n'.$$

Let $v_1, \cdots, v_6$ the center points of the six faces $F_1, \cdots, F_6$ of $[-\frac{1}{2}, \frac{1}{2}]^3$. We define

$$\tilde{w}(x + \varepsilon_n v_k) := \frac{1}{4}\sum_j w(x + \varepsilon_n z^j),$$

where z_j is a corner of the face with center v^k. Then we interpolate linearly on each of the 24 simplexes

$$\operatorname{co}\left(x, x + \varepsilon_n v^k, x + \varepsilon_n z^i, x + \varepsilon_n z^j\right)$$

with $|z^i - z^j| = 1, \left|z^i - v^k\right| = \left|z^j - v^k\right| = \frac{1}{\sqrt{2}}$. This defines a piecewise affine mapping $\tilde{w} \in W^{1,2}(\Omega_n^{\text{out}}; \mathbb{R}^3)$ which satisfies

$$\tilde{w}(x) = \fint_{Q(x)} \tilde{w}(\xi)\; d\xi \tag{2.8}$$

and

$$\tilde{w}(x + \varepsilon_n z^i) = \fint_{x + \varepsilon_n F^k} \tilde{w}(\zeta)\; d\zeta \tag{2.9}$$

for every face $x + \varepsilon_n F^k$ of $Q(x)$.

For the second interpolation define

$$V_n := \left(\bigcup_{x \in \Lambda_n} \left(x + \left[-\frac{\varepsilon_n}{2}, \frac{\varepsilon_n}{2} \right]^3 \right) \right)^\circ,$$

$$V_n^{\text{out}} := \left(\bigcup_{x \in \bar{\Lambda}_n} \left(x + \left[-\frac{\varepsilon_n}{2}, \frac{\varepsilon_n}{2} \right]^3 \right) \right)^\circ,$$

$$V_n^{\text{in}} := \left(\bigcup_{x \in \bar{\Lambda}_n \,:\, x' \in S_n} \left(x + \left[-\frac{\varepsilon_n}{2}, \frac{\varepsilon_n}{2} \right]^3 \right) \right)^\circ,$$

and let

$$\bar{w}(\xi) = w(x) \quad \text{for every } \xi \in x + \left(-\frac{\varepsilon_n}{2}, \frac{\varepsilon_n}{2} \right)^3, \ x \in \bar{\Lambda}_n.$$

Here we obtain a piecewise constant function $\bar{w} \in L^2(V_n^{\text{out}}; \mathbb{R}^3)$. Both interpolations have their own advantages. The first interpolation allows for an application of the results in [FJM06] while for the second one the discrete gradient can be extended to an almost everywhere defined piecewise constant function on Ω_n^{out}. This works as follows: For $\xi \in Q_n(x)$, $x \in \Lambda_n'$, we have

$$\xi + \varepsilon_n z^i \in \left(x + \varepsilon_n z^i \right) + \left(-\frac{\varepsilon_n}{2}, \frac{\varepsilon_n}{2} \right)^3.$$

We set

$$\begin{aligned} \bar{\partial}_i \bar{w}(\xi) &:= \bar{w}(\xi + \varepsilon_n z^i) - \frac{1}{8} \sum_{j=1}^{8} \bar{w}(\xi + \varepsilon_n z^j) \\ &= w(x + \varepsilon_n z^i) - \frac{1}{8} \sum_{j=1}^{8} w(x + \varepsilon_n z^j) \end{aligned}$$

which results in

$$\bar{\nabla} \bar{w}(\xi) = \bar{\nabla} w(x) \quad \text{whenever } \xi \in Q_n(x), x \in \Lambda_n'.$$

Further we have

$$\bar{\partial}_i \bar{w}(\xi) = \bar{D}_i w(\hat{x}) - \frac{1}{8} \sum_{k=1}^{8} \bar{D}_k w(\hat{x}) \tag{2.10}$$

whenever $\xi \in x + \left(-\frac{\varepsilon_n}{2}, \frac{\varepsilon_n}{2} \right)^3$ by $\bar{\partial}_i \bar{w}(\xi) = \bar{\partial}_i w(x)$ and (2.4).

While the linear interpolation results in a quite regular function the piecewise constant interpolation is tailor-made to pass from sums to integral terms. The rescaled versions of the interpolated functions are defined as

$$\begin{aligned} \tilde{\tilde{y}}(x) &= \tilde{w}(H_n x) \quad \text{on } \tilde{\Omega}_n^{\text{out}} = H_n^{-1} \Omega_n^{\text{out}}, \\ \bar{\bar{y}}(x) &= \bar{w}(H_n x) \quad \text{on } \tilde{V}_n^{\text{out}} = H_n^{-1} V_n^{\text{out}}. \end{aligned}$$

For the sake of completeness we also introduce $\tilde{\Omega}_n^{\mathrm{in}} = H_n^{-1}\Omega_n^{\mathrm{in}}$ and $\tilde{V}_n = H_n^{-1}V_n$. Next we make precise what it means that a sequence of discrete deformations converges to a limiting deformation. To do this we need to choose a suitable function space for these limiting deformations. It turns out that for thin films $L^2(\Omega;\mathbb{R}^3)$ is natural, while for ultrathin films $L^2\left(S\times\left(-\frac{1}{2(\nu-1)},\frac{2\nu-1}{2(\nu-1)}\right)\right)$ is a good choice. We extend a function $y\in L^2(\Omega;\mathbb{R}^3)$ by 0 outside of Ω.

Definition 2.1.2. *Let $y\in L^2(\Omega;\mathbb{R}^3)$ if $\nu_n\to\infty$ or $y\in L^2\left(S\times\left(-\frac{1}{2(\nu-1)},\frac{2\nu-1}{2(\nu-1)}\right)\right)$. Then y is called a limiting deformation if there exists a sequence of mappings $y_n:\tilde{\Lambda}_n\to\mathbb{R}^3$ such that*

$$\frac{\varepsilon_n^3}{h_n}\sum_{x\in\tilde{\Lambda}_n}\left|y_n(x)-\fint_{\left(-\frac{\varepsilon_n}{2},\frac{\varepsilon_n}{2}\right)^2\times\left(-\frac{\varepsilon_n}{2h_n},\frac{\varepsilon_n}{2h_n}\right)}y(x+\xi)\,d\xi\right|^2\underset{n\to\infty}{\longrightarrow}0.$$

This definition is independent of the particular extension that is chosen for y. Since some of the arguments below are slightly easier if y is extended by 0 we stick with this extension. The goal of this section is to show that limiting deformations do not depend on the interpolation scheme. More precisely for thin films:

Proposition 2.1.3. *Let $\nu_n\to\infty$. Let y_n be a sequence of lattice deformations and $y\in L^2(\Omega;\mathbb{R}^3)$. Then the following are equivalent:*

i) $\frac{\varepsilon_n^3}{h_n}\sum_{x\in\tilde{\Lambda}_n}\left|y_n(x)-\fint_{\left(-\frac{\varepsilon_n}{2},\frac{\varepsilon_n}{2}\right)^2\times\left(-\frac{\varepsilon_n}{2h_n},\frac{\varepsilon_n}{2h_n}\right)}y(x+\xi)\,d\xi\right|^2\underset{n\to\infty}{\longrightarrow}0,$

ii) $\bar{\bar{y}}_n\to y$ *in* $L^2(\Omega;\mathbb{R}^3)$,

iii) $\tilde{\tilde{y}}_n\to y$ *in* $L^2(\Omega;\mathbb{R}^3)$.

For ultrathin films there is a similar result:

Proposition 2.1.4. *Let $\nu_n\equiv\nu\in\mathbb{N}$. Let $y\in L^2(\Omega)$ such that y is continuous in x_3 and affine in x_3 on the intervals $\left(\frac{i-1}{\nu-1},\frac{i}{\nu-1}\right)$, $i=1,\dots,\nu-1$, and define $y^*(x)=y\left(x',\frac{i}{\nu-1}\right)$ whenever $x_3\in\left(\frac{2i-1}{2(\nu-1)},\frac{2i+1}{2(\nu-1)}\right)$, $i=0,\dots,\nu-1$. Then the following are equivalent:*

i) $\frac{\varepsilon_n^3}{h_n}\sum_{x\in\tilde{\Lambda}_n}\left|y_n(x)-\fint_{\left(-\frac{\varepsilon_n}{2},\frac{\varepsilon_n}{2}\right)^2\times\left(-\frac{\varepsilon_n}{2h_n},\frac{\varepsilon_n}{2h_n}\right)}y^*(x+\xi)\,d\xi\right|^2\underset{n\to\infty}{\longrightarrow}0,$

ii) $\bar{\bar{y}}_n\to y^*$ *in* $L^2\left(S\times\left(-\frac{1}{2(\nu-1)},\frac{2\nu-1}{2(\nu-1)}\right)\right)$,

iii) $\tilde{\tilde{y}}_n\to y$ *in* $L^2(\Omega)$.

We defer the proofs of these propositions to the end of this section since we need a few preliminaries.

Proposition 2.1.5. *(i) There are constants $c,C>0$ such that for every $x\in\tilde{\Lambda}_n'$*

$$c\int_{\tilde{Q}_n(x)}\left|\tilde{\bar{w}}(\xi)\right|^2\,d\xi\le\frac{\varepsilon_n^3}{h_n}\sum_{j=1}^{8}\left|w(x'+\varepsilon_n(z^j)',x_3+h_n^{-1}\varepsilon_n z_3^j)\right|^2\le C\int_{\tilde{Q}_n(x)}\left|\tilde{\bar{w}}(\xi)\right|^2\,d\xi.$$

(ii) There are constants $c, C > 0$ such that for every $x \in \tilde{\Lambda}'_n$

$$c\left|\bar{\nabla}_n w(x)\right|^2 \leq \fint_{\tilde{Q}_n(x)} \left|\nabla_n \tilde{\tilde{w}}(\xi)\right|^2 \, d\xi \leq C\left|\bar{\nabla}_n w(x)\right|^2.$$

Proof. (i) We consider without loss of generality $Q = [0,1]^3$ and the finite dimensional space of functions defined on the corners of Q

$$X \coloneqq \{y : \{0,1\}^3 \to \mathbb{R}^3\}.$$

The norms

$$\|w\|_1^2 = \sum_{x \in \{0,1\}^3} |w(x)|^2, \quad \|w\|_2^2 = \int_Q |\tilde{w}(\xi)|^2 \, d\xi$$

are equivalent. Hence there are $c, C > 0$ such that

$$c \int_{[0,1]^3} |\tilde{w}(\xi)|^2 \, d\xi \leq \sum_{x \in \{0,1\}^3} |w(x)|^2 \leq C \int_{[0,1]^3} |\tilde{w}(\xi)|^2 \, d\xi.$$

The general statement follows from a scaling and translation argument.

(ii) This is Lemma 3.2 in [BS22].

□

Lemma 2.1.6. *Let $y : \tilde{V}_n \to \mathbb{R}^3$ be a mapping which is constant on every $x + \left(-\frac{\varepsilon_n}{2}, \frac{\varepsilon_n}{2}\right)^2 \times \left(-\frac{\varepsilon_n}{2h_n}, \frac{\varepsilon_n}{2h_n}\right)$, $x \in \tilde{\Lambda}_n$. Then*

$$\|y\|_{L^2(\tilde{V}_n \setminus \Omega)} \leq \|y\|_{L^2(\Omega)}.$$

Proof. Without loss of generality we assume $h_n = 1$. Sinde Ω is a bounded Lipschitz domain by compactness there are open sets $U_1, \cdots, U_m$, $U \coloneqq \bigcup_{j=1}^m U_j$, and $\gamma_1, \cdots, \gamma_m : \mathbb{R}^2 \to \mathbb{R}$ Lipschitz mappings with Lipschitz constants L_j such that, after possible rotation,

- $\partial\Omega \subset U$,
- $U_j \cap \Omega = \{(x', x_3) \in U_j : x_3 < \gamma_j(x')\}$.

Let $x \in \Lambda_n$ such that $\left(x + \left[-\frac{\varepsilon_n}{2}, \frac{\varepsilon_n}{2}\right]^3\right) \cap \Omega^c \neq \emptyset$. Without loss of generality we can assume that $\left(x + \left[-\frac{\varepsilon_n}{2}, \frac{\varepsilon_n}{2}\right]^3\right) \subset U$. We claim that there is an $\alpha > 0$ independent of $x \in \Lambda_n$ such that

$$\left|\left(x + \left[-\frac{\varepsilon_n}{2}, \frac{\varepsilon_n}{2}\right]^3\right) \cap \Omega^c\right| \leq \alpha \left|\left(x + \left[-\frac{\varepsilon_n}{2}, \frac{\varepsilon_n}{2}\right]^3\right) \cap \Omega\right|.$$

Indeed, let $L > \max\{L_1, \cdots, L_m\}$ and consider the cone

$$\mathcal{C} \coloneqq \left\{(y', y_3) \in x + \left[-\frac{\varepsilon_n}{2}, \frac{\varepsilon_n}{2}\right]^3 : (y_3 - x_3) < -L\|y' - x'\|\right\}.$$

Then $\mathcal{C} \subset \Omega$ and there is a $\theta = \theta(L)$ such that $|\mathcal{C}| = \theta\varepsilon_n^3$. Hence we obtain

$$\frac{\left|\left(x + \left[-\frac{\varepsilon_n}{2}, \frac{\varepsilon_n}{2}\right]^3\right) \cap \Omega^c\right|}{\left|\left(x + \left[-\frac{\varepsilon_n}{2}, \frac{\varepsilon_n}{2}\right]^3\right) \cap \Omega\right|} \leq \frac{\varepsilon_n^3}{|\mathcal{C}|} = \frac{1}{\theta}.$$

The rest is an easy estimate. □

Summing over the inner points of the grid or the dual grid, respectively, we immediately obtain the following corollary.

Corollary 2.1.7. *There are constants $C_1, C_2, C_3, C_4 > 0$ such that*

(i)

$$\|\tilde{\bar{y}}\|_{L^2(\Omega_n^{\text{in}})} \leq C\|\bar{\bar{y}}_n\|_{L^2(\Omega)}.$$

(ii)

$$\|\bar{\bar{y}}\|_{L^2(V_n^{in})} \leq C_2\|\tilde{\bar{y}}\|_{L^2(\Omega)}.$$

(iii)

$$\|\bar{\bar{y}}\|_{L^2(V_n^{in})} \leq C_3 \frac{\varepsilon_n^3}{h_n} \sum_{x\in\tilde{\Lambda}_n} |y(x)|^2 .$$

(iv)

$$\|\tilde{\bar{y}}\|_{L^2(\Omega_n^{in})} \leq C_4 \frac{\varepsilon_n^3}{h_n} \sum_{x\in\tilde{\Lambda}_n} |y(x)|^2 .$$

Proof. This follows immediately from Proposition 2.1.5 and Lemma 2.1.6 by summing over the inner points of the grid or the dual grid, respectively. □

For $(x', x_3) \in \varepsilon_n\mathbb{Z}^2 \times \frac{\varepsilon_n}{h_n}\mathbb{Z}$ and $f \in L^2(\mathbb{R}^3)$ define

$$P_n f(\xi) = \fint_{x+\left(-\frac{\varepsilon_n}{2},\frac{\varepsilon_n}{2}\right)^2\times\left(-\frac{\varepsilon_n}{2h_n},\frac{\varepsilon_n}{2h_n}\right)} f(\omega)\, d\omega$$

for $\xi \in x + \left(-\frac{\varepsilon_n}{2},\frac{\varepsilon_n}{2}\right)^2 \times \left(-\frac{\varepsilon_n}{2h_n},\frac{\varepsilon_n}{2h_n}\right)$. It is $P_n \in L(L^2(\mathbb{R}^3))$ and $P_n f \to f$ in $L^2(\mathbb{R}^3)$ for every $f \in L^2(\mathbb{R}^3)$ if $\nu_n \to \infty$. Moreover it holds that

$$\|P_n f\|_{L^2(\mathbb{R}^3)} \leq \|f\|_{L^2(\mathbb{R}^3)} .$$

Also consider the same operator acting only on the in-plane variables, i.e. for $f \in L^2(\mathbb{R}^2)$ and $x' \in \varepsilon_n\mathbb{Z}^2$ let

$$P_n' f(\xi) = \fint_{x'+\left(-\frac{\varepsilon_n}{2},\frac{\varepsilon_n}{2}\right)^2} f(\omega)\, d\omega \quad \text{whenever} \quad \xi \in x' + \left(-\frac{\varepsilon_n}{2},\frac{\varepsilon_n}{2}\right)^2 .$$

We also need to control the behavior close to the boundary. For this we need to use the properties of the extension procedure given in [Sch09], [BS22]. For a boundary cell $x \in \partial\tilde{\Lambda}_n'$ we denote by $\mathcal{F}(x)$ the midpoints of the neighboring cells of $\tilde{Q}_n(x)$, where a map $y : \tilde{\Lambda}_n \to \mathbb{R}^3$ is already defined. By $\mathcal{B}(x)$ denote the corners of the neighboring cells of $\tilde{Q}_n(x)$, where y is already defined. In particular, c.f. Lemma 3.1 in [Sch09], we have

$$\left|\bar{\nabla} y(x) - Z\right|^2 \leq C \sum_{\omega\in\mathcal{F}(x)} \left|\bar{\nabla} y(\omega) - Z\right|^2 . \tag{2.11}$$

Lemma 2.1.8. *(i) Let $x \in \partial\tilde{\Lambda}_n'$ such that $y(x' + \varepsilon(z^i)', x_3 + \frac{\varepsilon_n}{h_n} z_3^i) \in \bar{\tilde{\Lambda}}_n \setminus \tilde{\Lambda}_n$. Then*

$$\left|y(x' + \varepsilon(z^i)', x_3 + \frac{\varepsilon_n}{h_n} z_3^i)\right|^2 \leq C \sum_{\eta \in \mathcal{B}(x)} |y(\eta)|^2 + C\varepsilon_n^2.$$

(ii) $\sum_{x \in \partial\tilde{\Lambda}_n'} \int_{\tilde{Q}_n(x)} \left|\tilde{\tilde{y}}_n\right|^2 \, d\xi \leq C \int_\Omega \left|\tilde{\tilde{y}}(\xi)\right|^2 \, d\xi + C\varepsilon_n^3$.

(iii) $\sum_{x \in \bar{\tilde{\Lambda}}_n \setminus \tilde{\Lambda}_n} \int_{\tilde{Q}_n(x)} |\bar{\bar{y}}|^2 \, d\xi \leq \int_\Omega |\bar{\bar{y}}|^2 \, d\xi + C\varepsilon_n^3$.

Proof. (i) Without loss of generality let $h_n = 1$. Let $x + \varepsilon z^j \in \Lambda_n$. Then, using (2.11) we obtain

$$\begin{aligned}
&\left|w(x + \varepsilon z^i)\right|^2 \\
&\leq C\left|w(x + \varepsilon_n z^i) - \langle y \rangle\right|^2 + C\left|w(x + \varepsilon_n z^j) - \langle y \rangle\right|^2 + C\left|w(x + \varepsilon z^j)\right|^2 \\
&\leq C\varepsilon^2 \left|\bar{\nabla} y(x) - Z\right|^2 + C\varepsilon_n^2 |Z|^2 + \left|w(x + \varepsilon_n z^j)\right|^2 \\
&\leq C\varepsilon_n^2 \sum_{\omega \in \mathcal{F}(x)} \left|\bar{\nabla} y(\omega) - Z\right|^2 + C\varepsilon_n^2 |Z|^2 + \left|w(x + \varepsilon_n z^j)\right|^2 \\
&\leq C \sum_{\omega \in \mathcal{B}(x)} |y(\omega)|^2 + C\varepsilon_n^2
\end{aligned}$$

(ii) Note that the number of elements in $\mathcal{B}(x)$ and $\mathcal{F}(x)$ is finite and bounded independently of n. Then, using Proposition 2.1.5 part (i) and the fact that the number of boundary cells is proportional to $h_n\varepsilon_n^{-2}$, the claim follows from (i) summing over all boundary cells.

(iii) This is proven similarly as (ii).

□

Corollary 2.1.9. *For a sequence of atomistic deformations $y_n : \tilde{\Lambda}_n \to \mathbb{R}^3$ Lemma 2.1.8 shows, given that the extension procedure of [Sch09] is applied, that the following conditions are equivalent.*

(i) $\tilde{\tilde{y}}_n$ *is bounded in* $L^2(\Omega)$*,*

(ii) $\bar{\bar{y}}_n$ *is bounded in* X*,*

(iii) $\sup_{n \in \mathbb{N}} \frac{\varepsilon_n^3}{h_n} \sum_{x \in \bar{\tilde{\Lambda}}_n} |y_n(x)|^2 < \infty$.

where $X = L^2(\Omega)$ if $\nu_n \to \infty$ and $X = L^2(S \times (-\frac{1}{2(\nu-1)}, \frac{2\nu-1}{2(\nu-1)}))$ if $\nu_n \equiv \nu \in \mathbb{N}$.

Lemma 2.1.10. *Let $\nu_n \to \infty$ and let $y \in L^2(\Omega)$ (extended by 0 outside of Ω).*

i) There is a $C > 0$ such that

$$\|\widetilde{\widetilde{P_n y}}\|_{L^2(\Omega)} \leq C \, \|y\|_{L^2(\Omega)}.$$

ii) $\widetilde{\widetilde{P_n y}} \to y$ *in* $L^2(\Omega)$.

Let $\nu_n \equiv \nu \in \mathbb{N}$ and let $y \in L^2(\Omega)$ (extended by 0 outside of Ω) which is continuous in x_3 and affine on the intervals $\left(\frac{i-1}{\nu-1}, \frac{i}{\nu-1}\right)$, $i = 1, \dots, \nu - 1$.

iii) There is a $C > 0$ such that

$$\|\widetilde{\widetilde{\widetilde{P_n y^*}}}\|_{L^2(\Omega)} \leq C \, \|y\|_{L^2(\Omega)} \, .$$

iv) $\widetilde{\widetilde{\widetilde{P_n y^*}}} \to y$ in $L^2(\Omega)$.

Proof. (i) Applying P_n to a function $y \in L^2(\Omega)$ defines a mapping $P_n y : \tilde{\Lambda}_n \to \mathbb{R}^3$. By Jensen's inequality there is a $C > 0$ such that for every $\xi \in \tilde{Q}_n(x), \; x \in \tilde{\Lambda}'_n$,

$$\left|\widetilde{\widetilde{P_n y}}(\xi)\right|^2 \leq C \sum_{j=1}^{8} \left|P_n y\left(x' + \varepsilon_n (z^i)', x_3 + h_n^{-1} \varepsilon_n z_3\right)\right|^2 .$$

Since y is extended by 0 outside of Ω we obtain

$$\int_\Omega \left|\widetilde{\widetilde{P_n y}}(\xi)\right|^2 \, d\xi \leq \sum_{x \in \tilde{\Lambda}'_n} \int_{\tilde{Q}_n(x)} \left|\widetilde{\widetilde{P_n y}}(\xi)\right|^2 \, d\xi$$

$$\leq C \frac{\varepsilon_n^3}{h_n} \sum_{x \in \tilde{\Lambda}_n} |P_n y(x)|^2 \leq C \, \|P_n y\|^2_{L^2(\mathbb{R}^3)} \leq C \, \|y\|^2_{L^2(\Omega)} \, .$$

(ii) First let $y \in C_c^\infty(\Omega)$. In particular y is uniformly continuous and therefore both, $P_n y$ and $\widetilde{\widetilde{P_n y}}$, converge uniformly to y. Let $y \in L^2(\Omega)$ and $\varepsilon > 0$. Choose $y_\varepsilon \in C_c^\infty(\Omega)$ such that $\|y - y_\varepsilon\|_{L^2(\Omega)} < \varepsilon$. From (i) we get

$$\begin{aligned}
&\|\widetilde{\widetilde{P_n y}} - y\|_{L^2(\Omega)} \\
&\leq \|\widetilde{\widetilde{P_n y}} - \widetilde{\widetilde{P_n y_\varepsilon}}\|_{L^2(\Omega)} + \|\widetilde{\widetilde{P_n y_\varepsilon}} - y_\varepsilon\|_{L^2(\Omega)} + \|y_\varepsilon - y\|_{L^2(\Omega)} \\
&\leq C\|y - y_\varepsilon\|_{L^2(\Omega)} + \|\widetilde{\widetilde{P_n y_\varepsilon}} - y_\varepsilon\|_{L^2(\Omega)} \\
&\leq \varepsilon + \|\widetilde{\widetilde{P_n y_\varepsilon}} - y_\varepsilon\|_{L^2(\Omega)}.
\end{aligned}$$

The last term tends to 0 since y_ε is smooth.

(iii) Literally as in (i) we get

$$\|\widetilde{\widetilde{\widetilde{P_n y^*}}}\|_{L^2(\Omega)} \leq C \, \|y^*\|_{L^2\left(S \times \left(-\frac{1}{2(\nu-1)}, \frac{2\nu-1}{2(\nu-1)}\right)\right)} \, .$$

Since we assume that y is continuous in x_3 and piecewise affine on the intervals $\left(\frac{i-1}{\nu-1}, \frac{i}{\nu-1}\right)$ there is a $C > 0$ (due to norm equivalence) such that for every $x' \in S$

$$\sum_{i=0}^{\nu-1} \left|y\left(x', \frac{i}{\nu - 1}\right)\right|^2 \leq C \int_0^1 |y(x', x_3)|^2 \, dx_3.$$

Thus we even have

$$\|\widetilde{\widetilde{\widetilde{P_n y^*}}}\|_{L^2(\Omega)} \leq C \, \|y^*\|_{L^2\left(S \times \left(-\frac{1}{2(\nu-1)}, \frac{2\nu-1}{2(\nu-1)}\right)\right)} \leq C \, \|y\|_{L^2(\Omega)} \, .$$

(iv) It is sufficient to show that $\widetilde{\widetilde{P_n y^*}} \to y$ in $L^2\left(S \times \left(\frac{i-1}{\nu-1}, \frac{i}{\nu-1}\right)\right)$. First we assume that the mappings $x' \mapsto y\left(x', \frac{i}{\nu-1}\right)$, $i = 0, \ldots, \nu-1$, are uniformly continuous with compact support in S and denote by ω_i their respective moduli of continuity. Let $x \in \tilde{\Lambda}'_n$ with $x_3 = \frac{2i-1}{2(\nu-1)}$ for some $i \in \{1, \ldots, \nu-1\}$. Let $q^1, \ldots, q^4$ be the corners of the bottom layer of $\tilde{Q}_n(x)$ and $q^5, \ldots, q^8$ be the corners of the top layer of $\tilde{Q}_n(x)$. Define $\Phi_n : \{q^1, \ldots, q^8\} \to \mathbb{R}^3$ by

$$\Phi_n(q^i) = \begin{cases} \fint_{x'+\left(-\frac{\varepsilon_n}{\varepsilon_n}, \frac{\varepsilon_n}{\varepsilon_n}\right)^2} y(\xi', \frac{i-1}{\nu-1})\, d\xi', & 1 \le i \le 4, \\ \fint_{x'+\left(-\frac{\varepsilon_n}{\varepsilon_n}, \frac{\varepsilon_n}{\varepsilon_n}\right)^2} y(\xi', \frac{i}{\nu-1})\, d\xi', & 5 \le i \le 8. \end{cases}$$

Then, for a suitable constant $C > 0$,

$$\begin{aligned} &\int_{S \times \left(\frac{i-1}{\nu-1}, \frac{i}{\nu-1}\right)} |\widetilde{\widetilde{P_n y^*}} - y|^2 \, d\xi \\ &\le \sum_{\substack{x \in \tilde{\Lambda}'_n \\ x_3 = \frac{2i-1}{2(\nu-1)}}} \int_{\tilde{Q}_n(x)} |\widetilde{\widetilde{P_n y^*}} - y|^2 \, d\xi \\ &\le C \sum_{\substack{x \in \tilde{\Lambda}'_n \\ x_3 = \frac{2i-1}{2(\nu-1)}}} \int_{\tilde{Q}_n(x)} |\widetilde{\widetilde{P_n y^*}} - \tilde{\tilde{\Phi}}_n|^2 \, d\xi + C \sum_{\substack{x \in \tilde{\Lambda}'_n \\ x_3 = \frac{2i-1}{2(\nu-1)}}} \int_{\tilde{Q}_n(x)} |\tilde{\tilde{\Phi}}_n - y|^2 \, d\xi \\ &=: (I) + (II). \end{aligned}$$

For term (I) we use that for every $j \in \{1, \ldots, 8\}$ there is an $l \in \{1, \ldots, 8\}$ such that $q^j = (x' + \varepsilon(z^l)', x_3 + h_n^{-1}\varepsilon_n z_3^l)$. Also note that $q_3^j \in \{\frac{i-1}{\nu-1}, \frac{i}{\nu-1}\}$. Thus for every $j \in \{1, \ldots, 8\}$

$$\begin{aligned} \left|P_n y^*(q^j) - \Phi_n(q^j)\right| &\le \fint_{x'+\left(-\frac{\varepsilon_n}{2}, \frac{\varepsilon_n}{2}\right)^2} \left|y(\xi' + \varepsilon_n(z^l)', q_3^j) - y(\xi', q_3^j)\right|^2 \, d\xi' \\ &\le \omega_{i-1}(\varepsilon_n) + \omega_i(\varepsilon_n). \end{aligned}$$

By Proposition 2.1.5 we obtain

$$\begin{aligned} (I) &\le \frac{\varepsilon_n^2}{\nu - 1} \sum_{\substack{x \in \tilde{\Lambda}'_n \\ x_3 = \frac{2i-1}{2(\nu-1)}}} \sum_{j=1}^{8} \left| (P_n y^* - \Phi_n) \left(x' + \varepsilon_n (z^l)', x_3 + \frac{\varepsilon_n}{h_n} z_3^l \right) \right|^2 \\ &\le C \left(\omega_{i-1}(\varepsilon_n) + \omega_i(\varepsilon_n) \right). \end{aligned} \tag{2.12}$$

For term (II) on the one hand we use that $\tilde{\tilde{\Phi}}_n$ and y are affine in the x_3 direction on $\left(\frac{i-1}{\nu-1}, \frac{i}{\nu-1}\right)$. On the other hand, for $j \in \{\frac{i-1}{\nu-1}, \frac{i}{\nu-1}\}$ and $\xi \in \tilde{Q}_n(x)$, we have $\tilde{\tilde{\Phi}}_n(\xi', j) = P'_n y(\xi', j)$. Thus we get

$$(II) \le C \sum_{\substack{x \in \tilde{\Lambda}'_n \\ x_3 = \frac{2i-1}{2(\nu-1)}}} \int_{\tilde{Q}_n(x)} \left| \tilde{\tilde{\Phi}}_n \left(\xi', \frac{i-1}{\nu-1} \right) - y \left(\xi', \frac{i-1}{\nu-1} \right) \right|^2 \, d\xi$$

$$+C \sum_{\substack{x\in\tilde{\Lambda}'_n\\ x_3=\frac{2i-1}{2(\nu-1)}}} \int_{\tilde{Q}_n(x)} \left|\tilde{\tilde{\Phi}}_n\left(\xi',\frac{i}{\nu-1}\right)-y\left(\xi',\frac{i}{\nu-1}\right)\right|^2\,d\xi$$

$$\leq \frac{C}{\nu-1}\int_S \left|P'_n y\left(\xi',\frac{i-1}{\nu-1}\right)-y\left(\xi',\frac{i-1}{\nu-1}\right)\right|^2\,d\xi'$$

$$+\frac{C}{\nu-1}\int_S \left|P'_n y\left(\xi',\frac{i}{\nu-1}\right)-y\left(\xi',\frac{i}{\nu-1}\right)\right|^2\,d\xi' \to 0 \qquad (2.13)$$

as $n\to\infty$. Combining (2.12) and (2.13) for every layer shows $\widetilde{\widetilde{\widetilde{P_n y^*}}}\to y$ in $L^2(\Omega)$ if y is uniformly continuous. For an arbitrary y one can argue like in (ii) with usage of (iii).

□

Now we have all the ingredients to prove Proposition 2.1.3 and Proposition 2.1.4.

Proof of Proposition 2.1.3. Assume that

$$\frac{\varepsilon_n^3}{h_n}\sum_{x\in\Lambda_n}\left|y_n(x)-\fint_{\left(-\frac{\varepsilon_n}{2},\frac{\varepsilon_n}{2}\right)^2\times\left(-\frac{\varepsilon_n}{2h_n},\frac{\varepsilon_n}{2h_n}\right)} y(x+\xi)\,d\xi\right|^2 \underset{n\to\infty}{\longrightarrow} 0$$

and consider the mappings

$$\tilde{\Lambda}_n\to\mathbb{R}^3,\quad x\mapsto y_n(x)-P_n y(x).$$

Extending this map suitably to $\tilde{\tilde{\Lambda}}_n$ shows that $\bar{\bar{y}}_n-\overline{\overline{P_n y}}$ is bounded in $L^2(\Omega)$. By Corollary 2.1.7 we have

$$\left\|\chi_{V_n^{\rm in}}(\bar{\bar{y}}_n-\overline{\overline{P_n y}})\right\|^2_{L^2(\Omega)} \leq C\frac{\varepsilon_n^3}{h_n}\sum_{x\in\tilde{\Lambda}_n}\left|y_n(x)-\fint_{\left(-\frac{\varepsilon_n}{2},\frac{\varepsilon_n}{2}\right)^2\times\left(-\frac{\varepsilon_n}{2h_n},\frac{\varepsilon_n}{2h_n}\right)} y(x+\xi)\,d\xi\right|^2 \underset{n\to\infty}{\longrightarrow} 0.$$

By Lemma A.1.8 we get $\|\bar{\bar{y}}_n-\overline{\overline{P_n y}}\|^2_{L^2(\Omega)}\to 0$ and, due to $\overline{\overline{P_n y}}=P_n y\to y$ in $L^2(\Omega)$, it follows that $\bar{\bar{y}}_n\to y$ in $L^2(\Omega)$.

Now suppose that $\bar{\bar{y}}_n\to y$ in $L^2(\Omega)$. Then

$$\frac{\varepsilon_n^3}{h_n}\sum_{x\in\tilde{\Lambda}_n}\left|y_n(x)-\fint_{\left(-\frac{\varepsilon_n}{2},\frac{\varepsilon_n}{2}\right)^2\times\left(-\frac{\varepsilon_n}{2h_n},\frac{\varepsilon_n}{2h_n}\right)} y(x+\xi)\,d\xi\right|^2$$

$$=\sum_{x\in\tilde{\Lambda}_n}\|\bar{\bar{y}}_n-P_n y\|^2_{L^2\left(x+\left(-\frac{\varepsilon_n}{2},\frac{\varepsilon_n}{2}\right)^2\times\left(-\frac{\varepsilon_n}{2h_n},\frac{\varepsilon_n}{2h_n}\right)\right)}$$

$$\leq C\,\|\bar{\bar{y}}_n-P_n y\|^2_{L^2(\Omega)} \underset{n\to\infty}{\longrightarrow} 0.$$

For the last estimate we have used that both, $\bar{\bar{y}}_n$ and $P_n y$ are constant on each $x+\left(-\frac{\varepsilon_n}{2},\frac{\varepsilon_n}{2}\right)^2\times\left(-\frac{\varepsilon_n}{2h_n},\frac{\varepsilon_n}{2h_n}\right)$, $x\in\tilde{\Lambda}_n$, i.e. we can estimate by Lemma 2.1.6

$$\|\bar{\bar{y}}_n-P_n y\|_{L^2\left(\left(x+\left(-\frac{\varepsilon_n}{2},\frac{\varepsilon_n}{2}\right)^2\times\left(-\frac{\varepsilon_n}{2h_n},\frac{\varepsilon_n}{2h_n}\right)\right)\cap\Omega^c\right)}$$

$$\leq \alpha \, \|\bar{\bar{y}}_n - P_n y\|_{L^2\left(\left(x+\left(-\frac{\varepsilon_n}{2},\frac{\varepsilon_n}{2}\right)^2\times\left(-\frac{\varepsilon_n}{2h_n},\frac{\varepsilon_n}{2h_n}\right)\right)\cap\Omega\right)}.$$

Further, by Corollary 2.1.7,

$$\|\chi_{\Omega_n^{\mathrm{in}}}(\tilde{\tilde{y}}_n - \widetilde{\widetilde{P_n y}})\|_{L^2(\Omega)} \leq C \, \|\bar{\bar{y}}_n - P_n y\|_{L^2(\Omega)} \underset{n\to\infty}{\longrightarrow} 0.$$

As $\chi_{\Omega_n^{\mathrm{in}}} \to 1$ boundedely in measure and both, $\tilde{\tilde{y}}_n$ and $\widetilde{\widetilde{P_n y}}$ are bounded in $L^2(\Omega)$ (c.f. Corollary 2.1.9), we get $\|\tilde{\tilde{y}}_n - \widetilde{\widetilde{P_n y}}\|_{L^2(\Omega)} \to 0$ by Lemma A.1.8. By Lemma 2.1.10 we obtain $\tilde{\tilde{y}}_n \to y$ in $L^2(\Omega)$.

Now suppose that $\tilde{\tilde{y}}_n \to y$ in $L^2(\Omega)$. Then

$$\begin{aligned} \frac{\varepsilon_n^3}{h_n} \sum_{x\in\tilde{\Lambda}_n} & |y_n(x) - P_n y(x)|^2 \\ &= \|\bar{\bar{y}}_n - P_n y\|^2_{L^2(\tilde{V}_n)} \\ &\leq \|\bar{\bar{y}}_n - P_n y\|^2_{L^2(\Omega)} \end{aligned} \tag{2.14}$$

by Lemma 2.1.6. To see that the last term tends to 0 we again use Corollary 2.1.7. Then

$$\left\|\chi_{V_n^{\mathrm{in}}}(\bar{\bar{y}}_n - P_n y)\right\|^2_{L^2(\Omega)} \leq \|\tilde{\tilde{y}}_n - \widetilde{\widetilde{P_n y}}\|_{L^2(\Omega)} \to 0$$

by Lemma 2.1.10. Since by Corollary 2.1.9 $\bar{\bar{y}}_n$ is bounded in $L^2(\Omega)$ we see that $\bar{\bar{y}}_n - P_n y \to 0$ in $L^2(\Omega)$. From (2.14) it follows that

$$\frac{\varepsilon_n^3}{h_n} \sum_{x\in\tilde{\Lambda}_n} |y_n(x) - P_n y(x)|^2 \to 0.$$

□

The proof of Proposition 2.1.4 is very similar, thus we do not give full details.

Proof of Proposition 2.1.4. First assume that

$$\frac{\varepsilon_n^3}{h_n} \sum_{x\in\tilde{\Lambda}_n} \Big| y_n(x) - \fint_{\left(-\frac{\varepsilon_n}{2},\frac{\varepsilon_n}{2}\right)^2\times\left(-\frac{\varepsilon_n}{2h_n},\frac{\varepsilon_n}{2h_n}\right)} y^*(x+\xi)\, d\xi \Big|^2 \underset{n\to\infty}{\longrightarrow} 0.$$

Since y^* is constant in the x_3 variable on the intervals $\left(\frac{2i-1}{2(\nu-1)}, \frac{2i+1}{2(\nu-1)}\right)$, $i = 0, \ldots, \nu-1$, this is equivalent to

$$\frac{\varepsilon_n^2}{\nu-1} \sum_{x'\in\bar{S}\cap\varepsilon_n\mathbb{Z}^2} \sum_{i=0}^{\nu-1} \Big| y_n\Big(x', \frac{i}{\nu-1}\Big) - \fint_{\left(-\frac{\varepsilon_n}{2},\frac{\varepsilon_n}{2}\right)^2} y^*\Big(x'+\xi', \frac{i}{\nu-1}\Big)\, d\xi' \Big|^2 \to 0. \tag{2.15}$$

We note that for every $i \in \{0, \ldots, \nu-1\}$

$$\Big\|\chi_{S_n}\big(\bar{\bar{y}}_n\big(\cdot, \frac{i}{\nu-1}\big) - P'_n y^*\big(\cdot, \frac{i}{\nu-1}\big)\big)\Big\|_{L^2(S)}$$

$$\leq \varepsilon_n^2 \sum_{x' \in \bar{S} \cap \varepsilon_n \mathbb{Z}^2} \left| y_n\big(x', \frac{i}{\nu - 1}\big) - \fint_{\left(-\frac{\varepsilon_n}{2}, \frac{\varepsilon_n}{2}\right)^2} y^*\big(x' + \xi', \frac{i}{\nu - 1}\big)\, d\xi' \right|^2 \to 0$$

by (2.15). Since $\bar{\bar{y}}_n$ is bounded in $L^2\big(S \times \big(-\frac{1}{2(\nu-1)}, \frac{2\nu-1}{2(\nu-1)}\big)\big)$ by Corollary 2.1.9 this yields $\bar{\bar{y}}_n \to y^*$ in $L^2\big(S \times \big(-\frac{1}{2(\nu-1)}, \frac{2\nu-1}{2(\nu-1)}\big)\big)$ by Lemma A.1.8.

On the contrary if $\bar{\bar{y}}_n \to y^*$ in $L^2\big(S \times \big(-\frac{1}{2(\nu-1)}, \frac{2\nu-1}{2(\nu-1)}\big)\big)$ we extend y by 0 outside of Ω to estimate like in the case $\nu_n \to \infty$

$$\begin{aligned} \frac{\varepsilon_n^3}{h_n} \sum_{x \in \tilde{\Lambda}_n} & \left| y_n(x) - \fint_{\left(-\frac{\varepsilon_n}{2}, \frac{\varepsilon_n}{2}\right)^2 \times \left(-\frac{\varepsilon_n}{2h_n}, \frac{\varepsilon_n}{2h_n}\right)} y^*(x + \xi)\, d\xi \right|^2 \\ & \leq \|\bar{\bar{y}}_n - P_n y^*\|_{L^2\left(S \times \left(-\frac{1}{2(\nu-1)}, \frac{2\nu-1}{2(\nu-1)}\right)\right)} \to 0 \end{aligned}$$

as $n \to \infty$.

If

$$\frac{\varepsilon_n^3}{h_n} \sum_{x \in \tilde{\Lambda}_n} \left| y_n(x) - \fint_{\left(-\frac{\varepsilon_n}{2}, \frac{\varepsilon_n}{2}\right)^2 \times \left(-\frac{\varepsilon_n}{2h_n}, \frac{\varepsilon_n}{2h_n}\right)} y^*(x + \xi)\, d\xi \right|^2 \xrightarrow[n \to \infty]{} 0$$

then also $\|\chi_{\Omega_n^{\text{in}}}(\tilde{\tilde{y}}_n - \widetilde{\widetilde{P_n y^*}})\|_{L^2(\Omega)} \to 0$ and therefore, due to the $L^2(\Omega)$-bound of $\tilde{\tilde{y}}_n$ also $\|\tilde{\tilde{y}}_n - \widetilde{\widetilde{P_n y^*}}\|_{L^2(\Omega)} \to 0$. By Lemma 2.1.10 this yiels $\tilde{\tilde{y}}_n \to y$ in $L^2(\Omega)$. For the remaining implication we have

$$\left\|\chi_{V_n^{\text{in}}}(\bar{\bar{y}}_n - P_n y^*)\right\|_{L^2\left(S \times \left(-\frac{1}{2(\nu-1)}, \frac{2\nu-1}{2(\nu-1)}\right)\right)} \leq C \|\tilde{\tilde{y}}_n - \widetilde{\widetilde{P_n y^*}}\|_{L^2(\Omega)} \to 0,$$

again by Lemma 2.1.10. □

2.2. The energy and the quadratic forms

2.2.1. The atomistic energy

For any matrix $F \in \mathbb{R}^{3\times 8}$ we define by $F^{(1)} := (F_{\cdot 1}, \cdots, F_{\cdot 4}) \in \mathbb{R}^{3\times 4}$ and $F^{(2)} := (F_{\cdot 5}, \cdots, F_{\cdot 8}) \in \mathbb{R}^{3\times 4}$ the matrices consisting of the first four respectively the last four columns of F.

Let $\vec{w}(x) = \frac{1}{\varepsilon_n} (w(x + \varepsilon_n z^1), \cdots, w(x + \varepsilon_n z^8)) \in \mathbb{R}^{3\times 8}$. We assume that the atomic interaction energy for a deformation can be written as

$$E_{\text{atom}}(w) = \sum_{x \in \Lambda_n'} W\left(x, \vec{w}(x)\right).$$

For $x \in \Lambda_n'$ close to the boundary of S it can happen that $x' + \varepsilon_n (z^i)'$ is not in S for some $i \in \{1, \ldots, 8\}$. To make E_{atom} well-defined $W(x, \cdot)$ should not depend on $x + \varepsilon_n z^i$ if $x' + \varepsilon_n (z^i)' \notin S$. If however this is not the case we assume that W is given by a homogeneous cell energy $W_{\text{cell}} : \mathbb{R}^{3\times 8} \to [0,\infty)$ together with homogeneous surface term $W_{\text{surf}} : \mathbb{R}^{3\times 4} \to [0,\infty)$. More precisely,

$$W\big(x,\vec{w}\big) = \begin{cases} W_{\text{cell}}\big(\vec{w}\big) & \text{if } \; x_3 \in \left(\frac{\varepsilon_n}{2}, h_n - \frac{\varepsilon_n}{2}\right), \\ W_{\text{cell}}\big(\vec{w}\big) + W_{\text{surf}}\big(\vec{w}^{(2)}\big) & \text{if } \; \nu_n \geq 3, \; x_3 = h_n - \frac{\varepsilon_n}{2,} \\ W_{\text{cell}}\big(\vec{w}\big) + W_{\text{surf}}\big(\vec{w}^{(1)}\big) & \text{if } \; \nu_n \geq 3, \; x_3 = \frac{\varepsilon_n}{2,} \\ W_{\text{cell}}\big(\vec{w}\big) + \sum_{i=1}^{2} W_{\text{surf}}\big(\vec{w}^{(i)}\big) & \text{if } \; \nu_n = 2, \; x_3 = \frac{\varepsilon_n}{2}. \end{cases} \tag{2.16}$$

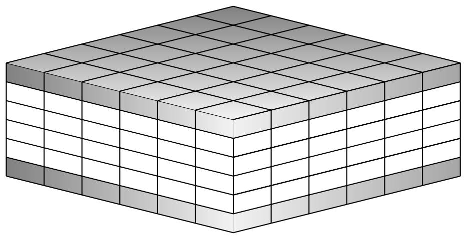

Figure 2.3: On the white cells there is only cell energy contribution. On the shaded cells additional surface terms need to be taken in account.

We assume that the cell energy W_{cell} satisfies the following properties:

$$W_{\text{cell}}(RA) = W_{\text{cell}}(A) \quad \text{for all } R \in SO(3), \tag{2.17}$$
$$W_{\text{cell}}(A + (c,\cdots,c)) = W_{\text{cell}}(A) \quad \text{for all } c \in \mathbb{R}^3, \tag{2.18}$$
$$W_{\text{cell}}(Z) = 0, \tag{2.19}$$
$$W_{\text{cell}} \text{ is } C^2 \text{ in a neighborhood of } Z. \tag{2.20}$$

Moreover we assume that there is a $c_0 > 0$ such that

$$W_{\text{cell}}(A) \geq c_0 \operatorname{dist}^2(A, SO(3)Z) \tag{2.21}$$

whenever $\sum_{i=1}^{8} A_{\cdot i} = 0$. Similarly the surface energy should satisfy

$$W_{\text{surf}}(RA) = W_{\text{surf}}(A) \quad \text{for all } R \in SO(3), \tag{2.22}$$
$$W_{\text{surf}}(A + (c,\cdots,c)) = W_{\text{surf}}(A) \quad \text{for all } c \in \mathbb{R}^3, \tag{2.23}$$
$$W_{\text{surf}}(Z^{(1)}) = W_{\text{surf}}(Z^{(2)}) = 0, \tag{2.24}$$
$$W_{\text{surf}} \text{ is } C^2 \text{ in a neighborhood of } Z^{(1)}. \tag{2.25}$$

Properties (2.17) and (2.22) are called frame indifference, this even implies $W(RZ) = W_{\text{cell}}(RZ) = W_{\text{surf}}(RZ^{(i)}) = 0$ for every $R \in SO(3)$. Further we assume that $W(x,\cdot)$ is C^2 in a neighborhood of Z for every $x \in \tilde{\Lambda}_n$. Together with (2.20) and (2.25) this allows for a Taylor expansion around Z or $Z^{(1)}$, respectively. Last we assume that $W(x,\cdot)$ is also invariant under rotations and translations for every $x \in \tilde{\Lambda}'_n$. Then we can replace $\vec{w}$ by $\bar{\nabla} w$ to write

$$E_{\text{atom}}(y) = \sum_{x \in \tilde{\Lambda}'_n} W(x, \bar{\nabla}_n y(x)),$$

where y is the rescaled version of w.

We need to make one last technical assumption. The derivatives of W_{cell} and W_{surf} should satisfy a linear growth condition

$$|DW_{\text{cell}}(A)| \leq C(1+|A|) \quad \text{for every } A \in \mathbb{R}^{3\times 8}, \tag{2.26}$$
$$|DW_{\text{surf}}(A)| \leq C(1+|A|) \quad \text{for every } A \in \mathbb{R}^{3\times 4}. \tag{2.27}$$

Further we assume that there is a constant $C > 0$ independent of $x \in \mathbb{R}^3$ such that

$$|D_F W(x, A)| \leq C(1+|A|). \tag{2.28}$$

In addition to the atomic interaction energy we consider an energy contribution from body forces $f_n : \tilde{\Lambda}_n \to \mathbb{R}^3$ depending only on the in-plane-variables and satisfying

$$\sum_{x\in\tilde{\Lambda}_n} f_n(x') = 0, \qquad \sum_{x\in\tilde{\Lambda}_n} f_n(x') \otimes x' = 0, \tag{2.29}$$

i.e. there is no first moment and no net force. Further we assume that the interpolations satisfy $h_n^{-3}\bar{f}_n \rightharpoonup f$ in $L^2(S)$, where $\bar{f}_n(\xi) = f_n(x)$ for every $\xi \in x + \left(-\frac{\varepsilon_n}{2}, \frac{\varepsilon_n}{2}\right)^2$. The body force part of the energy is given by

$$E_{\text{body}}(y) = \sum_{x\in\tilde{\Lambda}_n} f_n(x') \cdot y(x).$$

The overall energy per unit volume is given by

$$E_n(y) = \frac{\varepsilon_n^3}{h_n}\Bigg[\sum_{x\in\tilde{\Lambda}'_n} W(x, \bar{\nabla}_n(x)) + \sum_{x\in\tilde{\Lambda}_n} f_n(x')\cdot y(x)\Bigg]. \tag{2.30}$$

For $R \in SO(3)$ we define a corresponding energy with a 'rotated' force term

$$E_n^R(y) = \frac{\varepsilon_n^3}{h_n}\Bigg[\sum_{x\in\tilde{\Lambda}'_n} W(x, \bar{\nabla}_n(x)) + \sum_{x\in\tilde{\Lambda}_n} R^T f_n(x')\cdot y(x)\Bigg]. \tag{2.31}$$

We close this section by a simple consequence of frame indifference.

Lemma 2.2.1. *Let $W : \mathbb{R}^{n\times k} \to \mathbb{R}$ be a differentiable function which in addition is frame-indifferent. Then for every $R \in SO(n)$ and every $F \in \mathbb{R}^{n\times k}$*

$$DW(RF) = RDW(F).$$

Proof. Let $R \in SO(n)$ and $F, H \in \mathbb{R}^{n\times k}$. Then

$$\begin{aligned} DW(F)\colon H &= \frac{d}{dt}\Big|_{t=0} W(F+tH) \\ &= \frac{d}{dt}\Big|_{t=0} W(RF+tRH) \\ &= DW(RF)\colon RH \\ &= R^T DW(RF)\colon H \end{aligned}$$

and therefore $DW(RF)\colon H = RDW(F)\colon H$. □

2.2.2. The quadratic forms and their linearizations

For $A \in \mathbb{R}^{3\times 8}$ let
$$Q_{\text{cell}}(A) = D^2 W_{\text{cell}}(Z)[A, A]$$
with its linearization
$$\mathcal{L}A := \frac{1}{2} DQ_{\text{cell}}(A).$$
We associate a relaxed quadratic form on $\mathbb{R}^{3\times 8}$ given by
$$Q^{\text{rel}}_{\text{cell}}(A) := \min_{b\in\mathbb{R}^3} Q_{\text{cell}}(A + (b \otimes e_3)Z).$$

Since Q_{cell} is positive definite on $(\mathbb{R}^3 \otimes e_3)\, Z$ for every $A \in \mathbb{R}^{3\times 8}$ there is a unique $b(A) \in \mathbb{R}^3$ such that
$$Q^{\text{rel}}_{\text{cell}}\,(A) = Q_{\text{cell}}\,(A + (b(A) \otimes e_3)\, Z)\,.$$
This $b(A)$ is characterized by
$$0 = DQ_{\text{cell}}\,(A + (b(A) \otimes e_3)\, Z) : \ (c \otimes e_3)\, Z = 0$$
for every $c \in \mathbb{R}^3$, i.e. $DQ_{\text{cell}}\,(A + (b(A) \otimes e_3)\, Z) \perp (\mathbb{R}^3 \otimes e_3)\, Z$. Moreover, the map $A \mapsto b(A)$ is linear.

For $A \in \mathbb{R}^{2\times 2}$ let
$$Q_2(A) := Q^{\text{rel}}_{\text{cell}}\left(\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} Z\right)$$
and its linearization
$$\mathcal{L}_2 A := \frac{1}{2} DQ_2(A).$$
Analogously let
$$Q_{\text{surf}}(A) = D^2 W_{\text{surf}}(Z^{(1)})[A, A]$$
for $A \in \mathbb{R}^{3\times 4}$.

2.3. Some properties

We will give proofs of the following lemmas only for the cell energy. The proofs for the surface part are literally the same.

Lemma 2.3.1. *There is a neighborhood $\mathcal{U}$ of $SO(3)Z$ or $SO(3)Z^{(1)}$ respectively and a $C > 0$ such that for every $A \in \mathcal{U}$ it holds that*
$$W_{\text{cell}}\,(A) \le C \text{dist}^2\,(A, SO(3)Z)$$
and
$$W_{\text{surf}}\,(A) \le C \text{dist}^2\left(A, SO(3)Z^{(1)}\right).$$

Proof. Let $R \in SO(3)Z$ such that $|A - R| = \operatorname{dist}(A, SO(3)Z)$. By Taylor expansion we get

$$W_{\text{cell}}(A) = \frac{1}{2} D^2 W_{\text{cell}}(R)[A - R, A - R] + o(|A - R|^2)$$
$$\leq C\,|A - R|^2 = C \operatorname{dist}^2(A, SO(3)Z).$$

□

Lemma 2.3.2. *For $A \in \mathbb{R}^{3\times 3}_{\text{skew}}$ it holds that*

$$D^2 W_{\text{cell}}\,(Z)\,[AZ, AZ] = 0$$

as well as

$$D^2 W_{\text{surf}}\left(Z^{(1)}\right)\left[AZ^{(1)}, AZ^{(1)}\right] = 0.$$

Proof. For every $A \in \mathbb{R}^{3\times 3}$ we have

$$\operatorname{dist}(AZ, SO(3)Z) = \inf_{R \in SO(3)} |AZ - RZ|$$
$$\leq C \inf_{R \in SO(3)} |A - R| = C \operatorname{dist}(A, SO(3)).$$

If A is skew-symmetric then $\operatorname{dist}^2(Id + tA, SO(3)) \leq Ct^4$. Together with

$$W(Z + tAZ) = \frac{t^2}{2} D^2 W_{\text{cell}}(Z)[AZ, AZ] + o(t^2)$$

and Lemma 2.3.1 we get the inequality

$$\frac{t^2}{2} D^2 W_{\text{cell}}(Z)[AZ, AZ] + o(t^2)$$
$$= W(Z + tAZ) = W((Id + tA)Z)$$
$$\leq C \operatorname{dist}^2((Id + tA)Z, SO(3)Z)$$
$$\leq C \operatorname{dist}^2(Id + tA, SO(3)) \leq Ct^4.$$

Letting $t \to 0$ yields the claim. □

Corollary 2.3.3. *Let $A \in \mathbb{R}^{2\times 2}_{\text{skew}}$. Then $Q^{\text{rel}}_{\text{cell}}\left(\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} Z\right) = Q_2(A) = 0$.*

Proof.

$$0 \leq Q_2(A) = Q^{\text{rel}}_{\text{cell}}\left(\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} Z\right) \leq Q_{\text{cell}}\left(\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} Z\right) = 0.$$

□

There are two further corollaries:

Corollary 2.3.4. *For $A \in \mathbb{R}^{3\times 3}_{\text{skew}}$ it holds that $DQ_{\text{cell}}(AZ) = 0$ and $DQ_{\text{surf}}(AZ^{(1)}) = 0$.*

Proof. Due to $Q_{\text{cell}}(B) \geq 0$ for every $B \in \mathbb{R}^{3\times 8}$ every $AZ \in \mathbb{R}^{3\times 3}_{\text{skew}} Z$ is a local minimizer of Q_{cell}. □

Corollary 2.3.5. *For every $A \in \mathbb{R}^{3\times 8}$ and every $B \in \mathbb{R}^{3\times 3}_{\text{skew}}$*

$$Q_{\text{cell}}(A + BZ) = Q_{\text{cell}}(A), \tag{2.32}$$

$$Q_{\text{surf}}(A + BZ^{(1)}) = Q_{\text{surf}}(A). \tag{2.33}$$

In particular we have $Q^{\text{rel}}_{\text{cell}}(A) = \min_{b\in\mathbb{R}^3} Q_{\text{cell}}(A + \text{sym}(b \otimes e_3)Z)$.

Chapter 3

The stationary case

3.1. Introduction

The goal of this chapter is to prove a similar result as Theorem 1.1 in [MP08] which extends the Γ-convergence result of [FJM06]. Of course we seek for a convergence result of discrete equilibrium points. We begin by briefly reviewing the work of [BS22].

3.1.1. A review of the Γ-convergence result

Braun and Schmidt derived the von-Kármán theory from the nonlinear three-dimensional atomistic model introduced in the chapter before. In contrary to the continuum results of Friesecke, James and Müller the problem here is twofold: Both, the interatomic distance ε_n and the height of the plate h_n tend to 0 as $n \to \infty$. They studied two different regimes. For thin films, i.e. the number of layers $\nu_n = \frac{h_n}{\varepsilon_n} + 1 \to \infty$ (or equivalently $\frac{\epsilon_n}{h_n} \to 0$), they obtained the von-Kármán functional as a Γ-limit. In short the result is as follows, details are given below:

$$h_n^{-4} E_n \xrightarrow{\Gamma} E_{vK}, \tag{3.1}$$

where the von-Kármán functional $E_{vK} : H^1(S;\mathbb{R}^2) \times H^2(S) \times SO(3) \to \mathbb{R}$ is given by

$$\begin{aligned} E_{vK}(u, v, R^*) = \int_S \frac{1}{2} Q_2 \left(\frac{1}{2} \left(\nabla u' + \nabla' u^T + \nabla' v \otimes \nabla' v \right) \right) \\ + \frac{1}{24} Q_2 \left(\nabla'^2 v \right) + f(x') \cdot v(x') R^* e_3 \, dx'. \end{aligned} \tag{3.2}$$

If the ratio $\frac{\varepsilon_n}{h_n}$ is constant we have

$$h_n^{-4} E_n \xrightarrow{\Gamma} E_{vK}^{(\nu)} \tag{3.3}$$

with

$$\begin{aligned} E_{vK}^{(\nu)}(u, v, R^*) = \int_S \frac{1}{2} Q_{\text{cell}}^{\text{rel}} \left(\begin{pmatrix} G_1(x') & 0 \\ 0 & 0 \end{pmatrix} Z + \frac{1}{2(\nu-1)} G_3(x') \right) \\ + \frac{\nu(\nu-2)}{24(\nu-1)^2} Q_2 \left(G_2(x') \right) \\ + \frac{1}{\nu-1} Q_{\text{surf}} \left(\begin{pmatrix} G_1(x') & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} + \frac{\partial_{12} v(x')}{2(\nu-1)} M^{(1)} \right) \end{aligned}$$

$$+\frac{1}{4(\nu-1)}Q_{\text{surf}}\left(\begin{pmatrix} G_2(x') & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)}\right) + \frac{\nu}{\nu-1} f(x')\cdot v(x')R^* e_3 \, dx'. \tag{3.4}$$

Here

$$G_1(x') = \frac{1}{2}\left(\nabla u' + \nabla' u^T + \nabla' v \otimes \nabla' v\right),$$
$$G_2(x') = -\nabla'^2 v(x')$$

and

$$G_3(x') = \begin{pmatrix} G_2(x') & 0 \\ 0 & 0 \end{pmatrix} Z_- + \partial_{12} v(x') M,$$

where

$$M = \frac{1}{2} e_3 \otimes (+1, -1, +1, -1, +1, -1, +1, -1)$$

and $Z_- = (-Z^{(1)}, Z^{(2)})$.

Let us make a few remarks at this point:

(i) The map u corresponds to the in-plane displacement whereas the map v corresponds to the out-of-plane displacement.

(ii) The parameter $R \in SO(3)$ can be understood as a normalization parameter of the force. It an be chosen as $R = Id$ if suitable boundary conditions are prescribed.

(iii) For $\nu_n \equiv \nu \in \mathbb{N}$ we see additional surface contributions. This can be expected if we look at the discrete interaction energy given in (2.30) together with (2.16).

(iv) The surface terms are of order $\frac{1}{\nu}$. Hence the two model commute formally in the sense that sending $\nu \to \infty$ in the finite layer model we end up with the usual von-Kármán functional.

3.1.2. Definition of equilibrium points

Formally calculating the respective variations in u and v of the limiting functionals leads to the following notion of a weak solution.

Definition 3.1.1 (Definition of distributional solution). *Let $\nu_n \to \infty$. We say $(u, v) \in H^1(S; \mathbb{R}^2) \times H^2(S)$ is a distributional solution of the Euler-Lagrange-equations of E_{vk} if the following equations are satisfied:*

(i) For every $\phi \in C_c^\infty(S)$ it holds that

$$0 = \int_S \mathcal{L}_2\left(\frac{1}{2}\left(\nabla' u + (\nabla' u)^T + \nabla' v \otimes \nabla' v\right)\right) : \nabla' v \otimes \nabla' \phi + \frac{1}{12}\mathcal{L}_2\left((\nabla')^2 v\right) : (\nabla')^2 \phi + f(x')\cdot \phi(x') R^* e_3 \, dx'. \tag{3.5}$$

(ii) For every $\Psi \in C_c^\infty(S;\mathbb{R}^2)$ it holds that

$$0 = \int_S \mathcal{L}_2 \left(\frac{1}{2} \left(\nabla' u + (\nabla' u)^T + \nabla' v \otimes \nabla' v \right) \right) : \nabla' \Psi \; dx'. \tag{3.6}$$

Let $\nu_n \equiv \nu \in \mathbb{N}$. We say $(u, v) \in H^1(S;\mathbb{R}^2) \times H^2(S)$ is a distributional solution of the Euler-Lagrange-equations of $E_{vk}^{(\nu)}$ if the following equations are satisfied:

(i) For every $\phi \in C_c^\infty(S)$ it holds that

$$\begin{aligned}
0 = &\int_S \frac{1}{2} DQ_{\text{cell}}^{\text{rel}} \left(\begin{pmatrix} G_1 & 0 \\ 0 & 0 \end{pmatrix} Z + \frac{1}{2(\nu-1)} G_3 \right) : \begin{pmatrix} \nabla' v \otimes \nabla' \phi & 0 \\ 0 & 0 \end{pmatrix} Z \; dx' \\
&- \int_S \frac{1}{4(\nu-1)} DQ_{\text{cell}}^{\text{rel}} \left(\begin{pmatrix} G_1 & 0 \\ 0 & 0 \end{pmatrix} Z + \frac{1}{2(\nu-1)} G_3 \right) : \begin{pmatrix} \nabla'^2 \phi & 0 \\ 0 & 0 \end{pmatrix} Z_- \; dx' \\
&+ \int_S \frac{1}{4(\nu-1)} DQ_{\text{cell}}^{\text{rel}} \left(\begin{pmatrix} G_1 & 0 \\ 0 & 0 \end{pmatrix} Z + \frac{1}{2(\nu-1)} G_3 \right) : \partial_{12} \phi M \; dx' \\
&- \int_S \frac{\nu(\nu-2)}{24(\nu-1)^2} DQ_{\text{cell}}^{\text{rel}} \left(\begin{pmatrix} G_2 & 0 \\ 0 & 0 \end{pmatrix} Z \right) : \begin{pmatrix} \nabla'^2 \phi & 0 \\ 0 & 0 \end{pmatrix} Z \; dx' \\
&+ \int_S \frac{1}{\nu-1} DQ_{\text{surf}} \left(\begin{pmatrix} G_1 & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} + \frac{\partial_{12} v}{2(\nu-1)} M^{(1)} \right) : \begin{pmatrix} \nabla' v \otimes \nabla' \phi & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} \; dx' \\
&+ \int_S \frac{1}{2(\nu-1)^2} DQ_{\text{surf}} \left(\begin{pmatrix} G_1 & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} + \frac{\partial_{12} v}{2(\nu-1)} M^{(1)} \right) : \partial_{12} \phi M^{(1)} \; dx' \\
&- \int_S \frac{1}{4(\nu-1)} DQ_{\text{surf}} \left(\begin{pmatrix} G_2 & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} \right) : \begin{pmatrix} \nabla'^2 \phi & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} \; dx' \\
&+ \int_S \frac{\nu}{\nu-1} f(x') \cdot \phi(x') R^* e_3 \; dx'.
\end{aligned} \tag{3.7}$$

(ii) For every $\Psi \in C_c^\infty(S;\mathbb{R}^2)$ it holds that

$$\begin{aligned}
0 = &\int_S \frac{1}{2} DQ_{\text{cell}}^{\text{rel}} \left(\begin{pmatrix} G_1 & 0 \\ 0 & 0 \end{pmatrix} Z + \frac{1}{2(\nu-1)} G_3 \right) : \begin{pmatrix} \nabla' \Psi & 0 \\ 0 & 0 \end{pmatrix} Z \; dx' \\
&+ \int_S \frac{1}{\nu-1} DQ_{\text{surf}} \left(\begin{pmatrix} G_1 & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} + \frac{\partial_{12} v(x')}{2(\nu-1)} M^{(1)} \right) : \begin{pmatrix} \nabla' \Psi & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} \; dx'.
\end{aligned} \tag{3.8}$$

Definition 3.1.2 (Definition of discrete stationary point)**.** *A mapping $y : \tilde{\Lambda}_n \to \mathbb{R}^3$ is a stationary point of E_n if for every $\varphi : \tilde{\Lambda}_n \to \mathbb{R}^3$ it holds that*

$$0 = \frac{\varepsilon_n^3}{h_n} \left[\sum_{x \in \tilde{\Lambda}_n'} D_F W \left(x, \bar{\nabla}_n y(x) \right) : \bar{\nabla}_n \varphi(x) + \sum_{x \in \tilde{\Lambda}_n} f_n(x') \cdot \varphi(x) \right]. \tag{3.9}$$

In [MP08] only stationary points were considered that satisfy clamped boundary conditions. As shown in [LM09], Lemma 13, in this case the rigidity result in [FJM06] holds for $R_n = Id$ and $c_n = 0$, i.e. no rotation or translation is needed. However for discrete deformations treating boundary conditions is a bit cumbersome yet we still need to apply the rigidity results. For a stationary point y_n the corresponding point $\hat{y}_n = R_n^T y_n - c_n$ that satisfies the rigidity estimates in general is not a stationary point of E_n anymore. This is the reason why we introduced the functional with the rotated force term.

Lemma 3.1.3. *Let y be a stationary point of E_n. Let $R \in SO(3)$, $c \in \mathbb{R}^3$. Then the deformation $\hat{y} = R^T y + c$ is a stationary point of E_n^R.*

Proof. Let φ be an admissible test function. Then

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} E_n^R(\hat{y} + t\varphi) &= \frac{\varepsilon_n^3}{h_n}\Big[\sum_{x\in\tilde{\Lambda}_n'} D_F W\left(x, \bar{\nabla}_n \hat{y}(x)\right) : \bar{\nabla}_n \varphi(x) + \sum_{x\in\tilde{\Lambda}_n} \varphi(x)\cdot R^T f_n(x')\Big] \\
&= \frac{\varepsilon_n^3}{h_n}\Big[\sum_{x\in\tilde{\Lambda}_n'} R^T D_F W\left(x, \bar{\nabla}_n y(x)\right) : \bar{\nabla}_n \varphi(x) + \sum_{x\in\tilde{\Lambda}_n} (R\varphi(x))\cdot f_n(x')\Big] \\
&= \frac{\varepsilon_n^3}{h_n}\Big[\sum_{x\in\tilde{\Lambda}_n'} D_F W(x, \bar{\nabla}_n y(x)) \colon \bar{\nabla}_n (R\varphi(x)) + \sum_{x\in\tilde{\Lambda}_n} (R\varphi(x))\cdot f_n(x')\Big] = 0.
\end{aligned}$$

We have used the invariance of W under rotations and translations as well as the fact that $R\varphi$ is also an admissible test function. □

3.2. The main theorem

Now we are able to state the main theorem of this section.

Theorem 3.2.1. *Let W_{cell} satisfy (2.17) - (2.21) and W_{surf} satisfy (2.22) - (2.25) as well as (2.26) and (2.27). Let $f_n : \tilde{\Lambda}_n \to \mathbb{R}$ depend only on the in-plane-variables x' and satisfy (2.29) as well as $h_n^{-3}\bar{f}_n \rightharpoonup f$ in $L^2(S)$.*

Let y_n be a sequence of stationary points of E_n with

$$E_n(y_n) \le C h_n^4.$$

Let $R_n^* \in SO(3), c_n \in \mathbb{R}^3$ such that the normalized interpolated maps $\tilde{y}_n = (R_n^*)^T \tilde{\tilde{y}}_n - c_n$ satisfy the estimates and compactness results of Theorem 3.2.5. Then $\hat{y}_n = (R_n^*)^T y_n - c_n$ is a sequence of stationary points of $E_n^{R_n^*}$. Up to a subsequence it holds that

$$u_n(x') = h_n^{-2}\int_0^1 (\tilde{y}_n'(x', x_3) - x')\ dx_3 \rightharpoonup u \text{ in } W^{1,2}(S), \tag{3.10}$$

$$v_n(x') = h_n^{-1}\int_0^1 (\tilde{y}_n)_3\ dx_3 \to v \text{ in } W^{1,2}(S),\ v \in W^{2,2}(S), \tag{3.11}$$

$$R_n^* \to R^* \text{ in } SO(3). \tag{3.12}$$

The limiting pair (u, v) is a solution to (3.5) and (3.6) in the case $\nu_n \to \infty$ or of (3.7) and (3.8) in the case $\nu_n \equiv \nu \in \mathbb{N}$, respectively.

3.2.1. Some preliminary results

In this section we recall some important results which we will make use of. This includes the famous rigidity result of Friesecke, James and Müller as well as some results of Braun and Schmidt which ensure the application of these continuum results in our situation.

We start with the very basic rigidity result:

Theorem 3.2.2 ([FJM02], Theorem 3.1)**.** *Let U be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. There exists a constant $C(U)$ with the following property: For each $v \in W^{1,2}(U;\mathbb{R}^n)$ there is an associated rotation $R \in SO(n)$ such that*

$$\|\nabla v - R\|_{L^2(U)} \leq C(U) \left\|\operatorname{dist}^2(\nabla v, SO(n))\right\|_{L^2(U)}.$$

Theorem 3.2.2 can be used to obtain approximations by rotations in thin domains $\Omega_h = S \times (-\frac{h}{2}, \frac{h}{2})$. As usual we rescale to a fixed domain $\Omega = S \times \left(-\frac{1}{2}, \frac{1}{2}\right)$ with the notation $\nabla_h = (\nabla', h^{-1}\partial_3)$.

Theorem 3.2.3 ([FJM06], Theorem 6)**.** *Suppose that $S \subset \mathbb{R}^2$ is a Lipschitz domain and $\Omega = S \times \left(-\frac{1}{2}, \frac{1}{2}\right)$. Let $y \in W^{1,2}(\Omega;\mathbb{R}^3)$ and*

$$E = \int_\Omega \operatorname{dist}^2(\nabla_h y, SO(3))\, dx.$$

Then there exist maps $R : S \to SO(3)$ and $\tilde{R} : S \to \mathbb{R}^{3\times 3}$, with $|\tilde{R}| \leq C$, $\tilde{R} \in W^{1,2}(S;\mathbb{R}^{3\times 3})$ such that

$$\|\nabla_h y - R\|^2_{L^2(\Omega)} \leq CE, \tag{3.13}$$

$$\|R - \tilde{R}\|^2_{L^2(S)} \leq CE, \tag{3.14}$$

$$\|\nabla \tilde{R}\|^2_{L^2(S)} \leq \frac{C}{h^2}E, \tag{3.15}$$

$$\|R - \tilde{R}\|^2_{L^\infty(S)} \leq \frac{C}{h^2}E. \tag{3.16}$$

Moreover, there exists a constant rotation $\bar{Q} \in SO(3)$ such that

$$\left\|\nabla_h y - \bar{Q}\right\|^2_{L^2(\Omega)} \leq \frac{C}{h^2}E \tag{3.17}$$

and

$$\left\|R - \bar{Q}\right\|^2_{L^p(S)} \leq \frac{C_p}{h^2}E. \tag{3.18}$$

Proposition 3.2.4 ([BS22], Proposition 1)**.** *Consider a sequence y_n with*

$$E_n(y_n) \leq Ch_n^4.$$

Then

$$0 \leq \int_\Omega \operatorname{dist}^2\left(\nabla_n \tilde{\tilde{y}}_n, SO(3)\right)\, dx \leq Ch_n^4,$$

where $\tilde{\tilde{y}}_n \in W^{1,2}(\Omega)$ is the piecewise affine interpolation of y_n.

This proposition shows that the discrete energy bound implies the same energy bound for the piecewise affine interpolations. In particular we are able to use the continuum rigidity results of Friesecke, James and Müller as summarized in [BS22, Theorem 4.2]:

Theorem 3.2.5. *[BS22, Theorem 4.2] Let* $\hat{y}_n \in W^{1,2}(\Omega)$ *with* $\int_\Omega \operatorname{dist}^2(\nabla_n \hat{y}_n, SO(3))\, dx \leq Ch_n^4$. *Then there are maps* $R_n \in H^1(S;\mathbb{R}^{3\times 3})$ *such that* $R_n(x') \in SO(3)$ *for every* $x' \in S$, $\tilde{R}_n : S \to \mathbb{R}^{3\times 3}$ *with* $|\tilde{R}_n| \leq C$, $R_n^* \in SO(3)$, $c_n \in \mathbb{R}^3$ *as well as a* $u \in W^{1,2}(S;\mathbb{R}^2)$ *and a* $v \in W^{2,2}(S)$ *such that* $y_n = R_n^{*T}\hat{y}_n - c_n$ *satisfies*

$$\|\nabla_n y_n - R_n\|_{L^2(\Omega)}^2 \leq Ch_n^4, \tag{3.19}$$

$$\|R_n - \tilde{R}_n\|_{L^2(S)}^2 \leq Ch_n^4, \tag{3.20}$$

$$\|\nabla \tilde{R}\|_{L^2(S)}^2 \leq Ch_n^2, \tag{3.21}$$

$$\|\nabla_n y_n - Id\|_{L^2(\Omega)}^2 \leq Ch_n^2, \tag{3.22}$$

$$\int_\Omega (\nabla_n y_n)_{12} - (\nabla_n y_n)_{21} \; dx = 0. \tag{3.23}$$

And, up to extracting subsequences,

$$\frac{1}{h_n^2}\int_0^1 y_n' - x' \; dx_3 =: u_n \rightharpoonup u \quad \text{in } W^{1,2}(S;\mathbb{R}^2), \tag{3.24}$$

$$\frac{1}{h_n}\int_0^1 (y_n)_3 \; dx_3 =: v_n \to v \quad \text{in } W^{1,2}(S), \tag{3.25}$$

$$\frac{R_n - Id}{h_n} := A_n \to A = e_3 \otimes \nabla' v - \nabla' v \otimes e_3 \quad \text{in} \quad L^q(\Omega, \mathbb{R}^{3\times 3}) \quad \forall q < \infty \tag{3.26}$$

$$2\frac{\operatorname{sym}(R_n - Id)}{h_n^2} \to A^2 \quad \text{in} \quad L^p(S;\mathbb{R}^{3\times 3}) \quad \forall p < \infty, \tag{3.27}$$

$$\frac{R_n^T \nabla_n y_n - Id}{h_n^2} \rightharpoonup G \quad \text{in } L^2(\Omega, \mathbb{R}^{2\times 2}), \tag{3.28}$$

where the upper left 2×2 *submatrix* G'' *of* G *is given by*

$$G''(x) = G_1(x') + \left(x_3 - \frac{1}{2}\right) G_2(x'),$$

with

$$\operatorname{sym} G_1 = \frac{1}{2}\left(\nabla' u + \nabla' u^T + \nabla' v \otimes \nabla' v\right), \quad G_2 = -\left(\nabla'\right)^2 v.$$

Remark 3.2.6. *The maps in Theorem 3.2.5 are not uniquely determined. For example they can be chosen in the class* $C^\infty(S;\mathbb{R}^{3\times 3})$ *maps by smoothening and afterwards projecting to* $SO(3)$. *Alternatively they can also be chosen to be piecewise constant on two-dimensional cubes with side-length* h_n *as done in the proof of Theorem 4.1 in [FJM02].*

Proposition 3.2.7 ([BS22], Proposition 2)**.** *In the setting of Theorem 3.2.5, applied to* $\tilde{\tilde{y}}_n$ *and with* $\tilde{y}_n = R_n^{*T}\tilde{\tilde{y}}_n - c_n$, *we have*

$$\frac{1}{h_n^2}\left((\tilde{y}_n)' - x'\right) =: \hat{u}_n \rightharpoonup \hat{u} \quad \text{in } W^{1,2}(\Omega;\mathbb{R}^2), \tag{3.29}$$

$$\frac{1}{h_n}(\tilde{y}_n)_3 =: \hat{v}_n \rightharpoonup \hat{v} \quad \textit{in } W^{1,2}(\Omega), \tag{3.30}$$

where

$$\hat{u}(x) = u(x') - \left(x_3 - \frac{1}{2}\right)\nabla' v(x'), \tag{3.31}$$

$$\hat{v}(x) = v(x') + \left(x_3 - \frac{1}{2}\right). \tag{3.32}$$

Proposition 3.2.8 ([BS22], Proposition 3)**.** *Let y_n be a sequence with $E_n(y_n) \leq Ch_n^4$ and let $R_n^* \in SO(3)$ and $c_n \in \mathbb{R}^3$ such that $\tilde{y}_n = R_n^*\tilde{\tilde{y}}_n - c_n$ satisfies the estimates of Theorem 3.2.5. Assume that $R_n^* \to R^*$ and $v_n \to v$ in $W^{1,2}(S)$. Then*

$$\frac{\varepsilon_n}{h_n}E_{\text{body}}(y_n) \to \begin{cases} \int_S f(x') \cdot v(x')R^*e_3 \; dx', & \textit{if } \nu_n \to \infty, \\ \frac{\nu}{\nu-1}\int_S f(x') \cdot v(x')R^*e_3 \; dx', & \textit{if } \nu_n \equiv \nu \in \mathbb{N}. \end{cases}$$

3.2.2. The discrete strain

Let $\bar{y}_n = R_n^{*T}\bar{\bar{y}}_n - c_n$, where R_n^* and c_n are as in Theorem 3.2.1. As in the continuum case also in the atomistic case it is crucial to understand the limiting behavior of the discrete strains

$$\bar{G}_n = h_n^{-2}(R_n^T\bar{\nabla}_n\bar{y}_n - Z). \tag{3.33}$$

Due to its importance we give the respective theorem. In the case $\nu_n \equiv \nu \in \mathbb{N}$ a projection P is defined by

$$Pf(x', x_3) = \fint_{\frac{k-1}{\nu-1}}^{\frac{k}{\nu-1}} f(x', t)\; dt \quad \text{if } \frac{k-1}{\nu-1} \leq x_3 < \frac{k}{\nu-1}$$

resulting in piecewise constant maps in the x_3-direction on the intervals $\left[\frac{k-1}{\nu-1}, \frac{k}{\nu-1}\right)$. Then we have

Proposition 3.2.9 ([BS22], Proposition 4)**.** *Let $(y_n)_n$ satisfy $E_n(y_n) \leq Ch_n^4$ with*

$$\frac{1}{h_n^2}\left(R_n^T\nabla_n\tilde{y}_n - Id\right) \rightharpoonup G \quad \textit{in } L^2(\Omega, \mathbb{R}^{3\times 3}).$$

Then

$$\bar{G}_n \rightharpoonup \bar{G} = \begin{cases} GZ, & \textit{if } \nu_n \to \infty, \\ PGZ + \frac{1}{2(\nu-1)}G_3, & \textit{if } \nu_n \equiv \nu \in \mathbb{N}, \end{cases}$$

in $L^2(\Omega; \mathbb{R}^{3\times 8})$, where

$$G_3(x') = \begin{pmatrix} G_2(x') & 0 \\ 0 & 0 \end{pmatrix} Z_- + \partial_{12}vM,$$

$$M = \frac{1}{2} e_3 \otimes (+1, -1, +1, -1, +1, -1, +1, -1)\,,$$
$$Z_- = (-Z^{(1)}, Z^{(2)}).$$

The upper left 2×2 *matrix of* G *is of the form*

$$G''(x) = G_1(x') + \left(x_3 - \frac{1}{2}\right) G_2(x')$$

with

$$\text{sym } G_1 = \frac{1}{2}\left(\nabla' u + \nabla' u^T + \nabla' v \otimes \nabla' v\right),\ G_2(x') = -\left(\nabla'\right)^2 v.$$

Next we give some calculations that will be useful later.

Remark 3.2.10. *(i) For* $t \in \left[\frac{m-1}{\nu-1}, \frac{m}{\nu-1}\right)$ *we have*

$$P\left(id - \frac{1}{2}\right)(t) = \fint_{\frac{m-1}{\nu-1}}^{\frac{m}{\nu-1}} \left(s - \frac{1}{2}\right)\, ds = \frac{2m-\nu}{2(\nu-1)}.$$

(ii) $\sum_{m=1}^{\nu-1} \int_{\frac{m-1}{\nu-1}}^{\frac{m}{\nu-1}} P\left(id - \frac{1}{2}\right)(s)\, ds = 0$

(iii) $\int_0^{\frac{1}{\nu-1}} P\left(id - \frac{1}{2}\right)(s)\, ds = -\int_{\frac{\nu-2}{\nu-1}}^{1} P\left(id - \frac{1}{2}\right)(s)\, ds.$

Using these equalities we see that for $x' \in S$

$$\begin{aligned}&\int_0^{\frac{1}{\nu-1}} \begin{pmatrix} (PG)'' & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} + \frac{1}{2(\nu-1)} G_3^{(1)}(x')\, dx_3 \\ &+ \int_{\frac{\nu-2}{\nu-1}}^{1} \begin{pmatrix} (PG)'' & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} + \frac{1}{2(\nu-1)} G_3^{(2)}(x')\, dx_3 \\ &= \frac{1}{\nu-1}\left[\begin{pmatrix} 2G_1(x') & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} + \frac{\partial_{12} v(x')}{\nu-1} M^{(1)}\right]\end{aligned} \tag{3.34}$$

and

$$\begin{aligned}&\int_0^{\frac{1}{\nu-1}} \begin{pmatrix} (PG)'' & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} + \frac{1}{2(\nu-1)} G_3^{(1)}(x')\, dx_3 \\ &- \int_{\frac{\nu-2}{\nu-1}}^{1} \begin{pmatrix} (PG)'' & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} + \frac{1}{2(\nu-1)} G_3^{(2)}(x')\, dx_3 \\ &= -\frac{1}{\nu-1} \begin{pmatrix} G_2(x') & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)}.\end{aligned} \tag{3.35}$$

By definition of $\bar{G}_n$ we can decompose the discrete gradient into

$$\bar{\nabla}_n \bar{y}_n = R_n\left(Z + h_n^2 \bar{G}_n\right).$$

Remark 3.2.11. *If we choose test functions that vanish close to the lateral boundary of S we can rewrite* (3.9)*. Noting that the discrete gradients of the piecewise constant interpolated functions are piecewise constant on each $x+\left(-\frac{\varepsilon_n}{2},\frac{\varepsilon_n}{2}\right)^2\times\left(-\frac{\varepsilon_n}{2h_n},\frac{\varepsilon_n}{2h_n}\right)$, $x\in\tilde{\Lambda}_n$, we obtain from* (3.9) *along with consideration of the rotated force term and Lemma 2.2.1*

$$\begin{aligned}
0 &= \int_\Omega DW_{\text{cell}}\left(\bar{\nabla}_n\bar{y}_n\right) : \bar{\nabla}_n\bar{\bar{\varphi}}\ dx \\
&\quad+\int_S\int_0^{\frac{1}{\nu_n-1}} DW_{\text{surf}}\left(\left(\bar{\nabla}_n\bar{y}_n\right)^{(1)}\right) :\ \left(\bar{\nabla}_n\bar{\bar{\varphi}}\right)^{(1)}\ dx_3\ dx' \\
&\quad+\int_S\int_{\frac{\nu_n-2}{\nu_n-1}}^1 DW_{\text{surf}}\left(\left(\bar{\nabla}_n\bar{y}_n\right)^{(2)}\right) :\ \left(\bar{\nabla}_n\bar{\bar{\varphi}}\right)^{(2)}\ dx_3\ dx' \\
&\quad+\int_{\tilde{V}_n} R_n^{*T}\bar{\bar{f}}_n\cdot\bar{\bar{\varphi}}\ dx \\
&= \int_\Omega R_nDW_{\text{cell}}\left(Z+h_n^2\bar{G}_n\right) : \bar{\nabla}_n\bar{\bar{\varphi}}\ dx \\
&\quad+\int_S\int_0^{\frac{1}{\nu_n-1}} R_nDW_{\text{surf}}\left(Z^{(1)}+h_n^2\bar{G}_n^{(1)}\right) :\ \left(\bar{\nabla}_n\bar{\bar{\varphi}}\right)^{(1)}\ dx_3\ dx' \\
&\quad+\int_S\int_{\frac{\nu_n-2}{\nu_n-1}}^1 R_nDW_{\text{surf}}\left(Z^{(2)}+h_n^2\bar{G}_n^{(2)}\right) :\ \left(\bar{\nabla}_n\bar{\bar{\varphi}}\right)^{(2)}\ dx_3\ dx' \\
&\quad+\int_{\tilde{V}_n} R_n^{*T}\bar{\bar{f}}_n\cdot\bar{\bar{\varphi}}\ dx.
\end{aligned} \tag{3.36}$$

Regarding the linear growth conditions on DW_{cell} and DW_{surf} this suggests to define the mappings

$$\begin{aligned}
J^n &= h_n^{-2}DW_{\text{cell}}(Z+h_n^2\bar{G}_n), \\
J^{(1,n)} &= h_n^{-2}DW_{\text{surf}}(Z^{(1)}+h_n^2\bar{G}_n^{(1)}), \\
J^{(2,n)} &= h_n^{-2}DW_{\text{surf}}(Z^{(2)}+h_n^2\bar{G}_n^{(2)}).
\end{aligned}$$

Note that $J^{(2,n)}=h_n^{-2}DW_{\text{surf}}(Z^{(1)}+h_n^2\bar{G}_n^{(2)})$ by (2.23). It is easy to see that J^n, $J^{(1,n)}$, $J^{(2,n)}$ are bounded in $L^2(\Omega,\mathbb{R}^{3\times 8})$ or $L^2(\Omega,\mathbb{R}^{3\times 4})$ respectively. By Proposition A.1.2 their weak L^2-limits are given by

$$J := \mathcal{L}(\bar{G}) = D^2W_{\text{cell}}(Z)(\bar{G}) = \frac{1}{2}DQ_{\text{cell}}\left(\bar{G}\right), \tag{3.37}$$

$$J^{(1)} := D^2W_{\text{surf}}(Z^{(1)})(\bar{G}^{(1)}) = \frac{1}{2}DQ_{\text{surf}}\left(\bar{G}^{(1)}\right), \tag{3.38}$$

$$J^{(2)} := D^2W_{\text{surf}}(Z^{(1)})(\bar{G}^{(2)}) = \frac{1}{2}DQ_{\text{surf}}\left(\bar{G}^{(2)}\right). \tag{3.39}$$

Lemma 3.2.12. *There is an $M\in L^2(\Omega;\mathbb{R}^{3\times 8})$ such that*

$$h_n^{-2}D_FW(\cdot,Z+h_n^2\bar{G}_n(\cdot)) \rightharpoonup M$$

in $L^2(\Omega,\mathbb{R}^{3\times 8})$*. If* $\nu_n\to\infty$*, then for almost every* $x\in\Omega$

$$M(x)=\frac{1}{2}DQ_{\text{cell}}(\bar{G}(x)).$$

If $\nu_n\equiv\nu\in\mathbb{N}$*, then for almost every* $x'\in S$

$$M(x)=\begin{cases}J(x)+\left(J^{(1)}(x),0\right), & \textit{if } \nu\geq 3,\ x_3\in\left(0,\frac{1}{\nu-1}\right),\\ J(x), & \textit{if } \nu\geq 3,\ x_3\in\left(\frac{1}{\nu-1},\frac{\nu-2}{\nu-1}\right),\\ J(x)+\left(0,J^{(2)}(x)\right), & \textit{if } \nu\geq 3,\ x_3\in\left(0,\frac{\nu-2}{\nu-1}\right),\\ J(x)+\left(J^{(1)}(x),J^{(2)}(x)\right), & \textit{if } \nu=2.\end{cases}$$

Proof. By the estimate (2.28) there exists $M\in L^2(\Omega,\mathbb{R}^{3\times 8})$ such that (up to a subsequence) $h_n^{-2}D_FW(\cdot,Z+h_n^2\bar{G}_n(\cdot))\rightharpoonup M$ in $L^2(\Omega,\mathbb{R}^{3\times 8})$. It remains to identify this limit. Fix an arbitrary compact subset $S'\subset S$. For n large enough for every $x'\in S'$ the energy can be written as a sum of homogeneous cell and surface energies as in (2.16). In particular the convergences (3.37)-(3.39) remain true on the domain $\Omega'=S'\times(0,1)$. Now for $x'\in S'$ and $n\in\mathbb{N}$ large enough we can write

$$\begin{aligned} & h_n^{-2}D_FW(x,Z+h_n^2\bar{G}_n(x)) \\ &= \chi_{\{x_3\in\left(0,\frac{1}{\nu_n-1}\right)\}}h_n^{-2}\left(DW_{\text{cell}}\left(Z+h_n^2\bar{G}_n(x)\right)+\left(DW_{\text{surf}}\left(Z^{(1)}+h_n^2\bar{G}_n^{(1)}\right),0\right)\right) && (3.40)\\ &+\chi_{\{x_3\in\left(\frac{1}{\nu_n-1},\frac{\nu_n-2}{\nu_n-1}\right)\}}h_n^{-2}DW_{\text{cell}}\left(Z+h_n^2\bar{G}_n(x)\right) && (3.41)\\ &+\chi_{\{x_3\in\left(\frac{\nu_n-2}{\nu_n-1},1\right)\}}h_n^{-2}\left(DW_{\text{cell}}\left(Z+h_n^2\bar{G}_n(x)\right)+\left(0,DW_{\text{surf}}\left(Z^{(1)}+h_n^2\bar{G}_n^{(2)}\right)\right)\right). && (3.42)\end{aligned}$$

If $\nu_n\to\infty$ by Corollary A.1.6 the terms (3.40) and (3.42) converge weakly to 0. For (3.41) note that $\chi_{\{x_3\in\left(\frac{1}{\nu_n-1},\frac{\nu_n-2}{\nu_n-1}\right)\}}\to 1$ boundedely in measure and by Lemma A.1.4 and (3.37) the term (3.41) converges weakly to $D^2W_{\text{cell}}(Z)[\bar{G}]$. Hence we have

$$h_n^{-2}D_FW(\cdot,Z+h_n^2\bar{G}_n(\cdot))\rightharpoonup D^2W_{\text{cell}}(Z)[\bar{G}]=J$$

in $L^2(\Omega',\mathbb{R}^{3\times 8})$. In contrast for $\nu_n\equiv\nu\in\mathbb{N}$

$$h_n^{-2}D_FW(\cdot,Z+h_n^2\bar{G}_n(\cdot))\rightharpoonup\begin{cases}J(\cdot)+\left(J^{(1)}(\cdot),0\right) & \text{if } \nu_n\geq 3,\ x_3\in\left(0,\frac{1}{\nu-1}\right)\\ J(\cdot) & \text{if } \nu_n\geq 3,\ x_3\in\left(\frac{1}{\nu-1},\frac{\nu-2}{\nu-1}\right)\\ J(\cdot)+\left(0,J^{(2)}(\cdot)\right) & \text{if } \nu_n\geq 3,\ x_3\in\left(0,\frac{\nu-2}{\nu-1}\right)\\ J(\cdot)+\left(J^{(1)}(\cdot),J^{(2)}(\cdot)\right) & \text{if } \nu_n=2\end{cases}$$

in $L^2(\Omega',\mathbb{R}^{3\times 8})$. Since this convergence holds true for every compact subset $S'\subset S$ it is also true on Ω. □

3.2.3. Consequences of the discrete equilibrium equations

Throughout this section we assume that y_n is a sequence of low energy discrete stationary points, i.e. y_n satisfies (3.9), with the bound $E_n(y_n)\leq Ch_n^4$. In the previous chapter we have seen that this energy bound implies that J^n, $J^{(1,n)}$ and $J^{(2,n)}$ have weak limits in their respective spaces. Together with the equilibrium condition we are able to show that these limits are in some sense orthogonal to the space $(\mathbb{R}^3 \otimes e_3)Z$. This will turn out to be crucial to pass from the equilibrium equations to the limiting equations.

Lemma 3.2.13. *Let $\nu_n \to \infty$. Then for every $i \in \{1,2,3\}$ and almost every $x \in \Omega$ it holds that*

$$\sum_{j=1}^{4} J_{ij} = \sum_{j=5}^{8} J_{ij}.$$

Proof. Let φ be compactely supported in $S \times [0,1]$. Since $\bar{y}_n$ is a stationary point of $E_n^{R_n^*}$ we have by (3.36) and Remark 3.2.11 after multiplication by h_n^{-1}

$$\begin{aligned} 0 = &\int_\Omega R_n J^n \colon \left(h_n \bar{\nabla}_n \bar{\bar{\varphi}}\right) \, dx \\ &+ \int_S \int_0^{\frac{1}{\nu_n - 1}} R_n J^{(1,n)} \colon \left(h_n \bar{\nabla}_n \bar{\bar{\varphi}}\right)^{(1)} \, dx_3 \, dx' \\ &+ \int_S \int_{\frac{\nu_n - 2}{\nu_n - 1}}^{1} R_n J^{(2,n)} \colon \left(h_n \bar{\nabla}_n \bar{\bar{\varphi}}\right)^{(2)} \, dx_3 \, dx' \\ &+ \int_{\tilde{V}_n} \bar{\bar{\varphi}} \cdot (R_n^*)^T h_n^{-1} \bar{\bar{f}}_n(x') \, dx \\ \underset{n\to\infty}{\longrightarrow} &\ \frac{1}{2} \int_\Omega J \colon \left(-\partial_3 \varphi | \cdots | \partial_3 \varphi \cdots\right) \, dx. \end{aligned}$$

This convergence follows from

- $R_n \to Id$ boundedely in measure,
- $J_n \rightharpoonup J$ in $L^2(\Omega)$,
- $h_n(\bar{\nabla}_n \bar{\bar{\varphi}})_{\cdot j} \to -\frac{\partial_3 \varphi}{2}$ uniformly in Ω whenever $1 \le j \le 4$,
- $h_n(\bar{\nabla}_n \bar{\bar{\varphi}})_{\cdot j} \to \frac{\partial_3 \varphi}{2}$ uniformly in Ω whenever $5 \le j \le 8$.

The last two convergences follow directly from a Taylor expansion of (2.6). The force term vanishes since $h_n^{-1} \bar{f}_n \to 0$ in $L^2(S)$. The integrals with the surface terms involved vanish as well, since $h_n R_n J^{(i,n)} \colon \bar{\nabla}_n \bar{\bar{\varphi}}$ converges weakly in $L^1(\Omega)$. By equi-integrability we find that the surface terms tend to zero. Choosing $\varphi(x) = \left(\int_0^{x_3} \phi(x', t) \, dt\right) e_i$, $i \in \{1,2,3\}$, with $\phi \in C_c^\infty(\Omega)$ yields

$$0 = \sum_{j=1}^{4} J_{ij} - \sum_{j=5}^{8} J_{ij}$$

almost everywhere. □

The case $\nu_n \equiv \nu \in \mathbb{N}$ needs to be treated a bit differently since in the lowest and the uppermost layer the surface part cannot be neglected. This leads to different orthogonality conditions to $(\mathbb{R}^3 \otimes e_3)Z$ depending on the layers.

Lemma 3.2.14. *Let $\nu_n \equiv \nu \geq 3$. To first consider the bulk part let $m \in \{1, \cdots, \nu - 3\}$. Then for every $i \in \{1, 2, 3\}$ and almost every $x' \in S$ it holds that*

$$\sum_{l=1}^{4} J_{il}\left(x', \frac{2m+1}{2(\nu-1)}\right) = \sum_{l=5}^{8} J_{il}\left(x', \frac{2m+1}{2(\nu-1)}\right).$$

For the lowest and uppermost layer it holds that

$$\sum_{l=1}^{4}\left[J_{il}\left(x', \frac{1}{2(\nu-1)}\right) + J_{il}^{(1)}\left(x', \frac{1}{2(\nu-1)}\right)\right] = \sum_{l=5}^{8} J_{il}\left(x', \frac{1}{2(\nu-1)}\right),$$

respectively

$$\sum_{l=1}^{4} J_{il}\left(x', \frac{2\nu-3}{2(\nu-1)}\right) = \sum_{l=5}^{8} J_{il}\left(x', \frac{2\nu-3}{2(\nu-1)}\right) + \sum_{l=1}^{4} J_{il}^{(2)}\left(x', \frac{2\nu-3}{2(\nu-1)}\right).$$

If $\nu_n \equiv \nu = 2$ for every $i \in \{1, 2, 3\}$ and almost every $x' \in S$ it holds that

$$\begin{aligned}&\sum_{l=1}^{4}\left[J_{il}\left(x', \frac{1}{2(\nu-1)}\right) + J_{il}^{(1)}\left(x', \frac{1}{2(\nu-1)}\right)\right]\\ &= \sum_{l=5}^{8} J_{il}\left(x', \frac{1}{2(\nu-1)}\right) + \sum_{l=1}^{4} J_{il}^{(2)}\left(x', \frac{1}{2(\nu-1)}\right).\end{aligned}$$

Proof. Let $\phi_0, \cdots, \varphi_{\nu-1} \in C_c^\infty(S, \mathbb{R}^3)$. For $s \in \left[\frac{m-1}{\nu-1}, \frac{m}{\nu-1}\right)$ let

$$\begin{aligned}\varphi(x', s) = {} & m\phi_{m-1}(x') - (\nu-1)s\phi_{m-1}(x')\\ & +(\nu-1)s\phi_m(x') - (m-1)\phi_m(x').\end{aligned}$$

This choice of ϕ is an admissible test function and satisfies for every $m \in \{1, \cdots, \nu-1\}$

- $\varphi(x', \frac{m-1}{\nu-1}) = \phi_{m-1}(x')$,
- $\varphi(x', \frac{m}{\nu-1}) = \phi_m(x')$,
- φ interpolates linearly between two layers.

Further we have

$$\begin{aligned}& h_n \bar{\nabla}_n \varphi\left(x', \frac{2m-1}{2(\nu-1)}\right)_i\\ &= (\nu-1)\Bigg[\varphi\left(x' + \varepsilon_n (z^i)', \frac{2m-1}{2(\nu-1)} + \left(\frac{1}{\nu-1}\right) z_3^i\right)\\ &\quad -\frac{1}{8}\sum_{j=1}^{8}\varphi\left(x' + \varepsilon_n (z^j)', \frac{2m-1}{2(\nu-1)} + \left(\frac{1}{\nu-1}\right) z_3^j\right)\Bigg]\end{aligned}$$

$$\underset{n\to\infty}{\longrightarrow} \begin{cases} \frac{\nu-1}{2}\left(\varphi\left(x', \frac{m-1}{\nu-1}\right) - \varphi\left(x', \frac{m}{\nu-1}\right)\right), & 1 \le i \le 4, \\ \frac{\nu-1}{2}\left(\varphi\left(x', \frac{m}{\nu-1}\right) - \varphi\left(x', \frac{m-1}{\nu-1}\right)\right), & 5 \le i \le 8, \end{cases}$$
$$= \begin{cases} \frac{\nu-1}{2}\left(\phi_{m-1}(x') - \phi_m(x')\right), & 1 \le i \le 4, \\ \frac{\nu-1}{2}\left(\phi_m(x') - \phi_{m-1}(x')\right), & 5 \le i \le 8. \end{cases}$$

We first treat the lowest layer. Let $\chi \in C_c^\infty(S, \mathbb{R}^3)$ and set $\phi_0 = \chi$ and $\phi_l \equiv 0$ for every $l \neq 0$. Define φ as above. From the equilibrium equation with this φ we get after multiplication with h_n^{-1}

$$\begin{aligned} 0 = &\int_S \int_0^{\frac{1}{\nu-1}} R_n J^n \colon \left(h_n \bar{\nabla}_n \bar{\bar{\varphi}}\right)\, dx_3\, dx' \\ &+ \int_S \int_0^{\frac{1}{\nu-1}} R_n J^{(1,n)} \colon \left(h_n \bar{\nabla}_n \bar{\bar{\varphi}}^{(1)}\right)\, dx_3\, dx' \\ &+ \int_{\tilde{V}_n} \bar{\bar{\varphi}}\, (R_n^*)^T\, h_n^{-1} \bar{\bar{f}}_n\, dx \end{aligned}$$

$$\begin{aligned} \underset{n\to\infty}{\longrightarrow} &\int_S \int_0^{\frac{1}{\nu-1}} D^2 W_{\text{cell}}(Z) \left(\bar{G}\left(x', \frac{1}{2(\nu-1)}\right)\right) \colon \left(\frac{\nu-1}{2}\right)(\chi, \cdots, -\chi, \cdots)\, dx_3\, dx' \\ &+ \int_S \int_0^{\frac{1}{\nu-1}} D^2 W_{\text{surf}}(Z^{(1)}) \left(\bar{G}^{(1)}\left(x', \frac{1}{2(\nu-1)}\right)\right) \colon \left(\frac{\nu-1}{2}\right)(\chi, \cdots)\, dx_3\, dx' \\ = &\int_S \frac{1}{2} D^2 W_{\text{cell}}(Z) \left(\bar{G}\left(x', \frac{1}{2(\nu-1)}\right)\right) \colon (\chi, \cdots, -\chi, \cdots)\, dx' \\ &+ \int_S \frac{1}{2} D^2 W_{\text{surf}}(Z^{(1)}) \left(\bar{G}^{(1)}\left(x', \frac{1}{2(\nu-1)}\right)\right) \colon (\chi, \chi, \chi, \chi)\, dx' \end{aligned}$$

This implies for almost every $x' \in S$

$$\sum_{l=1}^{4} \left[J_{\cdot l}\left(x', \frac{1}{2(\nu-1)}\right) + J_{\cdot l}^{(1)}\left(x', \frac{1}{2(\nu-1)}\right)\right] = \sum_{l=5}^{8} J_{\cdot l}\left(x', \frac{1}{2(\nu-1)}\right).$$

Now let $\varphi_1 = \chi$ and $\varphi_l \equiv 0$ else. Together with the equality derived for the first layer we have

$$\begin{aligned} 0 = &\int_S \int_0^{\frac{1}{\nu-1}} R_n h_n^{-2} DW_{\text{cell}}(Z + h_n^2 \bar{G}_n) \colon h_n \bar{\nabla}_n \bar{\bar{\varphi}}\, dx_3\, dx' \\ &+ \int_S \int_0^{\frac{1}{\nu-1}} R_n h_n^{-2} DW_{\text{surf}}(Z^{(1)} + h_n^2 \bar{G}_n^{(1)}) \colon h_n \bar{\nabla}_n \bar{\bar{\varphi}}^{(1)}\, dx_3\, dx' \\ &+ \int_S \int_{\frac{1}{\nu-1}}^{\frac{2}{\nu-1}} R_n h_n^{-2} DW_{\text{cell}}(Z + h_n^2 \bar{G}_n^{(2)}) \colon h_n \bar{\nabla}_n \bar{\bar{\varphi}}\, dx_3\, dx' \\ &+ \int_{\tilde{V}_n} \bar{\bar{\varphi}}\, (R_n^*)^T\, h_n^{-1} \bar{\bar{f}}_n\, dx \end{aligned}$$

$$\underset{n\to\infty}{\longrightarrow} \int_S \frac{1}{2} D^2 W_{\text{cell}}(Z) \left(\bar{G}\left(x', \frac{1}{2(\nu-1)}\right)\right) \colon (-\chi, \cdots, \chi, \cdots)\, dx'$$

$$+\int_S \frac{1}{2} D^2 W_{\text{surf}}(Z^{(1)}) \left(\bar{G}^{(1)} \left(x', \frac{1}{2(\nu-1)} \right) \right) : (-\chi, -\chi, -\chi, -\chi) \ dx'$$
$$+\int_S \frac{1}{2} D^2 W_{\text{cell}}(Z) \left(\bar{G} \left(x', \frac{3}{2(\nu-1)} \right) \right) : (\chi, \cdots, -\chi, \cdots) \ dx'.$$

As shown for the lowest layer the first two terms sum up to 0 and therefore

$$\sum_{l=1}^{4} J_{il} \left(x', \frac{3}{2(\nu-1)} \right) = \sum_{l=5}^{8} J_{il} \left(x', \frac{3}{2(\nu-1)} \right).$$

Inductively this follows for every inner layer. Adding the terms for the lowest and the uppermost layer leads to the first equality stated in the lemma for $\nu = 3$. The case $\nu = 2$ is done by an analogous argument. □

Lemma 3.2.15. *Let $A \in \mathbb{R}^{3\times 8}$ and $b(A) \in \mathbb{R}^3$ such that*

$$Q_{\text{cell}}^{\text{rel}}(A) = Q_{cell}\left(A + (b(A) \otimes e_3) Z\right).$$

Then

$$DQ_{\text{cell}}^{\text{rel}}(A) = DQ_{\text{cell}}\left(A + (b(A) \otimes e_3) Z\right).$$

Proof. For $A, B \in \mathbb{R}^{3\times 8}$ it holds that

$$Q_{\text{cell}}^{\text{rel}}(A+B) = Q_{\text{cell}}^{\text{rel}}(A) + Q_{\text{cell}}^{\text{rel}}(B) + DQ_{\text{cell}}^{\text{rel}}(A) : B \tag{3.43}$$

and

$$\begin{aligned} &Q_{\text{cell}}^{\text{rel}}(A+B) \\ &= Q_{\text{cell}}\left(A + B + (b(A+B) \otimes e_3) Z\right) \\ &= Q_{\text{cell}}\left(A + (b(A) \otimes e_3) Z\right) + Q_{\text{cell}}\left(B + (b(B) \otimes e_3) Z\right) \\ &\quad + DQ_{\text{cell}}\left(A + (b(A) \otimes e_3) Z\right) : \left(B + (b(B) \otimes e_3) Z\right) \\ &= Q_{\text{cell}}^{\text{rel}}(A) + Q_{\text{cell}}^{\text{rel}}(B) + DQ_{\text{cell}}\left(A + (b(A) \otimes e_3) Z\right) : B. \end{aligned} \tag{3.44}$$

Combining (3.43) and (3.44) yields the claim. □

Lemma 3.2.16. *Let $F \in \mathbb{R}^{3\times 3}$ and $A \in \mathbb{R}^{3\times 8}$. If*

$$DQ_{\text{cell}}(FZ + A) \perp \left(\mathbb{R}^3 \otimes e_3\right) Z$$

then

$$DQ_{\text{cell}}^{\text{rel}}\left(\begin{pmatrix} F'' & 0 \\ 0 & 0 \end{pmatrix} Z + A \right) = DQ_{\text{cell}}(FZ + A).$$

Proof. Let $F = \begin{pmatrix} F'' & 0 \\ 0 & 0 \end{pmatrix} + F'$ and let $c = \begin{pmatrix} F_{13} + F_{31} \\ F_{23} + F_{32} \\ 2F_{33} \end{pmatrix}$. Then we have $\operatorname{sym} F' = \operatorname{sym}(c \otimes e_3)$ and, since DQ_{cell} vanishes on $\mathbb{R}^{3\times 3}_{\text{skew}} Z$,

$$DQ_{\text{cell}}(FZ + A)$$

$$= DQ_{\text{cell}}\left(\begin{pmatrix} F'' & 0 \\ 0 & 0 \end{pmatrix} Z + A + F'Z\right)$$
$$= DQ_{\text{cell}}\left(\begin{pmatrix} F'' & 0 \\ 0 & 0 \end{pmatrix} Z + A + \operatorname{sym}\left(c \otimes e_3\right) Z\right)$$
$$= DQ_{\text{cell}}\left(\begin{pmatrix} F'' & 0 \\ 0 & 0 \end{pmatrix} Z + A + \left(c \otimes e_3\right) Z\right).$$

Thus $DQ_{\text{cell}}\left(\begin{pmatrix} F'' & 0 \\ 0 & 0 \end{pmatrix} Z + A + \left(c \otimes e_3\right) Z\right) \perp (\mathbb{R}^3 \otimes e_3) Z$ and $c = b\left(\begin{pmatrix} F'' & 0 \\ 0 & 0 \end{pmatrix} Z + A\right)$. Hence by Lemma 3.2.15 it holds that

$$DQ^{\text{rel}}_{\text{cell}}\left(\begin{pmatrix} F'' & 0 \\ 0 & 0 \end{pmatrix} Z + A\right)$$
$$= DQ_{\text{cell}}\left(\begin{pmatrix} F'' & 0 \\ 0 & 0 \end{pmatrix} Z + A + \left(c \otimes e_3\right) Z\right)$$
$$= DQ_{\text{cell}}\left(FZ + A\right).$$

□

With the definition $Q_2(A) = Q^{\text{rel}}_{\text{cell}}\left(\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} Z\right)$ we often will use Lemma 3.2.16 in the following way:

Corollary 3.2.17. *Let $F \in \mathbb{R}^{3\times 3}$ such that $\mathcal{L}\left(FZ\right) \perp \left(\mathbb{R}^3 \otimes e_3\right) Z$. Then for every $B \in \mathbb{R}^{2\times 2}$ it holds that*

$$\mathcal{L}_2\left(F''\right) : B = \mathcal{L}\left(FZ\right) : \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} Z.$$

Proof. Let $B \in \mathbb{R}^{2\times 2}$. Then

$$DQ_2\left(F''\right) : B$$
$$= \frac{d}{dt}\Big|_{t=0} Q_2\left(F'' + tB\right)$$
$$= \frac{d}{dt}\Big|_{t=0} Q^{\text{rel}}_{\text{cell}}\left(\begin{pmatrix} F'' & 0 \\ 0 & 0 \end{pmatrix} Z + t\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}\right) Z$$
$$= DQ^{\text{rel}}_{\text{cell}}\left(\begin{pmatrix} F'' & 0 \\ 0 & 0 \end{pmatrix} Z\right) : \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} Z$$
$$= DQ_{\text{cell}}\left(FZ\right) : \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} Z.$$

The last equality follows from Lemma 3.2.16. □

Lemma 3.2.18. *Let $A \in \mathbb{R}^{3\times 8}$ such that*

$$\sum_{l=1}^{4} \left[DQ_{\text{cell}}(A)_{\cdot l} + DQ_{\text{surf}}(A^{(1)})_{\cdot l}\right] = \sum_{l=5}^{8} DQ_{\text{cell}}(A)_{\cdot l} \tag{3.45}$$

or

$$\sum_{l=1}^{4} DQ_{\text{cell}}(A)_{\cdot l} = \sum_{l=5}^{8} DQ_{\text{cell}}(A)_{\cdot l} + \sum_{l=1}^{4} DQ_{\text{surf}}(A^{(2)})_{\cdot l} \tag{3.46}$$

Then

$$DQ_{\text{cell}}(A) \perp \left(\mathbb{R}^3 \otimes e_3\right) Z.$$

Proof. Assume that (3.45) holds true, the statement for (3.46) is shown similarly. Let $Q : \mathbb{R}^{3\times 8} \to \mathbb{R}$ be a quadratic form defined by

$$Q(H) = Q_{\text{cell}}(H) + Q_{\text{surf}}(H^{(1)}).$$

Since Q_{cell} is positive definite on $(\mathbb{R}^3 \otimes e_3)\, Z$ and $Q_{\text{surf}} \geq 0$ this is also the case for Q. By assumption we have

$$\left[DQ_{\text{cell}}(A)^{(1)} + DQ_{\text{surf}}(A^{(1)}), DQ_{\text{cell}}(A)^{(2)}\right] \perp \left(\mathbb{R}^3 \otimes e_3\right) Z.$$

For $c \in \mathbb{R}^3$ we have

$$(c \otimes e_3)\, Z^{(1)} = -\,(c, c, c, c) \in \mathbb{R}^{3\times 4},$$

i.e. $t \mapsto Q_{\text{surf}}(A^{(1)} + t\,(c \otimes e_3)\, Z^{(1)})$ is constant. This implies

$$\begin{aligned}
0 = \frac{d}{dt}\Big|_{t=0} \left(Q_{\text{cell}}(A + t(c \otimes e_3)Z) + Q_{\text{surf}}(A^{(1)} + t(c \otimes e_3)Z^{(1)})\right) \\
= \frac{d}{dt}\Big|_{t=0} Q_{\text{cell}}(A + t(c \otimes e_3)Z) \\
= DQ_{\text{cell}}(A) : \ (c \otimes e_3)\, Z.
\end{aligned}$$

□

Symmetric properties of J^n

One of the main ingredients of the proof of the main theorem in [MP08] was the estimate

$$\left\| E^{(h)} - \left(E^{(h)}\right)^T \right\|_{L^1(\Omega;\mathbb{R}^{3\times 3})} \leq Ch^2$$

for the scaled stresses $E^{(h)} = \frac{1}{h^2} DW(Id + h^2 G^{(h)})$. Adapting the proof of this estimate leads to

Lemma 3.2.19. *The scaled stresses $J^n, J^{(1,n)}, J^{(2,n)}$ satisfy*

$$\begin{aligned}
\left\| J^n Z^T - Z\,(J^n)^T \right\|_{L^1(\Omega;\mathbb{R}^{3\times 3})} &\leq Ch_n^2, \\
\left\| J^{(1,n)} Z^{(1)^T} - Z^{(1)} J^{(1,n)^T} \right\|_{L^1(\Omega;\mathbb{R}^{3\times 3})} &\leq Ch_n^2, \\
\left\| J^{(2,n)} Z^{(2)^T} - Z^{(2)} J^{(2,n)^T} \right\|_{L^1(\Omega;\mathbb{R}^{3\times 3})} &\leq Ch_n^2.
\end{aligned}$$

Proof. We only show the first inequality. The others can be shown exactly the same way. Let $F \in \mathbb{R}^{3\times 8}$ and $H \in \mathbb{R}^{3\times 3}_{\text{skew}}$. By frame indifference we have

$$W_{\text{cell}}(\exp(tH)F) = W_{\text{cell}}(F)$$

for every $t \in \mathbb{R}$. Thus

$$0 = \frac{d}{dt}\Big|_{t=0} W_{\text{cell}}(\exp(tH)F) = DW_{\text{cell}}(F) \colon HF$$

and therefore

$$DW_{\text{cell}}(F)F^T \colon H = 0.$$

Since this holds for every skew symmetric $H \in \mathbb{R}^{3\times 3}$ we deduce that $DW_{\text{cell}}(F)F^T$ is symmetric. With $F = Z + h_n^2 \bar{G}_n$ we get

$$\begin{aligned} 0 = DW_{\text{cell}}(Z + h_n^2\bar{G}_n)(Z + h_n^2\bar{G}_n)^T \\ -(Z + h_n^2\bar{G}_n)DW_{\text{cell}}(Z + h_n^2\bar{G}_n)^T \\ = h_n^2 J^n Z^T + h_n^4 J^n \left(\bar{G}_n\right)^T \\ -h_n^2 Z \left(J^n\right)^T - h_n^4 \bar{G}_n \left(J^n\right)^T. \end{aligned}$$

Hence

$$\left| J^n Z^T - Z\left(J^n\right)^T \right| \le h_n^2 \left| J^n \bar{G}_n^T - \bar{G}_n \left(J^n\right)^T \right|.$$

Integrating and applying the Cauchy Schwarz inequality to the latter term yields the claim. □

Remark 3.2.20. *In particular the previous lemma implies for $i, j \in \{1, 2, 3\}$*

$$\left\| \sum_{l=1}^{8} \left[\left(J^n\right)_{il} z_j^l - \left(J^n\right)_{jl} z_i^l \right] \right\|_{L^1(\Omega)} \le Ch_n^2, \tag{3.47}$$

$$\left\| \sum_{l=1}^{4} \left[J_{il}^{(1,n)} z_j^l - J_{jl}^{(1,n)} z_i^l \right] \right\|_{L^1(\Omega)} \le Ch_n^2, \tag{3.48}$$

$$\left\| \sum_{l=1}^{4} \left[J_{il}^{(2,n)} z_j^{l+4} - J_{jl}^{(2,n)} z_i^{l+4} \right] \right\|_{L^1(\Omega)} \le Ch_n^2. \tag{3.49}$$

3.2.4. Outline of the proof

Due to the length of the proof of Theorem 3.2.1 we give a short overview here without caring about any details. We prove the cases $\nu_n \to \infty$ and $\nu_n \equiv \nu \in \mathbb{N}$ simultaneously and distinguish where needed.

The first part is denoted to derive the equations (3.6) and (3.8). This is quite straightforward: Choosing test functions of the form $\phi(x) = (\Psi(x'), 0)$ we will be able to directly pass to the limit $n \to \infty$ to obtain the equations.

The difficult part is the derivation of (3.5) and (3.7). Thus we split it into three parts. Fix a test function $\phi \in C_c^\infty(S)$.

In **Part 1** we test the discrete equation with $\varphi(x) = (0, 0, \phi(x'))$. If $\nu_n \to \infty$ we will show that

$$\begin{aligned} &\int_\Omega h_n^{-1} J^n : \bar{\nabla}_n \bar{\bar{\varphi}} \, dx \\ &+ \int_S \int_0^{\frac{1}{\nu_n - 1}} h_n^{-1} J^{(1,n)} : \left(\bar{\nabla}_n \bar{\bar{\varphi}}\right)^{(1)} \, dx_3 \, dx' \\ &+ \int_S \int_{\frac{\nu_n-2}{\nu_n-1}}^1 h_n^{-1} J^{(2,n)} : \left(\bar{\nabla}_n \bar{\bar{\varphi}}\right)^{(2)} \, dx_3 \, dx' \\ \underset{n\to\infty}{\longrightarrow} &- \int_S \mathcal{L}_2 \left(\frac{1}{2} \left(\nabla' u + \nabla' u^T + \nabla' v \otimes \nabla' v \right) \right) : \left(\nabla' v \otimes \nabla' \phi \right) \, dx' \\ &- \int_S f(x') \cdot \phi R^* e_3 \, dx'. \end{aligned} \tag{3.50}$$

Similarly for $\nu_n \equiv \nu \in \mathbb{N}$ we will show that

$$\begin{aligned} &\int_\Omega h_n^{-1} J^n : \bar{\nabla}_n \bar{\bar{\varphi}} \, dx \\ &+ \int_S \int_0^{\frac{1}{\nu_n - 1}} h_n^{-1} J^{(1,n)} : \left(\bar{\nabla}_n \bar{\bar{\varphi}}\right)^{(1)} \, dx_3 \, dx' \\ &+ \int_S \int_{\frac{\nu-2}{\nu-1}}^1 h_n^{-1} J^{(2,n)} : \left(\bar{\nabla}_n \bar{\bar{\varphi}}\right)^{(2)} \, dx_3 \, dx' \\ \underset{n\to\infty}{\longrightarrow} &- \int_S \frac{1}{2} DQ_{\text{cell}}^{\text{rel}} \left(\begin{pmatrix} \text{sym } G_1 & 0 \\ 0 & 0 \end{pmatrix} Z + \frac{1}{2(\nu-1)} G_3 \right) : \begin{pmatrix} \nabla' v \otimes \nabla' \phi & 0 \\ 0 & 0 \end{pmatrix} Z \, dx' \\ &- \int_S \frac{1}{\nu - 1} DQ_{\text{surf}} \left(\begin{pmatrix} \text{sym } G_1 & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} + \frac{\partial_{12} v}{2(\nu-1)} M^{(1)} \right) : \\ &\qquad \begin{pmatrix} \nabla' v \otimes \nabla' \phi & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} \, dx' \\ &- \int_S \frac{\nu}{\nu - 1} f(x') \cdot \phi R^* e_3 \, dx'. \end{aligned} \tag{3.51}$$

In **Part 2** we set $\varphi(x) = \begin{pmatrix} \nabla' \phi(x') \\ 0 \end{pmatrix} \left(x_3 - \frac{1}{2} \right)$. We will pass to the alternative discrete gradient to apply the product rule to this test function. With this approach we will show for $\nu_n \to \infty$

$$\begin{aligned} &\sum_{l=1}^{8} \int_\Omega h_n^{-1} \left(R_n J^n \right)_{\cdot l} \cdot z_3^l \begin{pmatrix} \nabla' \phi(\hat{x}') \\ 0 \end{pmatrix} \, dx \\ &+ \sum_{l=1}^{4} \int_S \int_0^{\frac{1}{\nu - 1}} h_n^{-1} \left(R_n J^{(1,n)} \right)_{\cdot l} \cdot z_3^l \begin{pmatrix} \nabla' \phi(\hat{x}') \\ 0 \end{pmatrix} \, dx_3 \, dx' \end{aligned}$$

$$+\sum_{l=1}^{4}\int_S\int_{\frac{\nu-2}{\nu-1}}^{1} h_n^{-1}\left(R_nJ^{(2,n)}\right)_{\cdot l}\cdot z_3^{l+4}\begin{pmatrix}\nabla'\phi(\hat{x}')\\0\end{pmatrix}\,dx_3\,dx'$$

$$\underset{n\to\infty}{\longrightarrow}\int_S\frac{1}{12}\mathcal{L}_2\left(\left(\nabla'\right)^2v\right):\left(\nabla'\right)^2\phi(x')\,dx'. \tag{3.52}$$

For $\nu_n\equiv\nu\in\mathbb{N}$ this will lead to

$$\sum_{l=1}^{8}\int_\Omega h_n^{-1}\left(R_nJ^n\right)_{\cdot l}\cdot z_3^{l}\begin{pmatrix}\nabla'\phi(\hat{x}')\\0\end{pmatrix}\,dx$$
$$+\sum_{l=1}^{4}\int_S\int_0^{\frac{1}{\nu-1}} h_n^{-1}\left(R_nJ^{(1,n)}\right)_{\cdot l}\cdot z_3^{l}\begin{pmatrix}\nabla'\phi(\hat{x}')\\0\end{pmatrix}\,dx_3\,dx'$$
$$+\sum_{l=1}^{4}\int_S\int_{\frac{\nu-2}{\nu-1}}^{1} h_n^{-1}\left(R_nJ^{(2,n)}\right)_{\cdot l}\cdot z_3^{l+4}\begin{pmatrix}\nabla'\phi(\hat{x}')\\0\end{pmatrix}\,dx_3\,dx'$$
$$\underset{n\to\infty}{\longrightarrow}-\int_S\frac{\nu(\nu-2)}{24(\nu-1)^2}DQ_{\text{cell}}^{\text{rel}}\left(\begin{pmatrix}G_2&0\\0&0\end{pmatrix}Z\right):\begin{pmatrix}\nabla'^2\phi&0\\0&0\end{pmatrix}Z\,dx'$$
$$-\int_S\frac{1}{4(\nu-1)}DQ_{\text{cell}}^{\text{rel}}\left(\begin{pmatrix}\text{sym }G_1&0\\0&0\end{pmatrix}Z+\frac{1}{2(\nu-1)}G_3(x')\right):\begin{pmatrix}\nabla'^2\phi&0\\0&0\end{pmatrix}Z_-\,dx'$$
$$-\int_S\frac{1}{4(\nu-1)}DQ_{\text{surf}}\left(\begin{pmatrix}G_2&0\\0&0\end{pmatrix}Z^{(1)}\right):\begin{pmatrix}\nabla'^2\phi&0\\0&0\end{pmatrix}Z^{(1)}\,dx'. \tag{3.53}$$

Part 3 is about to determine convergence of

$$\int_\Omega h_n^{-1}J^n:\bar{\nabla}_n\bar{\bar{\varphi}}\,dx$$
$$+\int_S\int_0^{\frac{1}{\nu-1}} h_n^{-1}J^{(1,n)}:\left(\bar{\nabla}_n\bar{\bar{\varphi}}\right)^{(1)}\,dx_3\,dx'$$
$$+\int_S\int_{\frac{\nu-2}{\nu-1}}^{1} h_n^{-1}J^{(2,n)}:\left(\bar{\nabla}_n\bar{\bar{\varphi}}\right)^{(2)}\,dx_3\,dx'$$
$$-\sum_{l=1}^{8}\int_\Omega h_n^{-1}\left(R_nJ^n\right)_{\cdot l}\cdot z_3^{l}\begin{pmatrix}\nabla'\phi(\hat{x}')\\0\end{pmatrix}\,dx_3\,dx'$$
$$-\sum_{l=1}^{4}\int_S\int_0^{\frac{1}{\nu-1}} h_n^{-1}\left(R_nJ^{(1,n)}\right)_{\cdot l}\cdot z_3^{l}\begin{pmatrix}\nabla'\phi(\hat{x}')\\0\end{pmatrix}\,dx_3\,dx'$$
$$-\sum_{l=1}^{4}\int_S\int_{\frac{\nu-2}{\nu-1}}^{1} h_n^{-1}\left(R_nJ^{(2,n)}\right)_{\cdot l}\cdot z_3^{l+4}\begin{pmatrix}\nabla'\phi(\hat{x}')\\0\end{pmatrix}\,dx_3\,dx', \tag{3.54}$$

where $\varphi(x)=(0,0,\phi(x'))$. In case $\nu_n\to\infty$ this converges to 0. As a consequence the limiting terms of (3.50) and (3.52) agree and (3.5) follows. In case $\nu_n\equiv\nu\in\mathbb{N}$ we show that

$$(3.54)\to\int_S\frac{1}{4(\nu-1)}DQ_{\text{cell}}^{\text{rel}}\left(\begin{pmatrix}\text{sym }G_1&0\\0&0\end{pmatrix}Z+\frac{1}{2(\nu-1)}G_3\right):\partial_{12}\phi M\,dx'$$

$$+\int_S \frac{1}{2(\nu-1)^2} DQ_{\text{surf}} \left(\begin{pmatrix} \text{sym } G_1 & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} + \frac{\partial_{12} v}{2(\nu-1)} M^{(1)} \right) : \partial_{12}\phi M^{(1)}\, dx'.$$

Together with the limiting terms of (3.51) and (3.53) we get (3.7).

3.2.5. Proof of the main theorem

After giving this short overview we focus on all of the mathematical details. We will follow exactly the structure given in the outline.

Proof of Theorem 3.2.1. Choose $R_n^* \in SO(3)$, $R_n \in C^\infty(S;\mathbb{R}^{3\times 3})$ and c_n according to Theorem 3.2.5. In particular we have

$$R_n(x') \in SO(3), \tag{3.55}$$

$$\|\nabla_n \tilde{y}_n - R_n\|_{L^2(\Omega)} \leq Ch_n^2, \tag{3.56}$$

$$\|\nabla' R_n\|_{L^2(\Omega)} \leq Ch_n, \tag{3.57}$$

$$\|R_n - Id\|_{W^{1,2}(\Omega)} \leq Ch_n. \tag{3.58}$$

By Lemma 3.1.3 the maps $\hat{y}_n$ are stationary points of $E_n^{R_n^*}$. The convergences (3.10) and (3.11) follow from Theorem 3.2.5.

The equations (3.6) **and** (3.8)

Let $\Psi \in C_c^\infty(S;\mathbb{R}^2)$ and let $\varphi(x) = (\Psi(x'), 0)$. Then φ is an admissible test function with

$$\nabla\varphi(x) = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\nabla'\Psi(x')} & 0 \\ & & 0 \\ \hline 0 & 0 & 0 \end{array} \right)$$

for every $x \in \Omega$. By Remark 3.2.11 testing the equilibrium equation with φ yields

$$\begin{aligned} 0 = &\int_\Omega R_n J^n : \bar{\nabla}_n \bar{\bar{\varphi}}(x)\, dx \\ &+ \int_S \int_0^{\frac{1}{\nu_n-1}} R_n J^{(1,n)} : \left(\bar{\nabla}_n \bar{\bar{\varphi}}\right)^{(1)}\, dx_3\, dx' \\ &+ \int_S \int_{\frac{\nu_n-2}{\nu_n-1}}^1 R_n J^{(2,n)} : \left(\bar{\nabla}_n \bar{\bar{\varphi}}\right)^{(2)}\, dx_3\, dx' \\ &+ \int_{\tilde{V}_n} h_n^{-2} \bar{\bar{f}}_n(x') R_n^* \bar{\bar{\varphi}}\, dx. \end{aligned} \tag{3.59}$$

Since φ is independent of the x_3 variable the interpolated discrete gradient $\bar{\nabla}_n \bar{\bar{\varphi}}$ converges uniformly to $\nabla\varphi Z$ and $R_n J^n \rightharpoonup J$ in $L^2(\Omega;\mathbb{R}^{3\times 8})$, c.f. (3.37). Passing to the limit in (3.59) we notice that the force term vanishes. If $\nu_n \to \infty$ this is also the case for the surface terms. Hence from (3.59), Lemma 3.2.13, Corollary 3.2.17, Proposition 3.2.9 and Corollary 2.3.3 we obtain

$$0 = \int_\Omega J : \nabla\varphi Z\, dx$$

$$
\begin{aligned}
&= \int_{\Omega} \mathcal{L}(GZ) \colon \nabla\varphi Z \, dx \\
&= \int_{S} \mathcal{L}_2 \left(\text{sym } G''\right) : \nabla'\Psi dx' \\
&= \int_{S} \mathcal{L}_2(\text{sym } G_1) \colon \nabla'\Psi \, dx',
\end{aligned}
$$

and (3.6) follows. If $\nu_n \equiv \nu \in \mathbb{N}$ in (3.59) only the force term vanishes and we obtain

$$
\begin{aligned}
0 = \int_{\Omega} D^2 W_{\text{cell}}(Z)\left(\bar{G}\right) : \begin{pmatrix} \nabla'\Psi(x') & 0 \\ 0 & 0 \end{pmatrix} Z \, dx \\
+ \int_S \int_0^{\frac{1}{\nu-1}} D^2 W_{\text{surf}}\left(Z^{(1)}\right) \left(\bar{G}^{(1)}\left(x', \frac{1}{2(\nu-1)}\right)\right) : \begin{pmatrix} \nabla'\Psi(x') & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} \, dx_3 \, dx' \\
+ \int_S \int_{\frac{\nu-2}{\nu-1}}^{1} D^2 W_{\text{surf}}\left(Z^{(1)}\right) \left(\bar{G}^{(2)}\left(x', \frac{2\nu-3}{2(\nu-1)}\right)\right) : \begin{pmatrix} \nabla'\Psi(x') & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} \, dx_3 \, dx'.
\end{aligned}
$$

In Lemma 3.2.14 and Lemma 3.2.18 we have shown that the assumptions of Lemma 3.2.16 are satisfied. Hence from the equality above, Proposition 3.2.9, Lemma 3.2.15 and Corollary 2.3.3 it follows that

$$
\begin{aligned}
0 = \int_{\Omega} \frac{1}{2} DQ_{\text{cell}}^{\text{rel}} \left(\begin{pmatrix} (PG)'' & 0 \\ 0 & 0 \end{pmatrix} Z + \frac{1}{2(\nu-1)} G_3\right) : \begin{pmatrix} \nabla'\Psi & 0 \\ 0 & 0 \end{pmatrix} Z \, dx \\
+ \int_S \int_0^{\frac{1}{\nu-1}} \frac{1}{2} DQ_{\text{surf}} \left(\begin{pmatrix} (PG)'' & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} + \frac{1}{2(\nu-1)} G_3^{(1)}\right) : \begin{pmatrix} \nabla'\Psi & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} \, dx_3 \, dx' \\
+ \int_S \int_{\frac{\nu-2}{\nu-1}}^{1} \frac{1}{2} DQ_{\text{surf}} \left(\begin{pmatrix} (PG)'' & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} + \frac{1}{2(\nu-1)} G_3^{(2)}\right) : \begin{pmatrix} \nabla'\Psi & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} \, dx_3 \, dx'.
\end{aligned}
\tag{3.60}
$$

For the surface terms by (3.34) we have

$$
\begin{aligned}
&\int_S \int_0^{\frac{1}{\nu-1}} \frac{1}{2} DQ_{\text{surf}} \left(\begin{pmatrix} (PG)'' & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} + \frac{1}{2(\nu-1)} G_3^{(1)}\right) \, dx_3 \, dx' \\
&+ \int_S \int_{\frac{\nu-2}{\nu-1}}^{1} \frac{1}{2} DQ_{\text{surf}} \left(\begin{pmatrix} (PG)'' & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} + \frac{1}{2(\nu-1)} G_3^{(2)}\right) \, dx_3 \, dx' \\
&= \int_S \frac{1}{2(\nu-1)} DQ_{\text{surf}} \left(\begin{pmatrix} 2\text{sym } G_1 & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} + \frac{\partial_{12} v}{\nu-1} M^{(1)}\right) \, dx'.
\end{aligned}
\tag{3.61}
$$

Combining (3.60) and (3.61) we obtain (3.8):

$$
\begin{aligned}
0 = \int_S \frac{1}{2} DQ_{\text{cell}}^{\text{rel}} \left(\begin{pmatrix} \text{sym } G_1 & 0 \\ 0 & 0 \end{pmatrix} Z + \frac{1}{2(\nu-1)} G_3\right) : \begin{pmatrix} \nabla'\Psi & 0 \\ 0 & 0 \end{pmatrix} Z \, dx' \\
+ \int_S \frac{1}{\nu-1} DQ_{\text{surf}} \left(\begin{pmatrix} \text{sym } G_1 & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} + \frac{\partial_{12} v}{2(\nu-1)} M^{(1)}\right) : \begin{pmatrix} \nabla'\Psi & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} \, dx'.
\end{aligned}
$$

The equations (3.5) **and** (3.7)

As the derivation of these equations is a bit more complicated, we split it into three parts.

Part 1

Let $\phi \in C_c^\infty(S)$ and $\varphi(x) = (0, 0, \phi(x'))$. Then

$$\bar{\nabla}_n \bar{\bar{\varphi}} \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \partial_1\phi & \partial_2\phi & 0 \end{pmatrix} Z$$

uniformly in Ω. Testing the equilibrium equation with φ and multiplying with h_n^{-3} yields

$$\begin{aligned} 0 = &\int_\Omega h_n^{-1} R_n J_n \colon \bar{\nabla}_n \bar{\bar{\varphi}}\ dx \\ &+ \int_S \int_0^{\frac{1}{\nu_n - 1}} h_n^{-1} R_n J^{(1,n)} \colon \left(\bar{\nabla}_n \bar{\bar{\varphi}}\right)^{(1)}\ dx_3\ dx' \\ &+ \int_S \int_{\frac{\nu_n - 2}{\nu_n - 1}}^1 h_n^{-1} R_n J^{(2,n)} \colon \left(\bar{\nabla}_n \bar{\bar{\varphi}}\right)^{(2)}\ dx_3\ dx' \\ &+ \int_{\tilde{V}_n} h_n^{-3} \bar{\bar{f}}_n(x') \cdot R_n^* \bar{\bar{\varphi}}\ dx. \end{aligned}$$

By Theorem 3.2.5 the sequence

$$A_n = \frac{R_n - Id}{h_n}$$

converges strongly to A in $L^q(\Omega)$ for any $q < \infty$ with

$$A = \begin{pmatrix} 0 & 0 & -\partial_1 v \\ 0 & 0 & -\partial_2 v \\ \partial_1 v & \partial_2 v & 0 \end{pmatrix}.$$

Via the relation $R_n = h_n A_n + Id$ we deduce

$$\begin{aligned} 0 = &\int_\Omega h_n^{-1} \left(h_n A_n + Id\right) J_n \colon \bar{\nabla}_n \bar{\bar{\varphi}}\ dx \\ &+ \int_S \int_0^{\frac{1}{\nu_n - 1}} h_n^{-1} \left(h_n A_n + Id\right) J^{(1,n)} \colon \left(\bar{\nabla}_n \bar{\bar{\varphi}}\right)^{(1)}\ dx_3\ dx' \\ &+ \int_S \int_{\frac{\nu_n - 2}{\nu_n - 1}}^1 h_n^{-1} \left(h_n A_n + Id\right) J^{(2,n)} \colon \left(\bar{\nabla}_n \bar{\bar{\varphi}}\right)^{(2)}\ dx_3\ dx' \\ &+ \int_{\tilde{V}_n} h_n^{-3} \bar{\bar{f}}_n(x') \cdot R_n^* \bar{\bar{\phi}}(x') e_3\ dx. \end{aligned}$$

Hence

$$\int_\Omega h_n^{-1} J^n \colon \bar{\nabla}_n \bar{\bar{\varphi}}\ dx$$

$$+\int_S\int_0^{\frac{1}{\nu_n-1}} h_n^{-1}J^{(1,n)}\colon\ \left(\bar{\nabla}_n\bar{\bar{\varphi}}\right)^{(1)}\ dx_3\ dx'$$
$$+\int_S\int_{\frac{\nu_n-2}{\nu_n-1}}^1 h_n^{-1}J^{(2,n)}\colon\ \left(\bar{\nabla}_n\bar{\bar{\varphi}}\right)^{(2)}\ dx_3\ dx'$$
$$=-\int_\Omega A_nJ^n\colon\bar{\nabla}_n\bar{\bar{\varphi}}\ dx \tag{3.62}$$
$$-\int_S\int_0^{\frac{1}{\nu_n-1}} A_nJ^{(1,n)}\colon\ \left(\bar{\nabla}_n\bar{\bar{\varphi}}\right)^{(1)}\ dx_3\ dx' \tag{3.63}$$
$$-\int_S\int_{\frac{\nu_n-2}{\nu_n-1}}^1 A_nJ^{(2,n)}\colon\ \left(\bar{\nabla}_n\bar{\bar{\varphi}}\right)^{(2)}\ dx_3\ dx' \tag{3.64}$$
$$-\int_{\tilde{V}_n} h_n^{-3}\bar{\bar{f}}_n\cdot R_n^*\bar{\bar{\phi}}e_3\ dx. \tag{3.65}$$

At this point we need to distinguish between $\nu_n\equiv\nu\in\mathbb{N}$ and $\nu_n\to\infty$. We first look at the latter case. Here the terms (3.63) and (3.64) converge to 0 by equi-integrability and therefore

$$\int_\Omega h_n^{-1}J^n\colon\bar{\nabla}_n\bar{\bar{\varphi}}\ dx$$
$$+\int_S\int_0^{\frac{1}{\nu_n-1}} h_n^{-1}J^{(1,n)}\colon\ \left(\bar{\nabla}_n\bar{\bar{\varphi}}\right)^{(1)}\ dx_3\ dx'$$
$$+\int_S\int_{\frac{\nu_n-2}{\nu_n-1}}^1 h_n^{-1}J^{(2,n)}\colon\ \left(\bar{\nabla}_n\bar{\bar{\varphi}}\right)^{(2)}\ dx_3\ dx'$$
$$\underset{n\to\infty}{\longrightarrow}-\int_\Omega AJ\colon\nabla\varphi Z\ dx-\int_S f(x')\phi R^*e_3\ dx'.$$

Due to the cyclic invariance of the trace we have

$$\begin{aligned}AJ\colon\nabla\varphi Z&=Tr\left(AJZ^T\nabla\varphi^T\right)=Tr\left(JZ^T\nabla\varphi^TA\right)\\&=J\colon A^T\nabla\varphi Z=J\colon\begin{pmatrix}\nabla'v\otimes\nabla'\phi&0\\0&0\end{pmatrix}Z,\end{aligned}$$

therefore, by (3.37), Proposition 3.2.9, Lemma 3.2.13 and Corollary 3.2.17

$$\begin{aligned}&\int_\Omega AJ\colon\nabla\varphi Z\ dx\\&=\int_\Omega AJZ^T\colon\nabla\varphi\ dx\\&=\int_\Omega\mathcal{L}\left(GZ\right)\colon\begin{pmatrix}\nabla'v\otimes\nabla'\phi&0\\0&0\end{pmatrix}Z\ dx\\&=\int_S\mathcal{L}_2\left(\text{sym }G_1\right)\colon\nabla'v\otimes\nabla'\phi\ dx'.\end{aligned}$$

Thus (3.50) follows.

Now we focus on the case $\nu_n \equiv \nu \in \mathbb{N}$. We cannot neglect the surface terms and obtain

$$\begin{aligned}
&\int_\Omega h_n^{-1} J^n : \bar{\nabla}_n \bar{\bar{\varphi}} \, dx \\
&+ \int_S \int_0^{\frac{1}{\nu_n - 1}} h_n^{-1} J^{(1,n)} : \left(\bar{\nabla}_n \bar{\bar{\varphi}}\right)^{(1)} \, dx_3 \, dx' \\
&+ \int_S \int_{\frac{\nu_n - 2}{\nu_n - 1}}^1 h_n^{-1} J^{(2,n)} : \left(\bar{\nabla}_n \bar{\bar{\varphi}}\right)^{(2)} \, dx_3 \, dx' \\
&\xrightarrow[n\to\infty]{} - \int_\Omega AJ : \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \partial_1 \phi & \partial_2 \phi & 0 \end{pmatrix} Z \, dx \\
&- \int_S \int_0^{\frac{1}{\nu - 1}} AJ^{(1)} : \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \partial_1 \phi & \partial_2 \phi & 0 \end{pmatrix} Z^{(1)} \, dx_3 \, dx' \\
&- \int_S \int_{\frac{\nu - 2}{\nu - 1}}^1 AJ^{(2)} : \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \partial_1 \phi & \partial_2 \phi & 0 \end{pmatrix} Z^{(1)} \, dx_3 \, dx' \\
&- \int_S \frac{\nu}{\nu - 1} f(x') \cdot R^* \phi e_3 \, dx'.
\end{aligned}$$

as $n \to \infty$. By Proposition 3.2.9, Lemma 3.2.18, Lemma 3.2.16, Corollary 2.3.3 and (3.34) we see that this term is equal to

$$\begin{aligned}
&- \int_S \frac{1}{2} DQ_{\text{cell}}^{\text{rel}} \left(\begin{pmatrix} \text{sym} G_1 & 0 \\ 0 & 0 \end{pmatrix} Z + \frac{1}{2(\nu - 1)} G_3 \right) : \begin{pmatrix} \nabla' v \otimes \nabla' \phi & 0 \\ 0 & 0 \end{pmatrix} Z \, dx' \\
&- \int_S \frac{1}{\nu - 1} DQ_{\text{surf}} \left(\begin{pmatrix} \text{sym} G_1 & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} + \frac{\partial_{12} v}{2(\nu - 1)} M^{(1)} \right) : \begin{pmatrix} \nabla' v \otimes \nabla' \phi & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} \, dx' \\
&- \int_S \frac{\nu}{\nu - 1} f(x') \cdot R^* \phi e_3 \, dx',
\end{aligned}$$

and (3.51) follows.

Part 2

Let $\eta \in C_c^\infty(S; \mathbb{R}^2)$ and

$$\varphi(x) = \begin{pmatrix} \eta(x') \\ 0 \end{pmatrix} \left(x_3 - \frac{1}{2} \right).$$

Later we will choose $\eta = \nabla' \phi$ with ϕ from part 1. But to keep notation clearer we remain temporarily with η. Again testing the equilibrium equation with φ we get

$$\begin{aligned}
0 = &\int_\Omega R_n J^n : \bar{\nabla}_n \bar{\bar{\varphi}} \, dx \\
&+ \int_S \int_0^{\frac{1}{\nu_n - 1}} R_n J^{(1,n)} : \left(\bar{\nabla}_n \bar{\bar{\varphi}}\right)^{(1)} \, dx_3 \, dx'
\end{aligned}$$

$$+\int_S \int_{\frac{\nu_n-2}{\nu_n-1}}^1 R_n J^{(2,n)} : \left(\bar{\nabla}_n \bar{\bar{\varphi}}\right)^{(2)} \ dx_3 \ dx' + \int_{\tilde{V}_n} h_n^{-2} \bar{\bar{f}}_n R_n^* \bar{\bar{\varphi}} \ dx. \tag{3.66}$$

With the product rule of the alternative discrete gradient we have

$$\bar{D}_l^n \varphi(\hat{x}) = \begin{pmatrix} \bar{D}_l^n \eta(\hat{x}') \\ 0 \end{pmatrix} \left(\hat{x}_3 + \frac{\varepsilon_n}{h_n} a_3^l - \frac{1}{2}\right) + h_n^{-1} a_3^l \begin{pmatrix} \eta(\hat{x}') \\ 0 \end{pmatrix}.$$

Exemplarily we look at one summand of the bulk part in (3.66), the surface terms work the same way:

$$\int_\Omega (R_n J^n)_{\cdot l} \cdot \bar{D}_l^n \varphi_n(\hat{x}) \ dx$$

$$= \int_\Omega (R_n J^n)_{\cdot l} \cdot \begin{pmatrix} \bar{D}_l^n \eta(x') \\ 0 \end{pmatrix} \left(\hat{x}_3 + \frac{\varepsilon_n}{h_n} a_3^l - \frac{1}{2}\right) \ dx \tag{3.67}$$

$$+ \int_\Omega (R_n J^n)_{\cdot l} \cdot h_n^{-1} a_3^l \begin{pmatrix} \eta(\hat{x}') \\ 0 \end{pmatrix} \ dx. \tag{3.68}$$

From (3.66) together with $\bar{\partial}_l^n \varphi = \bar{D}_l^n \varphi - \frac{1}{8}\sum_{k=1}^8 \bar{D}_k^n \varphi$ and $z_3^l = a_3^l - \frac{1}{8}\sum_{k=1}^8 a_3^k$ we obtain

$$\begin{aligned}
&\sum_{l=1}^8 \int_\Omega h_n^{-1} (R_n J^n)_{\cdot l} \cdot z_3^l \begin{pmatrix} \eta(\hat{x}') \\ 0 \end{pmatrix} \ dx \\
&+ \sum_{l=1}^4 \int_S \int_0^{\frac{1}{\nu-1}} h_n^{-1} \left(R_n J^{(1,n)}\right)_{\cdot l} \cdot z_3^l \begin{pmatrix} \eta(\hat{x}') \\ 0 \end{pmatrix} \ dx_3 \ dx' \\
&+ \sum_{l=1}^4 \int_S \int_{\frac{\nu-2}{\nu-1}}^1 h_n^{-1} \left(R_n J^{(2,n)}\right)_{\cdot l} \cdot z_3^{l+4} \begin{pmatrix} \eta(\hat{x}') \\ 0 \end{pmatrix} \ dx_3 \ dx' \\
&= -\sum_{l=1}^8 \Bigg[\int_\Omega (R_n J^n)_{\cdot l} \cdot \begin{pmatrix} \bar{D}_l^n \eta(\hat{x}') \\ 0 \end{pmatrix} \left(\hat{x}_3 + \frac{\varepsilon_n}{h_n} a_3^l - \frac{1}{2}\right) \ dx \\
&\quad - \frac{1}{8} \sum_{k=1}^8 \int_\Omega (R_n J^n)_{\cdot l} \cdot \begin{pmatrix} \bar{D}_k^n \eta(\hat{x}') \\ 0 \end{pmatrix} \left(\hat{x}_3 + \frac{\varepsilon_n}{h_n} a_3^k - \frac{1}{2}\right) \ dx \Bigg] \\
&- \sum_{l=1}^4 \Bigg[\int_S \int_0^{\frac{1}{\nu-1}} \left(R_n J^{(1,n)}\right)_{\cdot l} \cdot \begin{pmatrix} \bar{D}_l^n \eta(\hat{x}') \\ 0 \end{pmatrix} \left(\hat{x}_3 + \frac{\varepsilon_n}{h_n} a_3^l - \frac{1}{2}\right) \ dx_3 \ dx' \\
&- \frac{1}{8} \sum_{k=1}^8 \int_S \int_0^{\frac{1}{\nu-1}} \left(R_n J^{(1,n)}\right)_{\cdot l} \cdot \begin{pmatrix} \bar{D}_k^n \eta(\hat{x}') \\ 0 \end{pmatrix} \left(\hat{x}_3 + \frac{\varepsilon_n}{h_n} a_3^k - \frac{1}{2}\right) \ dx_3 \ dx' \Bigg] \\
&- \sum_{l=1}^4 \Bigg[\int_S \int_{\frac{\nu-2}{\nu-1}}^1 \left(R_n J^{(2,n)}\right)_{\cdot l} \cdot \begin{pmatrix} \bar{D}_{l+4}^n \eta(\hat{x}') \\ 0 \end{pmatrix} \left(\hat{x}_3 + \frac{\varepsilon_n}{h_n} a_3^{l+4} - \frac{1}{2}\right) \ dx_3 \ dx' \\
&- \frac{1}{8} \sum_{k=1}^8 \int_S \int_{\frac{\nu-2}{\nu-1}}^1 \left(R_n J^{(2,n)}\right)_{\cdot l} \cdot \begin{pmatrix} \bar{D}_k^n \eta(\hat{x}') \\ 0 \end{pmatrix} \left(\hat{x}_3 + \frac{\varepsilon_n}{h_n} a_3^k - \frac{1}{2}\right) \ dx_3 \ dx' \Bigg]
\end{aligned}$$

$$-\int_{\tilde{V}_n} h_n^{-2} \bar{\bar{f}}_n R_n^* \bar{\bar{\varphi}}\, dx. \tag{3.69}$$

If $\nu_n \to \infty$ the force term and the surface terms vanish. Hence, by (3.37), Lemma 3.2.16 and Corollary 3.2.17

$$
\begin{aligned}
(3.69) \underset{n\to\infty}{\longrightarrow} -\sum_{l=1}^{8} \int_\Omega \left(x_3 - \frac{1}{2}\right) J_{\cdot l} \cdot \begin{pmatrix} \nabla'\eta\left((a^l)' - \frac{1}{8}\sum_{k=1}^8 (a^k)'\right) \\ 0 \end{pmatrix} dx \\
= -\sum_{l=1}^{8} \int_\Omega \left(x_3 - \frac{1}{2}\right) J_{\cdot l} \cdot \begin{pmatrix} \nabla'\eta (z^l)' \\ 0 \end{pmatrix} dx \\
= -\int_\Omega \left(x_3 - \frac{1}{2}\right) J \colon \begin{pmatrix} \nabla'\eta & 0 \\ 0 & 0 \end{pmatrix} Z\, dx \\
= -\int_\Omega \left(x_3 - \frac{1}{2}\right) \mathcal{L}_2(G'') \colon \nabla'\eta(x')\, dx \\
= -\int_S \int_0^1 \left(x_3 - \frac{1}{2}\right)^2 \mathcal{L}_2(G_2) \colon \nabla'\eta(x')\, dx' \\
= \int_S \frac{1}{12} \mathcal{L}_2\left(\nabla'^2 v\right) : \nabla'\eta(x')\, dx'.
\end{aligned}
$$

For the special choice $\eta = \nabla'\phi$ we get the convergence (3.52). If $\nu_n \equiv \nu \in \mathbb{N}$

$$
\begin{aligned}
(3.69) \underset{n\to\infty}{\longrightarrow} -\sum_{l=1}^{8} \Bigg[\int_\Omega J_{\cdot l} \cdot \begin{pmatrix} \nabla'\eta\left(a^l\right)' \\ 0 \end{pmatrix} \left(\hat{x}_3 + \frac{1}{\nu-1} a_3^l - \frac{1}{2}\right) dx \\
-\frac{1}{8}\sum_{k=1}^{8} \int_\Omega J_{\cdot l} \cdot \begin{pmatrix} \nabla'\eta\left(a^k\right)' \\ 0 \end{pmatrix} \left(\hat{x}_3 + \frac{1}{\nu-1} a_3^k - \frac{1}{2}\right) dx \Bigg] \\
-\sum_{l=1}^{4} \Bigg[\int_S \int_0^{\frac{1}{\nu-1}} J_{\cdot\cdot l}^{(1)} \cdot \begin{pmatrix} \nabla'\eta\left(a^l\right)' \\ 0 \end{pmatrix} \left(\hat{x}_3 + \frac{1}{\nu-1} a_3^l - \frac{1}{2}\right) dx_3\, dx' \\
-\frac{1}{8}\sum_{k=1}^{8} \int_S \int_0^{\frac{1}{\nu-1}} J_{\cdot l}^{(1)} \cdot \begin{pmatrix} \nabla'\eta\left(a^k\right)' \\ 0 \end{pmatrix} \left(\hat{x}_3 + \frac{1}{\nu-1} a_3^k - \frac{1}{2}\right) dx_3\, dx' \Bigg] \\
-\sum_{l=1}^{4} \Bigg[\int_S \int_{\frac{\nu-2}{\nu-1}}^{1} J_{\cdot l}^{(2)} \cdot \begin{pmatrix} \nabla'\eta\left(a^{l+4}\right)' \\ 0 \end{pmatrix} \left(\hat{x}_3 + \frac{1}{\nu-1} a_3^{l+4} - \frac{1}{2}\right) dx_3\, dx' \\
-\frac{1}{8}\sum_{k=1}^{8} \int_S \int_{\frac{\nu-2}{\nu-1}}^{1} J_{\cdot l}^{(2)} \cdot \begin{pmatrix} \nabla'\eta\left(a^k\right)' \\ 0 \end{pmatrix} \left(\hat{x}_3 + \frac{1}{\nu-1} a_3^k - \frac{1}{2}\right) dx_3\, dx' \Bigg].
\end{aligned}
\tag{3.70}
$$

Note that, for $x_3 \in \left[\frac{m-1}{\nu-1}, \frac{m}{\nu-1}\right)$,

$$\hat{x}_3 + \frac{1}{\nu-1} a_3^l - \frac{1}{2} = \begin{cases} \frac{2m-\nu-1}{2(\nu-1)}, & l \in \{1,2,3,4\}, \\ \frac{2m-\nu+1}{2(\nu-1)}, & l \in \{5,6,7,8\} \end{cases}$$

for $m \in \{1, \cdots, \nu-1\}$. Then for $l \in \{1,2,3,4\}$

$$\begin{aligned} &\begin{pmatrix} \nabla'\eta\left(a^l\right)' \\ 0 \end{pmatrix}\left(\hat{x}_3 + \frac{1}{\nu-1}a_3^l - \frac{1}{2}\right) - \frac{1}{8}\sum_{k=1}^{8}\begin{pmatrix} \nabla'\eta\left(a^k\right)' \\ 0 \end{pmatrix}\left(\hat{x}_3 + \frac{1}{\nu-1}a_3^k - \frac{1}{2}\right) \\ &\qquad = \begin{pmatrix} \nabla'\eta\left(z^l\right)' \\ 0 \end{pmatrix}\frac{2m-\nu-1}{2(\nu-1)} - \frac{1}{8}\sum_{k=5}^{8}\begin{pmatrix} \nabla'\eta\left(a^k\right)' \\ 0 \end{pmatrix}\frac{1}{\nu-1} \end{aligned} \tag{3.71}$$

and for $l \in \{5,6,7,8\}$

$$\begin{aligned} &\begin{pmatrix} \nabla'\eta\left(a^l\right)' \\ 0 \end{pmatrix}\left(\hat{x}_3 + \frac{1}{\nu-1}a_3^l - \frac{1}{2}\right) - \frac{1}{8}\sum_{k=1}^{8}\begin{pmatrix} \nabla'\eta\left(a^k\right)' \\ 0 \end{pmatrix}\left(\hat{x}_3 + \frac{1}{\nu-1}a_3^k - \frac{1}{2}\right) \\ &\qquad = \begin{pmatrix} \nabla'\eta\left(z^l\right)' \\ 0 \end{pmatrix}\frac{2m-\nu+1}{2(\nu-1)} + \frac{1}{8}\sum_{k=1}^{4}\begin{pmatrix} \nabla'\eta\left(a^k\right)' \\ 0 \end{pmatrix}\frac{1}{\nu-1}. \end{aligned} \tag{3.72}$$

Thus, after multiplication with -1, the limiting terms in (3.70) are equal to

$$\begin{aligned} &\sum_{l=1}^{8}\int_S\sum_{m=1}^{\nu-1}\int_{\frac{m-1}{\nu-1}}^{\frac{m}{\nu-1}} J_{\cdot l}\cdot\begin{pmatrix} \nabla'\eta\left(z^l\right)' \\ 0 \end{pmatrix}\left(\frac{2m-\nu}{2(\nu-1)}\right)\,dx_3\,dx' \\ &-\sum_{l=1}^{4}\int_S\sum_{m=1}^{\nu-1}\int_{\frac{m-1}{\nu-1}}^{\frac{m}{\nu-1}} J_{\cdot l}\cdot\begin{pmatrix} \nabla'\eta\left(z^l\right)' \\ 0 \end{pmatrix}\frac{1}{2(\nu-1)}\,dx_3\,dx' \\ &+\sum_{l=5}^{8}\int_S\sum_{m=1}^{\nu-1}\int_{\frac{m-1}{\nu-1}}^{\frac{m}{\nu-1}} J_{\cdot l}\cdot\begin{pmatrix} \nabla'\eta\left(z^l\right)' \\ 0 \end{pmatrix}\frac{1}{2(\nu-1)}\,dx_3\,dx' \\ &-\sum_{l=1}^{4}\frac{1}{8}\sum_{k=5}^{8}\int_S\sum_{m=1}^{\nu-1}\int_{\frac{m-1}{\nu-1}}^{\frac{m}{\nu-1}} J_{\cdot l}\cdot\begin{pmatrix} \nabla'\eta\left(a^k\right)' \\ 0 \end{pmatrix}\frac{1}{\nu-1}\,dx_3\,dx' \\ &+\sum_{l=5}^{8}\frac{1}{8}\sum_{k=1}^{4}\int_S\sum_{m=1}^{\nu-1}\int_{\frac{m-1}{\nu-1}}^{\frac{m}{\nu-1}} J_{\cdot l}\cdot\begin{pmatrix} \nabla'\eta\left(a^k\right)' \\ 0 \end{pmatrix}\frac{1}{\nu-1}\,dx_3\,dx' \\ &+\sum_{l=1}^{4}\int_S\int_0^{\frac{1}{\nu-1}} J_{\cdot l}^{(1)}\cdot\begin{pmatrix} \nabla'\eta\left(z^l\right)' \\ 0 \end{pmatrix}\left(-\frac{1}{2}\right)\,dx_3\,dx' \\ &-\sum_{l=1}^{4}\frac{1}{8}\sum_{k=1}^{4}\int_S\int_0^{\frac{1}{\nu-1}} J_{\cdot l}^{(1)}\cdot\begin{pmatrix} \nabla'\eta\left(a^k\right)' \\ 0 \end{pmatrix}\left(\frac{1}{\nu-1}\right)\,dx_3\,dx' \\ &+\sum_{l=1}^{4}\int_S\int_{\frac{\nu-2}{\nu-1}}^{1} J_{\cdot l}^{(2)}\cdot\begin{pmatrix} \nabla'\eta\left(z^l\right)' \\ 0 \end{pmatrix}\left(\frac{1}{2}\right)\,dx_3\,dx' \\ &+\sum_{l=1}^{4}\frac{1}{8}\sum_{k=1}^{4}\int_S\int_{\frac{\nu-2}{\nu-1}}^{1} J_{\cdot l}^{(2)}\cdot\begin{pmatrix} \nabla'\eta\left(a^k\right)' \\ 0 \end{pmatrix}\left(\frac{1}{\nu-1}\right)\,dx_3\,dx'. \end{aligned}$$

We observe that

$$\frac{1}{\nu-1}\sum_{k=1}^{4}\begin{pmatrix} \nabla'\eta\left(a^k\right)' \\ 0 \end{pmatrix} = \frac{1}{\nu-1}\begin{pmatrix} \nabla'\eta\begin{pmatrix} 2 \\ 2 \end{pmatrix} \\ 0 \end{pmatrix} = \frac{1}{\nu-1}\sum_{k=5}^{8}\begin{pmatrix} \nabla'\eta\left(a^k\right)' \\ 0 \end{pmatrix}.$$

By Lemma 3.2.14 all the terms containing the factor $\frac{1}{\nu-1}$ sum up to 0. Therefore the remaining part of the limiting terms of (3.70) is given by

$$\begin{aligned}
&\sum_{l=1}^{8} \int_S \sum_{m=1}^{\nu-1} \int_{\frac{m-1}{\nu-1}}^{\frac{m}{\nu-1}} J_{\cdot l} \cdot \begin{pmatrix} \nabla'\eta \left(z^l\right)' \\ 0 \end{pmatrix} \left(\frac{2m-\nu}{2(\nu-1)}\right) \, dx_3 \, dx' \\
&- \sum_{l=1}^{4} \int_S \sum_{m=1}^{\nu-1} \int_{\frac{m-1}{\nu-1}}^{\frac{m}{\nu-1}} J_{\cdot l} \cdot \begin{pmatrix} \nabla'\eta \left(z^l\right)' \\ 0 \end{pmatrix} \frac{1}{2(\nu-1)} \, dx_3 \, dx' \\
&+ \sum_{l=5}^{8} \int_S \sum_{m=1}^{\nu-1} \int_{\frac{m-1}{\nu-1}}^{\frac{m}{\nu-1}} J_{\cdot l} \cdot \begin{pmatrix} \nabla'\eta \left(z^l\right)' \\ 0 \end{pmatrix} \frac{1}{2(\nu-1)} \, dx_3 \, dx' \\
&+ \sum_{l=1}^{4} \int_S \int_0^{\frac{1}{\nu-1}} J^{(1)}_{\cdot l} \cdot \begin{pmatrix} \nabla'\eta \left(z^l\right)' \\ 0 \end{pmatrix} \left(-\frac{1}{2}\right) \, dx_3 \, dx' \\
&+ \sum_{l=1}^{4} \int_S \int_{\frac{\nu-2}{\nu-1}}^{1} J^{(2)}_{\cdot l} \cdot \begin{pmatrix} \nabla'\eta \left(z^l\right)' \\ 0 \end{pmatrix} \left(\frac{1}{2}\right) \, dx_3 \, dx' \\
&= \int_S \frac{\nu(\nu-2)}{24(\nu-1)^2} DQ^{\text{rel}}_{\text{cell}} \left(\begin{pmatrix} G_2 & 0 \\ 0 & 0 \end{pmatrix} Z \right) : \begin{pmatrix} \nabla'\eta & 0 \\ 0 & 0 \end{pmatrix} Z \, dx' \\
&+ \int_S \frac{1}{4(\nu-1)} DQ^{\text{rel}}_{\text{cell}} \left(\begin{pmatrix} \text{sym } G_1 & 0 \\ 0 & 0 \end{pmatrix} Z + \frac{1}{2(\nu-1)} G_3 \right) : \begin{pmatrix} \nabla'\eta & 0 \\ 0 & 0 \end{pmatrix} Z_- \, dx' \\
&+ \int_S \frac{1}{4(\nu-1)} DQ_{\text{surf}} \left(\begin{pmatrix} G_2 & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} \right) : \begin{pmatrix} \nabla'\eta & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} \, dx'. \qquad (3.73)
\end{aligned}$$

The last equality can be seen as follows: First we look at the sum of the two terms of (3.73) which contain the factor $\frac{1}{2(\nu-1)}$. Because of part (ii) of Remark 3.2.10, Lemma 3.2.18, Lemma 3.2.15 and Lemma 3.2.16 we have

$$\begin{aligned}
&- \sum_{l=1}^{4} \int_S \sum_{m=1}^{\nu-1} \int_{\frac{m-1}{\nu-1}}^{\frac{m}{\nu-1}} J_{\cdot l} \cdot \begin{pmatrix} \nabla'\eta \left(z^l\right)' \\ 0 \end{pmatrix} \frac{1}{2(\nu-1)} \, dx_3 \, dx' \\
&+ \sum_{l=5}^{8} \int_S \sum_{m=1}^{\nu-1} \int_{\frac{m-1}{\nu-1}}^{\frac{m}{\nu-1}} J_{\cdot l} \cdot \begin{pmatrix} \nabla'\eta \left(z^l\right)' \\ 0 \end{pmatrix} \frac{1}{2(\nu-1)} \, dx_3 \, dx' \\
&= \frac{1}{2(\nu-1)} \int_S \sum_{m=1}^{\nu-1} \int_{\frac{m-1}{\nu-1}}^{\frac{m}{\nu-1}} J : \begin{pmatrix} \nabla'\eta & 0 \\ 0 & 0 \end{pmatrix} Z_- \, dx_3 \, dx' \\
&= \frac{1}{2(\nu-1)} \int_S \sum_{m=1}^{\nu-1} \int_{\frac{m-1}{\nu-1}}^{\frac{m}{\nu-1}} \frac{1}{2} DQ^{\text{rel}}_{\text{cell}} \left(\begin{pmatrix} (PG)'' & 0 \\ 0 & 0 \end{pmatrix} Z + \frac{1}{2(\nu-1)} G_3 \right) : \\
&\qquad \begin{pmatrix} \nabla'\eta & 0 \\ 0 & 0 \end{pmatrix} Z_- \, dx_3 \, dx' \\
&= \int_S \frac{1}{4(\nu-1)} DQ^{\text{rel}}_{\text{cell}} \left(\begin{pmatrix} \text{sym } G_1 & 0 \\ 0 & 0 \end{pmatrix} Z + \frac{1}{2(\nu-1)} G_3 \right) : \begin{pmatrix} \nabla'\eta & 0 \\ 0 & 0 \end{pmatrix} Z_- \, dx'.
\end{aligned}$$

To calculate the first term of (3.73) we note that

$$\frac{1}{\nu-1}\sum_{m=1}^{\nu-1}\frac{(2m-\nu)^2}{(2(\nu-1))^2}=\frac{\nu(\nu-2)}{12(\nu-1)^2}$$

and

$$\sum_{m=1}^{\nu-1}(2m-\nu)=0.$$

Thus

$$\begin{aligned}
&\sum_{l=1}^{8}\int_S\sum_{m=1}^{\nu-1}\int_{\frac{m-1}{\nu-1}}^{\frac{m}{\nu-1}} J_{\cdot l}\cdot\begin{pmatrix}\nabla'\eta\left(z^l\right)'\\ 0\end{pmatrix}\left(\frac{2m-\nu}{2(\nu-1)}\right)\,dx_3\,dx'\\
&\quad=\int_S\Bigg\{\left[\sum_{m=1}^{\nu-1}\frac{2m-\nu}{2(\nu-1)}\int_{\frac{m-1}{\nu-1}}^{\frac{m}{\nu-1}}P\left(id-\frac{1}{2}\right)(x_3)\,dx_3\right]\\
&\quad\quad\frac{1}{2}DQ_{\text{cell}}^{\text{rel}}\left(\begin{pmatrix}G_2&0\\0&0\end{pmatrix}Z\right):\begin{pmatrix}\nabla'\eta&0\\0&0\end{pmatrix}Z\,dx'\Bigg\}\\
&=\int_S\frac{1}{\nu-1}\sum_{m=1}^{\nu-1}\left(\frac{2m-\nu}{2(\nu-1)}\right)^2\frac{1}{2}DQ_{\text{cell}}^{\text{rel}}\left(\begin{pmatrix}G_2&0\\0&0\end{pmatrix}Z\right):\begin{pmatrix}\nabla'\eta&0\\0&0\end{pmatrix}Z\,dx'\\
&\quad=\int_S\frac{\nu(\nu-2)}{24(\nu-1)^2}DQ_{\text{cell}}^{\text{rel}}\left(\begin{pmatrix}G_2&0\\0&0\end{pmatrix}Z\right):\begin{pmatrix}\nabla'\eta&0\\0&0\end{pmatrix}Z\,dx'.
\end{aligned}$$

Finally, by (3.35)

$$\begin{aligned}
&+\sum_{l=1}^{4}\int_S\int_0^{\frac{1}{\nu-1}}J_{\cdot l}^{(1)}\cdot\begin{pmatrix}\nabla'\eta\left(z^l\right)'\\0\end{pmatrix}\left(-\frac{1}{2}\right)\,dx_3\,dx'\\
&\quad+\sum_{l=1}^{4}\int_S\int_{\frac{\nu-2}{\nu-1}}^{1}J_{\cdot l}^{(2)}\cdot\begin{pmatrix}\nabla'\eta\left(z^l\right)'\\0\end{pmatrix}\left(\frac{1}{2}\right)\,dx_3\,dx'\\
&=\frac{1}{4(\nu-1)}\int_S DQ_{\text{surf}}\left(\begin{pmatrix}G_2&0\\0&0\end{pmatrix}Z^{(1)}\right):\begin{pmatrix}\nabla'\eta&0\\0&0\end{pmatrix}Z^{(1)}\,dx'.
\end{aligned}$$

Summarized, letting $\eta=\nabla'\phi$ we obtain (3.53).

Part 3

It remains to determine the limit of (3.54). First we simplify the latter three terms of (3.54). A single summand can be written as

$$\begin{aligned}
&\int_\Omega h_n^{-1}R_nJ_{\cdot l}^n\cdot z_3^l\begin{pmatrix}\nabla'\phi(\hat{x}')\\0\end{pmatrix}\,dx\\
&\quad=\int_\Omega A_nJ_{\cdot l}^n\cdot z_3^l\begin{pmatrix}\nabla'\phi(\hat{x}')\\0\end{pmatrix}\,dx\\
&\quad+\int_\Omega h_n^{-1}J_{\cdot l}^n\cdot z_3^l\begin{pmatrix}\nabla'\phi(\hat{x}')\\0\end{pmatrix}\,dx.
\end{aligned}\tag{3.74}$$

Then we consider all terms of (3.74) that contain A_n and get for $\nu_n \to \infty$

$$\sum_{l=1}^{8} \int_\Omega A_n J^n_{\cdot l} \cdot z_3^l \begin{pmatrix} \nabla'\phi(\hat{x}') \\ 0 \end{pmatrix} dx$$
$$+ \sum_{l=1}^{4} \int_S \int_0^{\frac{1}{\nu_n - 1}} A_n J^{(1,n)}_{\cdot l} \cdot z_3^l \begin{pmatrix} \nabla'\phi(\hat{x}') \\ 0 \end{pmatrix} dx_3\, dx'$$
$$+ \sum_{l=1}^{4} \int_S \int_{\frac{\nu_n - 2}{\nu_n - 1}}^{1} A_n J^{(2,n)}_{\cdot l} \cdot z_3^{l+4} \begin{pmatrix} \nabla'\phi(\hat{x}') \\ 0 \end{pmatrix} dx_3\, dx'$$
$$\underset{n\to\infty}{\longrightarrow} \sum_{l=1}^{8} \int_\Omega J_{\cdot l} \cdot z_3^l A^T \begin{pmatrix} \nabla'\phi \\ 0 \end{pmatrix} dx = 0$$

due to (3.26) and Lemma 3.2.13, whereas for $\nu_n \equiv \nu \in \mathbb{N}$

$$\sum_{l=1}^{8} \int_\Omega A_n J^n_{\cdot l} \cdot z_3^l \begin{pmatrix} \nabla'\phi(\hat{x}') \\ 0 \end{pmatrix} dx$$
$$+ \sum_{l=1}^{4} \int_S \int_0^{\frac{1}{\nu_n - 1}} A_n J^{(1,n)}_{\cdot l} \cdot z_3^l \begin{pmatrix} \nabla'\phi(\hat{x}') \\ 0 \end{pmatrix} dx_3\, dx'$$
$$+ \sum_{l=1}^{4} \int_S \int_{\frac{\nu_n - 2}{\nu_n - 1}}^{1} A_n J^{(2,n)}_{\cdot l} \cdot z_3^{l+4} \begin{pmatrix} \nabla'\phi(\hat{x}') \\ 0 \end{pmatrix} dx_3\, dx'$$
$$\underset{n\to\infty}{\longrightarrow} \sum_{l=1}^{8} \int_\Omega J_{\cdot l} \cdot z_3^l A^T \begin{pmatrix} \nabla'\phi \\ 0 \end{pmatrix} dx$$
$$+ \sum_{l=1}^{4} \int_S \int_0^{\frac{1}{\nu - 1}} J^{(1)}_{\cdot l} \cdot z_3^l A^T \begin{pmatrix} \nabla'\phi \\ 0 \end{pmatrix}$$
$$+ \sum_{l=1}^{4} \int_S \int_{\frac{\nu - 2}{\nu - 1}}^{1} J^{(2)}_{\cdot l} \cdot z_3^{l+4} A^T \begin{pmatrix} \nabla'\phi \\ 0 \end{pmatrix} = 0$$

due to Lemma 3.2.14. Thus, instead of (3.54), it is sufficient to look at the limit of

$$\int_\Omega h_n^{-1} J^n \colon \bar{\nabla}_n \bar{\bar{\varphi}}\, dx$$
$$+ \int_S \int_0^{\frac{1}{\nu-1}} h_n^{-1} J^{(1,n)} \colon \left(\bar{\nabla}_n \bar{\bar{\varphi}}\right)^{(1)} dx_3\, dx'$$
$$+ \int_S \int_{\frac{\nu-2}{\nu-1}}^{1} h_n^{-1} J^{(2,n)} \colon \left(\bar{\nabla}_n \bar{\bar{\varphi}}\right)^{(2)} dx_3\, dx'$$
$$- \sum_{l=1}^{8} \int_\Omega h_n^{-1} J^n_{\cdot l} \cdot z_3^l \begin{pmatrix} \nabla'\phi(\hat{x}') \\ 0 \end{pmatrix} dx_3\, dx'$$
$$- \sum_{l=1}^{4} \int_S \int_0^{\frac{1}{\nu-1}} h_n^{-1} J^{(1,n)}_{\cdot l} \cdot z_3^l \begin{pmatrix} \nabla'\phi(\hat{x}') \\ 0 \end{pmatrix} dx_3\, dx'$$

$$-\sum_{l=1}^{4}\int_S\int_{\frac{\nu-2}{\nu-1}}^{1} h_n^{-1}J_{\cdot l}^{(2,n)}\cdot z_3^{l+4}\begin{pmatrix}\nabla'\phi(\hat{x}')\\ 0\end{pmatrix}\,dx_3\,dx'. \tag{3.75}$$

In the following we denote by $\tilde{x}$ the center of a cell $\tilde{x}+\left(-\frac{\varepsilon_n}{2},\frac{\varepsilon_n}{2}\right)^2\times\left(-\frac{\varepsilon_n}{2h_n},\frac{\varepsilon_n}{2h_n}\right)$, where $\tilde{x}\in\tilde{\Lambda}_n'$. Written as a sum and a subsequent Taylor expansion of the discrete derivative we get

$$
\begin{aligned}
&(3.75) = \\
&\sum_{l=1}^{8}\int_\Omega h_n^{-1}J_{\cdot l}^{n}\Bigg[\frac{1}{\varepsilon_n}\left(\phi\left(\tilde{x}'+\varepsilon_n\left(z^l\right)'\right)-\frac{1}{8}\sum_{j=1}^{8}\phi\left(\tilde{x}'+\varepsilon_n\left(z^j\right)'\right)\right)e_3-z_3^l\begin{pmatrix}\partial_1\phi(\hat{x}')\\ \partial_2\phi(\hat{x}')\\ 0\end{pmatrix}\Bigg]\,dx\\
&+\sum_{l=1}^{4}\int_S\int_0^{\frac{1}{\nu_n-1}} h_n^{-1}J_{\cdot l}^{(1,n)}\Bigg[\frac{1}{\varepsilon_n}\left(\phi\left(\tilde{x}'+\varepsilon_n\left(z^l\right)'\right)-\frac{1}{8}\sum_{j=1}^{8}\phi\left(\tilde{x}'+\varepsilon_n\left(z^j\right)'\right)\right)e_3\\
&\qquad -z_3^l\begin{pmatrix}\partial_1\phi(\hat{x}')\\ \partial_2\phi(\hat{x}')\\ 0\end{pmatrix}\Bigg]\,dx_3\,dx'\\
&+\sum_{l=1}^{4}\int_S\int_{\frac{\nu_n-2}{\nu_n-1}}^{1} h_n^{-1}J_{\cdot l}^{(2,n)}\Bigg[\frac{1}{\varepsilon_n}\left(\phi\left(\tilde{x}'+\varepsilon_n\left(z^{l+4}\right)'\right)-\frac{1}{8}\sum_{j=1}^{8}\phi\left(\tilde{x}'+\varepsilon_n\left(z^j\right)'\right)\right)e_3\\
&\qquad -z_3^l\begin{pmatrix}\partial_1\phi(\hat{x}')\\ \partial_2\phi(\hat{x}')\\ 0\end{pmatrix}\Bigg]\,dx_3\,dx'\\
&=\sum_{l=1}^{8}\int_\Omega h_n^{-1}J_{\cdot l}^{n}\cdot\Bigg[\Bigg(\nabla'\phi(\tilde{x}')\cdot\left(z^l\right)'+\frac{\varepsilon_n}{2}\nabla'^2\phi(\tilde{x}')\left[\left(z^l\right)',\left(z^l\right)'\right]\\
&\qquad -\frac{1}{8}\sum_{j=1}^{8}\frac{\varepsilon_n}{2}\nabla'^2\phi(\tilde{x}')\left[\left(z^j\right)',\left(z^j\right)'\right]+O(\varepsilon_n^2)\Bigg)e_3-z_3^l\begin{pmatrix}\partial_1\phi(\hat{x}')\\ \partial_2\phi(\hat{x}')\\ 0\end{pmatrix}\Bigg]\,dx\\
&+\sum_{l=1}^{4}\int_S\int_0^{\frac{1}{\nu_n-1}} h_n^{-1}J_{\cdot l}^{(1,n)}\cdot\Bigg[\Bigg(\nabla'\phi(\tilde{x}')\cdot\left(z^l\right)'+\frac{\varepsilon_n}{2}\nabla'^2\phi(\tilde{x}')\left[\left(z^l\right)',\left(z^l\right)'\right]\\
&\qquad -\frac{1}{8}\sum_{j=1}^{8}\frac{\varepsilon_n}{2}\nabla'^2\phi(\tilde{x}')\left[\left(z^j\right)',\left(z^j\right)'\right]+O(\varepsilon_n^2)\Bigg)e_3-z_3^l\begin{pmatrix}\partial_1\phi(\hat{x}')\\ \partial_2\phi(\hat{x}')\\ 0\end{pmatrix}\Bigg]\,dx_3\,dx'\\
&+\sum_{l=1}^{4}\int_S\int_{\frac{\nu_n-2}{\nu_n-1}}^{1} h_n^{-1}J_{\cdot l}^{(2,n)}\cdot\Bigg[\Bigg(\nabla'\phi(\tilde{x}')\cdot\left(z^l\right)'+\frac{\varepsilon_n}{2}\nabla'^2\phi(\tilde{x}')\left[\left(z^l\right)',\left(z^l\right)'\right]\\
&\qquad -\frac{1}{8}\sum_{j=1}^{8}\frac{\varepsilon_n}{2}\nabla'^2\phi(\tilde{x}')\left[\left(z^j\right)',\left(z^j\right)'\right]+O(\varepsilon_n^2)\Bigg)e_3-z_3^{l+4}\begin{pmatrix}\partial_1\phi(\hat{x}')\\ \partial_2\phi(\hat{x}')\\ 0\end{pmatrix}\Bigg]\,dx_3\,dx'.
\end{aligned}
$$

Note that

$$\nabla'^2\phi(x')\left[(z^i)',(z^i)'\right]=\begin{cases}\frac{1}{4}\left(\partial_{11}v(x')+2\partial_{12}v(x')+\partial_{22}v(x')\right), & i\in\{1,3,5,7\},\\ \frac{1}{4}\left(\partial_{11}v(x')-2\partial_{12}v(x')+\partial_{22}v(x')\right), & i\in\{2,4,6,8\},\end{cases}$$

hence we get

$$\nabla'^2\phi(x')\left[(z^i)',(z^i)'\right]-\frac{1}{8}\sum_{j=1}^{8}\nabla'^2\phi(x')\left[(z^j)',(z^j)'\right]=\frac{1}{2}\partial_{12}\phi(x')(-1)^{i+1}.$$

Rearranging the above terms leads to

$$(3.75)=$$

$$\sum_{l=1}^{8}\int_\Omega\frac{1}{2(\nu_n-1)}J^n_{\cdot l}\cdot\frac{1}{2}\partial_{12}\phi(\tilde{x}')(-1)^{l+1}e_3\;dx \tag{3.76}$$

$$+\sum_{l=1}^{4}\int_S\int_0^{\frac{1}{\nu_n-1}}\frac{1}{2(\nu_n-1)}J^{(1,n)}_{\cdot l}\cdot\frac{1}{2}\partial_{12}\phi(\tilde{x}')(-1)^{l+1}e_3\;dx_3\;dx' \tag{3.77}$$

$$+\sum_{l=1}^{4}\int_S\int_{\frac{\nu_n-2}{\nu_n-1}}^{1}\frac{1}{2(\nu_n-1)}J^{(2,n)}_{\cdot l}\frac{1}{2}\partial_{12}\phi(\tilde{x}')(-1)^{l+1}e_3\;dx_3\;dx' \tag{3.78}$$

$$+\sum_{l=1}^{8}\int_\Omega h_n^{-1}\left[J^n_{3l}\partial_1\phi(\tilde{x}')z_1^l+J^n_{3l}\partial_2\phi(\tilde{x}')z_2^l-J^n_{1l}\partial_1\phi(\hat{x}')z_3^l-J^n_{2l}\partial_2\phi(\hat{x}')z_3^l\right]\;dx \tag{3.79}$$

$$+\sum_{l=1}^{4}\int_S\int_0^{\frac{1}{\nu_n-1}}h_n^{-1}\Big[J^{(1,n)}_{3l}\partial_1\phi(\tilde{x}')z_1^l+J^{(1,n)}_{3l}\partial_2\phi(\tilde{x}')z_2^l$$

$$-J^{(1,n)}_{1l}\partial_1\phi(\hat{x}')z_3^l-J^{(1,n)}_{2l}\partial_2\phi(\hat{x}')z_3^l\Big]\;dx_3\;dx' \tag{3.80}$$

$$+\sum_{l=1}^{4}\int_S\int_{\frac{\nu_n-2}{\nu_n-1}}^{1}h_n^{-1}\Big[J^{(2,n)}_{3l}\partial_1\phi(\tilde{x}')z_1^{l+4}+J^{(2,n)}_{3l}\partial_2\phi(\tilde{x}')z_2^{l+4}$$

$$-J^{(2,n)}_{1l}\partial_1\phi(\hat{x}')z_3^{l+4}-J^{(2,n)}_{2l}\partial_2\phi(\hat{x}')z_3^{l+4}\Big]\;dx_3\;dx'+O(\varepsilon_n). \tag{3.81}$$

We determine the convergence of ((3.76) + (3.77) + (3.78)) and ((3.79) + (3.80) + (3.81)) separately. In case $\nu_n\to\infty$ the term ((3.76) + (3.77) + (3.78)) clearly vanishes. If $\nu_n\equiv\nu\in\mathbb{N}$ however by Proposition 3.2.9, Lemma 3.2.16,Corollary 2.3.3 and (3.34) we see that

$$(3.76)+(3.77)+(3.78)$$

$$\underset{n\to\infty}{\longrightarrow}\int_S\frac{1}{4(\nu-1)}DQ^{\mathrm{rel}}_{\mathrm{cell}}\left(\begin{pmatrix}\mathrm{sym}\;G_1 & 0\\ 0 & 0\end{pmatrix}Z+\frac{1}{2(\nu-1)}G_3\right):\partial_{12}\phi M\;dx'$$

$$+\int_S\frac{1}{2(\nu-1)^2}DQ_{\mathrm{surf}}\left(\begin{pmatrix}\mathrm{sym}\;G_1 & 0\\ 0 & 0\end{pmatrix}Z^{(1)}+\frac{\partial_{12}v}{2(\nu-1)}M^{(1)}\right):\partial_{12}\phi M^{(1)}\;dx'.$$

It remains to show that ((3.79) + (3.80) + (3.81)) vanishes as $n \to \infty$. First remember that we denoted by $\hat{x}$ the left lowest point of a cell $\tilde{x} + \left(-\frac{\varepsilon_n}{2}, \frac{\varepsilon_n}{2}\right) \times \left(-\frac{\varepsilon_n}{2h_n}, \frac{\varepsilon_n}{2h_n}\right)$. Thus

$$\hat{x}' - \tilde{x}' = -\frac{\varepsilon_n}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For instance we look only at (3.79) as the surface parts work just the same way again. A Taylor expansion yields

$$\begin{aligned}\partial_i\phi(\hat{x}') &= \partial_i\phi(\tilde{x}') + \nabla'\partial_i\phi(\tilde{x}') \cdot (\hat{x}' - \tilde{x}') + O(|\hat{x}' - \tilde{x}'|^2) \\ &= \partial_i\phi(\tilde{x}') - \frac{\varepsilon_n}{2}\nabla'\partial_i\phi(\tilde{x}') \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + O\left(\varepsilon_n^2\right).\end{aligned}$$

Hence for $i \in \{1, 2\}$ we have

$$\sum_{l=1}^{8} \int_\Omega h_n^{-1}\left(J_{3l}^n \partial_i\phi(\tilde{x}')z_i^l - J_{il}^n \partial_i\phi(\hat{x}')z_3^l\right)\, dx$$

$$= \sum_{l=1}^{8} \int_\Omega h_n^{-1}\left(J_{3l}^n \partial_i\phi(\tilde{x}')z_i^l - J_{il}^n \partial_i\phi(\tilde{x}')z_3^l\right)\, dx \tag{3.82}$$

$$-\sum_{l=1}^{8} \int_\Omega \frac{\varepsilon_n}{2h_n} J_{il}^n \nabla'\partial_i\phi(\tilde{x}') \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} z_3^l\, dx + O(h_n^{-1}\varepsilon_n^2). \tag{3.83}$$

In both, thin and ultrathin films, (3.82) tends to zero due to

$$\begin{aligned}&\left|\sum_{l=1}^{8} \int_\Omega h_n^{-1}\left(J_{3l}^n \partial_i\phi(\tilde{x}')z_i^l - J_{il}^n \partial_i\phi(\hat{x}')z_3^l\right)\, dx\right| \\ &\quad\le \int_\Omega h_n^{-1}\left|\sum_{l=1}^{8}\left(J_{3l}^n z_i^l - J_{il}^n z_3^l\right)\right| \|\partial_i\phi\|_\infty\, dx \\ &\quad\le Ch_n.\end{aligned}$$

For the last estimate we have used Lemma 3.2.19. If $\nu_n \to \infty$ term (3.83) tends to 0 too. If $\nu_n \equiv \nu \in \mathbb{N}$ however

$$(3.83) \underset{n\to\infty}{\longrightarrow} \sum_{l=1}^{8} \int_\Omega \frac{1}{2(\nu - 1)} J_{il} \nabla'\partial_i\phi(x') \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} z_3^l\, dx.$$

Note that

$$z_3^l = \begin{cases} -\frac{1}{2}, & 1 \le l \le 4, \\ +\frac{1}{2}, & 5 \le l \le 8, \end{cases}$$

hence

$$(3.79) + (3.80) + (3.81) \underset{n\to\infty}{\longrightarrow}$$

$$\sum_{i=1}^{2}\sum_{l=1}^{8}\int_S\int_0^{\frac{1}{\nu-1}}\frac{1}{2(\nu-1)}J_{il}\left(x',\frac{1}{2(\nu-1)}\right)\nabla'\partial_i\phi(x')\cdot\begin{pmatrix}1\\1\end{pmatrix}z_3^l\;dx_3\;dx'$$
$$+\sum_{i=1}^{2}\sum_{l=1}^{4}\int_S\int_0^{\frac{1}{\nu-1}}\frac{1}{2(\nu-1)}J_{il}^{(1)}\left(x',\frac{1}{2(\nu-1)}\right)\nabla'\partial_i\phi(x')\cdot\begin{pmatrix}1\\1\end{pmatrix}z_3^l\;dx_3\;dx'$$
$$+\sum_{i=1}^{2}\sum_{l=1}^{8}\int_S\int_{\frac{\nu-2}{\nu-1}}^{1}\frac{1}{2(\nu-1)}J_{il}\left(x',\frac{2\nu-3}{2(\nu-1)}\right)\nabla'\partial_i\phi(x')\cdot\begin{pmatrix}1\\1\end{pmatrix}z_3^l\;dx_3\;dx'$$
$$+\sum_{i=1}^{2}\sum_{l=1}^{4}\int_S\int_{\frac{\nu-2}{\nu-1}}^{1}\frac{1}{2(\nu-1)}J_{il}^{(2)}\left(x',\frac{2\nu-3}{2(\nu-1)}\right)\nabla'\partial_i\phi(x')\cdot\begin{pmatrix}1\\1\end{pmatrix}z_3^{l+4}\;dx_3\;dx'$$
$$+\sum_{i=1}^{2}\sum_{l=1}^{8}\sum_{m=2}^{\nu-2}\int_S\int_{\frac{m-1}{\nu-1}}^{\frac{m}{\nu-1}}\frac{1}{2(\nu-1)}J_{il}\left(x',\frac{2m-1}{2(\nu-1)}\right)\nabla'\partial_i\phi(x')\cdot\begin{pmatrix}1\\1\end{pmatrix}z_3^l\;dx_3\;dx'$$
$$=\sum_{i=1}^{2}\int_S\frac{1}{4(\nu-1)^2}\left[\sum_{l=1}^{4}J_{il}\left(x',\frac{1}{2(\nu-1)}\right)-\sum_{l=5}^{8}J_{il}\left(x',\frac{1}{2(\nu-1)}\right)\right.$$
$$\left.+\sum_{l=1}^{4}J_{il}^{(1)}\left(x',\frac{1}{2(\nu-1)}\right)\right]\nabla'\partial_i\phi(x')\cdot\begin{pmatrix}1\\1\end{pmatrix}\;dx'$$
$$+\sum_{i=1}^{2}\int_S\frac{1}{4(\nu-1)^2}\left[\sum_{l=1}^{4}J_{il}\left(x',\frac{2\nu-3}{2(\nu-1)}\right)-\sum_{l=5}^{8}J_{il}\left(x',\frac{2\nu-3}{2(\nu-1)}\right)\right.$$
$$\left.-\sum_{l=1}^{4}J_{il}^{(2)}\left(x',\frac{2\nu-3}{2(\nu-1)}\right)\right]\nabla'\partial_i\phi(x')\cdot\begin{pmatrix}1\\1\end{pmatrix}\;dx'$$
$$+\sum_{i=1}^{2}\int_S\sum_{m=2}^{\nu-2}\frac{1}{4(\nu-1)^2}\left[\sum_{l=1}^{4}J_{il}\left(x',\frac{2m-1}{2(\nu-1)}\right)-\sum_{l=5}^{8}J_{il}\left(x',\frac{2m-1}{2(\nu-1)}\right)\right]$$
$$\nabla'\partial_i\phi(x')\cdot\begin{pmatrix}1\\1\end{pmatrix}\;dx'$$
$$=0.$$

The terms sum up to 0 because of Lemma 3.2.14. In total we deduce that for $\nu_n\to\infty$ (3.54) converges to 0 and for $\nu_n\equiv\nu\in\mathbb{N}$

$$(3.54)\underset{n\to\infty}{\longrightarrow}\int_S\frac{1}{4(\nu-1)}DQ_{\text{cell}}^{\text{rel}}\left(\begin{pmatrix}\text{sym }G_1 & 0\\ 0 & 0\end{pmatrix}Z+\frac{1}{2(\nu-1)}G_3\right):\partial_{12}\phi M\;dx'$$
$$+\int_S\frac{1}{2(\nu-1)^2}DQ_{\text{surf}}\left(\begin{pmatrix}\text{sym }G_1 & 0\\ 0 & 0\end{pmatrix}Z^{(1)}+\frac{\partial_{12}v}{2(\nu-1)}M^{(1)}\right):\partial_{12}\phi M^{(1)}\;dx'.$$

Together with (3.51) and (3.53) we obtain (3.7). □

3.3. Fully clamped boundary conditions

So far our work only covers distributional solutions which means we kind of ignore boundary conditions. This is unsatisfactory since in mathematical elasticity we usually treat boundary value problems. It would be desirable to obtain clamped boundary conditions in the two dimensional limiting equations as it was done in the continuum case done by [MP08] or [MS09]. In fact our proof does apply to fully clamped boundary conditions with only a few modifications:

- We need to change the space for the definition of the weak solution, c.f. Definition 3.1.1. The right space for a weak solution (u, v) is $H_0^1(S;\mathbb{R}^2) \times H_0^2(S)$.
- We do not need to change the space of test functions used in Definition 3.1.1. By density it is sufficient to obtain the equations for every smooth test functions with compact support.
- We need to make sure that the displacements of piecewise affine interpolations of the three dimensional, stationary points vanish on ∂S. This is achieved by requiring $y_n(x) = (x', x_3 - \frac{h_n}{2})$ if x' is close to the boundary. Then clearly $u_{n|\partial S} = 0$ as well as $v_{n|\partial S} = 0$. Arguing as in [MP08] shows that the pair (u, v) lies in the right space.
- We do not need to normalize the sequence $\tilde{\tilde{y}}_n$, i.e. we can choose $R_n^* = Id$ and $c_n = 0$, as shown in [LM09]. Consequently there is no need to consider a rotated force term either.

3.4. An example

A basic mass-spring models with nearest and next-to-nearest interactions is given by

$$\begin{aligned} E_{\text{atom}}(w) = \frac{\alpha}{4} \sum_{\substack{x,x' \in \Lambda_n \\ |x-x'|=\varepsilon_n}} \left(\frac{|w(x) - w(x')|}{\varepsilon_n} - 1 \right)^2 \\ + \frac{\beta}{4} \sum_{\substack{x,x' \in \Lambda_n \\ |x-x'|=\sqrt{2}\varepsilon_n}} \left(\frac{|w(x) - w(x')|}{\varepsilon_n} - \sqrt{2} \right)^2 \\ = \sum_{x \in \Lambda_n'} W\big(x, \vec{w}(x)\big), \end{aligned}$$

if

$$\begin{aligned} W_{\text{cell}}\big(\vec{w}\big) = \frac{\alpha}{16} \sum_{\substack{1 \le i,j \le 8 \\ |z^i - z^j| = 1}} \left(|w_i - w_j| - 1\right)^2 \\ + \frac{\beta}{8} \sum_{\substack{1 \le i,j \le 8 \\ |z^i - z^j| = \sqrt{2}}} \left(|w_i - w_j| - \sqrt{2}\right)^2 \end{aligned}$$

and

$$W_{\text{surf}}(w_1, w_2, w_3, w_4) = \frac{\alpha}{8} \sum_{\substack{1 \le i,j \le 4 \\ |z^i - z^j| = 1}} \left(|w_i - w_j| - 1\right)^2 + \frac{\beta}{8} \sum_{\substack{1 \le i,j \le 8 \\ |z^i - z^j| = \sqrt{2}}} \left(|w_i - w_j| - \sqrt{2}\right)^2$$

It is desirable to cover such basic models as this is the case in the Γ-convergence result of Braun and Schmidt, yet we need to make little adaptions. In our model we assume that $D_F W(x, \cdot)$ exists everywhere, which in the example above is clearly not the case. Hence we look at models given by

$$E_{\text{atom}}(w) = \frac{\alpha}{4} \sum_{\substack{x, x' \in \Lambda_n \\ |x - x'| = \varepsilon_n}} \left(\frac{\varphi\left(w(x) - w(x')\right)}{\varepsilon_n} - 1 \right)^2 + \frac{\beta}{4} \sum_{\substack{x, x' \in \Lambda_n \\ |x - x'| = \sqrt{2}\varepsilon_n}} \left(\frac{\varphi\left(w(x) - w(x')\right)}{\varepsilon_n} - \sqrt{2} \right)^2,$$

where $\varphi \in C^1(\mathbb{R})$ with

(i) $\varphi(0) = 0$,

(ii) $\varphi(x) > 0$ for every $x \neq 0$,

(iii) $\|\varphi'\|_\infty \le C$ for some $C > 0$.

3.5. Summary

We close this section by putting our result into context. Theorem 3.2.1 extends the Γ-convergence result of Braun and Schmidt. However we need to make additional, physically problematic assumptions on the growth conditions on the derivative. These growth condition limit our analysis to very basic interaction models. In particular we cannot penalize strong compression. The same problem occurred in [MP08]. As a remedy Mora and Scardia proved in [MS09] a similar result under the growth condition

$$\left|DW(F)F^T\right| \le C(1 + W(F))$$

for every F with $\det F > 0$. It would be interesting if such a result is also possible in the discrete setting.

Further, as already mentioned, it would be desirable to treat clamped boundary conditions instead of fully clamped boundary conditions. At first glance this task seems to be rather harmless. Unfortunately it turned out to be way harder than expected and to the day of submission we could not find a solution to it yet.

Chapter 4

The time dependent case

4.1. Introduction

4.1.1. A quick review on the continuous case

In 2009 Abels, Mora and Mueller were able to extend the results of [MP08] to a dynamic model. Let $f \in L^2((0,\infty), L^2(S))$ and set $f^h(\tau, x) = h^3 f(\tau, x')e_3$. Consider a solution w of the dynamic equation of nonlinear elasticity

$$\partial_\tau^2 w - \mathrm{div}_x DW(\nabla w) = f^h \text{ in } [0, \tau_h] \times \Omega_h.$$

with $\Omega_h = S \times \left(-\frac{h}{2}, \frac{h}{2}\right)$. As usual the domain is rescaled $\Omega = S \times \left(-\frac{1}{2}, \frac{1}{2}\right)$ and in addition we rescale in time. The rescaled mappings

$$y^h(t, x) := w^h\left(\frac{t}{h}, x', hx_3\right)$$

solve the equation

$$h^2 \partial_t^2 y^h - \mathrm{div}_h DW(\nabla_h y^h) = h^3 g e_3 \text{ in } (0, T_h) \times \Omega \tag{4.1}$$

with $T_h = h\tau_h$ and $g(t, x') := f\left(\frac{t}{h}, x'\right)$. For $F \in H^1(\Omega; \mathbb{R}^3)$ the scaled divergence div_h is defined as

$$\mathrm{div}_h F \cdot e_i = \sum_{j=1,2} \partial_j F_{ij} + \frac{1}{h}\partial_3 F_{i3}, \quad i \in \{1, 2, 3\}.$$

In [AMM09] it was shown that, given certain energy estimates and initial conditions, the averaged in-plane and out-of-plane displacements

$$u^h(t, x') := h_n^{-2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\left(y^h\right)'(t, x) - x'\right) \, dx_3,$$
$$v^h(t, x') := h_n^{-1} \int_{-\frac{1}{2}}^{\frac{1}{2}} y_3^h(t, x) \, dx_3$$

of weak solutions of the dynamic equation of nonlinear elasticity converge in a suitable sense to maps (u, v). Note that the integration is here over the interval $(-\frac{1}{2}, \frac{1}{2})$ since the domain is $\Omega = S \times (-\frac{1}{2}, \frac{1}{2})$. The limiting displacement (u, v) is a weak solution to the dynamic von-Kármán plate equation

$$\begin{cases} \partial_t^2 v + \frac{1}{12}\mathrm{div}\left[\mathrm{div}\mathcal{L}_2\left(\nabla'^2 v\right)\right] - \mathrm{div}\left[\mathcal{L}_2\left(\mathrm{sym}\nabla' u + \frac{1}{2}\nabla' v \otimes \nabla' v\right)\nabla' v\right] = g, \\ \mathrm{div}\left[\mathcal{L}_2\left(\mathrm{sym}\nabla' u + \frac{1}{2}\nabla' v \otimes \nabla' v\right)\right] = 0 \end{cases} \tag{4.2}$$

in $[0, T] \times S$ with zero boundary conditions, i.e. $u_{|\partial S} = 0$, $v_{|\partial S} = 0$, $\nabla' v_{|\partial S} = 0$. For details on the assumptions, boundary conditions and energy bounds we refer to the article. The goal is to prove a similar result in the discrete setting.

4.1.2. The model and boundary considerations

We are working with the same model as in the stationary case. We refer to Chapter 2 for details. For reasons of simplicity we are modeling fully clamped plates. This means that we are looking only at deformations which satisfy the clamped boundary conditions on the whole lateral boundary of S in contrast to the clamped plates, where boundary conditions only were given on a subset Γ of ∂S with positive surface measure (c.f. [MP08]). The goal is to obtain

$$\tilde{\tilde{y}}(x) = \begin{pmatrix} x' \\ h_n\left(x_3 - \frac{1}{2}\right) \end{pmatrix} \tag{4.3}$$

whenever $x' \in \partial S$, where $\tilde{\tilde{y}}$ is the piecewise affine, rescaled interpolation of an extended atomistic deformation $w : \bar{\Lambda}_n \to \mathbb{R}^3$. Remember that we defined $S_n = \{x \in S \colon \mathrm{dist}(x, \partial S) > \sqrt{2}\varepsilon_n\}$. Let $\partial\Lambda_n = \{x \in \bar{\Lambda}_n \colon x' \notin S_n\}$ and $\mathrm{int}(\Lambda_n) := \bar{\Lambda}_n \setminus \partial\Lambda_n$.

Let

$$\mathcal{A}_n := \left\{ w : \bar{\Lambda}_n \to \mathbb{R}^3 \colon w(x) = \begin{pmatrix} x' \\ x_3 - \frac{h_n}{2} \end{pmatrix} \text{ for all } x \in \partial\Lambda_n \right\}. \tag{4.4}$$

By applying the piecewise affine interpolation scheme it is not hard to see that $\tilde{\tilde{y}}(x) = \tilde{w}(H_n x)$, $H_n = \mathrm{diag}\{1, 1, h_n\}$ satisfies the boundary conditions (4.3) whenever $w \in \mathcal{A}_n$.

The forces considered in this section will be a little less general compared to the stationary case. They will be of the form $g_n(t, x') = h_n^3 g(x') e_3$ for some function g specified in more detail in the next section.

We make one last remark on the spaces we will use in the following. By I_T we denote the interval $[0, T]$ for $T \in (0, \infty)$. If $T = \infty$ we set $I_T = [0, \infty)$. In the following we will work with the spaces $L^p_{\mathrm{loc}}(I_T; X)$, $p \in [1, \infty]$, for different Banach spaces X. In particular if $T < \infty$ the space $L^p_{\mathrm{loc}}(I_T; X)$ agrees with the space $L^p(I_T; X)$.

4.2. Weak solutions

First we give the notion of a weak solution to the limiting equations, which are the time-dependent von Kármán equations. We have to distinguish between $\nu_n \to \infty$ and $\nu_n \equiv \nu \in \mathbb{N}$. To make sense of the upcoming definitions we assume that the force term is sufficiently regular to have suitable integrability as well as point evaluation. This is certainly the case for $g \in L^2_{\mathrm{loc}}(0, T; W^{1,\infty}(S))$.

Definition 4.2.1. *Let $\nu_n \to \infty$. We say a pair (u, v) is a weak solution to (4.2) if $u \in L^\infty_{loc}(I_T; H^1(S;\mathbb{R}^2))$, $v \in L^\infty_{loc}(I_T; H^2(S)) \cap W^{1,\infty}_{loc}(I_T; L^2(S))$ and for every $T' \in I_T$ the following two equations are satisfied:*

$$\int_0^{T'}\int_S \partial_t v \partial_t \phi \; dx' \; dt - \int_0^{T'}\int_S \mathcal{L}_2\left(\text{sym } \nabla' u + \frac{1}{2}\nabla' v \otimes \nabla' v\right) : \nabla' v \otimes \nabla' \phi \; dx' \; dt$$
$$- \int_0^{T'}\int_S \frac{1}{12}\mathcal{L}_2\left(\nabla'^2 v\right) : \nabla'^2 \phi \; dx' \; dt$$
$$+ \int_0^{T'}\int_S g\phi \; dx' \; dt = 0 \quad (4.5)$$

for every $\phi \in L^2(0,T'; H^2_0(S)) \cap H^1_0(0,T'; L^2(S))$, and

$$\int_0^{T'}\int_S \mathcal{L}_2\left(\text{sym } \nabla' u + \frac{1}{2}\nabla' v \otimes \nabla' v\right) : \nabla' \Psi \; dx' \; dt = 0 \quad (4.6)$$

for every $\Psi \in L^2(0,T'; H^1_0(S;\mathbb{R}^2))$.

Definition 4.2.2. *Let $\nu_n \equiv \nu \in \mathbb{N}$. We say a pair (u, v) is a weak solution to the dynamic von-Kármán equations if $u \in L^\infty_{loc}(I_T; H^1(S;\mathbb{R}^2))$, $v \in L^\infty_{loc}(I_T; H^2(S)) \cap W^{1,\infty}_{loc}(I_T; L^2(S))$ and for every $T' \in I_T$ the following two equations are satisfied:*

$$\frac{\nu}{\nu-1}\int_0^{T'}\int_S \partial_t v \partial_t \phi \; dx' \; dt$$
$$- \int_0^{T'}\int_S \frac{1}{2} DQ^{\text{rel}}_{\text{cell}}\left(\begin{pmatrix} G_1 & 0 \\ 0 & 0 \end{pmatrix} Z + \frac{1}{2(\nu-1)} G_3\right) : \begin{pmatrix} \nabla' v \otimes \nabla' \phi & 0 \\ 0 & 0 \end{pmatrix} Z \; dx' \; dt$$
$$+ \int_0^{T'}\int_S \frac{1}{4(\nu-1)} DQ^{\text{rel}}_{\text{cell}}\left(\begin{pmatrix} G_1 & 0 \\ 0 & 0 \end{pmatrix} Z + \frac{1}{2(\nu-1)} G_3\right) : \begin{pmatrix} \nabla'^2 \phi & 0 \\ 0 & 0 \end{pmatrix} Z_- \; dx' \; dt$$
$$- \int_0^{T'}\int_S \frac{1}{4(\nu-1)} DQ^{\text{rel}}_{\text{cell}}\left(\begin{pmatrix} G_1 & 0 \\ 0 & 0 \end{pmatrix} Z + \frac{1}{2(\nu-1)} G_3\right) : \partial_{12}\phi M \; dx' \; dt$$
$$+ \int_0^{T'}\int_S \frac{\nu(\nu-2)}{24(\nu-1)^2} DQ^{\text{rel}}_{\text{cell}}\left(\begin{pmatrix} G_2 & 0 \\ 0 & 0 \end{pmatrix} Z\right) : \begin{pmatrix} \nabla'^2 \phi & 0 \\ 0 & 0 \end{pmatrix} Z \; dx' \; dt$$
$$- \int_0^{T'}\int_S \frac{1}{\nu-1} DQ_{\text{surf}}\left(\begin{pmatrix} G_1 & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} + \frac{\partial_{12} v}{2(\nu-1)} M^{(1)}\right) :$$
$$\begin{pmatrix} \nabla' v \otimes \nabla' \phi & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} \; dx' \; dt$$
$$- \int_0^{T'}\int_S \frac{1}{2(\nu-1)^2} DQ_{\text{surf}}\left(\begin{pmatrix} G_1 & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} + \frac{\partial_{12} v}{2(\nu-1)} M^{(1)}\right) : \partial_{12}\phi M^{(1)} \; dx' \; dt$$
$$+ \int_0^{T'}\int_S \frac{1}{4(\nu-1)} DQ_{\text{surf}}\left(\begin{pmatrix} G_2 & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)}\right) : \begin{pmatrix} \nabla'^2 \phi & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} \; dx' \; dt$$
$$+ \frac{\nu}{\nu-1}\int_0^{T'}\int_S g(t,x')\phi(t,x') \; dx' \; dt = 0 \quad (4.7)$$

for every $\phi \in L^2(0,T';H_0^2(S)) \cap H_0^1(0,T';L^2(S))$ *and*

$$0 = \int_0^{T'} \int_S \frac{1}{2} DQ_{\text{cell}}^{\text{rel}} \left(\begin{pmatrix} G_1 & 0 \\ 0 & 0 \end{pmatrix} Z + \frac{1}{2(\nu-1)} G_3 \right) : \begin{pmatrix} \nabla'\Psi & 0 \\ 0 & 0 \end{pmatrix} Z \, dx' \, dt$$

$$+ \frac{1}{\nu-1} \int_0^{T'} \int_S DQ_{\text{surf}} \left(\begin{pmatrix} G_1 & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} + \frac{\partial_{12} v}{2(\nu-1)} M^{(1)} \right) : \begin{pmatrix} \nabla'\Psi & 0 \\ 0 & 0 \end{pmatrix} dx' \, dt \quad (4.8)$$

for every $\Psi \in L^2(0,T';H_0^1(S;\mathbb{R}^2))$.

We also need the notion of a time dependent atomistic weak solution. Motivated by equation (4.1) we give the following definition:

Definition 4.2.3. *Let* $T \in (0,\infty]$. *We say that* $y_n \colon I_T \times \tilde{\Lambda}_n \to \mathbb{R}^3$ *is a time-dependent atomistic weak solution of* (4.1) *if for every* $x \in \tilde{\Lambda}_n$ *the time derivatives* $\partial_t y_n(t,x), \partial_t^2 y_n(t,x)$ *exist and lie in* $L^2_{loc}(I_T)$ *and for every* $T' \in (0,T)$

$$0 = h_n^2 \int_0^{T'} \frac{\varepsilon_n^3}{h_n} \sum_{x \in \tilde{\Lambda}_n} \partial_t y_n(t,x) \cdot \partial_t \varphi(t,x) \, dt$$

$$- \int_0^{T'} \frac{\varepsilon_n^3}{h_n} \sum_{x \in \tilde{\Lambda}_n'} D_F W\left(x, \bar{\nabla}_n y_n(t,x)\right) : \bar{\nabla}_n \varphi(t,x) \, dt$$

$$+ \int_0^{T'} \frac{\varepsilon_n^3}{h_n} \sum_{x \in \tilde{\Lambda}_n} h_n^3 g(t,x') \varphi_3(t,x) \, dt \quad (4.9)$$

for every $\varphi \colon I_T \times \tilde{\Lambda}_n \to \mathbb{R}^3$ *satisfying*

- $\varphi(\cdot,x) \in H_0^1(I_T)$ *for every* $x \in \tilde{\Lambda}_n$ *and*
- $\varphi(t,x) = 0$ *whenever* $x \in \partial\Lambda_n$.

We determine the Euler-Lagrange equation satisfied by a sufficiently regular solution according to Definition 4.2.3. Fix $x \in \tilde{\Lambda}_n$ and denote by

$$\mathcal{N}(x) = \left\{ \eta \in \tilde{\Lambda}_n' : \eta = x + \varepsilon_n((z^i)', h_n^{-1} z_3^i) \quad \text{for some } i \in \{1, \cdots, 8\} \right\}$$

the neighboring midpoints of cells of x, as long as they belong to $\tilde{\Lambda}_n'$. Let φ be an admissible test function such that $\varphi(t,z) = 0$ for every $z \neq x$ and for every t. Then by (4.9)

$$0 = h_n^2 \int_0^{T'} \frac{\varepsilon_n^3}{h_n} \partial_t y_n(t,x) \cdot \partial_t \varphi(t,x) \, dt$$

$$- \int_0^{T'} \frac{\varepsilon_n^3}{h_n} \sum_{\eta \in \mathcal{N}(x)} D_F W\left(\eta, \bar{\nabla}_n y_n(t,\eta)\right) : \bar{\nabla}_n \varphi(t,\eta) \, dt$$

$$+ \int_0^{T'} \frac{\varepsilon_n^3}{h_n} h_n^3 g(t,x') \varphi_3(t,x) \, dt.$$

In terms of the adjoint operator $\bar\nabla_n^*$ of $\bar\nabla_n$ (for the definition c.f. (2.3)) we get

$$\int_0^{T'} \left[-h_n^2\partial_t^2 y_n(t,x) - \bar\nabla_n^*\left(D_F W\left(\cdot,\bar\nabla_n y_n(t,\cdot)\right)\right)(x) + h_n^3 g(t,x')e_3\right]\cdot\varphi(t,x)\,dt = 0.$$

Thus y_n fulfills the high-dimensional system of ordinary differential equations

$$h_n^2\partial_t^2 y_n(t,x) = -\bar\nabla_n^*\left(D_F W\left(\cdot,\bar\nabla_n y_n(t,\cdot)\right)\right)(x) + h_n^3 g(t,x')e_3$$

where $x\in\tilde\Lambda_n$. These is precisely Newton's second law for motion. With suitable regularity assumptions on g and given initial conditions for y_n and $\partial_t y_n$ the Picard Lindeloef theorem guarantees existence and uniqueness for a time interval $(0,T_n)$ with $T_n>0$.

Remark 4.2.4. *For convenience we would like to compute* $\bar\nabla_n^*\left(D_F W(\cdot,\bar\nabla_n y(\cdot))\right)(x)$*. We assume for simplicity* $h_n=1$*, the general case follows by rescaling. Remember that for* $H:\Lambda_n'\to\mathbb{R}^{3\times 8}$ *and every* $\varphi:\bar\Lambda_n\to\mathbb{R}^3$ *the identity*

$$\sum_{x\in\bar\Lambda_n}\bar\nabla_n^* H(x)\cdot\varphi(x) = \sum_{\omega\in\Lambda_n'} H(\omega):\bar\nabla_n\varphi(\omega).$$

must hold. Let $x\in\Lambda_n$ *and* $\varphi(z)=0$ *for every* $z\in\bar\Lambda_n$ *with* $z\neq x$*. Then*

$$\begin{aligned}
&\sum_{\omega\in\Lambda_n'} D_F W(\omega,\bar\nabla_n y(\omega)):\bar\nabla_n\varphi(\omega)\\
&=\sum_{\omega\in\Lambda_n'}\sum_{i,j}\partial_{F_{ji}}W(\omega,\bar\nabla_n y(\omega))\bar\partial_i^n\varphi_j(\omega)\\
&=\frac{1}{\varepsilon_n}\sum_{\omega\in\Lambda_n'}\sum_{i,j}\partial_{F_{ji}}W(\omega,\bar\nabla_n y(\omega))\left(\varphi_j(\omega+\varepsilon_n z^i)-\frac{1}{8}\sum_{l=1}^{8}\varphi_j(\omega+\varepsilon_n z^l)\right)\\
&=\frac{1}{\varepsilon_n}\sum_{i,j}\sum_{\eta\in\mathcal{N}(x)}\partial_{F_{ji}}W(\eta,\bar\nabla_n y(\eta))\left(\varphi_j(\eta+\varepsilon_n z^i)-\frac{1}{8}\sum_{l=1}^{8}\varphi_j(\eta+\varepsilon_n z^l)\right)\\
&=\frac{1}{\varepsilon_n}\sum_{\substack{i\in\{1,\cdots 8\}:\\ x-\varepsilon_n z^i\in\Lambda_n'}}\sum_{j=1}^{3}\partial_{F_{ji}}W(x-\varepsilon_n z^i,\bar\nabla_n y(x-\varepsilon_n z^i))\varphi_j(x)\\
&\quad-\frac{1}{\varepsilon_n}\sum_{i,j}\sum_{\eta\in\mathcal{N}(x)}\partial_{F_{ji}}W(\eta,\bar\nabla_n y(\eta))\frac{1}{8}\sum_{l=1}^{8}\varphi_j(\eta+\varepsilon_n z^l)\\
&=\frac{1}{\varepsilon_n}\sum_{\substack{i\in\{1,\cdots 8\}:\\ x-\varepsilon_n z^i\in\Lambda_n'}}\sum_{j=1}^{3}\Bigg[\partial_{F_{ji}}W(x-\varepsilon_n z^i,\bar\nabla_n y(x-\varepsilon_n z^i))\\
&\quad-\frac{1}{8}\sum_{\substack{l\in\{1,\cdots 8\}:\\ x-\varepsilon_n z^l\in\Lambda_n'}}\partial_{F_{ji}}W(x-\varepsilon_n z^l,\bar\nabla_n y(x-\varepsilon_n z^l))\Bigg]\varphi_j(x)\\
&=\frac{1}{\varepsilon_n}\sum_{\substack{i\in\{1,\cdots 8\}:\\ x-\varepsilon_n z^i\in\Lambda_n'}}\Bigg[\partial_{F_{\cdot i}}W(x-\varepsilon_n z^i,\bar\nabla_n y(x-\varepsilon_n z^i))
\end{aligned}$$

$$-\frac{1}{8}\sum_{\substack{l\in\{1,\cdots 8\}:\\ x-\varepsilon_n z^l\in\Lambda_n'}} \partial_{F.i}W(x-\varepsilon_n z^l,\bar{\nabla}_n y(x-\varepsilon_n z^l))\Bigg]\cdot\varphi(x).$$

Therefore we see that

$$\begin{aligned}&\bar{\nabla}_n^*\left(D_F W(\cdot,\bar{\nabla}_n y(\cdot))\right)(x)\\ &=\frac{1}{\varepsilon_n}\sum_{\substack{i\in\{1,\cdots 8\}:\\ x-\varepsilon_n z^i\in\Lambda_n'}}\Bigg[\partial_{F.i}W(x-\varepsilon_n z^i,\bar{\nabla}_n y(x-\varepsilon_n z^i))\\ &-\frac{1}{8}\sum_{\substack{l\in\{1,\cdots 8\}:\\ x-\varepsilon_n z^l\in\Lambda_n'}}\partial_{F.i}W(x-\varepsilon_n z^l,\bar{\nabla}_n y(x-\varepsilon_n z^l))\Bigg]\cdot\varphi(x).\end{aligned}$$

corresponds to a discrete divergence. Of special interest are the following cases. If x is an interior atom, the result is

$$\begin{aligned}&\bar{\nabla}_n^*\left(D_F W(\cdot,\bar{\nabla}_n y(\cdot))\right)(x)\\ &=\frac{1}{\varepsilon_n}\sum_{i=1}^{8}\left[\partial_{F.i}W_{\text{cell}}(\bar{\nabla}_n y(x-\varepsilon_n z^i))-\frac{1}{8}\sum_{l=1}^{8}\partial_{F.i}W_{\text{cell}}(\bar{\nabla}_n y(x-\varepsilon_n z^l))\right].\end{aligned}$$

If x is an atom in the uppermost layer, i.e. $x_3=h_n$, and x' is away from the lateral boundary ∂S,

$$\begin{aligned}&\bar{\nabla}_n^*\left(D_F W(\cdot,\bar{\nabla}_n y(\cdot))\right)(x)\\ &=\frac{1}{\varepsilon_n}\sum_{i=5}^{8}\left(\partial_{F.i}W(x-\varepsilon_n z^i,\bar{\nabla}_n y(x-\varepsilon_n z^i))-\frac{1}{8}\sum_{l=5}^{8}\partial_{F.i}W(x-\varepsilon_n z^l,\bar{\nabla}_n y(x-\varepsilon_n z^l))\right)\\ &=\frac{1}{\varepsilon_n}\sum_{i=5}^{8}\Bigg[\partial_{F.i}W_{\text{cell}}(\bar{\nabla}_n y(x-\varepsilon_n z^i))+\partial_{F.(i-4)}W_{\text{surf}}\left(\left(\bar{\nabla}_n y(x-\varepsilon_n z^i)\right)^{(2)}\right)\\ &-\frac{1}{8}\sum_{l=5}^{8}\partial_{F.i}W_{\text{cell}}(\bar{\nabla}_n y(x-\varepsilon_n z^l))+\partial_{F.(i-4)}W_{\text{surf}}\left(\left(\bar{\nabla}_n y(x-\varepsilon_n z^l)\right)^{(2)}\right)\Bigg].\end{aligned}$$

If x is an atom in the lowest layer, i.e. $x_3=0$, and x' is away from the lateral boundary ∂S,

$$\begin{aligned}&\bar{\nabla}_n^*\left(D_F W(\cdot,\bar{\nabla}_n y(\cdot))\right)(x)\\ &=\frac{1}{\varepsilon_n}\sum_{i=1}^{4}\left(\partial_{F.i}W(x-\varepsilon_n z^i,\bar{\nabla}_n y(x-\varepsilon_n z^i))-\frac{1}{8}\sum_{l=1}^{4}\partial_{F.i}W(x-\varepsilon_n z^l,\bar{\nabla}_n y(x-\varepsilon_n z^l))\right)\\ &=\frac{1}{\varepsilon_n}\sum_{i=1}^{4}\Bigg[\partial_{F.i}W_{\text{cell}}(\bar{\nabla}_n y(x-\varepsilon_n z^i))+\partial_{F.i}W_{\text{surf}}\left(\bar{\nabla}_n y(x-\varepsilon_n z^i)^{(1)}\right)\\ &-\frac{1}{8}\sum_{l=1}^{4}\partial_{F.i}W_{\text{cell}}(\bar{\nabla}_n y(x-\varepsilon_n z^l))+\partial_{F.i}W_{\text{surf}}\left(\bar{\nabla}_n y(x-\varepsilon_n z^l)^{(1)}\right)\Bigg].\end{aligned}$$

4.3. Some preliminary results

The following proposition is a collection of the (for us) most relevant results of [FJM06] and [LM09] applied to mappings with time-dependence.

Proposition 4.3.1. *Let $y_n \in L^2(0,T;H^1(\Omega))$ be a sequence with*

$$\partial_t y_n \in L^2(0,T;L^2(\Omega;\mathbb{R}^3)),$$
$$\partial_{tt} y_n \in L^2(0,T;H^{-1}(\Omega;\mathbb{R}^3)),$$

such that for every $T' \in I_T$

$$\operatorname*{ess\,sup}_{t\in[0,T']} \int_\Omega |\partial_t y_n(t,x)|^2 \, dx \le C(T')h_n^2, \tag{4.10}$$

$$\operatorname*{ess\,sup}_{t\in[0,T']} \int_\Omega \operatorname{dist}^2(\nabla_n y_n(t,x), SO(3)) \, dx \le C(T')h_n^4. \tag{4.11}$$

and

$$y_n(t,x) = \begin{pmatrix} x' \\ h_n(x_3 - \frac{1}{2}) \end{pmatrix} \quad \text{for every } x' \in \partial S. \tag{4.12}$$

Then there is an approximating sequence $R_n \subset L^\infty_{loc}(I_T;H^1(S;\mathbb{R}^{3\times 3}))$ such that $R_n(t,x') \in SO(3)$ for almost every $(t,x') \in (0,T)\times S$ and

$$\operatorname*{ess\,sup}_{t\in[0,T']} \|\nabla_n y_n(t,\cdot) - R_n(t,\cdot)\|_{L^2(\Omega)} \le C(T')h_n^2, \tag{4.13}$$

$$\operatorname*{ess\,sup}_{t\in[0,T']} \|\nabla' R_n(t,\cdot)\|_{L^2(S)} \le C(T')h_n, \tag{4.14}$$

$$\operatorname*{ess\,sup}_{t\in[0,T']} \|R_n(t,\cdot) - Id\|_{H^1(S)} \le C(T')h_n \tag{4.15}$$

for every $T' \in I_T$. Moreover, the averaged scaled in- and out-of-plane displacements

$$u_n(t,x') := h_n^{-2} \int_0^1 (y_n(t,x))' - x' \, dx_3, \tag{4.16}$$

$$v_n(t,x') := h_n^{-1} \int_0^1 (y_n(t,x))_3 \, dx_3 \tag{4.17}$$

satisfy, up to a subsequence, the following convergence properties:

$$u_n \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty_{loc}(I_T;H^1(S;\mathbb{R}^2)), \tag{4.18}$$

$$v_n \to v \quad \text{in } L^\infty_{loc}(I_T;L^2(S)), \tag{4.19}$$

$$\partial_t v_n \overset{*}{\rightharpoonup} \partial_t v \quad \text{in } L^\infty_{loc}(I_T;L^2(S)). \tag{4.20}$$

The maps u and v satisfy the boundary conditions

$$u_{|\partial S} = 0, \;\; v_{|\partial S} = 0, \;\; \nabla' v_{|\partial S} = 0.$$

For the proof of the following proposition we refer to the step 2 in the proof of Theorem 2.1, [AMM09].

Proposition 4.3.2. *In the setting of Proposition 4.3.1 let*

$$A_n = h_n^{-1}\left(R_n - Id\right).$$

Then

$$A_n \overset{*}{\rightharpoonup} A = e_3 \otimes \nabla' v - \nabla' v \otimes e_3 \quad \text{in } L^\infty_{loc}(I_T; H^1(S; \mathbb{R}^{3\times 3})) \tag{4.21}$$

The map $h_n^{-2}\mathrm{sym}(R_n - Id)$ *is bounded in* $L^\infty_{loc}(I_T; L^p(S; \mathbb{R}^{3\times 3}))$ *for every* $p < \infty$*. Moreover* $A_n e_\alpha$ *is strongly compact in* $L^q_{loc}(I_T; L^p(S; \mathbb{R}^3))$ *for* $\alpha = 1, 2$ *and any* $1 \le p, q < \infty$*.*

We also have a time-dependent version of Proposition 3.2.7:

Proposition 4.3.3. *Let* y_n *satisfy the assumptions of Proposition 4.3.1 and define*

$$\hat{u}_n(t,x) := h_n^{-2}\left((y_n(t,x))' - x'\right),$$
$$\hat{v}_n(t,x) := h_n^{-1}\left(y_n(t,x)\right)_3.$$

Then

$$\hat{u}_n \overset{*}{\rightharpoonup} \hat{u} \quad \text{in } L^\infty_{loc}(I_T; H^1(\Omega; \mathbb{R}^2)), \tag{4.22}$$

$$\hat{v}_n \overset{*}{\rightharpoonup} \hat{v} \quad \text{in } L^\infty_{loc}(I_T; H^1(\Omega)), \tag{4.23}$$

where

$$\hat{u}(t,x) = u(t,x') - \left(x_3 - \frac{1}{2}\right)\nabla' v(t,x'), \tag{4.24}$$

$$\hat{v}(t,x) = v(t,x') + \left(x_3 - \frac{1}{2}\right). \tag{4.25}$$

Moreover it even holds that

$$\hat{v}_n \to \hat{v} \quad \text{in } L^\infty_{loc}(I_T; L^2(\Omega)). \tag{4.26}$$

Proof. By Korn's inequality (Proposition A.1.1) we have for $u \in W^{1,2}(\Omega; \mathbb{R}^2)$ the inequality

$$\|\begin{pmatrix} u \\ 0 \end{pmatrix} - Sx - c\|_{W^{1,2}(\Omega;\mathbb{R}^3)} \le \|\mathrm{sym}\ \nabla \begin{pmatrix} u \\ 0 \end{pmatrix}\|_{L^2(\Omega;\mathbb{R}^{3\times 3})}$$

for $S = \fint_\Omega \mathrm{skew}\ \nabla \begin{pmatrix} u \\ 0 \end{pmatrix}\ dx$ and $c = \fint_\Omega \begin{pmatrix} u \\ 0 \end{pmatrix}\ dx$. Hence

$$\|\hat{u}_n(t)\|_{W^{1,2}(\Omega;\mathbb{R}^2)} \le C\Bigg(\|\mathrm{sym}\ \nabla' \hat{u}_n(t)\|_{L^2(\Omega;\mathbb{R}^{2\times 2})} + \|\partial_3 \hat{u}_n(t)\|_{L^2(\Omega;\mathbb{R}^2)}$$
$$\left|\int_\Omega \mathrm{skew}\ \nabla' \hat{u}_n(t)\ dx\right| + \left|\int_\Omega \hat{u}_n(t)\ dx\right|\Bigg)$$

for almost every t. By (4.13) and Proposition 4.3.2 we recognize that sym $\nabla'\hat{u}_n$ is bounded in $L^\infty_{\text{loc}}(I_T; L^2(\Omega;\mathbb{R}^{2\times 2}))$. The sequence $\left|\int_\Omega \hat{u}_n(t)\ dx\right|$ is bounded in $L^\infty_{\text{loc}}(I_T)$ due to (4.18). Note that $\partial_3 (\hat{u}_n)_i = h_n^{-1} (\nabla_n y_n - Id)_{i3}$ for $i = 1, 2$ and the boundedness in $L^\infty_{\text{loc}}(I_T; L^2(\Omega;\mathbb{R}^2))$ follows from (4.13) and (4.15). To bound $\left|\int_\Omega \text{skew } \nabla'\hat{u}_n(t)\ dx\right|$ we use that

$$\operatorname*{ess\,sup}_{t\in[0,T']} \left|\int_\Omega \text{skew } \nabla'\hat{u}_n(t)\ dx\right| = \operatorname*{ess\,sup}_{t\in[0,T']} \left|\int_S \text{skew } \nabla' u_n(t)\ dx'\right| \leq C(T')$$

by (4.18). This yields boundedness of $\hat{u}_n$ in $L^\infty_{\text{loc}}(I_T; H^1(\Omega;\mathbb{R}^2))$. It remains to identify the limit. Since

$$\int_0^1 \hat{u}_n\ dx_3 \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty_{\text{loc}}(I_T; H^1(S;\mathbb{R}^2))$$

and, for $i = 1, 2$,

$$\partial_3 (\hat{u}_n)_i = h_n^{-1} (\nabla_n y_n - Id)_{i3} \overset{*}{\rightharpoonup} -\partial_i v \quad \text{in } L^\infty_{\text{loc}}(I_T; L^2(\Omega))$$

by (4.13) and Proposition 4.3.2. Therefore the limit is identified by (4.24) and subsequences were not needed.

Next we show that $\hat{v}_n$ is bounded in $L^\infty_{\text{loc}}(I_T; L^2(\Omega))$: From (4.13) and (4.15) we get the bound

$$\operatorname*{ess\,sup}_{t\in[0,T']} \|\nabla\hat{v}_n(t)\|_{L^2(\Omega)} \leq C(T'). \tag{4.27}$$

Since $v_n \to v$ in $L^\infty_{\text{loc}}(0,T; L^2(S))$ and $\fint_\Omega \hat{v}_n\ dx = \frac{1}{|\Omega|}\int_S v_n\ dx'$ we deduce by Poincaré's inequality

$$\begin{aligned} \|\hat{v}_n(t)\|_{L^2(\Omega)} &\leq \left\|\hat{v}_n(t) - \fint_\Omega \hat{v}_n(t)\right\|_{L^2(\Omega)} + \left\|\fint_\Omega \hat{v}_n(t)\right\|_{L^2(\Omega)} \\ &\leq C\left(\|\nabla\hat{v}_n(t)\|_{L^2(\Omega)} + \|v_n(t)\|_{L^2(S)}\right) \\ &\leq C(T') \end{aligned}$$

by (4.27) and (4.19). Thus there is a $\hat{v} \in L^\infty_{\text{loc}}(I_T; H^1(\Omega))$ such that (up to a subsequence) $\hat{v}_n \overset{*}{\rightharpoonup} \hat{v}$ in $L^\infty_{\text{loc}}(I_T; H^1(\Omega))$. To identify $\hat{v}$ note that by (4.19)

$$\int_0^1 \hat{v}_n\ dx_3 \to v$$

and by (4.13) and (4.15)

$$\partial_3\hat{v}_n \to 1$$

both in $L^\infty_{\text{loc}}(I_T; L^2(S))$. Hence the limit $\hat{v}$ is given by (4.25).

Finally the convergence (4.26) follows from Lemma A.2.5, since, due to (4.10), the mappings $\partial_t\hat{v}_n$ are bounded in $L^\infty_{\text{loc}}(I_T; L^2(\Omega))$. □

The next lemma can be seen as a weaker analogue to Proposition 2.1.3 and Proposition 2.1.4 for weak-∗-convergence. Note that we additionally assume that both interpolations of the sequence converge. Thus we only have to identify the limit.

Lemma 4.3.4. *Let $y_n : [0,T] \times \tilde{\Lambda}_n \to \mathbb{R}^3$ be a sequence of discrete deformations.*

(i) Let $\nu_n \to \infty$ and assume that $\tilde{\tilde{y}}_n \overset{*}{\rightharpoonup} y$ and $\bar{\bar{y}}_n \overset{*}{\rightharpoonup} z$ both in $L^\infty(0,T;L^2(\Omega))$. Then $y = z$.

(ii) Let $\nu_n \equiv \nu \in \mathbb{N}$ and assume that $\tilde{\tilde{y}}_n \overset{*}{\rightharpoonup} y$ in $L^\infty(0,T;L^2(\Omega))$ and $\bar{\bar{y}}_n \overset{*}{\rightharpoonup} z$ in $L^\infty\left(0,T;L^2\left(S \times \left(-\frac{1}{2(\nu-1)}, \frac{2\nu-1}{2(\nu-1)}\right)\right)\right)$. Then $z = y^*$, where $y^*(x',x_3) = y(x', \frac{i}{\nu-1})$ for $x_3 \in \left(\frac{2i-1}{2(\nu-1)}, \frac{2i+1}{2(\nu-1)}\right)$, $i = 0, \dots, \nu-1$.

Proof. (i) By Lemma A.2.1 it is sufficient to consider test functions of the form $\varphi(t,x) = \eta(t)\chi_Q(x) \cdot e_i$ with $\eta \in C_c^\infty(0,T)$ and a cube $Q = \prod_{i=1}^3 [a_i, b_i) \subset\subset \Omega$. Let

$$Q^n = \bigcup_{\substack{x \in \tilde{\Lambda}'_n: \\ \tilde{Q}_n(x) \cap Q \neq \emptyset}} \tilde{Q}_n(x) \setminus Q$$

and

$$\hat{Q}_n = \left(\prod_{i=1}^2 [a_i - 3\varepsilon_n, b_i + 3\varepsilon_n)\right) \times \left[a_3 - 3\frac{\varepsilon_n}{h_n}, b_n + 3\frac{\varepsilon_n}{h_n}\right) \setminus Q.$$

Clearly $|Q_n|, |\hat{Q}_n| \to 0$ as $n \to \infty$. For the interpolation $\tilde{\tilde{y}}$ of a mapping $y : \tilde{\Lambda}_n \to \mathbb{R}^3$ by (2.8) it holds that $⨍_{\tilde{Q}_n(x)} \tilde{\tilde{y}}\, d\xi = \bar{\bar{y}}(x) = \frac{1}{8}\sum_{i=1}^8 y\left(x' + \varepsilon(z^i)', x_3 + \frac{\varepsilon_n}{h_n} z_3^i\right)$ for $x \in \tilde{\Lambda}'_n$. Thus for a constant $C > 0$ we can estimate

$$\begin{aligned}
&\left|\int_Q \tilde{\tilde{y}}\, d\xi - \int_Q \bar{\bar{y}}\, d\xi\right| \\
&= \left| \sum_{\substack{x \in \tilde{\Lambda}'_n \\ \tilde{Q}_n(x) \cap Q \neq \emptyset}} \int_{\tilde{Q}_n(x)} \tilde{\tilde{y}}\, d\xi - \int_{Q^n} \tilde{\tilde{y}}\, d\xi - \int_Q \bar{\bar{y}}\, d\xi \right| \\
&= \left| \frac{\varepsilon_n^3}{h_n} \sum_{\substack{x \in \tilde{\Lambda}'_n \\ \tilde{Q}_n(x) \cap Q \neq \emptyset}} ⨍_{\tilde{Q}_n(x)} \tilde{\tilde{y}}\, d\xi - \int_{Q^n} \tilde{\tilde{y}}\, d\xi - \int_Q \bar{\bar{y}}\, d\xi \right| \\
&= \left| \frac{\varepsilon_n^3}{h_n} \sum_{\substack{x \in \tilde{\Lambda}'_n \\ \tilde{Q}_n(x) \cap Q \neq \emptyset}} \frac{1}{8}\sum_{l=1}^8 y\left(x' + \varepsilon_n(z^l)', x_3 + \frac{\varepsilon_n}{h_n} z_3^l\right) - \int_Q \bar{\bar{y}}\, d\xi - \int_{Q^n} \tilde{\tilde{y}}\, d\xi \right| \\
&\le \left|\int_{Q^n} \tilde{\tilde{y}}\, d\xi\right| + C\left|\int_{\hat{Q}_n} \bar{\bar{y}}\, d\xi\right|.
\end{aligned}$$

Applying this estimate yields

$$\left|\int_0^T \int_\Omega \tilde{\tilde{y}}_n \varphi\, dx\, dt - \int_0^T \int_\Omega z\varphi\, dx\, dt\right|$$

$$
\begin{aligned}
&= \left| \int_0^T \eta(t) \int_Q \tilde{\tilde{y}}_n \cdot e_i \, dx \, dt - \int_0^T \int_\Omega z\varphi \, dx \, dt \right| \\
&\leq \left| \int_0^T \eta(t) \int_Q \bar{\bar{y}}_n \cdot e_i \, dx \, dt - \int_0^T \int_\Omega z\varphi \, dx \, dt \right| \\
&\qquad + \left| \int_0^T \eta(t) \int_{Q^n} \tilde{\tilde{y}}_n \cdot e_i \, dx \, dt \right| \\
&\qquad + C \left| \int_0^T \eta(t) \int_{\hat{Q}_n} \bar{\bar{y}}_n \cdot e_i \, dx \, dt \right| \\
&= \left| \int_0^T \int_\Omega \bar{\bar{y}}_n \varphi \, dx \, dt - \int_0^T \int_\Omega z\varphi \, dx \, dt \right| \\
&\qquad + \left| \int_0^T \eta(t) \int_{Q^n} \tilde{\tilde{y}}_n \cdot e_i \, dx \, dt \right| \\
&\qquad + C \left| \int_0^T \eta(t) \int_{\hat{Q}_n} \bar{\bar{y}}_n \cdot e_i \, dx \, dt \right| \to 0
\end{aligned}
$$

as $n \to \infty$. Hence $\tilde{\tilde{y}}_n \overset{*}{\rightharpoonup} z$ in $L^\infty(0,T;L^2(\Omega))$ and $z = y$ must hold.

(ii) With the isomorphism $L^2\left(S \times \left(-\frac{1}{2(\nu-1)}, \frac{2\nu-1}{2(\nu-1)}\right)\right) \cong \bigoplus_{i=0}^{\nu-1} L^2(S)$ for piecewise constant functions in x_3 on the intervals $\left(\frac{2i-1}{2(\nu-1)}, \frac{2i+1}{2(\nu-1)}\right)$, $i = 0, \ldots, \nu-1$, we can show similarly as in (i) that $\tilde{\tilde{y}}_n(\cdot, \frac{i}{\nu-1}) \overset{*}{\rightharpoonup} z_i$ in $L^\infty(0,T;L^2(S))$ for $i = 0, \ldots, \nu-1$. Define $P'_n \colon L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ by

$$
P'_n f(\xi') = \fint_{x' + \left(-\frac{\varepsilon_n}{2}, \frac{\varepsilon_n}{2}\right)^2} f(y') \, dy' \quad \text{whenever } \xi' \in x' + \left(-\frac{\varepsilon_n}{2}, \frac{\varepsilon_n}{2}\right)^2
$$

with $x' \in \varepsilon_n \mathbb{Z}^2$. By Lemma A.2.6 also $P'_n \tilde{\tilde{y}}_n(\cdot, \frac{i}{\nu-1}) \overset{*}{\rightharpoonup} z_i$ in $L^\infty(0,T;L^2(S))$. Moreover for almost every $x' \in S$ the map $x_3 \mapsto P'_n \tilde{\tilde{y}}_n(x', x_3)$ is affine on the intervals $\left(\frac{i-1}{\nu-1}, \frac{i}{\nu-1}\right)$, $i = 1, \ldots, \nu-1$, by Lemma 4.3.5. For $i \in \{1, \ldots, \nu-1\}$ let $\varphi(t,x) = \eta(t)\chi(x)$ with $\eta \in C_c^\infty(0,T)$ and $\chi \in C_c^\infty(S \times \left(\frac{i-1}{\nu-1}, \frac{i}{\nu-1}\right))$. Then

$$
\begin{aligned}
&\int_0^T \int_{S \times \left(\frac{i-1}{\nu-1}, \frac{i}{\nu-1}\right)} P'_n \tilde{\tilde{y}}_n \varphi \, dx \, dt \\
&= \int_0^T \int_{S \times \left(\frac{i-1}{\nu-1}, \frac{i}{\nu-1}\right)} \left[P'_n \tilde{\tilde{y}}_n \left(t, x', \frac{i-1}{\nu-1}\right) + \right. \\
&\left. (\nu - 1) \left(P'_n \tilde{\tilde{y}}_n \left(t, x', \frac{i}{\nu-1}\right) - P'_n \tilde{\tilde{y}}_n \left(t, x', \frac{i-1}{\nu-1}\right) \right) \left(x_3 - \frac{i-1}{\nu-1} \right) \right] \varphi(t,x) \, dx \, dt \\
&\underset{n\to\infty}{\longrightarrow} \int_0^T \int_{S \times \left(\frac{i-1}{\nu-1}, \frac{i}{\nu-1}\right)} \left[z_{i-1} + (\nu-1)(z_i - z_{i-1}) \left(x_3 - \frac{i-1}{\nu-1} \right) \right] \varphi(t,x).
\end{aligned}
$$

Thus $z_i = y(\cdot, \frac{i}{\nu-1})$ for every $i \in \{0, \ldots, \nu-1\}$.

□

Lemma 4.3.5. *Let $Q = [0,1]^3$ and $y : \{0,1\}^3 \to \mathbb{R}$ with its piecewise affine interpolation $\tilde{y}$. Let $Q' = [0,1]^2$. Then the map $x_3 \mapsto \int_{Q'} \tilde{y}(x', x_3)\ dx'$ is affine on the interval $(0,1)$.*

Proof. Denote by $z_1, \dots, z_4$ the corners of the bottom layer and by $z_5, \dots, z_8$ the corners of the top layer of Q. Let $y_i = y(z_i)$. First we assume that $y_1 = y_2 = y_3 = y_4$ and $y_5 = y_6 = y_7 = y_8$. Then the map $y \colon \{0,1\}^3 \to \mathbb{R}$ is affine and consequently the interpolation $\tilde{y}$ is affine on Q. For an arbitrary mapping y we note that permuting the corners on the lower part of the cube as well as the upper part of the cube does not change the value of the integral $\int_{Q'} \tilde{y}(x', x_3)\ dx'$, as we only permute the simplices used in the interpolation. Let $\tilde{y}^i$ be the interpolation after rotating the base face and the top face of the cube i times per 90 degrees, see Figure 4.1. For $x_3 \in [0,1]$ this leads to

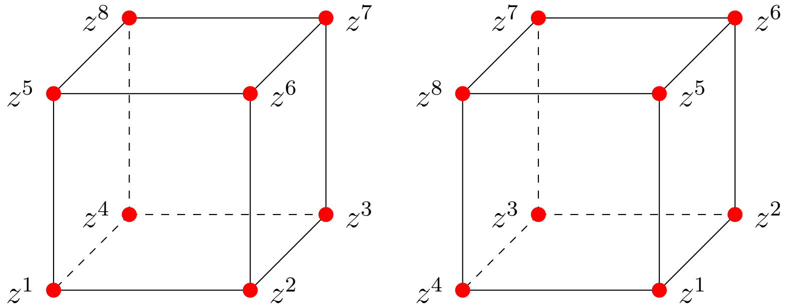

Figure 4.1: One rotation of the cube. The value of the integral of the interpolation does not change.

$$\int_{Q'} \tilde{y}(x', x_3)\ dx' = \frac{1}{4}\sum_{i=1}^{4} \int_{Q'} \tilde{y}^i(x', x_3)\ dx' = \int_{Q'} \tilde{s}(x', x_3)\ dx', \tag{4.28}$$

with $s(z) = \frac{1}{4}\sum_{i=1}^{4} y^i(z)$ for $z \in \{z_1, \dots, z_8\}$. Now

$$s(z_i) = \begin{cases} \frac{1}{4}\sum_{j=1}^{4} y(z_j), & i \in \{1,2,3,4\}, \\ \frac{1}{4}\sum_{j=5}^{8} y(z_j), & i \in \{5,6,7,8\} \end{cases}$$

and, due to the previous case, the map $x_3 \mapsto \int_{Q'} \tilde{s}(x', x_3)\ dx'$ is affine. But then, by (4.28), the map $x_3 \mapsto \int_{Q'} \tilde{y}(x', x_3)\ dx'$ is affine too. □

Corollary 4.3.6. *Let $y_n : I_T \times \tilde{\Lambda}_n \to \mathbb{R}^3$ be a sequence of discrete deformations such that the interpolations $\tilde{y}_n$ satisfy the assumptions of Proposition 4.3.1. Let*

$$\bar{\bar{v}}_n(t,x) := h_n^{-1} \bar{\bar{y}}_n(t,x)_3. \tag{4.29}$$

If $\nu_n \to \infty$

$$\bar{\bar{v}}_n \to \hat{v} \quad in\ L^\infty_{loc}(I_T; L^2(\Omega)). \tag{4.30}$$

If $\nu_n \equiv \nu \in \mathbb{N}$

$$\bar{\bar{v}}_n \to \hat{v}^* \quad in\ L^\infty_{loc}\Big(I_T; L^2\big(S \times (-\frac{1}{2(\nu-1)}, \frac{2\nu-1}{2(\nu-1)})\big)\Big). \tag{4.31}$$

Proof. This is an immediate consequence of (4.26) together with Proposition 2.1.3 or Proposition 2.1.4. □

4.4. The discrete, time-dependent strain

Wee need to find convergence of the discrete strain

$$\bar{G}_n = h_n^{-2}\left(R_n^T \bar{\nabla}_n \bar{\bar{y}}_n - Z\right).$$

As we will do later in the main theorem we assume that

$$\operatorname*{ess\,sup}_{t\in[0,T']} E_n(y_n(t)) \le C(T')h_n^4 \quad \text{for all } T' \in (0,T],$$

by Proposition 3.2.4 this implies

$$\operatorname*{ess\,sup}_{t\in[0,T']} \int_\Omega \operatorname{dist}^2\left(\nabla_n \tilde{\bar{y}}_n(t), SO(3)\right)\, dx \le C(T')h_n^4.$$

Thus we can use the results of [AMM09] proven in Step 4 of their Theorem 2.1: The sequence

$$G_n \coloneqq h_n^{-2}\left(R_n^T \nabla_n \tilde{\bar{y}}_n - Id\right)$$

converges weakly-* to some G in $L^\infty_{\mathrm{loc}}(I_T; L^2(\Omega, \mathbb{R}^{3\times 3}))$. The upper 2×2 matrix G'' of G is affine in x_3, i.e.

$$G''(x,t) = G_1(t,x') + \left(x_3 - \frac{1}{2}\right) G_2(t,x')$$

with $\operatorname{sym} G_1 = \frac{1}{2}\left(\nabla' u + \nabla' u^T + \nabla' v \otimes \nabla' v\right)$ and $G_2 = -\nabla'^2 v$. The mappings u and v are the ones from Proposition 4.3.1.

We again use the projections P_n to piecewise constant functions defined by

$$P_n f(x) = \fint_{\tilde{Q}_n(x)} f(\xi)\, d\xi, \quad x \in \tilde{\Lambda}'_n,$$

on $\tilde{Q}_n(x)$ and P defined by

$$Pf(x) = \fint_{\frac{k-1}{\nu-1}}^{\frac{k}{\nu-1}} f(x',t)\, dt \quad \text{if } \frac{k-1}{\nu-1} \le x_3 \le \frac{k}{\nu-1},\ k \in \{1,\dots,\nu-1\}$$

if $\nu_n \equiv \nu \in \mathbb{N}$ and $P = Id$ if $\nu_n \to \infty$. We remind that for every $f \in L^2(\Omega)$ we have $P_n f \to Pf$ in $L^2(\Omega)$.

Proposition 4.4.1. *Let $(y_n)_n$ be a sequence of discrete deformations such that for every $T' \in (0,T)$*

$$\operatorname*{ess\,sup}_{t\in[0,T']} E_n(y_n(t)) \le C(T')h_n^4 \tag{4.32}$$

and the interpolations $\tilde{\bar{y}}_n$ satisfy the assumptions of Proposition 4.3.1. Let

$$h_n^{-2}\left(R_n^T \nabla_n \tilde{\bar{y}}_n - Id\right) \overset{*}{\rightharpoonup} G \quad \text{in } L^\infty_{loc}(I_T; L^2(\Omega, \mathbb{R}^{3\times 3})).$$

Then

$$\bar{G}_n \overset{*}{\rightharpoonup} \bar{G} \quad \text{in } L^\infty_{loc}(I_T; L^2(\Omega, \mathbb{R}^{3\times 8})),$$

where

$$\bar{G} = \begin{cases} GZ & \text{if } \nu_n \to \infty, \\ PGZ + \frac{1}{2(\nu-1)} G_3 & \text{if } \nu_n \equiv \nu \in \mathbb{N}. \end{cases}$$

Since the proof is essentially the same as in [BS22] we will omit detailed calculations and focus instead on why all of the convergences remain true in the time-dependent setting within the respective function spaces. We start with the boundedness of $\bar{G}_n$ in $L^\infty_{\text{loc}}(I_T; L^2(\Omega; \mathbb{R}^{3\times 8}))$.

Lemma 4.4.2. *In the setting of Proposition 4.4.1 the sequence $\bar{G}_n$ is bounded in $L^\infty_{loc}(I_T; L^2(\Omega; \mathbb{R}^{3\times 8}))$.*

Proof. At this point let us remind that by the energy bound (4.32) it follows from [Theorem 6, [FJM06]] that there is an approximating sequence $R_n \subset L^\infty_{\text{loc}}(I_T; H^1(S; \mathbb{R}^{3\times 3}))$ such that $R_n(t, x') \in SO(3)$ for almost every $(t, x') \in (0, T) \times S$ and

$$\operatorname*{ess\,sup}_{t\in[0,T']} \left\|\nabla_n \tilde{\tilde{y}}_n(t,\cdot) - R_n(t,\cdot)\right\|_{L^2(\Omega)} \leq C(T')h_n^2, \tag{4.33}$$

$$\operatorname*{ess\,sup}_{t\in[0,T']} \|\nabla' R_n(t,\cdot)\|_{L^2(S)} \leq C(T')h_n, \tag{4.34}$$

$$\operatorname*{ess\,sup}_{t\in[0,T']} \|R_n(t,\cdot) - Id\|_{H^1(S)} \leq C(T')h_n \tag{4.35}$$

for every $T' \in (0, T)$. Note that (c.f. Remark 3.2.6) the maps $R_n(t, \cdot)$ can be chosen piecewise constant on two-dimensional cubes of length h_n. As $\varepsilon_n \leq h_n$ in particular they can be chosen piecewise constant on cubes $\tilde{Q}_n(x)$ for $x \in \tilde{\Lambda}'_n$. This is crucial to pass from $\bar{\nabla}_n \bar{\bar{y}}_n$ to $\nabla_n \tilde{\tilde{y}}$ and apply the results for the approximating sequence. By part (ii) of Proposition 2.1.5 and (4.33) we have for $x \in \tilde{\Lambda}'_n$

$$\begin{aligned}
\int_{\tilde{Q}_n(x)} \left|\bar{G}_n(t,\xi)\right|^2 \, d\xi = h_n^{-4} \int_{\tilde{Q}_n(x)} &\left|\bar{\nabla}_n \bar{\bar{y}}_n(t,\xi) - R_n(t,\xi')Z\right|^2 \, d\xi \\
&= h_n^{-4} \frac{\varepsilon_n^3}{h_n} \left|\bar{\nabla}_n \bar{\bar{y}}_n(x) - R_n(t,x')Z\right|^2 \\
\leq C h_n^{-4} \int_{\tilde{Q}_n(x)} &\left|\nabla_n \tilde{\tilde{y}}_n(t,\xi) - R_n(t,\xi)\right|^2 \, d\xi \\
&\leq C(T').
\end{aligned}$$

Summing over all cubes yields the claim. □

Proof of Proposition 4.4.1. Compactness follows from Lemma 4.4.2, i.e. there is a $\bar{G} \in L^\infty_{\text{loc}}(I_T; L^2(\Omega, \mathbb{R}^{3\times 8}))$ such that, up to a subsequence, $\bar{G}_n \overset{*}{\rightharpoonup} \bar{G}$ in $L^\infty_{\text{loc}}(I_T; L^2(\Omega, \mathbb{R}^{3\times 8}))$. It remains to identify this limit. As $R_n \to Id$ boundedely in measure on $(0, T') \times \Omega$ for every $T' \in (0, T)$ we have by Lemma A.2.4

$$R_n \bar{G}_n = h_n^{-2}(\bar{\nabla}_n \bar{\bar{y}}_n - R_n Z) \overset{*}{\rightharpoonup} \bar{G}$$

in $L^\infty_{\text{loc}}(I_T; L^2(\Omega; \mathbb{R}^{3\times 8}))$ as well as

$$h_n^{-2}(\nabla_n \tilde{\tilde{y}}_n - R_n) \overset{*}{\rightharpoonup} G$$

in $L^\infty_{\text{loc}}(I_T; L^2(\Omega; \mathbb{R}^{3\times 3}))$. We distinguish between the affine parts and the non-affine parts. A $b \in \mathbb{R}^8$ is called affine if it is in the linear span of the vectors $b^0, \cdots, b^3$ with $b^0 = (1, \cdots, 1)$

and $b^i = Z^T e_i$ for $i \in \{1,2,3\}$. All vectors orthogonal to the affine vectors are called non-affine. They are characterized by $\sum_{i=1}^8 b_i = 0$ and $Zb = 0$. We start with the affine parts. For every $n \in \mathbb{N}$ we have $R_n\bar{G}_n b^0 = 0$ and therefore $\bar{G}b^0 = 0$. By Lemma A.2.6 (for the calculations see [BS22], Proposition 4) for $i = 1,2$ it holds that

$$\begin{aligned} P_n[R_n\bar{G}_n]b^i &= \frac{2}{h_n^2} P_n[\partial_i \tilde{\bar{y}}_n - R_n e_i] \\ &\overset{*}{\rightharpoonup} 2PGe_i = PGZb^i, \end{aligned}$$

whereas for $i = 3$

$$\begin{aligned} P_n[R_n\bar{G}_n]b^3 &= \frac{2}{h_n^2} P_n[h_n^{-1}\partial_3 \tilde{\bar{y}}_n - R_n e_3] \\ &\overset{*}{\rightharpoonup} 2PGe_3 = PGZb^3. \end{aligned}$$

Summarized for every affine vector b we have $\bar{G}b = PGZb$.

For a non-affine vector b we write $b^T = \left(b^{(1)^T}, b^{(2)^T}\right)$ with $b^{(i)} \in \mathbb{R}^4$. Further we consider the two-dimensional difference operator

$$\bar{\nabla}_n^{2d} f(x) := \frac{1}{\varepsilon_n}\left(f(x' + \varepsilon_n (z^i)', x_3) - \frac{1}{4}\sum_{j=1}^4 f(x' + \varepsilon_n (z^j)', x_3) \right)_{i=1,2,3,4}.$$

By $(\bar{\nabla}_n^{2d})^*$ we denote its adjoint determined by the relation

$$\sum_{x\in\bar{\tilde{\Lambda}}_n} \bar{\nabla}_n^{2d} f(x) \colon H(x) = \sum_{\omega\in\tilde{\Lambda}_n'} f(\omega)\cdot\left((\bar{\nabla}_n^{2d})^* H(\cdot)\right)(\omega)$$

for every $H : \tilde{\Lambda}_n' \to \mathbb{R}^{3\times 4}$. Then (for the calculations see [BS22], (44) and (45))

$$R_n\bar{G}_n(t,x)b = h_n^{-2}\bar{\nabla}_n \bar{\bar{y}}_n(t,x)b$$
$$= h_n^{-2}\left(\bar{\nabla}_n^{2d}\bar{\bar{y}}_n(t, x + \frac{\varepsilon_n}{2h_n}e_3) - \bar{\nabla}_n^{2d}\bar{\bar{y}}_n(t, x - \frac{\varepsilon_n}{2h_n}e_3)\right) b^{(2)} \tag{4.36}$$
$$+ h_n^{-2}\bar{\nabla}_n^{2d}\bar{\bar{y}}_n(t, x - \frac{\varepsilon_n}{2h_n}e_3)\left(b^{(1)} + b^{(2)}\right). \tag{4.37}$$

We treat (4.36) and (4.37) separately and start with (4.37). Let $\varphi(t,x) = \eta(t)\Psi(x)$ with $\eta \in C_c^\infty(0,T')$ and $\Psi \in C_c^\infty(\Omega;\mathbb{R}^3)$. Then, with $Z^{2d} = \left((z^1)', (z^2)', (z^3)', (z^4)'\right) \in \mathbb{R}^{2\times 4}$,

$$\left(\bar{\nabla}_n^{2d}\right)^* \varphi(t,x) \to -\nabla'\varphi(t,x)Z^{2d}$$

uniformly and therefore for $i \in \{1,2\}$, by Proposition 4.3.3, Lemma 4.3.4 and due to $Z^{2d}(b^{(1)} + b^{(2)}) = 0$,

$$\begin{aligned} &h_n^{-2} e_i^T \int_0^{T'} \int_\Omega \bar{\nabla}_n^{2d}(\bar{\bar{y}}_n - \bar{\bar{\mathrm{id}}})(t, x - \frac{\varepsilon_n}{2h_n}e_3)\left(b^{(1)} + b^{(2)}\right)\varphi(t,x)\,dx\,dt \\ &= h_n^{-2} e_i^T \int_0^{T'} \int_\Omega (\bar{\bar{y}}_n - \bar{\bar{\mathrm{id}}})(t, x - \frac{\varepsilon_n}{2h_n}e_3)\left(\bar{\nabla}_n^{2d}\right)^* \varphi(t,x)(b^{(1)} + b^{(2)})\,dx\,dt \end{aligned}$$

$$\to -\int_0^{T'}\int_\Omega \hat{u}_i(t,\tilde{x})\nabla'\varphi(t,x)Z^{2d}\left(b^{(1)}+b^{(2)}\right)\,dx\,dt = 0, \tag{4.38}$$

where $\tilde{x}=x$ if $\nu_n\to\infty$ and $\tilde{x}=\left(x',\frac{\lfloor(\nu-1)x_3\rfloor}{\nu-1}\right)$. For $i=3$ we have

$$\begin{aligned}
&h_n^{-2}e_3^T\int_0^{T'}\int_\Omega \bar{\nabla}_n^{2d}\bar{\bar{y}}_n\left(t,x-\frac{\varepsilon_n}{2h_n}e_3\right)\left(b^{(1)}+b^{(2)}\right)\varphi(t,x)\,dx\,dt\\
&=\frac{1}{(\nu_n-1)\varepsilon_n}\int_0^{T'}\int_\Omega \frac{(\bar{\bar{y}}_n)_3\left(t,x-\frac{\varepsilon_n}{2h_n}e_3\right)}{h_n}\left(\left(\bar{\nabla}_n^{2d}\right)^*\varphi(t,x)\right.\\
&\qquad\left.+\nabla_n'\varphi(t,x)Z^{2d}\right)\left(b^{(1)}+b^{(2)}\right)\,dx\,dt.
\end{aligned}\tag{4.39}$$

Now

$$\begin{aligned}
&\frac{1}{\varepsilon_n}\left(\left(\bar{\nabla}_n^{2d}\right)^*\varphi(t,x)+\nabla_n'\varphi(t,x)Z^{2d}\right)\\
&\to\left(\frac{1}{2}\nabla'^2\varphi(t,x)\left[(z^i)',(z^i)'\right]-\frac{1}{8}\sum_{j=1}^4\nabla'^2\varphi(t,x)\left[(z^j)',(z^j)'\right]\right)_{i=1,2,3,4}
\end{aligned}\tag{4.40}$$

uniformly. By Proposition 4.3.3 together with Proposition 2.1.3 if $\nu_n\to\infty$ from (4.39) we get

$$h_n^{-2}e_3^T\int_0^{T'}\int_\Omega \bar{\nabla}_n^{2d}\bar{\bar{y}}_n\left(t,x-\frac{\varepsilon_n}{2h_n}e_3\right)\left(b^{(1)}+b^{(2)}\right)\varphi(t,x)\,dx\,dt\to 0. \tag{4.41}$$

For $\nu_n\equiv\nu\in\mathbb{N}$ instead by Proposition 4.3.3 and Proposition 2.1.4 it follows from (4.39) and (4.40) that

$$\begin{aligned}
&h_n^{-2}e_3^T\int_0^{T'}\int_\Omega \bar{\nabla}_n^{2d}\bar{\bar{y}}_n\left(t,x-\frac{\varepsilon_n}{2h_n}e_3\right)\left(b^{(1)}+b^{(2)}\right)\varphi(t,x)\,dx\,dt\\
&\to\frac{1}{\nu-1}\int_0^{T'}\int_\Omega\left(\frac{1}{2}\nabla'^2 v\left[(z^i)',(z^i)'\right]\right)_{i=1,2,3,4}\left(b^{(1)}+b^{(2)}\right)\varphi(t,x)\,dx\,dt.
\end{aligned}\tag{4.42}$$

For (4.36) let φ as above. Then, after repeating the calculation done in [BS22] for fixed t and integrating afterwards,

$$\begin{aligned}
&\int_0^{T'}\int_\Omega h_n^{-2}\left(\bar{\nabla}_n^{2d}\bar{\bar{y}}_n(t,x+\frac{\varepsilon_n}{2h_n}e_3)-\bar{\nabla}_n^{2d}\bar{\bar{y}}_n(t,x-\frac{\varepsilon_n}{2h_n}e_3)\right)b^{(2)}\cdot\varphi(t,x)\,dx\,dt\\
&=\frac{\varepsilon_n}{h_n}\int_0^{T'}P_n\tilde{A}_n(t,x)e_3\cdot\left(\bar{\nabla}_n^{2d}\right)^*\varphi(t,x)b^{(2)}\,dx\,dt
\end{aligned}$$

with $\tilde{A}_n=\frac{\nabla_n\tilde{y}_n-Id}{h_n}$. Since $\tilde{A}_n\overset{*}{\rightharpoonup}A=e_3\otimes\nabla'v-\nabla'v\otimes e_3$ in $L^\infty_{\mathrm{loc}}(I_T;L^2(\Omega;\mathbb{R}^{3\times3}))$ by (4.13) and (4.21) we have $P_n\tilde{A}_n\overset{*}{\rightharpoonup}PA$ by Lemma A.2.6 and thus

$$\int_0^{T'}\int_\Omega h_n^{-2}\left(\bar{\nabla}_n^{2d}\bar{\bar{y}}_n(t,x+\frac{\varepsilon_n}{2h_n}e_3)-\bar{\nabla}_n^{2d}\bar{\bar{y}}_n(t,x-\frac{\varepsilon_n}{2h_n}e_3)\right)b^{(2)}\cdot\varphi(t,x)\,dx\,dt\to 0 \tag{4.43}$$

if $\nu_n \to \infty$ and

$$\int_0^{T'} \int_\Omega h_n^{-2} \left(\bar{\nabla}_n^{2d} \bar{\bar{y}}_n(t, x + \frac{\varepsilon_n}{2h_n} e_3) - \bar{\nabla}_n^{2d} \bar{\bar{y}}_n(t, x - \frac{\varepsilon_n}{2h_n} e_3) \right) b^{(2)} \cdot \varphi(t,x) \, dx \, dt$$

$$\to -\frac{1}{\nu - 1} \int_0^{T'} \int_\Omega PAe_3 \cdot \nabla' \varphi(t,x) Z^{2d} b^{(2)} \, dx \, dt$$

$$= -\frac{1}{\nu - 1} \int_0^{T'} \int_\Omega \begin{pmatrix} \nabla'^2 v(t,x') Z^{2d} b^{(2)} \\ 0 \end{pmatrix} \cdot \varphi(t,x) \, dt \, dx. \qquad (4.44)$$

This finishes the investigations of the relevant convergences. Summarized for every non-affine $b \in \mathbb{R}^8$ in case $\nu_n \to \infty$ we get $\bar{G}b = 0$ by (4.38), (4.41) and (4.43). If $\nu_n \equiv \nu \in \mathbb{N}$ by (4.38), (4.42) and (4.44) we obtain $\bar{G}b = (PGZ + \frac{1}{2(\nu-1)} G_3)b$ for every non-affine $b \in \mathbb{R}^8$ after repeating the calculations of [BS22] for fixed t and integrating in time afterwards. Thus for every $b \in \mathbb{R}^8$ it holds that $\bar{G}b = GZ$ if $\nu_n \to \infty$ and $\bar{G}b = (PGZ + \frac{1}{2(\nu-1)} G_3)b$ for every $b \in \mathbb{R}^8$. □

Once again we define

$$J^n(t,x) := h_n^{-2} DW_{\text{cell}}(Z + h_n^2 \bar{G}_n(t,x)), \qquad (4.45)$$

$$J^{(1,n)}(t,x) := h_n^{-2} DW_{\text{surf}}(Z^{(1)} + h_n^2 \bar{G}_n^{(1)}(t,x)), \qquad (4.46)$$

$$J^{(2,n)}(t,x) := h_n^{-2} DW_{\text{surf}}(Z^{(1)} + h_n^2 \bar{G}_n^{(2)}(t,x)). \qquad (4.47)$$

It is an immediate consequence from the growth conditions on DW_{cell} and DW_{surf} that all of these mappings are bounded in $L^\infty_{\text{loc}}(I_T; L^2(\Omega; \mathbb{R}^{3\times 8}))$. By Proposition A.2.3 we have the convergences

$$J^n \overset{*}{\rightharpoonup} J := D^2 W_{\text{cell}}(Z)[\bar{G}] \quad \text{in } L^\infty_{\text{loc}}(I_T; L^2(\Omega; \mathbb{R}^{3\times 8})), \qquad (4.48)$$

$$J^{(1,n)} \overset{*}{\rightharpoonup} J^{(1)} := D^2 W_{\text{surf}}(Z^{(1)})[\bar{G}^{(1)}] \quad \text{in } L^\infty_{\text{loc}}(I_T; L^2(\Omega; \mathbb{R}^{3\times 4})), \qquad (4.49)$$

$$J^{(2,n)} \overset{*}{\rightharpoonup} J^{(2)} := D^2 W_{\text{surf}}(Z^{(1)})[\bar{G}^{(2)}] \quad \text{in } L^\infty_{\text{loc}}(I_T; L^2(\Omega; \mathbb{R}^{3\times 4})). \qquad (4.50)$$

Remark 4.4.3. *It is useful to write the weak form of the equations of motion* (4.9) *in terms of* J^n, $J^{(1,n)}$, *and* $J^{(2,n)}$. *Let* $\varphi(t,x) = \eta(t)\phi(x)$ *such that* $\phi \in C^\infty(\Omega)$ *is compactly supported in* $S \times [0,1]$ *and* $\eta \in C_c^\infty(0,T')$. *After point evaluation on the grid points* $x \in \tilde{\Lambda}_n$ *and subsequent interpolation with Lemma 2.2.1 we get from* (4.9) *for large enough* n

$$\begin{aligned} 0 = &\int_0^{T'} \int_{\tilde{V}_n} \partial_t \bar{\bar{y}}_n \cdot \partial_t \bar{\bar{\varphi}} \, dx \, dt \\ &- \int_0^{T'} \int_\Omega R_n J^n : \bar{\nabla}_n \bar{\bar{\varphi}} \, dx \, dt \\ &- \int_0^{T'} \int_{S \times \left(0, \frac{1}{\nu_n - 1}\right)} R_n J^{(1,n)} : \left(\bar{\nabla}_n \bar{\bar{\varphi}}\right)^{(1)} \, dx \, dt \\ &- \int_0^{T'} \int_{S \times \left(\frac{1}{\nu_n - 1}, 1\right)} R_n J^{(2,n)} : \left(\bar{\nabla}_n \bar{\bar{\varphi}}\right)^{(2)} \, dx \, dt \\ &+ h_n \int_0^{T'} \int_{\tilde{V}_n} \bar{\bar{g}} \bar{\bar{\varphi}}_3. \end{aligned} \qquad (4.51)$$

Corollary 4.4.4. *There is an* $M \in L^\infty_{loc}(I_T; L^2(\Omega;\mathbb{R}^{3\times 8}))$ *such that*

$$h_n^{-2} D_F W(\cdot, Z + h_n^2 \bar{G}_n(\cdot)) \overset{*}{\rightharpoonup} M \quad \text{in } L^\infty_{loc}(I_T; L^2(\Omega;\mathbb{R}^{3\times 8})).$$

If $\nu_n \to \infty$ *it holds that* $M = J$ *whereas for* $\nu_n \equiv \nu \in \mathbb{N}$

$$M(t,x) = \begin{cases} J(t,x) + \big(J^{(1)}(t,x), 0\big) & \text{if } \nu \geq 3,\ x_3 \in \big(0, \frac{1}{\nu-1}\big), \\ J(t,x) & \text{if } \nu \geq 3,\ x_3 \in \big(\frac{1}{\nu-1}, \frac{\nu-2}{\nu-1}\big), \\ J(t,x) + \big(0, J^{(2)}(t,x)\big) & \text{if } \nu \geq 3,\ x_3 \in \big(0, \frac{\nu-2}{\nu-1}\big), \\ J(t,x) + \big(J^{(1)}(t,x), J^{(2)}(t,x)\big) & \text{if } \nu = 2. \end{cases}$$

Proof. This is proven identically as in the stationary case. □

4.4.1. Consequences of the equations of motion

Throughout this section we assume that y_n is a sequence of discrete deformations satisfying (4.9) as well as the energy bounds

$$\operatorname*{ess\,sup}_{t\in[0,T']} E_n(y_n(t)) \leq C(T') h_n^4, \tag{4.52}$$

$$\operatorname*{ess\,sup}_{t\in[0,T']} \frac{\varepsilon_n^3}{h_n} \sum_{x\in\tilde{\Lambda}_n} |\partial_t y_n(t,x)|^2 \leq C(T') h_n^2 \tag{4.53}$$

for every $T' \in (0,T)$.

Proposition 4.4.5. *Let* $\nu_n \to \infty$. *For almost every* $(t,x) \in I_T \times \Omega$ *it holds that*

$$\sum_{l=1}^4 J_{\cdot l}(t,x) = \sum_{l=5}^8 J_{\cdot l}(t,x).$$

Proof. Let $T' \in I_T$ and $\varphi(t,x) = \eta(t)\Psi(x)$ with $\eta \in C_c^\infty(0,T')$ and $\Psi \in C^\infty(\Omega;\mathbb{R}^3)$ compactely supported in $S \times [0,1]$. Then

$$h_n \bar{\nabla}_n \bar{\bar{\varphi}} \to \left(-\frac{\partial_3\varphi}{2}, \cdots, \frac{\partial_3\varphi}{2}, \cdots\right) \tag{4.54}$$

uniformly and therefore also in $L^1(0,T';L^2(\Omega;\mathbb{R}^3))$. Then, by (4.53), (4.54) and Corollary 4.4.4

$$\begin{aligned} 0 = h_n \int_0^{T'} \int_{S\times\left(-\frac{1}{2(\nu_n-1)}, \frac{2\nu_n-1}{2(\nu_n-1)}\right)} \partial_t \bar{\bar{y}}_n \partial_t \bar{\bar{\varphi}} \, dx \, dt \\ - \int_0^{T'} \int_\Omega h_n^{-2} D_F W(x, \bar{\nabla}_n \bar{\bar{y}}_n) \colon h_n \bar{\nabla}_n \bar{\varphi} \, dx \, dt \\ + h_n^2 \int_0^{T'} \int_{\tilde{V}_n} \bar{\bar{g}} \bar{\bar{\varphi}}_3 \, dx \, dt \end{aligned}$$

$$\underset{n\to\infty}{\longrightarrow} -\frac{1}{2}\int_0^{T'}\int_\Omega J\colon\ (-\partial_3\varphi,\cdots,\partial_3\varphi,\cdots)\ dx\ dt.$$

By Lemma A.2.1 we obtain for every $\chi \in L^1(0,T';L^2(\Omega))$

$$0 = \int_0^{T'}\int_\Omega J\colon\ (-\chi,\cdots,\chi\cdots)\ dx\ dt.$$

□

Proposition 4.4.6. *Let $\nu_n \equiv \nu \geq 3$. Let $m \in \{1,\cdots,\nu-3\}$. Then for every $i \in \{1,2,3\}$ and almost every $(t,x') \in I_T \times S$ it holds that*

$$\sum_{l=1}^{4} J_{il}\left(t,x',\frac{2m+1}{2(\nu-1)}\right) = \sum_{l=5}^{8} J_{il}\left(t,x',\frac{2m+1}{2(\nu-1)}\right).$$

For the lowest and uppermost layer it holds that

$$\sum_{l=1}^{4}\left[J_{il}\left(t,x',\frac{1}{2(\nu-1)}\right) + J_{il}^{(1)}\left(t,x',\frac{1}{2(\nu-1)}\right)\right] = \sum_{l=5}^{8} J_{il}\left(t,x',\frac{1}{2(\nu-1)}\right)$$

as well as

$$\sum_{l=1}^{4} J_{il}\left(t,x',\frac{2\nu-3}{2(\nu-1)}\right) = \sum_{l=5}^{8} J_{il}\left(t,x',\frac{2\nu-3}{2(\nu-1)}\right) + \sum_{l=1}^{4} J_{il}^{(2)}\left(t,x',\frac{2\nu-3}{2(\nu-1)}\right).$$

Let $\nu_n \equiv \nu = 2$. Then for every $i \in \{1,2,3\}$ and almost every $(t,x') \in I_T \times S$ it holds that

$$\sum_{l=1}^{4}\left[J_{il}\left(t,x',\frac{1}{2(\nu-1)}\right) + J_{il}^{(1)}\left(t,x',\frac{1}{2(\nu-1)}\right)\right]$$
$$= \sum_{l=5}^{8} J_{il}\left(t,x',\frac{1}{2(\nu-1)}\right) + \sum_{i=1}^{4} J_{il}^{(2)}\left(t,x',\frac{1}{2(\nu-1)}\right).$$

Proof. Let $\nu \geq 3$. Let $\phi_0,\cdots,\phi_{\nu-1}$ be functions of the form

$$\phi_i(t,x) = \eta_i(t)\chi_i(x')$$

with $\eta_i \in C_c^\infty(0,T')$ and $\chi_i \in C_c^\infty(S;\mathbb{R}^3)$. For $s \in \left[\frac{m-1}{\nu-1},\frac{m}{\nu-1}\right)$, $m = 1,\ldots,\nu-1$, we interpolate linearly between the layers, i.e. we set

$$\begin{aligned}\varphi(t,x',s) = &\, m\phi_{m-1}(t,x') - (\nu-1)s\phi_{m-1}(t,x')\\ &+(\nu-1)s\phi_m(t,x') - (m-1)\phi_m(t,x').\end{aligned}$$

Then

$$h_n\bar{\nabla}_n\varphi\left(t,x',\frac{2m-1}{2(\nu-1)}\right)_i$$

$$\underset{n\to\infty}{\longrightarrow} \begin{cases} \frac{\nu-1}{2}\left(\phi_{m-1}(t,x') - \phi_m(t,x')\right) & 1 \leq i \leq 4 \\ \frac{\nu-1}{2}\left(\phi_m(t,x') - \phi_{m-1}(t,x')\right) & 5 \leq i \leq 8 \end{cases} \tag{4.55}$$

uniformly. First we look at the lowest layer. Let $\eta \in C_c^\infty(0,T')$, $\chi \in C_c^\infty(S;\mathbb{R}^3)$ and $\phi_0(t,x') = \eta(t)\chi(x')$. For $l \neq 0$ let $\phi_l \equiv 0$. Then, by (4.53), (4.55) and Lemma A.2.4,

$$\begin{aligned} 0 = h_n \int_0^{T'} \int_{S\times\left(-\frac{1}{2(\nu-1)},\frac{2\nu-1}{2(\nu-1)}\right)} \partial_t \bar{\bar{y}}_n \partial_t \bar{\bar{\varphi}}\; dx\; dt \\ - \int_0^{T'} \int_{S\times\left(0,\frac{1}{\nu-1}\right)} R_n J^n \colon \left(h_n \bar{\nabla}_n \bar{\bar{\varphi}}\right)\; dx\; dt \\ - \int_0^{T'} \int_{S\times\left(0,\frac{1}{\nu-1}\right)} R_n J^{(1,n)} \colon \left(h_n \bar{\nabla}_n \bar{\bar{\varphi}}\right)^{(1)}\; dx\; dt \\ + h_n^2 \int_0^{T'} \int_{\tilde{V}_n} \bar{\bar{\varphi}} \bar{\bar{g}}\; dx\; dt \\ \underset{n\to\infty}{\longrightarrow} -\frac{\nu-1}{2} \int_0^{T'} \int_{S\times\left(0,\frac{1}{\nu-1}\right)} J\left(t,x',\frac{1}{2(\nu-1)}\right) \colon \left(\phi_0,\cdots,\phi_0,-\phi_0,\cdots,-\phi_0\right)\; dx\; dt \\ -\frac{\nu-1}{2} \int_0^{T'} \int_{S\times\left(0,\frac{1}{\nu-1}\right)} J^{(1)}\left(t,x',\frac{1}{2(\nu-1)}\right) \colon \left(\phi_0,\phi_0,\phi_0,\phi_0\right)\; dx\; dt. \end{aligned}$$

Therefore, for almost every $t \in (0,T')$

$$\sum_{l=1}^{4} \left(J_{\cdot l}\left(t,\cdot,\frac{1}{2(\nu-1)}\right) + J^{(1)}\left(t,\cdot,\frac{1}{2(\nu-1)}\right)\right) = \sum_{l=5}^{8} J_{\cdot l}\left(t,\cdot,\frac{1}{2(\nu-1)}\right)$$

in $L^2(S;\mathbb{R}^{3\times 8})$. Analogously setting $\phi_1(t,x') = \eta(t)\chi(x')$ and $\phi_i = 0$ else yields

$$\begin{aligned} 0 = \frac{\nu-1}{2} \int_0^{T'} \int_{S\times\left(0,\frac{1}{\nu-1}\right)} J\left(t,x',\frac{1}{2(\nu-1)}\right) : \left(-\phi_1,\cdots,-\phi_1,\phi_1,\cdots,\phi_1\right)\; dx\; dt \\ + \frac{\nu-1}{2} \int_0^{T'} \int_{S\times\left(0,\frac{1}{\nu-1}\right)} J^{(1)}\left(t,x',\frac{1}{2(\nu-1)}\right) : \left(-\phi_1,-\phi_1,-\phi_1,-\phi_1\right)\; dx\; dt \\ + \frac{\nu-1}{2} \int_0^{T'} \int_{S\times\left(\frac{1}{\nu-1},\frac{2}{\nu-1}\right)} J\left(t,x',\frac{3}{2(\nu-1)}\right) : \left(\phi_1,\cdots,\phi_1,-\phi_1,\cdots,-\phi_1\right)\; dx\; dt. \end{aligned}$$

The first two terms sum up to zero as seen before which directly gives for almost every $t \in (0,T')$

$$\sum_{l=1}^{4} J_{\cdot l}\left(t,\cdot,\frac{3}{2(\nu-1)}\right) = \sum_{l=5}^{8} J_{\cdot l}\left(t,\cdot,\frac{3}{2(\nu-1)}\right)$$

in $L^2(S;\mathbb{R}^{3\times 8})$. Proceeding like that yields the claimed identities. The case $\nu = 2$ can be treated analogously. □

Also in the time dependent setting we have the following symmetry properties.

Lemma 4.4.7. *The scaled stresses J^n, $J^{(1,n)}$ and $J^{(2,n)}$ satisfy*

$$\operatorname*{ess\,sup}_{t\in[0,T']} \left\| J^n(t)Z^T - Z(J^n(t)^T)\right\|_{L^1(\Omega;\mathbb{R}^{3\times 3})} \leq C(T')h_n^2, \tag{4.56}$$

$$\operatorname*{ess\,sup}_{t\in[0,T']} \left\| J^{(1,n)}(t)Z^{(1)^T} - Z^{(1)}(J^{(1,n)}(t)^T)\right\|_{L^1(\Omega;\mathbb{R}^{3\times 3})} \leq C(T')h_n^2, \tag{4.57}$$

$$\operatorname*{ess\,sup}_{t\in[0,T']} \left\| J^{(2,n)}(t)Z^{(2)^T} - Z^{(2)}(J^{(2,n)}(t)^T)\right\|_{L^1(\Omega;\mathbb{R}^{3\times 3})} \leq C(T')h_n^2. \tag{4.58}$$

Proof. For the bulk part the proof of Lemma 3.2.19 shows that for fixed t the left-hand-side is bounded by $(\|J^n(t,\cdot)\|_{L^2(\Omega;\mathbb{R}^{3\times 8})} + \|\bar{G}_n(t,\cdot)\|_{L^2(\Omega;\mathbb{R}^{3\times 8})})h_n^2$. Integrating in t yields the claim. For the surface terms we proceed analogously. □

4.5. The main result in the time dependent case

Theorem 4.5.1. *Let $w_n^{(0)}, w_n^{(1)} : \tilde{\Lambda}_n \to \mathbb{R}^3$ be two sequences of lattice deformations satisfying*

$$\frac{1}{2}\frac{\varepsilon_n^3}{h_n}\sum_{x\in\tilde{\Lambda}_n} \left|w_n^{(1)}(x)\right|^2 + \frac{\varepsilon_n^3}{h_n}\sum_{x\in\tilde{\Lambda}_n'} W(x, \bar{\nabla}_n w_n^{(0)}(x)) \leq Ch_n^4. \tag{4.59}$$

as well as the boundary and compatibility conditions

$$w_n^{(0)}(x) = \begin{pmatrix} x' \\ h_n\left(x_3 - \frac{1}{2}\right)\end{pmatrix} \quad \text{on } \partial\tilde{\Lambda}_n,$$
$$w_n^{(1)}(x) = 0 \quad \text{on } \partial\tilde{\Lambda}_n.$$

Let $T \in (0,\infty]$ and $g \in L^2(I_T; W^{1,\infty}(S)) \cap C^0(I_T, L^\infty(S))$. Let (y_n) be a sequence of solutions to the system of ordinary differential equations

$$\begin{cases} h_n^2\partial_t^2 y_n(t,x) = -\bar{\nabla}_n^*\left(D_F W\left(\cdot, \bar{\nabla}_n y_n(t,\cdot)\right)\right)(x) + h_n^3 g(t,x')e_3 & \text{in } (0,T_n)\times \operatorname{int}(\tilde{\Lambda}_n), \\ y_n(t,x) = \left(x', h_n(x_3-\frac{1}{2})\right) & \text{on } (0,T_n)\times\partial\tilde{\Lambda}_n \\ y_n(0,x) = w_n^{(0)}(x) & \text{in } \tilde{\Lambda}_n, \\ \partial_t y_n(0,x) = h_n^{-1}w_n^{(1)}(x) & \text{in } \tilde{\Lambda}_n, \end{cases} \tag{ODE}$$

where T_n is the maximal time of existence. Then $T_n = T$ for all $n \in \mathbb{N}$.

Let

$$u_n(t,x') := h_n^{-2}\int_0^1 \left(\tilde{\bar{y}}_n'(t,x',x_3) - x'\right)\, dx_3,$$
$$v_n(t,x') := h_n^{-1}\int_0^1 \left(\tilde{\bar{y}}_n(t,x',x_3)\right)_3\, dx_3.$$

There exist $u \in L^\infty_{loc}(I_T; H^1(S;\mathbb{R}^2))$ *and* $v \in L^\infty_{loc}(I_T; H^2(S)) \cap W^{1,\infty}_{loc}(0,T;L^2(S))$ *such that, up to a subsequence,*

$$u_n \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty_{loc}(I_T; H^1(S;\mathbb{R}^2)), \tag{4.60}$$

$$v_n \to v \quad \text{in } L^\infty_{loc}(I_T; L^2(S)), \tag{4.61}$$

$$\partial_t v_n \overset{*}{\rightharpoonup} \partial_t v \quad \text{in } L^\infty_{loc}(I_T; L^2(S)). \tag{4.62}$$

The maps u *and* v *satisfy the boundary conditions*

$$u_{|\partial S} = 0, \;\; v_{|\partial S} = 0, \;\; \nabla' v_{|\partial S} = 0. \tag{4.63}$$

Moreover the mappings

$$t \mapsto v(t): \quad I_T \to H^2(S),$$
$$t \mapsto \partial_t v(t): \quad I_T \to L^2(S)$$

are weakly continuous. The pair (u, v) *satisfies the equations* (4.5), (4.6) *if* $\nu_n \to \infty$, *the equations* (4.7), (4.8) *if* $\nu_n \equiv \nu \in \mathbb{N}$ *and the initial conditions*

$$v(0,x') = w_3^{(0)}(x'), \tag{4.64}$$

$$\partial_t v(0,x') = w_3^{(1)}(x') \tag{4.65}$$

for almost every $x' \in S$, *where*

$$\frac{1}{h_n}\int_0^1 \left(\tilde{\bar{w}}_n^{(0)}(\cdot, x_3)\right)_3 \, dx_3 \to w_3^{(0)} \quad \text{in } L^2(S),$$
$$\frac{1}{h_n^2}\int_0^1 \left(\tilde{\bar{w}}_n^{(1)}(\cdot, x_3)\right)_3 \, dx_3 \rightharpoonup w_3^{(1)} \quad \text{in } H^1(S).$$

4.5.1. Outline of the proof

In *Step 1* we will derive an energy bound which will be used in *Step 2* to show that the solutions to (ODE) exist up to time T. *Step 3* and *Step 4* deal with the decomposition of the discrete gradient and the convergence of the displacements. In *Step 5* we will show that the equations (4.6) and (4.8) hold true. *Step 7* is to show the weak continuity statements as well as that v satisfies the stated initial conditions. *Step 6*, the derivation of equations (4.5) and (4.7), is the most complex part. Thus we split it in 3 parts. Let $\phi(t,x') = \mu(t)\chi(x')$ with $\mu \in C_c^\infty(0,T')$ and $\chi \in C_c^\infty(S)$. In **Part 1** let $\varphi(t,x) = (0,0,\phi(t,x'))$. The goal of this part is to show

$$\int_0^{T'}\int_\Omega h_n^{-1} J^n : \bar{\nabla}_n \bar{\bar{\varphi}} \; dx \; dt$$
$$+ \int_0^{T'}\int_{S\times\left(0,\frac{1}{\nu_n-1}\right)} h_n^{-1} J^{(1,n)} : \left(\bar{\nabla}_n \bar{\bar{\varphi}}\right)^{(1)} \; dx \; dt$$

$$+\int_0^{T'}\int_{S\times\left(\frac{\nu_n-2}{\nu_n-1},1\right)} h_n^{-1}J^{(2,n)}:\ \left(\bar{\nabla}_n\bar{\bar{\varphi}}\right)^{(2)}\ dx\ dt$$

$$\underset{n\to\infty}{\longrightarrow}\int_0^{T'}\int_S \partial_t v\partial_t\phi\ dx'\ dt$$

$$-\int_0^{T'}\int_S \mathcal{L}_2\left(\mathrm{sym}\ \nabla' u+\frac{1}{2}\nabla' v\otimes\nabla' v\right):\nabla' v\otimes\nabla'\phi\ dx'\ dt$$

$$+\int_0^{T'}\int_S g\phi\ dx'\ dt. \tag{4.66}$$

for $\nu_n\to\infty$ and

$$\int_0^{T'}\int_\Omega h_n^{-1}J^n:\bar{\nabla}_n\bar{\bar{\varphi}}\ dx\ dt$$

$$+\int_0^{T'}\int_{S\times\left(0,\frac{1}{\nu-1}\right)} h_n^{-1}J^{(1,n)}:\ \left(\bar{\nabla}_n\bar{\bar{\varphi}}\right)^{(1)}\ dx\ dt$$

$$+\int_0^{T'}\int_{S\times\left(\frac{\nu-2}{\nu-1},1\right)} h_n^{-1}J^{(2,n)}:\ \left(\bar{\nabla}_n\bar{\bar{\varphi}}\right)^{(2)}\ dx\ dt$$

$$\underset{n\to\infty}{\longrightarrow}\frac{\nu}{\nu-1}\int_0^{T'}\int_S \partial_t v\partial_t\phi\ dx'\ dt$$

$$-\int_0^{T'}\int_S \frac{1}{2}DQ_{\mathrm{cell}}^{\mathrm{rel}}\left(\begin{pmatrix}\mathrm{sym}\ G_1 & 0\\ 0 & 0\end{pmatrix}Z+\frac{1}{2(\nu-1)}G_3\right):\begin{pmatrix}\nabla' v\otimes\nabla'\phi & 0\\ 0 & 0\end{pmatrix}Z\ dx'\ dt$$

$$-\int_0^{T'}\int_S \frac{1}{\nu-1}DQ_{\mathrm{surf}}\left(\begin{pmatrix}\mathrm{sym}\ G_1 & 0\\ 0 & 0\end{pmatrix}Z^{(1)}+\frac{\partial_{12}v}{2(\nu-1)}M^{(1)}\right):$$

$$\begin{pmatrix}\nabla' v\otimes\nabla'\phi & 0\\ 0 & 0\end{pmatrix}Z^{(1)}\ dx'\ dt$$

$$+\frac{\nu}{\nu-1}\int_0^{T'}\int_S g\phi dx'\ dt. \tag{4.67}$$

for $\nu_n\equiv\nu\in\mathbb{N}$.

In **Part 2** we will show that for $\nu_n\to\infty$ (where $\hat{x}$ is such that $\tilde{Q}_n(x)=\hat{x}+\left(-\frac{\varepsilon_n}{2},\frac{\varepsilon_n}{2}\right)\times\left(-\frac{\varepsilon_n}{2h_n},\frac{\varepsilon_n}{2h_n}\right)$)

$$\sum_{l=1}^{8}\int_0^{T'}\int_\Omega h_n^{-1}\left(R_nJ^n\right)_{\cdot l}\cdot z_3^l\begin{pmatrix}\nabla'\phi(t,\hat{x}')\\ 0\end{pmatrix}\ dx\ dt$$

$$+\sum_{l=1}^{4}\int_0^{T'}\int_{S\times\left(0,\frac{1}{\nu_n-1}\right)} h_n^{-1}\left(R_nJ^{(1,n)}\right)_{\cdot l}\cdot z_3^l\begin{pmatrix}\nabla'\phi(t,\hat{x}')\\ 0\end{pmatrix}\ dx\ dt$$

$$+\sum_{l=1}^{4}\int_0^{T'}\int_{S\times\left(\frac{\nu_n-2}{\nu_n-1},1\right)} h_n^{-1}\left(R_nJ^{(2,n)}\right)_{\cdot l}\cdot z_3^{l+4}\begin{pmatrix}\nabla'\phi(t,\hat{x}')\\ 0\end{pmatrix}\ dx\ dt$$

$$\underset{n\to\infty}{\longrightarrow} \int_0^{T'} \int_S \frac{1}{12} \mathcal{L}_2 \left(\nabla'^2 v\right) : \nabla'^2 \phi(t,x') \; dx' \; dt. \tag{4.68}$$

and for $\nu_n \equiv \nu \in \mathbb{N}$

$$\begin{aligned}
&\sum_{l=1}^{8} \int_0^{T'} \int_\Omega h_n^{-1} \left(R_n J^n\right)_{\cdot l} \cdot z_3^l \begin{pmatrix} \nabla'\phi(t,\hat{x}') \\ 0 \end{pmatrix} \; dx \; dt \\
&+ \sum_{l=1}^{4} \int_0^{T'} \int_{S\times\left(0,\frac{1}{\nu_n-1}\right)} h_n^{-1} \left(R_n J^{(1,n)}\right)_{\cdot l} \cdot z_3^l \begin{pmatrix} \nabla'\phi(t,\hat{x}') \\ 0 \end{pmatrix} \; dx \; dt \\
&+ \sum_{l=1}^{4} \int_0^{T'} \int_{S\times\left(\frac{\nu_n-2}{\nu_n-1},1\right)} h_n^{-1} \left(R_n J^{(2,n)}\right)_{\cdot l} \cdot z_3^{l+4} \begin{pmatrix} \nabla'\phi(t,\hat{x}') \\ 0 \end{pmatrix} \; dx \; dt \\
&\underset{n\to\infty}{\longrightarrow} - \int_0^{T'} \int_S \frac{\nu(\nu-2)}{24(\nu-1)^2} DQ_{\text{cell}}^{\text{rel}} \left(\begin{pmatrix} G_2 & 0 \\ 0 & 0 \end{pmatrix} Z \right) : \begin{pmatrix} \nabla'^2\phi & 0 \\ 0 & 0 \end{pmatrix} Z \; dx' \; dt \\
&- \int_0^{T'} \int_S \frac{1}{4(\nu-1)} DQ_{\text{cell}}^{\text{rel}} \left(\begin{pmatrix} \text{sym } G_1 & 0 \\ 0 & 0 \end{pmatrix} Z + \frac{1}{2(\nu-1)} G_3 \right) : \begin{pmatrix} \nabla'^2\phi & 0 \\ 0 & 0 \end{pmatrix} Z_- \; dx' \; dt \\
&- \int_0^{T'} \int_S \frac{1}{4(\nu-1)} DQ_{\text{surf}} \left(\begin{pmatrix} G_2 & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} \right) : \begin{pmatrix} \nabla'^2\phi & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} \; dx' \; dt.
\end{aligned} \tag{4.69}$$

In **Part 3** we determine the convergence of

$$\begin{aligned}
&\int_0^{T'} \int_\Omega h_n^{-1} J^n : \bar{\nabla}_n \bar{\bar{\varphi}} \; dx \; dt \\
&+ \int_0^{T'} \int_{S\times\left(0,\frac{1}{\nu_n-1}\right)} h_n^{-1} J^{(1,n)} : \left(\bar{\nabla}_n \bar{\bar{\varphi}}\right)^{(1)} \; dx \; dt \\
&+ \int_0^{T'} \int_{S\times\left(\frac{\nu_n-2}{\nu_n-1},1\right)} h_n^{-1} J^{(2,n)} : \left(\bar{\nabla}_n \bar{\bar{\varphi}}\right)^{(2)} \; dx \; dt \\
&- \sum_{l=1}^{8} \int_0^{T'} \int_\Omega h_n^{-1} \left(R_n J^n\right)_{\cdot l} \cdot z_3^l \begin{pmatrix} \nabla'\phi(t,\hat{x}') \\ 0 \end{pmatrix} \; dx \; dt \\
&- \sum_{l=1}^{4} \int_0^{T'} \int_{S\times\left(0,\frac{1}{\nu_n-1}\right)} h_n^{-1} \left(R_n J^{(1,n)}\right)_{\cdot l} \cdot z_3^l \begin{pmatrix} \nabla'\phi(t,\hat{x}') \\ 0 \end{pmatrix} \; dx \; dt \\
&- \sum_{l=1}^{4} \int_0^{T'} \int_{S\times\left(\frac{\nu_n-2}{\nu_n-1},1\right)} h_n^{-1} \left(R_n J^{(2,n)}\right)_{\cdot l} \cdot z_3^{l+4} \begin{pmatrix} \nabla'\phi(t,\hat{x}') \\ 0 \end{pmatrix} \; dx \; dt.
\end{aligned} \tag{4.70}$$

For $\nu_n \to \infty$ we will prove that (4.70) $\to 0$ which yields (4.5). For $\nu_n \equiv \nu \in \mathbb{N}$ we have

$$\begin{aligned}
(4.70) \to &\int_0^{T'} \int_S \frac{1}{4(\nu-1)} DQ_{\text{cell}}^{\text{rel}} \left(\begin{pmatrix} \text{sym } G_1 & 0 \\ 0 & 0 \end{pmatrix} Z + \frac{1}{2(\nu-1)} G_3 \right) : \partial_{12}\phi M \; dx' \; dt \\
+ &\int_0^{T'} \int_S \frac{1}{2(\nu-1)^2} DQ_{\text{surf}} \left(\begin{pmatrix} \text{sym } G_1 & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} + \frac{\partial_{12} v}{2(\nu-1)} M^{(1)} \right) : \partial_{12}\phi M^{(1)} \; dx' \; dt
\end{aligned} \tag{4.71}$$

which implies (4.7).

4.5.2. Proof of the main theorem

We will follow the structure given in the outline. At this point we remark that solutions y_n of (ODE) in particular satisfy (4.9).

Proof of Theorem 4.5.1. Step 1: An energy bound. From the bound on the initial data we will show the inequalities

$$\operatorname*{ess\,sup}_{t\in[0,T']} \frac{\varepsilon_n^3}{h_n} \sum_{x\in\tilde\Lambda_n'} W\left(x, \bar\nabla_n y_n(t,x)\right) \le C(T')h_n^4, \tag{4.72}$$

$$\operatorname*{ess\,sup}_{t\in[0,T']} \frac{\varepsilon_n^3}{h_n} \sum_{x\in\tilde\Lambda_n} |\partial_t y_n(t,x)|^2 \le C(T')h_n^2 \tag{4.73}$$

for every $T' \in (0,T)$. We may assume for a moment that $T_n \le T'$. Now we show the energy bound which will eventually yield that $T_n \ge T'$ must hold for every $n \in \mathbb{N}$. Multiplying (ODE) with $\partial_t y_n(t,x)$ and summing over all $x \in \tilde\Lambda_n$ we deduce

$$\frac{d}{dt}\left(\frac{h_n^2}{2} \sum_{x\in\tilde\Lambda_n} |\partial_t y_n(t,x)|^2 + \sum_{x\in\tilde\Lambda_n'} W(x, \bar\nabla_n y_n(t,x)) \right) = h_n^3 \sum_{x\in\tilde\Lambda_n} g(t,x')\,(\partial_t y_n(t,x))_3\,.$$

Thus we get

$$\begin{aligned}
&\frac{h_n^2}{2} \sum_{x\in\tilde\Lambda_n} |\partial_t y_n(t,x)|^2 + \sum_{x\in\tilde\Lambda_n'} W\left(x, \bar\nabla_n y_n(t,x)\right)\\
&= \frac{h_n^2}{2} \sum_{x\in\tilde\Lambda_n} |\partial_t y_n(0,x)|^2 + \sum_{x\in\tilde\Lambda_n} W(x, \bar\nabla_n y_n(0,x))\\
&\qquad + \int_0^t h_n^3 \sum_{x\in\tilde\Lambda_n} g(s,x')\,(\partial_t y_n(s,x))_3 \; ds
\end{aligned} \tag{4.74}$$

for every $0 \le t \le T_n$. From (4.59) and (4.74) we deduce using Young's inequality in the form $|ab| \le h\frac{a^2}{2} + \frac{b^2}{2h}$

$$\begin{aligned}
&\frac{h_n^2}{2}\frac{\varepsilon_n^3}{h_n} \sum_{x\in\tilde\Lambda_n} |\partial_t y_n(t,x)|^2 + \frac{\varepsilon_n^3}{h_n} \sum_{x\in\tilde\Lambda_n'} W\left(x, \bar\nabla_n y_n(t,x)\right)\\
&\le Ch_n^4 + \frac{h_n^4}{2} \int_0^t \frac{\varepsilon_n^3}{h_n} \sum_{x\in\tilde\Lambda_n} |g(s,x')|^2 \; ds + \frac{h_n^2}{2} \int_0^t \frac{\varepsilon_n^3}{h_n} \sum_{x\in\tilde\Lambda_n} |\partial_t y_n(s,x)|^2 \; ds\\
&= Ch_n^4 + \frac{h_n^4}{2} \int_0^t \int_{\tilde V_n} |\bar{\bar g}(s,x')|^2 \; dx \; ds + \frac{h_n^2}{2} \int_0^t \frac{\varepsilon_n^3}{h_n} \sum_{x\in\tilde\Lambda_n} |\partial_t y_n(s,x)|^2 \; ds
\end{aligned}$$

$$\leq Ch_n^4 + \frac{h_n^2}{2} \int_0^t \frac{\varepsilon_n^3}{h_n} \sum_{x \in \tilde{\Lambda}_n} |\partial_t y_n(s,x)|^2 \, ds. \tag{4.75}$$

Thus

$$\frac{\varepsilon_n^3}{h_n} \sum_{x \in \tilde{\Lambda}_n} |\partial_t y_n(t,x)|^2 \leq Ch_n^2 + \int_0^t \frac{\varepsilon_n^3}{h_n} \sum_{x \in \tilde{\Lambda}_n} |\partial_t y_n(s,x)|^2 \, ds$$

and applying Gronwall's inequality yields

$$\frac{\varepsilon_n}{h_n} \sum_{x \in \tilde{\Lambda}_n} |\partial_t y_n(t,x)|^2 \leq Ch_n^2 + Ch_n^2 \exp(T_n) \leq Ch_n^2(1 + \exp(T')) = C(T')h_n^2.$$

Integration leads to

$$\int_0^{T_n} \frac{\varepsilon_n^3}{h_n} \sum_{x \in \tilde{\Lambda}_n} |\partial_t y_n(t,x)|^2 \, dt \leq C(T')h_n^2.$$

Together with (4.75) this immediately implies the inequalities (4.72) and (4.73) up to time T_n. Since $\partial_t y_n(t,x) = 0$ für $x \in \partial\tilde{\Lambda}_n$ we get by Corollary 2.1.7 the same energy bound for the piecewise affine interpolations, i.e. it holds that

$$\operatorname*{ess\,sup}_{t \in [0,T_n]} \int_\Omega \left|\partial_t \tilde{\bar{y}}_n(x)\right|^2 \, dx \leq C(T_n)h_n^2. \tag{4.76}$$

Step 2: Existence time of the solutions.

Evidently (4.73) rules out a finite time blow-up for $\partial_t y_n$, i.e. for every $n \in \mathbb{N}$ there is a constant $C(n)$ such that

$$\operatorname*{ess\,sup}_{t \in [0,T']} \|\partial_t y_n(t,\cdot)\|_{l^\infty(\tilde{\Lambda}_n)} \leq C(n). \tag{4.77}$$

It follows from (4.72) and the lower bound of W that

$$\operatorname*{ess\,sup}_{t \in [0,T_n]} \left\|\bar{\nabla}_n y_n(t,\cdot)\right\|_\infty \leq C(n),$$

and equivalently (with a different constant)

$$\operatorname*{ess\,sup}_{t \in [0,T_n]} \left\|\bar{D}_n y_n(t,\cdot)\right\|_\infty \leq C(n). \tag{4.78}$$

Now fix an arbitrary $x_0 \in \partial\tilde{\Lambda}_n$. For $x \in \tilde{\Lambda}_n$ we can find l elements $x_1, \cdots, x_l \in \tilde{\Lambda}_n$, where l is independent of x, such that

- $x = x_1$,
- $x_0 = x_l$,
- $x_i, x_{i+1} \in Q_n(\omega)$ for some $\omega \in \tilde{\Lambda}'_n$.

Together with (4.78) the third property implies

$$|y_n(t, x_{i+1}) - y_n(t, x_i)| \le C\varepsilon_n.$$

The (fully) clamped boundary conditions then imply $|y_n(t,x)| \le C(n)$ and therefore

$$\operatorname*{ess\,sup}_{t\in[0,T_n]} \|y_n(t,\cdot)\|_{l^\infty(\tilde\Lambda_n)} \le C(n). \tag{4.79}$$

Together with (4.77) this implies $T_n = T$.

Step 3: Decomposition of deformation gradient.
By (4.72) and the choice of the force term we have for every $T' \in (0,T)$

$$\operatorname*{ess\,sup}_{t\in[0,T']} E_n(y_n(t)) \le C(T')h_n^4.$$

By Proposition 3.2.4 this induces

$$\operatorname*{ess\,sup}_{t\in[0,T']} \int_\Omega \operatorname{dist}^2(\nabla_n \tilde{\bar{y}}_n(t,x), SO(3))\ dx \le C(T')h_n^4.$$

Together with (4.76) by Proposition 4.3.1 we find an approximating sequence $R_n \subset L^\infty_{\text{loc}}(0,T;H^1(S;\mathbb{R}^{3\times3}))$ such that $R_n(t,x') \in SO(3)$ for almost every $(t,x') \in (0,T)\times S$ and

$$\operatorname*{ess\,sup}_{t\in[0,T']} \left\|\nabla_n\tilde{\bar{y}}_n(t,\cdot) - R_n(t,\cdot)\right\|_{L^2(\Omega)} \le C(T')h_n^2, \tag{4.80}$$

$$\operatorname*{ess\,sup}_{t\in[0,T']} \|\nabla' R_n(t,\cdot)\|_{L^2(S)} \le C(T')h_n, \tag{4.81}$$

$$\operatorname*{ess\,sup}_{t\in[0,T']} \|R_n(t,\cdot) - Id\|_{H^1(S)} \le C(T')h_n \tag{4.82}$$

for every $T' \in I_T$. Here we have used that the deformations y_n satisfy the fully clamped boundary conditions (otherwise we would need an additional rotation and a translation). The estimates (4.80), (4.81) and (4.82) have been shown in [AMM09].

Step 4: Convergence of the displacements and boundary conditions
The convergence of the displacements u_n and v_n as well as the boundary conditions (4.63) are contained in Proposition 4.3.1.

Step 5: Derivation of equations (4.6) *and* (4.8). Let $\Psi(t,x') = \eta(t)\chi(x')$ with $\eta \in C_c^\infty(0,T')$ and $\chi \in C_c^\infty(S;\mathbb{R}^2)$. With $\varphi(t,x) := (\Psi(t,x'),0)$ we have by (4.9)

$$\begin{aligned} 0 = h_n^2 &\int_0^{T'} \int_{S\times\left(-\frac{1}{2(\nu-1)},\frac{2\nu-1}{2(\nu-1)}\right)} \partial_t\bar{\bar{y}}_n(t,x)\cdot\partial_t\bar{\bar{\varphi}}(t,x)\ dx\ dt \\ &- \int_0^{T'} \int_\Omega D_F W\left(x, \bar\nabla_n\bar{\bar{y}}_n(t,x)\right) : \bar\nabla_n\bar{\bar{\varphi}}(t,x)\ dx\ dt. \end{aligned}$$

Note that the force term does not appear here because of $\varphi_3 = 0$. After multiplication with h_n^{-2} and decomposition of the discrete gradient we get

$$
\begin{aligned}
0 &= \int_0^{T'} \int_{S\times\left(-\frac{1}{2(\nu_n-1)},\frac{2\nu_n-1}{2(\nu_n-1)}\right)} \partial_t \bar{\bar{y}}_n(t,x)\cdot \partial_t \bar{\bar{\varphi}}(t,x)\; dx\; dt \\
&\quad - \int_0^{T'} \int_\Omega h_n^{-2} R_n D_F W\left(x, Z + h_n^2 \bar{G}_n\right) : \bar{\nabla}_n \bar{\bar{\varphi}}\; dx\; dt \\
&= \int_0^{T'} \int_{S\times\left(-\frac{1}{2(\nu_n-1)},\frac{2\nu_n-1}{2(\nu_n-1)}\right)} \partial_t \bar{\bar{y}}_n(t,x)\cdot \partial_t \bar{\bar{\varphi}}(t,x)\; dx\; dt \qquad (4.83)\\
&\quad - \int_0^{T'} \int_\Omega J^n \colon \bar{\nabla}_n \bar{\bar{\varphi}}\; dx\; dt \qquad (4.84)\\
&\quad - \int_0^{T'} \int_{S\times\left(0,\frac{1}{\nu_n-1}\right)} J^{(1,n)} \colon \left(\bar{\nabla}_n \bar{\bar{\varphi}}\right)^{(1)}\; dx\; dt \qquad (4.85)\\
&\quad - \int_0^{T'} \int_{S\times\left(\frac{\nu_n-2}{\nu_n-1},1\right)} J^{(2,n)} \colon \left(\bar{\nabla}_n \bar{\bar{\varphi}}\right)^{(2)}\; dx\; dt \qquad (4.86)
\end{aligned}
$$

The term (4.83) vanishes as $n\to\infty$ because of (4.73). For term (4.84) note that $J^n \overset{*}{\rightharpoonup} J$ in $L^\infty_{\text{loc}}(I_T; L^2(\Omega;\mathbb{R}^{3\times 8}))$ and $\bar{\nabla}_n \bar{\bar{\varphi}} \to \nabla\varphi Z$ in $L^1_{\text{loc}}(I_T; L^2(\Omega;\mathbb{R}^{3\times 8}))$, hence we get

$$
(4.84) \to -\int_0^{T'} \int_\Omega J\colon \nabla\varphi Z\; dx\; dt.
$$

If $\nu_n\to\infty$ the terms (4.85) and (4.86) vanish as $n\to\infty$: Exemplarily for (4.85) we estimate

$$
\begin{aligned}
&\left| \int_0^{T'} \int_{S\times\left(0,\frac{1}{\nu_n-1}\right)} J^{(1,n)}(t,x) \colon \left(\bar{\nabla}_n \bar{\bar{\varphi}}(t,x)\right)^{(1)}\; dx\; dt \right| \\
&\qquad \le C \int_0^{T'} \int_{S\times\left(0,\frac{1}{\nu-1}\right)} \left|J^{(1,n)}(t,x)\right|\; dx\; dt \\
&\le C \int_0^{T'} \left\|J^{(1,n)}(t)\right\|_{L^2(\Omega;\mathbb{R}^{3\times 4})} \left| S\times\left(0,\frac{1}{\nu_n-1}\right)\right|\; dt \\
&\le CT' \left\|J^{(1,n)}\right\|_{L^\infty(0,T';L^2(\Omega;\mathbb{R}^{3\times 4}))} |S| \left(\frac{1}{\nu_n-1}\right) \underset{n\to\infty}{\longrightarrow} 0.
\end{aligned}
$$

Term (4.86) can be treated analogously. If $\nu_n \equiv \nu \in \mathbb{N}$

$$
\begin{aligned}
(4.85) &\to -\int_0^{T'} \int_{S\times\left(0,\frac{1}{\nu-1}\right)} J^{(1)} \colon \nabla\varphi Z^{(1)}\; dx\; dt,\\
(4.86) &\to -\int_0^{T'} \int_{S\times\left(\frac{\nu-2}{\nu-1},1\right)} J^{(2)} \colon \nabla\varphi Z^{(2)}\; dx\; dt.
\end{aligned}
$$

In total for $\nu_n \to \infty$ we obtain by Proposition 4.4.1, Proposition 4.4.5, Lemma 3.2.16 and Corollary 3.2.17,

$$0 = \int_0^{T'} \int_S \mathcal{L}_2 \left(\text{sym } \nabla' u + \frac{1}{2} \nabla' v \otimes \nabla' v \right) : \nabla' \Psi \, dx' \, dt.$$

By part (i) of Lemma A.2.1 this holds for every $\Psi \in L^2(0, T'; H_0^1(S; \mathbb{R}^2))$ and we have (4.6).

For $\nu_n \equiv \nu \in \mathbb{N}$ by Proposition 4.4.1, Proposition 4.4.6, Lemma 3.2.18, Lemma 3.2.16 and (3.34)

$$0 = \int_0^{T'} \int_S \frac{1}{2} DQ_{\text{cell}}^{\text{rel}} \left(\begin{pmatrix} \text{sym } G_1 & 0 \\ 0 & 0 \end{pmatrix} Z + \frac{1}{2(\nu - 1)} G_3 \right) : \begin{pmatrix} \nabla' \Psi & 0 \\ 0 & 0 \end{pmatrix} Z \, dx' \, dt$$
$$+ \int_0^{T'} \int_S \frac{1}{\nu - 1} DQ_{\text{surf}} \left(\begin{pmatrix} \text{sym } G_1 & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} + \frac{\partial_{12} v}{2(\nu - 1)} M^{(1)} \right) : \begin{pmatrix} \nabla' \Psi & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} \, dx' \, dt$$

for every $\Psi \in L^2(0, T'; H_0^1(S))$, which is (4.8).

Step 6: Derivation of equations (4.5) *and* (4.7).

Part 1: Let $\phi(t, x') = \mu(t)\chi(x')$ with $\mu \in C_c^\infty(0, T')$, $\chi \in C_c^\infty(S)$, and define $\varphi(t, x) = (0, 0, \phi(t, x'))$. Then

$$\begin{aligned}
0 = \int_0^{T'} \int_{S \times \left(-\frac{1}{2(\nu_n - 1)}, \frac{2\nu_n - 1}{2(\nu_n - 1)}\right)} & \partial_t \bar{\bar{y}}_n \cdot \partial_t \bar{\bar{\varphi}} \, dx \, dt \\
- \int_0^{T'} \int_\Omega & R_n J^n : \bar{\nabla}_n \bar{\bar{\varphi}} \, dx \, dt \\
- \int_0^{T'} \int_{S \times \left(0, \frac{1}{\nu_n - 1}\right)} & R_n J^{(1,n)} : \left(\bar{\nabla}_n \bar{\bar{\varphi}} \right)^{(1)} \, dx \, dt \\
- \int_0^{T'} \int_{S \times \left(\frac{\nu_n - 2}{\nu_n - 1}, 1\right)} & R_n J^{(2,n)} : \left(\bar{\nabla}_n \bar{\bar{\varphi}} \right)^{(2)} \, dx \, dt \\
+ \int_0^{T'} \int_{S \times \left(-\frac{1}{2(\nu_n - 1)}, \frac{2\nu_n - 1}{2(\nu_n - 1)}\right)} & h_n \bar{\bar{g}} \bar{\bar{\varphi}}_3 \, dx \, dt.
\end{aligned}$$

Letting $A_n := \frac{R_n - Id}{h_n}$ and multiplying with h_n^{-1} leads to

$$\begin{aligned}
0 = \int_0^{T'} \int_{S \times \left(-\frac{1}{2(\nu_n - 1)}, \frac{2\nu_n - 1}{2(\nu_n - 1)}\right)} & h_n^{-1} \partial_t \left(\bar{\bar{y}}_n \right)_3 \partial_t \bar{\bar{\phi}} \, dx \, dt \\
- \int_0^{T'} \int_\Omega & A_n J^n : \bar{\nabla}_n \bar{\bar{\varphi}} \, dx \, dt \\
- \int_0^{T'} \int_{S \times \left(0, \frac{1}{\nu_n - 1}\right)} & A_n J^{(1,n)} : \left(\bar{\nabla}_n \bar{\bar{\varphi}} \right)^{(1)} \, dx \, dt \\
- \int_0^{T'} \int_{S \times \left(\frac{\nu_n - 2}{\nu_n - 1}, 1\right)} & A_n J^{(2,n)} : \left(\bar{\nabla}_n \bar{\bar{\varphi}} \right)^{(2)} \, dx \, dt
\end{aligned}$$

$$
\begin{aligned}
&-\int_0^{T'}\int_\Omega h_n^{-1} J^n : \bar{\nabla}_n \bar{\bar{\varphi}}\; dx\; dt \\
&-\int_0^{T'}\int_{S\times\left(0,\frac{1}{\nu_n-1}\right)} h_n^{-1} J^{(1,n)} : \left(\bar{\nabla}_n \bar{\bar{\varphi}}\right)^{(1)}\; dx\; dt \\
&-\int_0^{T'}\int_{S\times\left(\frac{\nu_n-2}{\nu_n-1},1\right)} h_n^{-1} J^{(2,n)} : \left(\bar{\nabla}_n \bar{\bar{\varphi}}\right)^{(2)}\; dx\; dt \\
&+\frac{\nu_n}{\nu_n-1}\int_0^{T'}\int_S \bar{\bar{g}}\bar{\bar{\phi}}\; dx'\; dt,
\end{aligned}
$$

thus

$$
\begin{aligned}
&\int_0^{T'}\int_\Omega h_n^{-1} J^n : \bar{\nabla}_n \bar{\bar{\varphi}}\; dx\; dt \\
&+\int_0^{T'}\int_{S\times\left(0,\frac{1}{\nu_n-1}\right)} h_n^{-1} J^{(1,n)} : \left(\bar{\nabla}_n \bar{\bar{\varphi}}\right)^{(1)}\; dx\; dt \\
&+\int_0^{T'}\int_{S\times\left(\frac{\nu_n-2}{\nu_n-1},1\right)} h_n^{-1} J^{(2,n)} : \left(\bar{\nabla}_n \bar{\bar{\varphi}}\right)^{(2)}\; dx\; dt \\
=&\int_0^{T'}\int_{S\times\left(-\frac{1}{2(\nu_n-1)},\frac{2\nu_n-1}{2(\nu_n-1)}\right)} h_n^{-1}\partial_t\left(\bar{\bar{y}}_n\right)_3\partial_t\bar{\bar{\phi}}\; dx\; dt \qquad (4.87)\\
&-\int_0^{T'}\int_\Omega A_n J^n : \bar{\nabla}_n \bar{\bar{\varphi}}\; dx\; dt \qquad (4.88)\\
&-\int_0^{T'}\int_{S\times\left(0,\frac{1}{\nu_n-1}\right)} A_n J^{(1,n)} : \left(\bar{\nabla}_n \bar{\bar{\varphi}}\right)^{(1)}\; dx\; dt \qquad (4.89)\\
&-\int_0^{T'}\int_{S\times\left(\frac{\nu_n-2}{\nu_n-1},1\right)} A_n J^{(2,n)} : \left(\bar{\nabla}_n \bar{\bar{\varphi}}\right)^{(2)}\; dx\; dt \qquad (4.90)\\
&+\frac{\nu_n}{\nu_n-1}\int_0^{T'}\int_S \bar{\bar{g}}\bar{\bar{\phi}}\; dx'\; dt. \qquad (4.91)
\end{aligned}
$$

We determine the convergence of these terms separately. First look at (4.91): For almost every $t \in (0, T')$ the estimate

$$
\left|g(t,x') - g(t,\xi')\right| \le C(t)\left|x'-\xi'\right| \quad \forall x',\xi' \in S
$$

holds true, where $C(t)$ denotes the Lipschitz constant of $g(t)$. Further there is a constant $C > 0$ such that $C(t) \le C\,\|g(t)\|_{W^{1,\infty}(S)}$. Thus we can estimate

$$
\begin{aligned}
&\left|\int_0^{T'}\int_S \bar{\bar{g}}\bar{\bar{\phi}}\; dx'\; dt - \int_0^{T'}\int_S g\phi\; dx'\; dt\right| \\
\le&\left|\int_0^{T'}\int_S (\bar{\bar{g}}-g)\bar{\bar{\phi}}\; dx'\; dt\right| + \left|\int_0^{T'}\int_S g(\bar{\bar{\phi}}-\phi)\; dx'\; dt\right|
\end{aligned}
$$

$$\leq C\varepsilon_n \int_0^{T'} \|g(t)\|_{W^{1,\infty}(S)} \; dt + C\varepsilon_n \int_0^{T'} \|g(t)\|_{L^1(S)} \; dt$$

and get

$$(4.91) \underset{n\to\infty}{\longrightarrow} \begin{cases} \int_0^{T'} \int_S g\phi \; dx' \; dt & \text{if } \nu_n \to \infty, \\ \frac{\nu}{\nu-1} \int_0^{T'} \int_S g\phi \; dx' \; dt & \text{if } \nu_n \equiv \nu \in \mathbb{N}. \end{cases}$$

For the terms (4.88) - (4.90) we note that $\left(\bar{\nabla}_n \varphi(t,x)\right)_{ij} = 0$ whenever $i \in \{1,2\}$ and $\left(\bar{\nabla}_n \varphi(t,x)\right)_{3j} = \bar{\partial}_j^n \phi(t,x')$ for $j = 1,\ldots,8$ and therefore

$$\begin{aligned} A_n J^n \colon \bar{\nabla}_n \varphi(t,x) &= \sum_{l=1}^{8} (A_n J^n)_{3l} \, \bar{\partial}_l^n \phi(t,x') \\ &= \sum_{l=1}^{8} \left(\sum_{k=1}^{3} (A_n)_{3k} J_{kl}^n \right) \bar{\partial}_l^n \phi(t,x') \\ &= \sum_{l=1}^{8} \left(\sum_{k=1}^{2} (A_n)_{3k} J_{kl}^n \right) \bar{\partial}_l^n \phi(t,x') \\ &\qquad + \sum_{l=1}^{8} (A_n)_{33} J_{3l}^n \bar{\partial}_l^n \phi(t,x'), \end{aligned}$$

$$\begin{aligned} A_n J^{(1,n)} \colon \left(\bar{\nabla}_n \varphi(t,x)\right)^{(1)} &= \sum_{l=1}^{4} \left(\sum_{k=1}^{2} (A_n)_{3k} J_{kl}^{(1,n)} \right) \bar{\partial}_l^n \phi(t,x') \\ &\qquad + \sum_{l=1}^{4} (A_n)_{33} J_{3l}^{(1,n)} \bar{\partial}_l^n \phi(t,x') \end{aligned}$$

as well as

$$\begin{aligned} A_n J^{(2,n)} \colon \left(\bar{\nabla}_n \varphi(t,x)\right)^{(2)} &= \sum_{l=1}^{4} \left(\sum_{k=1}^{2} (A_n)_{3k} J_{kl}^{(2,n)} \right) \bar{\partial}_{l+4}^n \phi(t,x') \\ &\qquad + \sum_{l=1}^{4} (A_n)_{33} J_{3l}^{(2,n)} \bar{\partial}_{l+4}^n \phi(t,x'). \end{aligned}$$

By Proposition 4.3.2 we have that $A_n e_i \to A e_i$ for $A = e_3 \otimes \nabla' v - \nabla' v \otimes e_3$ strongly in $L^q_{\text{loc}}(I_T; L^p(S,\mathbb{R}^3))$ for $i = 1,2$ and any $1 \leq p,q < \infty$ as well as

$$\text{sym } A_n \to 0 \quad \text{strongly in } L^\infty_{\text{loc}}(I_T; L^r(S,\mathbb{R}^{3\times 3}))$$

for all $r < \infty$. In particular this implies that $(A_n)_{33} \to 0$ strongly in $L^\infty_{\text{loc}}(I_T; L^r(S))$ and

$$(4.88) \underset{n\to\infty}{\longrightarrow} -\int_0^{T'} \int_\Omega AJ \colon \nabla\varphi Z \; dx \; dt$$

$$= -\int_0^{T'}\int_\Omega J\colon \begin{pmatrix} \nabla'\phi\otimes\nabla' v & 0 \\ 0 & 0 \end{pmatrix} Z\ dx\ dt.$$

The terms (4.89) and (4.90) vanish in case $\nu_n \to \infty$ by equi-integrability. In case $\nu_n \equiv \nu \in \mathbb{N}$ we argue as for (4.88) and obtain

$$(4.89) \to -\int_0^{T'}\int_{S\times\left(0,\frac{1}{\nu-1}\right)} AJ^{(1)}\colon \nabla\varphi Z^{(1)}\ dx\ dt,$$
$$(4.90) \to -\int_0^{T'}\int_{S\times\left(\frac{\nu-2}{\nu-1},1\right)} AJ^{(2)}\colon \nabla\varphi Z^{(2)}\ dx\ dt.$$

For term (4.87) we note that $\bar{\bar{v}}_n \to \hat{v}$ in $L^\infty_{\text{loc}}(I_T; L^2(\Omega))$ by Corollary 4.3.6 if $\nu_n \to \infty$, where $\hat{v}(t,x) = v(t,x') + \left(x_3 - \frac{1}{2}\right)$. If $\nu_n \equiv \nu \in \mathbb{N}$ the convergence $\bar{\bar{v}}_n \to \hat{v}^*$ in $L^\infty_{\text{loc}}\left(I_T; L^2\left(S\times\left(-\frac{1}{2(\nu-1)}, \frac{2\nu-1}{2(\nu-1)}\right); \mathbb{R}^2\right)\right)$ holds true with $\hat{v}^*(t,x',x_3) = \hat{v}(t,x',\frac{i}{\nu-1})$ whenever $x_3 \in \left(\frac{2i-1}{2(\nu-1)}, \frac{2i+1}{2(\nu-1)}\right)$, $i = 1,\ldots,\nu-1$.

Moreover $\partial_t\bar{\bar{v}}_n$ is bounded in $L^\infty_{\text{loc}}(I_T; L^2(\Omega))$ or in $L^\infty_{\text{loc}}\left(I_T; L^2\left(S\times\left(-\frac{1}{2(\nu-1)}, \frac{2\nu-1}{2(\nu-1)}\right)\right)\right)$, respectively, and therefore $\partial_t\bar{\bar{v}}_n \overset{*}{\rightharpoonup} \partial_t\hat{v}$ or $\partial_t\bar{\bar{v}}_n \overset{*}{\rightharpoonup} \partial_t\hat{v}^*$ in the respective space.

Together with $\partial_t\bar{\bar{\phi}} \to \partial_t\phi$ in $L^\infty_{\text{loc}}(0,T; L^2(S))$ we deduce that

$$(4.87) \to \begin{cases} \int_0^{T'}\int_S \partial_t v\partial_t\phi\ dx'\ dt & \text{if } \nu_n \to \infty, \\ \frac{\nu}{\nu-1}\int_0^{T'}\int_S \partial_t v\partial_t\phi\ dx'\ dt & \text{if } \nu_n \to \infty \end{cases}$$

as $n \to \infty$. Combining the convergences of (4.87), (4.88), (4.89), (4.90) and (4.91) we obtain (4.66) for $\nu_n \to \infty$ by Proposition 4.4.1, Proposition 4.4.5, Lemma 3.2.16 and Corollary 3.2.17.

For $\nu_n \equiv \nu \in \mathbb{N}$ we get (4.67) by Proposition 4.4.1, Proposition 4.4.6, Lemma 3.2.18, Lemma 3.2.16 and (3.34).

Part 2: Let $\varphi(t,x) = \begin{pmatrix} \eta(t,x') \\ 0 \end{pmatrix}\left(x_3 - \frac{1}{2}\right)$ with $\eta(t,x') = \begin{pmatrix} \partial_1\phi(t,x') \\ \partial_2\phi(t,x') \end{pmatrix}$. Then

$$0 = \int_0^{T'}\int_{S\times\left(-\frac{1}{2(\nu_n-1)}, \frac{2\nu_n-1}{2(\nu_n-1)}\right)} \partial_t\bar{\bar{y}}_n\cdot\partial_t\bar{\bar{\varphi}}\ dx\ dt \tag{4.92}$$
$$-\int_0^{T'}\int_\Omega R_n J^n\colon \bar{\nabla}_n\bar{\bar{\varphi}}\ dx\ dt \tag{4.93}$$
$$-\int_0^{T'}\int_{S\times\left(0,\frac{1}{\nu_n-1}\right)} R_n J^{(1,n)}\colon \left(\bar{\nabla}_n\bar{\bar{\varphi}}\right)^{(1)}\ dx'\ dt \tag{4.94}$$
$$-\int_0^{T'}\int_{S\times\left(\frac{\nu_n-2}{\nu_n-1},1\right)} R_n J^{(2,n)}\colon \left(\bar{\nabla}_n\bar{\bar{\varphi}}\right)^{(2)}\ dx'\ dt. \tag{4.95}$$

The term (4.92) tends to 0 by the energy inequality (4.73). The treatment of (4.93) - (4.95) is done as in the stationary case: Remember that $\hat{x}$ is the left-lower corner of a cell, i.e.

$$x + \left(-\frac{\varepsilon_n}{2}, \frac{\varepsilon_n}{2}\right)^2 \times \left(-\frac{\varepsilon_n}{2h_n}, \frac{\varepsilon_n}{2h_n}\right) = \hat{x} + (0, \varepsilon_n)^2 \times \left(0, \varepsilon_n h_n^{-1}\right)$$

for $x \in \tilde{\Lambda}_n'$. For example we look at term (4.93):

$$\int_0^{T'} \int_\Omega R_n J^n : \bar{\nabla}_n \bar{\bar{\varphi}} \; dx \; dt$$

$$= \sum_{l=1}^{8} \left[\int_0^{T'} \int_\Omega (R_n J^n)_{.l} \cdot \bar{D}_l^n \varphi(t, \hat{x}) \; dx \; dt - \frac{1}{8} \sum_{k=1}^{8} \int_0^{T'} \int_\Omega (R_n J^n)_{.l} \cdot \bar{D}_k^n \varphi(t, \hat{x}) \; dx \; dt \right]$$

with

$$\bar{D}_l^n \varphi(t, \hat{x}) = \begin{pmatrix} \bar{D}_l^n \eta(t, \hat{x}') \\ 0 \end{pmatrix} \left(\hat{x}_3 + \frac{\varepsilon_n}{h_n} a_3^l - \frac{1}{2} \right) + h_n^{-1} a_3^l \begin{pmatrix} \eta(t, \hat{x}') \\ 0 \end{pmatrix}.$$

Together with the same decomposition of terms (4.94) and (4.95) we obtain

$$\begin{aligned}
&\sum_{l=1}^{8} \int_0^{T'} \int_\Omega (R_n J^n)_{.l} \cdot h_n^{-1} z_3^l \begin{pmatrix} \eta(t, \hat{x}') \\ 0 \end{pmatrix} \; dx \; dt \\
&+ \sum_{l=1}^{4} \int_0^{T'} \int_{S \times \left(0, \frac{1}{\nu_n - 1}\right)} R_n J_{.l}^{(1,n)} \cdot h_n^{-1} z_3^l \begin{pmatrix} \eta(t, \hat{x}') \\ 0 \end{pmatrix} \; dx \; dt \\
&+ \sum_{l=1}^{4} \int_0^{T'} \int_{S \times \left(\frac{\nu_n - 2}{\nu_n - 1}, 1\right)} R_n J_{.l}^{(2,n)} \cdot h_n^{-1} z_3^{l+4} \begin{pmatrix} \eta(t, \hat{x}') \\ 0 \end{pmatrix} \; dx \; dt \\
&= \int_0^{T'} \int_{S \times \left(-\frac{1}{2(\nu_n - 1)}, \frac{2\nu_n - 1}{2(\nu_n - 1)}\right)} \partial_t \bar{\bar{y}}_n \partial_t \bar{\bar{\varphi}} \; dx \; dt \qquad (4.96) \\
&- \sum_{l=1}^{8} \left[\int_0^{T'} \int_\Omega R_n J_{.l}^n \cdot \begin{pmatrix} \bar{D}_l^n \eta(t, \hat{x}') \\ 0 \end{pmatrix} \left(\hat{x}_3 + \frac{\varepsilon_n}{h_n} a_3^l - \frac{1}{2} \right) \; dx \; dt \right. \\
&\left. - \frac{1}{8} \sum_{k=1}^{8} \int_0^{T'} \int_\Omega R_n J_{.l}^n \cdot \begin{pmatrix} \bar{D}_k^n \eta(t, \hat{x}') \\ 0 \end{pmatrix} \left(\hat{x}_3 + \frac{\varepsilon_n}{h_n} a_3^k - \frac{1}{2} \right) \; dx \; dt \right] \qquad (4.97) \\
&- \sum_{l=1}^{4} \left[\int_0^{T'} \int_{S \times \left(0, \frac{1}{\nu_n - 1}\right)} R_n J_{.l}^{(1,n)} \cdot \begin{pmatrix} \bar{D}_l^n \eta(t, \hat{x}') \\ 0 \end{pmatrix} \left(\hat{x}_3 + \frac{\varepsilon_n}{h_n} a_3^l - \frac{1}{2} \right) \; dx \; dt \right. \\
&\left. - \frac{1}{8} \sum_{k=1}^{8} \int_0^{T'} \int_{S \times \left(0, \frac{1}{\nu_n - 1}\right)} R_n J_{.l}^{(1,n)} \cdot \begin{pmatrix} \bar{D}_k^n \eta(t, \hat{x}') \\ 0 \end{pmatrix} \left(\hat{x}_3 + \frac{\varepsilon_n}{h_n} a_3^k - \frac{1}{2} \right) \; dx \; dt \right] \qquad (4.98) \\
&- \sum_{l=1}^{4} \left[\int_0^{T'} \int_{S \times \left(\frac{\nu_n - 2}{\nu_n - 1}, 1\right)} R_n J_{.l}^{(2,n)} \cdot \begin{pmatrix} \bar{D}_{l+4}^n \eta(t, \hat{x}') \\ 0 \end{pmatrix} \left(\hat{x}_3 + \frac{\varepsilon_n}{h_n} a_3^{l+4} - \frac{1}{2} \right) \; dx \; dt \right. \\
&\left. - \frac{1}{8} \sum_{k=1}^{8} \int_0^{T'} \int_{S \times \left(\frac{\nu_n - 2}{\nu_n - 1}, 1\right)} R_n J_{.l}^{(2,n)} \cdot \begin{pmatrix} \bar{D}_k^n \eta(t, \hat{x}') \\ 0 \end{pmatrix} \left(\hat{x}_3 + \frac{\varepsilon_n}{h_n} a_3^k - \frac{1}{2} \right) \; dx \; dt \right]. \qquad (4.99)
\end{aligned}$$

We need to determine convergence of (4.96) + (4.97) + (4.98) + (4.99). Independent of the number of layers term (4.96) vanishes by (4.73). In case $\nu_n \to \infty$ the terms (4.98) and

(4.99) clearly vanish by equi-integrability and dominated convergence. We have

$$\begin{pmatrix} \bar{D}_l^n \eta(t,\hat{x}') \\ 0 \end{pmatrix} \left(\hat{x}_3 + \frac{\varepsilon_n}{h_n} a_3^k - \frac{1}{2}\right) \underset{n\to\infty}{\longrightarrow} \left(x_3 - \frac{1}{2}\right) \begin{pmatrix} \nabla'\eta(t,x)(a^l)' \\ 0 \end{pmatrix}$$

uniformly on $(0,T') \times \Omega$ and therefore, together with the convergence of (4.92) and $R_n J^n \overset{*}{\rightharpoonup} J$ in $L^\infty_{\mathrm{loc}}(0,T;L^2(\Omega;\mathbb{R}^{3\times 8}))$ it follows that

$$\begin{aligned}
&(4.96) + (4.97) + (4.98) + (4.99) \underset{n\to\infty}{\longrightarrow} \\
&-\sum_{l=1}^{8} \int_0^{T'} \int_\Omega \left(x_3 - \frac{1}{2}\right) J_{\cdot l} \cdot \begin{pmatrix} \nabla'^2\phi(t,x')(z^l)' \\ 0 \end{pmatrix} dx\ dt \\
&= -\int_0^{T'} \int_\Omega \left(x_3 - \frac{1}{2}\right) J \colon \begin{pmatrix} \nabla'^2\phi(t,x') & 0 \\ 0 & 0 \end{pmatrix} Z\ dx\ dt \\
&= -\int_0^{T'} \int_\Omega \left(x_3 - \frac{1}{2}\right) \mathcal{L}_2\left(G''\right) : \nabla'^2\phi(t,x')\ dx\ dt \\
&= -\int_0^{T'} \int_S \int_0^1 \left(x_3 - \frac{1}{2}\right)^2 \mathcal{L}_2(G_2) \colon \nabla'^2\phi(t,x')\ dx_3\ dx'\ dt \\
&= -\int_0^{T'} \int_S \frac{1}{12} \mathcal{L}_2\left(-\nabla'^2 v\right) : \nabla'^2\phi(t,x')\ dx'\ dt.
\end{aligned} \tag{4.100}$$

Again we have used Proposition 4.4.1, Proposition 4.4.5, Lemma 3.2.16 and Corollary 3.2.17.

In case $\nu_n \equiv \nu \in \mathbb{N}$

$$\begin{aligned}
&(4.96) + (4.97) + (4.98) + (4.99) \underset{n\to\infty}{\longrightarrow} \\
&-\sum_{l=1}^{8} \Bigg[\int_0^{T'} \int_\Omega J_{\cdot l} \cdot \begin{pmatrix} \nabla'\eta\left(a^l\right)' \\ 0 \end{pmatrix} \left(\hat{x}_3 + \frac{1}{\nu - 1} a_3^l - \frac{1}{2}\right) dx\ dt \\
&-\frac{1}{8} \sum_{k=1}^{8} \int_0^{T'} \int_\Omega J_{\cdot l} \cdot \begin{pmatrix} \nabla'\eta\left(a^k\right)' \\ 0 \end{pmatrix} \left(\hat{x}_3 + \frac{1}{\nu - 1} a_3^k - \frac{1}{2}\right) dx\ dt \Bigg] \\
&-\sum_{l=1}^{4} \Bigg[\int_0^{T'} \int_{S\times\left(0,\frac{1}{\nu-1}\right)} J^{(1)}_{\cdot l} \cdot \begin{pmatrix} \nabla'\eta\left(a^l\right)' \\ 0 \end{pmatrix} \left(\hat{x}_3 + \frac{1}{\nu - 1} a_3^l - \frac{1}{2}\right) dx\ dt \\
&-\frac{1}{8} \sum_{k=1}^{8} \int_0^{T'} \int_{S\times\left(0,\frac{1}{\nu-1}\right)} J^{(1)}_{\cdot l} \cdot \begin{pmatrix} \nabla'\eta\left(a^k\right)' \\ 0 \end{pmatrix} \left(\hat{x}_3 + \frac{1}{\nu - 1} a_3^k - \frac{1}{2}\right) dx\ dt \Bigg] \\
&-\sum_{l=1}^{4} \Bigg[\int_0^{T'} \int_{S\times\left(\frac{\nu-2}{\nu-1},1\right)} J^{(2)}_{\cdot l} \cdot \begin{pmatrix} \nabla'\eta\left(a_3^{l+4}\right) \\ 0 \end{pmatrix} \left(\hat{x}_3 + \frac{1}{\nu - 1} a_3^{l+4} - \frac{1}{2}\right) dx\ dt \\
&-\frac{1}{8} \sum_{k=1}^{8} \int_0^{T'} \int_{S\times\left(\frac{\nu-2}{\nu-1},1\right)} J^{(2)}_{\cdot l} \cdot \begin{pmatrix} \nabla'\eta\left(a^k\right)' \\ 0 \end{pmatrix} \left(\hat{x}_3 + \frac{1}{\nu - 1} a_3^k - \frac{1}{2}\right) dx\ dt \Bigg] \\
&= -\int_0^{T'} \int_S \frac{\nu(\nu-2)}{24(\nu-1)^2} DQ^{\mathrm{rel}}_{\mathrm{cell}} \left(\begin{pmatrix} G_2 & 0 \\ 0 & 0 \end{pmatrix} Z \right) : \begin{pmatrix} \nabla'^2\phi & 0 \\ 0 & 0 \end{pmatrix} Z\ dx'\ dt
\end{aligned}$$

$$
-\int_0^{T'}\int_S \frac{1}{4(\nu-1)} DQ_{\text{cell}}^{\text{rel}}\left(\begin{pmatrix} \text{sym } G_1 & 0 \\ 0 & 0 \end{pmatrix} Z + \frac{1}{2(\nu-1)} G_3\right) : \begin{pmatrix} \nabla'^2\phi & 0 \\ 0 & 0 \end{pmatrix} Z_- \, dx' \, dt
$$
$$
-\int_0^{T'}\int_S \frac{1}{4(\nu-1)} DQ_{\text{surf}}\left(\begin{pmatrix} G_2 & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)}\right) : \begin{pmatrix} \nabla'^2\phi & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} \, dx' \, dt, \tag{4.101}
$$

where for the last equality we again need to repeat all the calculations done in the stationary case for a fixed t using Proposition 4.4.1, Proposition 4.4.6, Lemma 3.2.18, Lemma 3.2.16, (3.34) and (3.35). For details of the calculations we refer to Part 2 in the proof of Theorem 3.2.1.

Part 3: Finally we need to determine the limit of

$$
\begin{aligned}
&\int_0^{T'}\int_\Omega h_n^{-1} J^n : \bar{\nabla}_n \bar{\bar{\varphi}} \, dx \, dt \\
&+\int_0^{T'}\int_{S\times\left(0,\frac{1}{\nu_n-1}\right)} h_n^{-1} J^{(1,n)} : \left(\bar{\nabla}_n \bar{\bar{\varphi}}\right)^{(1)} \, dx \, dt \\
&+\int_0^{T'}\int_{S\times\left(\frac{\nu_n-2}{\nu_n-1},1\right)} h_n^{-1} J^{(2,n)} : \left(\bar{\nabla}_n \bar{\bar{\varphi}}\right)^{(2)} \, dx \, dt \\
&-\sum_{l=1}^{8}\int_0^{T'}\int_\Omega h_n^{-1} \left(R_n J^n\right)_{\cdot l} \cdot z_3^l \begin{pmatrix} \nabla'\phi(t,\hat{x}') \\ 0 \end{pmatrix} dx \, dt \\
&-\sum_{l=1}^{4}\int_0^{T'}\int_{S\times\left(0,\frac{1}{\nu_n-1}\right)} h_n^{-1} \left(R_n J^{(1,n)}\right)_{\cdot l} \cdot z_3^l \begin{pmatrix} \nabla'\phi(t,\hat{x}') \\ 0 \end{pmatrix} dx \, dt \\
&-\sum_{l=1}^{4}\int_0^{T'}\int_{S\times\left(\frac{\nu_n-2}{\nu_n-1},1\right)} h_n^{-1} \left(R_n J^{(2,n)}\right)_{\cdot l} \cdot z_3^{l+4} \begin{pmatrix} \nabla'\phi(t,\hat{x}') \\ 0 \end{pmatrix} dx \, dt
\end{aligned} \tag{4.102}
$$

where $\varphi(t,x) = (0,0,\phi(t,x'))$. We need to simplify these terms. First goal is to get rid of the R_n in the latter three terms of the sum. We consider exemplarily a single term and write it as

$$
\begin{aligned}
&\int_0^{T'}\int_\Omega h_n^{-1} \left(R_n J^n\right)_{\cdot l} \cdot z_3^l \begin{pmatrix} \nabla'\phi(t,\hat{x}') \\ 0 \end{pmatrix} dx \, dt \\
&= \int_0^{T'}\int_\Omega \left(A_n J^n\right)_{\cdot l} \cdot z_3^l \begin{pmatrix} \nabla'\phi(t,\hat{x}') \\ 0 \end{pmatrix} dx \, dt \\
&+ \int_0^{T'}\int_\Omega h_n^{-1} \left(J^n\right)_{\cdot l} \cdot z_3^l \begin{pmatrix} \nabla'\phi(t,\hat{x}') \\ 0 \end{pmatrix} dx \, dt.
\end{aligned}
$$

We want to show that the terms containing A_n tend to 0 for $n \to \infty$. Doing the same decomposition with the surface terms we obtain by collecting the terms which include A_n

$$
\sum_{l=1}^{8}\int_0^{T'}\int_\Omega \left(A_n J^n\right)_{\cdot l} \cdot z_3^l \begin{pmatrix} \nabla'\phi(t,\hat{x}') \\ 0 \end{pmatrix} dx \, dt
$$

$$+\sum_{l=1}^{4}\int_0^{T'}\int_{S\times\left(0,\frac{1}{\nu_n-1}\right)}\left(A_nJ^{(1,n)}\right)_{\cdot l}\cdot z_3^l\begin{pmatrix}\nabla'\phi(t,\hat{x}')\\0\end{pmatrix}dx\ dt$$
$$+\sum_{l=1}^{4}\int_0^{T'}\int_{S\times\left(\frac{\nu_n-2}{\nu_n-1},1\right)}\left(A_nJ^{(2,n)}\right)_{\cdot l}\cdot z_3^{l+4}\begin{pmatrix}\nabla'\phi(t,\hat{x}')\\0\end{pmatrix}dx\ dt \tag{4.103}$$

Since $A_ne_i\to Ae_i$ for $i=1,2$ in $L^q_{\text{loc}}(I_T;L^p(S))$ for all $1\le p,q<\infty$ by Proposition 4.3.2 in case $\nu_n\to\infty$

$$(4.103)\underset{n\to\infty}{\longrightarrow}\sum_{l=1}^{8}\int_0^{T'}\int_\Omega J_{\cdot l}\cdot z_3^lA^T\begin{pmatrix}\nabla'\phi(t,x')\\0\end{pmatrix}dx\ dt=0$$

since $J\perp(\mathbb{R}^3\otimes e_3)\,Z$. If $\nu_n\equiv\nu\in\mathbb{N}$

$$(4.103)\underset{n\to\infty}{\longrightarrow}\sum_{l=1}^{8}\int_0^{T'}\int_\Omega J_{\cdot l}\cdot z_3^lA^T\begin{pmatrix}\nabla'\phi(t,x')\\0\end{pmatrix}dx\ dt$$
$$+\sum_{l=1}^{4}\int_0^{T'}\int_{S\times\left(0,\frac{1}{\nu-1}\right)}J^{(1)}_{\cdot l}\cdot z_3^lA^T\begin{pmatrix}\nabla'\phi(t,x')\\0\end{pmatrix}dx\ dt$$
$$+\sum_{l=1}^{4}\int_0^{T'}\int_{S\times\left(\frac{\nu-2}{\nu-1}\right),1}J^{(2)}_{\cdot l}\cdot z_3^{l+4}A^T\begin{pmatrix}\nabla'\phi(t,x')\\0\end{pmatrix}dx\ dt=0$$

by Proposition 4.4.6. Therefore, instead of (4.102), it is sufficient to consider the limiting behavior of

$$\int_0^{T'}\int_\Omega h_n^{-1}J^n:\bar{\nabla}_n\bar{\bar{\varphi}}\ dx\ dt$$
$$\int_0^{T'}\int_{S\times\left(0,\frac{1}{\nu_n-1}\right)}h_n^{-1}J^{(1,n)}:\ \left(\bar{\nabla}_n\bar{\bar{\varphi}}\right)^{(1)}\ dx\ dt$$
$$\int_0^{T'}\int_{S\times\left(\frac{\nu_n-2}{\nu_n-1},1\right)}h_n^{-1}J^{(2,n)}:\ \left(\bar{\nabla}_n\bar{\bar{\varphi}}\right)^{(2)}\ dx\ dt$$
$$-\sum_{l=1}^{8}\int_0^{T'}\int_\Omega h_n^{-1}J^n_{\cdot l}\cdot z_3^l\begin{pmatrix}\nabla'\phi(t,\hat{x}')\\0\end{pmatrix}dx\ dt$$
$$-\sum_{l=1}^{4}\int_0^{T'}\int_{S\times\left(0,\frac{1}{\nu_n-1}\right)}h_n^{-1}J^{(1,n)}_{\cdot l}\cdot z_3^l\begin{pmatrix}\nabla'\phi(t,\hat{x}')\\0\end{pmatrix}dx\ dt$$
$$-\sum_{l=1}^{4}\int_0^{T'}\int_{S\times\left(\frac{\nu_n-2}{\nu_n-1},1\right)}h_n^{-1}J^{(2,n)}_{\cdot l}\cdot z_3^{l+4}\begin{pmatrix}\nabla'\phi(t,\hat{x}')\\0\end{pmatrix}dx\ dt \tag{4.104}$$

Using Lemma 4.4.7 this is done almost literally as in the stationary case. Hence for the calculations we refer to Part 3 in the proof of Theorem 3.2.1. For $\nu_n\to\infty$ we get (4.104) $\to 0$. Combining this with (4.66) and (4.68) we see that equation (4.5) holds for every ϕ of the form $\phi(t,x')=\eta(t)\chi(x')$ with $\eta\in C_c^\infty((0,T'))$, $\chi\in C_c^\infty(S)$.

For $\nu_n \equiv \nu \in \mathbb{N}$ we get

$$(4.104) \to \int_0^{T'} \int_S \frac{1}{4(\nu-1)} DQ_{\text{cell}}^{\text{rel}} \left(\begin{pmatrix} \text{sym } G_1 & 0 \\ 0 & 0 \end{pmatrix} Z + \frac{1}{2(\nu-1)} G_3 \right) : \partial_{12}\phi M \, dx' \, dt$$
$$+ \int_0^{T'} \int_S \frac{1}{2(\nu-1)^2} DQ_{\text{surf}} \left(\begin{pmatrix} \text{sym } G_1 & 0 \\ 0 & 0 \end{pmatrix} Z^{(1)} + \frac{\partial_{12} v}{2(\nu-1)} M^{(1)} \right) : \partial_{12}\phi M^{(1)} \, dx' \, dt.$$

Together with (4.66) and (4.100) it follows that equation (4.7) holds because of (4.67) and (4.101). Finally, by Lemma A.2.1, both equations (4.5) and (4.7) hold true for every $\phi \in L^2(0,T;H_0^2(S)) \cap H_0^1(0,T;L^2(S))$.

Step 7: Weak continuity and the initial conditions.
From inequality (4.59) it follows that, up to a subsequence,

$$\frac{1}{h_n^2} \int_0^1 \left(\tilde{\tilde{w}}_n^{(1)}(\cdot, x_3)\right)_3 \, dx_3 \rightharpoonup w_3^{(1)} \quad \text{in } L^2(S)$$

for some $w_3^{(1)} \in L^2(S)$. Note that, since $w_n^{(0)} \in \mathcal{A}_n$, by ([LM09], Lemma 13) and (4.59)

$$\frac{1}{h_n} \int_0^1 \left(\tilde{\tilde{w}}_n^{(0)}(\cdot, x_3)\right)_3 \, dx_3 \to w_3^{(0)} \quad \text{in } H^1(S)$$

for some $w_3^{(0)} \in H^1(S)$. Since $v_n, v \in W^{1,\infty}(0,T';L^2(S)) \hookrightarrow C([0,T'];L^2(S))$ we get for almost every $x' \in S$

$$\begin{aligned} w_3^{(0)}(x') &= \lim_{n\to\infty} h_n^{-1} \int_0^1 \left(\tilde{\tilde{w}}_n^{(0)}(x', x_3)\right)_3 \, dx_3 \\ &= \lim_{n\to\infty} h_n^{-1} \int_0^1 \left(\tilde{\tilde{y}}_n(0, x', x_3)\right)_3 \, dx_3 \\ &= v(0,x') \end{aligned}$$

which is (4.64). In order to derive the initial condition (4.65) consider the continuous in-plane-projection operator $P_n' \colon L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ defined by

$$P_n' f(\xi') = \fint_{x' + \left(-\frac{\varepsilon_n}{2}, \frac{\varepsilon_n}{2}\right)^2} f(y) \, dy \quad \text{whenever } \xi' \in x' + \left(-\frac{\varepsilon_n}{2}, \frac{\varepsilon_n}{2}\right)^2$$

and $x' \in \varepsilon_n \mathbb{Z}^2$. If $f \in L^2(\mathbb{R}^2)$ is constant on each $x' + \left(-\frac{\varepsilon_n}{2}, \frac{\varepsilon_n}{2}\right)^2$ by Lemma A.2.7 for every $\varphi \in L^2(S)$ it holds that

$$\int_{\mathbb{R}^2} f P_n' \varphi \, dx' = \int_{\mathbb{R}^2} f \varphi \, dx'. \tag{4.105}$$

Let $\phi \in C_c^\infty((0,T') \times S)$ and consider the test functions

$$\varphi_1(t,x) = (0,0,\phi(t,x')),$$

$$\varphi_2(t,x) = \left(x_3 - \frac{1}{2}\right)\left(\begin{array}{c} \nabla'\phi(t,x') \\ 0 \end{array}\right).$$

Let

$$v^n(t,x') = h_n^{-1}\int_{-\frac{1}{2(\nu_n-1)}}^{\frac{2\nu_n-1}{2(\nu_n-1)}} (\bar{\bar{y}}_n)_3(t,x',x_3)\ dx_3$$

and

$$q^n(t,x') = \int_{-\frac{1}{2(\nu_n-1)}}^{\frac{2\nu_n-1}{2(\nu_n-1)}} \overline{\overline{\left(x_3 - \frac{1}{2}\right)}}\bar{\bar{y}}_n'(t,x)\ dx_3.$$

Writing the difference of the weak form of the equations of motions for y_n with φ_1 and φ_2 in terms of P_n' we get by (4.105) and $R_n = Id + h_n A_n$

$$\begin{aligned}
&\int_0^{T'}\int_S \partial_t v^n \partial_t \phi\ dx'\ dt - \int_0^{T'}\int_S \sum_{\alpha=1}^2 \partial_t q_\alpha^n \partial_t \partial_\alpha \phi\ dx'\ dt \\
&= \int_0^{T'}\int_S \partial_t v^n \partial_t P_n' \phi\ dx'\ dt - \int_0^{T'}\int_S \sum_{\alpha=1}^2 \partial_t q_\alpha^n \partial_t P_n' \partial_\alpha \phi\ dx'\ dt \\
&= \int_0^{T'}\int_\Omega A_n J^n : \bar{\nabla}_n P_n' \varphi_1\ dx\ dt \\
&+ \int_0^{T'}\int_{S\times\left(0,\frac{1}{\nu_n-1}\right)} A_n J^{(1,n)} :\ \left(\bar{\nabla}_n P_n' \varphi_1\right)^{(1)}\ dx\ dt \\
&+ \int_0^{T'}\int_{S\times\left(\frac{\nu_n-2}{\nu_n-1},1\right)} A_n J^{(2,n)} :\ \left(\bar{\nabla}_n P_n' \varphi_1\right)^{(2)}\ dx\ dt \\
&- \frac{\nu_n}{\nu_n-1}\int_0^{T'}\int_S \bar{\bar{g}} P_n' \phi\ dx'\ dt \\
&+ \int_0^{T'}\int_\Omega h_n^{-1} J^n : \bar{\nabla}_n P_n' \varphi_1\ dx\ dt \\
&+ \int_0^{T'}\int_{S\times\left(0,\frac{1}{\nu_n-1}\right)} h_n^{-1} J^{(1,n)} :\ \left(\bar{\nabla}_n P_n' \varphi_1\right)^{(1)}\ dx\ dt \\
&+ \int_0^{T'}\int_{S\times\left(\frac{\nu_n-2}{\nu_n-1},1\right)} h_n^{-1} J^{(2,n)} :\ \left(\bar{\nabla}_n P_n' \varphi_1\right)^{(2)}\ dx\ dt \\
&- \int_0^{T'}\int_\Omega R_n J^n : \bar{\nabla}_n \overline{\overline{P_n'\varphi_2}}\ dx\ dt \\
&- \int_0^{T}\int_{S\times\left(0,\frac{1}{\nu_n-1}\right)} R_n J^{(1,n)} :\ \left(\bar{\nabla}_n \overline{\overline{P_n'\varphi_2}}\right)^{(1)}\ dx\ dt \\
&- \int_0^{T}\int_{S\times\left(\frac{\nu_n-2}{\nu_n-1},1\right)} R_n J^{(2,n)} :\ \left(\bar{\nabla}_n \overline{\overline{P_n'\varphi_2}}\right)^{(2)}\ dx\ dt \qquad (4.106)
\end{aligned}$$

The goal is to obtain an estimate

$$\left| \int_0^{T'} \int_S \partial_t v^n \partial_t \phi \; dx' \; dt - \int_0^{T'} \int_S \sum_{\alpha=1}^{2} \partial_t q_\alpha^n \partial_t \partial_\alpha \phi \; dx' \; dt \right| \leq C \left\| \phi \right\|_{L^2(0,T';H_0^4(S))} . \qquad (4.107)$$

First we look at the terms of (4.106) which contain the mappings A_n. We have

$$\begin{aligned} &\left| \int_0^{T'} \int_\Omega A_n J^n : \bar{\nabla}_n P_n' \varphi_1 \; dx \; dt \right| \\ &\leq \int_0^{T'} \int_\Omega \left| A_n J^n \right| \left| \bar{\nabla}_n P_n' \varphi_1 \right| \; dx \; dt \\ &\leq C \int_0^{T'} \left\| \nabla' \phi(t, \cdot) \right\|_{L^\infty(S)} \int_\Omega \left| A_n J^n \right| \; dx \; dt \\ &\leq C \int_0^{T'} \left\| \phi(t) \right\|_{H_0^4(S)} \left\| A_n J^n \right\|_{L^1(\Omega)} \; dt \\ &\leq C \left\| A_n J^n \right\|_{L^2(0,T';L^1(\Omega))} \left\| \phi \right\|_{L^2(0,T';H_0^4(S))} \\ &\leq C \left\| \phi \right\|_{L^2(0,T';H_0^4(S))} \end{aligned} \qquad (4.108)$$

and a similar estimate holds for the surface-terms which involve A_n. Clearly

$$\left| \int_0^{T'} \int_S \bar{\bar{g}} P_n' \phi \; dx' \; dt \right| \leq C \left\| \phi \right\|_{L^2(0,T';H_0^4(S))} . \qquad (4.109)$$

It remains to bound

$$\begin{aligned} &\left| \int_0^{T'} \int_\Omega h_n^{-1} J^n : \bar{\nabla}_n P_n' \varphi_1 \; dx \; dt \right. \\ &+ \int_0^{T'} \int_{S \times \left(0, \frac{1}{\nu_n - 1}\right)} h_n^{-1} J^{(1,n)} : \; \left(\bar{\nabla}_n P_n' \varphi_1 \right)^{(1)} \; dx \; dt \\ &+ \int_0^{T'} \int_{S \times \left(\frac{\nu_n - 2}{\nu_n - 1}, 1\right)} h_n^{-1} J^{(2,n)} : \; \left(\bar{\nabla}_n P_n' \varphi_1 \right)^{(2)} \; dx \; dt \\ &- \int_0^{T'} \int_\Omega R_n J^n : \bar{\nabla}_n \overline{\overline{P_n' \varphi_2}} \; dx \; dt \\ &- \int_0^{T} \int_{S \times \left(0, \frac{1}{\nu_n - 1}\right)} R_n J^{(1,n)} : \; \left(\bar{\nabla}_n \overline{\overline{P_n' \varphi_2}} \right)^{(1)} \; dx \; dt \\ &\left. - \int_0^{T} \int_{S \times \left(\frac{\nu_n - 2}{\nu_n - 1}, 1\right)} R_n J^{(2,n)} : \; \left(\bar{\nabla}_n \overline{\overline{P_n' \varphi_2}} \right)^{(2)} \; dx \; dt \right| \end{aligned} \qquad (4.110)$$

Here we have to be careful because of the factor h_n^{-1}. We get control of these terms with the help of Lemma 4.4.7. To apply this lemma we need to do some estimates on the derivatives of the test functions. For $x \in \tilde{\Lambda}'_n$ and $\xi \in x + \left(-\frac{\varepsilon_n}{2}, \frac{\varepsilon_n}{2}\right)^2 \times \left(-\frac{\varepsilon_n}{2h_n}, \frac{\varepsilon_n}{2h_n}\right)$ there are $\eta_j(y) \in x' + \left(-\frac{\varepsilon_n}{2}, \frac{\varepsilon_n}{2}\right)^2$, $j = 1, \ldots, 8$, such that

$$\bar{\partial}_i^n P'_n \phi(t, \xi') = \bar{\partial}_i^n P'_n \phi(t, x')$$

$$= \frac{1}{\varepsilon_n} \left(\fint_{x' + \varepsilon_n (z^i)' + \left(-\frac{\varepsilon_n}{2}, \frac{\varepsilon_n}{2}\right)^2} \phi(t, y)\, dy - \frac{1}{8} \sum_{j=1}^{8} \fint_{x' + \varepsilon_n (z^j)' + \left(-\frac{\varepsilon_n}{2}, \frac{\varepsilon_n}{2}\right)^2} \phi(t, y)\, dy \right)$$

$$= \frac{1}{\varepsilon_n} \left(\fint_{\left(-\frac{\varepsilon_n}{2}, \frac{\varepsilon_n}{2}\right)^2} \phi(t, x' + \varepsilon_n (z^i)' + y) - \frac{1}{8} \sum_{j=1}^{8} \phi(t, x' + \varepsilon_n (z^j)' + y)\, dy \right)$$

$$= \frac{1}{\varepsilon_n} \Bigg(\varepsilon_n \nabla' \phi(t, x') \cdot (z^i)' + \fint_{\left(-\frac{\varepsilon_n}{2}, \frac{\varepsilon_n}{2}\right)^2} \frac{1}{2} D^2 \phi(t, \eta_i(y))[\varepsilon_n (z^i)' + y, \varepsilon_n (z^i)' + y] -$$
$$\frac{1}{8} \sum_{j=1}^{8} \frac{1}{2} D^2 \phi(t, \eta_j(y))[\varepsilon_n (z^j)' + y, \varepsilon_n (z^j)' + y]\, dy \Bigg)$$

$$= \nabla' \varphi(t, \xi') \cdot (z^i)'$$
$$+ \nabla' \varphi(t, x') \cdot (z^i)' - \nabla' \varphi(t, \xi') \cdot (z^i)' \tag{4.111}$$
$$+ \frac{1}{\varepsilon_n} \Bigg(\fint_{\left(-\frac{\varepsilon_n}{2}, \frac{\varepsilon_n}{2}\right)^2} \frac{1}{2} D^2 \phi(t, \eta_i(y))[\varepsilon_n (z^i)' + y, \varepsilon_n (z^i)' + y] -$$
$$\frac{1}{8} \sum_{j=1}^{8} \frac{1}{2} D^2 \phi(t, \eta_j(y))[\varepsilon_n (z^j)' + y, \varepsilon_n (z^j)' + y]\, dy \Bigg) \tag{4.112}$$

It is not hard to see that there is a constant $C = C(T')$ such that

$$|(4.111)| + |(4.112)| \le C \left\| D^2 \phi(t) \right\|_{L^\infty(S)}. \tag{4.113}$$

Further, using the discrete product rule, there is an $\eta_i(y) \in x' + \left(-\frac{\varepsilon_n}{2}, \frac{\varepsilon_n}{2}\right)^2$ such that

$$\bar{D}_i^n \left(\overline{\overline{\left(x_3 - \frac{1}{2} \right) P'_n \nabla' \phi(t, \hat{x}')}} \right)$$

$$= h_n^{-1} a_3^i \fint_{\hat{x}' + \varepsilon_n (a^i)' + \left(-\frac{\varepsilon_n}{2}, \frac{\varepsilon_n}{2}\right)^2} \nabla' \phi(t, y)\, dy$$
$$+ \left(\hat{x}_3 - \frac{1}{2} \right) \frac{1}{\varepsilon_n} \fint_{\left(-\frac{\varepsilon_n}{2}, \frac{\varepsilon_n}{2}\right)^2} \nabla' \phi(t, \hat{x}' + \varepsilon_n (a^i)' + y) - \nabla' \phi(t, \hat{x}' + y)\, dy$$
$$= h_n^{-1} a_3^i \nabla' \phi(t, \xi')$$
$$+ h_n^{-1} a_3^i \fint_{\hat{x}' + \varepsilon_n (a^i)' + \left(-\frac{\varepsilon_n}{2}, \frac{\varepsilon_n}{2}\right)^2} \left(\nabla' \phi(t, y) - \nabla' \phi(t, \xi') \right)\, dy \tag{4.114}$$
$$+ \left(\hat{x}_3 - \frac{1}{2} \right) \frac{1}{\varepsilon_n} \fint_{\left(-\frac{\varepsilon_n}{2}, \frac{\varepsilon_n}{2}\right)^2} D^2 \phi(t, \eta_i(y))[\varepsilon_n (a^i)' + y, \varepsilon_n (a^i)' + y]\, dy. \tag{4.115}$$

Again, there is a $C = C(T')$ such that

$$|(4.114)| + |(4.115)| \leq C \left\| D^2\phi(t) \right\|_{L^\infty(S)}. \tag{4.116}$$

Now we have all the ingredients to bound (4.110): For the first three terms we use estimate (4.113). With help of the calculation for the estimate (4.116) we write

$$\begin{aligned}
&\int_0^{T'} \int_\Omega R_n J^n : \bar{\nabla}_n \overline{\overline{P'_n \varphi_2}} \, dx \, dt \\
&= \sum_{l=1}^{8} \int_0^{T'} \int_\Omega R_n J^n_{\cdot l} \cdot \bar{D}^n_l \overline{\overline{P'_n \varphi_2}} \, dx \, dt \\
&-\frac{1}{8} \sum_{k=1}^{8} \int_0^{T'} \int_\Omega R_n J^n_{\cdot l} \cdot \bar{D}^n_k \overline{\overline{P'_n \varphi_2}} \, dx \, dt \\
&= h_n^{-1} \sum_{l=1}^{8} \int_0^{T'} \int_\Omega J^n_{\cdot l} \cdot \begin{pmatrix} \nabla'\phi \\ 0 \end{pmatrix} z_3^l \, dx \, dt \\
&+ \sum_{l=1}^{8} \int_0^{T'} \int_\Omega A_n J^n_{\cdot l} \cdot \begin{pmatrix} \nabla'\phi \\ 0 \end{pmatrix} z_3^l \, dx \, dt + \omega(t),
\end{aligned}$$

where $\omega(t)$ collects the remaining terms and can be bounded by $|\omega(t)| \leq C \|D^2\phi(t)\|_{L^\infty(S)}$. The same can be done for the surface terms. Combined with (4.113) and (4.116) we finally obtain using Lemma 4.4.7 and the embedding $H^2(S) \hookrightarrow C_b(\overline{S})$

$$\begin{aligned}
(4.110) \leq \Bigg| & h_n^{-1} \sum_{l=1}^{8} \int_0^{T'} \int_\Omega \sum_{k=1}^{2} J^n_{3l} z_k^l \partial_k \phi - \sum_{i=1}^{2} J^n_{il} z_3^l \partial_i \phi \, dx \, dt \\
&+ h_n^{-1} \sum_{l=1}^{4} \int_0^{T'} \int_{S \times \left(0, \frac{1}{\nu_n - 1}\right)} \sum_{k=1}^{2} J^{(1,n)}_{3l} z_k^l \partial_k \phi - \sum_{i=1}^{2} J^{(1,n)}_{il} z_3^l \partial_i \phi \, dx \, dt \\
& h_n^{-1} \sum_{l=1}^{4} \int_0^{T'} \int_{S \times \left(\frac{\nu_n - 2}{\nu_n - 1}, 1\right)} \sum_{k=1}^{2} J^{(2,n)}_{3l} z_k^{l+4} \partial_k \phi - \sum_{i=1}^{2} J^{(2,n)}_{il} z_3^{l+4} \partial_i \phi \, dx \, dt \Bigg| \\
&+ C \int_0^{T'} \left\| D^2\phi(t) \right\|^2_{L^\infty(S)} \, dt \\
&\leq C \|\phi\|_{L^2(0,T';H_0^4(S))}.
\end{aligned}$$

Together with (4.108) and (4.109) this yields (4.107) and we get the boundedness of the sequence $\partial_t^2 v^n + \sum_{\alpha=1}^{2} \partial_t^2 \partial_\alpha q_\alpha^n$ in $L^2(0,T';H^{-4}(S))$. The partial derivatives $\partial_\alpha q_\alpha^n$, $\alpha = 1,2$, are understood as distributional derivatives here. Since for the distributional derivative $\partial^\alpha f$ of an $L^2(S)$-function f we always have the estimate $\|\partial_\alpha f\|_{H^{-1}(S)} \leq \|f\|_{L^2(S)}$ from (4.73) it follows that

$$\partial_t \partial_\alpha q_\alpha^n \to 0 \quad \text{in } L^\infty(0,T';H^{-1}(S)).$$

By Lemma 4.3.4 and Corollary 4.3.6 it holds that

$$\partial_t v^n \overset{*}{\rightharpoonup} \begin{cases} \partial_t v & \text{if } \nu_n \to \infty, \\ \frac{\nu}{\nu-1} \partial_t v & \text{if } \nu_n \equiv \nu \in \mathbb{N} \end{cases}$$

in $L^\infty(0,T';L^2(S))$ and therefore also in $L^\infty(0,T';H^{-1}(S))$, hence we get

$$\partial_t v^n + \sum_{\alpha=1}^{2} \partial_t\partial_\alpha q_\alpha^n \overset{*}{\rightharpoonup} \begin{cases} \partial_t v & \text{if } \nu_n \to \infty, \\ \frac{\nu}{\nu-1}\partial_t v & \text{if } \nu_n \equiv \nu \in \mathbb{N} \end{cases}$$

in $L^\infty(0,T';H^{-1}(S))$. Since the embedding

$$L^\infty(0,T';H^{-1}(S)) \cap H^1(0,T';H^{-4}(S)) \hookrightarrow C([0,T'];H^{-4}(S))$$

is compact (c.f. [Sim86]) we deduce

$$\frac{\nu_n - 1}{\nu_n}\left(\partial_t v^n(0,\cdot) + \sum_{\alpha=1}^{2}\partial_t\partial_\alpha q_\alpha^n(0,\cdot)\right) \to \partial_t v(0,\cdot)$$

strongly in $H^{-4}(S)$. The prefactor $\frac{\nu_n-1}{\nu_n}$ is to avoid case distinction. Let $\phi \in C_c^\infty(S)$, then

$$\begin{aligned} &\int_S w_3^{(1)}(x')\phi(x')\ dx' \\ &= \lim_{n\to\infty}\frac{\nu_n-1}{\nu_n}\int_S h_n^{-1}\left(\bar{\bar{w}}_n^{(2)}\right)_3\phi(x')\ dx' \\ &= \lim_{n\to\infty}\frac{\nu_n-1}{\nu_n}\int_S \partial_t v^n(0,x')\phi(x')\ dx' \\ &= \lim_{n\to\infty}\frac{\nu_n-1}{\nu_n}\int_S\left(\partial_t v^n(0,x') + \sum_{\alpha=1}^{2}\partial_t\partial_\alpha q_\alpha^n(0,x')\right)\phi(x')\ dx' \\ &\quad -\frac{\nu_n-1}{\nu_n}\lim_{n\to\infty}\int_S\sum_{\alpha=1}^{2}\partial_t\partial_\alpha q_\alpha^n(0,x')\phi(x')\ dx' \\ &= \int_S \partial_t v(0,x')\phi(x')\ dx', \end{aligned}$$

and (4.65) follows.

Finally we show the weak continuity of $t \mapsto \partial_t v : I_T \to L^2(S)$. Let $(t_n)_n \subset I_T$ and $t \in I_T$ such that $t_n \to t$. Since $\partial_t v \in L^\infty_{\text{loc}}(I_T;L^2(S))$ the sequence $(\partial_t v(t_n))_{n\in\mathbb{N}}$ is bounded in $L^2(S)$ and therefore converges weakly to some $f \in L^2(S)$. But since $\partial_t v \in C([0,T'];H^{-4}(S))$ we have $\partial_t v(t_n) \to \partial_t v(t)$ in $H^{-4}(S)$. Hence $f = \partial_t v(t)$ and weak continuity follows.

Similarly we can show weak continuity of the map $t \mapsto v(t) : I_T \to H^2(S)$. For every $T' \in I_T$ we have $v \in W^{1,\infty}(0,T';L^2(S)) \hookrightarrow C([0,T'];L^2(S))$ and $v \in L^\infty(0,T';H^2(S))$. Thus the sequence $(v(t_n))_n$ is bounded in $H^2(S)$, i.e. $v(t_n) \rightharpoonup f$ for some $f \in H_0^2(S)$. Further $v(t_n) \to v(t)$ in $L^2(S)$, hence $v = f$. □

4.6. Summary

Starting with solutions from Newtons equations of motion for particle systems we have shown that the averaged in- and out-of-plane displacements converge, up to a subsequence, to a weak solution of the dynamical von Kármán equations for both, thin and ultrathin films. Our result covers basic interaction potentials for nearest and next-to-nearest interactions of atoms given in section 3.4. Note that our analysis includes an existence result for weak solutions of the time-dependent von Kármán equations. Yet again it would be desirable to get similar results without the growth condition of the derivative.

Appendix A

Analytical lemmas

A.1. Analytical time-independent results

Proposition A.1.1 (Korn's inequality, [Cia10], [GSN86])**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and*

$$E_p(\Omega) \coloneqq \{\, u \in L^p(\Omega;\mathbb{R}^n) \,\} : \text{sym } \nabla u \in L^p(\Omega;\mathbb{R}^{n\times n}).$$

Then for $u \in E_p(\Omega)$ it holds that

$$\|u\|^p_{W^{1,p}(\Omega;\mathbb{R}^n)} \le C_p \int_\Omega |u|^p + |\text{sym } \nabla u|^p \ dx,$$

$$\min\left\{ \|u - Ax - b\|^p_{W^{1,p}(\Omega;\mathbb{R}^n)} : A \in \mathbb{R}^{n\times n}_{\text{skew}},\ b \in \mathbb{R}^n \right\} \le C \|\text{sym } \nabla u\|^p_{L^p(\Omega;\mathbb{R}^{n\times n})} .$$

If $\Gamma \subset \partial\Omega$ has positive surface measure then for every $u \in E_p(\Omega)$ with $u|_\Gamma = 0$

$$\|\nabla u\|_{L^p(\Omega;\mathbb{R}^{n\times n})} \le \|\text{sym } \nabla u\|_{L^p(\Omega;\mathbb{R}^{n\times n})} .$$

Proposition A.1.2 ([MP08], Proposition 2.3)**.** *Let $E \subset \mathbb{R}^n$ be a bounded, measurable set, $1 \le p < \infty$. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a function which is differentiable at 0 and satisfies for every $a \in \mathbb{R}^n$ the inequality*

$$|f(a)| \le C |a| .$$

Let $z^\delta \rightharpoonup z$ in $L^p(E;\mathbb{R}^n)$. Then

$$\frac{1}{\delta} f(\delta z^{(\delta)}) \rightharpoonup Df(0)z \text{ in } L^p(E).$$

Definition A.1.3 (Convergence (boundedely) in measure)**.** *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $f_n, f : X \to \mathbb{R}$ be measurable functions. We say f_n converges in measure to f if for every $\varepsilon > 0$ it holds that*

$$\mu\left(\{x \in \Omega\colon\ |f_n(x) - f(x)| > \varepsilon\}\right) \underset{n\to\infty}{\longrightarrow} 0.$$

We say f_n converges to f boundedly in measure if f_n converges to f in measure and in addition it holds that $f_n, f \in L^\infty(\Omega)$ and $\sup_{n\in\mathbb{N}} \|f_n\|_{L^\infty(\Omega)} < \infty$.

Lemma A.1.4. *Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space and let $p \in [1, \infty)$. Let $f_n, f : \Omega \to \mathbb{R}$ be measurable functions such that $f_n \to f$ boundedely in measure. Let $g_n \rightharpoonup g$ in $L^p(\Omega)$. Then $f_n g_n \rightharpoonup fg$ in $L^p(\Omega)$.*

Proof. First consider $p = 1$. Let $\phi \in L^\infty(\Omega)$ and $\varepsilon > 0$. Then

$$\begin{aligned} &\left| \int_\Omega f_n g_n \phi \, d\mu - \int_\Omega f g \phi \, d\mu \right| \\ \leq &\left| \int_\Omega f_n g_n \phi \, d\mu - \int_\Omega f g_n \phi \, d\mu \right| + \left| \int_\Omega f g_n \phi \, d\mu - \int_\Omega f g \phi \, d\mu \right| \end{aligned} \tag{A.1}$$

The second term vanishes for $n \to \infty$ since $f\phi \in L^\infty(\Omega)$, so we remain with the first term. Let $\varepsilon > 0$. Then

$$\begin{aligned} &\left| \int_\Omega f_n g_n \phi \, d\mu - \int_\Omega f g_n \phi \, d\mu \right| \\ \leq &\left| \int_{\{|f_n - f| > \varepsilon\}} (f_n - f) \, g_n \phi \, d\mu \right| + \left| \int_{\{|f_n - f| < \varepsilon\}} (f_n - f) \, g_n \phi \, d\mu \right| \\ &\leq C \int_{\{|f_n - f| > \varepsilon\}} |g_n| \, d\mu + C\varepsilon \sup_{n \in \mathbb{N}} \|g_n\|_{L^1(\Omega)} \\ &\underset{n\to\infty}{\longrightarrow} C\varepsilon. \end{aligned}$$

The convergence $\int_{\{|f_n - f| > \varepsilon\}} |g_n| \, d\mu \to 0$ is due to the equiintegrability of the sequence g_n. The case $p > 1$ works almost the same choosing test functions $\phi \in C_C^\infty(\Omega)$. □

Theorem A.1.5 (Egorov's theorem, [EG15])**.** *Let μ be a measure on $\mathbb{R}^n$ and suppose $f_k : \mathbb{R}^n \to \mathbb{R}^m$ are μ-measurable. Assume also $A \subset \mathbb{R}^n$ is μ-measurable, with $\mu(A) < \infty$, and*

$$f_k \to f \quad \mu\text{-a.e on } A.$$

Then for each $\varepsilon > 0$ there exists a μ-measurable set $B \subset A$ such that

(i) $\mu(A \setminus B) < \varepsilon$ and

(ii) $f_k \to f$ uniformly on B.

Corollary A.1.6. *Let $\Omega \subset \mathbb{R}^n$ with $\lambda^n(\Omega) < \infty$, $g \in L^1_{loc}(\Omega)$ and $f_k : \mathbb{R}^n \to \mathbb{R}^m$ measurable functions such that*

(i) $\sup_{k\in\mathbb{N}} \|f_k\|_{L^p(\Omega)} < \infty$ for $p \in (1, \infty)$ and

(ii) $f_k \to g$ a.e. in Ω.

Then $f_k \rightharpoonup g$ in $L^p(\Omega)$.

Proof. Let $f \in L^p(\Omega)$ such that, up to a subsequence, $f_k \rightharpoonup f$. Then the subsequence $(f_k)_k$ is equiintegrable. Let $\varepsilon > 0$ and $\delta > 0$ such that

$$\int_A |g| \, dx + \sup_k \int_A |f_k| \, dx < \varepsilon$$

whenever $\lambda^n(A) < \delta$. Let $\varphi \in C_c^\infty(\Omega)$. By Egorov's theorem there is a set A_δ such that $f_k \to g$ uniformly on A_δ and $\lambda^n(\Omega \setminus A_\delta) < \delta$. Togehter with the weak convergence of f_k we obtain

$$\left| \int_\Omega (f - g)\varphi \right|$$

$$
\begin{aligned}
&\leq \left|\int_\Omega f\varphi - \int_\Omega f_k\varphi\right| + \left|\int_\Omega f_k\varphi - \int_\Omega g\varphi\right| \\
&\leq \left|\int_\Omega f\varphi - \int_\Omega f_k\varphi\right| + \left|\int_{A_\delta} f_k\varphi - \int_{A_\delta} g\varphi\right| + \left|\int_{\Omega\setminus A_\delta} f_k\varphi - \int_{\Omega\setminus A_\delta} g\varphi\right| \\
&\leq \left|\int_\Omega f\varphi - \int_\Omega f_k\varphi\right| + \left|\int_{A_\delta} f_k\varphi - \int_{A_\delta} g\varphi\right| + \|\varphi\|_{L^\infty(\Omega)} \left(\int_{\Omega\setminus A_\delta} |g| + \sup_k \int_{\Omega\setminus A_\delta} |f_k|\right) \\
&\leq \left|\int_\Omega f\varphi - \int_\Omega f_k\varphi\right| + \left|\int_{A_\delta} f_k\varphi - \int_{A_\delta} g\varphi\right| + \|\varphi\|_{L^\infty(\Omega)}\,\varepsilon \\
&\leq C\varepsilon.
\end{aligned}
$$

for k large enough. Since the limit does not depend on the chosen subsequence the convergence holds for the entire sequence. □

Remark A.1.7. *The proof of Corollary A.1.6 relies on the fact that from boundedness of $(f_k)_k$ in $L^p(\Omega)$ we find a weakly convergent subsequence. If $p = 1$ Corollary A.1.6 also holds true if we additionally require equiintegrability of the sequence $(f_k)_k$.*

Lemma A.1.8. *Let $p > 1$. Let $g_n \to 1$ converge boundedely in measure on Ω and (f_n) be a bounded sequence in $L^p(\Omega)$ such that $g_n f_n \to f$ in $L^p(\Omega)$. Then $f_n \to f$ in $L^p(\Omega)$.*

Proof. Let $\varepsilon > 0$ Then

$$
\begin{aligned}
\|f_n - f\|_{L^p(\Omega)}^p &\leq C\int_\Omega |f_n - g_n f_n|^p \, dx + C\int_\Omega |g_n f - f|^p \, dx \\
= C\int_{\{|1-g_n|<\varepsilon\}} |1-g_n|^p\,|f_n|^p \, dx &+ C\int_{\{|1-g_n|\geq\varepsilon\}} |f_n|^p \, dx + \int_\Omega |g_n f - f|^p \, dx \\
&\leq C\varepsilon^p
\end{aligned}
$$

for n large enough due to equi-integrability of the sequence (f_n). □

A.2. Analytical time-dependent results

We start by collecting well-known density results for Bochner or Bochner-Sobolev spaces:

Lemma A.2.1. *Let X be a Banach-space and $\mathcal{D} \subset X$ be a dense subset. Let $1 \leq p < \infty$ and $I \subset \mathbb{R}$ be an open interval.*

(i) The set

$$
M := \left\{ \sum_{i=1}^n \eta_i d_i \colon n \in \mathbb{N},\ \eta_i \in C_c^\infty(I),\ d_i \in \mathcal{D} \right\} \subset L^p(I;X)
$$

is dense in $L^p(I;X)$.

(ii) The embedding

$$\mathcal{C} := \left\{ \sum_{i=1}^{n} \eta_i d_i \colon n \in \mathbb{N},\ \eta_i \in C^\infty(\bar{I}),\ d_i \in \mathcal{D} \right\} \hookrightarrow W^{1,p}(I, X)$$

is dense.

(iii) The embedding

$$\mathcal{F} := \left\{ \sum_{i=1}^{n} \eta_i d_i \colon n \in \mathbb{N},\ \eta_i \in C_c^\infty(I),\ d_i \in \mathcal{D} \right\} \hookrightarrow W_0^{1,p}(I, X)$$

is dense.

We do not give the full proof here but remark, that for (i) one uses the density of simple functions. (ii) and (iii) rely on the embedding $W^{1,p}(I;X) \hookrightarrow C(I;X)$.

Lemma A.2.2. *Let X be a separable, reflexive Banach space. Let $(x_n)_n \subset L^\infty(0,T;X')$ such that $x_n \overset{*}{\rightharpoonup} x$ in $L^\infty(0,T;X')$ and $x_n(t) \rightharpoonup y(t)$ in X' for almost every $t \in (0,T)$. Then $x = y$ in $L^\infty(0,T;X')$.*

Proof. Let $\varphi \in L^1(0,T;X)$. Then

$$\begin{aligned} \langle y, \varphi \rangle_{L^1(0,T;X)} &= \int_0^T \langle y(t), \varphi(t) \rangle_X \, dt \\ &= \int_0^T \lim_{n\to\infty} \langle \varphi(t), x_n(t) \rangle_X \, dt \\ &= \lim_{n\to\infty} \int_0^T \langle \varphi(t), x_n(t) \rangle_X \, dt \\ &= \langle x, \varphi \rangle_{L^1(0,T;X)}. \end{aligned}$$

The penultimate step follows from dominated convergence, since x_n is bounded in $L^\infty_{\mathrm{loc}}(0,T;X')$ and $|\langle \varphi(t), x_n(t) \rangle_X| \leq \|x_n(t)\| \, \|\varphi(t)\|$. ☐

The next proposition is a version of Proposition 2.3 in [MP08] which is adapted for the time dependent case.

Proposition A.2.3. *Let $E \subset \mathbb{R}^n$ be a bounded, measurable set, $1 < p < \infty$. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a function which is differentiable at 0 and satisfies for every $a \in \mathbb{R}^n$ the inequality*

$$|f(a)| \leq C\, |a| .$$

Let $z^\delta \overset{*}{\rightharpoonup} z$ in $L^\infty(0,T;L^p(E;\mathbb{R}^n))$. Then

$$\frac{1}{\delta} f(\delta z^{(\delta)}) \overset{*}{\rightharpoonup} Df(0)z \text{ in } L^\infty\left(0,T;L^p(E;\mathbb{R}^n)\right).$$

Proof. We can assume that $Df(0) = 0$, otherwise consider the function $g(x) = f(x) - Df(0)x$. Let

$$\omega(\delta) \coloneqq \sup_{|a|\le\sqrt{\delta}} \frac{|f(a)|}{|a|}.$$

Combining Taylor's Theorem, $f(0) = 0$ and the assumption $Df(0) = 0$ we get

$$w(\delta) \to 0$$

as $\delta \to 0$. Let

$$A_{t,\delta} \coloneqq \left\{ x \in E\colon \left|z^{(\delta)}(t,x)\right| \ge \frac{1}{\sqrt{\delta}} \right\}.$$

By the boundedness of the sequence $z^{(\delta)}$ in $L^\infty(0,T;L^p(E))$ it holds that $|A_{t,\delta}| \to 0$ for $\delta \to 0$. Let q such that $\frac{1}{p} + \frac{1}{q} = 1$ and $g \in L^1(0,T;L^q(E;\mathbb{R}^n))$. Then

$$\begin{aligned} &\left|\int_0^T \int_E g \cdot \frac{1}{\delta} f(\delta z^{(\delta)})\ dx\ dt\right| \\ \le &\int_0^T \int_{E\setminus A_{t,\delta}} |g| \frac{1}{\delta} \left|f(\delta z^{(\delta)})\right|\ dx\ dt + \int_0^T \int_{A_{t,\delta}} |g| \frac{1}{\delta} \left|f(\delta z^{(\delta)})\right|\ dx\ dt. \end{aligned}$$

We need to show that both terms tend to zero. Using $f(0) = 0$ leads to

$$\begin{aligned} \int_0^T \int_{E\setminus A_{t,\delta}} |g| \frac{1}{\delta} \left|f(\delta z^{(\delta)})\right|\ dx\ dt &= \int_0^T \int_{E\setminus A_{t,\delta}\cap\left\{ z^{(\delta)}\neq 0\right\}} \frac{\left|f(\delta z^{(\delta)})\right|}{\left|\delta z^{(\delta)}\right|} |g| \left|z^{(\delta)}\right|\ dx\ dt \\ &\le \int_0^T \int_{E\setminus A_{t,\delta}\cap\left\{ z^{(\delta)}\neq 0\right\}} \omega(\delta) |g| \left|z^{(\delta)}\right|\ dx\ dt \\ &\le \omega(\delta) \|g\|_{L^1(0,T;L^q(E))} \left\|z^{(\delta)}\right\|_{L^\infty(0,T;L^p(E))} \\ &\le \omega(\delta) \|g\|_{L^1(0,T;L^q(E))} \sup_\delta \left\|z^\delta\right\|_{L^\infty(0,T;L^p(E))} \\ &\le C\omega(\delta) \to 0. \end{aligned}$$

Since $|A_{t,\delta}| \to 0$ and $|f(a)| \le C|a|$ we conclude for the second term

$$\begin{aligned} &\int_0^T \int_{A_{t,\delta}} |g| \frac{1}{\delta} \left|f(\delta z^{(\delta)})\right|\ dx\ dt \\ &\le C \int_0^T \int_{A_{t,\delta}} |g| \frac{1}{\delta} \delta \left|z^{(\delta)}\right|\ dx\ dt \\ \le C \left\|z^{(\delta)}\right\|_{L^\infty(0,T;L^p(E))} &\int_0^T \|g\|_{L^q(A_{t,\delta})}\ dt \\ &\to 0 \end{aligned}$$

as $\delta \to 0$, due to dominated convergence with dominating function $t \mapsto \|g(t)\|_{L^q(E)}$. □

Lemma A.2.4. *Let $f_n \to f$ boundedely in measure on $(0,T) \times \Omega$ and $g_n \overset{*}{\rightharpoonup} g$ in $L^\infty(0,T;L^p(\Omega))$ for $p > 1$. Then $f_n g_n \overset{*}{\rightharpoonup} fg$ in $L^\infty(0,T;L^p(\Omega))$.*

Proof. Let $q = p' = \frac{p}{p-1}$. By Lemma A.2.1 functions of the form $\sum_{i=1}^{n} \eta_i d_i$, $\eta_i \in C_c^\infty(0,T)$, $d_i \in C_c^\infty(\Omega)$ are dense in $L^1(0,T;L^q(\Omega))$. Let ϕ be such a function. Then

$$\left| \int_0^T \int_\Omega f_n g_n \phi \; dx \; dt - \int_\Omega f g \phi \; dx \; dt \right|$$

$$\leq \left| \int_0^T \int_\Omega f_n g_n \phi \; dx \; dt - \int_0^T \int_\Omega f g_n \phi \; dx \; dt \right| \tag{A.2}$$

$$+ \left| \int_0^T \int_\Omega f g_n \phi \; dx \; dt - \int_0^T \int_\Omega f g \phi \; dx \; dt \right| \tag{A.3}$$

It is $\phi f \in L^1(0,T;L^q(\Omega))$ since $f \in L^\infty((0,T)\times\Omega)$ and $\phi \in C_c^\infty((0,T)\times\Omega)$. Therefore (A.2) vanishes for $n \to \infty$. For the second term let $\varepsilon > 0$. We have $g_n \rightharpoonup g$ in $L^p((0,T)\times\Omega)$ as $L^\infty(0,T;L^p(\Omega)) \hookrightarrow L^p(0,T;L^p(\Omega)) \cong L^p((0,T)\times\Omega)$. In particular the sequence $(g_n)_n$ is equi-integrable. Therefore

$$\begin{aligned}
\text{(A.3)} \leq & \left| \int_{\{|f_n - f| > \varepsilon\}} (f_n - f)\, g_n \phi \; d(t,x) \right| + \left| \int_{\{|f_n - f| < \varepsilon\}} (f_n - f)\, g_n \phi \; d(t,x) \right| \\
& \leq C \int_{\{|f_n - f| > \varepsilon\}} |g_n| \;\; d(t,x) + C\varepsilon \sup_{n\in\mathbb{N}} \|g_n\|_{L^\infty(0,T;L^p(\Omega))} \\
& \leq C\varepsilon
\end{aligned}$$

for n large enough. □

Lemma A.2.5 (Aubin-Lions-lemma, [Sim86]). *Let X, Y, Z be Banach spaces such that $X \subset Y \subset Z$. The embedding $X \hookrightarrow Y$ is compact and the embedding $Y \hookrightarrow Z$ is continuous. For $1 \leq p, q \leq \infty$ let*

$$W = \{u \in L^p(0,T;X) \colon \partial_t u \in L^q(0,T;Z)\}.$$

(i) If $p < \infty$ the embedding $W \hookrightarrow L^p(0,T;Y)$ is compact.

(ii) If $p = \infty$ and $q > 1$ the embedding $W \hookrightarrow C([0,T],Y)$ is compact.

Lemma A.2.6. *Let H be a Hilbert space and $P_n \subset L(H)$ orthogonal projections such that $P_n x \to Px$ for all $x \in H$. Further let $x_n \overset{*}{\rightharpoonup} x$ in $L^\infty(0,T;H)$. Then $P_n x_n \overset{*}{\rightharpoonup} Px$ in $L^\infty(0,T;H)$.*

Proof. Let $\varphi \in L^1(0,T;H)$. Then

$$\begin{aligned}
\int_0^T \langle P_n x_n, \varphi \rangle \; dt &= \int_0^T \langle x_n, P_n \varphi \rangle \; dt \\
= \int_0^T \langle x_n, P_n\varphi - P\varphi \rangle \; dt &+ \int_0^T \langle x_n, P\varphi \rangle \; dt \\
&\underset{n\to\infty}{\longrightarrow} \int_0^T \langle Px, \varphi \rangle \; dt.
\end{aligned}$$

Here we have used dominated convergence to see that the first term vanishes. □

Lemma A.2.7. *Let H be a Hilbert space and $U \subset H$ be a closed subspace with its respective orthogonal projection $P_U : H \to U$. For every $x \in U$ and every $y \in H$ it holds that $\langle x, P_U y \rangle = \langle x, y \rangle$.*

Proof.

$$\langle x, P_U y \rangle = \langle P_u x, y \rangle = \langle x, y \rangle.$$

□

Appendix B

Notation

Throughout the thesis we try to stick to standard notation used in mathematical analysis. Yet, as in every field of mathematics, there is also some special notation used in mathematical elasticity theory for thin objects.

- For $x \in \mathbb{R}^3$ we write $x = (x', x_3)$ with $x' = (x_1, x_2) \in \mathbb{R}^2$.
- If $A \in \mathbb{R}^{n\times m}$ we write $A_{\cdot l}$ for the l-th column and $A_{k\cdot}$ for the k-th row.
- sym F is the symmetric part of a matrix, i.e. $\operatorname{sym} F = \frac{1}{2}(F + F^T)$.
- skew F is the skew-symmetric part of a matrix, i.e. $\operatorname{skew} F = \frac{1}{2}(F - F^T)$.
- F'' is the upper 2×2 submatrix of $F \in \mathbb{R}^{3\times 3}$.
- For matrices $A, B \in \mathbb{R}^{m\times n}$ the inner product is given by $A\colon B = Tr\left(A^T B\right)$.
- The *deformation gradient* of a function $y : \mathbb{R}^3 \to \mathbb{R}^3$ at $x \in \mathbb{R}^3$ is denoted by $\nabla y(x) \in \mathbb{R}^{3\times 3}$ with $(\nabla y(x))_{ij} = \partial_j y_i(x)$.
- By $\nabla' y(x)$ we denote the matrix containing the in-plane-derivatives $\partial_1 y(x)$ and $\partial_2 y(x)$.
- $\nabla'^2 v$ is the Hessian matrix of $v : \mathbb{R}^n \to \mathbb{R}$.
- $(a \otimes b)_{ij} = a_i b_j$ for $a, b \in \mathbb{R}^n$.
- $|U|$ is the measure of a measurable set $U \subset \mathbb{R}^n$.

Bibliography

[AC04] R. Alicandro and Marco Cicalese. A general integral representation result for continuum limits of discrete energies with superlinear growth. *SIAM J. Math. Analysis*, 36:1–37, 01 2004.

[AMM09] Helmut Abels, Maria Mora, and Stefan Müller. The time-dependent von kármán plate equation as a limit of 3d nonlinear elasticity. *Calculus of Variations and Partial Differential Equations*, 41, 12 2009.

[BBL07] Xavier Blanc, Claude Bris, and Pierre Lions. Atomistic to continuum limits for computational materials science. *http://dx.doi.org/10.1051/m2an:2007018*, 41, 03 2007.

[Bra02] Andrea Braides. *Gamma-Convergence for Beginners.* Oxford University Press, 07 2002.

[BS13] Julian Braun and Bernd Schmidt. On the passage from atomistic systems to nonlinear elasticity theory for general multi-body potentials with p-growth. *Networks and Heterogeneous Media [electronic only]*, 4, 12 2013.

[BS16] Julian Braun and Bernd Schmidt. Existence and convergence of solutions of the boundary value problem in atomistic and continuum nonlinear elasticity theory. *Calculus of Variations and Partial Differential Equations*, 55(5):125, Sep 2016.

[BS22] Julian Braun and Bernd Schmidt. An atomistic derivation of von-kármán plate theory. *Networks and Heterogeneous Media*, 17(4):613–644, 2022.

[CDKM06] Sergio Conti, Georg Dolzmann, Bernd Kirchheim, and Stefan Müller. Sufficient conditions for the validity of the cauchy-born rule close to so(n). *Journal of the European Mathematical Society*, 8:515–539, 2006.

[Cia10] Philippe Ciarlet. On korn's inequality. *Chin. Ann. Math. B*, 31:607–618, 09 2010.

[DR95] Hervé Le Dret and Annie Raoult. The nonlinear membrane model as variational limit of nonlinear three-dimensional elasticity. *Journal de Mathématiques Pures et Appliquées*, 74:549–578, 1995.

[EG15] Lawrence Evans and Ronald Gariepy. *Measure Theory and Fine Properties of Functions, Revised Edition.* 04 2015.

[Eri08] J.L. Ericksen. On the cauchy—born rule. *Mathematics and Mechanics of*

Solids, 13(3-4):199–220, 2008.

[FJM02] Gero Friesecke, Richard James, and Stefan Müller. A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity. *Communications on Pure and Applied Mathematics, v.55, 1461-1506 (2002)*, 55, 11 2002.

[FJM06] Gero Friesecke, Richard James, and Stefan Müller. A hierarchy of plate models derived from nonlinear elasticity by gamma-convergence. *Archive for Rational Mechanics and Analysis*, 180:183–236, 05 2006.

[FT02] Friesecke and Theil. Validity and failure of the cauchy-born hypothesis in a two-dimensional mass-spring lattice. *Journal of Nonlinear Science*, 12(5):445–478, Oct 2002.

[GSN86] G. Geymonat, P. Suquet, and J. C. Nedelec. Functional spaces for norton-hoff materials. *Mathematical Methods in the Applied Sciences*, 8(1):206–222, 1986.

[LM09] Myriam Lecumberry and Stefan Müller. Stability of slender bodies under compression and validity of the von karman theory. *Archive for Rational Mechanics and Analysis*, 193:255–310, 01 2009.

[Mas93] G.D. Maso. *An Introduction to G-Convergence*. Progress in Nonlinear Differential Equations and Their Applications. Birkhäuser Boston, 1993.

[Mie23] Alexander Mielke. Non-equilibrium steady states as saddle points and EDP-convergence for slow-fast gradient systems. *Journal of Mathematical Physics*, 64(12):123502, 12 2023.

[MM08] M. G. Mora and S. Müller. Convergence of equilibria of three-dimensional thin elastic beams. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, 138(4):873–896, 2008.

[MMR06] Maria Mora, Stefan Mueller, and Maximilian Rinck. Convergence of equilibria of planar thin elastic beams. *Indiana University Mathematics Journal*, 56, 05 2006.

[MNP02] G. Dal Maso, M. Negri, and D. Percivale. Linearized elasticity as γ-limit of finite elasticity. *Set-Valued Analysis*, 10(2):165–183, Jun 2002.

[Mon03] R. Monneau. Justification of the nonlinear kirchhoff-love theory of plates as the application of a new singular inverse method. *Archive for Rational Mechanics and Analysis*, 169(1):1–34, Aug 2003.

[MP08] S. Müller and M. Pakzad. Convergence of equilibria of thin elastic plates - the von karman case. *Communications in Partial Differential Equations, v.33, 1018-1032 (2008)*, 33, 06 2008.

[MS09] Maria Mora and Lucia Scardia. Convergence of equilibria of thin elastic plates under physical growth conditions for the energy density. *Journal of Differential Equations*, 252, 02 2009.

[Mül17] Stefan Müller. *Mathematical Problems in Thin Elastic Sheets: Scaling Limits, Packing, Crumpling and Singularities*, pages 125–193. Springer International Publishing, Cham, 2017.

[Sch06] Bernd Schmidt. A derivation of continuum nonlinear plate theory from atomistic models. *Multiscale Modeling & Simulation*, 5(2):664–694, 2006.

[Sch09] Bernd Schmidt. On the derivation of linear elasticity from atomistic models, 2009.

[Sim86] Jacques Simon. Compact sets in the space lp(o,t; b). *Annali di Matematica Pura ed Applicata*, 146(1):65–96, Dec 1986.

[SZ23] Bernd Schmidt and Jiří Zeman. A bending-torsion theory for thin and ultrathin rods as a **Γ**-limit of atomistic models. *Multiscale Modeling & Simulation*, 21(4):1717–1745, 2023.