Our Daily Platonism

Lessons From a Theory of Pure Finite Sets

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The first lesson that can be extracted from this paper is that (ontological) Platonism has at least in some parts (in this paper, mainly the theory of pure finite sets) an intuitional – quasi-visual – foundation, which fact makes nominalism a rather questionable position.

The second lesson is that (standard) axiomatic set theory is, to a very large extent, intuitionally well-founded, since all the axioms of axiomatic set theory – with the exception, of course, of *the axiom of infinity* – can be proven in the intuitionally well-founded, *true* (hence logically consistent) purely logical theory of pure finite sets. The question of the consistency of axiomatic set theory, therefore, boils down to the question whether the axiom of infinity is logically consistent with the rest of its axioms. This seems to be the case, although, of course, in the theory of pure finite sets *the negation* of any principle postulating infinite sets can be proven. Thus, the independence of the axiom of infinity from the rest of the axioms of axiomatic set theory is demonstrable on the basis of the true theory of pure finite sets, and all that is perhaps problematic about axiomatic set theory is precisely *the axiom of infinity* – if it is added (as it must be in order to have *axiomatic set theory*) to the quite unproblematic rest of the axioms.

The third lesson is that elementary arithmetic can be founded – demonstrably in a perfectly intuitional way – on the basis of the intuitionally well-founded, *true* (hence logically consistent) purely logical theory of pure finite sets (which theory does without infinite sets, but nevertheless holds that there are infinitely many finite sets). Thus, logicism regarding elementary arithmetic is fully vindicated.

The fourth lesson is that the presented theory of pure finite sets, including elementary arithmetic, is deductively and semantically complete and consistent – without conflict with the Gödelian results.

I. The language L and a first interpretation of it

Pure closed brackets

[1] <> is a pure closed bracket.

[2] If β is a pure closed bracket, then <β> is a pure closed bracket.

[3] If β and β´ are pure closed brackets, then <β, β´> is a pure closed bracket.

[4] If β_1 … β_n are pure closed brackets, with $n \geq 3$, then <β_1, …, β_n> is a pure closed bracket.

[5] Pure closed brackets are only expressions according to [1] – [4].

Note that a closed bracket is an object consisting of a "left bracket" – its left border – and a "right bracket" – its right border, and of whatever is in between. In view of the appearances of the left and right borders of pure closed brackets as defined above, and in order to avoid ambiguity, the left and right borders of pure closed bracket will not be called "brackets" here, but "angles".

Accordingly, pure closed brackets are graphical objects; the first of infinitely many are these: <>. <<>>. <<<>>>. <<>, <<>>>. … Pure closed brackets are *universals*, namely *graphical type-objects* (each of which may have many occurrences, instantiations, tokens; this is why pure closed brackets are *universals*). In addition, they are *ideal* proper names of certain abstract objects: *pure finite sequences*: Each pure finite sequence has exactly one pure closed bracket as a proper name of it – one that perfectly depicts it; each pure closed bracket is a proper name of exactly one pure finite sequence, perfectly depicting it. Indeed, one might simply identify pure finite sequences and pure closed brackets, which would make the pure finite sequences themselves, just like their ideal proper names, *graphical* type-objects – universals that, in a sense, can be *seen*.

Consider now a formal language L of first-order predicate-logic-with-identity-and-definite-descriptions, which is enriched by the pure closed brackets as proper names and by the two-place predicate (υ ∈ υ´) ["υ" and "υ´" being schematic letters for variables]. And consider an axiomatization of – classical – first-order predicate-logic-with-identity-and-definite-descriptions formulated in L. As schematic representations of pure closed brackets, the expressions β, β´, β´´, … (with or without attached numerical indices) will serve; they will also serve as quantifiable metalinguistic variables for pure closed brackets. As schematic representations of sentences of L, the expressions σ, σ´, σ´´, … (with or without attached numerical indices) will serve.

For what follows in this section and the next sections the subsequent explanations are certainly not otiose (they all refer to the language L):

"$\beta'[\beta_1, \beta_2]$" means that at a certain location (no matter how deeply embedded) within the pure closed bracket β´, read from left to right, first the pure closed bracket β_1 occurs, then a comma, and then the pure closed bracket β_2; $\beta'[\beta_2, \beta_1]$ is the pure closed bracket that results from $\beta'[\beta_1, \beta_2]$ when one reverses the order of β_1 and β_2 at *that* location in β´.

"$\beta'[\beta_1, \beta_1]$" means that at a certain location within the pure closed bracket β´, read from left to right, first the pure closed bracket β_1 occurs, then a comma, and then again the pure closed bracket β_1; $\beta'[\beta_1]$ is the pure closed bracket that results from $\beta'[\beta_1, \beta_1]$ when the two occurrences of β_1 and the occurrence of the comma are replaced by one occurrence of β_1 at that location in β´.

"<…β´…>" means that at a certain location within a pure closed bracket – no matter how deeply embedded that location may be – the pure closed bracket β´ occurs. Note that <…<…β´…>…> and <…<…<…β´…>…>…> etc. are special cases of <…β´…> (and <…<…<…β´…>…>…> is a special case of <…<…β´…>…>), just as <β´> and <β, β´> and <β´, β> and <β, β´, β´´> etc. are special cases of <…β´…>: cases where β´ occurs, so to speak, "on the surface" of the pure closed bracket.

"σ[β]" means that at certain locations (it may be just one location) within the sentence σ the pure closed bracket β occurs; ∀υσ[υ] is any *sentence* that results from σ[β] when one replaces β by a new variable υ at those locations in σ and puts ∀υ in front of the result.[1]

Important note: If σ[β] is a *closed sentence*, σ[υ] is a monadic predicate (*if* ∀υσ[υ] is a *closed sentence*). However, *used propositionally* in deductions (for making assumptions, or drawing conclusions from assumptions), also σ[υ] counts as a sentence; for the variable υ is then being used as a singular term with arbitrary designation. *Propositionally used* predicates (with one or more free variables) are *open sentences*; all other sentences are *closed sentences*.

If one is applying first-order predicate-logic-with-identity-and-definite-descriptions to the (complete) realm of *pure finite sequences*, with pure closed brackets as *ideal* proper names (*ideal* in the sense just described) of pure finite sequences, then, given the special relationship between pure closed brackets and pure finite sequences, two logical principles (precisely speaking, *schemata* of logical principles) are salient additions to first-order predicate-logic-with-identity-and-definite-descriptions:

$PL_{seq}1$: If β and β´ are different pure closed brackets, then $\vdash \beta \neq \beta'$.

$PL_{seq}2$: If for every pure closed bracket β: $\vdash \sigma[\beta]$, then $\vdash \forall\upsilon\sigma[\upsilon]$.

[1] For ∀υσ[υ] to be a sentence, the variable υ must not already occur somewhere in σ [= σ[β]]; it must be a *new* variable.

And of course there are, under the envisaged interpretation of L, more additions to first-order predicate-logic-with-identity-and-definite-descriptions than merely these two. However, I omit the further logical principles that need to be added concerning the identity of pure finite sequences and concerning the predicate (υ ∈ υ´) – "υ is an immediate constituent / an element of υ´" – when it is being applied to *pure finite sequences*; for this paper does not focus on pure finite sequences (as is already announced by its subtitle).

II. The logical theory of pure finite sets

When, in L, one is talking about and quantifying over *precisely* the (complete) realm of *pure finite sets* (*instead* of the realm of pure finite sequences), pure closed brackets can still be used as *best* proper names of the pure finite sets (every pure closed bracket naming and *depicting* – though usually *not perfectly* depicting – exactly one pure finite set, and every pure finite set being named by a pure closed bracket that depicts it – but usually by more than one such bracket).

Note that *only one* pure finite set is designated by exactly one pure closed bracket and perfectly depicted by it: the empty set, designated and depicted by "<>" and by no other closed pure bracket.

Under this interpretation of L, one cannot identify the pure closed brackets – meaning the *types* (type-objects), not the *tokens* (inscriptions of type-objects) – with the items in the universe of discourse; for one cannot abstract from – disregard – the sequential order of the comma-separated pure closed brackets within a pure closed bracket, and one cannot abstract from any repetition of a pure closed bracket among them. However, given this other interpretation of L, the resulting additional *logical principles*[2] – logical principles to be added

[2] Why are those principles *logical* principles? Because they can be taken to be true *without* assuming anything special about the universe of discourse of L. It is, indeed, natural to assume that the universe of discourse of L are the pure finite sets (so interpreted, L becomes L_{set}). *Yet* one can take the principles $PL_{set}1$ – $PL_{set}4$ to be the mere result of L's containing rather special – namely, compositionally structured and "speaking" – proper names (their individual compositional structures enable them to "speak"), which, to boot, are assumed to name every object in the universe of discourse of L. *How* – according to which rules – those proper names *speak* – *via* their individual compositional structures – can be read off $PL_{set}1$, $PL_{set}3$, and $PL_{set}4$. And that they name every object in the universe of discourse of L is expressed by $PL_{set}2$. Thus, it does not really matter what it is in the universe of discourse of L that is being (classically) referred to and (classically) quantified over; what is true according to $PL_{set}1$ – $PL_{set}4$ is true already due to the intrinsic syntactic-semantic nature of L (no reference to any particular universe of discourse is needed for this): it is *logically* true.

to those of first-order predicate-logic-with-identity-and-definite descriptions – are the following (all of which are presented via *schemata*; in the schemata, it does not matter which "β"-based schematic symbols are being used as long as the depicted schematic structure remains the same):

$PL_{set}1$:

[a] $\vdash \beta'[\beta_1, \beta_2] = \beta'[\beta_2, \beta_1]$

[b] $\vdash \beta'[\beta_1, \beta_1] = \beta'[\beta_1]$

[c] $\vdash \beta' \neq <\ldots\beta'\ldots>$.

Note that the following two (schemata of) principles are not axiomatic; they are, obviously, just special cases of [c]:

[d] $\vdash \beta_1 \neq <\beta_1, \beta_2> \wedge \beta_2 \neq <\beta_1, \beta_2>$

[e] $\vdash \beta_i \neq <\beta_1, \ldots, \beta_n>$ – for any $n \geq 3$, with $1 \leq i \leq n$.

$PL_{set}2$: If for every pure closed bracket β: $\vdash \sigma[\beta]$, then $\vdash \forall\upsilon\sigma[\upsilon]$.[3]

$PL_{set}3$:

[a] $\vdash \beta \in <\beta>$

[b] $\vdash \beta \in <\beta, \beta'>$; $\vdash \beta' \in <\beta, \beta'>$

[c] $\vdash \beta_i \in <\beta_1, \ldots, \beta_n>$ – for any $n \geq 3$, with $1 \leq i \leq n$.

$PL_{set}4$:

[a] $\vdash \beta \notin <>$

[b] $\vdash \beta' \neq \beta \supset \beta' \notin <\beta>$ [*equivalently*: $\vdash \beta' \in <\beta> \supset \beta' = \beta$]

[c] $\vdash \beta'' \neq \beta \wedge \beta'' \neq \beta' \supset \beta'' \notin <\beta, \beta'>$ [*equivalently*: $\vdash \beta'' \in <\beta, \beta'> \supset \beta'' = \beta \vee \beta'' = \beta'$]

[d] $\vdash \beta' \neq \beta_1 \wedge \ldots \wedge \beta' \neq \beta_n \supset \beta' \notin <\beta_1, \ldots, \beta_n>$ [*equivalently*: $\vdash \beta' \in <\beta_1, \ldots, \beta_n> \supset \beta' = \beta_1 \vee \ldots \vee \beta' = \beta_n$] – for any $n \geq 3$.

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[3] Though $PL_{set}2$ is typographically identical to $PL_{seq}2$, the two principles are semantically different; for, *now*, we are talking about and quantifying over pure finite sets, and neither talking about nor quantifying over pure finite sequences.

First-order predicate-logic-with-identity-and-definite-descriptions *plus* $PL_{set}1$ – $PL_{set}4$ is the set-theoretic logical theory PL_{set}.

It is a great convenience to have conventional rules for saving parentheses in formulas:

Outer parentheses – parentheses around formulas that are not in the context of a formula – can be omitted, and parentheses in $\wedge$-chains and in $\vee$-chains can be omitted.

$=$, $\neq$, $\in$, $\notin$ bind equally strong (syntactically), but stronger than $\wedge$, which in turn binds stronger than $\vee$, which in turn binds stronger than $\supset$, which in turn binds stronger than $\equiv$. Parentheses around expressions formed by the mentioned operators and predicate-symbols can be omitted in observance of the (just-presented) order of binding-strength.

Note that a parenthesis immediately after a quantifier [Qυ(…)] or immediately after the negation-symbol [$\neg$(…)] must not be omitted.

Pure closed brackets which differ typographically never refer to the same pure finite *sequence* (cf. $PL_{seq}1$), but in the presently considered interpretation of L they often do refer to the same pure finite *set*. In fact, there is a decision procedure for finding out whether the pure closed brackets β_1 and β_2 do, or do not, refer to the same pure finite set:

If, (1), by employing $PL_{set}1$, [a] and [b], both β_1 and β_2 can be transformed into the same pure closed bracket β^* *in normal form*, then, obviously, β_1 and β_2 refer to the same pure finite set. If, (2), β_1 and β_2 cannot be thus transformed [in other words: if every pure closed bracket in normal form generable from β_1 is different from every pure closed bracket in normal form generable from β_2], then β_1 and β_2 do not refer to the same pure finite set.

It is, however, in case (2), *not obvious* that β_1 and β_2 do not refer to the same pure finite set. Suppose that every pure closed bracket in normal form generable from β_1 is different from every pure closed bracket in normal form generable from β_2; let β_1^* be a pure closed bracket in normal form generable (in the way described after this note) from β_1, and let β_2^* be a pure closed bracket in normal form generable from β_2. *Assume*, notwithstanding the supposition just made, that β_1^* and β_2^* refer to the same pure finite set. It seems – but remains to be proven – that under this assumption the pure closed brackets β_1^* and β_2^* are, (a), typographically identical to each other, or, (b), permutational typographic variants of each other (because somewhere in them the pure closed brackets *with the same* **A**-*number x and the same* B-*number y* are not in the same sequential order; concerning this, see the expositions following this note). In both cases, there is – *contrary* to what is being supposed – a pure closed bracket in normal form that is generable (as described further below) both from β_1 and from β_2. Thus, under the supposition made above, the assumption that β_1^* and β_2^* refer to the same pure finite set must be false; but then, under that same supposition, β_1 and β_2 do not refer to the same pure finite set (since $\vdash \beta_1 = \beta_1^*$ and $\vdash \beta_2 = \beta_2^*$ are provable in PL_{set}).

A pure closed bracket β* *in normal form* evolves from any pure closed bracket β by, *first*, rearranging (from left to right) the order of the *immediate constituent-occurrences* (if there are any[4]) in *every* pure closed bracket enclosed in β (*everywhere* it is enclosed in β), not excluding β itself, on the basis of $PL_{set}1$, [a], *doing so* by implementing the following ordering-schema:

The **A**-number is the number of occurrences of "," in the relevant *immediate constituent-occurrence*. The B-number is the number of occurrences of "<" *plus* the number of occurrences of ">" in the relevant *immediate constituent-occurrence*:

1.1 **A**-number: 0; B-number: 2;
1.2 **A**-number: 0; B-number: 4;
1.3 **A**-number: 0; B-number: 6;
1.4 **A**-number: 0; B-number: 8;
…
1.*n* **A**-number: 0; B-number: 2*n*.

2.1 **A**-number: 1; B-number: 2;
2.2 **A**-number: 1; B-number: 4;
2.3 **A**-number: 1; B-number: 6;
2.4 **A**-number: 1; B-number: 8;
…
2.*n* **A**-number number: 1; B-number: 2*n*.

3.1 **A**-number: 2; B-number: 2;
3.2 **A**-number: 2; B-number: 4;
3.3 **A**-number: 2; B-number: 6;
3.4 **A**-number: 2; B-number: 8;
…
3.*n* **A**-number: 2; B-number: 2*n*;
etc., etc.

[4] <> has *no* immediate constituent-occurrence in it. <<>> has *one* immediate constituent-occurrence in it: <>. <<<>>> has *one* immediate constituent-occurrence in it: <<>>. <<>, <<>>> has *two* immediate constituent-occurrences in it: first <>, and second <<>>. <<>, <>> has *two* immediate constituent-occurrences in it: first <>, and second <>. If β' has the form <β_1, β_2, β_3 …>, then β_1 is the first immediate constituent-occurrence in β, β_2 the second, β_3 the third … .

If parts of this ordering-schema do not apply – or even *cannot* apply (as, for example, **2**.1 and **2**.2, and **3**.1 – **3**.3) – they are simply skipped. If it happens that in the rearranging according to the above ordering-schema two or more immediate constituent-occurrences claim the same position, then they are simply put side by side (a comma between them).

Second, once the rearranging is done, multiple side-by-side occurrences of any pure closed bracket in the pure closed bracket obtained by the rearranging are each reduced to *one* occurrence of that (first-mentioned) pure closed bracket – in accordance with $PL_{set}1$, [b]. Then, if necessary, the resulting pure closed bracket is again subjected to rearranging on the basis of $PL_{set}1$, [a], in accordance with the above ordering-schema, and if necessary, the reduction of multiple side-by-side occurrences to *one* occurrence, on the basis of $PL_{set}1$, [b], is repeated (to the extent feasible) – *and so on*. This rearranging-and-reduction process will, however, come to an end (every pure closed bracket is, after all, a finite entity), and the pure closed bracket that *ultimately* results from it is a pure closed bracket β^* *in normal form*, with $\vdash \beta = \beta^*$ (provable in PL_{set}).

Here is an example. The pure closed bracket from which a pure closed bracket in normal form is to be generated is this:

<<<>, <>>, <<<>>>, <<>>>.

Then the result of the first rearranging (by $PL_{set}1$, [a], obeying the above-presented ordering-schema) is this:

<<<>>, <<<>>>, <<>, <>>>.

Then the result of the first reduction of multiples (by $PL_{set}1$, [b]) is this:

<<<>>, <<<>>>, <<>>>.

Then the result of the second rearranging is this:

<<<>>, <<>>, <<<>>>>.

Then the result of the second reduction of multiples is this:

<<<>>, <<<>>>>.

And this is a pure closed bracket in normal form and a (indeed, *the*) normal form of <<<>, <>>, <<<>>>, <<>>>.

According to the described procedure, a pure closed bracket β^* *in normal form* is for a given pure closed bracket β not necessarily *uniquely objectively determined*: objectively, there may be for β two (or more) typographically different pure closed brackets in normal form, β^*_1 and β^*_2 (with $\vdash \beta = \beta^*_1$ and $\vdash \beta = \beta^*_2$). The reason for this is that two (or more) pure closed brackets may have the same A-number and the same B-number and yet differ typographically otherwise than merely in the order of the immediate constituent-occurrences in them. Here is an example: <<>, <<<<<>>>>>> and <<<>>, <<<<>>>>>.[5] In the rearranging, such pure closed brackets will land side by side, because they have the same **A**-number and the same B-number (in the example the **A**-number is 1, and the B-number 14); however, which one of them – having been put side by side – comes first is not objectively determined, but up to arbitrary (subjective) choice. Thus – objectively – there may result two (or more) pure closed brackets in normal form from a given pure closed bracket.

Consider, for example, the following pure closed bracket:

<<<<<<<>>>>>, <<>>>, <<<>>>, <<<<<>>>>, <>>>.

One pure closed bracket in normal form that results from this pure closed bracket is this:

<<<<>>>, <<>, <<<<<>>>>>>, <<<>>, <<<<>>>>>>.

But, *objectively*, another pure closed bracket in normal form resulting from it is this:

<<<<>>>, <<<>>, <<<<>>>>>, <<>, <<<<<>>>>>>>.

Note that *comma-free* pure closed brackets – pure closed brackets with **A**-number *zero* (that is, <>, <<>>, <<<>>>, <<<<>>>>, …) – are *automatically* in normal form.

III. PL_{set}-provable set-theoretical principles

Familiarity with the principles of first-order predicate-logic-with-identity-and-definite-descriptions is assumed in this paper, and *no*, or only rudimentary, explicit commentary will be provided when these principles are deductively employed in the proofs that follow; in other words, readers are more or less expected to be able to see for themselves which of those principles is the one that is being used in a given deductive step.[6] However, this policy regarding commentary will certainly not be followed in this paper when one or the other of the *additional* logical principles $PL_{set}1$ – $PL_{set}4$ (and more later) comes into play. And a few remarks with general pertinence may be helpful:

First, in the deductions that follow, rules of logical inference (all justifiable for first-order predicate-logic) will play a large role, such as: (**1**) If σ' is deducible from Γ and σ, then $\sigma \supset \sigma'$

[5] The outermost markers (outermost *angles*) are colored red in order to make the pure closed brackets more readable. Later, green color will be used for the same purpose *inside* pure closed brackets.

[6] Since the logic of definite descriptions is not so familiar as the rest of first-order logic, let it be stated that all principles of the logic of definite descriptions to be employed in this paper can be shown to be deducible from the following logical axiom-schema (and the principles of some complete axiomatization of first-order predicate-logic-with-identity): $\vdash \exists^{=1}\upsilon\sigma[\upsilon] \supset \sigma[\iota\upsilon\sigma[\upsilon]]$.

is deducible from Γ. (**2**) If σ'' is deducible from Γ and σ, and also from Γ and σ', then σ'' is deducible from Γ and $\sigma \vee \sigma'$. (**3**) If σ' is deducible from Γ and σ, and σ is deducible from Γ, then σ' is deducible from Γ. (**4**) If $\neg\sigma$ / $\sigma' \wedge \neg\sigma'$ is deducible from Γ and σ, then $\neg\sigma$ is deducible from Γ. *Etc. etc.*

Second, in these rules the expressions referring schematically to sentences of the underlying symbolic language [for example, "σ" or "$\sigma \supset \sigma'$"], or to finite sequences of such sentences [for example, "Γ"], can be uniformly complemented by "the truth of," thus making those rules more explicit; *there is no change in content*. For example, the more explicit version of (**1**) is this: (**1´**) If the truth of σ' is deducible from the truth of Γ and the truth of σ, then the truth of $\sigma \supset \sigma'$ is deducible from the truth of Γ.

Third, in the deductions that follow, if Γ is a sequence of sentences of which the *logical* truth is provable [in PL_{set}, and, later, in PLN_{set}] and σ' – in other words, the truth of σ' – is deducible from [the truth of] Γ, then the *logical* truth of σ' is provable, in other words, $\vdash \sigma'$ is provable. [Do not read "$\vdash$ (σ is provable)," but read "($\vdash \sigma$) is provable"!]

Fourth, as far as the language L_{set} is concerned, and later the language LN_{set} (and not yet the provability-systems PL_{set} and, later, PLN_{set}), truth and logical truth coincide in these languages simply due to the languages' syntactic-semantic nature. (And in any case, logical truth entails truth unconditionally.) Consequently, provability of truth and provability of logical truth coincide in all provability-systems based on the mentioned languages.

Fifth, it is, moreover, taken for granted that for every closed sentence σ of L_{set} and, later, LN_{set}, either σ or $\neg\sigma$ is true.

On the basis of first-order predicate-logic-with-identity-and-definite-descriptions and the *additional* logical principles $PL_{set}1$ – $PL_{set}4$, the following theorems are provable (the initial "$\vdash$" indicates the logical validity – logical truth – of the contents subsequent to it).

<u>PL_{set}Th1: $\vdash \forall z(z \notin z)$</u>

Proof: Assume $\beta' \in \beta'$ is true. β' is typographically either "$<>$", or $<\beta_1>$ or $<\beta_1, \beta_2>$ or $<\beta_1, \ldots, \beta_n>$ for some pure closed brackets $\beta_1, \ldots, \beta_n$ ($n \geq 3$).

In the first case, $<> \in <>$ would be true – which (*logically*) cannot be because it contradicts $PL_{set}4$, [a].

In the second case, $<\beta_1> \in <\beta_1>$ would be true, hence by $PL_{set}4$, [b]: $<\beta_1> = \beta_1$ would be true – which cannot be because it contradicts $PL_{set}1$, [c].

In the third case, $<\beta_1, \beta_2> \in <\beta_1, \beta_2>$ would be true, hence by $PL_{set}4$, [c]: $<\beta_1, \beta_2> = \beta_1 \vee <\beta_1, \beta_2> = \beta_2$ would be true – which cannot be because it contradicts $PL_{set}1$, [d] (a corollary of $PL_{set}1$, [c]).

In the fourth (and last) case, $<\beta_1, \ldots, \beta_n> \in <\beta_1, \ldots, \beta_n>$ would be true, hence by $PL_{set}4$, [d]: $<\beta_1, \ldots, \beta_n> = \beta_1 \vee \ldots \vee <\beta_1, \ldots, \beta_n> = \beta_n$ would be true – which cannot be because it contradicts $PL_{set}1$, [e] (a corollary of $PL_{set}1$, [c]).

In all four possible cases of what β' might typographically be, $\beta' \in \beta'$ cannot be true (because in each of these cases it contradicts the logically true axioms). Therefore: $\vdash \beta' \notin \beta'$, for every pure closed bracket β'. Therefore: $\vdash \forall z(z \notin z)$ by $PL_{set}2$.

<u>PL_{set}Th2: $\vdash \neg\exists z\forall x(x \in z)$</u>

Proof: Suppose $\forall x(x \in \beta)$ is true; therefore: $\beta \in \beta$ is true – contradicting Th$PL_{set}1$. Therefore: $\vdash \neg\forall x(x \in \beta)$, for every pure closed bracket β. Therefore: $\vdash \forall z\neg\forall x(x \in z)$ by $PL_{set}2$, and hence $\vdash \neg\exists z\forall x(x \in z)$.

<u>PL_{set}Th3: $\vdash \neg\exists z(z \in \beta) \equiv \beta = <>$</u>

Proof:

(**i**) Assume $\neg\exists z(z \in \beta)$ is true. β is typographically either “$<>$”, or $<\beta_1>$ or $<\beta_1, \beta_2>$ or $<\beta_1, \ldots, \beta_n>$ for some pure closed brackets $\beta_1, \ldots, \beta_n$ ($n \geq 3$). If β is typographically $<\beta_1>$ or $<\beta_1, \beta_2>$ or $<\beta_1, \ldots, \beta_n>$, then $\beta_1 \in \beta$ is true (by $PL_{set}3$), and therefore: $\exists z(z \in \beta)$ is true – contradicting the assumption. Therefore: β is typographically “$<>$”, and hence: $\beta = <>$ is true. Thus: $\vdash \neg\exists z(z \in \beta) \supset \beta = <>$.

[Or alternatively: β is typographically either “$<>$”, or $<\beta_1>$ or $<\beta_1, \beta_2>$ or $<\beta_1, \ldots, \beta_n>$ for some pure closed brackets $\beta_1, \ldots, \beta_n$ ($n \geq 3$). If β is typographically $<\beta_1>$ or $<\beta_1, \beta_2>$ or $<\beta_1, \ldots, \beta_n>$, then $\beta_1 \in \beta$ is true (by $PL_{set}3$), and therefore: $\exists z(z \in \beta)$ is true. If β is typographically “$<>$”, then $\beta = <>$ is true. Therefore: $\vdash \exists z(z \in \beta) \vee \beta = <>$, or in other words: $\vdash \neg\exists z(z \in \beta) \supset \beta = <>$.]

(**ii**) Assume $\beta = <>$ is true. By $PL_{set}4$, [a]: $\vdash \beta' \notin <>$ (schematically, that is: no matter which pure closed bracket β' is), hence by $PL_{set}2$: $\vdash \forall z(z \notin <>)$, and hence (logically) $\vdash \neg\exists z(z \in <>)$; and hence – because β = <> is assumed to be true – it follows by substitution of identicals that ¬∃z(z ∈ β) is true. Thus: ⊦ β = <> ⊃ ¬∃z(z ∈ β).

On the basis of (**i**) and (**ii**) it follows: ⊦ ¬∃z(z ∈ β) ≡ β = <>.

PL_{set}Th3.1: ⊦ ∀x(¬∃z(z ∈ x) ≡ x = <>)

Proof: From PL_{set}Th3 by PL_{set}2 (since β in PL_{set}Th3 schematically represents just *any* pure closed bracket).

PL_{set}Th3.2: ⊦ ∃z(z ∈ β) ≡ β ≠ <>

Proof: From PL_{set}Th3 by elementary logic.

What now follows is the proof of *the principle of extensionality* – followed by the proofs of other set-theoretical principles, the greater part of which have the status of axioms in axiomatic set theory:

PL_{set}Th4: ⊦ ∀x∀y(∀z(z ∈ x ≡ z ∈ y) ⊃ x = y)

Proof: The proposition to be proven follows by PL_{set}2 from

Lemma: ⊦ ∀z(z ∈ β ≡ z ∈ β´) ⊃ β = β´, for all pure closed brackets β and β´.

Proof of the Lemma: Let β and β´ be arbitrary pure closed brackets. Assume ∀z(z ∈ β ≡ z ∈ β´) is true. It follows that β and β´ designate pure finite sets that have the same number of elements.[7]

If β and β´ designate pure finite sets that have *no* element, then (by PL_{set}Th3) β = <> is true and β´ = <> is true, and therefore: β = β´ is true.

If β and β´ designate pure finite sets that have (exactly) *one* element, then (as can be proved: see footnote 8) β = <β´´> is true and β´ = <β´´´> is true – for some pure closed bracket β´´ and some pure closed bracket β´´´.[8] Therefore [due to the assumption that ∀z(z ∈ β ≡ z ∈ β´) is

[7] Predicates that have *the same instantiations* [like z ∈ β and z ∈ β´, *given* the truth of ∀z(z ∈ β ≡ z ∈ β´)] also have *the same number of instantiations*. For predicates with *finitely many* instantiations, which are the predicates we are here concerned with, this logical truth can be expressed in a language of first-order predicate-logic-with-identity-and-definite-descriptions, hence in L, since quantifiers of the form $\exists^{=n}\upsilon$ ("There are exactly *n* υ") can be defined in that language for any natural number *n* (*n* ≥ 0).

[8] Quite in general: Suppose the set designated by β has exactly *n* elements (*n* ≥ 1). β has a certain typographic shape: <$\beta´´_1$, …, $\beta´´_k$>. *k* cannot be smaller than *n*; otherwise, the set designated by β would have less than *n*

true]: $\forall z(z \in <\beta''> \equiv z \in <\beta'''>)$ is true. Now, $\beta'' \in <\beta''>$ is true, according to $PL_{set}3$, [a]. Therefore: $\beta'' \in <\beta'''>$ is true, and hence $\beta'' = \beta'''$ is true (because of $PL_{set}4$, [b]), and hence (by substitution of identicals and $\vdash <\beta'''> = <\beta'''>$): $<\beta''> = <\beta'''>$ is true, and therefore: $\beta = \beta'$ is true.

If β and β' designate pure finite sets that have *two* elements, then $\beta = <\beta_1, \beta_2>$ is true and $\beta' = <\beta'_1, \beta'_2>$ is true – for some pure closed brackets $\beta_1, \beta_2, \beta'_1, \beta'_2$ with $\beta_1 \neq \beta_2$ and $\beta'_1 \neq \beta'_2$ being true (see footnote 8). Therefore [because $\forall z(z \in \beta \equiv z \in \beta')$ is assumed to be true]: $\forall z(z \in <\beta_1, \beta_2> \equiv z \in <\beta'_1, \beta'_2>)$ is true. Now, $\beta_1 \in <\beta_1, \beta_2>$ and $\beta_2 \in <\beta_1, \beta_2>$ are true according to $PL_{set}3$, [b]. Therefore: $\beta_1 \in <\beta'_1, \beta'_2>$ and $\beta_2 \in <\beta'_1, \beta'_2>$ are true, and hence $\beta_1 = \beta'_1 \vee \beta_1 = \beta'_2$ and $\beta_2 = \beta'_1 \vee \beta_2 = \beta'_2$ are true (because of $PL_{set}4$, [c]), and hence (i) $\beta_1 = \beta'_1 \wedge \beta_2 = \beta'_1$ is true, or (ii) $\beta_1 = \beta'_1 \wedge \beta_2 = \beta'_2$, or (iii) $\beta_1 = \beta'_2 \wedge \beta_2 = \beta'_1$, or (iv) $\beta_1 = \beta'_2 \wedge \beta_2 = \beta'_2$. *Alternatives* (i) and (iv) are excluded because of $\beta_1 \neq \beta_2$. *Alternative* (ii) implies the truth of $<\beta_1, \beta_2> = <\beta'_1, \beta'_2>$, and hence the truth of $\beta = \beta'$. *Alternative* (iii) implies the truth of $<\beta_1, \beta_2> = <\beta'_2, \beta'_1>$; now, by $PL_{set}1$, [a], we have: $<\beta'_2, \beta'_1> = <\beta'_1, \beta'_2>$ is true; therefore, $<\beta_1, \beta_2> = <\beta'_1, \beta'_2>$ is true, and therefore: $\beta = \beta'$ is true.

If β and β' both designate pure finite sets that have n elements, with $n \geq 3$, then $\beta = <\beta_1, \ldots, \beta_n>$ is true and $\beta' = <\beta'_1, \ldots, \beta'_n>$ true – for some pure closed brackets $\beta_1, \ldots, \beta_n, \beta'_1, \ldots, \beta'_n$ with $\beta_i \neq \beta_j$ and $\beta'_i \neq \beta'_j$ being true for every i and j such that i is not identical to j and $1 \leq i,j \leq n$ (see footnote 8). Therefore [because $\forall z(z \in \beta \equiv z \in \beta')$ is assumed to be true]: $\forall z(z \in <\beta_1, \ldots, \beta_n> \equiv z \in <\beta'_1, \ldots, \beta'_n>)$ is true. Now, $\beta_1 \in <\beta_1, \ldots, \beta_n>$ and … and $\beta_n \in <\beta_1, \ldots, \beta_n>$ are true according to $PL_{set}3$, [c]. Therefore: $\beta_1 \in <\beta'_1, \ldots, \beta'_n>$ and … and $\beta_n \in <\beta'_1, \ldots, \beta'_n>$ are true, and hence $\beta_1 = \beta'_1 \vee \ldots \vee \beta_1 = \beta'_n$ and … and $\beta_n = \beta'_1 \vee \ldots \vee \beta_n = \beta'_n$ are true (because of $PL_{set}4$, [d]). Let $\beta'(\beta_i)$ be *the pure closed bracket* β'' *out of* $\beta'_1, \ldots, \beta'_n$ *which is such that* $\beta_i = \beta''$ *is true*.

There is one and only one *such* pure closed bracket for β_i ($1 \leq i \leq n$) because $\beta_i = \beta'_1 \vee \ldots \vee \beta_i = \beta'_n$ is true for every i with $1 \leq i \leq n$, and because $\beta'_i \neq \beta'_j$ is true for every i and j such that i is not identical to j and $1 \leq i,j \leq n$. Moreover,

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elements (by $PL_{set}3$, [c], and $PL_{set}4$, [d]) – contradicting the assumption. But k might be greater than n; in that case, the members of groups of the $\beta''_1, \ldots, \beta''_k$ – of at least one such group – must designate the same pure finite set as is designated by one of the group-members: in such a manner as to conform to the initial assumption that the number of the elements of the set designated by β is n. On this basis, using substitution of identicals and $PL_{set}1$, [a] and [b], it turns out that β, which is typographically identical to $<\beta''_1, \ldots, \beta''_k>$, designates the same set as $<\beta_1, \ldots, \beta_n>$, in other words: that $\beta = <\beta_1, \ldots, \beta_n>$ is true, the pure closed brackets $\beta_1, \ldots, \beta_n$ being among the pure closed brackets $\beta''_1, \ldots, \beta''_k$ and each of the $\beta_1, \ldots, \beta_n$ designating a pure finite set that the others in $\beta_1, \ldots, \beta_n$ do not designate.

if β_i and β_j are pure closed brackets out of $\beta_1, \ldots, \beta_n$ with *i* not identical to *j* ($1 \leq i,j \leq n$), then $\beta'(\beta_i) \neq \beta'(\beta_j)$ is true. [*Otherwise*, we would have the truth of $\beta_i = \beta_j$ – which follows from the truth of $\beta_i = \beta'(\beta_i)$, $\beta'(\beta_i) = \beta'(\beta_j)$, and $\beta_j = \beta'(\beta_j)$ – contradicting the already established truth of $\beta_i \neq \beta_j$.] Moreover, all immediate constituent-occurrences in $<\beta'_1, \ldots, \beta'_n>$ are used as $\beta'(\beta_i)$ for some *i* with $1 \leq i \leq n$, because there are precisely as many constituent-occurrences in $<\beta'_1, \ldots, \beta'_n>$ as there are in $<\beta_1, \ldots, \beta_n>$ (namely, *n*).

Consider $<\beta'(\beta_1), \ldots, \beta'(\beta_n)>$. $<\beta_1, \ldots, \beta_n> = <\beta'(\beta_1), \ldots, \beta'(\beta_n)>$ is true, due to substitution of identicals, because $\beta_1 = \beta'(\beta_1)$ and … and $\beta_n = \beta'(\beta_n)$ are true. And $<\beta'_1, \ldots, \beta'_n> = <\beta'(\beta_1), \ldots, \beta'(\beta_n)>$ is also true, due to $PL_{set}1$, [a], because the pure closed bracket $<\beta'(\beta_1), \ldots, \beta'(\beta_n)>$ is a mere permutation of the pure closed bracket $<\beta'_1, \ldots, \beta'_n>$ (regarding the immediate constituent-occurrences in $<\beta'_1, \ldots, \beta'_n>$), if, indeed, it is not the very same as the latter). Therefore, $<\beta_1, \ldots, \beta_n> = <\beta'_1, \ldots, \beta'_n>$ is true, and hence $\beta = \beta'$ is true.

Thus, in every possible case under the assumption of (the truth of) $\forall z(z \in \beta \equiv z \in \beta')$ it turns out that $\beta = \beta'$ is true. This establishes the *Lemma* (since β and β´ schematically represent just *any* pure closed brackets) – and concludes the proof (due to $PL_{set}2$).

<u>PL_{set}Th5: $\vdash \forall x \forall y \exists z \forall u(u \in z \equiv u = x \vee u = y)$</u> – which is *the pair-set principle*.

Proof: $\vdash \beta'' \in <\beta, \beta'> \supset \beta'' = \beta \vee \beta'' = \beta'$ by $PL_{set}4$, [c]; $\vdash \beta'' = \beta \vee \beta'' = \beta' \supset \beta'' \in <\beta, \beta'>$ by $PL_{set}3$, [b] (and the logic of identity, etc.). Therefore (elementarily): $\vdash \beta'' \in <\beta, \beta'> \equiv \beta'' = \beta \vee \beta'' = \beta'$. Therefore by $PL_{set}2$: $\vdash \forall u(u \in <\beta, \beta'> \equiv u = \beta \vee u = \beta')$ [since β´´ schematically represents just *any* pure closed bracket]. Therefore (elementarily): $\vdash \exists z \forall u(u \in z \equiv u = \beta \vee u = \beta')$. Therefore by $PL_{set}2$ (twice employed): $\vdash \forall x \forall y \exists z \forall u(u \in z \equiv u = x \vee u = y)$ [since β and β´ schematically represent just *any* pure closed brackets].

The following two definitions of expressions for describing pure closed brackets in schemata are needed for the next proof:

Def0: $<\_> =_{Def} <>$; $<\_, \ldots> =_{Def} <\ldots>$; $<\ldots, \_> =_{Def} <\ldots>$.

Def1: $\in(<>) =_{Def} \_$; $\in(<\beta>) =_{Def} \beta$; $\in(<\beta, \beta'>) =_{Def} \beta, \beta'$; $\in(<\beta_1, \ldots, \beta_n>) =_{Def} \beta_1, \ldots, \beta_n$ ($n \geq 3$).

<u>PL_{set}Th6: $\vdash \forall x \forall y \exists z \forall u(u \in z \equiv u \in x \vee u \in y)$</u> – which is *the union-set principle.*

Proof: The proposition to be proven follows along the lines displayed in the proof of the pair-set principle (see “Therefore by $PL_{set}2$ …”) from

Lemma: $\vdash \beta'' \in <\in(\beta), \in(\beta')> \equiv \beta'' \in \beta \vee \beta'' \in \beta'$.

Proof of the Lemma: β is typographically either "$<>$", or $<\beta_1>$ or $<\beta_1, \beta_2>$ or $<\beta_1, \ldots, \beta_n>$ for some pure closed brackets $\beta_1, \ldots, \beta_n$ ($n \geq 3$). And β' is typographically either "$<>$", or $<\beta'_1>$ or $<\beta'_1, \beta'_2>$ or $<\beta'_1, \ldots, \beta'_k>$ for some pure closed brackets $\beta'_1, \ldots, \beta'_k$ ($k \geq 3$). The 16 possible combinations are the following:

1. Both β and β' are typographically "$<>$", and thus it is true: $<\in(\beta), \in(\beta')> = <\in(<>), \in(<>)> = <\_, \_> = <\_> = <>$ [according to Def1 and Def0].

2. β is typographically "$<>$" and β' typographically $<\beta'_1>$, and thus it is true: $<\in(\beta), \in(\beta')> = <\in(<>), \in(<\beta'_1>)> = <\_, \beta'_1> = <\beta'_1>$ [according to Def1 and Def0].

3. β is typographically "$<>$" and β' typographically $<\beta'_1, \beta'_2>$, and thus it is true: $<\in(\beta), \in(\beta')> = <\in(<>), \in(<\beta'_1, \beta'_2>)> = <\_, \beta'_1, \beta'_2> = <\beta'_1, \beta'_2>$ [according to Def1 and Def0].

4. β is typographically "$<>$" and β' typographically $<\beta'_1, \ldots, \beta'_k>$ [$k \geq 3$], and thus it is true: $<\in(\beta), \in(\beta')> = <\in(<>), \in(<\beta'_1, \ldots, \beta'_k>)> = <\_, \beta'_1, \ldots, \beta'_k> = <\beta'_1, \ldots, \beta'_k>$ [according to Def1 and Def0].

5. β is typographically $<\beta_1>$ and β' typographically "$<>$", and thus it is true: $<\in(\beta), \in(\beta')> = <\in(<\beta_1>), \in(<>)> = <\beta_1, \_> = <\beta_1>$ [according to Def1 and Def0].

6. β is typographically $<\beta_1>$ and β' typographically $<\beta'_1>$, and thus it is true: $<\in(\beta), \in(\beta')> = <\in(<\beta_1>), \in(<\beta'_1>)> = <\beta_1, \beta'_1>$ [according to Def1].

7. β is typographically $<\beta_1>$ and β' typographically $<\beta'_1, \beta'_2>$, and thus it is true: $<\in(\beta), \in(\beta')> = <\in(<\beta_1>), \in(<\beta'_1, \beta'_2>)> = <\beta_1, \beta'_1, \beta'_2>$ [according to Def1].

8. β is typographically $<\beta_1>$ and β' typographically $<\beta'_1, \ldots, \beta'_k>$ [$k \geq 3$], and thus it is true: $<\in(\beta), \in(\beta')> = <\in(<\beta_1>), \in(<\beta'_1, \ldots, \beta'_k>)> = <\beta_1, \beta'_1, \ldots, \beta'_k>$ [according to Def1].

9. β is typographically $<\beta_1, \beta_2>$ and β' typographically "$<>$", and thus it is true: $<\in(\beta), \in(\beta')> = <\in(<\beta_1, \beta_2>), \in(<>)> = <\beta_1, \beta_2, \_> = <\beta_1, \beta_2>$ [according to Def1 and Def0].

10. β is typographically $<\beta_1, \beta_2>$ and β' typographically $<\beta'_1>$, and thus it is true: $<\in(\beta), \in(\beta')> = <\in(<\beta_1, \beta_2>), \in(<\beta'_1>)> = <\beta_1, \beta_2, \beta'_1>$ [according to Def1].

11. β is typographically $<\beta_1, \beta_2>$ and β' typographically $<\beta'_1, \beta'_2>$, and thus it is true: $<\in(\beta), \in(\beta')> = <\in(<\beta_1, \beta_2>), \in(<\beta'_1, \beta'_2>)> = <\beta_1, \beta_2, \beta'_1, \beta'_2>$ [according to Def1].

12. β is typographically $<\beta_1, \beta_2>$ and β' typographically $<\beta'_1, \ldots, \beta'_k>$ [$k \geq 3$], and thus it is true: $<\in(\beta), \in(\beta')> = <\in(<\beta_1, \beta_2>), \in(<\beta'_1, \ldots, \beta'_k>)> = <\beta_1, \beta_2, \beta'_1, \ldots, \beta'_k>$ [according to Def1].

13. β is typographically $<\beta_1, \ldots, \beta_n>$ $[n \geq 3]$ and β' typographically "$<>$", and thus it is true: $<\in(\beta), \in(\beta')> = <\in(<\beta_1, \ldots, \beta_n>), \in(<>)> = <\beta_1, \ldots, \beta_n, \_> = <\beta_1, \ldots, \beta_n>$. [according to Def1 and Def0].

14. β is typographically $<\beta_1, \ldots, \beta_n>$ $[n \geq 3]$ and β' typographically $<\beta'_1>$, and thus it is true: $<\in(\beta), \in(\beta')> = <\in(<\beta_1, \ldots, \beta_n>), \in(<\beta'_1>)> = <\beta_1, \ldots, \beta_n, \beta'_1>$ [according to Def1].

15. β is typographically $<\beta_1, \ldots, \beta_n>$ $[n \geq 3]$ and β' typographically $<\beta'_1, \beta'_2>$, and thus it is true: $<\in(\beta), \in(\beta')> = <\in(<\beta_1, \ldots, \beta_n>), \in(<\beta'_1, \beta'_2>)> = <\beta_1, \ldots, \beta_n, \beta'_1, \beta'_2>$ [according to Def1].

16. β is typographically $<\beta_1, \ldots, \beta_n>$ $[n \geq 3]$ and β' typographically $<\beta'_1, \ldots, \beta'_k>$ $[k \geq 3]$, and thus it is true: $<\in(\beta), \in(\beta')> = <\in(<\beta_1, \ldots, \beta_n>), \in(<\beta'_1, \ldots, \beta'_k>)> = <\beta_1, \ldots, \beta_n, \beta'_1, \ldots, \beta'_k>$ [according to Def1].

Now, assume the truth of $\beta'' \in <\in(\beta), \in(\beta')>$; hence $<\in(\beta), \in(\beta')> \neq <>$ is true (on the basis of $PL_{set}4$, [a]). In all the remaining 15 cases of the 16 possible cases of $<\in(\beta), \in(\beta')>$ being a pure closed bracket (all these cases are listed above), it follows that $\beta'' \in \beta \vee \beta'' \in \beta'$ is true.

Consider, as a paradigm, the last in the list of the 16 cases. In that case, the truth of $\beta'' \in <\in(\beta), \in(\beta')>$ is the truth of $\beta'' \in <\beta_1, \ldots, \beta_n, \beta'_1, \ldots, \beta'_k>$ $[n \geq 3, k \geq 3]$. It follows by $PL_{set}4$, [d]: $\beta'' = \beta_1 \vee \ldots \vee \beta'' = \beta_n \vee \beta'' = \beta'_1 \vee \ldots \vee \beta'' = \beta'_k$ is true, and hence by $PL_{set}3$, [c] (and the logic of identity, etc.): $\beta'' \in <\beta_1, \ldots, \beta_n> \vee \beta'' \in <\beta'_1, \ldots, \beta'_k>$ is true, and therefore (under case 16): $\beta'' \in \beta \vee \beta'' \in \beta'$ is true.

Thus: $\vdash \beta'' \in <\in(\beta), \in(\beta')> \supset \beta'' \in \beta \vee \beta'' \in \beta'$. Assume, conversely, the truth of $\beta'' \in \beta \vee \beta'' \in \beta'$. Case 1 is again logically excluded (since $\beta'' \in \beta \vee \beta'' \in \beta'$ is assumed to be true, β and β' cannot both be "$<>$" – in view of $PL_{set}4$, [a]). In all the remaining 15 cases of the 16 possible cases of $<\in(\beta), \in(\beta')>$ being a pure closed bracket, it follows (from the assumption) that $\beta'' \in <\in(\beta), \in(\beta')>$ is true.

Consider again, as a paradigm, the last in the list of the 16 cases. In that case, the truth of $\beta'' \in \beta \vee \beta'' \in \beta'$ is the truth of $\beta'' \in <\beta_1, \ldots, \beta_n> \vee \beta'' \in <\beta'_1, \ldots, \beta'_k>$ $[n \geq 3, k \geq 3]$. It follows by $PL_{set}4$, [d]: $\beta'' = \beta_1 \vee \ldots \vee \beta'' = \beta_n \vee \beta'' = \beta'_1 \vee \ldots \vee \beta'' = \beta'_k$ is true, and hence by $PL_{set}3$, [c] (and the logic of identity, etc.): $\beta'' \in <\beta_1, \ldots, \beta_n, \beta'_1, \ldots, \beta'_k>$ is true, and therefore (under case 16): $\beta'' \in <\in(\beta), \in(\beta')>$ is true.

Thus also: $\vdash \beta'' \in \beta \vee \beta'' \in \beta' \supset \beta'' \in <\in(\beta), \in(\beta')>$. And in total: $\vdash \beta'' \in <\in(\beta), \in(\beta')> \equiv \beta'' \in \beta \vee \beta'' \in \beta'$. This establishes the *Lemma* – and concludes the proof (in view of what has been said at its beginning).

The subsequent two further definitions of expressions for describing pure closed brackets in schemata are needed for the proof that follows below:

Def2:

$\subseteq(<>) =_{Def} <<>>$ [*configuration*: 1; and 1 immediate constituent-occurrence];

$\subseteq(<\beta>) =_{Def} <<>, <\beta>>$ [*configuration*: 1, 1; and 2 immediate constituent-occurrences];

$\subseteq(<\beta, \beta'>) =_{Def} <<>, <\beta>, <\beta'>, <\beta, \beta'>>$ [*configuration*: 1, 2, 1; and 4 immediate constituent-occurrences].

Def3:

$\subseteq(<\beta_1, \ldots, \beta_n>)$ with $n \geq 3$ is constructed as follows. The general method is best picked up by considering particular cases:

For $n = 3$, $<\beta_1, \ldots, \beta_n>$ is $<\beta_1, \beta_2, \beta_3>$. Then there are 4 levels:

level 0: $<>$;

level 1: $<\beta_1>, <\beta_2>, <\beta_3>$;

level 2: $<\beta_1, \beta_2>, <\beta_1, \beta_3>, <\beta_2, \beta_3>$;

level 3: $<\beta_1, \beta_2, \beta_3>$;

configuration (that is, *number of items on each relevant level*: from level 0 to level 3): 1, 3, 3, 1.

$\subseteq(<\beta_1, \beta_2, \beta_3>) =_{Def} <<>, <\beta_1>, <\beta_2>, <\beta_3>, <\beta_1, \beta_2>, <\beta_1, \beta_3>, <\beta_2, \beta_3>, <\beta_1, \beta_2, \beta_3>>$ [8 immediate constituent-occurrences between the red angles "$<$" and "$>$"; color is being used for better readability].

For $n = 4$, $<\beta_1, \ldots, \beta_n>$ is $<\beta_1, \beta_2, \beta_3, \beta_4>$. Then there are 5 levels:

level 0: $<>$;

level 1: $<\beta_1>, <\beta_2>, <\beta_3>, \beta_4>$;

level 2: $<\beta_1, \beta_2>, <\beta_1, \beta_3>, <\beta_1, \beta_4>, <\beta_2, \beta_3>, <\beta_2, \beta_4>, <\beta_3, \beta_4>$;

level 3: $<\beta_1, \beta_2, \beta_3>, <\beta_1, \beta_2, \beta_4>, <\beta_1, \beta_3, \beta_4,>, <\beta_2, \beta_3, \beta_4>$;

level 4: $<\beta_1, \beta_2, \beta_3, \beta_4>$;

configuration (from level 0 to level 4): 1, 4, 6, 4, 1.

$\subseteq(<\beta_1, \beta_2, \beta_3, \beta_4>) =_{Def} <<>, <\beta_1>, <\beta_2>, <\beta_3>, \beta_4>, <\beta_1, \beta_2>, <\beta_1, \beta_3>, <\beta_1, \beta_4>, <\beta_2, \beta_3>, <\beta_2, \beta_4>, <\beta_3, \beta_4>, <\beta_1, \beta_2, \beta_3>, <\beta_1, \beta_2, \beta_4>, <\beta_1, \beta_3, \beta_4,>, <\beta_2, \beta_3, \beta_4>, <\beta_1, \beta_2, \beta_3, \beta_4>>$ [16 immediate constituent-occurrences between the red angles "$<$" and "$>$"].

For $n = 5$, $<\beta_1, \ldots, \beta_n>$ is $<\beta_1, \beta_2, \beta_3, \beta_4, \beta_5>$. Then there are 6 levels:

level 0: $<>$;

level 1: $<\beta_1>, <\beta_2>, <\beta_3>, <\beta_4>, <\beta_5>$;

level 2: $<\beta_1, \beta_2>, <\beta_1, \beta_3>, <\beta_1, \beta_4>, <\beta_1, \beta_5>, <\beta_2, \beta_3>, <\beta_2, \beta_4>, <\beta_2, \beta_5>, <\beta_3, \beta_4>, <\beta_3, \beta_5>, <\beta_4, \beta_5>$;

level 3: $<\beta_1, \beta_2, \beta_3>, <\beta_1, \beta_2, \beta_4>, <\beta_1, \beta_2, \beta_5>, <\beta_1, \beta_3, \beta_4>, <\beta_1, \beta_3, \beta_5>, <\beta_1, \beta_4, \beta_5>, <\beta_2, \beta_3, \beta_4>, <\beta_2, \beta_3, \beta_5>, <\beta_2, \beta_4, \beta_5>, <\beta_3, \beta_4, \beta_5>$;

level 4: $<\beta_1, \beta_2, \beta_3, \beta_4>, <\beta_1, \beta_2, \beta_3, \beta_5>, <\beta_1, \beta_2, \beta_4, \beta_5>, < \beta_1, \beta_3, \beta_4, \beta_5>, < \beta_2, \beta_3, \beta_4, \beta_5>$;

level 5: $<\beta_1, \beta_2, \beta_3, \beta_4, \beta_5>$;

configuration: 1, 5, 10, 10, 5, 1.

$\subseteq(<\beta_1, \beta_2, \beta_3, \beta_4, \beta_5>) =_{Def} <<>, <\beta_1>, <\beta_2>, <\beta_3>, <\beta_4>, <\beta_5>, <\beta_1, \beta_2>, <\beta_1, \beta_3>, <\beta_1, \beta_4>, <\beta_1, \beta_5>, <\beta_2, \beta_3>, <\beta_2, \beta_4>, <\beta_2, \beta_5>, <\beta_3, \beta_4>, <\beta_3, \beta_5>, <\beta_4, \beta_5>, <\beta_1, \beta_2, \beta_3>, <\beta_1, \beta_2, \beta_4>, <\beta_1, \beta_2, \beta_5>, <\beta_1, \beta_3, \beta_4>, <\beta_1, \beta_3, \beta_5>, <\beta_1, \beta_4, \beta_5>, <\beta_2, \beta_3, \beta_4>, <\beta_2, \beta_3, \beta_5>, <\beta_2, \beta_4, \beta_5>, <\beta_3, \beta_4, \beta_5>, <\beta_1, \beta_2, \beta_3, \beta_4>, <\beta_1, \beta_2, \beta_3, \beta_5>, <\beta_1, \beta_2, \beta_4, \beta_5>, < \beta_1, \beta_3, \beta_4, \beta_5>, < \beta_2, \beta_3, \beta_4, \beta_5>, <\beta_1, \beta_2, \beta_3, \beta_4, \beta_5>>$ [32 immediate constituent-occurrences between the red angles “<” and “>”].

And so on.

In other words: Given $<\beta_1, \ldots, \beta_n>$ with $n \geq 3$, *the first thing to do* for obtaining $\subseteq(<\beta_1, \ldots, \beta_n>)$ is to generate, to the extent possible, out of the immediate constituent-occurrences of $<\beta_1, \ldots, \beta_n>$, i.e., out of $\beta_1, \ldots, \beta_n$, the singles, pairs, …, n-tuples [triples, quadruples, quintuples, …] *without* a repetition of any of those immediate constituent-occurrences – that is, without a repetition *which is new* (not already there in $\beta_1, \ldots, \beta_n$) – and *without* generating different items that are mere permutations of each other. Moreover, the members of the items thus generated, *if* these items have more than one member, are to be separated from their neighbor-member(s) by a comma. *The second thing to do* is to put each of the generated item “within the angles”: “<” at the beginning of the item, and “>” at the end. *The third thing to do* is to list all the pure closed brackets produced at this point, not forgetting to add “<>” to the list. The list should be prepared in such a way that it is headed by “<>”, followed by the pure closed brackets with *one* immediate constituent-occurrence, followed by those with *two* immediate constituent-occurrences, …, followed by the one with n immediate constituent-occurrences. (Further regulations that will bring about a unique determination of the manner in which the list is prepared are left to combinatorial methods.) *The fourth, and last, thing to do* is to put the complete list “within the angles”: “<” at the beginning and “>” at the end.

Let it be noted, as an aside, that the six *configurations* displayed (three in Def2, three in Def3) form the beginning of *Pascal's triangle*:

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

… … … … … … … …

The configuration for $\subseteq(\langle\beta_1, \ldots, \beta_n\rangle)$ with $n = 6$ [7 levels: level 0 – level 6; and 64 immediate constituent-occurrences] follows suit; it is this:

1 6 15 20 15 6 1

And the configurations for $\subseteq(\langle\beta_1, \ldots, \beta_n\rangle)$ with $n \geq 7$, too, follow suit.

<u>$PL_{set}Th7$: $\vdash \forall x\exists z\forall u(u \in z \equiv \forall w(w \in u \supset w \in x))$</u> – which is *the power-set principle.*

Proof: The proposition to be proven is proven by first proving

Lemma: $\vdash \beta'' \in \subseteq(\beta') \equiv \forall w(w \in \beta'' \supset w \in \beta')$. Then, this proven, one obtains $\vdash \forall u(u \in \subseteq(\beta') \equiv \forall w(w \in u \supset w \in \beta'))$ by $PL_{set}2$ (since β'' represents just *any* pure closed bracket), and hence $\vdash \exists z\forall u(u \in z \equiv \forall w(w \in u \supset w \in \beta'))$, and therefore by $PL_{set}2$: $\vdash \forall x\exists z\forall u(u \in z \equiv \forall w(w \in u \supset w \in x))$ (since β' represents just *any* pure closed bracket).

Proof of the Lemma: β' is typographically either “$\langle\rangle$”, or $\langle\beta_1\rangle$ or $\langle\beta_1, \beta_2\rangle$ or $\langle\beta_1, \ldots, \beta_n\rangle$ for some pure closed brackets $\beta_1, \ldots, \beta_n$ ($n \geq 3$).

If β' is typographically “$\langle\rangle$”, then, in view of Def2, what is in question, as a step towards obtaining $\vdash \beta'' \in \subseteq(\beta') \equiv \forall w(w \in \beta'' \supset w \in \beta')$, is $\vdash \beta'' \in \langle\langle\rangle\rangle \equiv \forall w(w \in \beta'' \supset w \in \langle\rangle)$; and this latter equivalence is certainly provable: From the assumed truth of $\beta'' \in \langle\langle\rangle\rangle$ one obtains by $PL_{set}4$, [b]: $\beta'' = \langle\rangle$ is true, and hence $\forall w(w \in \beta'' \supset w \in \langle\rangle)$ is true [since, trivially, $\vdash \forall w(w \in \langle\rangle \supset w \in \langle\rangle)$]. Conversely, from the assumed truth of $\forall w(w \in \beta'' \supset w \in \langle\rangle)$ one obtains by $\vdash \neg\exists w(w \in \langle\rangle)$ (a corollary of $PL_{set}4$, [a], via $PL_{set}2$): $\neg\exists w(w \in \beta'')$ is true, and hence by $PL_{set}Th3$: $\beta'' = \langle\rangle$ is true, and therefore by $PL_{set}3$, [a] (and the logic of identity): $\beta'' \in \langle\langle\rangle\rangle$ is true.

If β' is typographically $\langle\beta_1\rangle$, then, in view of Def2, what is in question, as a step towards obtaining $\vdash \beta'' \in \subseteq(\beta') \equiv \forall w(w \in \beta'' \supset w \in \beta')$, is $\vdash \beta'' \in \langle\langle\rangle, \langle\beta_1\rangle\rangle \equiv \forall w(w \in \beta'' \supset w \in \langle\beta_1\rangle)$; and this latter equivalence is certainly provable:

(**i**) From the assumed truth of $\beta'' \in \langle\langle\rangle, \langle\beta_1\rangle\rangle$ one obtains by $PL_{set}4$, [c]: $\beta'' = \langle\rangle \vee \beta'' = \langle\beta_1\rangle$ is true, and hence $\forall w(w \in \beta'' \supset w \in \langle\beta_1\rangle)$ is true [for $\forall w(w \in \langle\rangle \supset w \in \langle\beta_1\rangle)$ and $\forall w(w \in \langle\beta_1\rangle \supset w \in \langle\beta_1\rangle)$ are both true[9]];

(**ii**) from the assumed truth of $\forall w(w \in \beta'' \supset w \in \langle\beta_1\rangle)$ one obtains via $PL_{set}4$, [b], and $PL_{set}2$ [which together provide $\vdash \forall w(w \in \langle\beta_1\rangle \supset w = \beta_1)$]: $\forall w(w \in \beta'' \supset w = \beta_1)$ is true, and therefore:

[9] $\forall w(w \in \langle\rangle \supset w \in \langle\beta_1\rangle)$ is true in virtue of $\vdash \neg\exists w(w \in \langle\rangle)$ – a corollary of $PL_{set}4$, [a], via $PL_{set}2$.

$\beta'' = \langle\rangle \vee \beta'' = \langle\beta_1\rangle$ is true,[10] and therefore by $PL_{set}3$, [b] (and the logic of identity, etc.): $\beta'' \in \langle\langle\rangle, \langle\beta_1\rangle\rangle$ is true.

If β' is typographically $\langle\beta_1, \beta_2\rangle$, then, in view of Def2, what is in question [for obtaining $\vdash \beta'' \in \subseteq(\beta') \equiv \forall w(w \in \beta'' \supset w \in \beta')$] is $\vdash \beta'' \in \langle\langle\rangle, \langle\beta_1\rangle, \langle\beta_2\rangle, \langle\beta_1, \beta_2\rangle\rangle \equiv \forall w(w \in \beta'' \supset w \in \langle\beta_1, \beta_2\rangle)$; and this latter equivalence is certainly provable:

(**i**) From the assumed truth of $\beta'' \in \langle\langle\rangle, \langle\beta_1\rangle, \langle\beta_2\rangle, \langle\beta_1, \beta_2\rangle\rangle$ one obtains by $PL_{set}4$, [d]: $\beta'' = \langle\rangle \vee \beta'' = \langle\beta_1\rangle \vee \beta'' = \langle\beta_2\rangle \vee \beta'' = \langle\beta_1, \beta_2\rangle$ is true, and hence $\forall w(w \in \beta'' \supset w \in \langle\beta_1, \beta_2\rangle)$ is true [for $\forall w(w \in \langle\rangle \supset w \in \langle\beta_1, \beta_2\rangle)$, $\forall w(w \in \langle\beta_1\rangle \supset w \in \langle\beta_1, \beta_2\rangle)$, $\forall w(w \in \langle\beta_2\rangle \supset w \in \langle\beta_1, \beta_2\rangle)$, and $\forall w(w \in \langle\beta_1, \beta_2\rangle \supset w \in \langle\beta_1, \beta_2\rangle)$ are true[11]];

(**ii**) from the assumed truth of $\forall w(w \in \beta'' \supset w \in \langle\beta_1, \beta_2\rangle)$ one obtains via $PL_{set}4$, [c], and $PL_{set}2$: $\forall w(w \in \beta'' \supset w = \beta_1 \vee w = \beta_2)$ is true, and therefore: $\beta'' = \langle\rangle \vee \beta'' = \langle\beta_1\rangle \vee \beta'' = \langle\beta_2\rangle \vee \beta'' = \langle\beta_1, \beta_2\rangle$ is true,[12] and therefore by $PL_{set}3$, [c] (and the logic of identity, etc.): $\beta'' \in \langle\langle\rangle, \langle\beta_1\rangle, \langle\beta_2\rangle, \langle\beta_1, \beta_2\rangle\rangle$ is true.

If β' is typographically $\langle\beta_1, \ldots, \beta_n\rangle$ ($n \geq 3$), then the steps of deduction for reaching $\vdash \beta'' \in \subseteq(\langle\beta_1, \ldots, \beta_n\rangle) \equiv \forall w(w \in \beta'' \supset w \in \langle\beta_1, \ldots, \beta_n\rangle)$ are, *mutatis mutandis*, analogous to those already considered for reaching $\vdash \beta'' \in \subseteq(\langle\beta_1, \beta_2\rangle) \equiv \forall w(w \in \beta'' \supset w \in \langle\beta_1, \beta_2\rangle)$ [or, *in other words*, analogous to those for reaching $\vdash \beta'' \in \langle\langle\rangle, \langle\beta_1\rangle, \langle\beta_2\rangle, \langle\beta_1, \beta_2\rangle\rangle \equiv \forall w(w \in \beta'' \supset w \in \langle\beta_1, \beta_2\rangle)$]. Consider, *as a paradigm*, the demonstration of the simplest case: β' is typographically $\langle\beta_1, \beta_2, \beta_3\rangle$; one needs to establish, (**i**), $\vdash \beta'' \in \subseteq(\langle\beta_1, \beta_2, \beta_3\rangle) \supset \forall w(w \in \beta'' \supset w \in \langle\beta_1, \beta_2, \beta_3\rangle)$, and (**ii**), $\vdash \forall w(w \in \beta'' \supset w \in \langle\beta_1, \beta_2, \beta_3\rangle) \supset \beta'' \in \subseteq(\langle\beta_1, \beta_2, \beta_3\rangle)$. I omit going into

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[10] If $\forall w(w \in \beta'' \supset w = \beta_1)$ is true, then the pure finite set designated by β'' has no element, or has exactly one element, namely, the pure finite set designated by β_1. In the first case, $\beta'' = \langle\rangle$ is true because of $PL_{set}Th3$; in the second case $\beta'' = \langle\beta_1\rangle$ is true because, in that case, $\forall x(x \in \beta'' \equiv x \in \langle\beta_1\rangle)$ can be shown to be true and we have $PL_{set}Th4$ (*the principle of extensionality*).

[11] The deductive mechanics for showing this are obvious. For illustration, consider $\forall w(w \in \langle\beta_2\rangle \supset w \in \langle\beta_1, \beta_2\rangle)$: From the assumed truth of $\beta''' \in \langle\beta_2\rangle$ by $PL_{set}4$, [b]: $\beta''' = \beta_2$ is true; by $PL_{set}3$, [b]: $\vdash \beta_2 \in \langle\beta_1, \beta_2\rangle$; hence (by the logic of identity) $\beta''' \in \langle\beta_1, \beta_2\rangle$ is true. Therefore: $\vdash \beta''' \in \langle\beta_2\rangle \supset \beta''' \in \langle\beta_1, \beta_2\rangle$. Hence by $PL_{set}2$: $\vdash \forall w(w \in \langle\beta_2\rangle \supset w \in \langle\beta_1, \beta_2\rangle)$.

[12] If $\forall w(w \in \beta'' \supset w = \beta_1 \vee w = \beta_2)$ is true, then the pure finite set designated by β'' has no element; *or* has exactly one element, namely, the pure finite set designated by β_1, or the pure finite set designated by β_2;*or* has exactly two elements, namely, the pure finite set designated by β_1 (but not by β_2) and the pure finite set designated by β_2 (but not by β_1). In the first case, $\beta'' = \langle\rangle$ is true because of $PL_{set}Th3$; in the second case, $\beta'' = \langle\beta_1\rangle$ is true because, in that case, $\forall x(x \in \beta'' \equiv x \in \langle\beta_1\rangle)$ can be shown to be true and we have $PL_{set}Th4$; in the third case, $\beta'' = \langle\beta_2\rangle$ is true because, in that case, $\forall x(x \in \beta'' \equiv x \in \langle\beta_2\rangle)$ can be shown to be true and we have $PL_{set}Th4$; in the fourth case, $\beta'' = \langle\beta_1, \beta_2\rangle$ is true because, in that case, $\forall x(x \in \beta'' \equiv x \in \langle\beta_2, \beta_1\rangle)$ can be shown to be true and we have $PL_{set}Th4$.

the deductive steps for establishing (**i**): *the essential invariant structure* of these steps was presented in the deduction of $\vdash \beta'' \in \subseteq(<\beta_1, \beta_2>) \supset \forall w(w \in \beta'' \supset w \in <\beta_1, \beta>)$ [see, above, the deduction of $\vdash \beta'' \in <<>, <\beta_1>, <\beta_2>, <\beta_1, \beta_2>> \supset \forall w(w \in \beta'' \supset w \in <\beta_1, \beta_2>)$], and that invariant structure is easily adapted to the requirements of the new situation. But I *do* go into the deductive steps for establishing (**ii**) because, although *the essential invariant structure* of these latter steps was presented in the deduction of $\forall w(w \in \beta'' \supset w \in <\beta_1, \beta_2>) \supset \beta'' \in \subseteq(<\beta_1, \beta_2>)$ [see, above, the deduction of $\vdash \forall w(w \in \beta'' \supset w \in <\beta_1, \beta_2>) \supset \beta'' \in <<>, <\beta_1>, <\beta_2>, <\beta_1, \beta_2>>$], that invariant structure is not quite so easily adapted to the requirements of the new situation: From the assumed truth of $\forall w(w \in \beta'' \supset w \in <\beta_1, \beta_2, \beta_3>)$ one obtains via $PL_{set}4$, [d], and $PL_{set}2$: $\forall w(w \in \beta'' \supset w = \beta_1 \vee w = \beta_2 \vee w = \beta_3)$ is true, *and therefore*: $\beta'' = <> \vee \beta'' = <\beta_1> \vee \beta'' = <\beta_2> \vee \beta'' = <\beta_3> \vee \beta'' = <\beta_1, \beta_2> \vee \beta'' = <\beta_1, \beta_3> \vee \beta'' = <\beta_2, \beta_3> \vee \beta'' = <\beta_1, \beta_2, \beta_3>$ is true. The justification of *the foregoing deductive step* is the following (and compare footnotes 10 and 12):

If $\forall w(w \in \beta'' \supset w = \beta_1 \vee w = \beta_2 \vee w = \beta_3)$ is true, then the pure finite set designated by β'' has no element [case 1];
or has exactly one element, namely, the pure finite set designated by β_1 [case 2], or the pure finite set designated by β_2 [case 3], or the pure finite set designated by β_3 [case 4];
or has exactly two elements, namely, the pure finite set designated by β_1 and the pure finite set designated by β_2 [case 5], or the pure finite set designated by β_1 and the pure finite set designated by β_3 [case 6], or the pure finite set designated by β_2 and the pure finite set designated by β_3 [case 7];
or has exactly three elements, namely, the pure finite set designated by β_1 and the pure finite set designated by β_2 and the pure finite set designated by β_3 [case 8].

In case 1, $\beta'' = <>$ is true because of $PL_{set}Th3$; in case 2, $\beta'' = <\beta_1>$ is true because, in that case, $\forall x(x \in \beta'' \equiv x \in <\beta_1>)$ can be shown to be true and we have $PL_{set}Th4$; in case 3, $\beta'' = <\beta_2>$ is true because, in that case, $\forall x(x \in \beta'' \equiv x \in <\beta_2>)$ can be shown to be true and we have $PL_{set}Th4$; in case 4, $\beta'' = <\beta_3>$ is true because, in that case, $\forall x(x \in \beta'' \equiv x \in <\beta_3>)$ can be shown to be true and we have $PL_{set}Th4$; in case 5, $\beta'' = <\beta_1, \beta_2>$ is true because, in that case, $\forall x(x \in \beta'' \equiv x \in <\beta_1, \beta_2>)$ can be shown to be true and we have $PL_{set}Th4$; in case 6, $\beta'' = <\beta_1, \beta_3>$ is true because, in that case, $\forall x(x \in \beta'' \equiv x \in <\beta_1, \beta_3>)$ can be shown to be true and we have $PL_{set}Th4$; in case 7, $\beta'' = <\beta_2, \beta_3>$ is true because, in that case, $\forall x(x \in \beta'' \equiv x \in <\beta_2, \beta_3>)$ can be shown to be true and we have $PL_{set}Th4$; in case 8, $\beta'' = <\beta_1, \beta_2, \beta_3>$ is true because, in that case, $\forall x(x \in \beta'' \equiv x \in <\beta_1, \beta_2, \beta_3>)$ can be shown to be true and we have $PL_{set}Th4$.

But *how* – for example – is $\forall x(x \in \beta'' \equiv x \in <\beta_1, \beta_2, \beta_3>)$ shown to be true, presupposing, first, that $\forall w(w \in \beta'' \supset w = \beta_1 \vee w = \beta_2 \vee w = \beta_3)$ is true and, second, that the pure finite set designated by β'' has exactly three elements – namely (due to the first presupposition), the pure finite set designated by β_1 and the pure finite set designated by β_2 and the pure finite set designated by β_3? It is easy to establish the truth of $\forall x(x \in \beta'' \supset x \in <\beta_1, \beta_2, \beta_3>)$: the truth of $\forall x(x \in \beta'' \supset x = \beta_1 \vee x = \beta_2 \vee x = \beta_3)$ is presupposed and $\vdash \forall x(x = \beta_1 \vee x = \beta_2 \vee x = \beta_3 \supset x \in <\beta_1, \beta_2, \beta_3>)$ is a consequence of $PL_{set}3$, [c], the logic of identity, etc., and $PL_{set}2$; the two generalities together logically yield what is desired. What is not so easily established is the truth of *the converse*: the truth of $\forall x(x \in <\beta_1, \beta_2, \beta_3> \supset x \in \beta'')$. The full (the combined) presupposition is this: $\exists x_1 \exists x_2 \exists x_3[x_1 \in \beta'' \wedge x_2 \in \beta'' \wedge x_3 \in \beta'' \wedge x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge x_2 \neq x_3 \wedge \forall y(y \in \beta'' \supset y = x_1 \vee y = x_2 \vee y = x_3)] \wedge \forall w(w \in \beta'' \supset w = \beta_1 \vee w = \beta_2 \vee w = \beta_3)$ is true. Now assume that $x \in <\beta_1, \beta_2, \beta_3>$ is true; hence by $PL_{set}4$, [d]:[13] $x = \beta_1 \vee x = \beta_2 \vee x = \beta_3$ is true. It follows from the presupposition (within the existentially quantified formula): $(x_1 = \beta_1 \vee x_1 = \beta_2 \vee x_1 = \beta_3) \wedge (x_2 = \beta_1 \vee x_2 = \beta_2 \vee x_2 = \beta_3) \wedge (x_3 = \beta_1 \vee x_3 = \beta_2 \vee x_3 = \beta_3)$ is true, hence (in view of $x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge x_2 \neq x_3$): $x_1 = \beta_1 \wedge x_2 = \beta_2 \wedge x_3 = \beta_3 \vee x_1 = \beta_1 \wedge x_2 = \beta_3 \wedge x_3 = \beta_2 \vee x_1 = \beta_2 \wedge x_2 = \beta_1 \wedge x_3 = \beta_3 \vee x_1 = \beta_2 \wedge x_2 = \beta_3 \wedge x_3 = \beta_1 \vee x_1 = \beta_3 \wedge x_2 = \beta_1 \wedge x_3 = \beta_2 \vee x_1 = \beta_3 \wedge x_2 = \beta_2 \wedge x_3 = \beta_1$ is true. And therefore $x \in \beta''$ is true, no matter whether $x = \beta_1$ or $x = \beta_2$ or $x = \beta_3$ is true (it is already established that one of these three is true), *considering* that $x_1 \in \beta'' \wedge x_2 \in \beta'' \wedge x_3 \in \beta''$ is true. [For example: Let $x = \beta_1$ be true; hence, in virtue of the truth of *the long disjunction*, $x = x_1 \vee x = x_2 \vee x = x_3$ is true; and therefore $x \in \beta''$ is true because of the truth of $x_1 \in \beta'' \wedge x_2 \in \beta'' \wedge x_3 \in \beta''$.] The truth of $x \in <\beta_1, \beta_2, \beta_3> \supset x \in \beta''$ has now been deduced (on the basis of the presuppositions), and therefore also the truth of $\forall x(x \in <\beta_1, \beta_2, \beta_3> \supset x \in \beta'')$ (and see footnote 20 concerning $\forall$-*generalization*, which is the rule of logical inference that has just been employed and which is quite independent from $PL_{set}2$). The task set – above, in this paragraph – is completed.

No doubt: the truth of $\beta'' = <> \vee \beta'' = <\beta_1> \vee \beta'' = <\beta_2> \vee \beta'' = <\beta_3> \vee \beta'' = <\beta_1, \beta_2> \vee \beta'' = <\beta_1, \beta_3> \vee \beta'' = <\beta_2, \beta_3> \vee \beta'' = <\beta_1, \beta_2, \beta_3>$ has been logically deduced from the relevant assumption. From $\beta'' = <> \vee \beta'' = <\beta_1> \vee \beta'' = <\beta_2> \vee \beta'' = <\beta_3> \vee \beta'' = <\beta_1, \beta_2> \vee \beta'' = <\beta_1, \beta_3> \vee \beta'' = <\beta_2, \beta_3> \vee \beta'' = <\beta_1, \beta_2, \beta_3>$ being true, one obtains by $PL_{set}3$, [c] (and the logic of identity, etc.): $\beta'' \in <<>, <\beta_1>, <\beta_2>, <\beta_3>, <\beta_1, \beta_2>, <\beta_1, \beta_3>, <\beta_2, \beta_3>, <\beta_1, \beta_2, \beta_3>>$ is true, or in other words (because of Def3): $\beta'' \in \subseteq(<\beta_1, \beta_2, \beta_3>)$ is true. And therewith one has all that is needed for establishing (**ii**).

Having established (**i**) and (**ii**), $\vdash \beta'' \in \subseteq(<\beta_1, \beta_2, \beta_3>) \equiv \forall w(w \in \beta'' \supset w \in <\beta_1, \beta_2, \beta_3>)$ is established – the proof-procedure for it being a *paradigm* for showing the much more general $\vdash \beta'' \in \subseteq(<\beta_1, \ldots, \beta_n>) \equiv \forall w(w \in \beta'' \supset w \in <\beta_1, \ldots, \beta_n>)$ with $n \geq 3$. It is a *perfect* paradigm because for any $n \geq 3$ the steps of deduction are *essentially the same structurally*; all complications are solely due to how the essential invariant structure of the deductive steps must be filled in accordance with increasing n.

[13] Precisely speaking, it is the following particular consequence (via $PL_{set}2$) of $PL_{set}4$, [d], that justifies the deductive step to be taken: $\vdash \forall x(x \in <\beta_1, \beta_2, \beta_3> \supset x = \beta_1 \vee x = \beta_2 \vee x = \beta_3)$.

It is, therefore, safe to conclude: In *all* possible cases of what the pure closed bracket β' might typographically be one obtains: $\vdash \beta'' \in \subseteq(\beta') \equiv \forall w(w \in \beta'' \supset w \in \beta')$. This concludes the proof of the *Lemma*, and the entire proof (in view of what has been said at its beginning).

PL_{set}Th8: $\vdash \exists\upsilon'\forall\upsilon(\upsilon \in \upsilon' \equiv \sigma[\upsilon])$, $\sigma[\upsilon]$ being a predicate with finitely many instantiations [with respect to υ, and among the pure finite sets] and with υ' not in $\sigma[\upsilon]$ – which is *the principle of finite comprehension*.

Proof: $\sigma[\upsilon]$ – being a predicate with finitely many instantiations – either has no instantiation [with respect to υ, and among the pure finite sets], or exactly one instantiation, or exactly two instantiations, or exactly *n* instantiations for some $n \geq 3$.

In the first case, consider $<>$; clearly, $\forall\upsilon(\upsilon \in <> \equiv \sigma[\upsilon])$ is true in that case [in view of $\vdash \neg\exists z(z \in <>)$, already repeatedly employed] and hence $\exists\upsilon'\forall\upsilon(\upsilon \in \upsilon' \equiv \sigma[\upsilon])$ is true.

In the second case, the sole instantiation of $\sigma[\upsilon]$ is named [*logically necessarily* named, given the syntactic-semantic nature of L_{set}] by some pure closed bracket β. Consider $<\beta>$; clearly, $\forall\upsilon(\upsilon \in <\beta> \equiv \sigma[\upsilon])$ is true, and hence $\exists\upsilon'\forall\upsilon(\upsilon \in \upsilon' \equiv \sigma[\upsilon])$ is true.

In the third case, the two instantiations of $\sigma[\upsilon]$ are named [*logically necessarily* named, given the syntactic-semantic nature of L_{set}], by, respectively, a pure closed bracket β_1 and a pure closed bracket β_2. Consider $<\beta_1, \beta_2>$; clearly, $\forall\upsilon(\upsilon \in <\beta_1, \beta_2> \equiv \sigma[\upsilon])$ is true, and hence $\exists\upsilon'\forall\upsilon(\upsilon \in \upsilon' \equiv \sigma[\upsilon])$ is true.

In case the truth of $\forall\upsilon(\upsilon \in <\beta_1, \beta_2> \equiv \sigma[\upsilon])$ is not evident enough to readers: *For one thing*, $\vdash \forall\upsilon(\upsilon \in <\beta_1, \beta_2> \equiv \upsilon = \beta_1 \vee \upsilon = \beta_2)$ by PL_{set}4, [c], and PL_{set}3, [b] (and the logic of identity, etc.), and PL_{set}2; *for another thing*, $\forall\upsilon(\sigma[\upsilon] \equiv \upsilon = \beta_1 \vee \upsilon = \beta_2)$ is true due to assumption (under the third case). Therefore (logically): $\forall\upsilon(\upsilon \in <\beta_1, \beta_2> \equiv \sigma[\upsilon])$ is true.

In the fourth case, the *n* instances of $\sigma[\upsilon]$ are named [*logically necessarily* named, given the syntactic-semantic nature of L_{set}], by, respectively, a pure closed bracket β_1 and … and a pure closed bracket β_n. Consider $<\beta_1, …, \beta_n>$; clearly, $\forall\upsilon(\upsilon \in <\beta_1, …, \beta_n> \equiv \sigma[\upsilon])$ is true, and hence $\exists\upsilon'\forall\upsilon(\upsilon \in \upsilon' \equiv \sigma[\upsilon])$ is true.

In all possible cases, $\exists\upsilon'\forall\upsilon(\upsilon \in \upsilon' \equiv \sigma[\upsilon])$ is true, which means: $\vdash \exists\upsilon'\forall\upsilon(\upsilon \in \upsilon' \equiv \sigma[\upsilon])$.

That every pure finite set (every item in the universe of discourse) is named by some pure closed bracket is part of the syntactic-semantic nature of L_{set}. Is this part completely expressed by PL_{set}2, or is it not completely expressed by it? The answer is “Yes,” as emerges from the following considerations:

Consider a schema (of principles) which is similar to $PL_{set}2$ and just as correct, but, at least prima facie, also significantly different from $PL_{set}2$,

$PL_{set}2^*$: If for every pure closed bracket β: $\sigma[\beta]$ is true, then $\forall\upsilon\sigma[\upsilon]$ is true.

Consider, then, a specialization of this schema,

$PL_{set}2^{**}$: If for every pure closed bracket β: $\beta \neq x$ is true, then $\forall y(y \neq x)$ is true – where x is a pure finite set.

Since the schema $PL_{set}2^*$ is correct, its specialization $PL_{set}2^{**}$ is just as correct. Now, $\forall y(y \neq x)$ is not true; it therefore follows according to $PL_{set}2^{**}$: For some pure closed bracket β: $\beta \neq x$ is not true; in other words: For some pure closed bracket $\beta = x$ is true; in other words: Some pure closed bracket β names [or: designates] x.

Thus, $PL_{set}2^*$ captures the part of the syntactic-semantic nature of L_{set} which is in question, because $PL_{set}2^{**}$, a specialization of $PL_{set}2^*$, does so. But $PL_{set}2$ captures that part no less than $PL_{set}2^*$. To see this, we must merely take into account that $PL_{set}2^{**}$ follows not only from $PL_{set}2^*$ but also from $PL_{set}2$; this is so because

(a) if $\beta \neq x$ is true, then $\beta \neq x$ is logically true, i.e., if $\beta \neq x$ is true, then $\vdash \beta \neq x$ – no matter which pure closed bracket β is; and

(b) if $\forall y(y \neq x)$ is logically true, then $\forall y(y \neq x)$ is true, i.e., if $\vdash \forall y(y \neq x)$, then $\forall y(y \neq x)$ is true.

It is indubitable that (b) is correct, and it is just as indubitable that (a) is correct: given the syntactic-semantic nature of L_{set}, β *with logical necessity* names the pure finite set it names, and *with logical necessity* it does not name any pure finite set it does not name. And from the following specialization of $PL_{set}2$,

$PL_{set}2'$: If for every pure closed bracket β: $\vdash \beta \neq x$, then $\vdash \forall y(y \neq x)$,

we get, via (a) and (b), $PL_{set}2^{**}$.

$PL_{set}Th9$: $\vdash \forall x(x \in <\beta_1> \equiv x = \beta_1)$; $\vdash \forall x(x \in <\beta_1, \beta_2> \equiv x = \beta_1 \vee x = \beta_2)$; $\vdash \forall x(x \in <\beta_1, \ldots, \beta_n> \equiv x = \beta_1 \vee \ldots \vee x = \beta_n)$, for $n \geq 3$

Proof: $PL_{set}Th9$ is a matter of $PL_{set}4$, $PL_{set}3$, and $PL_{set}2$. Consider, as a proof-paradigm, the proof of $\vdash \forall x(x \in <\beta_1, \beta_2> \equiv x = \beta_1 \vee x = \beta_2)$ [cf. the principle just considered in a note within the previous proof]. The proof of $\vdash \forall x(x \in <\beta_1, \beta_2> \equiv x = \beta_1 \vee x = \beta_2)$ is a part of the proof of $PL_{set}Th5$.

$PL_{set}Th10$: $\vdash \forall x(x \neq <> \supset \exists y(y \in x \wedge \neg\exists z(z \in x \wedge z \in y)))$ – which is *the principle of foundation*.

Proof: The proposition to be proven is proven by first proving

Lemma: $\vdash \beta \neq <> \supset \exists y(y \in \beta \wedge \neg\exists z(z \in \beta \wedge z \in y))$. This proven, one obtains $\vdash \forall x(x \neq <> \supset \exists y(y \in x \wedge \neg\exists z(z \in x \wedge z \in y)))$ by $PL_{set}2$ (since β represents just *any* pure closed bracket).

Proof of the Lemma: β is typographically either “$<>$”, or $<\beta_1>$ or $<\beta_1, \beta_2>$ or $<\beta_1, \ldots, \beta_n>$ for some pure closed brackets $\beta_1, \ldots, \beta_n$ ($n \geq 3$).

First case: β is typographically “$<>$”, and, therefore, $\beta \neq <> \supset \exists y(y \in \beta \wedge \neg\exists z(z \in \beta \wedge z \in y))$ amounts typographically to $<> \neq <> \supset \exists y(y \in <> \wedge \neg\exists z(z \in <> \wedge z \in y))$ – and $\vdash <> \neq <> \supset \exists y(y \in <> \wedge \neg\exists z(z \in <> \wedge z \in y))$ is a triviality.

Second case: β is typographically $<\beta_1>$, and, therefore, $\beta \neq <> \supset \exists y(y \in \beta \wedge \neg\exists z(z \in \beta \wedge z \in y))$ amounts typographically to $<\beta_1> \neq <> \supset \exists y(y \in <\beta_1> \wedge \neg\exists z(z \in <\beta_1> \wedge z \in y))$, which in turn amounts logically to $\exists y(y \in <\beta_1> \wedge \neg\exists z(z \in <\beta_1> \wedge z \in y))$ [since $\vdash <\beta_1> \neq <>$, considering $PL_{set}3$, [a], and $PL_{set}4$, [a]]. And we have: $\vdash \exists y(y \in <\beta_1> \wedge \neg\exists z(z \in <\beta_1> \wedge z \in y))$ because of (**i**) $\vdash \beta_1 \in <\beta_1>$ and (**ii**) $\vdash \neg\exists z(z \in <\beta_1> \wedge z \in \beta_1)$:

(**i**) $\vdash \beta_1 \in <\beta_1>$ by $PL_{set}3$, [a].

(**ii**) Suppose $\beta' \in <\beta_1> \wedge \beta' \in \beta_1$ is true; hence by $PL_{set}4$, [b]: $\beta' = \beta_1 \wedge \beta' \in \beta_1$ is true, and hence $\beta_1 \in \beta_1$ is true – contradicting $PL_{set}Th1$. Therefore: $\vdash \neg(\beta' \in <\beta_1> \wedge \beta' \in \beta_1)$, hence by $PL_{set}2$: $\vdash \forall z\neg(z \in <\beta_1> \wedge z \in \beta_1)$, and hence $\vdash \neg\exists z(z \in <\beta_1> \wedge z \in \beta_1)$.

(**i**) and (**ii**) together yield (logically) $\vdash \exists y(y \in <\beta_1> \wedge \neg\exists z(z \in <\beta_1> \wedge z \in y))$.

Third case: β is typographically $<\beta_1, \beta_2>$, and, therefore, $\beta \neq <> \supset \exists y(y \in \beta \wedge \neg\exists z(z \in \beta \wedge z \in y))$ amounts typographically to $<\beta_1, \beta_2> \neq <> \supset \exists y(y \in <\beta_1, \beta_2> \wedge \neg\exists z(z \in <\beta_1, \beta_2> \wedge z \in y))$, which in turn amounts logically to $\exists y(y \in <\beta_1, \beta_2> \wedge \neg\exists z(z \in <\beta_1, \beta_2> \wedge z \in y))$ [since $\vdash <\beta_1, \beta_2> \neq <>$ by $PL_{set}3$, [b], and $PL_{set}4$, [a]]. And we have: $\vdash \exists y(y \in <\beta_1, \beta_2> \wedge \neg\exists z(z \in <\beta_1, \beta_2> \wedge z \in y))$ because (**i**) $\vdash \beta_1 \in <\beta_1, \beta_2> \wedge \beta_2 \in <\beta_1, \beta_2>$ and (**ii**) $\vdash \neg\exists z(z \in <\beta_1, \beta_2> \wedge z \in \beta_1) \vee \neg\exists z(z \in <\beta_1, \beta_2> \wedge z \in \beta_2)$:

(**i**) $\vdash \beta_1 \in <\beta_1, \beta_2> \wedge \beta_2 \in <\beta_1, \beta_2>$ by $PL_{set}3$, [b].

(**ii**) Suppose $\exists z(z \in <\beta_1, \beta_2> \wedge z \in \beta_1) \wedge \exists z(z \in <\beta_1, \beta_2> \wedge z \in \beta_2)$ is true; hence $\exists z((z = \beta_1 \vee z = \beta_2) \wedge z \in \beta_1) \wedge \exists z((z = \beta_1 \vee z = \beta_2) \wedge z \in \beta_2)$ is true [due to $\vdash \forall z(z \in <\beta_1, \beta_2> \equiv z = \beta_1 \vee z = \beta_2)$ – $PL_{set}Th9$], hence (logically) $(\exists z(z = \beta_1 \wedge z \in \beta_1) \vee \exists z(z = \beta_2 \wedge z \in \beta_1)) \wedge (\exists z(z = \beta_1 \wedge z \in \beta_2) \vee \exists z(z = \beta_2 \wedge z \in \beta_2))$ is true, and hence (logically) $(\beta_1 \in \beta_1 \vee \beta_2 \in \beta_1) \wedge (\beta_1 \in \beta_2 \vee \beta_2 \in \beta_2)$ is true. Now, $\vdash \beta_1 \notin \beta_1$ and $\vdash \beta_2 \notin \beta_2$ because of $PL_{set}Th1$. And therefore – under the initial assumption that $\exists z(z \in <\beta_1, \beta_2> \wedge z \in \beta_1) \wedge \exists z(z \in <\beta_1, \beta_2> \wedge z \in \beta_2)$ is true – $\beta_1 \in \beta_2 \wedge \beta_2 \in \beta_1$ is true, which is, however, *impossible* [that is, $\vdash \neg(\beta_1 \in \beta_2 \wedge \beta_2 \in \beta_1)$]. Thus: $\vdash \neg\exists z(z \in <\beta_1, \beta_2> \wedge z \in \beta_1) \vee \neg\exists z(z \in <\beta_1, \beta_2> \wedge z \in \beta_2)$.

(**i**) and (**ii**) together yield (logically) $\vdash \exists y(y \in <\beta_1, \beta_2> \wedge \neg\exists z(z \in <\beta_1, \beta_2> \wedge z \in y))$.

Why is the truth of $\beta_1 \in \beta_2 \wedge \beta_2 \in \beta_1$ *impossible*? – If $\beta_1 \in \beta_2$ is true, then there is a pure closed bracket $\beta+$ such that $\beta+ = \beta_2$ is true and in which (i.e., in $\beta+$) an occurrence of β_1 is an immediate constituent-occurrence [if $\beta_1 \in \beta_2$ is true, then – in view of $PL_{set}4$ and in view of the possible typographic shapes of β_2 that are consistent with the truth of $\beta_1 \in \beta_2$ –the pure finite set designated by β_1 must be identical with the pure finite set designated by some immediate constituent-occurrence in β_2;[14] replace that constituent-occurrence by an occurrence of β_1; the result is $\beta+$, with $\beta+ = \beta_2$ being true via substitution of identicals *salva veritate*]; thus β_1 occurs in $\beta+$, and one may represent $\beta+$ by $<\ldots\beta_1\ldots>$. Now, if $\beta_2 \in \beta_1$ is also true, then [by the same argumentation as has just been advanced] there is also a pure closed bracket $\beta{++}$ such that $\beta{++} = \beta_1$ is true and in which (i.e., in $\beta{++}$) an occurrence of β_2 is an immediate constituent-occurrence; thus β_2 occurs in $\beta{++}$, and one may represent $\beta{++}$ by $<\ldots\beta_2\ldots>$. Thus, if both $\beta_1 \in \beta_2$ and $\beta_2 \in \beta_1$ are true, $\beta_1 = \beta{++} = <\ldots\beta_2\ldots> = <\ldots\beta+\ldots> = <\ldots<\ldots\beta_1\ldots>\ldots>$ is true, hence $\beta_1 = <\ldots<\ldots\beta_1\ldots>\ldots>$ is true – contradicting $PL_{set}1$, [c]. – Obviously, the above argumentation is not limited to a particular β_1 and a particular β_2 but shows quite generally that the truth of $\beta' \in \beta'' \wedge \beta'' \in \beta'$ is *impossible*, no matter which pure closed brackets are concerned.

Fourth case: β is typographically $<\beta_1, \ldots, \beta_n>$ [$n \geq 3$], and, therefore, $\beta \neq <> \supset \exists y(y \in \beta \wedge \neg\exists z(z \in \beta \wedge z \in y))$ amounts typographically to $<\beta_1, \ldots, \beta_n> \neq <> \supset \exists y(y \in <\beta_1, \ldots, \beta_n> \wedge \neg\exists z(z \in <\beta_1, \ldots, \beta_n> \wedge z \in y))$, which in turn amounts logically to $\exists y(y \in <\beta_1, \ldots, \beta_n> \wedge \neg\exists z(z \in <\beta_1, \ldots, \beta_n> \wedge z \in y))$ [since $\vdash <\beta_1, \ldots, \beta_n> \neq <>$ by $PL_{set}3$, [c], and $PL_{set}4$, [a]]. And we have: $\vdash \exists y(y \in <\beta_1, \ldots, \beta_n> \wedge \neg\exists z(z \in <\beta_1, \ldots, \beta_n> \wedge z \in y))$ because (**i**) $\vdash \beta_1 \in <\beta_1, \ldots, \beta_n> \wedge \ldots \wedge \beta_n \in <\beta_1, \ldots, \beta_n>$ and (**ii**) $\vdash \neg\exists z(z \in <\beta_1, \ldots, \beta_n> \wedge z \in \beta_1) \vee \ldots \vee \neg\exists z(z \in <\beta_1, \ldots, \beta_n> \wedge z \in \beta_n)$:

(**i**) $\vdash \beta_1 \in <\beta_1, \ldots, \beta_n> \wedge \ldots \wedge \beta_n \in <\beta_1, \ldots, \beta_n>$ by $PL_{set}3$, [c].

(**ii**) Suppose $\exists z(z \in <\beta_1, \ldots, \beta_n> \wedge z \in \beta_1) \wedge \ldots \wedge \exists z(z \in <\beta_1, \ldots, \beta_n> \wedge z \in \beta_n)$ is true; hence $\exists z((z = \beta_1 \vee \ldots \vee z = \beta_n) \wedge z \in \beta_1) \wedge \ldots \wedge \exists z((z = \beta_1 \vee \ldots \vee z = \beta_n) \wedge z \in \beta_n)$ is true [due to $\vdash \forall z(z \in <\beta_1,\ldots, \beta_n> \equiv z = \beta_1 \vee \ldots z = \beta_n)$ – $PL_{set}9Th$], hence (logically) $(\exists z(z = \beta_1 \wedge z \in \beta_1) \vee \ldots \vee \exists z(z = \beta_n \wedge z \in \beta_1)) \wedge \ldots \wedge (\exists z(z = \beta_1 \wedge z \in \beta_n) \vee \ldots \vee \exists z(z = \beta_n \wedge z \in \beta_n))$ is true, and hence (logically) $(\beta_1 \in \beta_1 \vee \ldots \vee \beta_n \in \beta_1) \wedge \ldots \wedge (\beta_1 \in \beta_n \vee \ldots \vee \beta_n \in \beta_n)$ is true.

For $n = 3$, the (hypothetical) truth of $(\beta_1 \in \beta_1 \vee \ldots \vee \beta_n \in \beta_1) \wedge \ldots \wedge (\beta_1 \in \beta_n \vee \ldots \vee \beta_n \in \beta_n)$ amounts to the truth of

$(\beta_1 \in \beta_1 \vee \beta_2 \in \beta_1 \vee \beta_3 \in \beta_1) \wedge (\beta_1 \in \beta_2 \vee \beta_2 \in \beta_2 \vee \beta_3 \in \beta_2) \wedge (\beta_1 \in \beta_3 \vee \beta_2 \in \beta_3 \vee \beta_3 \in \beta_3)$, in other words (because of $PL_{set}Th1$): to the truth of

$(\beta_2 \in \beta_1 \vee \beta_3 \in \beta_1) \wedge (\beta_1 \in \beta_2 \vee \beta_3 \in \beta_2) \wedge (\beta_1 \in \beta_3 \vee \beta_2 \in \beta_3)$,

hence to the truth of

[14] Any occurrence of a pure closed bracket designates the same pure finite set as the bracket itself.

$[(\beta_2 \in \beta_1 \vee \beta_3 \in \beta_1) \wedge \beta_1 \in \beta_2 \vee (\beta_2 \in \beta_1 \vee \beta_3 \in \beta_1) \wedge \beta_3 \in \beta_2] \wedge (\beta_1 \in \beta_3 \vee \beta_2 \in \beta_3)$,

hence to the truth of

$[\beta_2 \in \beta_1 \wedge \beta_1 \in \beta_2 \vee \beta_3 \in \beta_1 \wedge \beta_1 \in \beta_2 \vee \beta_2 \in \beta_1 \wedge \beta_3 \in \beta_2 \vee \beta_3 \in \beta_1 \wedge \beta_3 \in \beta_2] \wedge (\beta_1 \in \beta_3 \vee \beta_2 \in \beta_3)$,

hence to the truth of

$[\beta_2 \in \beta_1 \wedge \beta_1 \in \beta_2 \wedge \beta_1 \in \beta_3 \vee \beta_3 \in \beta_1 \wedge \beta_1 \in \beta_2 \wedge \beta_1 \in \beta_3 \vee \beta_2 \in \beta_1 \wedge \beta_3 \in \beta_2 \wedge \beta_1 \in \beta_3 \vee \beta_3 \in \beta_1 \wedge \beta_3 \in \beta_2 \wedge \beta_1 \in \beta_3] \vee [\beta_2 \in \beta_1 \wedge \beta_1 \in \beta_2 \wedge \beta_2 \in \beta_3 \vee \beta_3 \in \beta_1 \wedge \beta_1 \in \beta_2 \wedge \beta_2 \in \beta_3 \vee \beta_2 \in \beta_1 \wedge \beta_3 \in \beta_2 \wedge \beta_2 \in \beta_3 \vee \beta_3 \in \beta_1 \wedge \beta_3 \in \beta_2 \wedge \beta_2 \in \beta_3]$,

hence – after elimination of the disjuncts that contain an *impossibility* of the form $\beta' \in \beta'' \wedge \beta'' \in \beta'$ – to the truth of

$\beta_2 \in \beta_1 \wedge \beta_3 \in \beta_2 \wedge \beta_1 \in \beta_3 \vee \beta_3 \in \beta_1 \wedge \beta_1 \in \beta_2 \wedge \beta_2 \in \beta_3$.

It is, however, *impossible* that $\beta_2 \in \beta_1 \wedge \beta_3 \in \beta_2 \wedge \beta_1 \in \beta_3 \vee \beta_3 \in \beta_1 \wedge \beta_1 \in \beta_2 \wedge \beta_2 \in \beta_3$ is true.

For $n = 4$, the (hypothetical) truth of $(\beta_1 \in \beta_1 \vee \ldots \vee \beta_n \in \beta_1) \wedge \ldots \wedge (\beta_1 \in \beta_n \vee \ldots \vee \beta_n \in \beta_n)$ *ultimately* – *after* the (relevant adaptation of the) procedure described for $n = 3$, using the ∈-*impossibilities* already established [established at and before stage $n = 3$, namely: It is impossible that sentences of the following forms $\beta' \in \beta'$ and $\beta' \in \beta'' \wedge \beta'' \in \beta'$ and $\beta'' \in \beta' \wedge \beta''' \in \beta'' \wedge \beta' \in \beta''' \vee \beta''' \in \beta' \wedge \beta' \in \beta'' \wedge \beta'' \in \beta'''$ are true[15]] – comes down to the truth of $[\beta_2 \in \beta_1 \wedge \beta_3 \in \beta_2 \wedge \beta_4 \in \beta_3 \wedge \beta_1 \in \beta_4] \vee [\beta_4 \in \beta_1 \wedge \beta_1 \in \beta_2 \wedge \beta_2 \in \beta_3 \wedge \beta_3 \in \beta_4]$ (square brackets inserted for easier readability).

It is, however, *impossible* that $[\beta_2 \in \beta_1 \wedge \beta_3 \in \beta_2 \wedge \beta_4 \in \beta_3 \wedge \beta_1 \in \beta_4] \vee [\beta_4 \in \beta_1 \wedge \beta_1 \in \beta_2 \wedge \beta_2 \in \beta_3 \wedge \beta_3 \in \beta_4]$ is true.

And so on (*mutatis mutandis*) for $n = 5$, $n = 6$, $n = 7$, etc., at each stage making use of all the ∈-*impossibilities* that have already been established as impossibilities – for all pure closed brackets – before coming to that stage. The ultimate step at each stage is always (for $n \geq 4$) the same (*generically*):

[15] Note that the truth of $\beta'' \in \beta' \wedge \beta''' \in \beta'' \wedge \beta' \in \beta''' \vee \beta''' \in \beta' \wedge \beta' \in \beta'' \wedge \beta'' \in \beta'''$ is *impossible* – or in other words: $\vdash \neg(\beta'' \in \beta' \wedge \beta''' \in \beta'' \wedge \beta' \in \beta''' \vee \beta''' \in \beta' \wedge \beta' \in \beta'' \wedge \beta'' \in \beta''')$ – if and only if both the truth of $\beta'' \in \beta' \wedge \beta''' \in \beta'' \wedge \beta' \in \beta'''$ and the truth of $\beta''' \in \beta' \wedge \beta' \in \beta'' \wedge \beta'' \in \beta'''$ is *impossible*, or in other words: iff $\vdash \neg(\beta'' \in \beta' \wedge \beta''' \in \beta'' \wedge \beta' \in \beta''')$ and $\vdash \neg(\beta''' \in \beta' \wedge \beta' \in \beta'' \wedge \beta'' \in \beta''')$.

It is, however, *impossible* that $[\beta_2 \in \beta_1 \wedge \beta_3 \in \beta_2 \ldots \wedge \beta_n \in \beta_{n-1} \wedge \beta_1 \in \beta_n] \vee [\beta_n \in \beta_1 \wedge \beta_1 \in \beta_2 \wedge \beta_2 \in \beta_3 \ldots \wedge \beta_{n-1} \in \beta_n]$ is true [in addition to the above-established "It is, however, *impossible* that $\beta_2 \in \beta_1 \wedge \beta_3 \in \beta_2 \wedge \beta_1 \in \beta_3 \vee \beta_3 \in \beta_1 \wedge \beta_1 \in \beta_2 \wedge \beta_2 \in \beta_3$ is true"].

Thus, assuming the truth of $\exists z(z \in <\beta_1, \ldots, \beta_n> \wedge z \in \beta_1) \wedge \ldots \wedge \exists z(z \in <\beta_1, \ldots, \beta_n> \wedge z \in \beta_n)$ leads via the consequent hypothetical truth of $(\beta_1 \in \beta_1 \vee \ldots \vee \beta_n \in \beta_1) \wedge \ldots \wedge (\beta_1 \in \beta_n \vee \ldots \vee \beta_n \in \beta_n)$ for all $n \geq 3$ to an $\in$-*impossibility*. Therefore: $\vdash \neg\exists z(z \in <\beta_1, \ldots, \beta_n> \wedge z \in \beta_1) \vee \ldots \vee \neg\exists z(z \in <\beta_1, \ldots, \beta_n> \wedge z \in \beta_n)$.
(**i**) and (**ii**) together yield (logically) $\vdash \exists y(y \in <\beta_1, \ldots, \beta_n> \wedge \neg\exists z(z \in <\beta_1, \ldots, \beta_n> \wedge z \in y))$, for all $n \geq 3$.

Thus, in all possible cases (of the typographic shape of β) we have $\vdash \beta \neq <> \supset \exists y(y \in \beta \wedge \neg\exists z(z \in \beta \wedge z \in y))$. This concludes the proof of the *Lemma*, and thereby the entire proof (in view of what has been said at its beginning).

But why is not only the truth of $\beta_1 \in \beta_1$ and of $\beta_1 \in \beta_2 \wedge \beta_2 \in \beta_1$ *impossible*, as we have already seen, no matter which pure closed brackets β_1 and β_2 may be, but *impossible* also the truth of $\beta_2 \in \beta_1 \wedge \beta_3 \in \beta_2 \wedge \beta_1 \in \beta_3 \vee \beta_3 \in \beta_1 \wedge \beta_1 \in \beta_2 \wedge \beta_2 \in \beta_3$ and of $[\beta_2 \in \beta_1 \wedge \beta_3 \in \beta_2 \wedge \beta_4 \in \beta_3 \wedge \beta_1 \in \beta_4] \vee [\beta_4 \in \beta_1 \wedge \beta_1 \in \beta_2 \wedge \beta_2 \in \beta_3 \wedge \beta_3 \in \beta_4]$, and quite generally the truth of $[\beta_2 \in \beta_1 \wedge \beta_3 \in \beta_2 \ldots \wedge \beta_n \in \beta_{n-1} \wedge \beta_1 \in \beta_n] \vee [\beta_n \in \beta_1 \wedge \beta_1 \in \beta_2 \wedge \beta_2 \in \beta_3 \ldots \wedge \beta_{n-1} \in \beta_n]$, for all $n > 3$? Why are all sentences of any of these forms $\in$-*impossibilities*? The all-sufficient answer is that they all contradict $PL_{set}1$, [c]: $\vdash \beta' \neq <\ldots\beta'\ldots>$. For seeing this, it may help to visualize *the situations of circularity* that all those sentences describe:

(A) Distribute the *n* pure closed brackets $\beta_1, \ldots, \beta_n$ clockwise in their given (but arbitrary) order on the periphery of a dial – for merely esthetic reasons with one and the same, or roughly one and the same, distance (on the periphery) of each pure closed bracket from its predecessor in the clockwise direction, like this: If $n = 1$, then β_1 at 12 o'clock; if $n = 2$, then β_1 at 12 o'clock and β_2 at 6 o'clock; if $n = 3$, then β_1 at 12 o'clock, β_2 at 4 o'clock, and β_3 at 8 o'clock; if $n = 4$, then β_1 at 12 o'clock, β_2 at 3 o'clock, β_3 at 6 o'clock, and β_4 at 9 o'clock; and so on.

(B) Draw, along the periphery of the dial, clockwise-directed bent arrows from each pure closed bracket to the next one following it in the clockwise direction on the periphery (if $n = 1$, then the next pure closed bracket following β_1 in the clockwise direction on the periphery is β_1).

(C) Also draw, along the periphery of the dial, *counter*clockwise-directed bent arrows from one pure closed bracket to the next one following it in the *counter*clockwise direction on the periphery (if $n = 1$, then the next pure closed bracket following β_1 in the *counter*clockwise direction on the periphery is β_1).

The arrows, of course, represent the $\in$-relation. Note that for $n = 1$ the $\in$-chain that is represented by $\beta_1 \rightarrow \beta_1$ (clockwise) is in no way different from the $\in$-chain that is represented by $\beta_1 \leftarrow \beta_1$ (counterclockwise); it's just $\beta_1 \in \beta_1$. The same is true for $n = 2$: the $\in$-chain that is represented by $\beta_1 \rightarrow \beta_2 \rightarrow \beta_1$ (clockwise) is in no way different from the $\in$-chain that is represented by $\beta_1 \leftarrow \beta_2 \leftarrow \beta_1$ (counterclockwise); it's just $\beta_1 \in \beta_2 \wedge \beta_2 \in \beta_1$. For

$n = 3$, however, the $\in$-chain that is represented by $\beta_1 \to \beta_2 \to \beta_3 \to \beta_1$ (clockwise) is not the same $\in$-chain as the one that is represented by $\beta_1 \leftarrow \beta_2 \leftarrow \beta_3 \leftarrow \beta_1$ (counterclockwise), because $\beta_2 \in \beta_1 \wedge \beta_3 \in \beta_2 \wedge \beta_1 \in \beta_3$ (which is represented by $\beta_1 \leftarrow \beta_2 \leftarrow \beta_3 \leftarrow \beta_1$) says something else than $\beta_3 \in \beta_1 \wedge \beta_1 \in \beta_2 \wedge \beta_2 \in \beta_3$ (which is represented by $\beta_1 \to \beta_2 \to \beta_3 \to \beta_1$).[16] *Mutatis mutandis* the same is true for any $n > 3$.

Now, consider as a paradigm the situation for $n = 3$. The truth of $\beta_2 \in \beta_1 \wedge \beta_3 \in \beta_2 \wedge \beta_1 \in \beta_3 \vee \beta_3 \in \beta_1 \wedge \beta_1 \in \beta_2 \wedge \beta_2 \in \beta_3$ is – it is asserted – *impossible*. What must be shown (to make good this assertion) is both $\vdash \neg(\beta_2 \in \beta_1 \wedge \beta_3 \in \beta_2 \wedge \beta_1 \in \beta_3)$ and $\vdash \neg(\beta_3 \in \beta_1 \wedge \beta_1 \in \beta_2 \wedge \beta_2 \in \beta_3)$.[17] Suppose, *first*, $\beta_2 \in \beta_1 \wedge \beta_3 \in \beta_2 \wedge \beta_1 \in \beta_3$ is true; hence $\beta_1 = <\ldots\beta_2\ldots>$ is true, and $\beta_2 = <\ldots\beta_3\ldots>$, and $\beta_3 = <\ldots\beta_1\ldots>$;[18] hence (by substitution of identicals) $\beta_1 = <\ldots\beta_2\ldots>$ and $\beta_2 = <\ldots<\ldots\beta_1\ldots>\ldots>$ are true, and therefore (again by substitution of identicals) $\beta_1 = <\ldots<\ldots<\ldots\beta_1\ldots>\ldots>\ldots>$ is true – contradicting $PL_{set}1$, [c]. This establishes $\vdash \neg(\beta_2 \in \beta_1 \wedge \beta_3 \in \beta_2 \wedge \beta_1 \in \beta_3)$. Suppose, *second*, $\beta_3 \in \beta_1 \wedge \beta_1 \in \beta_2 \wedge \beta_2 \in \beta_3$ is true; hence $\beta_1 = <\ldots\beta_3\ldots>$ is true, and $\beta_2 = <\ldots\beta_1\ldots>$, and $\beta_3 = <\ldots\beta_2\ldots>$; hence $\beta_2 = <\ldots\beta_1\ldots>$ and $\beta_1 = <\ldots<\ldots\beta_2\ldots>\ldots>$ are true, and therefore $\beta_1 = <\ldots<\ldots<\ldots\beta_1\ldots>\ldots>\ldots>$ is true – contradicting $PL_{set}1$, [c]. This establishes $\vdash \neg(\beta_3 \in \beta_1 \wedge \beta_1 \in \beta_2 \wedge \beta_2 \in \beta_3)$.

<u>PL_{set}Th11: $\vdash \forall x(\exists z'(z' \in x) \wedge \forall u(u \in x \supset \exists z'(z' \in u)) \supset \exists z \forall u(u \in x \supset \exists z'(z' \in u \wedge z' \in z)))$</u> – which is *the principle of choice*.

Proof: The proposition to be proven follows by $PL_{set}2$ from

Lemma: $\vdash \exists z'(z' \in \beta) \wedge \forall u(u \in \beta \supset \exists z'(z' \in u)) \supset \exists z \forall u(u \in \beta \supset \exists z'(z' \in u \wedge z' \in z))$.

Proof of the Lemma: Assume that $\exists z'(z' \in \beta) \wedge \forall u(u \in \beta \supset \exists z'(z' \in u))$ is true; hence β is typographically $<\beta_1>$ or $<\beta_1, \beta_2>$ or $<\beta_1, \ldots, \beta_n>$ for some pure closed brackets $\beta_1, \ldots, \beta_n$ with $n \geq 3$.

Since $\exists z'(z' \in \beta)$ is true – according to assumption –, $\beta \neq <>$ is true because of PL_{set}Th3.2, and hence β cannot be typographically "<>".

First case: β is typographically $<\beta_1>$; hence β_1 is typographically [identical to] $<<\ldots>_{1\beta 1}\ldots>$ [the notation will be explained immediately].

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[16] Note that $\beta_2 \in \beta_1 \wedge \beta_3 \in \beta_2 \wedge \beta_1 \in \beta_3$ is logically equivalent to $\beta_1 \in \beta_3 \wedge \beta_3 \in \beta_2 \wedge \beta_2 \in \beta_1$, while $\beta_3 \in \beta_1 \wedge \beta_1 \in \beta_2 \wedge \beta_2 \in \beta_3$ is logically equivalent to $\beta_1 \in \beta_2 \wedge \beta_2 \in \beta_3 \wedge \beta_3 \in \beta_1$.

[17] Cf. footnote 15.

[18] Let the justification of the truth of these three identity-sentences be paradigmatically given for the first one of them: β_1 is some pure closed bracket *with* immediate constituent-occurrences (otherwise $\beta_2 \in \beta_1$ could not be true – due to PLset4, [a]). Since $\beta_2 \in \beta_1$ is true, it then follows (because of PL_{set}Th9, in view of the possible typographic shapes of β_1) that there is a pure closed bracket $\beta+$ such that $\beta_2 = \beta+$ is true *and* such that $\beta+$ is an immediate constituent-occurrence in β_1. $<\ldots\beta_2\ldots>$ is what results from β_1 by replacing that occurrence of $\beta+$ by an occurrence of β_2. Then, in view of the truth of $\beta_2 = \beta+$, $\beta_1 = <\ldots\beta_2\ldots>$ results as true via substitution of identicals *salva veritate* (which is a law of first-order predicate-logic-with identity-and-definite-descriptions).

According to assumption, $\forall u(u \in \beta \supset \exists z'(z' \in u))$ is true, hence (under the first case) $\forall u(u \in <\beta_1> \supset \exists z'(z' \in u))$ is true, hence – because of $\vdash \beta_1 \in <\beta_1>$ ($PL_{set}3$, [a]) – we have: $\exists z'(z' \in \beta_1)$ is true. $<...>_{1\beta 1}$ represents *the first immediate constituent-occurrence in* β_1.[19]

Second case: β is typographically $<\beta_1, \beta_2>$; hence β_1 is typographically $<<...>_{1\beta 1}...>$, and β_2 is typographically $<<...>_{1\beta 2}...>$.

According to assumption, $\forall u(u \in \beta \supset \exists z'(z' \in u))$ is true, hence (under the second case) $\forall u(u \in <\beta_1, \beta_2> \supset \exists z'(z' \in u))$ is true, hence – because of $\vdash \beta_1 \in <\beta_1, \beta_2>$ and $\vdash \beta_2 \in <\beta_1, \beta_2>$ ($PL_{set}3$, [b]) – we have: $\exists z'(z' \in \beta_1)$ is true and $\exists z'(z' \in \beta_1)$ is true. $<...>_{1\beta 1}$ represents *the first immediate constituent-occurrence in* β_1, and $<...>_{1\beta 2}$ represents *the first immediate constituent-occurrence in* β_2.

Third case: β is typographically $<\beta_1, ..., \beta_n>$ ($n \geq 3$); hence β_1 is typographically $<<...>_{1\beta 1}...>$ and … and β_n is typographically $<<...>_{1\beta n}...>$.

According to assumption, $\forall u(u \in \beta \supset \exists z'(z' \in u))$ is true, hence (under the third case) $\forall u(u \in <\beta_1, ..., \beta_n> \supset \exists z'(z' \in u))$ is true, hence because of $\vdash \beta_1 \in <\beta_1, ..., \beta_n>$ and … and $\vdash \beta_n \in <\beta_1, ..., \beta_n>$ ($PL_{set}3$, [c]) we have: $\exists z'(z' \in \beta_1)$ is true and … and $\exists z'(z' \in \beta_n)$ is true. $<...>_{1\beta 1}$ represents *the first immediate constituent-occurrence in* β_1 and … and $<...>_{1\beta n}$ represents *the first immediate constituent-occurrence in* β_n. [For example, if $n = 3$, then $<\beta_1, ..., \beta_n>$ is typographically $<\beta_1, \beta_2, \beta_3>$, and $\exists z'(z' \in \beta_1)$, $\exists z'(z' \in \beta_2)$, $\exists z'(z' \in \beta_3)$ are true, and $<...>_{1\beta 1}$, $<...>_{1\beta 2}$, $<...>_{1\beta 3}$ represent *the first immediate constituent-occurrence in*, respectively, $\beta_1, \beta_2, \beta_3$.]

In the *first case*, sel(β) is sel($<\beta_1>$), which is defined as $<<...>_{1\beta 1}>$.
In the *second case*, sel(β) is sel($<\beta_1, \beta_2>$), which is defined as $<<...>_{1\beta 1}, <...>_{1\beta 2}>$.
In the *third case*, sel(β) is sel($<\beta_1, ..., \beta_n>$), which is defined as $<<...>_{1\beta 1}, ..., <...>_{1\beta n}>$ [for example, sel($<\beta_1, \beta_2, \beta_3>$) is defined as $<<...>_{1\beta 1}, <...>_{1\beta 2}, <...>_{1\beta 3}>$, sel($<\beta_1, \beta_2, \beta_3, \beta_4>$) is defined as $<<...>_{1\beta 1}, <...>_{1\beta 2}, <...>_{1\beta 3}, <...>_{1\beta 4}>$].
Now, as can easily be seen:
$\forall u(u \in <\beta_1> \supset \exists z'(z' \in u \wedge z' \in sel(<\beta_1>)))$ is true;
$\forall u(u \in <\beta_1, \beta_2> \supset \exists z'(z' \in u \wedge z' \in sel(<\beta_1, \beta_2>)))$ is true;
$\forall u(u \in <\beta_1, ..., \beta_n> \supset \exists z'(z' \in u \wedge z' \in sel(<\beta_1, ..., \beta_n>)))$ is true (with $n \geq 3$).

Focusing paradigmatically on this last assertion: Suppose $u \in <\beta_1, ..., \beta_n>$ is true, hence by $PL_{set}Th9$: $u = \beta_1 \vee ... \vee u = \beta_n$ is true. sel($<\beta_1, ..., \beta_n>$) is defined as $<<...>_{1\beta 1}, ..., <...>_{1\beta n}>$, and by $PL_{set}3$, [c]: $<...>_{1\beta 1} \in <<...>_{1\beta 1}, ..., <...>_{1\beta n}>$ and … and $<...>_{1\beta n} \in <<...>_{1\beta 1}, ..., <...>_{1\beta n}>$ are true. Moreover, $<...>_{1\beta 1} \in \beta_1$ and … and $<...>_{1\beta n} \in \beta_n$ are true. [For all

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[19] β_1 has the typographic shape $<...>$, and since $\exists z'(z' \in \beta_1)$ is true, $<...>$ cannot here be "$<>$" (due to $PL_{set}Th3.2$). Thus, there must be immediate constituent-occurrences in β_1 (at least one such), and of course one of them must be *the first one*: the one following immediately after the initial "<" of β_1.

j with $3 \leq j \leq n$: β_j, being non-empty, has a first constituent-occurrence, represented by $<\ldots>_{1\beta j}$; that $<\ldots>_{1\beta j} \in \beta_j$ is true follows by $PL_{set}3$.] Thus:

(1) If $u = \beta_1$ is true, then $<\ldots>_{1\beta 1} \in u \wedge <\ldots>_{1\beta 1} \in <<\ldots>_{1\beta 1}, \ldots, <\ldots>_{1\beta n}>$ is true, and hence $\exists z'(z' \in u \wedge z' \in <<\ldots>_{1\beta 1}, \ldots, <\ldots>_{1\beta n}>)$ is true;

… … …

(n) if $u = \beta_n$ is true, then $<\ldots>_{1\beta n} \in u \wedge <\ldots>_{1\beta n} \in <<\ldots>_{1\beta 1}, \ldots, <\ldots>_{1\beta n}>$ is true, and hence, $\exists z'(z' \in u \wedge z' \in <<\ldots>_{1\beta 1}, \ldots, <\ldots>_{1\beta n}>)$ is true.

Thus: If $u = \beta_1 \vee \ldots \vee u = \beta_n$ is true, then $\exists z'(z' \in u \wedge z' \in <<\ldots>_{1\beta 1}, \ldots, <\ldots>_{1\beta n}>)$ is true.

Therefore: If $u \in <\beta_1, \ldots, \beta_n>$ is true, then $\exists z'(z' \in u \wedge z' \in <<\ldots>_{1\beta 1}, \ldots, <\ldots>_{1\beta n}>)$ is true [since the truth of $u = \beta_1 \vee \ldots \vee u = \beta_n$ follows from the truth of $u \in <\beta_1, \ldots, \beta_n>$ – as already established]. Therefore: If $u \in <\beta_1, \ldots, \beta_n>$ is true, then $\exists z'(z' \in u \wedge z' \in sel(<\beta_1, \ldots, \beta_n>))$ is true [by the definition of $sel(<\beta_1, \ldots, \beta_n>)$]. Therefore: $u \in <\beta_1, \ldots, \beta_n> \supset \exists z'(z' \in u \wedge z' \in sel(<\beta_1, \ldots, \beta_n>))$ is true, and hence: $\forall u(u \in <\beta_1, \ldots, \beta_n> \supset \exists z'(z' \in u \wedge z' \in sel(<\beta_1, \ldots, \beta_n>)))$ is true.[20]

The *three assertions* (listed above) are the results, following from the initial assumption, for each of the three possibilities of what β can typographically amount to under that assumption. Thus, following from the initial assumption, $\forall u(u \in \beta \supset \exists z'(z' \in u \wedge z' \in sel(\beta)))$ is true, and hence $\exists z \forall u(u \in \beta \supset \exists z'(z' \in u \wedge z' \in z))$ is true.

It has now been shown:

$\vdash \exists z'(z' \in \beta) \wedge \forall u(u \in \beta \supset \exists z'(z' \in u)) \supset \exists z \forall u(u \in \beta \supset \exists z'(z' \in u \wedge z' \in z))$.

This concludes the proof of the *Lemma*, and thereby the entire proof (in consideration of what has been said at its beginning).

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[20] Note that this step of generalization has nothing to do with $PL_{set}2$. It is simply due to $\forall$-*generalization*: a rule of logical inference which is justifiable for first-order predicate-logic (already without any additives): If (the truth of) $\sigma[\upsilon]$ is deducible from (the truth of) Γ [according to the axiomatized logic that is being used], then (the truth of) $\forall \upsilon \sigma[\upsilon]$ is deducible from (the truth of) Γ – where υ is a variable which occurs free in $\sigma[\upsilon]$, but does not occur free in Γ, Γ being a (finite) series of assumptions (possibly axiomatic). Note that Γ may be empty or may be completely "cuttable"; in that case, $\forall$-*generalization* boils down to the following rule: If (the truth of) $\sigma[\upsilon]$ is *provable* [according to the axiomatized logic that happens to be "in charge" and, perhaps, also according to non-logical *axioms*], then (the truth of) $\forall \upsilon \sigma[\upsilon]$ is *provable* [according to that logic and, perhaps, non-logical axioms]. Note also that the parenthetical "(the truth of)" can be uniformly replaced by the non-parenthetical "the *logical* truth of," or: "$\vdash$" – and the validity of $\forall$-*generalization* is not affected. (And note: the *logical* truth of sentence $\sigma[\upsilon]$ is provable if and only if the truth of $\sigma[\upsilon]$ is provable according to the *employed* axiomatized *logic* [here PL_{set}, later PLN_{set}], *nothing else needed*.)

PL$_{set}$Th12: $\vdash \forall\upsilon''\exists\upsilon'\forall\upsilon(\upsilon \in \upsilon' \equiv \exists\upsilon'''(\upsilon''' \in \upsilon'' \wedge \upsilon = \varphi[\upsilon''']))$, with υ' not in $\exists\upsilon'''(\upsilon''' \in \upsilon'' \wedge \upsilon = \varphi[\upsilon'''])$, and υ not in $\varphi[\upsilon''']$ – which is *the principle of replacement*.[21]

Proof: PL$_{set}$Th12 is a consequence of PL$_{set}$Th8 and PL$_{set}$2, *because*, for every pure closed bracket β, $\exists\upsilon'''(\upsilon''' \in \beta \wedge \upsilon = \varphi[\upsilon'''])$ is a predicate with finitely many instantiations [with respect to υ, and among the pure finite sets]. (I omit the proof of the assertion after "*because*.")

The following theorem one may consider *the heart* of the principle of replacement, since a pure finite set as asserted to exist (for each pure finite set) by PL$_{set}$Th12 is obtained by simply removing "the unnecessary elements" from a pure finite set as asserted to exist (for each pure finite set) by PL$_{set}$Th12.1. Moreover, the proof of PL$_{set}$Th12.1 below displays the central move in a *direct* proof of PL$_{set}$Th12 (a proof not via PLsetTh8); it is *replacing* $<\beta_1>$, $<\beta_1, \beta_2>$, $<\beta_1, \ldots, \beta_n>$ [$n \geq 3$] by $<\varphi[\beta_1]>$, respectively $<\varphi[\beta_1], \varphi[\beta_2]>$, respectively $<\varphi[\beta_1], \ldots, \varphi[\beta_n]>$, and "$<>$", trivially, by "$<>$".

PL$_{set}$Th12.1: $\vdash \forall\upsilon''\exists\upsilon'\forall\upsilon(\upsilon \in \upsilon'' \supset \varphi[\upsilon] \in \upsilon')$, υ' not in $\varphi[\upsilon]$

Proof: $\vdash \forall\upsilon''\exists\upsilon'\forall\upsilon(\upsilon \in \upsilon'' \supset \varphi[\upsilon] \in \upsilon')$ is a consequence by PL$_{set}$2 of the following

Lemma: $\vdash \exists\upsilon'\forall\upsilon(\upsilon \in \beta \supset \varphi[\upsilon] \in \upsilon')$, since β represents just any pure closed bracket.

Proof of the Lemma: β is typographically either "$<>$", or $<\beta_1>$ or $<\beta_1, \beta_2>$ or $<\beta_1, \ldots., \beta_n>$ for some pure closed brackets $\beta_1, \ldots, \beta_n$ with $n \geq 3$. $\varphi[\upsilon]$ is a functional expression – so to speak: a singular term with a free variable υ (which is automatically free in $\varphi[\upsilon]$ if it does not occur at all in $\varphi[<>]$) – which functional expression may well be defined with the help of *the operator of definite description*: ι (in the following manner: $\varphi[\upsilon] =_{Def} \iota\upsilon'''\sigma[\upsilon][\upsilon''']$).

In case β *is typographically* "$<>$", $\exists\upsilon'\forall\upsilon(\upsilon \in \beta \supset \varphi[\upsilon] \in \upsilon')$ amounts to $\exists\upsilon'\forall\upsilon(\upsilon \in <> \supset \varphi[\upsilon] \in \upsilon')$; since $\vdash \neg\exists z(z \in <>)$ (as repeatedly noted), $\vdash \forall\upsilon(\upsilon \in <> \supset \varphi[\upsilon] \in <>)$ is a trivial consequence, and hence also $\vdash \exists\upsilon'\forall\upsilon(\upsilon \in <> \supset \varphi[\upsilon] \in \upsilon')$, that is (under the case considered): $\vdash \exists\upsilon'\forall\upsilon(\upsilon \in \beta \supset \varphi[\upsilon] \in \upsilon')$.

[21] Like *the principle of finite comprehension* (see PL$_{set}$Th8), *the principle of replacement* is not a logically true sentence but a *schema* for logically true sentences. In presenting these schemata, it is tacitly understood that all of their (infinitely many) instantiations are syntactically well-formed. (It cannot be, for example, that in an instantiation two different occurrences of quantifiers bind the very same occurrence of a variable.)

In case β *is typographically* $<\beta_1>$, and assuming that $\beta' \in \beta$ is true, we have by $PL_{set}4$, [b]: $\beta' = \beta_1$ is true. Consider $<\varphi[\beta_1]>$; $\varphi[\beta_1] \in <\varphi[\beta_1]>$ is true by $PL_{set}3$, [a].[22] Consequently, by the substitution of identicals: $\varphi[\beta'] \in <\varphi[\beta_1]>$. What has now been shown is this: $\vdash \beta' \in \beta \supset \varphi[\beta'] \in <\varphi[\beta_1]>$, hence by $PL_{set}2$: $\vdash \forall\upsilon(\upsilon \in \beta \supset \varphi[\upsilon] \in <\varphi[\beta_1]>)$, and hence logically: $\vdash \exists\upsilon'\forall\upsilon(\upsilon \in \beta \supset \varphi[\upsilon] \in \upsilon')$.

In case β *is typographically* $<\beta_1, \beta_2>$, and assuming that $\beta' \in \beta$ is true, we have by $PL_{set}4$, [c]: $\beta' = \beta_1 \vee \beta' = \beta_2$ is true. Consider $<\varphi[\beta_1], \varphi[\beta_2]>$; $\varphi[\beta_1] \in <\varphi[\beta_1], \varphi[\beta_2]>$ and $\varphi[\beta_2] \in <\varphi[\beta_1], \varphi[\beta_2]>$ are true by $PL_{set}3$, [b]. Consequently, by the substitution of identicals (and propositional logic): $\varphi[\beta'] \in <\varphi[\beta_1], \varphi[\beta_2]>$ is true. What has now been shown is this: $\vdash \beta' \in \beta \supset \varphi[\beta'] \in <\varphi[\beta_1], \varphi[\beta_2]>$, hence by $PL_{set}2$: $\vdash \forall\upsilon(\upsilon \in \beta \supset \varphi[\upsilon] \in <\varphi[\beta_1], \varphi[\beta_2]>)$, and hence logically: $\vdash \exists\upsilon'\forall\upsilon(\upsilon \in \beta \supset \varphi[\upsilon] \in \upsilon')$.

In case β *is typographically* $<\beta_1, \ldots, \beta_n>$ ($n \geq 3$), and assuming that $\beta' \in \beta$ is true, we have by $PL_{set}4$, [d]: $\beta' = \beta_1 \vee \ldots \vee \beta' = \beta_n$ is true. Consider $<\varphi[\beta_1], \ldots, \varphi[\beta_n]>$; $\varphi[\beta_1] \in <\varphi[\beta_1], \ldots, \varphi[\beta_n]>$ and … and $\varphi[\beta_n] \in <\varphi[\beta_1], \ldots, \varphi[\beta_n]>$ are true by $PL_{set}3$, [c]. Consequently, by the substitution of identicals (and propositional logic): $\varphi[\beta'] \in <\varphi[\beta_1], \ldots, \varphi[\beta_n]>$ is true. What has now been shown is this: $\vdash \beta' \in \beta \supset \varphi[\beta'] \in <\varphi[\beta_1], \ldots, \varphi[\beta_n]>$, hence by $PL_{set}2$: $\vdash \forall\upsilon(\upsilon \in \beta \supset \varphi[\upsilon] \in <\varphi[\beta_1], \ldots, \varphi[\beta_n]>)$, and hence logically: $\vdash \exists\upsilon'\forall\upsilon(\upsilon \in \beta \supset \varphi[\upsilon] \in \upsilon')$.

Thus, in all cases of the typographical shape of β we have $\vdash \exists\upsilon'\forall\upsilon(\upsilon \in \beta \supset \varphi[\upsilon] \in \upsilon')$. This concludes the proof of the *Lemma*, and the entire proof (in view of what has been said at its beginning).

In the theory of pure finite sets, it cannot be shown that there are pure *infinite* sets; on the contrary, whatever example of such a set one may have in mind, it can be shown in the theory of pure finite sets that there is no such thing. Consider, for example, the *simplest* pure infinite set: the set which is such that the elements of it are the following: $<>$, $<<>>$, $<<<>>>$, $<<<<>>>>$, $<<<<<>>>>>$, … and so on *ad infinitum* in the following manner (and only in that manner): at each further step, each (immediately) succeeding pure closed bracket acquires *two more angles* than the (immediately) preceding pure closed bracket, *one* angle – opening to the right –

[22] Precisely speaking, it is the following particular consequence (via $PL_{set}2$) of $PL_{set}3$, [a], that justifies the deductive step taken: $\vdash \forall x(x \in <x>)$. *Mutatis mutandis* the same observation applies to the invocation of $PL_{set}3$, [b], and $PL_{set}3$, [c], a little later in this proof. In these two cases, the directly justifying general principles are: $\vdash \forall x\forall y(x \in <x, y> \wedge y \in <x, y>)$, and $\vdash \forall x_1 \ldots \forall x_n(x_1 \in <x_1, \ldots, x_n> \wedge \ldots \wedge x_n \in <x_1, \ldots, x_n>)$ for any $n \geq 3$.

at the left end of the preceding pure closed bracket, and *one* angle – opening to the left – at the right end of the preceding pure closed bracket. According to the theory of pure finite sets, there is no such thing – which is not surprising, since for the theory of pure finite sets *everything* is a pure *finite* set. Accordingly, one can prove:

<u>PL_{set}Th13: $\vdash \neg\exists x(<> \in x \wedge \forall y(y \in x \supset <y> \in x))$</u>

Proof: Assume $<> \in \beta \wedge \forall y(y \in \beta \supset <y> \in \beta)$ is true for some pure closed bracket β. β must have the typographic form $<\beta_1, \ldots, \beta_n>$ (for some $n \geq 1$; otherwise, it wouldn't be a pure *closed* bracket). Therefore: Since, according to assumption, $<> \in \beta$ is true, $<> = \beta_1 \vee \ldots \vee <> = \beta_n$ must be true (because of PL_{set}4); and since, according to assumption (as a consequence of it), $<<>> \in \beta$ is also true, $<<>> = \beta_1 \vee \ldots \vee <<>> = \beta_n$ must be true, too (because of PL_{set}4); and since, according to assumption (as a further consequence of it), $<<<>>> \in \beta$ is also true, $<<<>>> = \beta_1 \vee \ldots \vee <<<>>> = \beta_n$ must be true, too (because of PL_{set}4) – and so on *ad infinitum*: *Whichever pure closed bracket* β''' *of the form* <…<>…> [in the limiting case: *of the form* <>] *we are looking at,* $\beta''' = \beta_1 \vee \ldots \vee \beta''' = \beta_n$ *must be true*.

Now, because of PL_{set}1, [c], for any *typographically different* pure closed brackets β´ and β´´ of the form <…<>…> ["<" being reiterated in β´ *k* times ($k \geq 0$) to the left of "<>", but in β´´ *k´* times (with $k' \geq 0$ *and* $k' \neq k$); and ">" being reiterated in β´ *k* times to the right of "<>", but in β´´ *k´* times], we have: $\beta' \neq \beta''$ *is true*; for, *either* β´ is typographically-properly included in β´´ and $\beta'' = <\ldots\beta'\ldots>$ is true, and at the same time, according to PL_{set}1, [c], $\beta' \neq <\ldots\beta'\ldots>$ is true[23]; *or* β´´ is typographically-properly included in β´ and $\beta' = <\ldots\beta''\ldots>$ is true, and at the same time, according to PL_{set}1, [c], $\beta'' \neq <\ldots\beta''\ldots>$ is true.

Consider, then, *n*+1 *typographically different* pure closed brackets of the form <…<>…>: $<\ldots<>\ldots>_1$, and … and $<\ldots<>\ldots>_n$ and $<\ldots<>\ldots>_{n+1}$. These *n*+1 typographically different pure closed brackets designate *n*+1 different pure finite sets (as has just been shown). Hence $<\ldots<>\ldots>_j = \beta_1 \vee \ldots \vee <\ldots<>\ldots>_j = \beta_n$ cannot be true for every *j* with $1 \leq j \leq n+1$ – contradicting a conclusion previously reached (the one italicized above), and thereby contradicting the initial assumption.

[23] According to PL_{set}1, [c], we have: $\beta' \neq <\ldots\beta'\ldots>$ is *logically* true, in other words: $\vdash \beta' \neq <\ldots\beta'\ldots>$; but *truth*, of course, follows from *logical truth*.

It has now been shown: $\vdash \neg(<> \in \beta \wedge \forall y(y \in \beta \supset <y> \in \beta))$, and therefore because of $PL_{set}2$ (since β represents just any pure closed bracket): $\vdash \forall x \neg(<> \in x \wedge \forall y(y \in x \supset <y> \in x))$, and therefore (logically): $\vdash \neg\exists x(<> \in x \wedge \forall y(y \in x \supset <y> \in x))$.

IV. The logical theory of elementary arithmetic

$PL_{set}Th13$ notwithstanding, elementary arithmetic can be founded within the theory of pure finite sets in the following manner. *The first step is this*:

Ideal number-names

[1] “<>” is an ideal number-name.

[2] If β is an ideal number-name, then $<\beta>$ is an ideal number-name.

[3] Ideal number-names are only pure closed brackets according to [1] and [2].

The expressions just defined – forming a subgroup of the pure closed brackets, namely: the *comma-free* pure closed brackets, in short: *the comma-frees* – are *ideal* proper names for the natural numbers (here taken to be *the whole positive numbers*, hence including *zero*; there is no good reason, really, to consider *zero* as “non-natural”): Each natural number has exactly one comma-free as a proper name of it – one that depicts it *relatively perfectly* (that is: in the best way a pure closed bracket can depict a natural number); each comma-free is a proper name of exactly one natural number, relatively perfectly depicting it. The natural number designated and relatively perfectly depicted by a comma-free is $k/2 - 1$, where k is the number of angle-occurrences in the comma-free.

As is well-known, there are many – indeed, infinitely many – ways of subsuming the natural numbers under the pure finite sets. The way here chosen is *the simplest way*. Less simple is the following way:

<>; <<>>; <<>, <<>>>; <<>, <<>>, <<<>>>>; <<>, <<>>, <<<>>>, <<<<>>>>>; …

Here, each natural number n has exactly n elements – a feature that can be advantageous for certain applications. Even less simple – but nevertheless *the most popular way* of subsuming the natural numbers under the pure finite sets – is the following way:

<>; <<>>; <<>, <<>>>; <<>, <<>>, <<>, <<>>>>; <<>, <<>>, <<>, <<>>>, <<>, <<>>, <<>, <<>>>>>; …

Here, each natural number n has exactly n elements, and its elements are precisely all of its predecessors.

The natural numbers are best regarded as type-objects (or in other words: as non-predicative universals), and, so regarded, one can *pictorially represent* – indeed, relatively perfectly depict – each natural number by the *comma-free* [by the comma-free pure closed bracket: a *graphical* type-object] which is its proper name. In this manner, natural numbers – though universals – can, in a sense (no doubt, in a more tenuous sense than pure finite sequences), be *seen*.

In contrast to the *relatively perfect pictorial representation* of the natural numbers by their respective ideal proper names, their *identification* with their respective ideal proper names is unfeasible. The reason for this is that the theory of elementary arithmetic is embedded in the theory of pure finite sets; and therefore, for example, “<<>>”, “<<>, <>>”, “<<>, <>, <>>”, “<<>, <>, <>, <>>”, and so on (*ad infinitum*), will all – because of $PL_{set}1$, [b] – name the same object: the number 1. Clearly, either every one of these names would have to be identical with the number 1 – which cannot be (because the names are obviously non-identical) – *or none of them can be identical with the number* 1.[24]

The second step is this: The predicate $N(\upsilon)$ – “υ is a natural number” – is added to the formal language L [L being the language of first-order predicate-logic-with-identity-and-definite-descriptions that is enriched, (a), by the pure closed brackets as proper names and, (b), by the two-place predicate $\upsilon \in \upsilon'$], which addition turns L into LN. Then PLN_{set} is first-order predicate-logic-with-identity-and-definite-descriptions *plus* $PL_{set}1$, $PL_{set}2$, $PL_{set}3$, $PL_{set}4$ (now being referred to LN_{set} instead of just to L_{set}), *plus* the following two further schemata:

$PLN_{set}1$: $\vdash$ N(<…<>…>), no matter how often “<” is reiterated on the left side of “<>” [*k*-times, with $k \geq 0$], and no matter how often “>” is reiterated the same number of times on the right side of “<>”.

[$PLN_{set}1$ *equivalently* formulated: $\vdash N(\beta)$, for *every ideal number-name* β.]

$PLN_{set}2$: If for *every ideal number-name* β: $\vdash \sigma[\beta]$, then $\vdash \forall\upsilon(N(\upsilon) \supset \sigma[\upsilon])$.

[$PLN_{set}2$ *equivalently* formulated: If for *all comma-frees* β: $\vdash \sigma[\beta]$, then $\vdash \forall\upsilon(N(\upsilon) \supset \sigma[\upsilon])$.]

[24] However, if one develops elementary arithmetic within the theory of pure finite *sequences* (instead of the theory of pure finite *sets*), then nothing stands in the way of identifying, for example, the number 1 with its ideal proper name “<<>>” – the number 1 being the pure finite sequence <<>>, which, in turn, is being identified with its ideal proper name “<<>>”.

In effect, the proof of PL_{set}Th13 already contains the proof of

PLN_{set}Th1: $\vdash \beta' \neq \beta''$, for all typographically different ideal number-names β' and β''

Proof: This follows by the definition of ideal number-names and PL_{set}1, [c] (see the proof of PL_{set}Th13, second paragraph).

PLN_{set}1 and PLN_{set}Th1 together entail that there is a one-to-one correspondence between certain pure closed brackets – namely, *the ideal number-names* – and the natural numbers, which are nothing else than precisely the pure finite sets that are designated by those pure closed brackets. And PLN_{set}2, in fact, is "not entirely" axiomatic – so to speak – but can be seen to be a consequence of PL_{set}2 under the background-assumption *that every natural number is designated by some ideal number-name (logically) necessarily, and that* $\vdash N(\beta')$ *or* $\vdash \neg N(\beta')$ *for every pure closed bracket* β':

Which assumption – subsequently called "*the background assumption*" – is not much of an *assumption*, its truth being guaranteed by the syntactic-semantic nature of LN_{set}: an extension of L_{set} by the mere addition of the predicate $N(\upsilon)$, with *some* of the pure finite sets that constitute the universe of discourse of L_{set} being regarded [in LN_{set}, which has the same universe of discourse as L_{set}] as the natural numbers, namely, those pure finite sets that are designated – each one *necessarily* (due to the syntactic-semantic nature of LN_{set}) – by *the comma-frees*, i.e., *the ideal number-names*, among the pure closed brackets.

As a mere specialization of PL_{set}2 we have: (A) If for every pure closed bracket β': $\vdash N(\beta') \supset \sigma[\beta']$, then $\vdash \forall\upsilon(N(\upsilon) \supset \sigma[\upsilon])$. Moreover we have: (B) If for *all* comma-*frees* β: $\vdash \sigma[\beta]$, then for every pure closed bracket β': $\vdash N(\beta') \supset \sigma[\beta']$. Obviously, PLN_{set}2 follows from (B) together with (A). And (B) is seen to be true as follows: Assume that for all comma-frees β: $\vdash \sigma[\beta]$, *the deduction-assumption*, and let β' be any pure closed bracket. Now, $\vdash N(\beta')$ or $\vdash \neg N(\beta')$ by *the background-assumption*.[25] If $\vdash \neg N(\beta')$, then trivially: $\vdash N(\beta') \supset \sigma[\beta']$. If $\vdash N(\beta')$, then, since due to *the background-assumption* every natural number is designated by some ideal number-name (by some *comma-free*) *necessarily*, we have: $\vdash \beta' = \beta^*$ for some comma-free β^*; and by *the deduction-assumption* applied to β^* we have: $\vdash \sigma[\beta^*]$; hence $\vdash \sigma[\beta']$ [by the logic of logi-

[25] In Section V, this is proved with the help of PLN_{set}2 (see (III) and the proof of (III) in Section V). Here – in proceedings that aim at a derivation of PLN_{set}2 from PL_{set}2 – it needs to be *assumed*.

cally necessary – i.e., logically true – identity], and therefore trivially: $\vdash N(\beta') \supset \sigma[\beta']$ – which concludes the proof of (B) (that is, the deduction of (B) from *the background-assumption*).

Continuing with the expositions:

Def4: $\underline{0} =_{Def} <>$; $\underline{1} =_{Def} <<>>$; $\underline{2} =_{Def} <<<>>>$; $\underline{3} =_{Def} <<<<>>>>$; $\underline{4} =_{Def} <<<<<>>>>>$; …[26]

Def5: $nf(\beta) =_{Def} <\beta>$

The Arabic numerals defined by Def4 are underlined because they are a part of LN, whereas the corresponding non-underlined numerals are a part of the metalanguage of LN.

With these definitions in place and *the first Peano-axiom* – $\vdash N(\underline{0})$ – being an obvious consequence of $PLN_{set}1$ and Def4, the remaining four Peano-axioms are proven as follows:

$\underline{PLN_{set}Th2: \vdash \forall x(N(x) \supset N(nf(x))}$ – which is *the second Peano-axiom*.

Proof: For every ideal number-name β, $\vdash N(<\beta>)$ – which is a consequence of $PLN_{set}1$ and the definition of ideal number-names. Therefore by Def5: For every ideal number-name β, $\vdash N(nf(\beta))$. Therefore by $PLN_{set}2$: $\vdash \forall x(N(x) \supset N(nf(x))$.

$\underline{PLN_{set}Th3: \vdash \neg\exists z(0 = nf(z))}$

Proof: Suppose $0 = nf(\beta)$ is true, in other words (by Def4 and Def5): $<> = <\beta>$ is true; however, this contradicts the combination of $PL_{set}3$, [a], and $PL_{set}4$, [a]. Thus we have: $\vdash 0 \neq nf(\beta)$, and hence by $PL_{set}2$ (since β represents just any pure closed bracket): $\vdash \forall z(0 \neq nf(z))$, and hence (logically): $\vdash \neg\exists z(0 = nf(z))$.

$\underline{PLN_{set}Th3.1: \vdash \forall z(N(z) \supset nf(z) \neq 0)}$ – which is *the third Peano-axiom*.

Proof: From $PLN_{set}Th3$ by fundamental logic (which, in this paper, is first-order predicate-logic-with-identity-and-definite-descriptions).

$\underline{PLN_{set}Th4: \vdash \forall x\forall y(x \neq y \supset nf(x) \neq nf(y))}$

[26] Note that the number designated by $<\ldots<>\ldots>$ is always the number of the lefthand angles of $<\ldots<>\ldots>$ *minus* 1.

Proof: Suppose $\beta \neq \beta' \wedge nf(\beta) = nf(\beta')$ is true; hence $<\beta> = <\beta'>$ is true (by Def5). Now, $\beta \in <\beta>$ is true by $PL_{set}3$, [a]. Consequently (by the logic of identity), $\beta \in <\beta'>$ is true, and therefore (because of $PL_{set}4$, [b]): $\beta = \beta'$ is true – contradicting the initial supposition. Thus: $\vdash \neg(\beta \neq \beta' \wedge nf(\beta) = nf(\beta'))$, or in other words (as logically equivalent): $\vdash \beta \neq \beta' \supset nf(\beta) \neq nf(\beta')$. Since both β and β' represent just any pure closed bracket, it follows by $PL_{set}2$ [twice employed]: $\vdash \forall x \forall y(x \neq y \supset nf(x) \neq nf(y))$.

$PLN_{set}Th4.1$: $\vdash \forall x \forall y(N(x) \wedge N(y) \wedge nf(x) = nf(y) \supset x = y)$ – which is *the fourth Peano-axiom*.
Proof: From $PLN_{set}4$ by fundamental logic.

$PLN_{set}Th5$: If $\vdash \sigma[0] \wedge \forall \upsilon(N(\upsilon) \wedge \sigma[\upsilon] \supset \sigma[nf(\upsilon)])$, then $\vdash \forall \upsilon(N(\upsilon) \supset \sigma[\upsilon])$ – which is *the principle of complete induction* and *the fifth Peano-axiom*.
Proof: Suppose $\vdash \sigma[0] \wedge \forall \upsilon(N(\upsilon) \wedge \sigma[\upsilon] \supset \sigma[nf(\upsilon)])$, in other words (by employing the definitions): $\vdash \sigma[<>] \wedge \forall \upsilon(N(\upsilon) \wedge \sigma[\upsilon] \supset \sigma[<\upsilon>])$. For every ideal number-name β: $\vdash N(\beta)$ (because of $PLN_{set}1$, given the definition of the ideal number-names). Therefore, for each ideal number-name β, it follows logically from the initial supposition (recursively, in finitely many structurally completely repetitive steps): $\vdash \sigma[\beta]$. Thus: For every ideal number-name β: $\vdash \sigma[\beta]$. Therefore by $PLN_{set}2$: $\vdash \forall \upsilon(N(\upsilon) \supset \sigma[\upsilon])$.

Consider, furthermore, the following definition:

Def6: $(\beta + \beta') =_{Def} \beta[<>/\beta']$, where β and β' are ideal number-names, and $\beta[<>/\beta']$ is the ideal number-name that results from β by replacing "<>" in β by β' ("<>" being *the eye of* β).

Def6 defines the arithmetical operation of *addition* for all natural numbers (and of course one can supplement some arbitrary stipulation that defines *addition* also for cases where one of the terms of addition is *not* a natural number). Here is an example of the application of Def4 and Def6: (5 + 3) = (<<<<<<>>>>>> + <<<<>>>>) = <<<<<<<<<<>>>>>>>>>> = 8.

Since all ideal number-names – that is, all *comma-frees* – are symmetrical with respect to "their "eye," one need write only the first half of each ideal number-name. Thus, the above identity-sequence can also be written as follows: (5 +3) = (<<<<<< + <<<<) = <<<<<<<<<< = 8.

The following definition, finally, gives us, for all natural numbers, the arithmetical operation of *multiplication*:

Def7: $(\beta \times <>) =_{Def} <>$; $(\beta \times nf(\beta')) =_{Def} ((\beta \times \beta') + \beta)$

In contrast to Def6, which is a (so-called) *explicit* definition, Def7 is a (so-called) *recursive* definition. But, of course, addition can also be defined *recursively*, as follows: $(\beta + <>) =_{Def} \beta$; $(\beta + nf(\beta')) =_{Def} nf((\beta + \beta'))$.

The last three definitions are written *without* applying pertinent conventions for saving parentheses: (i) outer parentheses can be omitted; (ii) by convention, $\times$ binds stronger than + (and parentheses can be omitted accordingly); (iii) in "nf((…))" the second pair of parentheses can be omitted. Applying (i) – (iii), the two recursive definitions get to look like this:

$\beta + <> =_{Def} \beta$; $\beta + nf(\beta') =_{Def} nf(\beta + \beta')$

$\beta \times <> =_{Def} <>$; $\beta \times nf(\beta') =_{Def} \beta \times \beta' + \beta$.

V. The question of the deductive completeness of PLN_{set}

Consider the sentences of LN_{set} that have either the form $\beta' = \beta$, or the form $\beta' \in \beta$, or the form $N(\beta)$ (β and β' being any pure closed brackets). The following three *assertions of specialized deductive completeness* for PLN_{set} can be seen to be true:

(I) $\vdash \beta' = \beta$ or $\vdash \beta' \neq \beta$ is provable in PLN_{set} for all pure closed brackets β and β'.

(II) $\vdash \beta' \in \beta$ or $\vdash \beta' \notin \beta$ is provable in PLN_{set} for all pure closed brackets β and β'.

(III) $\vdash N(\beta)$ or $\vdash \neg N(\beta)$ is provable in PLN_{set} for all pure closed brackets β.

The "or" in these assertions can be strengthened to "either – , or –"; for PLN_{set} is (simpliciter) *deductively consistent*: There is no sentence σ of LN_{set} whatsoever such that both $\vdash \sigma$ and $\vdash \neg\sigma$ are provable in PLN_{set}. This will be proven in Section VI.

Let it be presupposed – to be proved later in this section – *that* (I) *is true*. Then, beginning with (III), we have:

The proof of (III)

(a) If β is *a comma-free*, then $\vdash N(\beta)$ by $PLN_{set}1$, and hence $\vdash N(\beta)$ is provable in PLN_{set}.

(b) If β is *not* a comma-free, then

either (i): $\vdash \beta = \beta^*$ is provable in PLN_{set} for some β^* which is a comma-free; and hence by $PLN_{set}1$ and the logic of [logically true] identity: $\vdash N(\beta)$ is provable in PLN_{set};

or (ii): $\vdash \beta = \beta^*$ is *not* provable in PLN_{set} for *any* β^* which is a comma-free; and hence by the (presupposed) truth of (I): $\vdash \beta \neq \beta^*$ is provable in PLN_{set} for *any* β^* which is a comma-free; and therefore by $PLN_{set}2$: $\vdash \forall x(N(x) \supset \beta \neq x)$ is provable in PLN_{set}, and consequently [by the basic logic]: $\vdash \neg N(\beta)$ is provable in PLN_{set}.

All possible cases having been considered, it is evident that $\vdash N(\beta)$ or $\vdash \neg N(\beta)$ is provable in PLN_{set} for all pure closed brackets β.

In Section IV, before proving the Peano-axioms, $PLN_{set}2$ was derived from the rest of the axioms of PLN_{set} (in particular, from $PL_{set}2$) and *the background-assumption:* Every natural number is designated by some ideal number-name (logically) necessarily, and $\vdash N(\beta')$ or $\vdash \neg N(\beta')$ for every pure closed bracket β'. Now, in Section V, *the second part* of *the background-assumption* has just been derived from $PLN_{set}2$ and the rest of the axioms of PLN_{set} (in particular, from $PLN_{set}1$ – *and* from $PL_{set}1$ that, as will be seen below, is employed in the proof of (I), which latter assertion is here, for the time being, *presupposed* as true).

Moreover – the truth of (I) being presupposed (as stated before) – we have:

The proof of (II)

(**a**) If β is typographically “$<>$”, then $\vdash \beta' \notin \beta$ by $PL_{set}4$, [a], and hence $\vdash \beta' \notin \beta$ is provable in PLN_{set}.

(**b**) If β is typographically $<\beta_1>$, then

either (i): $\vdash \beta' = \beta_1$ is provable in PLN_{set}; and hence by $PL_{set}3$, [a], and the logic of [necessary] identity: $\vdash \beta' \in <\beta_1>$ – that is, $\vdash \beta' \in \beta$ – is provable in PLN_{set};

or (ii): $\vdash \beta' = \beta_1$ is *not* provable in PLN_{set}; and hence by the (presupposed) truth of (I): $\vdash \beta' \neq \beta_1$ is provable in PLN_{set}, and hence by $PL_{set}4$, [b]: $\vdash \beta' \notin <\beta_1>$ – that is, $\vdash \beta' \notin \beta$ – is provable in PLN_{set}.

(**c**) If β is typographically $<\beta_1, \beta_2>$, then *either* (i): $\vdash \beta' = \beta_1$ *or* $\vdash \beta' = \beta_2$ is provable in PLN_{set}; and hence by $PL_{set}3$, [b], and the logic of identity, etc.: $\vdash \beta' \in <\beta_1, \beta_2>$ – that is, $\vdash \beta' \in \beta$ – is provable in PLN_{set};

or (ii): *neither* $\vdash \beta' = \beta_1$ *nor* $\vdash \beta' = \beta_2$ is provable in PLN_{set}; and hence by the (presupposed) truth of (I): both $\vdash \beta' \neq \beta_1$ and $\vdash \beta' \neq \beta_2$ are provable in PLN_{set}, hence also $\vdash \beta' \neq \beta_1 \wedge \beta' \neq \beta_2$ is provable in PLN_{set}, and hence by $PL_{set}4$, [c]: $\vdash \beta' \notin <\beta_1, \beta_2>$ – that is, $\vdash \beta' \notin \beta$ – is provable in PLN_{set}.

(**d**) If β is typographically $<\beta_1, \ldots, \beta_n>$, with $n \geq 3$, then

either (i): $\vdash \beta' = \beta_1$ *or* … *or* $\vdash \beta' = \beta_n$ is provable in PLN_{set}; and hence by $PL_{set}3$, [c], and the logic of identity, etc.: $\vdash \beta' \in <\beta_1, \ldots, \beta_n>$ – that is, $\vdash \beta' \in \beta$ – is provable in PLN_{set};

or (ii): *neither* $\vdash \beta' = \beta_1$ *nor* … *nor* $\vdash \beta' = \beta_n$ is provable in PLN_{set}; and hence by the (presupposed) truth of (I): $\vdash \beta' \neq \beta_1$ and … and $\vdash \beta' \neq \beta_n$ are provable in PLN_{set}, hence also $\vdash \beta' \neq \beta_1 \wedge \ldots \wedge \beta' \neq \beta_n$ is provable in PLN_{set}, and hence by $PL_{set}4$, [d]: $\vdash \beta' \notin <\beta_1, \ldots, \beta_n>$ – that is, $\vdash \beta' \notin \beta$ – is provable in PLN_{set}.

All possible cases having been considered, it is evident that $\vdash \beta' \in \beta$ or $\vdash \beta' \notin \beta$ is provable in PLN_{set} for all pure closed brackets β and β'.

It remains to be shown that (I) – the truth of which was presupposed in the proofs of (III) and (II) – is, indeed, true. For showing this, one needs to have recourse to *the normal form(s)* of any pure closed bracket, which notion was explained in Section II. As it turns out, there is a *reservation* to the here-presented proof of (I).

The proof of (I)

In a sentence of the form $\beta' = \beta$

either β' and β are typographically identical, and in that case $\vdash \beta' = \beta$ is (trivially) provable in PLN_{set};

or β' and β are typographically non-identical; in this case

either β' and β can be transformed [according to the $PL_{set}1$-based procedure described in Section II] into the same pure closed bracket in normal form – that is, among the normal forms for β' there is at least one which is also a normal form for β – and consequently $\vdash \beta' = \beta$ is provable in PLN_{set};

or β' and β cannot be transformed into the same pure closed bracket in normal form – that is, among the normal forms for β' there is none which is also a normal form of β – and *consequently* $\vdash \beta' \neq \beta$ is provable in PLN_{set}.

The *reservation* touched on above concerns the consequence relationship that has just been indicated by a use of "*consequently*"; that *reservation* is this: It has not been proved here (and will not be proved here) that if among the normal forms for β' there is none which is also a normal form for β, that then $\vdash \beta' \neq \beta$ is provable in PLN_{set}; in the here-presented proof of (I), the obtaining of the consequence relationship in question is, strictly speaking, *presupposed*.

All possible cases having been considered, it is evident that $\vdash \beta' = \beta$ or $\vdash \beta' \neq \beta$ is provable in PLN_{set} for all pure closed brackets β and β'.

The assertions (I), (II), and (III) of specialized deductive completeness for PLN_{set} have now been shown to be true. Consider, then, *the assertion of general deductive completeness* for PLN_{set}:

(IV) $\vdash \sigma$ or $\vdash \neg\sigma$ is provable in PLN_{set} for all *closed* sentences σ of LN_{set}.

Why only "for all *closed* sentences of LN_{set}," and not "for all sentences of LN_{set}"? Consider the following *open* sentences of LN_{set}: "$x = \underline{1}$" and "$x \neq \underline{1}$", and suppose $\vdash x = \underline{1}$ or $\vdash x \neq \underline{1}$ were provable in LN_{set}. Then, according to $\forall$-*generalization* (see footnote 20), $\vdash \forall x(x = \underline{1})$ or $\vdash \forall x(x \neq \underline{1})$ would be provable in LN_{set}; but, of course, neither $\vdash \forall x(x = \underline{1})$ nor $\vdash \forall x(x \neq \underline{1})$ is provable in LN_{set}.

Proof of (IV)

Basis of induction: The closed sentences of LN_{set} with the logical degree 0 [the logical degree of a sentence being the number of occurrences of logical operators in the sentence] either have the form $\beta' = \beta$, or the form $\beta' \in \beta$, or the form $N(\beta)$ (β and β' being any pure closed brackets). It follows from the conjunction of (I), (II), and (III) that for all closed sentences σ of LN_{set} with the logical degree 0: $\vdash \sigma$ or $\vdash \neg\sigma$ is provable in PLN_{set}.

Step of induction: Let (IV) be proven for all closed sentences σ' of LN_{set} with a logical degree $\leq$ *n* [this is *the induction assumption*], and let σ be a closed sentence of LN_{set} with the logical degree n+1. All logical operators other than $\neg$, $\supset$, and $\forall$ are defined in L (in the familiar ways),[27] and hence in LN [and of course in L_{set}, L_{seq}, LN_{set}, the subscript indicating the interpretation L or LN is being given]; therefore, consider LN_{set} purely with sentences containing at most the logical operators $\neg$, $\supset$, and $\forall$ [all *defined* logical operators having been replaced by the expressions defining them].

Now, σ must either be typographically identical with $\neg\sigma'$ (σ´ being a closed sentence of LN_{set}), or with $\sigma' \supset \sigma''$ (σ´ and σ´´ being closed sentences of LN_{set}), or with $\forall\upsilon\sigma'[\upsilon]$ ($\sigma'[\beta]$ being a closed sentence of LN_{set} for every pure closed bracket β).

In the first case: $\vdash \sigma'$ or $\vdash \neg\sigma'$ is provable in PLN_{set} (according to *the induction-assumption*) because σ´ is a closed sentence of LN_{set} and the logical degree of $\sigma' \leq n$.

If $\vdash \sigma'$ is provable in PLN_{set}, then (elementarily) $\vdash \neg\neg\sigma'$ is provable in PLN_{set}, and hence $\vdash \neg\sigma$ is provable in PLN_{set} because (under the first case) σ is typographically $\neg\sigma'$.

If $\vdash \neg\sigma'$ is provable in PLN_{set}, then $\vdash \sigma$ is provable in PLN_{set} because σ is typographically $\neg\sigma'$ (under the first case).

It therefore follows: $\vdash \sigma$ or $\vdash \neg\sigma$ is provable in PLN_{set}.

In the second case: $\vdash \sigma'$ or $\vdash \neg\sigma'$ is provable in PLN_{set}, *and* $\vdash \sigma''$ or $\vdash \neg\sigma''$ is provable in PLN_{set} (according to *the induction-assumption*), because σ´ and σ´´ are closed sentences of LN_{set} and the logical degree of σ´ and of $\sigma'' \leq n$.

If $\vdash \sigma''$ is provable in PLN_{set}, then (by elementary logic) $\vdash \sigma' \supset \sigma''$ is provable in PLN_{set}, and hence $\vdash \sigma$ is provable in PLN_{set} because (under the second case) σ is typographically $\sigma' \supset \sigma''$.

If If $\vdash \neg\sigma'$ is provable in PLN_{set}, then (by elementary logic) $\vdash \sigma' \supset \sigma''$ is provable in PLN_{set}, and hence $\vdash \sigma$ is provable in PLN_{set} because (under the second case) σ is typographically $\sigma' \supset \sigma''$.

If $\vdash \sigma'$ and $\vdash \neg\sigma''$ are provable in PLN_{set}, then (by elementary logic) $\vdash \neg(\sigma' \supset \sigma'')$ is provable in PLN_{set}, and hence $\vdash \sigma$ is provable in PLN_{set} because (under the second case) σ is typographically $\sigma' \supset \sigma''$.

It therefore follows: $\vdash \sigma$ or $\vdash \neg\sigma$ is provable in PLN_{set} (all possibilities under the second case having been covered).

[27] Note that ι, the operator of definite description, is defined in L, in a well-known way, as follows: $\sigma'[\iota\upsilon\sigma[\upsilon]] =_{Def} \exists^{=1}\upsilon\sigma[\upsilon] \wedge \exists\upsilon'(\sigma[\upsilon'] \wedge \sigma'[\upsilon']) \vee \neg\exists^{=1}\upsilon\sigma[\upsilon] \wedge \sigma'[<>]$. $\exists^{=1}\upsilon\sigma[\upsilon]$, in turn, is defined in L as follows: $\exists^{=1}\upsilon\sigma[\upsilon] =_{Def} \exists\upsilon(\sigma[\upsilon] \wedge \forall\upsilon'(\sigma[\upsilon'] \supset \upsilon' = \upsilon))$.

In the third case: Either (i) ⊦ $\sigma'[\beta]$ is provable in PLN_{set} for every pure closed bracket β; or (ii) ⊦ $\sigma'[\beta']$ is not provable in PLN_{set} for some pure closed bracket β', hence ⊦ $\neg\sigma'[\beta']$ is provable in PLN_{set} (according to *the induction-assumption*) because $\sigma'[\beta']$ is a closed sentence of LN_{set} and the logical degree of $\sigma'[\beta'] \leq n$ (for, according to *the induction-assumption*, ⊦ $\sigma'[\beta']$ or ⊦ $\neg\sigma'[\beta']$ is provable in PLN_{set}, and [under (ii)] ⊦ $\sigma'[\beta']$ is not provable in PLN_{set}).
If (i) is the case, then ⊦ $\forall\upsilon\sigma'[\upsilon]$ is provable in PLN_{set} by $PL_{set}2$, and hence ⊦ σ is provable in PLN_{set} because (under the third case) σ is typographically $\forall\upsilon\sigma'[\upsilon]$.
If (ii) is the case, then (by elementary logic) ⊦ $\neg\forall\upsilon\sigma'[\upsilon]$ is provable in PLN_{set}, and hence ⊦ $\neg\sigma$ is provable in PLN_{set} because (under the third case) σ is typographically $\forall\upsilon\sigma'[\upsilon]$.
It therefore follows (again): ⊦ σ or ⊦ $\neg\sigma$ is provable in PLN_{set}.

Together, the *basis of induction* and the *step of induction* of this proof *by complete induction* establish (IV) – in other words: *the general deductive completeness of* PLN_{set}.

Obviously, for reaching this result the logical principle $PL_{set}2$ is crucial. Its near relative $PLN_{set}2$ is (a version of) the well-known logical rule of *ω-completeness* [here: for the natural numbers being taken to be among the pure finite sets], and $PL_{set}2$ can fittingly be called *ω-completeness for pure finite sets*. Here are a few questions and answers:

Question 1: Are logical rules like $PL_{set}2$ and $PLN_{set}2$ – *rules of ω-completeness* – intuitionally well-founded? – *Answer*: Yes, they are intuitionally perfectly justified if they are correct. They are correct if there is a logical system Σ [for the respective formal language] for naming *all* the entities of a certain sort φ [each φ-entity bearing its Σ-name with logical necessity]. Then it is evident:

If $\sigma[\nu]$ is logically true *with each* name ν out of Σ which is substituted for the free variable υ in $\sigma[\upsilon]$, then the following is logically true: $\sigma[\upsilon]$ is true *of every* φ-entity.

Thus, also the following logical rule is intuitionally perfectly justified:

If if $\sigma[\nu]$ is logically true *with each* name ν out of the Arabic numeral system which is substituted for the free variable υ in $\sigma[\upsilon]$, then the following is logically true: $\sigma[\upsilon]$ is true *of every* natural number,
or in other words:
If for *every Arabic numeral* α: ⊦ $\sigma[\alpha]$, then ⊦ $\forall\upsilon(N(\upsilon) \supset \sigma[\upsilon])$.

This rule is perfectly justified because the Arabic numeral system [in a given formal language] is a logical system for naming all the natural numbers [each natural number bearing its Arabic-numeral-name with logical necessity].

Question 2: Does not the provability of (IV) contradict Gödel's proof of the deductive incompleteness of arithmetic? – *Answer*: No, it does not. Gödel's result was reached for deductive systems *without* a rule of ω-completeness, be it as a basic principle or as a derived one.

Question 3: Is there a reason for preferring (formal) deductive systems *without* a rule of ω-completeness, be it as a basic principle or as a derived one, to systems *with* such a rule? – *Answer*: If the number of the φ-entities is greater than countably infinite, then no rule of ω-completeness for the φ-entities, or anything equivalent to such a rule, can be correct – because there is no logical system Σ for naming all the φ-entities. If the number of the φ-entities is finite, then a rule of ω-completeness [or in view of the finiteness of the number of the φ-entities *more appropriately*: rule of "ω-completeness"] for the φ-entities is derivable within any deductive system in which φ is expressible by a predicate and which contains first-order predicate-logic-with-identity. Now, if the number of the φ-entities is countably infinite, then there is no reason not to include a rule of ω-completeness, be it as basic or as derived, in a deductive system which is *about* the φ-entities (perhaps among other entities it is *about*) – *provided* one can find a logical system Σ [for the respective formal language] for naming *all* the φ-entities [each φ-entity bearing its Σ-name with logical necessity].

Question 4: But is not a rule of ω-completeness for φ-entities whose number is countably infinite an *infinitary* logical rule, that is, a logical rule with (countably) infinitely many premises, and should not such rules be avoided in, indeed, any deductive system? – *Answer*: As far as I can see, a rule of ω-completeness has just *one* premise. However, admittedly, that premise is formulated with the help of *metalinguistic quantification* ["for every pure closed bracket β", "for every ideal number-name / for all comma-frees β", "for every Arabic numeral α", …], and if one does not grant oneself the use of metalinguistic quantification in the codification of a deductive system for a given object-language, then, indeed, a rule of ω-completeness can only be formulated *infinitarily*: as a logical rule with infinitely many premises, and hence can be formulated only *incompletely* in the regard of *descriptive explicitness*; for one cannot write down infinitely man premises. However (in turn), why should one not grant oneself the use of metalinguistic quantification in the codification of a deductive system for a given object-language *if such use is opportune* (for example, for the concise and completely explicit unequivocal formulation of an indubitably correct logical rule)?

As is amply illustrated in this paper, *in applications* the demonstration of the *one* premise of a rule of ω-completeness – which single premise we are dealing with if we avail ourselves of metalinguistic quantification – can, although it is *just one* premise, involve us in an infinite deduction: a deduction with infinitely many steps (each having a metalinguistic trait). *But*, so what? If the infinite deduction can be *seen* to be *logically gapless* although not every step in it has actually been taken (nor can actually be taken), everything is all right; at least this is so if matters are considered, as is only fair, in the light of an epistemic ambition that merely aims at *sufficiently justified knowledge* of the truth, and not at knowledge in a yet stricter sense.

How can it be that an infinite deduction is seen to be logically gapless although not every step of it has been taken, nor can be taken? – It can be seen to be logically gapless via the use of *deduction-paradigms*: deductions actually made – parts of the infinite deduction – which are *paradigmatic* for infinitely many deductions not actually made: all the remaining parts of the infinite deduction. From a deduction-paradigm it can be seen that the deductions not actually made consist in mere repetitions-cum-modifications of the very pattern presented by the deduction-paradigm, the modifications being, moreover, foreseeable.

Question 5: If all this were right, would it not considerably diminish the epistemological importance of Gödel's proof of the deductive incompleteness of arithmetic? – *Answer*: Yes, it would. The pure closed brackets constitute a mirror of truth for the pure finite sets, and for the natural numbers among them: Via certain *graphical* type-objects – namely, *the pure closed brackets* – certain other objects – namely, *the pure finite sets* and, in particular, *the natural numbers* – are comprehensively cognized, and *fundamentally* cognized by being, in a sense, *seen* (namely, in the instantiations of the pure closed brackets: in the tokens of these types on the printed page). This can be so because the former objects (the pure closed brackets: graphical type-objects, hence universals) serve as proper names of the latter objects (the pure finite sets) in the following manner: Every one of the former objects designates and pictorially represents exactly one of the latter, and every one of the latter objects is designated and pictorially represented by one of the former (*usually* not by *exactly one* of the former; however, different ways of pictorial representation of one and the same pure finite set differ only due to permutations and repetitions of immediate constituent-occurrences in some pure closed bracket or other within a pure closed bracket that designates the object).

VI. The question of the semantic completeness and semantic (and deductive) consistency of PLN$_{\text{set}}$

A few final points need to be made. *Deductive* completeness is not ipso facto *semantic* completeness. What about the *semantic* completeness of the theory of pure finite sets and of elementary arithmetic as it is formulated by PLN$_{\text{set}}$? And, indeed, what about the *semantic* consistency of PLN$_{\text{set}}$? *Deductive* consistency is not ipso facto *semantic* consistency.

Consider the following four assertions:

(IV) $\vdash \sigma'$ or $\vdash \neg\sigma$ is provable in PLN$_{\text{set}}$ for all closed sentences σ of LN$_{\text{set}}$ [the deductive completeness for PLN$_{\text{set}}$, *proven in the preceding section*].
(V) $\vdash \sigma$ or $\vdash \neg\sigma$ is not provable in PLN$_{\text{set}}$ for all closed sentences σ of LN$_{\text{set}}$ [the deductive consistency for PLN$_{\text{set}}$, *yet unproven*].
(VI) For all closed sentences σ of LN$_{\text{set}}$: if $\vdash \sigma$, then $\vdash \sigma$ is provable in PLN$_{\text{set}}$ [the semantic completeness of PLN$_{\text{set}}$, *yet unproven*].
(VII) For all closed sentences σ of LN$_{\text{set}}$: if $\vdash \sigma$ is provable in PLN$_{\text{set}}$, then $\vdash \sigma$ [the semantic consistency of PLN$_{\text{set}}$, *yet unproven*].

On the basis of (IV) and (VII), (V) and (VI) are easily proven:

Proof of (V)

Suppose (for *reductio*) both $\vdash \sigma$ and $\vdash \neg\sigma$ are provable in PLN$_{\text{set}}$ for a closed sentences σ of LN$_{\text{set}}$;[28] hence by (VII): $\vdash \sigma$ and $\vdash \neg\sigma$, that is: $\vdash \sigma \wedge \neg\sigma$ [in other words: $\sigma \wedge \neg\sigma$ is logically true] – which is absurd.

[28] With σ being a closed sentence, $\neg\sigma$ is also one.

Proof of (VI)

Suppose (for *reductio*) that ⊦ σ, and that ⊦ σ is *not* provable in PLN_{set}, σ being a closed sentence of LN_{set}; hence by (IV): ⊦ ¬σ is provable in PLN_{set}; hence by (VII): ⊦ ¬σ. Consequently, ⊦ σ ∧ ¬σ – which is absurd.

What remains to be shown is (VII):

Proof of (VII)

The addition of $PL_{set}1$, $PL_{set}2$, $PL_{set}3$, $PL_{set}4$, $PLN_{set}1$, $PLN_{set}2$ and of the definitions Def0 – Def7 to first-order predicate-logic-with-identity-and-definite-descriptions leads to a system that in its axioms maintains, and in its theorems preserves, *logical truth*. (This fact cannot be made plainer than it is already.) That system is PLN_{set}, and we have for all closed sentences σ of LN_{set}: if ⊦ σ is provable in PLN_{set}, then ⊦ σ.

Note that (V) and (VII) – *the consistency-statements for* PLN_{set} [(VII) being the deductively stronger one, (V) the deductively weaker] – are deductively independent of *the completeness-statements for* PLN_{set}: (IV) and (VI) [(IV) being, relative to (VII), the deductively stronger one, (VI) the deductively weaker]. Thus, they may be relied on in deductions that negate the truth of the completeness-statements for PLN_{set}. In fact, there is such a deduction, and *prima facie* it "wreaks havoc" (fortunately only prima facie).

In view of the results of Kurt Gödel, it *seems* that PLN_{set} (including elementary arithmetic) enables the comprehensive encoding of metalinguistic sentences about sentences of LN_{set} *as sentences of* LN_{set}. Most interestingly, *apparently* there must be a closed sentence σ* of LN_{set} which states (in virtue of the encoding) *that* σ* *is not provable in* PLN_{set}, and of which the negation, ¬σ*, states *that* σ* *is provable in* PLN_{set}. Consider now:

(**i**) Assume: σ* is not true, hence: ¬σ* is true [LN_{set} is such that either σ´ or ¬σ´ is true, for any closed sentence σ´ of LN_{set}], *in other words*: σ* is provable in PLN_{set}; hence (due to the nature of PLN_{set}, where [the truth of] σ´ is provable if and only if ⊦ σ´ is provable): ⊦ σ* is provable in PLN_{set}; hence by (VII): ⊦ σ*, and hence: σ* is true – contradicting the assumption. Therefore: σ* is true. And, due to the nature of LN_{set}, we certainly also have: ⊦ σ*.

(**ii**) Since σ* is true (as has just been shown), [the truth of] σ* is not provable in PLN_{set} (for σ* states *that* σ* *is not provable in* PLN_{set}); and therefore: ⊦ σ* is not provable in PLN_{set} (since, due to the nature of PLN_{set}, [the truth of] σ´ is provable if and only if ⊦ σ´ is provable, for any closed sentence σ´ of LN_{set}).

Thus, (VI) is *apparently* being refuted here: there is an apparent counterexample to (VI), namely σ* [in view of (**i**) and (**ii**)]. But (VI) has been proven above! Moreover, also (IV) is *apparently* being refuted here, considering that (VI) follows from (IV) together with (VII) and considering that (VII) is quite unassailable. But (IV) has been proven in the previous section! – What is wrong here?

An assumption is wrong here. There simply is *no* sentence σ* of LN_{set} which states (in virtue of an encoding) *that* σ* *is not provable in* PLN_{set}, and of which the negation, ¬σ*, states *that σ* is provable in* PLN_{set}. The contrary impression is produced by thinking Gödelian techniques to be applicable to PLN_{set}; but, in fact, they are inapplicable to PLN_{set}. In the original Gödelian procedures, the metalinguistic predicate of provability for the arithmetical deductive system under consideration – which is matched, via encoding, by an arithmetical predicate: a predicate of the system – *does not accommodate infinitary proofs*: proofs with infinitely many steps. However, the metalinguistic provability predicate for PLN_{set} *does* accommodate infinitary (and yet perfectly manageable) proofs; see the *Questions-and-Answers* in the previous section.