FIDELITY AND ENTANGLEMENT OF A SPATIALLY EXTENDED LINEAR THREE-QUBIT REGISTER

ROLAND DOLL*, MARTIJN WUBS†, SIGMUND KOHLER*
and PETER HÄNGGI*

*Institut für Physik, Universität Augsburg,
Universitätsstraße 1, D-86135 Augsburg, Germany

†The Niels Bohr Institute, Niels Bohr International
Academy & QUANTOP,
Blegdamsvej 17, DK-2100 Copenhagen, Denmark

We study decoherence of a three-qubit array coupled to substrate phonons. Assuming an initial three-qubit entangled state that would be decoherence-free for identical qubit positions, allows us to focus on non-Markovian effects of the inevitable spatial qubit separation. It turns out that the coherence is most affected when the qubits are regularly spaced. Moreover, we find that up to a constant scaling factor, two-qubit entanglement is not influenced by the presence of the third qubit, even though all qubits interact via the phonon field.

1. Introduction

A major obstable on the way towards a working quantum computer is decoherence: the interaction of the qubits with their environment reduces the indispensable quantum coherence of the quantum states. Several strategies are pursued to beat decoherence. An active strategy is quantum error correction, which requires a redundant encoding of a logical qubit by several physical qubits.[1–3] Standard error correction protocols presuppose that all physical qubits couple individually to uncorrelated baths. A passive strategy is the use of decoherence-free subspaces (DFS).[4–7] There, one logical qubit is encoded by several physical qubits in such a way that the logical qubit states do not couple to the environment. Ideal DFSs occur when physical qubits couple via a collective coordinate to a common bath.

For solid-state qubits, the coupling to substrate phonons often is a relevant source of decoherence, in particular for charge qubits in quantum dots.[8] Whether these qubits experience correlated or uncorrelated noise depends on their distance in relation to the coherence length of the phonons, the sound velocity, the cutoff frequency and also on the dimensionality of the substrate. In Ref. 9, this dependence has been worked out by studying pure dephasing of a two-qubit state with an initial entanglement that is decoherence-free if both qubits couple to the environment at the same position.[10] If the qubits are spatially separated, however, this behavior changes: the entanglement decays until the transit time of a sound wave from one qubit to the other is reached. However, if the qubits are embedded in a quasi-one-dimensional environment, the noise at the two positions eventually becomes sufficiently correlated to bring decoherence to a standstill. In this way, a decoherence-poor subspace can emerge. Similar results can be found for the decoherence of entangled states of a regularly spaced qubit array.[11]

This work is motivated by two questions: First, do imperfections in the regular alignment of the qubits involve additional decoherence, or, put differently, how do irregularities in the spatial extension of a qubit register influence the collective decoherence properties? And second, is the entanglement of a qubit pair affected by its indirect interaction with a third qubit via the environment?

2. Qubits Coupled to a Bosonic Field

As sketched in Fig. 1, we consider a linear arrangement of three qubits at positions $x_1 = 0$, $x_2 = a + \delta$, and $x_3 = 2a$, i.e. the nearest-neighbor separations $x_{12} = a + \delta$ and $x_{23} = a - \delta$. To elaborate on the impact of spatially correlated noise we assume the qubit array to be embedded in a channel-like structure, as may be realized in carbon nanotubes or in linear ion traps. Thus, we treat the bosonic environment as effectively one-dimensional. The total Hamiltonian modelling this situation reads $H = H_{\mathrm{q}} + H_{\mathrm{qb}} + H_{\mathrm{b}}$, where $H_{\mathrm{q}} = \sum_{\nu=1}^{3} \hbar\Omega_\nu \sigma_{\nu z}/2$ describes three qubits with energy splittings $\hbar\Omega_\nu$ with $\sigma_{\nu z}$ being a Pauli matrix for qubit $\nu = 1, 2, 3$. Note that there is no direct interaction between the qubits. The bosonic field described by $H_{\mathrm{b}} = \sum_k \hbar\omega_k b_k^\dagger b_k$ consists of modes k with energies $\hbar\omega_k$ and the respective annihilation and creation operators b_k and $b_k^\dagger$. We assume a linear dispersion relation $\omega_k = c|k|$ with sound velocity c. The transit time of a field distortion between the qubits ν and ν' is then $t_{\nu\nu'} = x_{\nu\nu'}/c$ with $x_{\nu\nu'} = |x_\nu - x_{\nu'}|$. Qubit ν couples linearly via the operator X_ν to the field, so that the coupling Hamiltonian reads

$$H_{\mathrm{qb}} = \hbar \sum_{\nu=1}^{2} X_\nu \xi_\nu \,, \tag{1}$$

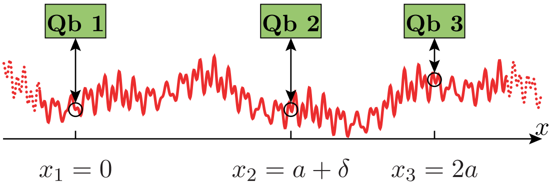


Fig. 1. Sketch of three qubits (green boxes) in a linear arrangement at positions x_ν, $\nu = 1, 2, 3$, with the distances $a + \delta$ and $a - \delta$. They interact via a coupling to the bosonic field (red line).

with $\xi_\nu = \xi(x_\nu) = \sum_k g_k e^{ikx_\nu}(b_k + b^\dagger_{-k})$ the bosonic field operator at the respective qubit position x_ν. We assume the microscopic coupling constants g_k to be real-valued, isotropic, and identical for all qubits, i.e. $g_{k\nu} = g_k$ and $g_{-k} = g_k$. They determine the spectral properties of the bath and show up in the spectral density $J(\omega) = \sum_k g_k^2 \delta(\omega - ck)$. Here we consider an Ohmic spectral density[12]

$$J(\omega) = \alpha\, \omega e^{-\omega/\omega_c}, \tag{2}$$

where the dimensionless parameter α denotes the overall coupling strength and ω_c the cutoff frequency of the bath spectrum.

The dynamics for the total density operator R of the qubits plus the environment is governed by the Liouville-von Neumann equation

$$i\hbar \frac{d}{dt}\widetilde{R}(t) = \big[\widetilde{H}_{\mathrm{qb}}(t), \widetilde{R}(t)\big]. \tag{3}$$

The tilde denotes the interaction-picture representation with respect to $H_0 = H_\mathrm{q} + H_\mathrm{b}$, i.e. $\tilde{A}(t) = U_0^\dagger(t) A\, U_0(t)$, with time-evolution operator $U_0(t) = \exp\{-iH_0 t/\hbar\}$. We assume that at time $t = 0$, the qubits can be prepared in a well-defined initial state, uncorrelated with the thermal bath. This constitutes an initial condition of the Feynman-Vernon type, where the total initial density matrix $\tilde{R}(0)$ is a direct product of a qubit and bath density operator, $\tilde{R}(0) = \tilde{\rho}(0) \otimes \rho_\mathrm{b}^\mathrm{eq}$. The canonical ensemble of the bosons at temperature T is denoted by $\rho_\mathrm{b}^\mathrm{eq} = \exp(-H_\mathrm{b}/k_\mathrm{B}T)/Z$, with Z the partition function. We are interested in the reduced density matrix of the qubits $\tilde{\rho}(t) = tr_\mathrm{b}\, \tilde{R}(t)$, where tr_b denotes the trace over the bath variables.

3. Dephasing and Entanglement Decay

In order to exemplify the impact of a spatial qubit separation on decoherence, we consider as the initial state the three-qubit entangled W state

$$|W\rangle = \frac{1}{\sqrt{3}}\big(\,|100\rangle + |010\rangle + |001\rangle\,\big), \tag{4}$$

i.e. $\tilde{\rho}(0) = |W\rangle\langle W|$, with the computational basis $\{|n_1 n_2 n_3\rangle\}$ where $\sigma_{\nu z}|n_1 n_2 n_3\rangle = (-1)^{n_\nu}|n_1 n_2 n_3\rangle$ and $n_\nu = 0, 1$. Our motivation to focus on the initial state (4) is twofold: first, W states play an important role in several protocols for quantum information processing,[13–15] so that the entanglement dynamics after their preparation is relevant in itself. Second, the W state is special since it stays robust under collective dephasing, i.e. for vanishing qubit separations ($x_1 = x_2 = x_3$).[10,11]

Pure phase noise will also be assumed in the following, in which case the coupling operators in Eq. (1) become $X_\nu = \sigma_{\nu z}$. As a consequence, the interaction-picture qubit operators remain time-independent, $\widetilde{X}_\nu(t) = X_\nu$. The exact time evolution of the reduced density operator can then be obtained, e.g. by a direct solution of the Liouville-von Neumann equation (3).[11] Amazingly, the exact result can even be obtained with an *approximative* time-local master equation approach already in second order of the qubit-field coupling α.[16] It turns out that the density matrix elements in the basis $\{|n_1 n_2 n_3\rangle\}$ at time t are proportional to their initial values. Thus, all density matrix elements that are initially zero remain zero, so that for the state $|W\rangle$, the dissipative quantum dynamics is restricted to the states

$$|1\rangle = |100\rangle, \quad |2\rangle = |010\rangle, \quad |3\rangle = |001\rangle, \tag{5}$$

i.e. at most nine out of 64 density matrix elements are nonvanishing. Initially they are all equal, i.e. $\rho_{jj'}(0) = \langle j|\rho(0)|j'\rangle = 1/3$, with $j, j' = 1, 2, 3$. They evolve as

$$\tilde{\rho}_{jj'}(t) = \frac{1}{3}\exp\{-\Lambda_{jj'}(t) + i[\phi_j(t) - \phi_{j'}(t)]\}, \tag{6}$$

where the real part $\Lambda_{jj'}(t)$ of the exponent accounts for the time-dependent amplitude damping of the matrix element; indirect interactions among qubits via the environment give rise to a time-dependent frequency shift and the concomitant phase $\phi_j(t) - \phi_{j'}(t)$ with $\phi_j(t) \equiv \varphi_j(t) - \Delta\Omega_j t$. Here $\Delta\Omega_j$ is a static frequency renormalization $\Omega_j \to \Omega_j + \Delta\Omega_j$ for qubit j and $\varphi_j(t)$ describes its onset; cf. Ref. 17. Henceforth we work in the interaction picture with respect to the renormalized energies. With the scaled temperature $\theta = k_{\mathrm{B}}T/\hbar\omega_{\mathrm{c}}$ and the scaled times $\tau = \omega_{\mathrm{c}}t$ and $\tau_{\nu\nu'} = \omega_{\mathrm{c}}t_{\nu\nu'}$, the density matrix elements become $\tilde{\rho}_{jj'}(\tau) = \exp\{-\Lambda_{jj'}(\tau) + i[\varphi_j(\tau) - \varphi_{j'}(\tau)]\}/3$[11] with the phases

$$\varphi_j(\tau) = -\frac{\alpha}{2}\sum_{\nu,\nu'=1}^{3}(-1)^{\delta_{j\nu}+\delta_{j\nu'}}\sum_{\pm}\arctan[\tau \pm \tau_{\nu\nu'}], \tag{7}$$

and the amplitude damping $\exp\{-\Lambda_{jj'}(\tau)\} = f(\tau, \tau_{jj'})/f(\tau, 0)$, where

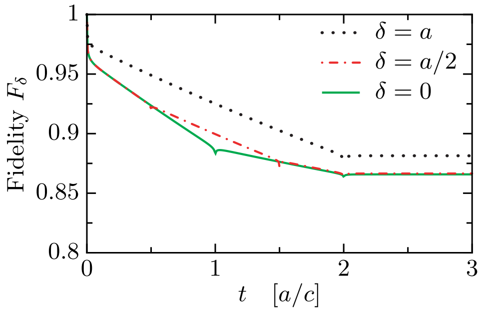
$$f(\tau, \tau') = \frac{|\Gamma(\theta[1 - i\tau'])|^{16\alpha}}{|\Gamma^2(\theta[1 + i(\tau - \tau')])\Gamma^2(\theta[1 + i(\tau - \tau')])(1 + \tau^2[1 - i\tau']^{-2})|^{4\alpha}}. \tag{8}$$

As expected for pure dephasing, populations are preserved: $\tilde{\rho}_{jj}(t) = \tilde{\rho}_{jj}(0)$. This implies that neither the qubits nor the total system will reach thermal equilibrium. Nevertheless decoherence does occur since relative phases between eigenstates will be randomized so that off-diagonal density matrix elements decay.

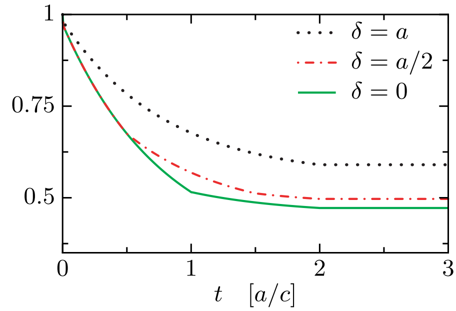
Only in the absence of the environment ($\alpha = 0$), the qubits remain in the W state (4). As a measure for the deviation from this "ideal" output state $\rho(0) = |W\rangle\langle W|$, we employ the fidelity $F(t) = tr\{\tilde{\rho}(t)\tilde{\rho}(0)\}$,[18] which in our case reads

$$F_\delta(t) = \frac{1}{3}\sum_{j,j'=1}^{3}\tilde{\rho}_{jj'}(t) = \frac{1}{3} + \frac{2}{9}\sum_{j<j'} e^{-\Lambda_{jj'}(t)}\cos[\varphi_j(t) - \varphi_{j'}(t)]\,. \tag{9}$$

The index δ refers to the displacement of the middle qubit $\nu = 2$. The time evolution of the fidelity is shown in Figs. 2(a) and 2(b) for two different temperatures. Clearly, the fidelity decay is slowed down whenever a transit time $t = t_{jj'}$ is reached. At time $t = t_{13}$, when the field has also enabled communication between the two outer qubits 1 and 3, decoherence even comes to a standstill! Note that other initial states may lead to complete dephasing.[17] The fidelity saturates to a finite value $F_\delta(t > t_{13}) = F_\delta(\infty)$ which is larger the lower the temperature. For a fixed



(a) $\theta = 10^{-4}$



(b) $\theta = 10^{-3}$

Fig. 2. Time evolution of the fidelity (9) for the temperatures $\theta = k_{\mathrm{B}}T/\hbar\omega_{\mathrm{c}} = 10^{-4}$ (a) and $\theta = 10^{-3}$ (b), the qubit-field coupling strength $\alpha = 0.001$, and various displacements δ of the middle qubit (see Fig. 1).

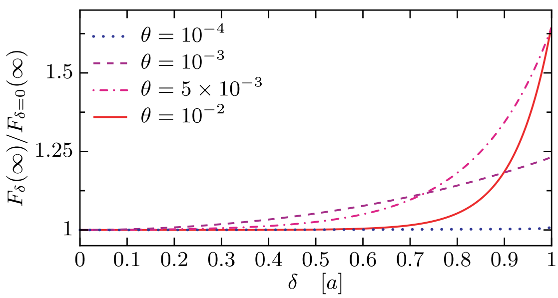


Fig. 3. Final fidelity $F_\delta(\infty)$ as a function of the displacement δ of the middle qubit, scaled to the value for equidistant qubit arrangement $\delta = 0$ for various temperatures $\theta = k_{\mathrm{B}}T/\hbar\omega_{\mathrm{c}}$ and coupling strength $\alpha = 0.001$.

temperature, the stable fidelity increases if the middle qubit is displaced away from the central position and is maximal for $\delta = \pm a$, i.e. if qubit 2 becomes co-located with qubit 1 or 3. The fidelity gain as a function of the asymmetry δ/a is shown in Fig. 3.

The explanation of this behavior of $F_\delta(\infty)$ follows from Eq. (9). The individual coherences $\tilde{\rho}_{\nu\nu'}$ decay approximately exponentially and with the same decay rate,[11] but stop decaying at different times $|x_\nu - x_{\nu'}|/c$. Hence for the fidelity it pays off to displace the middle qubit, thereby stopping the decay of $\tilde{\rho}_{23}$ at an earlier time $(a-\delta)/c$ at the expense of stopping the decay of $\tilde{\rho}_{12}$ only at time $(a+\delta)/c$.

For the extreme cases $\delta = 0$ and $\delta = a$, the fidelity decreases monotonously with temperature, but for $0 < \delta < a$ an optimal temperature exists for which the fidelity gain is maximal. This behavior is related to the fact that for all temperatures $\tilde{\rho}_{12}(t) = \tilde{\rho}_{23}(t)$ when $\delta = 0$, and $\tilde{\rho}_{13}(t) = \tilde{\rho}_{23}(t)$ when $\delta = a$. Only in the general asymmetric configuration do all three coherences stop decaying at different times.

The initial W state is special in the sense that it exhibits bipartite entanglement between any qubit pair. But does the middle qubit affect the entanglement decay of the outer ones in any way? The reduced density matrix of qubits 1 and 3 is

$$\mathrm{tr}_2\tilde{\rho}(t) = \frac{1}{3}\left(|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10|\right) + \tilde{\rho}_{13}(t)|10\rangle\langle 01| + \tilde{\rho}_{31}(t)|01\rangle\langle 10|. \quad (10)$$

It depends on time only through the coherence $\tilde{\rho}_{13}(t) = \tilde{\rho}^*_{31}(t)$ and thus, according to Eq. (6), only on the transit time $t_{13} = 2a/c$ between the outer qubits. The entanglement of the outer qubits is therefore *independent* of the displacement δ of the middle qubit! Indeed, for their concurrence,[19] we find $C[tr_2\,\tilde{\rho}(t)] = 2|\tilde{\rho}_{13}(t)|$. The same dynamics is found for the two-qubit concurrence when only the outer two qubits had been present, with the two-qubit W state (i.e. the robust Bell state) as their initial state.[9] Surprisingly, the only effect of the presence of the middle qubit is a rescaling of the concurrence of the outer two by a factor 2/3.

4. Conclusion

We have studied the pure dephasing of three spatially separated qubits in an Ohmic environment. Dephasing can be incomplete when starting in a W state, not only for symmetrically spaced qubits. For fixed separation of the outer two qubits, the final fidelity is even larger the more "clustered" the qubits are. Surprisingly, the middle qubit does not affect the dynamics of the concurrence of the outer two.

Acknowledgments

This work has been supported by DFG via SFB 631. P. Hänggi and S. Kohler acknowledge funding by the DFG excellence cluster "Nanosystems Initiative Munich".

References

1. P. W. Shor, *Phys. Rev. A* **52** (1995) R2493.
2. J. Kempe, Séminaire poincaré **2** (2005) 1.
3. A. W. Steane, in *Proc. Int. School of Physics "Enrico Fermi", CLXII, Quantum Computers, Algorithms and Chaos*, eds. G. Casati, D. L. Shepelyansky and P. Zoller (IOS Press, Amsterdam, 2006), p. 1.
4. G. M. Palma, K.-A. Suominen and A. K. Ekert, *Proc. Roy. Soc. London, Ser. A* **452** (1996) 567.
5. P. Zanardi and M. Rasetti, *Phys. Rev. Lett.* **79** (1997) 3306.
6. D. A. Lidar, I. L. Chuang and K. B. Whaley, *Phys. Rev. Lett.* **81** (1998) 2594.
7. D. A. Lidar and K. B. Whaley, in *Irreversible Quantum Dynamics*, eds. F. Benatti and R. Floreanini (Springer, Berlin, 2003), pp. 83–120.
8. T. Hayashi, T. Fujisawa, H. D. Cheong, Y. H. Jeong and Y. Hirayama, *Phys. Rev. Lett.* **91** (2003) 226804.
9. R. Doll, M. Wubs, P. Hänggi and S. Kohler, *Europhys. Lett.* **76** (2006) 547.
10. T. Yu and J. H. Eberly, *Phys. Rev. B* **66** (2002) 193306.
11. R. Doll, M. Wubs, P. Hänggi and S. Kohler, *Phys. Rev. B* **76** (2007) 045317.
12. A. J. Leggett, S. Chakravarty, A. T. Dorsey, M. P. A. Fisher, A. Garg and W. Zwerger, *Rev. Mod. Phys.* **59** (1987) 1.

13. J. Joo, Y.-J. Park, S. Oh and J. Kim, *New J. Phys.* **5** (2003) 136.
14. Y.-J. Han, Y.-S. Zhang and G.-C. Guo, *Phys. Lett. A* **295** (2002) 61.
15. P. Agrawal and A. Pati, *Phys. Rev. A* **74** (2006) 062320.
16. R. Doll, D. Zueco, M. Wubs, S. Kohler and P. Hänggi, *Chem. Phys.* (2007), doi:10.1016/j.chemphys.2007.09.003.
17. D. Solenov, D. Tolkunov and V. Privman, *Phys. Rev. B* **75** (2007) 035134.
18. J. F. Poyatos, J. I. Cirac and P. Zoller, *Phys. Rev. Lett.* **78** (1997) 390.
19. W. K. Wootters, *Phys. Rev. Lett.* **80** (1998) 2245.