SIAM J. Control Optim.
Vol. 62, No. 4, pp. 2387–2411

CHAIN CONTROLLABILITY OF LINEAR CONTROL SYSTEMS*

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Abstract. For linear control systems with bounded control range, chain controllability properties are analyzed. It is shown that there exists a unique chain control set and that it equals the sum of the control set around the origin and the center Lyapunov space of the homogeneous part. For the proof, the linear control system is extended to a bilinear control system on an augmented state space. This system induces a control system on projective space. For the associated control flow, attractor-repeller decompositions are used to show that the control system on projective space has a unique chain control set that is not contained in the equator. It is given by the image of the chain control set of the original linear control system.

Key words. linear control system, chain control set, attractor, Poincaré sphere

MSC codes. 93B05, 37B20

DOI. 10.1137/23M1626347

1. Introduction.

We study generalized controllability properties of linear control systems on $\mathbb{R}^n$ with control restrictions of the form

$$\dot{x}(t) = Ax(t) + Bu(t),\, u \in \mathcal{U}, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$, and the set of control functions is defined by

$$\mathcal{U} = \{u \in L^\infty(\mathbb{R}, \mathbb{R}^m)\,|u(t) \in U \text{ for almost all } t \in \mathbb{R}\}; \tag{1.2}$$

here, the control range U is a compact and convex neighborhood of $0 \in \mathbb{R}^m$. We denote the solution for initial condition $x(0) = x_0 \in \mathbb{R}^n$ and control $u \in \mathcal{U}$ by $\varphi(t, x_0, u), t \in \mathbb{R}$.

The main topic of this paper is chain controllability properties (cf. Definition 2.11). They constitute a weaker version of approximate controllability in infinite time and may be difficult to distinguish from it in numerical computations. Here, small jumps in the trajectories are allowed, and hence, chain controllability is not a physical notion. In the theory of dynamical systems, analogous constructions (going back to Rufus Bowen and Charles Conley) have been quite successful in order to describe the limit behavior as time tends to infinity for complicated flows.

Due to the control restriction, the familiar linear algebra tools of linear systems theory have to be complemented by methods from dynamical systems theory. Control system (1.1) defines a flow $\Phi : \mathbb{R} \times \mathcal{U} \times \mathbb{R}^n \to \mathcal{U} \times \mathbb{R}^n, \Phi_t(u, x_0) = (u(t + \cdot), \varphi(t, x_0, u))$ on the state space $\mathcal{U} \times \mathbb{R}^n$. Here, the set $\mathcal{U}$ of control functions is endowed with the weak* topology of $L^\infty(\mathbb{R}, \mathbb{R}^m)$, which makes it into a compact metrizable space, and the flow Φ, called the control flow, becomes continuous (cf. the comments after (2.6)). An important tool for us is the extension of the linear control system on $\mathbb{R}^n$

*Received by the editors December 22, 2023; accepted for publication (in revised form) June 3, 2024; published electronically August 19, 2024.
https://doi.org/10.1137/23M1626347
Funding: The second author was partially supported by CNPq grant 309409/2023-3. The third author was supported by CNPq grant 88887.669835/2022-00.
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to a bilinear control system on $\mathbb{R}^n \times \mathbb{R}$ and an ensuing projection to projective space $\mathbb{P}^n$. This provides us with a compactification of the state space and at the same time introduces convenient linear structures. In fact, the affine control flow Φ on the state space $\mathcal{U}\times\mathbb{R}^n$ associated with linear control system (1.1) is extended to a linear control flow Φ^1 on the vector bundle $\mathcal{U} \times \mathbb{R}^n \times \mathbb{R}$ over the base space $\mathcal{U}$. The projection to $\mathbb{P}^n$ allows us to use attractor-repeller decompositions that clarify the relation between the chain control set (the maximal chain controllable subset) and the control set (the maximal approximately controllable subset) in $\mathbb{R}^n$. Furthermore, the powerful Selgrade decomposition can be applied to the linear control flow Φ^1, revealing further relations between the central Selgrade bundle $\mathcal{V}_c^1$ (i.e., the unique Selgrade bundle where the last component in $\mathcal{U} \times \mathbb{R}^n \times \mathbb{R}$ is nontrivial) and the chain control sets on $\mathbb{P}^n$. In the theory of nonlinear differential equations, analogous extensions of the state space and ensuing projections to the sphere bear the name of the Poincaré sphere; cf. Perko [12]. The Selgrade decomposition of linear flows on vector bundles (cf., e.g., Salamon and Zehnder [13], Colonius and Kliemann [4, Theorem 5.2.5]) provides the finest decomposition into exponentially separated subbundles. It is constructed by the finest Morse decomposition of the induced flow on the projective bundle $\mathcal{U} \times \mathbb{P}^n$, where the Morse sets are the maximal chain transitive subsets.

Many techniques and results for chain controllability rely on compactness properties. Hence, a major source of difficulties in our case is the fact that the state space $\mathbb{R}^n$ is not compact. The construction using the Poincaré sphere admits the description of the system behavior near infinity.

For chain transitivity and chain recurrence in the topological theory of flows on metric spaces, consult Alongi and Nelson [1]. Background on control sets, chain control sets, and control flows is given in Colonius and Kliemann [4, 5], Kawan [10], and also in Desheng Li [11], who proposes an approach to chain controllability via differential inclusions. A recent paper [8] by da Silva analyzes compact chain control sets of linear control systems on connected Lie groups. Ayala, da Silva, and Mamani [3] explicitly compute control sets for linear control systems in two-dimensional cases. In Colonius and Santana [7], the approach of the present paper is used for affine flows on vector bundles, and some applications to affine control systems are given.

The main results of this paper are Theorem 4.8 and Theorem 5.3. The former theorem characterizes the chain control set E as the sum of the center Lyapunov space for A (i.e., the sum of the generalized eigenspaces for eigenvalues with vanishing real part) and the closure of the control set around the origin. On the way, we need that the chain recurrent set of the flow for a linear autonomous differential equation coincides with the center Lyapunov space; cf. Theorem 3.2. This is presumably known, but we could not find a reference in the literature; hence, we include a proof. Theorem 5.3 characterizes the central Selgrade bundle $\mathcal{V}_c^1$. A consequence presented in Corollary 5.4 is that the chain control set E_c^1 on the projective Poincaré sphere $\mathbb{P}^n$ coincides with the closure of the image of the chain control set E.

Section 2 contains preliminary results on control systems and recalls notions and results from the topological theory of flows on metric spaces, including Selgrade's theorem for linear flows on vector bundles. It is noteworthy that we can simplify the notion of chain control sets: It is not necessary to require that, for every $x\in E$, there exists $u\in\mathcal{U}$ with $\varphi(t,x,u)\in E$ for all $t\in\mathbb{R}$ since this holds for every maximal chain controllable set; cf. Remark 2.12. Section 3 proves that the center Lyapunov space of an autonomous linear differential equation is chain transitive, yielding first results on chain controllability. Section 4 provides the characterization of the chain control set E in $\mathbb{R}^n$. Section 5 analyzes the induced system on the Poincaré sphere, and section 6 presents two examples.

Notation: The control $u(t) \equiv 0$ in $\mathcal{U}$ is denoted by $0_{\mathcal{U}}$. The trivial subspace $\{0\} \subset \mathbb{R}^n$ is denoted by 0_n, and 0_1 is abbreviated by 0. The letter $\mathbb{K}$ denotes $\mathbb{R}$ or $\mathbb{C}$.

2. Preliminaries.

The first subsection presents elements of the topological theory of flows on metric spaces. Subsection 2.2 describes control sets and chain control sets for control-affine systems and recalls the notion of control flows. Subsection 2.3 derives some properties of control sets for linear control systems.

2.1. Flows on metric spaces.

For the following concepts for flows on metric spaces, consult Alongi and Nelson [1] and Colonius and Kliemann [4, Appendix B], [5].

A conjugacy of flows Ψ and Ψ' on complete metric spaces X and X', respectively, is a homeomorphism $h : X \to X'$ with $h(\Psi(t,x)) = \Psi'(t, h(x))$ for all $x \in X, t \in \mathbb{R}$. Where convenient, we also write $\Psi_t(x) = \Psi(t,x)$.

For ε, $T > 0$, an (ε, T)-chain ζ for Ψ from x to y is given by $k \in \mathbb{N}, T_0, \ldots, T_{k-1} \geq T$, and $x_0 = x, \ldots, x_{k-1}, x_k = y \in X$ with $d(\Psi(T_i, x_i), x_{i+1}) < \varepsilon$ for $i = 0, \ldots, k-1$. For $x \in X$, the ω-limit set is $\omega(x) = \{y \in X \,|\, \exists t_k \to \infty : \Psi(t_k, x) \to y\}$ and the α-limit set is $\alpha(x) = \{y \in X \,|\, \exists t_k \to -\infty : \Psi(t_k, x) \to y\}$. The (forward) chain limit set for Ψ is $\Omega(x) = \{y \in X \,|\, \text{for all } \varepsilon, T > 0 \, \exists (\varepsilon, T)\text{-chain from } x \text{ to } y\}$. The backward chain limit set $\Omega^*(x)$ of x is the chain limit set of x for the time-reversed flow $\Psi^*(t,x) := \Psi(-t, x), t \in \mathbb{R}, x \in X$. A point $x \in X$ is chain recurrent if $x \in \Omega(x)$, and a nonvoid set Y is called chain transitive if $y \in \Omega(x)$ for all $x, y \in Y$. Observe that any nonvoid subset of a chain transitive set is chain transitive and (cf. [1, Proposition 2.7.10]) a set is chain transitive if and only if its closure is chain transitive. If X is chain transitive for a flow on X, then the flow is also called chain transitive. On a compact metric space, the maximal chain transitive sets, called the chain recurrent components, are the connected components of the chain recurrent set (i.e., the set of chain recurrent points) and the flow restricted to a chain recurrent component is chain transitive.

PROPOSITION 2.1. *For any flow Ψ on X, the forward and backward chain limit sets are related by $y \in \Omega(x)$ if and only if $x \in \Omega^*(y)$.*

Proof. Let $y \in \Omega(x)$ and $\varepsilon, T > 0$. The difficulty in the proof lies in the fact that, in the last step of an (ε, T)-chain from x to y, there is a jump to y. By continuity of the flow, there is $\delta \in (0, \varepsilon)$ such that $d(y,z) < \delta$ implies that $d(\Psi(-T, y), \Psi(-T, z)) < \varepsilon$. Consider a $(\delta, 2T)$ chain from x to y given by $T_0, \ldots, T_{k-1} \geq 2T, x = x_0, x_1, \ldots, x_k = y$. We construct an (ε, T)-chain from y to x for the time-reversed flow Ψ^* in the following way:

Let $T_0^* = T, T_1^* = T_{k-1} - T \geq T, T_2^* = T_{k-2}, \ldots, T_k^* = T_0$, and

$$x_0^* = y,\, x_1^* = \Psi_{-T}\left(\Psi_{T_{k-1}}(x_{k-1})\right),\, x_2^* = \Psi_{T_{k-2}}(x_{k-2}), \ldots,\, x_k^* = \Psi_{T_0}(x_0),\, x_{k+1}^* = x.$$

Since $d(y, \Psi_{T_{k-1}}(x_{k-1})) < \delta$, it follows that

$$\begin{aligned} d\left(\Psi^*_{T_0^*}(x_0^*), x_1^*\right) &= d\left(\Psi_{-T}(y), \Psi_{-T}\left(\Psi_{T_{k-1}}(x_{k-1})\right)\right) < \varepsilon, \\ d\left(\Psi^*_{T_i^*}(x_i^*), x_{i+1}^*\right) &= d\left(x_{k-i}, \Psi_{T_{k-i-1}}(x_{k-i-1})\right) < \varepsilon \text{ for } i \in \{1, \ldots, k-1\}, \\ d\left(\Psi^*_{T_k^*}(x_k^*), x_{k+1}^*\right) &= d\left(\Psi_{-T_0}(\Psi_{T_0}(x_0)), x_0\right) = 0. \end{aligned}$$

Thus, $x \in \Omega^*(y)$. The other implication follows analogously since $\Psi = (\Psi^*)^*$. ∎

PROPOSITION 2.2.
(i) *For a flow* Ψ *on* X*, the chain limit sets* $\Omega(x)$ *and* $\Omega^*(x), x \in X$ *are invariant.*
(ii) *Any maximal chain transitive set* Y *of* Ψ *is invariant and coincides with* $\Omega(x) \cap \Omega^*(x)$ *for all* $x \in Y$.

Proof. (i) By Proposition 2.1, it suffices to prove that $\Psi_t(y) \in \Omega(x)$ for all $t \in \mathbb{R}$ and all $y \in \Omega(x)$. Fix $t \in \mathbb{R}$, and let $\varepsilon, T > 0$. By continuity, there is $\delta > 0$ such that

$$d(z,y) < \delta \text{ implies that } d\left(\Psi_t(z), \Psi_t(y)\right) < \varepsilon.$$

Pick a (δ, S) chain from x to y with $S + t \geq T$. Then, the final piece of this chain satisfies $d\left(\Psi_{T_{k-1}}(x_{k-1}), y\right) < \delta$, implying that

$$d\left(\Psi_{t+T_{k-1}}(x_{k-1}), \Psi_t(y)\right) = d\left(\Psi_t(\Psi_{T_{k-1}}(x_{k-1})), \Psi_t(y)\right) < \varepsilon.$$

Hence, we obtain an (ε, T)-chain from x to $\Psi_t(y)$.

(ii) By (i), it suffices to prove that $Y = \Omega(x) \cap \Omega^*(x)$ for all $x \in Y$. Let $y \in Y$. Then, $y \in \Omega(x)$ and $x \in \Omega(y)$; hence, by Proposition 2.1, it follows that $y \in \Omega^*(x)$. For the converse inclusion, note that any point $y \in \Omega(x) \cap \Omega^*(x)$ satisfies $y \in \Omega(x)$ and $x \in \Omega(y)$; hence, the chain transitive set $\Omega(x) \cap \Omega^*(x)$ is contained in the maximal chain transitive set containing x. ∎

Let X be compact. An attractor is a compact invariant set $\mathcal{A}$ such that $\mathcal{A} \subset \mathrm{int} N$ for a set N with

$$\mathcal{A} = \omega(N) := \{y \in X \,|\, \exists t_k \to \infty \ \exists x_k \in N : \Psi_{t_k}(x_k) \to y\}. \tag{2.1}$$

The complementary repeller is $\mathcal{A}^* := \{y \in X \,|\, \omega(x) \cap \mathcal{A} = \varnothing\}$. It is also compact invariant and has the property that $\mathcal{A}^* \subset \mathrm{int} N^*$ for a set N^* with

$$\mathcal{A}^* = \omega^*(N^*) := \{y \in X \,|\, \exists t_k \to -\infty \ \exists x_k \in N : \Psi_{t_k}(x_k) \to y\}. \tag{2.2}$$

The relation to the chain recurrent set is given by the following theorem (cf. [4, Theorem B.2.26]).

THEOREM 2.3. *For a flow on a compact metric space* X*, the chain recurrent set coincides with* $\bigcap(\mathcal{A} \cup \mathcal{A}^*)$*, where the intersection is taken over all attractors* $\mathcal{A}$.

A related concept are Morse decompositions, introduced next. Note first that a compact subset $K \subset X$ is called isolated invariant for Ψ if $\Psi_t(x) \in K$ for all $x \in K$ and all $t \in \mathbb{R}$ and there exists a set N with $K \subset \mathrm{int}\, N$ such that $\Psi_t(x) \in N$ for all $t \in \mathbb{R}$ implies that $x \in K$.

DEFINITION 2.4. *A Morse decomposition of a flow* Ψ *on a compact metric space* X *is a finite collection* $\{\mathcal{M}_i \,|\, i = 1, \ldots, \ell\}$ *of nonvoid, pairwise disjoint, and compact isolated invariant sets such that,*
(i) *for all* $x \in X$*, the limit sets satisfy* $\omega(x), \alpha(x) \subset \bigcup_{i=1}^{\ell} \mathcal{M}_i$ *and,*
(ii) *supposing that there are* $\mathcal{M}_{j_0}, \mathcal{M}_{j_1}, \ldots, \mathcal{M}_{j_k}$ *and* $x_1, \ldots, x_k \in X \setminus \bigcup_{i=1}^{\ell} \mathcal{M}_i$ *with* $\alpha(x_i) \subset \mathcal{M}_{j_{i-1}}$ *and* $\omega(x_i) \subset \mathcal{M}_{j_i}$ *for* $i = 1, \ldots, k$*, then* $\mathcal{M}_{j_0} \neq \mathcal{M}_{j_k}$.

The elements of a Morse decomposition are called Morse sets. A Morse decomposition $\{\mathcal{M}_1, \ldots, \mathcal{M}_k\}$ *is called finer than a Morse decomposition* $\{\mathcal{M}'_1, \ldots, \mathcal{M}'_{k'}\}$*, if, for all* $j \in \{1, \ldots, k'\}$*, there is* $i \in \{1, \ldots, k\}$ *with* $\mathcal{M}_i \subset \mathcal{M}'_j$.

An order is defined by the relation $\mathcal{M}_i \preceq \mathcal{M}_j$ if there are indices $j_0, \ldots, j_k$ with $\mathcal{M}_i = \mathcal{M}_{j_0}, \mathcal{M}_j = \mathcal{M}_{j_k}$ and points $x_{j_i} \in X$ with

$$\alpha(x_{j_i}) \subset \mathcal{M}_{j_i - 1} \text{ and } \omega(x_{j_i}) \subset \mathcal{M}_{j_i} \text{ for } i = 1, \ldots, k.$$

The following theorem relates chain recurrent components and Morse decompositions; cf. [5, Theorem 8.3.3].

THEOREM 2.5. *For a flow on a compact metric space, there exists a finest Morse decomposition if and only if the chain recurrent set has only finitely many connected components. Then, the Morse sets coincide with the chain recurrent components.*

Remark 2.6. For continuous maps f on locally compact metric spaces X, Hurley [9] proposes modifying the definition of chain recurrence by considering, instead of a constant $\varepsilon > 0$, continuous functions $\varepsilon : X \to (0,\infty)$. A strong $\varepsilon(\cdot)$-chain consists of $x_0, x_1, \ldots, x_k$ with $d(f(x_j), x_{j+1}) < \varepsilon(f(x_j))$ for all j, and a point $x \in X$ is strongly chain recurrent if, for each such function $\varepsilon(\cdot)$, there is a strong $\varepsilon(\cdot)$-chain from x to x. This approach does not appear appropriate in our context, mainly due to the fact that we will use the compactification provided by the Poincaré sphere $\mathbb{P}^n$.

We will consider vector bundles $\mathcal{V} = B \times \mathbb{R}^n$, where B is a compact metric base space. A linear flow $\Psi = (\theta, \psi)$ on $B \times \mathbb{R}^n$ is a flow of the form

$$\Psi : \mathbb{R} \times B \times \mathbb{R}^n \to B \times \mathbb{R}^n,\ \Psi_t(b,x) = (\theta_t b, \psi(t,b,x)) \text{ for } (t,b,x) \in \mathbb{R} \times B \times \mathbb{R}^n,$$

where θ is a flow on the base space B and $\psi(t,b,x)$ is linear in x; i.e., $\psi(t,b,\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 \psi(t,b,x_1) + \alpha_2 \psi(t,b,x_2)$ for $\alpha_1, \alpha_2 \in \mathbb{R}$ and $x_1, x_2 \in \mathbb{R}^n$. A closed subset $\mathcal{V}$ of $B \times \mathbb{R}^n$ that intersects each fiber $\{b\} \times \mathbb{R}^n, b \in B$ in a linear subspace of constant dimension is a subbundle. Denote the projection $\mathbb{R}^n \setminus \{0\} \to \mathbb{P}^{n-1}$ as well as the corresponding map $B \times (\mathbb{R}^n \setminus \{0\}) \to B \times \mathbb{P}^{n-1}$ by the letter $\mathbb{P}$. A linear flow Ψ induces a flow $\mathbb{P}\Psi$ on the projective bundle $B \times \mathbb{P}^{n-1}$, which is a compact metric space; a metric on $\mathbb{P}^{n-1}$ can be obtained by defining, for elements $p_1 = \mathbb{P}x, p_2 = \mathbb{P}y$,

$$d(p_1, p_2) = \min\left\{ \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|, \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| \right\}, \tag{2.3}$$

and a metric on $B \times \mathbb{P}^{n-1}$ is defined by taking the maximum of the distances in B and $\mathbb{P}^{n-1}$.

For a linear flow Ψ, two nontrivial invariant subbundles $(\mathcal{V}^+, \mathcal{V}^-)$ with $B \times \mathbb{R}^n = \mathcal{V}^+ \oplus \mathcal{V}^-$ (a Whitney sum) are exponentially separated if there are $c, \mu > 0$ with

$$\left\| \Psi(t,b,x^+) \right\| \le c e^{-\mu t} \left\| \Psi(t,b,x^-) \right\|, t \ge 0 \text{ for } (b, x^\pm) \in \mathcal{V}^\pm,\ \left\| x^+ \right\| = \left\| x^- \right\|. \tag{2.4}$$

The following is Selgrade's theorem for linear flows; cf. [5, Theorem 9.2.5], and [4, Theorem 5.1.4] for the result on exponential separation.

THEOREM 2.7. *Let $\Psi = (\theta, \psi) : \mathbb{R} \times B \times \mathbb{R}^n \to B \times \mathbb{R}^n$ be a linear flow on the vector bundle $B \times \mathbb{R}^n$ with chain transitive flow θ on the base space B. Then, the projected flow $\mathbb{P}\Psi$ on $B \times \mathbb{P}^{n-1}$ has the linearly ordered chain recurrent components $\mathcal{M}_1 \preceq \cdots \preceq \mathcal{M}_\ell, 1 \le \ell \le n$. These components form the finest Morse decomposition for $\mathbb{P}\Psi$. The lifts of the Morse sets $\mathcal{M}_i$*

$$\mathcal{V}_i = \mathbb{P}^{-1}\mathcal{M}_i := \{(b,x) \in B \times \mathbb{R}^n \,|x \ne 0 \Rightarrow (b, \mathbb{P}x) \in \mathcal{M}_i \}$$

are subbundles, called the Selgrade bundles. They form the decomposition $B \times \mathbb{R}^n = \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_\ell$. This Selgrade decomposition is the finest decomposition into exponentially separated subbundles: For any pair of exponentially separated subbundles $(\mathcal{V}^+, \mathcal{V}^-)$, there is $1 \le j < \ell$ with

$$\mathcal{V}^+ = \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_j \text{ and } \mathcal{V}^- = \mathcal{V}_{j+1} \oplus \cdots \oplus \mathcal{V}_\ell .$$

Conversely, subbundles $\mathcal{V}^+$ and $\mathcal{V}^-$ defined in this way are exponentially separated.

2.2. Control sets, chain control sets, and control flows. Consider control-affine systems of the form

$$\dot{x}(t) = X_0(x(t)) + \sum_{i=1}^{m} u_i(t)X_i(x(t)),\ u \in \mathcal{U}, \tag{2.5}$$

where $X_0, X_1, \ldots, X_m$ are smooth (C^∞-) vector fields on a smooth manifold M and $\mathcal{U}$ is defined by (1.2). We assume that, for every control $u \in \mathcal{U}$ and every initial state $x(0) = x_0 \in M$, there exists a unique (Carathéodory) solution $\varphi(t, x_0, u), t \in \mathbb{R}$.

The control flow associated with control system (2.5) is the flow on $\mathcal{U} \times M$ defined by

$$\Phi : \mathbb{R} \times \mathcal{U} \times M \to \mathcal{U} \times M, \Phi_t(u, x) = (\theta_t u, \varphi(t, x, u)), \tag{2.6}$$

where $\theta_t u = u(t + \cdot)$ is the right shift on $\mathcal{U}$. Note that $\Phi_{t+s} = \Phi_t \circ \Phi_s$ for $t, s \in \mathbb{R}$. The space $\mathcal{U}$ is a compact metrizable space with respect to the weak* topology of $L^\infty(\mathbb{R}, \mathbb{R}^m)$ (we fix such a metric) and the shift flow θ is continuous; cf. Kawan [10, Proposition 1.15]. The flow Φ is continuous; cf. [10, Proposition 1.17].

For $x \in M$, the controllable set $\mathbf{C}(x)$ and the reachable set $\mathbf{R}(x)$ are defined as

$$\mathbf{C}(x) = \{ y \in M \,|\, \exists u \in \mathcal{U}\ \exists T > 0 : \varphi(T, y, u) = x \},$$
$$\mathbf{R}(x) = \{ y \in M \,|\, \exists u \in \mathcal{U}\ \exists T > 0 : y = \varphi(T, x, u) \},$$

respectively. The accessibility rank condition is defined as

$$\dim \mathcal{LA}\{X_0, X_1, \ldots, X_m\}(x) = \dim M \text{ for all } x \in M; \tag{2.7}$$

here, $\mathcal{LA}\{X_0, X_1, \ldots, X_m\}(x)$ is the subspace of the tangent space $T_x M$ corresponding to the vector fields, evaluated in x, in the Lie algebra generated by $X_0, X_1, \ldots, X_m$. The controllable sets $\mathbf{C}(x)$ are the reachable sets of the time-reversed system

$$\dot{x}(t) = -X_0(x(t)) - \sum_{i=1}^{m} u_i(t)X_i(x(t)),\ u \in \mathcal{U}. \tag{2.8}$$

The following definition introduces sets of complete approximate controllability.

Definition 2.8. *A nonvoid set $D \subset M$ is called a control set of system* (2.5) *if it has the following properties:* (i) *for all $x \in D$, there is a control $u \in \mathcal{U}$ such that $\varphi(t, x, u) \in D$ for all $t \geq 0$;* (ii) *for all $x \in D$, one has $D \subset \overline{\mathbf{R}(x)}$; and* (iii) *$D$ is maximal with these properties; that is, if $D' \supset D$ satisfies conditions* (i) *and* (ii)*, then $D' = D$.*

We recall some properties of control sets; cf. Colonius and Kliemann [4, Chapter 3].

Remark 2.9. If the intersection of two control sets is nonvoid, the maximality property (iii) implies that they coincide. If (2.7) holds, then, by [4, Lemma 3.2.13(i)], $\overline{D} = \overline{\text{int}(D)}$ and $D = \mathbf{C}(x) \cap \overline{\mathbf{R}(x)}$ for all $x \in \text{int}(D)$ and $\text{int}(D) \subset \mathbf{R}(x)$ for all $x \in D$.

Next, we introduce a notion of controllability allowing for (small) jumps between pieces of trajectories. Here, we fix a metric d on M.

Definition 2.10. *Let $x, y \in M$. For $\varepsilon, T > 0$, a controlled (ε, T)-chain ζ from x to y is given by $k \in \mathbb{N}$, $x_0 = x, x_1, \ldots, x_k = y \in M$, $u_0, \ldots, u_{k-1} \in \mathcal{U}$, and $t_0, \ldots, t_{k-1} \geq T$ with*

$$d(\varphi(t_j,x_j,u_j),x_{j+1})<\varepsilon \text{ for all } j=0,\dots,k-1.$$

If, for every $\varepsilon,T>0$, there is a controlled (ε,T)-chain from x to y, the point x is chain controllable to y. The chain reachable set from x and the chain controllable set to x are

$$\mathbf{R}^c(x)=\{y\in M\,|x \text{ is chain controllable to } y\},$$
$$\mathbf{C}^c(x)=\{y\in M\,|y \text{ is chain controllable to } x\},$$

respectively. If a nonvoid set $F\subset M$ has the property that x is chain controllable to y for all $x,y\in F$, the set F is said to be chain controllable.

Note that a set F is chain controllable if and only if $F\subset\mathbf{R}^c(x)$ for all $x\in F$. Observe also that chain controllable sets are a generalized version of limit sets for time tending to infinity. In analogy to control sets, we define chain control sets as maximal chain controllable sets.

DEFINITION 2.11. *A nonvoid set $E\subset M$ is called a chain control set of system* (2.5) *if, for all $x,y\in E$ and $\varepsilon,T>0$, there is a controlled (ε,T)-chain from x to y and E is maximal with this property.*

Since the concatenation of two controlled (ε,T)-chains again yields a controlled (ε,T)-chain, two chain control sets coincide if their intersection is nonvoid.

Remark 2.12. The definition of chain control sets given, e.g., in Colonius and Kliemann [4] requires in addition to the chain controllability property that, for every $x\in E$, there is a control function $u\in\mathcal{U}$ such that $\varphi(t,x,u)\in E$ for all $t\in\mathbb{R}$. Theorem 2.14 will show that this property holds for every maximal chain controllable set. Thus, it is not necessary to impose this condition. The paper by Li [11, Proposition 6.8] treats control systems as particular cases of differential inclusions. Hence, the shift flow on the space of control functions plays no role. The definition of chain control sets (cf. [11, Definition 6.7]) also does not require invariance. The results on chain transitivity properties and Morse decompositions apply to compact invariant sets of control systems of the form $\dot{x}=f(x,u)$, where $f(x,U)$ is convex and the control range U is compact.

We note the following properties of chain controllability.

PROPOSITION 2.13.

(i) *For control system* (2.5)*, the chain controllable set $\mathbf{C}^c(x)$ to x coincides with the chain reachable set from x for the time-reversed system* (2.8)*.*

(ii) *Every chain controllable set F is contained in a chain control set.*

(iii) *The chain controllable sets $\mathbf{C}^c(x)$ and the chain reachable sets $\mathbf{R}^c(x)$ are closed.*

Proof.

(i) This follows using the same arguments as for Proposition 2.1.

(ii) Define E as the union of all chain controllable sets F' containing F. Then, E is a maximal chain controllable set and hence a chain control set containing F.

(iii) Let $y\in\overline{\mathbf{R}^c(x)}$, and fix $\varepsilon,T>0$. There is $y_1\in\mathbf{R}^c(x)$ with $d(y,y_1)<\varepsilon/2$. The last piece of a controlled $(\varepsilon/2,T)$-chain from x to y_1 satisfies

$$d(\varphi(T_{k-1},x_{k-1},u_{k-1}),y)\le d(\varphi(T_{k-1},x_{k-1},u_{k-1}),y_1)+d(y_1,y)<\varepsilon/2+\varepsilon/2=\varepsilon.$$

Thus, we obtain a controlled (ε,T)-chain from x to y, and hence, $\mathbf{R}^c(x)$ is closed. The assertion for $\mathbf{C}^c(x)$ follows using (i). ∎

The following theorem clarifies the relations between chain control sets and chain reachable and controllable sets.

THEOREM 2.14.

(i) *For all $y\in\mathbf{C}^c(x)$ and all $u\in\mathcal{U}$, it follows that $\varphi(t,y,u)\in\mathbf{C}^c(x)$ for all $t<0$. Furthermore, there exists $u\in\mathcal{U}$ such that $\varphi(t,y,u)\in\mathbf{C}^c(x)$ for all $t>0$.*

(ii) *For all $y\in\mathbf{R}^c(x)$ and all $v\in\mathcal{U}$, it follows that $\varphi(t,y,v)\in\mathbf{R}^c(x)$ for all $t>0$. Furthermore, there exists $v\in\mathcal{U}$ such that $\varphi(t,y,v)\in\mathbf{R}^c(x)$ for all $t<0$.*

(iii) *Let E be a chain control set. Then, it follows for all $x\in E$ that $E=\mathbf{R}^c(x)\cap\mathbf{C}^c(x)$ and that there exists $u\in\mathcal{U}$ such that $\varphi(t,x,u)\in E$ for all $t\in\mathbb{R}$.*

Proof. (i) Recall that the set $\mathcal{U}$ is a compact metric space in the weak* topology and that the map $\varphi:\mathbb{R}\times M\times\mathcal{U}\to M$ is continuous. Let $y\in\mathbf{C}^c(x)$. First, note that $\varphi(t,y,u)\in\mathbf{C}^c(x)$ for all $t<0$ and $u\in\mathcal{U}$. In fact, for any $\varepsilon,T>0$, one gets a controlled (ε,T)-chain from y to x by concatenating the trajectory from $\varphi(t,y,u)$ to y with a controlled (ε,T)-chain from y to x.

In the case of positive time, consider, for sequences $\varepsilon^i\to0$ and $T^i\to\infty$, controlled (ε^i,T^i)-chains ζ^i from y to x. Let the first pieces of the chains ζ^i be given by $\varphi(s,y,u_1^i), s\in[0,T_1^i]$ with $T_1^i\geq T^i$. Without loss of generality, u_1^i converges to some control $u\in\mathcal{U}$. We claim that $\varphi(t,y,u)\in\mathbf{C}^c(x)$ for all $t>0$.

For the proof, fix $t>0$. We construct for all $\varepsilon,T>0$ a controlled (ε,T)-chain from $\varphi(t,y,u)$ to x. By compactness of $\mathcal{U}$ and continuity, there is $\delta>0$ such that

$$d(\varphi(t,y,u),z)<\delta \text{ implies that } d(\varphi(T,\varphi(t,y,u),v),\varphi(T,z,v))<\varepsilon \text{ for all } v\in\mathcal{U}.$$

For i large enough, we obtain $\varepsilon_i<\varepsilon, T^i\geq 3T$, and $d(\varphi(t,y,u_1^i),\varphi(t,y,u))<\delta$.

Using the flow property of Φ, we replace the initial piece of the chain ζ^i by two pieces given by

$$x_0=\varphi(t,y,u), x_1=\varphi(T+t,y,u_1^i)=\varphi(T,\varphi(t,y,u_1^i),u_1^i(t+\cdot))$$

with times $T_0=T, T_1=T_1^i-T-t\geq T$ and controls $u_1^i(t+\cdot)$ and $u_1^i(T+t+\cdot)$. We find that

$$d(\varphi(T,x_0,u_1^i(t+\cdot)),x_1)=d(\varphi(T,\varphi(t,y,u),u_1^i(t+\cdot)),\varphi(T,\varphi(t,y,u_1^i),u_1^i(t+\cdot)))<\varepsilon,$$
$$\varphi(T_1,x_1,u_1^i(T+t+\cdot))=\varphi(T_1^i-T-t,\varphi(T+t,y,u_1^i),u_1^i(T+t+\cdot))=\varphi(T_1^i,y,u_1^i).$$

Thus, we get a controlled (ε,T)-chain from $\varphi(t,y,u)$ to x.

The proof of (ii) follows by time reversal using Proposition 2.13(i). It remains to prove (iii). By definition, E is a maximal chain controllable set. Let $x,y\in E$. Then, $y\in\mathbf{R}^c(x)$ and $x\in\mathbf{R}^c(y)$ or, equivalently, $y\in\mathbf{C}^c(x)$, showing that $E\subset\mathbf{R}^c(x)\cap\mathbf{C}^c(x)$. For the converse inclusion, note that the same argument shows that $\mathbf{R}^c(x)\cap\mathbf{C}^c(x)$ is a chain controllable set and hence contained in E.

Let $y\in E=\mathbf{R}^c(x)\cap\mathbf{C}^c(x)$. By assertions (i) and (ii), there are controls $u,v\in\mathcal{U}$ such that $\varphi(t,y,u)\in\mathbf{C}^c(x)$ for all $t<0$ and $\varphi(t,y,v)\in\mathbf{R}^c(x)$ for all $t>0$. Then, the control

$$w(t):=\begin{cases} v(t) & \text{for} \quad t\leq 0,\\ u(t) & \text{for} \quad t>0\end{cases}$$

yields $\varphi(t,y,w)\in\mathbf{R}^c(x)\cap\mathbf{C}^c(x)$ for all $t\in\mathbb{R}$. ∎

The relation between chain control sets and the control flow Φ defined in (2.6) is explained in the following theorem.

THEOREM 2.15. *Let $\mathcal{E} \subset \mathcal{U} \times M$ be a maximal chain transitive set for the control flow Φ. Then, $\{x \in M \,|\, \exists u \in \mathcal{U} : (u,x) \in \mathcal{E}\}$ is a chain control set. Conversely, if $E \subset M$ is a chain control set, then*

$$\mathcal{E} := \{(u,x) \in \mathcal{U} \times M \,|\, \varphi(t,x,u) \in E \text{ for all } t \in \mathbb{R}\} \tag{2.9}$$

is a maximal chain transitive set

Proof. This follows from Colonius and Kliemann [4, Theorem 4.3.11] or Kawan [10, Proposition 1.24(iv)]. Note that the proofs given there do not use compactness properties. These results assume that $\mathcal{E}$ is a maximal invariant chain transitive set, and the employed definition of chain control sets requires that, for every $x \in E$, there is a control $u \in \mathcal{U}$ with $\varphi(t,x,u) \in E$ for all $t \in \mathbb{R}$. Proposition 2.13 shows that every maximal chain controllable set has this property, and Proposition 2.2(ii) shows that every maximal chain transitive set of a flow is invariant. Thus, it is not necessary to impose these invariance assumptions. ∎

2.3. Control sets for linear control systems. Consider linear control systems of the form (1.1), where we now allow $A \in \mathbb{K}^{n\times n}, B \in \mathbb{K}^{n\times m}$, and $U \subset \mathbb{K}^m$.

The decomposition of $\mathbb{K}^n$ into the sum of the unstable, center, and stable subspaces of A is given by

$$\mathbb{K}^n = L^+ \oplus L^0 \oplus L^-, \tag{2.10}$$

where L^+, L^0, and L^- are the Lyapunov spaces for positive, vanishing, and negative Lyapunov exponents, respectively. Hence, L^+, L^0, and L^- are the sums of the generalized eigenspaces for eigenvalues with positive, vanishing, and negative real part, respectively. The center-unstable and the center-stable subspaces are $L^{+,0} = L^+ \oplus L^0$ and $L^{-,0} = L^- \oplus L^0$, respectively. We denote by $\pi^\pm : \mathbb{K}^n \to L^\pm$ the associated projections along $L^{\mp,0}$ and by $\pi^h : \mathbb{K}^n \to L^+ \oplus L^-$ the projection along L^0.

THEOREM 2.16. *Assume that (A,B) is controllable; i.e., the reachable subspace $\mathcal{C} := \mathrm{Im}[B\ AB \cdots A^{n-1}B]$ coincides with $\mathbb{K}^n$.*

(i) *The controllable set to $0 \in \mathbb{K}^n$ and the reachable set from $0 \in \mathbb{K}^n$ are*

$$\mathbf{C}(0) = (\mathbf{C}(0) \cap L^+) \oplus L^{-,0} \text{ and } \mathbf{R}(0) = L^{+,0} \oplus (\mathbf{R}(0) \cap L^-),$$

respectively, where $\mathbf{C}(0) \cap L^+$ is bounded, convex, and open relative to L^+ and $\mathbf{R}(0) \cap L^-$ is bounded, convex, and open relative to L^-.

(ii) *There is a unique control set D with nonvoid interior* int$D$. *It is convex and contains the origin $0 \in \mathbb{K}^n$ in the interior and $\overline{\mathrm{int} D} = \overline{D}$.*

(iii) *The control set satisfies*

$$D = \mathbf{C}(0) \cap \overline{\mathbf{R}(0)} = \big(\mathbf{C}(0) \cap L^+\big) \oplus L^0 \oplus (\overline{\mathbf{R}(0)} \cap L^-).$$

(iv) *For all $x \in D$ and $y \in \mathrm{int} D$, there are $t > 0$ and $u \in \mathcal{U}$ with $\varphi(t,x,u) = y$.*

(v) *For every $x \in \overline{D}$ and $t > 0$, there is $u \in \mathcal{U}$ with $\varphi(t,x,u) \in \overline{D}$.*

Proof. (i) The result on $\mathbf{R}(0)$ is proved in Sontag [14, Corollary 3.6.7], and the result for $\mathbf{C}(0)$ follows by time reversal. Assertion (ii) follows by Colonius and Kliemann [4, Example 3.2.16] and Remark 2.9, which also implies (iv). For assertion (iii), use decomposition (2.10) and (i) and (ii) to see that

$$D = \mathbf{C}(0) \cap \overline{\mathbf{R}(0)} = (\mathbf{C}(0) \cap L^+) \oplus L^0 \oplus (\overline{\mathbf{R}(0)} \cap L^-).$$

For assertion (v), note that, by (ii), there are $x_k \in \mathrm{int}D$ with $x_k \to x$, and by (iv), one finds $t_k > 0, u_k \in \mathcal{U}$ with $\varphi(t_k, x_k, u_k) \in D$. Since the t_k may be chosen bounded, the compactness of $\mathcal{U}$ and continuity of φ imply that there are $t > 0$ and $u \in \mathcal{U}$ with $\varphi(t, x, u) \in \overline{D}$. ∎

Next, we use these results for general linear systems by applying them to the restriction to the reachable subspace $\mathcal{C}$.

COROLLARY 2.17. *Consider a linear control system of the form* (1.1).

(i) *There is a control set D_0 with $0 \in D_0$, and it satisfies*

$$D_0 = \left(\mathbf{C}(0) \cap L^+\right) \oplus \left(L^0 \cap \mathcal{C}\right) \oplus \left(\overline{\mathbf{R}(0)} \cap L^-\right),$$

where $\overline{\mathbf{R}(0)} \cap L^- \subset \mathcal{C}$ is compact and $\mathbf{C}(0) \cap L^+ \subset \mathcal{C}$ is bounded and open in the relative topology of $L^+ \cap \mathcal{C}$.

(ii) *The origin $0 \in \mathbb{K}^n$ is in the interior $\mathrm{int}_{\mathcal{C}} D_0$ of D_0 relative to $\mathcal{C}$, which is dense in D_0, and, for all $x \in D_0$ and $y \in \mathrm{int}_{\mathcal{C}} D_0$, there are $t > 0$ and $u \in \mathcal{U}$ with $y = \varphi(t, x, u)$.*

Proof. It is clear that there is a unique control set D_0 containing $0 \in \mathbb{K}^n$. The subspace $\mathcal{C}$ is closed, and $\varphi(t, x, u) \in \mathcal{C}$ for all $t \in \mathbb{R}, x \in \mathcal{C}$, and $u \in \mathcal{U}$. By invariance of the subspace $\mathcal{C}$, the control set, the controllable set, and the reachable set satisfy $D_0 \subset \overline{\mathbf{C}(0)} \cup \overline{\mathbf{R}(0)} \subset \mathcal{C}$. Thus, the assertions follow from Theorem 2.16. ∎

We also note the following lemma.

LEMMA 2.18.

(i) *The control set D_0^- containing 0 of the system induced on L^-,*

$$\dot{y}(t) = Ay(t) + \pi^- Bu(t),\ u \in \mathcal{U}, \tag{2.11}$$

coincides with the projection $\pi^- \mathbf{R}(0)$.

(ii) *The control set D_0^+ containing 0 of the system induced on L^+,*

$$\dot{y}(t) = Ay(t) + \pi^+ Bu(t),\ u \in \mathcal{U}, \tag{2.12}$$

coincides with the projection $\pi^+ \mathbf{C}(0)$.

(iii) *Consider the system induced on the subspace $L^+ \oplus L^-$*

$$\dot{y}(t) = A^h y(t) + \pi^h Bu(t),\ u \in \mathcal{U}, \tag{2.13}$$

where $A^h := A_{|L^+ \oplus L^-}$. Then, A^h is hyperbolic; i.e., $\mathrm{spec}(A^h) \cap \imath\mathbb{R} = \varnothing$. The reachable set $\mathbf{R}^h(0)$, the controllable set $\mathbf{C}^h(0)$, and the control set D_0^h containing 0 of (2.13) satisfy $\mathbf{R}^h(0) = \pi^h \mathbf{R}(0), \mathbf{C}^h(0) = \pi^h \mathbf{C}(0)$, and $D_0^h = \pi^h D_0$.

Proof. (i) Since L^- and $L^{+,0}$ are A-invariant, it follows, for $x \oplus y$ with $x \in L^-, y \in L^{+,0}$, that $Ax \in L^-, Ay \in L^{+,0}$, and hence, $\pi^- A(x \oplus y) = Ax = A\pi^-(x \oplus y)$. This shows that $\pi^- A = A\pi^-$. For $x \in \pi^- \mathbf{R}(0)$, there are $T > 0$ and $u \in \mathcal{U}$ with

$$x = \pi^- x = \pi^- \int_0^T e^{A(T-s)} Bu(s) ds = \int_0^T e^{A(T-s)} \pi^- Bu(s) ds \in \mathbf{R}^-(0).$$

Conversely, if $x \in \mathbf{R}^-(0)$, there are $T > 0$ and $u \in \mathcal{U}$ with

$$x = \int_0^T e^{A(T-s)} \pi^- Bu(s) ds = \pi^- \int_0^T e^{A(T-s)} Bu(s) ds \in \pi^- \mathbf{R}(0).$$

Since $A_{|L^-}$ is asymptotically stable, it follows from Corollary 2.17(i) that

$$D_0^- = \overline{\mathbf{R}^-(0)} = \overline{\pi^- \mathbf{R}(0)} = \pi^- \overline{\mathbf{R}(0)}.$$

(ii) The proof follows analogously.

(iii) Suppose first that (A,B) is controllable. Then, it follows that $0 \in \mathrm{int} D_0$, and hence, $D_0 = \overline{\mathbf{R}(0)} \cap \mathbf{C}(0)$, showing that $0 \in \mathrm{int}\mathbf{R}(0)$ and $0 \in \mathrm{int}\mathbf{C}(0)$. Since the projection $\pi^h : \mathbb{K}^n \to L^+ \oplus L^-$ along L^0 is open, it follows that

$$0 \in \mathrm{int}\left(\pi^h \mathbf{R}(0)\right) \text{ and } 0 \in \mathrm{int}\left(\pi^h \mathbf{C}(0)\right).$$

The same arguments as in (i) and (ii) imply that $\mathbf{R}^h(0) = \pi^h \mathbf{R}(0)$ and $\mathbf{C}^h(0) = \pi^h \mathbf{C}(0)$. This shows that $0 \in \mathrm{int}\overline{\mathbf{R}^h(0)}$ and $0 \in \mathrm{int}\mathbf{C}^h(0)$, implying that

$$D_0^h = \overline{\mathbf{R}^h(0)} \cap \mathbf{C}^h(0) = \overline{\pi^h \mathbf{R}(0)} \cap \pi^h \mathbf{C}(0) = \pi^h \left(\overline{\mathbf{R}(0)} \cap \mathbf{C}(0)\right) = \pi^h D_0.$$

In the general case, the sets $\overline{\mathbf{R}(0)}, \mathbf{C}(0)$, and the control set D_0 containing 0 are contained in the controllable subspace $\mathcal{C}$, and D_0 has nonvoid interior relative to $\mathcal{C}$. Observe also that $\ker \pi^h = L^0$; hence, $D_0 \subset D_0^h + L^0$, and similarly for $\mathbf{R}(0)$ and $\mathbf{C}(0)$. ◻

3. Chain transitivity for autonomous linear differential equations. We prove that the chain transitive set of the flow for autonomous linear differential equations coincides with the center Lyapunov space. This also yields first results on chain controllability properties of the linear control system (1.1).

The following lemma contains the crux of the argument. Let $\psi : \mathbb{R} \times \mathbb{K}^n \to \mathbb{K}^n, \psi(t,x) = e^{At}x, t \in \mathbb{R}, x \in \mathbb{K}^n$ be the flow of the differential equation $\dot{x} = Ax$ for $A \in \mathbb{K}^{n \times n}$.

LEMMA 3.1. *If all eigenvalues of the matrix A have null real parts, then, for all $x, y \in \mathbb{K}^n$ and all $\varepsilon, T > 0$, there is an (ε, T)-chain of the flow ψ from x to y.*

Proof. We first prove this lemma for the complex case $\mathbb{K} = \mathbb{C}$ using induction over the dimension n. The lemma holds trivially for $n = 0$ since $\mathbb{C}^0$ consists of a single point. Assume that the assertion holds for all $A \in \mathbb{C}^{(n-1)\times(n-1)}$. Since $\mathbb{C}$ is algebraically closed, A has some eigenvector v associated to an eigenvalue $\mu \in \mathbb{C}$, and, by hypothesis, $\mathrm{Re}\,\mu = 0$. There is $r > 0$ such that $r\mu = 2j\pi\imath$ for some $j \in \mathbb{Z}$. It follows that, for all $\lambda \in \mathbb{C}$,

$$e^{rA}\lambda v = \lambda e^{2j\pi\imath} v = \lambda v.$$

Thus, the restriction of ψ to the linear span $\langle v \rangle$ of v is periodic. We claim that, for all $\lambda_1 v, \lambda_2 v \in \langle v \rangle, \lambda_1, \lambda_2 \in \mathbb{C}$, and all $\varepsilon, T > 0$, there is an (ε, T)-chain from $\lambda_1 v$ to $\lambda_2 v$; in fact, consider periodic solutions through $\lambda_1 v + \frac{i}{j}(\lambda_2 - \lambda_1)v, i \in \{0, 1, \dots, j\}$, where $j \in \mathbb{N}$ is large enough such that

$$\left\| \lambda_1 v + \frac{i}{j}(\lambda_2 - \lambda_1)v - \left[\lambda_1 v + \frac{i+1}{j}(\lambda_2 - \lambda_1)v\right] \right\| = \frac{1}{j}\left|\lambda_2 - \lambda_1\right| \|v\| < \varepsilon.$$

Staying on each periodic solution long enough such that the time is greater than T and jumping from one periodic solution to the next yields an (ε, T)-chain from $\lambda_1 v$ to $\lambda_2 v$.

Now, let $W \subset \mathbb{C}^n$ be a subspace such that $\mathbb{C}^n = W \oplus \langle v \rangle$. There are linear transformations $A_1 : W \to W$ and $A_2 : W \to \langle v \rangle$ such that $Aw = A_1 w + A_2 w$. We obtain, for $w \in W$,

$$\frac{d}{dt}\psi(t,w) = A\psi(t,w) = A_1\psi(t,w) + A_2\psi(t,w) \text{ with } A_2\psi(t,w) \in \langle v \rangle.$$

The variation-of-parameters formula for A_1 shows that

$$\psi(t,w) = e^{At}w = e^{tA_1}w + \widehat{\psi}(t,w) \text{ with } \widehat{\psi}(t,w) := \int_0^t e^{(t-s)A_1} A_2 \psi(s,w) ds.$$

On the other hand, $\psi(t,\lambda v) = e^{At}\lambda v = e^{t\mu}\lambda v$, and hence, by linearity,

$$\psi(t, w + \lambda v) = e^{tA_1}w + \widehat{\psi}(t,w) + e^{t\mu}\lambda v. \tag{3.1}$$

Let $x \in \mathbb{C}^n$. We will show that $0 \in \Omega(x)$ and $x \in \Omega(0)$. This implies the assertion since any two points x, y can be connected by concatenating a chain from x to 0 with a chain from 0 to y.

Let $\varepsilon, T > 0$, and write $x = w \oplus \lambda v \in \mathbb{C}^n$. By the induction hypothesis for all $\varepsilon, T > 0$, there is an (ε, T)-chain from w to 0 given by $w_0 = w, w_1, \ldots, w_k = 0$ and $T_0, T_1, \ldots, T_{k-1} \geq T$ with

$$\left\| e^{T_i A_1} w_i - w_{i+1} \right\| < \varepsilon \text{ for } i = 0, \ldots, k-1.$$

Next, we construct an (ε, T)-chain from $x = w \oplus \lambda v \in \mathbb{C}^n$ to 0. Define recursively

$$v_0 = \lambda v, v_{i+1} = \widehat{\psi}(T_i, w_i) + e^{T_i \mu} v_i \in \langle v \rangle \text{ for } i = 1, \ldots, k-1.$$

Hence, by (3.1),

$$\begin{aligned} &e^{T_i A}(w_i + v_i) - (w_{i+1} + v_{i+1}) \\ &\quad = e^{T_i A_1} w_i + \widehat{\psi}(T_i, w_i) + e^{T_i \mu} v_i - w_{i+1} - \widehat{\psi}(T_i, w_i) - e^{T_i \mu} v_i = e^{T_i A_1} w_i - w_{i+1}. \end{aligned} \tag{3.2}$$

For $x_i := w_i + v_i, i = 0, \ldots, k$, it follows that

$$\left\| e^{T_i A} x_i - x_{i+1} \right\| = \left\| e^{T_i A_1} w_i - w_{i+1} \right\| < \varepsilon.$$

Thus, $x_0 = x, x_1, \ldots, x_k = v_k$ and $T_0, T_1, \ldots, T_{k-1} \geq T$ is an (ε, T)-chain from x to $v_k \in \langle v \rangle$. Concatenating this with an (ε, T)-chain from v_k to 0 in $\langle v \rangle$, we obtain the desired chain.

The inclusion $x \in \Omega(0)$ follows from considering the time-reversed system and Proposition 2.1.

Now, assume $\mathbb{K} = \mathbb{R}$. The previous result yields that, for all $x, y \in \mathbb{R}^n \subset \mathbb{C}^n$ and for all $\varepsilon, T > 0$, there are (ε, T)-chains in $\mathbb{C}^n$ from x to y. Consider such a chain given by $z_0 = x, z_1, \ldots, z_k = y$ in $\mathbb{C}^n$ and $T_0, \ldots, T_{k-1} \geq T$ with

$$\left\| e^{T_i A} z_i - z_{i+1} \right\| < \varepsilon \text{ for } i = 0, \ldots, k-1.$$

Since the entries of A are real, it follows that, for all i,

$$\left\| e^{T_i A} \operatorname{Re} z_i - \operatorname{Re} z_{i+1} \right\| = \left\| \operatorname{Re}\left(e^{T_i A} z_i - z_{i+1} \right) \right\| \leq \left\| e^{T_i A} z_i - z_{i+1} \right\| < \varepsilon.$$

This shows that one obtains an (ε, T)-chain on $\mathbb{R}^n$ from $z_0 = x$ to $z_k = y$. ∎

A consequence of this lemma is the following theorem characterizing the chain recurrent set of linear autonomous differential equations.

THEOREM 3.2. *For a differential equation $\dot{x}(t) = Ax(t)$ with $A \in \mathbb{K}^{n\times n}$, the chain recurrent set of its flow $\psi(t,x)$ coincides with the center Lyapunov space L^0.*

Proof. By Lemma 3.1, the set L^0 is chain transitive. It remains to show that any chain recurrent point is in L^0. Since all norms in $\mathbb{K}^n$ are equivalent, we can assume without loss in generality that $L^+ \oplus L^- = \left(L^0\right)^{\perp}$, which implies that

$$\|v\| \le \|v + w\| \text{ for all } v \in L^+ \oplus L^-, w \in L^0. \tag{3.3}$$

Let $x \in \mathbb{K}^n$ be a chain recurrent point. Thus, for all $\varepsilon, T > 0$, there is an (ε,T)-chain $x_0 = x, x_1, \dots, x_k = x$ with $T_0, T_1, \dots, T_{k-1} \ge T$. Write

$$x = v \oplus w \text{ and } x_i = v_i \oplus w_i \text{ with } v, v_i \in L^+ \oplus L^- \text{ and } w, w_i \in L^0.$$

Then, it follows from (3.3) that, for all i,

$$\left\|e^{T_iA}v_i - v_{i+1}\right\| \le \left\|e^{T_iA}v_i + e^{T_iA}w_i - v_{i+1} - w_{i+1}\right\| = \left\|e^{T_iA}x_i - x_{i+1}\right\| < \varepsilon.$$

This shows that $v_0 = v, v_1, \dots, v_k = v$ is an (ε,T)-chain from v to v for the flow restricted to $L^+ \oplus L^-$. Since $\varepsilon, T > 0$ are arbitrary and this flow is hyperbolic, it follows that $v = 0$. In fact, Antunez, Mantovani, and Varão [2, Corollary 2.11] shows this for discrete linear flows, and similar arguments can be used for continuous linear flows. ∎

Recall from subsection 2.1 the definition of chain limit sets $\Omega(x)$ for flows.

PROPOSITION 3.3. *For the flow ψ on $\mathbb{K}^n$, the center-unstable subspace and the center-stable subspace satisfy $L^{+,0} \subset \Omega(0)$ and $L^{-,0} \subset \Omega^*(0)$, respectively.*

Proof. Let $x = x_+ \oplus x_0 \in L^{+,0}$ be arbitrary, where $x_+ \in L^+$ and $x_0 \in L^0$, and fix $\varepsilon, T > 0$. For $S \ge T$ sufficiently large, $\|e^{-SA}x_+\| < \varepsilon$. Since L^0 is A-invariant, it follows that $e^{-2SA}x_0 \in L^0$. Hence, by chain transitivity of L^0, there is an (ε,T) chain $0 = y_0, y_1, \dots, y_k = e^{-2SA}x_0$ with times $T_0, T_1, \dots, T_{k-1} \ge T$. Define an (ε,T)-chain from 0 to x by $x_0 = y_0 = 0, x_1 = y_1, \dots, x_k = y_k = e^{-2SA}x_0, x_{k+1} = e^{-SA}\left(x_+ + x_0\right), x_{k+2} = x_+ + x_0 = x$ with times $T_0, \dots, T_{k-1}, T_k = T_{k+1} = S$. This holds since

$$\begin{aligned}
\|\psi(T_i,x_i) - x_{i+1}\| &= \|\psi(T_i,y_i) - y_{i+1}\| < \varepsilon \text{ for } i = 0, \dots, k-1,\\
\|\psi(T_k,x_k) - x_{k+1}\| &= \left\|e^{SA}e^{-2SA}x_0 - \left(e^{-SA}x_+ + e^{-SA}x_0\right)\right\| = \left\|e^{-SA}x_+\right\| < \varepsilon,\\
\|\psi(T_{k+1},x_{k+1}) - x_{k+2}\| &= \left\|e^{SA}\left(e^{-SA}x_+ + e^{-SA}x_0\right) - (x_+ + x_0)\right\| = 0.
\end{aligned}$$

The assertion for $L^{-,0}$ follows by considering the time-reversed system. ∎

The following simple example illustrates that, for $T \to \infty$, the considered chains may become unbounded. Hence, in the noncompact state space $\mathbb{R}^n$, it is not sufficient to consider chain transitivity restricted to compact subsets of $\mathbb{R}^n$.

Example 3.4. For the differential equation

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

the x-axis consists of equilibria. In the open upper half-plane, all trajectories move to the right in parallel to the x-axis, and in the open lower half-plane, they move to the left. The state space $\mathbb{R}^2$ is chain transitive, and for $T \to \infty$, the required (ε,T)-chains become unbounded.

We return to control system (1.1). The following proposition relates the control set D_0 containing 0 and the chain reachable and chain controllable sets to the spectral subspaces of the homogeneous part $\dot{x} = Ax$.

PROPOSITION 3.5. *Consider a linear control system of the form* (1.1).
(i) *The inclusions* $\overline{D_0}+L^{-,0}\subset \mathbf{C}^c(0)$ *and* $\overline{D_0}+L^{+,0}\subset \mathbf{R}^c(0)$ *hold.*
(ii) *The set* $\overline{D_0}+L^0$ *is chain controllable.*

Proof. (i) First, we prove the following claim:

$$\mathbf{R}(z)\cap L^{-,0}\neq\varnothing \text{ for } z\in \operatorname{int}_{\mathcal{C}}D_0+L^{-,0} \text{ and } \mathbf{C}(z)\cap L^{+,0}\neq\varnothing \text{ for } z\in \operatorname{int}_{\mathcal{C}}D_0+L^{+,0}.$$

Write $z=x+y$ with $x\in \operatorname{int}_{\mathcal{C}}D_0$ and $y\in L^{\pm,0}$. By linearity, it follows that

$$\varphi(t,z,u)=\varphi(t,x,u)+\varphi(t,y,0) \text{ for all } t\in\mathbb{R} \text{ and } u\in\mathcal{U}.$$

Since $0,x\in \operatorname{int}_{\mathcal{C}}D_0$, Corollary 2.17(ii) implies that there are $T_1,T_2\geq 0$ and $u_1,u_2\in\mathcal{U}$ with $\varphi(T_1,x,u_1)=0$ and $\varphi(-T_2,x,u_2)=0$. Using that $L^{\pm,0}$ is invariant for the control $u\equiv 0$, one finds that

$$\varphi(T_1,z,u_1)=\varphi(T_1,y,0)\in L^{-,0} \text{ if } z\in \operatorname{int}_{\mathcal{C}}D_0+L^{-,0},$$
$$\varphi(-T_2,z,u_2)=\varphi(-T_2,y,0)\in L^{+,0} \text{ if } z\in \operatorname{int}_{\mathcal{C}}D_0+L^{+,0}.$$

This implies the claim. Observe that we can take the times T_1,T_2 arbitrarily large when we extend u_1 and u_2 by the control $u\equiv 0$.

Next, we show that $\overline{D_0}+L^{-,0}\subset \mathbf{C}^c(0)$. Since $\mathcal{C}$ is a closed subspace and the set D_0 is contained in the closure of $\operatorname{int}_{\mathcal{C}}D_0$, it follows that $\overline{D_0}=\overline{\operatorname{int}_{\mathcal{C}}D_0}$. Recalling that $\mathbf{R}^c(0)$ is closed by Proposition 2.13(iii), one sees that it suffices to prove that $\operatorname{int}_{\mathcal{C}}D_0+L^{-,0}\subset \mathbf{C}^c(x)$. Let $z\in \operatorname{int}_{\mathcal{C}}D_0+L^{-,0}$, and fix $\varepsilon,T>0$. By the claim, there is a point $y\in L^{-,0}$, which can be reached from z at a time greater than T. By Proposition 3.3, there is a controlled (ε,T)-chain (with control $u\equiv 0$) from $y\in L^{-,0}$ to 0. Concatenating this with the trajectory segment from z to y, one obtains a controlled (ε,T)-chain from z to 0.

For the inclusion $\overline{D_0}+L^{+,0}\subset \mathbf{R}^c(0)$, it similarly suffices to show that $\operatorname{int}_{\mathcal{C}}D_0+L^{+,0}$ is contained in the closed set $\mathbf{R}^c(0)$. Let $z\in \operatorname{int}_{\mathcal{C}}D_0+L^{+,0}$, and fix $\varepsilon,T>0$. By the claim, there is a point $y\in L^{+,0}$ such that z can be reached from y at a time greater than T. By Proposition 3.3, there is a controlled (ε,T)-chain from 0 to y. Concatenation of these controlled chains yields a controlled (ε,T)-chain from 0 to z.

(ii) This follows from assertion (i) since

$$\overline{D_0}+L^0=(\overline{D_0}+L^{+,0})\cap(\overline{D_0}+L^{-,0})\subset \mathbf{R}^c(0)\cap\mathbf{C}^c(0). \qquad \square$$

4. The chain control set in $\mathbb{R}^n$. In this section we first recall results from Colonius and Santana [7] on the projection to the Poincaré sphere. Together with attractor-repeller decompositions, this will enable us to prove that the chain controllable set $\overline{D_0}+L_0$ in Proposition 3.5 actually coincides with the chain control set.

Linear control systems of the form (1.1) on $\mathbb{R}^n$ can be lifted to bilinear control systems with states $(x(t),y(t))$ in $\mathbb{R}^n\times\mathbb{R}=\mathbb{R}^{n+1}$ by

$$\dot{x}(t)=Ax(t)+y(t)Bu(t),\ \dot{y}(t)=0,\ u\in\mathcal{U}. \tag{4.1}$$

The solutions for initial condition $(x(0),y(0))=(x_0,r)\in\mathbb{R}^n\times\mathbb{R}$ may be written as

$$\varphi^1(t,x_0,r,u)=\left(e^{At}x_0+r\int_0^t e^{A(t-s)}Bu(s)ds,r\right),\ t\geq 0.$$

Observe that, for $r = 0$, one has $\varphi^1(t,x,0,u) = (\psi(t,x),0)$, and for $r = 1$, one has $\varphi^1(t,x,1,u) = (\varphi(t,x,u),1)$ for all $t \in \mathbb{R}, x \in \mathbb{R}^n, u \in \mathcal{U}$. Hence, on the hyperplane $\mathbb{R}^n \times \{1\} \subset \mathbb{R}^{n+1}$, one obtains a copy of control system (1.1).

Define subsets of $\mathbb{R}^{n+1}$, of projective space $\mathbb{P}^n$, and of the unit sphere $\mathbb{S}^n$ by

$$\begin{aligned} &\mathbb{R}^{n+1,0} = \mathbb{R}^n \times \{0\}, \mathbb{R}^{n+1,1} = \mathbb{R}^n \times (\mathbb{R} \setminus \{0\}), \\ &\mathbb{P}^{n,0} = \{\mathbb{P}(x,0) \,|\, x \in \mathbb{R}^n\}, \mathbb{P}^{n,1} = \{\mathbb{P}(x,r) \,|\, x \in \mathbb{R}^n, r \neq 0\}, \\ &\mathbb{S}^{n,+} := \{(x,r) \in \mathbb{S}^n \,|\, x \in \mathbb{R}^n, r > 0\}, \mathbb{S}^{n,0} = \{(x,0) \in \mathbb{S}^n \,|\, x \in \mathbb{R}^n\}, \end{aligned} \tag{4.2}$$

respectively. Observe that $\mathbb{P}^{n,1} = \{\mathbb{P}(x,1) \,|\, x \in \mathbb{R}^n\}$ can be identified with the "northern" hemisphere $\mathbb{S}^{n,+}$ of $\mathbb{S}^n$ and $\mathbb{P}^{n,0}$ is the projection of the equator $\mathbb{S}^{n,0}$.

DEFINITION 4.1. *The projective Poincaré sphere is $\mathbb{P}^n$, and the projective Poincaré bundle is $\mathcal{U} \times \mathbb{P}^n$.*

Control system (4.1) induces a control flow Φ^1 on $\mathcal{U} \times \mathbb{R}^{n+1}$ defined by

$$\Phi^1_t(u,x,r) = (u(t+\cdot), \varphi^1(t,x,r,u)), t \in \mathbb{R}, (x,r) \in \mathbb{R}^{n+1}, u \in \mathcal{U}. \tag{4.3}$$

The maps between the fibers $\{u\} \times \mathbb{R}^{n+1}$ are linear, and hence, Φ^1 is a linear flow: For every $u \in \mathcal{U}$ and $\alpha, \beta \in \mathbb{R}, (x_1,r_1), (x_2,r_2) \in \mathbb{R}^n \times \mathbb{R}$,

$$\alpha\varphi^1(t,x_1,r_1,u) + \beta\varphi^1(t,x_2,r_2,u) = \varphi^1(t, \alpha x_1 + \beta x_2, \alpha r_1 + \beta r_2, u).$$

The projection of system (4.1) to projective space yields a projective control flow $\mathbb{P}\Phi^1$ on the projective Poincaré bundle $\mathcal{U} \times \mathbb{P}^n$. The subsets $\mathcal{U} \times \mathbb{P}^{n,0}$ and $\mathcal{U} \times \mathbb{P}^{n,1}$ are invariant under the flow $\mathbb{P}\Phi^1$. The following proposition shows some relations between the control flows on $\mathcal{U} \times \mathbb{R}^n$ and on the projective Poincaré bundle.

PROPOSITION 4.2.
(i) *Every $(u,x) \in \mathcal{U} \times \mathbb{R}^n$ satisfies the equality $\Phi^1_t(u,x,0) = (u(t+\cdot), \varphi^1(t,x,0,u)) = (u(t+\cdot), \psi(t,x), 0), t \in \mathbb{R}$, and the projective map*

$$h^0 : \mathcal{U} \times \mathbb{P}^{n-1} \to \mathcal{U} \times \mathbb{P}^{n,0}, h^0(u, \mathbb{P}x) = (u, \mathbb{P}(x,0)) \tag{4.4}$$

is a conjugacy of the flow on $\mathcal{U} \times \mathbb{P}^{n-1}$ given by $(u,p) \mapsto (u(t+\cdot), \mathbb{P}\psi(t,p))$ and the flow $\mathbb{P}\Phi^1$ restricted to $\mathcal{U} \times \mathbb{P}^{n,0}$.

(ii) *The map*

$$h^1 : \mathcal{U} \times \mathbb{R}^n \to \mathcal{U} \times \mathbb{P}^{n,1}, (u,x) \mapsto (u, \mathbb{P}(x,1)) \tag{4.5}$$

is a conjugacy of the flows Φ on $\mathcal{U} \times \mathbb{R}^n$ and $\mathbb{P}\Phi^1$ restricted to $\mathcal{U} \times \mathbb{P}^{n,1}$.

(iii) *For $\varepsilon, T > 0$, any (ε,T)-chain in $\mathcal{U} \times \mathbb{R}^n$ is mapped by h^1 onto a $(2\varepsilon,T)$-chain in $\mathcal{U} \times \mathbb{P}^{n,1}$; hence, any chain transitive set $C \subset \mathcal{U} \times \mathbb{R}^n$ is mapped onto a chain transitive set $h^1(C) \subset \mathcal{U} \times \mathbb{P}^{n,1}$.*

(iv) *For a subset $C \subset \mathcal{U} \times \mathbb{R}^n$, the set $\{x \in \mathbb{R}^n \,|\, (u,x) \in C$ for some $u \in \mathcal{U}\}$ is bounded if and only if $\overline{h^1(C)} \cap (\mathcal{U} \times \mathbb{P}^{n,0}) = \varnothing$.*

Proof. Assertion (i) holds since $\varphi^1(t,x,0,u) = (\psi(t,x),0), t \in \mathbb{R}, x \in \mathbb{R}^n, u \in \mathcal{U}$, implying (4.4) for the corresponding projective flows. Assertions (ii), (iii), and (iv) are a special case of Colonius and Santana [7, Proposition 9]. ∎

Applying Selgrade's theorem, Theorem 2.7, to the linear control flow Φ^1, one obtains the following results (cf. [7, Corollary 32, Remark 34]). They will be sharpened below.

THEOREM 4.3. *Consider a linear control system of the form* (1.1) *on* $\mathbb{R}^n$ *and its lift* (4.1) *to* $\mathbb{R}^{n+1}$.

(i) *The linear flow* $(\theta_t u, \psi(t,x))$ *on* $\mathcal{U}\times\mathbb{R}^n$ *defined by the linear part* $\dot{x}=Ax$ *of* (1.1) *has the Selgrade decomposition*

$$\mathcal{U}\times\mathbb{R}^n=\bigoplus_{i=1}^{\ell}\mathcal{V}_i \text{ with } \mathcal{V}_i=\mathcal{U}\times L(\lambda_i),$$

where $L(\lambda_i)$ *are the Lyapunov spaces of* A *for the Lyapunov exponents* λ_i.

(ii) *For the lifted flow* Φ^1 *in* (4.3), *the Selgrade decomposition has the form*

$$\mathcal{U}\times\mathbb{R}^{n+1}=\mathcal{V}_1^{\infty}\oplus\cdots\oplus\mathcal{V}_{\ell^+}^{\infty}\oplus\mathcal{V}_c^1\oplus\mathcal{V}_{\ell^++\ell^0+1}^{\infty}\oplus\cdots\oplus\mathcal{V}_{\ell}^{\infty} \tag{4.6}$$

for some numbers $\ell^+,\ell^0\geq 0$ *with* $\ell^++\ell^0\leq\ell$ *and*

$$\mathcal{V}_i^{\infty}:=\mathcal{V}_i\times 0=\mathcal{U}\times L(\lambda_i)\times 0\subset\mathcal{U}\times\mathbb{R}^{n+1,0}, i\in\{1,\dots,\ell^+\}\cup\{\ell^++\ell^0+1,\dots,\ell\}.$$

(iii) *The central Selgrade bundle* $\mathcal{V}_c^1$ *satisfies*

$$\mathcal{V}_c^1\cap(\mathcal{U}\times\mathbb{R}^n\times 0)=\bigoplus_{i=\ell^++1}^{i=\ell^++\ell^0}\mathcal{V}_i^{\infty}:=\mathcal{V}_c^{\infty} \text{ and } \dim\mathcal{V}_c^1=1+\dim\mathcal{V}_c^{\infty}.$$

The indices $i\in\{\ell^++1,\dots,\ell^++\ell^0\}$ *are the indices such that* $h^1(\mathcal{U}\times L(\lambda_i))=\mathcal{U}\times\mathbb{P}(L(\lambda_i)\times\{1\})\subset\mathcal{U}\times\mathbb{P}^n$ *is a chain transitive set.*

(iv) *If* A *is hyperbolic (i.e.,* $\mathrm{spec}A\cap\imath\mathbb{R}=\varnothing$*), the central Selgrade bundle is the line bundle*

$$\mathcal{V}_c^1=\{(u,-re(u,0),r)\in\mathcal{U}\times\mathbb{R}^n\times\mathbb{R}\,|\,u\in\mathcal{U},r\in\mathbb{R}\},$$

where $e(u,t),t\in\mathbb{R}$, *is the unique bounded solution of* (1.1) *for* $u\in\mathcal{U}$ *and the projection* $\mathcal{M}_c^1=\mathbb{P}\mathcal{V}_c^1$ *satisfies* $\mathcal{M}_c^1\subset\mathbb{P}^{n,1}$.

Theorem 4.3 implies the following consequences for chain control sets (cf. [7, Theorem 35]).

COROLLARY 4.4. *Consider a linear control system of the form* (1.1) *on* $\mathbb{R}^n$.

(i) *For the induced control system on the projective Poincaré sphere* $\mathbb{P}^n$, *there exists a unique chain control set* E_c^1 *with* $E_c^1\cap\mathbb{P}^{n,1}\neq\varnothing$. *It is given by*

$$E_c^1=\{\mathbb{P}x\in\mathbb{P}^n\,\big|\,\exists u\in\mathcal{U}:(u,\mathbb{P}x)\in\mathcal{M}_c^1\}.$$

Furthermore, E_c^1 *contains* $\mathbb{P}(L(0)\times 0)$ *and the image* $\mathbb{P}(0_n,1)$ *of the north pole* $(0_n,1)$.

(ii) *For the unique chain control set* E *in* $\mathbb{R}^n$ *of the linear control system* (1.1), *the image* $\mathbb{P}(E\times\{1\})$ *in the projective Poincaré sphere* $\mathbb{P}^n$ *is contained in* E_c^1.

(iii) *If* A *is hyperbolic, then the chain control set* E_c^1 *equals* $\mathbb{P}(E\times\{1\})$, *it is a compact subset of* $\mathbb{P}^{n,1}$, *and, for every* $u\in\mathcal{U}$, *there is a unique element* $x\in E$ *with* $\varphi(t,x,u)\in E$ *for all* $t\in\mathbb{R}$.

The next proposition identifies a subbundle occurring in assertion (iii) of Theorem 4.3 (in Theorem 5.3, we will show that it is the only one). It is the bundle for the center Lyapunov space of A, and it is convenient to denote it by $\mathcal{V}_0:=\mathcal{U}\times L(0)$.

PROPOSITION 4.5. *The inclusion* $\mathcal{V}_0^\infty := \mathcal{U}\times L(0)\times 0 \subset \mathcal{V}_c^1$ *holds.*

Proof. Theorem 3.2 shows that the subspace $L^0 = L(0)$ is chain transitive for the flow ψ. This implies by Proposition 4.2(iii) that $\mathbb{P}\left(L^0\times\{1\}\right)$ is chain transitive for the projectivized flow $\mathbb{P}\psi$ on the compact space $\mathbb{P}^n$. The set $\mathcal{U}$ is also compact and chain transitive for the shift flow θ. By Alongi and Nelson [1, Theorem 2.7.18], for both flows, it suffices to consider chains with all jump times equal to 1. This implies that $h^1(\mathcal{U}\times L(0)) = \mathcal{U}\times\mathbb{P}(L(0)\times\{1\})$ is chain transitive since the restriction of the flow $\mathbb{P}\Phi^1$ is a product flow, noting that the flow on $\mathbb{P}(L(0)\times\{1\})$ does not depend on the element in $\mathcal{U}$. It follows that $\mathcal{U}\times\mathbb{P}(L(0)\times 0)$ is also chain transitive, and hence, $\mathbb{P}\mathcal{V}_0^\infty \subset \mathbb{P}\mathcal{V}_c^1$. ◻

In order to determine the chain control set E, we prepare the following results.

PROPOSITION 4.6. *Assume that* $\mathbb{R}^n = L^-$*; hence, the homogeneous part* $\dot{x} = Ax$ *is asymptotically stable. Then, the following assertions hold.*

(i) *The control set* D_0 *containing* $0\in\mathbb{R}^n$ *is compact and given by* $D_0 = \overline{\mathbf{R}(0)}$*.*

(ii) *The image* $\mathbb{P}(D_0\times\{1\})$ *on the projective Poincaré sphere is a compact control set contained in* $\mathbb{P}^{n,1}$ *for the induced system on* $\mathbb{P}^n$*.*

(iii) *The set* $\mathbb{P}(D_0\times\{1\})$ *is chain controllable for the induced system on* $\mathbb{P}^n$*.*

(iv) *The set* $\mathcal{A}\subset\mathcal{U}\times\mathbb{P}^{n,1}$ *defined by*

$$\mathcal{A} := \{(u,\mathbb{P}(x,1))\in\mathcal{U}\times\mathbb{P}^n \,|\, \mathbb{P}(\varphi(t,x,u),1)\in\mathbb{P}(D_0\times\{1\}) \text{ for all } t\in\mathbb{R}\} \tag{4.7}$$

is chain transitive for the control flow $\mathbb{P}\Phi^1$*.*

(v) $\mathcal{A}$ *is an attractor, and its complementary repeller is the set* $\mathcal{A}^* = \mathcal{U}\times\mathbb{P}^{n,0}$*.*

(vi) *The set* $\mathbb{P}(D_0\times\{1\})\subset\mathbb{P}^{n,1}$ *is the unique chain control set in* $\mathbb{P}^n$*.*

Proof.

(i) This follows by the characterization of the control set D_0 in Corollary 2.17.

(ii) This holds since, by Proposition 4.2(ii), the map h^1 is a conjugacy and D_0 is compact.

(iii) The set D_0 is a control set with nonvoid interior relative to the controllability subspace $\mathcal{C}$. Since h^1 is a conjugacy, it follows that $\mathbb{P}(D_0\times\{1\})$ is a control set with nonvoid interior for the system on $\mathbb{P}^n$ restricted to $\mathbb{P}(\mathcal{C}\times\{1\})$. Then, Kawan [10, Proposition 1.24(ii)] implies that $\mathbb{P}(D_0\times\{1\})$ is contained in a chain control set of the system restricted to $\mathbb{P}(\mathcal{C}\times\{1\})$ and hence is contained in a chain control set of the system on $\mathbb{P}^n$. It follows that $\mathbb{P}(D_0\times\{1\})$ is contained in a chain control set and hence chain controllable.

(iv) By [10, Proposition 1.24(iv)], the lift of a chain control set is a chain transitive set for the control flow; hence, by (iii), the set $\mathcal{A}$ in (4.7) is chain transitive. Furthermore, $\mathcal{A}$ is an invariant compact subset of $\mathcal{U}\times\mathbb{P}^{n,1}$ for the flow $\mathbb{P}\Phi_t^1(u,y) = (\theta_t u, \mathbb{P}\varphi^1(t,y,u))$.

(v) Note that $\mathcal{A}^*$ is also compact and invariant since $\mathbb{P}^{n,0}$ is invariant.

Claim: Every $(u,y)\in(\mathcal{U}\times\mathbb{P}^n)\setminus(\mathcal{A}\cup\mathcal{A}^*)$ satisfies $\omega(u,y)\subset\mathcal{A}$ and $\alpha(u,y)\subset\mathcal{A}^*$.

According to Colonius and Kliemann [4, Lemma B.2.14], the disjoint compact invariant sets $\mathcal{A}$ and $\mathcal{A}^*$ form an attractor-repeller pair if and only if (a) $(u,y)\notin\mathcal{A}^*$ implies that $\left\{\mathbb{P}\Phi_t^1(u,y)\,|\,t\geq 0\right\}\cap N \neq \varnothing$ for every neighborhood N of $\mathcal{A}$ and (b) $(u,y)\notin\mathcal{A}$ implies that $\left\{\mathbb{P}\Phi_t^1(u,y)\,|\,t\leq 0\right\}\cap N\neq\varnothing$ for every neighborhood N of $\mathcal{A}^*$. Since the limit sets are nonvoid in the compact space $\mathcal{U}\times\mathbb{P}^n$, these conditions hold if the claim is proved. Thus, assertion (v) will follow.

In order to prove the claim, let $(u,y)\notin\mathcal{A}\cup\mathcal{A}^*$. Then, y can be written as $y = \mathbb{P}(x,1)$, and, by linearity,

$$\varphi^1(t,x,1,u) = (\varphi(t,0,u) + e^{tA}x, 1) = \varphi^1(t,0,1,u) + \varphi^1(t,x,0,0_{\mathcal{U}}).$$

Since $e^{At}x \to 0$ for $t \to \infty$, it follows that

$$\begin{aligned} \lim_{t\to\infty} d(\mathbb{P}\Phi^1_t(u,y), \mathbb{P}\Phi^1_t(u,\mathbb{P}(0,1))) &= \lim_{t\to\infty} d(\left(\theta_t u, \mathbb{P}\varphi^1(t,x,1,u)\right), (\theta_t u, \mathbb{P}\varphi^1(t,0,1,u)) \\ &= 0. \end{aligned} \tag{4.8}$$

By (i), it follows that $\varphi(t,0,u) \in \mathbf{R}(0) \subset D_0$ for all $t \geq 0$. Define $u^+ \in \mathcal{U}$ by $u^+(t) := 0$ for $t < 0$ and $u^+(t) := u(t)$ for $t \geq 0$. Then, $(u^+, \mathbb{P}(0,1)) \in \mathcal{A}$ and $d(\mathbb{P}\Phi^1_t(u^+, \mathbb{P}(0,1)), \mathbb{P}\Phi^1_t(u, \mathbb{P}(0,1))) \to 0$ for $t \to \infty$. Together with (4.8), this yields $d(\mathbb{P}\Phi^1_t(u,y), \mathcal{A}) \to 0$ for $t \to \infty$; hence, $\omega(u,y) \subset \mathcal{A}$.

In order to prove that $\alpha(u,y) \subset \mathcal{A}^*$, we may assume that $\mathbb{P}(x,1) \notin \mathbb{P}(D_0 \times 1)$. It suffices to show that $\|\varphi(-t,x,u)\| \to \infty$ for $t \to \infty$. Assume, by contradiction, that $\varphi(-t_i,x,u)$ remains bounded for a sequence $t_i \to \infty$. Write $x_i = \varphi(-t_i,x,u)$ and $u_i = u(\cdot - t_i)$ for all $i \in \mathbb{N}$. Then,

$$x = \varphi(t_i,x_i,u_i) = \varphi(t_i,x_i,0) + \varphi(t_i,0,u_i) = e^{t_iA}x_i + \varphi(t_i,0,u_i).$$

Note that $e^{t_iA}x_i \to 0$ since A is asymptotically stable and $\{x_i, i \in \mathbb{N}\}$ is bounded. Therefore, $\varphi(t_i,0,u_i) \to x$ and $x \in \overline{\mathbf{R}(0)} = D_0$, which is a contradiction.

(vi) A consequence of (iv), (v), and Theorem 2.3 is that the chain transitive sets of $\mathbb{P}\Phi^1$ are either contained in $\mathcal{U} \times \mathbb{P}^{n,0}$ or coincide with the compact set $\mathcal{A} \subset \mathcal{U} \times \mathbb{P}^{n,1}$. Thus, $\mathcal{A}$ is a maximal chain transitive set, and, by Theorem 2.15, $\mathbb{P}(D_0 \times \{1\})$ is a chain control set. □

The following proposition for completely unstable systems is analogous.

PROPOSITION 4.7. *Assume that $\mathbb{R}^n = L^+$; hence, the homogeneous part is completely unstable. Then, the following assertions hold.*

(i) *The control set D_0 containing $0 \in \mathbb{R}^n$ is open and has compact closure satisfying $\overline{D_0} = \mathbf{C}(0)$.*

(ii) *The image $\mathbb{P}(D_0 \times \{1\})$ is an open control set with compact closure*

$$\overline{\mathbb{P}(D_0 \times \{1\})} = \mathbb{P}(\overline{D_0} \times \{1\}) \subset \mathbb{P}^{n,1}$$

for the induced system on $\mathbb{P}^n$.

(iii) *The set $\overline{\mathbb{P}(D_0 \times \{1\})}$ is chain controllable for the induced system on $\mathbb{P}^n$.*

(iv) *The set $\mathcal{A}^* \subset \mathcal{U} \times \mathbb{P}^{n,1}$ defined by*

$$\mathcal{A}^* := \{(u, \mathbb{P}(x,1)) \in \mathcal{U} \times \mathbb{P}^n \,|\, \mathbb{P}(\varphi(t,x,u),1) \in \mathbb{P}(D_0 \times \{1\}) \text{ for all } t \in \mathbb{R}\}$$

is chain transitive for the control flow $\mathbb{P}\Phi^1$.

(v) *$\mathcal{A}^*$ is a repeller, and its complementary attractor is $\mathcal{A} = \mathcal{U} \times \mathbb{P}^{n,0}$.*

(vi) *The set $\overline{\mathbb{P}(D_0 \times \{1\})} \subset \mathbb{P}^{n,1}$ is the unique chain control set in $\mathbb{P}^{n,1}$.*

Proof. This follows by time reversal from Proposition 4.6. Observe that, by time reversal, closed control sets become open control sets and attractor-repeller pairs exchange their roles. □

Next, we combine the previous propositions to obtain the following main result of this section.

THEOREM 4.8. *The unique chain control set E in $\mathbb{R}^n$ of the linear control system (1.1) is given by $E = \overline{D_0} + L^0$, where D_0 is the control set containing 0 and L^0 is the center Lyapunov space of A.*

Proof. Along with control system (1.1) on $\mathbb{R}^n$, consider the induced control systems on L^- and on L^+, which are described in Lemma 2.18. By Corollary 2.17 and Lemma 2.18(i) and (ii), the control set D_0 can be written as $D_0 = D^- \oplus (L^0 \cap \mathcal{C}) \oplus D^+$, where $D^- \subset L^-$ is the control set containing 0 of system (2.11) and $D^+ \subset L^+$ is the control set containing 0 of system (2.12).

By Proposition 4.6(vi), the unique chain control set in $\mathbb{P}^{n^-,1}$ is the compact set $\mathbb{P}(D^- \times \{1\})$. This implies that $\overline{D^-}$ is the chain control set in L^-. Since distances are decreased under the map π^-, the chain control set E in $\mathbb{R}^n$ satisfies $\pi^- E \subset \overline{D^-}$; hence, $E \subset \overline{D^-} \oplus L^0 \oplus L^+$. By Lemma 2.18, $\pi^- \mathbf{R}(0) = D^-$. It follows that

$$E \subset \overline{\pi^- \mathbf{R}(0)} \oplus L^0 \oplus L^+.$$

Analogously, one shows that the chain control set E in $\mathbb{R}^n$ is contained in $L \oplus L^0 \oplus \overline{\pi^+ \mathbf{C}(0)}$. Using Theorem 2.16(iii), it follows that E is contained in

$$\left(\overline{\pi^- \mathbf{R}(0)} \oplus L^0 \oplus L^+\right) \cap \left(L^- \oplus L^0 \oplus \overline{\pi^+ \mathbf{C}(0)}\right) = \overline{\pi^- \mathbf{R}(0)} \oplus L^0 \oplus \overline{\pi^+ \mathbf{C}(0)} = \overline{D_0} + L^0.$$

Since, by Proposition 3.5, $\overline{D_0} + L^0$ is chain controllable equality holds. □

Remark 4.9. The arguments above provide an alternative proof of the claim that E is the unique chain control set in $\mathbb{R}^n$.

The following corollary to Theorem 4.8 presents another description of the chain control set.

COROLLARY 4.10. *For the linear control system* (1.1)*, the chain control set E is*

$$E = (\overline{\mathbf{C}(0)} \cap L^+) \oplus L^0 \oplus (\overline{\mathbf{R}(0)} \cap L^-) = (\mathbf{C}^c(0) \cap L^-) \oplus L^0 \oplus (\mathbf{R}^c(0) \cap L^-).$$

Proof. Corollary 2.17 implies that

$$E = \overline{D_0} + L^0 = \left(\overline{\mathbf{C}(0)} \cap L^+\right) \oplus L^0 \oplus (\mathbf{R}(0) \cap L^-).$$

By Proposition 3.5, the chain reachable set $\mathbf{R}^c(0)$ contains $L^+ \oplus L^0$ and the chain controllable set $\mathbf{C}^c(0)$ contains $L^0 \oplus L^-$. Then, decomposition (2.10) shows that

$$\mathbf{R}^c(0) = L^+ \oplus L^0 \oplus \left(\mathbf{R}^c(0) \cap L^-\right) \text{ and } \mathbf{C}^c(0) = (\mathbf{C}^c(0) \cap L^+) \oplus L^0 \oplus L^-.$$

Combining this with Theorem 2.14, one finds that

$$E = \mathbf{C}^c(0) \cap \mathbf{R}^c(0) = (\mathbf{C}^c(0) \cap L^-) \oplus L^0 \oplus (\mathbf{R}^c(0) \cap L^-).$$ □

Comparing this to the characterization of the control set D_0 in Corollary 2.17, one sees that the difference between the control set D_0 and the chain control set E is that D_0 contains the summand $L^0 \cap \mathcal{C}$, while, for E, the whole center subspace L^0 is a summand and the closure of $\mathbf{C}(0) \cap L^+$ is taken.

5. The system on the Poincaré sphere. In this section, we deduce a formula for the central Selgrade bundle and show that the chain control set on the Poincaré sphere coincides with the closure of the image of the chain control set on $\mathbb{R}^n$.

By Theorem 4.3, we know that the central Selgrade bundle $\mathcal{V}_c^1$ for (1.1) is the unique Selgrade bundle not contained in $\mathcal{U} \times \mathbb{R}^{n+1,0}$; cf. (4.2). The projection

$\mathcal{M}_c^1 = \mathbb{P}\mathcal{V}_c^1 \subset \mathcal{U} \times \mathbb{P}^n$ is the unique chain recurrent component of $\mathbb{P}\Phi^1$ with $\mathcal{M}_c^1 \cap (\mathcal{U} \times \mathbb{P}^{n,1}) \neq \varnothing$. Furthermore,

$$E_c^1 = \{p \in \mathbb{P}^n \,|\, \exists u \in \mathcal{U} : (u,p) \in \mathcal{M}_c^1\}$$

is the unique chain control set not contained in the equator $\mathbb{P}^{n,0}$ of the Poincaré sphere $\mathbb{P}^n$, i.e., having nonvoid intersection with $\mathbb{P}^{n,1}$.

We will exploit the decomposition of $\mathbb{R}^n$ into the hyperbolic part $L^+ \oplus L^-$ and L^0. Fixing a basis, we identify the subspace $L^+ \oplus L^-$ with $\mathbb{R}^{n^h}$ where $n^h = \dim(L^+ \oplus L^-)$. The corresponding Poincaré sphere is $\mathbb{P}^{n^h}$. The next proposition provides basic information on the behavior of the induced control system (2.13) on $L^+ \oplus L^-$.

PROPOSITION 5.1.

(i) *The chain control set E^h for the induced control system* (2.13) *on $L^+ \oplus L^-$ is compact, and $E^h = \overline{D_0^h}$, where $D_0^h = \pi^h D_0$ is the control set containing* 0.

(ii) *There exists a unique chain control set E_c^h having nonvoid intersection with $\mathbb{P}^{n^h,1}$ for the induced control system on $\mathbb{P}^{n^h}$. It is a compact subset of $\mathbb{P}^{n^h,1}$ and coincides with $\overline{\mathbb{P}(E^h \times \{1\})} = \overline{\mathbb{P}(D_0^h \times \{1\})}$.*

(iii) *Consider the flows $\mathbb{P}\Phi^1$ on $\mathcal{U} \times \mathbb{P}^n$ and $\mathbb{P}\Phi^{h,1}$ on $\mathcal{U} \times \mathbb{P}^{n^h}$. The chain recurrent components $\mathcal{M}_c^1 = \mathbb{P}\mathcal{V}_c^1$ and $\mathcal{M}_c^{h,1} = \mathbb{P}\mathcal{V}_c^{h,1}$ with $\mathcal{M}_c^1 \cap \mathbb{P}^{n,1} \neq \varnothing$ and $\mathcal{M}_c^{h,1} \cap \mathbb{P}^{n^h,1} \neq \varnothing$ are the lifts $\mathcal{E}_c^1$ and $\mathcal{E}_c^{h,1}$ of the chain control sets E_c^1 and $E_c^{h,1}$, respectively.*

(iv) *The central Selgrade bundle for the flow $\Phi^{h,1}$ on $\mathcal{U} \times (L^+ \oplus L^-) \times \mathbb{R}$ is the line bundle*

$$\mathcal{V}_c^{h,1} = \left\{(u, -re(u,0), r) \in \mathcal{U} \times \mathbb{R}^{n^h} \times \mathbb{R} \,|\, u \in \mathcal{U}, r \in \mathbb{R}\right\},$$

where $e(u,t), t \in \mathbb{R}$, is the unique bounded solution of (2.13) *for $u \in \mathcal{U}$, and the projection satisfies $\mathcal{M}_c^{h,1} = \mathbb{P}\mathcal{V}_c^{h,1} \subset \mathbb{P}^{n^h,1}$.*

Proof. Assertion (i) follows from the characterization of the chain control set in Theorem 4.8 using that A^h is hyperbolic and by Lemma 2.18(iii). Assertion (iii) follows by (i) and Corollary 4.4(iii). Furthermore, (iii) holds since the lifts of chain control sets are chain recurrent components by Theorem 2.15, and (iv) holds by Theorem 4.3(iv). ∎

Define a map of the projective Poincaré spheres by

$$P : \mathbb{P}^n \to \mathbb{P}^{n^h} : P(\mathbb{P}(x,r)) := \mathbb{P}(\pi^h x, r) \text{ for } (0,0) \neq (x,r) \in \mathbb{R}^n \times \mathbb{R}. \tag{5.1}$$

PROPOSITION 5.2.

(i) *The chain control sets E_c^1 and $E_c^{h,1}$ of the induced control systems on the Poincaré spheres $\mathbb{P}^n$ and $\mathbb{P}^{n^h}$, respectively, satisfy*

$$P(E_c^1) \subset E_c^{h,1} = \overline{\mathbb{P}(E^h \times \{1\})} = \overline{\mathbb{P}(D_0^h \times \{1\})}.$$

(ii) *The central Selgrade bundles $\mathcal{V}_c^1$ and $\mathcal{V}_c^{h,1}$ of the flow Φ^1 on $\mathcal{U} \times \mathbb{R}^n \times \mathbb{R}$ and the flow $\Phi^{h,1}$ on $\mathcal{U} \times (L^+ \oplus L^-) \times \mathbb{R}$, respectively, satisfy*

$$\mathcal{V}_c^1 \subset \{(u,x,r) \in \mathcal{U} \times \mathbb{R}^n \times \mathbb{R} \,|\, (u, \pi^h x, r) \in \mathcal{V}_c^{h,1}\}.$$

Proof.

(i) The map $\mathbb{R}^n \times \mathbb{R} \to (L^+ \oplus L^-) \times \mathbb{R} : (x,r) := \subsetneq \pi^h x, r)$ is a projection decreasing the norm; hence, the map P decreases the distance. Thus, controlled

(ε,T)-chains in $\mathbb{P}^n$ are mapped to controlled (ε,T)-chains in $\mathbb{P}^{n^h}$, showing that the image $P(E_c^1)$ is contained in a chain control set. Since $E_c^1 \cap \mathbb{P}^{n,1} \neq \varnothing$, it follows that this chain control set has nonvoid intersection with $\mathbb{P}^{n^h,1}$ and hence coincides with $E_c^{h,1}$. The equalities hold by Proposition 5.1(ii).

(ii) This follows from (i) and Proposition 5.1(iii). ∎

The following theorem characterizes the central Selgrade bundle $\mathcal{V}_c^1$ and, equivalently, the chain control set E_c^1 on the projective Poincaré sphere $\mathbb{P}^n$. Recall that $\mathcal{V}_c^{h,1}$ is the central Selgrade bundle of the hyperbolic part and $\mathcal{V}_0^\infty = \mathcal{U} \times L^0 \times 0$ with the center Lyapunov space $L^0 = L(0)$.

THEOREM 5.3.

(i) *The central Selgrade bundle $\mathcal{V}_c^1$ of the control flow Φ^1 on $\mathcal{U}\times\mathbb{R}^n\times\mathbb{R}$ associated with control system* (1.1) *is given by*

$$\mathcal{V}_c^1 = \mathcal{V}_c^{h,1} \oplus \mathcal{V}_0^\infty = \left\{(u, -re(u,0)+x, r) \in \mathcal{U}\times\mathbb{R}^n\times\mathbb{R} \,\middle|\, u\in\mathcal{U}, x\in L^0, r\in\mathbb{R}\right\}, \tag{5.2}$$

where $e(u,t), t\in\mathbb{R}$ is the unique bounded solution for u of the induced system (2.13) *on $L^+\oplus L^-$. The dimension is $\dim \mathcal{V}_c^1 = 1+\dim L^0$.*

(ii) *The chain control set E_c^1 on the Poincaré sphere $\mathbb{P}^n$ is given by*

$$E_c^1 = \overline{\{\mathbb{P}(-e(u,0)+v,1) \,|\, u\in\mathcal{U}, v\in L^0\}}.$$

Proof. Assertion (ii) is a consequence of (i). The second equality in (i) holds since, by Proposition 5.1(iv), the central Selgrade bundle $\mathcal{V}_c^{h,1}$ of the hyperbolic part is

$$\mathcal{V}_c^{h,1} = \left\{(u, -re(u,0), r) \in \mathcal{U}\times\mathbb{R}^{n^h}\times\mathbb{R} \,|\, u\in\mathcal{U}, r\in\mathbb{R}\right\}.$$

It remains to prove the first equality in (i).

Step 1. We claim that

$$\mathcal{V}' := \mathcal{V}_c^{h,1}\oplus\mathcal{V}_0^\infty = \left\{(u, -re(u,0)+x, r)\in\mathcal{U}\times\mathbb{R}^n\times\mathbb{R} \,\middle|\, u\in\mathcal{U}, x\in L^0, r\in\mathbb{R}\right\}$$

defines a subbundle. This holds if it is closed and the fibers are linear with constant dimension; cf. Colonius and Kliemann [4, Lemma B.1.13]. In fact, consider for $u\in\mathcal{U}$ the fiber

$$V_u' = \left\{(-re(u,0)+x, r)\in\mathbb{R}^n\times\mathbb{R} \,\middle|\, x\in L^0, r\in\mathbb{R}\right\},$$

and let $(-r_ie(u,0)+x_i, r_i)\in V_u'$ for $i=1,2$. Then, for all $\alpha,\beta\in\mathbb{R}$,

$$\begin{aligned}&\alpha(-r_1e(u,0)+x_1, r_1)+\beta(-r_2e(u,0)+x_2, r_2)\\ &\quad = (-(\alpha r_1+\beta r_2)e(u,0)+(\alpha x_1+\beta x_2), \alpha r_1+\beta r_2)\in V_u'.\end{aligned}$$

The dimension is $\dim V_u' = 1+\dim L^0$. Furthermore, if $(u_k, -r_ke(u_k,0)+x_k, r_k)\to(v,y,r)$ for $k\to\infty$, it follows that $u_k\to u$ and $r_k\to r$. Then (cf. Colonius and Santana [6, Theorem 2.5 and proof of Theorem 3.1]), it follows that the unique bounded solutions at time $t=0$ given by $e(u_k,0)$ converge to $e(u,0)$; hence, $-r_ke(u_k,0)+x_k\to re(u,0)+x$ for an element $x\in L^0$. This shows that $(v,y,r)=(u,-re(u,0)+x,r)\in V_u'$, and hence, $\mathcal{V}'$ is a subbundle.

Step 2. The subbundles satisfy $\mathcal{V}_c^1 \subset \mathcal{V}'$.

Write $x = \pi^h x + \pi^0 x$ for $x \in \mathbb{R}^n$. By Proposition 5.2(ii), every $(u,x,r) \in \mathcal{V}_c^1$ satisfies

$$(u,\pi^h x, r) \in \mathcal{V}_c^{h,1} = \left\{(u,-re(u,0),r) \in \mathcal{U} \times (L^+ \oplus L^-) \times \mathbb{R} \,|\, u \in \mathcal{U}, r \in \mathbb{R}\right\},$$

implying that $(u,x,r) = (u,\pi^h x + \pi^0 x, r) = (u, -re(u,0) + \pi^0 x, r) \in \mathcal{V}'$.

Step 3. Also, the converse inclusion $\mathcal{V}' \subset \mathcal{V}_c^1$ holds; hence, $\mathcal{V}' = \mathcal{V}_c^1$.

For the proof, let $(u,y,r) \in \mathcal{V}' = \mathcal{V}_c^{h,1} \oplus \mathcal{V}_0^\infty$. Observe first that, by Proposition 4.5, the inclusion $\mathcal{V}_0^\infty \subset \mathcal{V}_c^1$ holds. If

$$\begin{aligned} \mathcal{V}_c^{h,1} &= \{(u,-re(u,0),r) \in \mathcal{U} \times \mathbb{R}^n \times \mathbb{R} \,|\, r \in \mathbb{R}\} \\ &\subset \left\{(u,\pi^h x, r) \in \mathcal{U} \times \mathbb{R}^n \times \mathbb{R} \,\middle|\, (u,x,r) \in \mathcal{V}_c^1\right\} =: \pi^h \mathcal{V}_c^1, \end{aligned} \tag{5.3}$$

the assertion follows since, by linearity, this implies that

$$(u,y,r) = (u,\pi^h y + \pi^0 y, r) \in \mathcal{V}_c^{h,1} \oplus \mathcal{V}_0^\infty \subset \pi^h \mathcal{V}_c^1 \oplus \mathcal{V}_0^\infty \subset \mathcal{V}_c^1.$$

Inclusion (5.3) is equivalent to

$$\mathcal{M}_c^{h,1} \subset \left\{(u,\mathbb{P}\left(\pi^h x, 1\right)) \in \mathcal{U} \times \mathbb{P}^n \,\middle|\, (u,\mathbb{P}(x,r)) \in \mathcal{M}_c^1\right\}. \tag{5.4}$$

By Proposition 5.1(iii), the chain transitive components $\mathcal{M}_c^{h,1}$ and $\mathcal{M}_c^1$ are the lifts $\mathcal{E}_c^{h,1}$ and $\mathcal{E}_c^1$ of the chain control sets $E_c^{h,1}$ and E_c^1, respectively. Thus, (5.4) is, with $P : \mathbb{P}^n \to \mathbb{P}^{n^h}$ defined by (5.1), equivalent to

$$E_c^{h,1} \subset P(E_c^1) = \left\{\mathbb{P}(\pi^h x, 1) \,\middle|\, \mathbb{P}(x,1) \in E_c^1\right\}. \tag{5.5}$$

For the proof of inclusion (5.5), recall that, by Corollary 4.4(iii), $E_c^{h,1} = \overline{\mathbb{P}(E^h \times \{1\})}$, and by Proposition 5.1(i), the chain control set E^h and the control sets D_0^h and D_0 are related by

$$E^h = \overline{D_0^h}, \text{ and } \pi^h(D_0) = D_0^h; \text{ hence, } E^h = \overline{\pi^h(D_0)}.$$

It follows that

$$E_c^{h,1} = \overline{\mathbb{P}(E^h \times \{1\})} = \overline{\mathbb{P}(\overline{\pi^h(D_0)} \times \{1\})}. \tag{5.6}$$

The points in $(\mathrm{int} D_0) \times \{1\} \subset \mathbb{R}^n \times \mathbb{R}$ are controllable to each other. Hence, the same is true for the points in $\mathbb{P}((\mathrm{int} D_0) \times \{1\}) \subset \mathbb{P}^n$, and it follows that $\mathbb{P}(D_0 \times \{1\}) \subset E_c^1$; thus, $x \in D_0$ implies that $\mathbb{P}(x,1) \in E_c^1$. This shows that

$$\mathbb{P}(\pi^h(D_0) \times \{1\}) = \left\{\mathbb{P}(\pi^h x, 1) \,|\, x \in D_0\right\} \subset \left\{\mathbb{P}(\pi^h x, 1) \,\middle|\, \mathbb{P}(x,1) \in E_c^1\right\}.$$

Taking closures and using (5.6), inclusion (5.5) follows. This concludes the proof of Theorem 5.3. ∎

Finally, we compare the image $\mathbb{P}(E \times \{1\})$ of the chain control set in $\mathbb{R}^n$ and the chain control set E_c^1 on the Poincaré sphere on $\mathbb{P}^n$.

COROLLARY 5.4. *The chain control set E_c^1 on the Poincaré sphere coincides with the closure of the image of the chain control set E in $\mathbb{R}^n$, $E_c^1 = \overline{\mathbb{P}(E \times \{1\})}$.*

Proof. By Corollary 4.4(ii), we already know that $\mathbb{P}(E\times\{1\})\subset E_c^1$. Since chain control sets are closed, it only remains to prove the converse inclusion. First, we show that the unique bounded solution of (2.13) for control $u\in\mathcal{U}$ satisfies $e(u,\cdot)\subset E^h$, where E^h is the chain control set of (2.13). Consider the set

$$\left\{y\in\mathbb{R}^n\,\middle|\,\exists t_k\to\infty:\varphi^h(t_k,e(u,0),u)=e(u,t_k)\to y\right\},$$

where $\varphi^h(t,x,u), t\in\mathbb{R}$ denotes the solution of (2.13). This set is nonvoid since $e(u,\cdot)$ is bounded and, as an ω-limit set, it is contained in a chain control set and hence in E^h. Analogously,

$$\varnothing\neq\left\{y\in\mathbb{R}^n\,\middle|\,\exists t_k\to-\infty:\varphi^h(t_k,e(u,0),u)\to y\right\}\subset E^h.$$

It follows that $e(u,t), t\in\mathbb{R}$, is contained in E^h. By Corollary 2.17, we know, for the control sets $D_0\subset\mathbb{R}^n$ and $D_0^h\subset L^+\oplus L^-$ containing the origin, that

$$D_0=\overline{\mathbf{R}(0)}\cap\mathbf{C}(0)\text{ and }D_0^h=\overline{\mathbf{R}^h(0)}\cap\mathbf{C}^h(0).$$

By Lemma 2.18(ii), $\mathbf{R}(0)\subset\mathbf{R}^h(0)+L^0$, and hence, $\mathbf{R}(0)+L^0=\mathbf{R}^h(0)+L^0$. Analogously, it follows that $\mathbf{C}(0)+L^0=\mathbf{C}^h(0)+L^0$, and hence,

$$D_0+L^0=\left(\overline{\mathbf{R}(0)}\cap\mathbf{C}(0)\right)+L^0=\left(\overline{\mathbf{R}^h(0)}+L^0\right)\cap\left(\mathbf{C}^h(0)+L^0\right)=D_0^h+L^0.$$

By Theorem 4.8, the chain control sets are $E=\overline{D_0}+L^0$ and $E^h=\overline{D_0^h}$, and we obtain

$$E=\overline{D_0}+L^0=\overline{D_0^h}+L^0=E^h+L^0.$$

By Theorem 5.3(ii), it follows that

$$E_c^1=\overline{\mathbb{P}\left\{(-e(u,0)+x,1)\,|\,u\in\mathcal{U},x\in L^0\right\}}\subset\overline{\mathbb{P}\left\{(E^h+L^0)\times\{1\}\right\}}=\overline{\mathbb{P}\left(E\times\{1\}\right)}.$$

This concludes the proof of the corollary. ∎

6. Examples. In this section, we present two examples illustrating the results above. In the first example, the matrix A is nonhyperbolic and the controllability subspace is a proper subspace.

Example 6.1. Consider the two-dimensional system

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{pmatrix}0&0\\0&-1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}u(t),\; u(t)\in[-1,1].$$

The controllability subspace is $\mathcal{C}=\mathrm{Im}[B,AB]=0\times\mathbb{R}$. The control set D_0 containing $0\in\mathbb{R}^2$ is given by $D_0=0\times[-1,1]\subset\mathcal{C}=0\times\mathbb{R}$ with nonvoid interior in $\mathcal{C}$. The center subspace of A is $L^0=L(0)=\mathbb{R}\times 0$, and, by Theorem 4.8, the chain control set is

$$E=\overline{D_0}+L^0=(0\times[-1,1])+(\mathbb{R}\times 0)=\mathbb{R}\times[-1,1].$$

Consider the subbundles $\mathcal{V}_0=\mathcal{U}\times L(0)=\mathcal{U}\times(\mathbb{R}\times 0)$ and (cf. Proposition 5.1(iv))

$$\mathcal{V}_c^{h,1}=\left\{(u,-re(u,0),r)\in\mathcal{U}\times\mathbb{R}\times\mathbb{R}\,|\,u\in\mathcal{U},r\in\mathbb{R}\right\},$$

where $e(u,t), t\in\mathbb{R}$, is the unique bounded solution for $u\in\mathcal{U}$ of the hyperbolic part. By Theorem 5.3(i), the central Selgrade bundle is

$$\mathcal{V}_c^1=\mathcal{V}_c^{h,1}\oplus\mathcal{V}_0^\infty=\left\{(u,-re(u,0)+x,0,r)\,|\,u\in\mathcal{U},x\in\mathbb{R},r\in\mathbb{R}\right\}.$$

A little computation shows that the unique bounded solution of $\dot{y}(t) = -y(t) + u(t)$ is

$$e(u,t) := \int_{-\infty}^{t} e^{-(t-s)} u(s) ds, t \in \mathbb{R}. \tag{6.1}$$

Thus, it follows that

$$\mathcal{V}_c^1 = \left\{ \left(u, x, -r \int_{-\infty}^{0} e^s u(s) ds, r \right) \middle| u \in \mathcal{U}, x \in \mathbb{R}, r \in \mathbb{R} \right\} \subset \mathcal{U} \times \mathbb{R}^2.$$

Corollary 5.4 shows that the chain control set E_c^1 on the projective Poincaré sphere is

$$E_c^1 = \mathbb{P}(E \times \{1\}) = \mathbb{P}(\mathbb{R} \times [-1,1] \times \{1\}) \subset \mathbb{P}^2.$$

The next example is controllable, and the matrix A is hyperbolic.

Example 6.2. Consider, with hyperbolic matrix A, the system given by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(t), u(t) \in [-1,1]. \tag{6.2}$$

The controllability subspace is $\mathcal{C} = \mathbb{R}^2$, and inspection of the phase portrait shows that the control set is $D_0 = (-1,1) \times [-1,1]$.

By Theorem 4.8, the chain control set is

$$E = \overline{D_0} + L^0 = \overline{D_0} = [-1,1] \times [-1,1].$$

By Theorem 5.3, the central Selgrade bundle is

$$\mathcal{V}_c^1 = \mathcal{V}_c^{h,1} = \left\{ (u, -re(u,0), r) \,\middle|\, x \in L^0, r \in \mathbb{R} \right\} \subset \mathcal{U} \times \mathbb{R}^2,$$

where $e(u,t), t \in \mathbb{R}$ is the bounded solution of (6.2) for $u \in \mathcal{U}$. The bounded solution of $\dot{y}(t) = -y(t) + u(t)$ is given by (6.1). The bounded solution of $\dot{x}(t) = x(t) + u(t)$ is $x_b(t) := -\int_{-\infty}^{-t} e^{t+s} u(-s) ds, t \in \mathbb{R}$. Thus,

$$\mathcal{V}_c^1 = \left\{ \left(u, r \int_{-\infty}^{0} e^s u(-s) ds, -r \int_{-\infty}^{0} e^s u(s) ds, r \right) \middle| u \in \mathcal{U}, r \in \mathbb{R} \right\}.$$

Corollary 5.4 shows that the chain control set E_c^1 on the projective Poincaré sphere is

$$E_c^1 = \left\{ \mathbb{P} \left(\int_{-\infty}^{0} e^s u(-s) ds, -\int_{-\infty}^{0} e^s u(s) ds, 1 \right) \middle| u \in \mathcal{U} \right\} = \mathbb{P}(E \times \{1\}).$$

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