Social Choice and Welfare (2025) 64:915–959
https://doi.org/10.1007/s00355-024-01562-x

ORIGINAL PAPER

Network creation with homophilic agents

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Received: 25 July 2023 / Accepted: 7 October 2024 / Published online: 5 November 2024

Abstract

Network Creation Games are an important framework for understanding the formation of real-world networks. These games usually assume a set of indistinguishable agents strategically buying edges at a uniform price, which leads to the formation of a network among them. However, in real life, agents are heterogeneous and their relationships often display a bias towards similar agents, say of the same ethnic group. This homophilic behavior on the agent level can then lead to the emergent global phenomenon of social segregation. We study Network Creation Games with multiple types of homophilic agents and non-uniform edge cost, introducing two models focusing on the perception of same-type and different-type neighbors, respectively. Despite their different initial conditions, both our theoretical and experimental analysis show that both the composition and segregation strength of the resulting stable networks are very similar, indicating a robust structure of social networks under homophily.

1 Introduction

Networks play an eminent role in today's world. They are crucial for our energy supply (power grid networks), our information exchange (the Internet and the World Wide Web), and our social relationships (friendship networks, email exchange, or co-author networks). There exists an abundance of approaches to provide formal frameworks for modeling networks, see, for example, the books by Jackson (2010) and Newman

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(2018). In many of these models, the nodes of the network correspond to agents that strategically create connections, which is particularly suitable for our main focus of modeling social networks. One such stream of research considers variants of the *Network Creation Game (NCG)* as proposed by Fabrikant et al. (2003). There, selfish agents create edges to form a network among themselves. Forming edges is costly and hence agents try to create only the most useful edges. On the other hand, the force that causes agents to form edges at all is well-connectivity within the network, captured by a desire to occupy a central position.

The NCG is a stylized model of social interaction, providing valuable insight to agents' decision processes when interacting with each other. However, it is important to refine the basic model to spotlight specific details of this decision making. In this sense, we study network creation under the additional assumption that agents are separated into various *types* that model ethnic groups or affiliations.

Our goal is to contribute a new perspective on the simple causes that lead to the segregation of a society, similar to Schelling's checkerboard model for residential segregation (Schelling 1969, 1971). Therefore, our agents' cost functions have a bias towards the creation of relationships with agents of the same type. Specifically, we study two models based on two seemingly orthogonal treatments of other agents. In the first model, agents incur a fixed cost for every created edge and a variable cost that only depends on the number of edges towards same-type agents. In the second model, edges towards different-type agents are initially more expensive but their cost drops with an inverse linear decay. Both models give a different point-of-view on the same underlying principle, namely homophily of agents, i.e., the tendency to form connections with like-minded people. This is often summarized with the proverb "birds of a feather flock together", a dominant intrinsic force repeatedly observed in the creation of social networks, see the survey by McPherson et al. (2001) for an overview over the extensive sociological research on homophily in social networks. While our first model expresses homophily explicitly by an increasing comfort among friends, the second model incorporates homophily indirectly by accounting for a decreasing effort of integration once first contact is established. The latter paradigm is closely related to the well-known effect in social sciences called the "contact hypothesis" which states that stereotypes and prejudices between ethnic groups can be weakened by intensified contact (Allport et al. 1954; Amir 1969; Dovidio et al. 2003).

We measure the desirability of networks by means of stability. Since we consider social networks, we assume a bilateral model where two agents have to cooperate to connect. Consequently, we use pairwise stability (Jackson and Wolinsky 1996) as solution concept, rather than, for instance, Nash stability which is more appropriate for unilateral models.

Interestingly, we find an almost identical structure of stable networks for both models. This hints at a robust structure of networks created under homophily incentives. Figure 1 provides a qualitative picture of our theoretical results and displays typical stable networks. Our games are parameterized by $\alpha > 0$, a parameter that governs the cost of buying edges. Naturally, networks get sparser the higher the edge cost. A very small edge cost causes extremely high connectivity, as indicated by the fully connected network on the left of the cost range. For moderately small edge cost, we provide characterizations of stable networks which are all highly segregated. We inter-

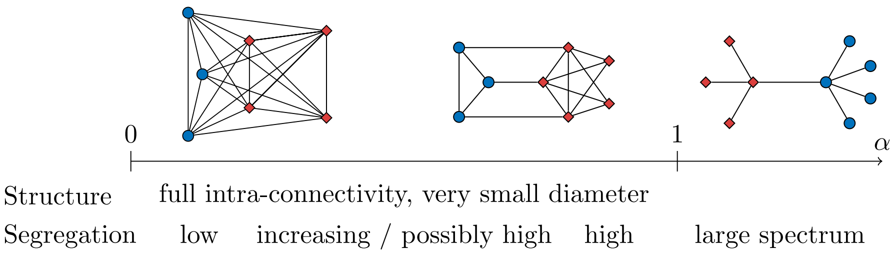

Fig. 1 Qualitative picture of stable networks. The blue circles and red diamonds represent two different types of agents. A more extensive version of this picture including exact parameters and a split view of our two models is provided in Fig. 13 in Sect. 7

pret this as identifying a sweet spot of high sensitivity towards agent types. A typical stable network at this cost range is displayed in the middle of Fig. 1: all agents of one type are necessarily exposed to agents of the other type, but the edges connecting agent types form a matching between different agent types. Clearly, if agents of the same type are highly connected, this causes a high segregation strength. For larger edge cost, stability causes a large spectrum of networks to form with respect to segregation strength. For instance, networks can be highly segregated or highly integrated trees. One example, a tree in which each agent type forms a star such that only their centers are connected, is provided at the right of Fig. 1.

We accompany our theoretical findings with an average-case analysis by detailed simulations that give insight in typical stable networks for $\alpha \geq 1$ where our theory postulates the most diverse picture of stable networks. For this, we consider a simple distributed dynamics, where agents perform improvements towards stable networks. It would be plausible if a generally high edge cost causes less distinction of agent types. While this is sometimes confirmed, we also identify contrasting tendencies towards extreme segregation. An important driver for the different behavior is the initial segregation level. In fact, segregation may remain low if it is low initially. As a policy, this suggests that segregation can be avoided by a high initial effort towards integration even if policy makers do not interact subsequently.

2 Related work

In the original NCG, the cost of every edge is α, where α is a parameter of the game that allows adjusting the tradeoff between the agents' cost for creating edges and their cost for the centrality in the network, e.g., the sum of distances to all other nodes. Stable networks always exist. In particular, for $\alpha < 1$, only cliques are stable, whereas for $1 \leq \alpha < n$ stars, other trees and also non-tree networks can be stable (Mamageishvili et al. 2015). For $\alpha \geq n$ it is conjectured that all stable networks are trees and a recent line of works has proven this for $\alpha > 2n$ (Àlvarez and Messegué 2017; Bilò and Lenzner 2020; Dippel and Vetta 2024). Bilateral NCGs with uniform edge price have been introduced by Corbo and Parkes (2005) and recently this framework was extended by Friedrich et al. (2023a). In addition, variants of the NCG with non-uniform edge cost have been studied: a version where edges of differing quality can be bought (Cord-Landwehr et al. 2014), and NCGs where the edge cost depends on the node degrees (Chauhan et al. 2017), on the length of the edges in a geometric setting (Bilò et al. 2024), or on the hop-distance of the endpoints (Bilò et al. 2021). The latter is motivated by social networks, and bilateral edge formation with pairwise stability as a solution concept is considered.

The NCG variant by Meirom et al. (2014) features different types of agents and different but fixed edge costs for each agent type.

Closest to our work is the model proposed by Martí and Zenou (2017) that is a variant of the connections model (Jackson and Wolinsky 1996) with different types of agents. Similar to our model, the cost for maintaining an other-type connection depends on the homogeneity of the neighborhoods of the involved agents. In contrast to us, the cost for same-type edges is fixed and the distance cost is defined differently. The authors study the existence and structure of equilibria but do not investigate segregation quantitatively. The latter has been done by Henry et al. (2011) using a stochastic process that starts with a randomly drawn network with nodes of different types. Then edges are randomly rewired with a built-in bias toward favoring same-type edges. As the main result, the authors show that the network strongly segregates over time, even if the built-in bias is very low.

Residential segregation has recently received a lot of attention by a stream of research developing a game-theoretic framework based on Schelling's checkerboard model (see, e.g., Chauhan et al. 2018; Agarwal et al. 2021; Echzell et al. 2019; Bilò et al. 2022b; Kanellopoulos et al. 2021; Bullinger et al. 2021; Bilò et al. 2023). There, agents of several types strategically select positions on a given *fixed* network and they individually aim for having at least a τ-fraction of same-type neighbors, for some $0 < \tau \leq 1$.

Also, certain classes of coalition formation games have a similar flavor. In hedonic diversity games (Bredereck et al. 2019; Boehmer and Elkind 2020; Darmann 2021; Brandt et al. 2023), there are two types of agents and the utility of an agent within some coalition depends on the type distribution of her coalition. Moreover, there exist classes of hedonic games, where the preferences depend on distinguishing friends and enemies (see, e.g., Dimitrov et al. 2006; Kerkmann et al. 2020).

3 Preliminaries and model

We consider a set $V = \{1, \dots, n\}$ of n agents partitioned into $k \geq 2$ disjoint *types*. The set of types is denoted by $\mathcal{T}$, and for every type $T \in \mathcal{T}$, let V_T be the set of agents of type T, i.e., $V = \bigcup_{T \in \mathcal{T}} V_T$ and $V_T \cap V_{T'} = \emptyset$ for $T, T' \in \mathcal{T}$, with $T \neq T'$. For an agent $u \in V$, we denote by $\mathcal{T}(u)$ her type, i.e., we have that $u \in V_{\mathcal{T}(u)}$. Given a type $T \in \mathcal{T}$, let $n_T = |V_T|$ denote the number of agents of type T. We identify types with colors and assume that there are specific types B and R of *blue* and *red* agents, respectively, which are associated with an agent type having the smallest and largest number of agents, respectively. Thus, for every type $T \in \mathcal{T}$, we have $n_B \leq n_T \leq n_R$. In particular, with exactly two agent types we have precisely a blue minority and a red majority type.

In a network creation game, agents buy edges to eventually form a network. This leads to an undirected graph $G = (V, E)$ whose vertices are the agent set and whose edges are the established links between the agents. To speak about these networks, we now introduce some graph-theoretic concepts and notation. Therefore, consider an undirected graph $G = (V, E)$ together with vertices $u, v \in V$. We denote the (potential) edge between u and v by uv (whether it is present or not). For two agents $u, v \in V$, the edge uv is said to be *monochromatic* if u and v are of the same type, and *bichromatic*, otherwise. If $uv \in E$, we use the notation $G - uv := (V, E \setminus \{uv\})$, otherwise we use $G + uv := (V, E \cup \{uv\})$. Further, let $N_G(u) := \{v \in V : uv \in E\}$ denote the *neighborhood* of u in G, let $\deg_G(u) := |N_G(u)|$ be the *degree* of u in G, i.e., the size of its neighborhood, and let $\mathrm{d}_G(u, v)$ be the *distance* from u to v in G, i.e., the length of a shortest path from u to v in G. The *diameter* of G is defined as $\mathrm{diam}(G) := \max_{u,v \in V} \mathrm{d}_G(u, v)$, i.e., the maximum length of any shortest path in G. Finally, a useful measure for the centrality of a vertex in a network is its distance to a set of vertices. Given a subset $V' \subseteq V$ of vertices, let $\mathrm{d}_G(u, V') := \sum_{v \in V'} \mathrm{d}_G(u, v)$ denote the sum of distances from u to all vertices in V'. Also, given a subset of agents $C \subseteq V$, we denote by $G[C]$ the subgraph of G *induced by* C, i.e., $G[C] := (C, F)$, where $F = \{uv \in E : u, v, \in C\}$.

Before formally defining our network creation model, we introduce notation for some relevant special types of graphs. By $\mathbf{K}_n = (V, E)$ we denote the *complete* graph in which all possible edges are present, i.e., $E = \{uv : u, v, \in V\}$. Further, we denote by $\mathbf{S}_n = (V, E)$ a *star* graph, i.e., a graph for which there exists $u \in V$ such that $E = \{uv : v \in V \setminus \{u\}\}$. We also consider networks for the special case of two agent types. By $\mathbf{DS}_n = (V, E)$ we denote a graph for which there exist two agents $u \in V_B$ and $v \in V_R$ such that $E = \{uv\} \cup \{uw : w \in V_B\} \cup \{vw : w \in V_R\}$. Moreover, by $\mathbf{DSX}_n = (V, E)$ we denote a graph for which there exist two agents $u \in V_B$ and $v \in V_R$ such that $E = \{uv\} \cup \{uw : w \in V_R\} \cup \{vw : w \in V_B\}$. We refer to $\mathbf{DS}_n$ and $\mathbf{DSX}_n$ as a *double star* and a *double star with switched centers*, respectively.

Network creation games with homophilic agents We study network creation within a cost-oriented bilateral model à la Corbo and Parkes (2005), where the agent cost is separated into a neighborhood cost encompassing the cost of sponsoring edges and a distance cost encompassing the cost of the agents' centrality. In both our models, a created network G has a *distance cost* for agent u of $\mathrm{d}_G(u) := \mathrm{d}_G(u, V)$, i.e., the sum of agent u's distances to all other agents. The neighborhood cost is different in our two models and will be specified in the definition of our network creation games.

To model the cost dependency on the types of neighbors, we define the set of same-type agents in the neighborhood of agent $u \in V_T$ as $\mathrm{F}_G(u) := V_T \cap N_G(u)$. We will sometimes call the set of same-type neighbors of an agent her *friends* and her other-type neighbors, defined as $\mathrm{E}_G(u) := N_G(u) \setminus \mathrm{F}_G(u)$, as *enemies*. We denote the cardinalities of these sets by $\mathrm{f}_G(u) := |\mathrm{F}_G(u)|$ and $\mathrm{e}_G(u) := |\mathrm{E}_G(u)|$, respectively.

We now define our network creation games. A *network creation game with increasing comfort among friends* (ICF-NCG) with cost parameter $\alpha > 0$ is a network creation game where the neighborhood cost is given by

$$a_G^{ICF}(u) = \deg_G(u) \cdot \alpha \left(1 + \frac{1}{\mathrm{f}_G(u) + 1}\right),$$

i.e., there is a fixed cost of α for every edge and an additional cost that decreases with an increasing number of same-type neighbors. If each agent is of a different type, then $\mathrm{f}_G(u) = 0$ for all agents, and therefore each edge has a uniform cost. In this case, the game is equivalent to the single-type bilateral network creation game by Corbo and Parkes (2005).

A *network creation game with decreasing effort of integration* (DEI-NCG) with cost parameter $\alpha > 0$ is a network creation game where the neighborhood cost is given by

$$a_G^{DEI}(u) = \alpha\left(\deg_G(u) + \sum_{j=1}^{\mathrm{e}_G(u)} \frac{1}{j}\right).$$

Hence, there is a fixed edge cost of α for every edge to an agent in the neighborhood together with a harmonically decreasing additional cost for edges towards other-type agents. Note that the sum is empty for $\mathrm{e}_G(u) = 0$, and therefore, the DEI-NCG with a single type of agents is identical to the single-type bilateral network creation game by Corbo and Parkes (2005).

For the neighborhood cost, we omit the superscript indicating the type of network creation game, whenever this is clear from the context. Also, for both our models, we define the *cost of an agent u in a network G* as $c_G(u) := a_G(u) + \mathrm{d}_G(u)$.

The cost functions mimic the two effects that we want to model, namely a general homophilic behavior via the ICF-NCG and diminishing prejudices with intensified contact via the DEI-NCG. In both models, edge costs have a similar decay structure and identical range of $[\alpha, 2\alpha]$. In the ICF-NCG, the cost of edges is 2α for each edge if an agent has no friends, and the edge cost is approaching α when the number of neighboring same-type agents is growing. In the DEI-NCG, the cost of edges to friends is always α and the variable cost only affects other-type agents, where we approach α with a harmonic decay starting at a cost of 2α for the first other-type agent.

Measures for desirable networks We analyze networks by the incentives of agents to maintain the network in terms of stability and by the diversity of their neighborhood with respect to other agent types. An agent $u \in V$ is said to be *discontent* in network G if

(i) there exists a neighbor $v \in N_G(u)$ such that $c_G(u) > c_{G-uv}(u)$, i.e., v would benefit from unilaterally severing the edge to v or
(ii) There exists a non-neighbor $v \notin N_G(u)$ such that $c_G(u) > c_{G+uv}(u)$ and $c_G(v) > c_{G+uv}(v)$, i.e., u and her neighbor v can bilaterally create an edge to decrease their respective cost.

An agent that is not *discontent* in network G is said to be *content* in network G. Following Jackson and Wolinsky (1996), a network G is said to be *pairwise stable* if every agent is content in network G.

An agent severing an edge or creating an edge to decrease her and, in the case of edge creation, her new neighbor's cost is said to perform a *better response*. A better response that decreases an agent's cost the most is called a *best response*. If a better response is executed, we also speak of a *deviation*. In our simulations, we will study dynamics where agents iteratively preform best responses and we refer to such dynamics as *best response dynamics*.

Connectivity is an important aspect in network analysis. With multiple agent types, the internal connectivity per type deserves special consideration. Formally, a network $G = (V, E)$ is said to be *fully intra-connected* if, for every pair $u, v \in V$ of same-type agents, it holds that $uv \in E$. Furthermore, network G is *fully connected* if G is complete.

For the evaluation of diversity, we consider two segregation measures. Given a network $G = (V, E)$, its *local segregation*, denoted by $LS(G)$, is defined as the average fraction of agents of the same type, i.e.,

$$LS(G) = \frac{1}{|V|} \sum_{u \in V} \frac{\mathrm{f}_G(u)}{\deg_G(u)}.$$

The *global segregation*, called $GS(G)$, is the proportion of monochromatic edges, i.e.,

$$GS(G) = \frac{\sum_{u \in V} \mathrm{f}_G(u)}{2|E|}.$$

Note that $\frac{1}{2} \sum_{u \in V} \mathrm{f}_G(u)$ is the number of monochromatic edges, i.e., in the numerator of $GS(G)$, we count each such edge twice. Both segregation measures are designed in a way that higher values indicate higher segregation. The range is between 0, which occurs in networks where all neighbors are from different types and 1, which happens in networks where all edges are between same-type agents. Because of the distance cost, all reasonable networks are connected and their segregation measures are well-defined.[1] This also means that networks with more than one agent type will typically not lead to a segregation measure of exactly 0 or 1 (a value of 1 is even impossible because we need at least one bichromatic edge). In our worst-case examples, these extreme values will, however, occur in an asymptotic sense, i.e., if the number of agents tends to infinity.

The segregation measures *LS* and *GS* are (related to) standard measures in social sciences to capture the agents' *exposure* (Massey and Denton 1988). The segregation measure *LS* is used by Paolillo and Lorenz (2018) and *GS* is used in the simulation framework Netlogo (Wilensky 1997) and by Zhang (2011).

Finally, the minimum willingness of an agent to integrate can be evaluated by checking if she entertains any bichromatic edge. Therefore, we call an agent *curious* if she is part of a bichromatic edge. Similarly, a type of agents is said to be *curious* if it solely consists of curious agents. Note that this concept is related to the *degree of integration*, which is identical to the number of curious agents and has been studied in game-theoretic models for residential segregation (Agarwal et al. 2021; Bilò et al. 2022a; Friedrich et al. 2023b).

[1] In fact, all networks considered in our paper are connected. To make *LS* and *GS* well-defined for isolated vertices, one can use the convention that $\frac{0}{0} := 1$, which is based on the idea that isolated vertices are fully segregated.

4 Increasing comfort among friends

In this section, we perform our theoretical analysis of the ICF-NCG. Unless explicitly stated otherwise, all statements hold for an arbitrary number of types. All missing proofs here and in the subsequent sections can be found in the appendix.

We start by gathering some statements concerning structural properties and simple pairwise stable networks. Their proof follows by a careful analysis of the cost difference after the creation and deletion of edges.

Proposition 4.1 *For the ICF-NCG the following hold:*

1. *If $\alpha < \frac{6}{7}$, then every pairwise stable network is fully intra-connected.*
2. *If $\alpha < \frac{4}{3}$, then* $\mathrm{diam}(G) \leq 2$ *for every pairwise stable network G. In particular, network G contains a curious type.*
3. *Let $\alpha < 1$, let G be a pairwise stable network, and let $C \subseteq V$ such that every agent in C is curious and $C \subseteq V_T$, for some type $T \in \mathcal{T}$. Then, network $G[C]$ is a clique. In particular, every curious type of agents is fully intra-connected.*
4. *If $\alpha \leq \frac{n_B}{n_B+1}$, then the complete network $\mathbf{K}_n$ is pairwise stable. Moreover for $\alpha < \min\{\frac{6}{7}, \frac{n_B}{n_B+1}\}$, the network $\mathbf{K}_n$ is the unique pairwise stable network.*
5. *If $\alpha \geq 1$, then the star $\mathbf{S}_n$ is pairwise stable.*
6. *If $\alpha \geq \frac{4}{3}$, then the double star $\mathbf{DS}_n$ is pairwise stable.*

Notably, all bounds for α obtained in Proposition 4.1 are tight. In fact:

1. If $\alpha \geq \frac{6}{7}$, then a star with a red center agent, one blue leaf agent, and three red leaf agents is pairwise stable but not fully intra-connected.
2. If $\alpha \geq \frac{4}{3}$, then the double star is pairwise stable but has a diameter of 3 and none of the types is curious.
3. If $\alpha \geq 1$, then any star network is pairwise stable, but all agents of types different from the center agent's type are curious but not directly connected with each other.
4. If $\alpha > \frac{n_B}{n_B+1}$, then every blue agent in a complete network has an incentive to sever a bichromatic edge.
5. If $\alpha < 1$, then every pair of agents from a type different than the center agent's type in a star have an incentive to together create an edge.
6. If $\alpha < \frac{4}{3}$, then two leaves of differnt types of a double star have an incentive to create an edge.

Moreover, the uniqueness in Proposition 4.1(4.1) excludes the parameter range $\frac{6}{7} \leq \alpha \leq \frac{n_B}{n_B+1}$, which can only happen for sufficiently many blue agents. In fact, there the uniqueness ceases to hold, as we show in the next example.

Example 4.2 Consider the ICF-NCG with two agent types. Let $n_B \geq 6$ and $\frac{6}{7} \leq \alpha \leq \frac{n_R}{n_R+1}$. We fix a specific red agent $r^* \in V_R$ and consider the network $G = (V, E)$ with $E = \{vw\colon v, w \in V_R\} \cup \{vr^*\colon v \in V \setminus \{r^*\}\}$, i.e., the red type is fully intra-connected and there is a special agent r^* to which all agents are connected. The structure of this network is depicted in Fig. 2.

If $\frac{6}{7} \leq \alpha \leq \frac{n_B}{n_B+1}$, it is even possible to interchange the roles of the two agent types. Pairwise stability of this network follows by straightforward computations. ◁

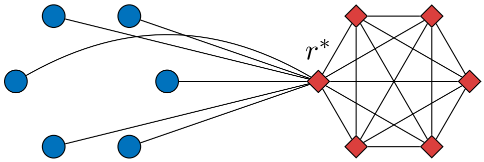

Fig. 2 Pairwise stable network for $\frac{6}{7} \le \alpha \le \frac{n_R}{n_R+1}$ with $n_B = 6$ and $n_R = 6$ blue and red agents, respectively

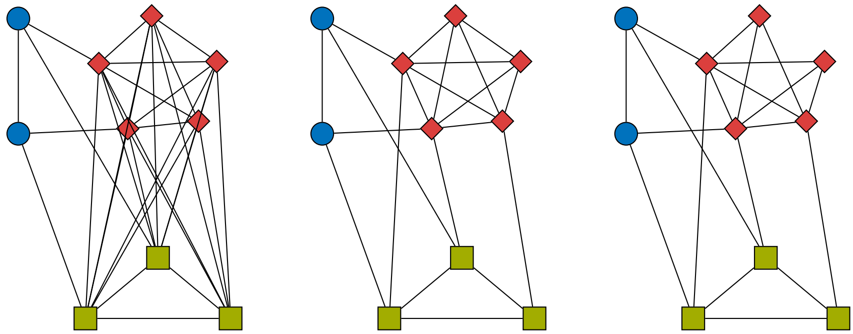

Fig. 3 Illustration of the proof of Proposition 4.3. We consider an ICF-NCG with 3 types containing 2, 3, and 5 agents, respectively. Hence, we consider the parameter range $\frac{n_B}{n_B+1} = \frac{2}{3} \le \alpha < 1$. The pairwise stable networks are dependent on the thresholds $\tau_2 = \frac{3}{4}$ and $\tau = \frac{12}{13}$. We then find the pairwise stable networks for $\frac{n_B}{n_B+1} \le \alpha < \tau_2$ (left), $\tau_2 \le \alpha < \tau$ (middle), and $\tau \le \alpha < 1$ (right)

For the existence of stable networks, we still have to consider the intermediate parameter range $\frac{n_B}{n_B+1} < \alpha < 1$. We fill this gap by identifying two similar types of stable networks for this range.

Proposition 4.3 *In the ICF-NCG, a pairwise stable network exists for every* $\frac{n_B}{n_B+1} \le \alpha < 1$.

Proof Consider an instance of the ICF-NCG and let $\frac{n_B}{n_B+1} \le \alpha < 1$. Assume that we have ordered the types in increasing size, i.e., $\mathcal{T} = \{T_1, \dots, T_k\}$, where $T_1 = B$, $T_k = R$ and $n_{T_1} \le \dots \le n_{T_k}$. Suppose that $V_{T_j} = \{t_j^1, \dots, t_j^{n_{T_j}}\}$. We will define a stable network for α dependent on several thresholds for α. In particular, there is a threshold $\tau = \frac{n_{T_{k-1}}(n_{T_{k-1}}+1)}{n_{T_{k-1}}(n_{T_{k-1}}+1)+1}$. In addition, we consider further threshold values. Let $2 \le j \le k-1$, and define $\tau_j = \frac{n_{T_j}}{n_{T_j}+1}$. Note that $\frac{n_B}{n_B+1} \le \tau_2 \le \tau_3 \le \dots \le \tau_{k-1} < \tau < 1$ as $n_{T_{k-1}}(n_{T_{k-1}}+1) > n_{T_{k-1}}$.

We define the network $G = (V, E)$ with edges given as follows:

- $\{t_j^i, t_j^\ell\} \in E$ for $1 \le j \le k$, $1 \le i < \ell \le \min\{n_{T_j}, n_{T_{k-1}}\}$,
- $\{t_j^i, t_\ell^i\} \in E$ for $1 \le j < \ell \le k$, $1 \le i \le n_{T_j}$,
- $\{t_k^i, t_k^\ell\} \in E$ for $1 \le i \le n_{T_{k-1}}$ and $n_{T_{k-1}} + 1 \le \ell \le n_{T_k}$,
- for each $2 \le j \le k-1$, if $\alpha < \tau_j$, then $\{t_j^i, t_\ell^m\} \in E$ for $j < \ell \le k$, $1 \le i \le n_{T_j}$, and $1 \le m \le n_{T_\ell}$,

- if $\alpha < \tau$, then $\{t_k^i, t_k^\ell\} \in E$ for $n_{T_{k-1}} + 1 \leq i < \ell \leq n_{T_k}$, and
- no further edges are in E.

The different cases for the network G are illustrated in Fig. 3.

We claim that G is pairwise stable. First, we show that no agent can sever an edge. Let $1 \leq j \leq k$, $1 \leq i \leq n_{T_{k-1}}$, and $n_{T_{k-1}} + 1 \leq \ell, m \leq n_{T_k}$.

If agent t_j^i severs an edge to an agent of her type, the distance cost is increased by 1 while the neighborhood cost is decreased by $\alpha\left(1 + \frac{\mathrm{f}_G(u) - \deg_G(u) + 1}{(\mathrm{f}_G(u)+1)\mathrm{f}_G(u)}\right) \leq \alpha < 1$ (which can be computed with the aid of the update formula in Lemma A.1 in the appendix).

Next, we show that no agent can sever a bichromatic edge between an agent in V_{T_j} an agent of type T_p, for $j + 1 \leq p \leq k$. First, for $j = 1$, agent t_1^i cannot sever a bichromatic edge, because then the distance to the adjacent neighbor increases by 2 while the neighborhood cost is decreased by $\alpha\left(1 + \frac{1}{n_{T_1}}\right) < 2\alpha < 2$. For the same reason, the unique neighbor of agent t_1^i in V_{T_p}, for $2 \leq p \leq k$, cannot sever the edge to agent t_1^i. Next, consider the case that $2 \leq j \leq k-1$. If $\alpha < \tau_j$, then severing an edge to a neighbor in V_{T_p}, for $2 \leq p \leq k$, is not profitable because this increases the distance cost by 1 while saving only a neighborhood cost of $\alpha\left(1 + \frac{1}{n_{T_j}}\right) < \tau_j\left(1 + \frac{1}{n_{T_j}}\right) = 1$. The neighbors in V_{T_p} have (weakly) more friends and would save even less neighborhood cost by severing the respective edge. In the case $\alpha \geq \tau_j$, there is again a unique neighbor of type T_p and the case is analogous to the case for agents of type T_1. Thus, we have considered all bichromatic edges. The red agent t_k^ℓ cannot sever the edge towards agent t_k^i, because this improves the neighborhood cost by less than 2 while it increases the distance to both t_k^i and t_1^i by 1 each. Finally, consider the case that $\alpha < \tau$. Then, agent t_k^ℓ cannot sever the edge $t_k^\ell t_k^m$, for $\ell \neq m$. Indeed, this would increase the distance cost by 1 while saving a neighborhood cost of $\alpha\left(1 + \frac{1}{n_{T_k}(n_{T_k}-1)}\right) \leq \alpha\left(1 + \frac{1}{(n_{T_{k-1}}+1)n_{T_{k-1}}}\right)$. Here, we use that such an edge can only exist if $n_{T_k} \geq n_{T_{k-1}} + 1$. Hence, the total increase in cost is at least $1 - \alpha\left(1 + \frac{1}{(n_{T_{k-1}}+1)n_{T_{k-1}}}\right) = 1 - \alpha\frac{n_{T_{k-1}}(n_{T_{k-1}}+1)+1}{(n_{T_{k-1}}+1)n_{T_{k-1}}} > 1 - \tau\frac{n_{T_{k-1}}(n_{T_{k-1}}+1)+1}{(n_{T_{k-1}}+1)n_{T_{k-1}}} = 0$.

Next, we show that it is also not possible to create edges. As a first step, we show that agents cannot create bichromatic edges. Let therefore $1 \leq j < p \leq k$ and let $1 \leq i \leq n_{T_j}$ and $1 \leq \ell \leq n_{T_p}$ with $i \neq j$. Note that the edge $t_j^i t_p^\ell$ is present if $\alpha < \tau_j$ and $j \geq 2$. Hence, we assume that $\alpha \geq \tau_j$ if $j \geq 2$. Then, t_j^i does not benefit from creating the edge $t_j^i t_p^\ell$. Indeed, this decreases her distance cost by exactly 1 while it increases her neighborhood cost by $\alpha\frac{n_{T_j}+1}{n_{T_j}} \geq 1$. Here, we use that $\alpha \geq \frac{n_B}{n_B+1}$ if $j = 1$ and $\alpha \geq \tau_j$ if $j \geq 2$. It remains the case of missing edges between red agents for large edge cost. Assume therefore $\alpha \geq \tau$ and let $n_{T_{k-1}} + 1 \leq i, \ell \in n_{T_k}$. Adding the edge $t_k^i t_k^\ell$ decreases the distance cost for t_k^i by 1 while increasing her neighborhood cost by $\alpha\left(1 + \frac{1}{n_{T_{k-1}}(n_{T_{k-1}}+1)}\right) \geq \tau\frac{n_{T_{k-1}}(n_{T_{k-1}}+1)+1}{(n_{T_{k-1}}+1)n_{T_{k-1}}} = 1$. Hence, creating this edge is not beneficial for agent t_k^i.

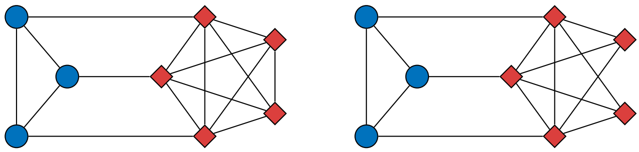

Fig. 4 Pairwise stable networks for $\frac{n_B}{n_B+1} \leq \alpha < \tau$ (left) and $\tau \leq \alpha < 1$ (right)

Overall, we have found stable networks for α in the desired range. $\square$

Combining Propositions 4.1 and 4.3, we immediately obtain the following theorem.

Theorem 4.4 *In the ICF-NCG pairwise stable networks always exist.*

Next, we want to have a closer look at the case of $k = 2$ agent types. An illustration of the stable networks constructed in Proposition 4.3 is provided in Fig. 4. In particular, the construction only depends on the threshold $\tau = \frac{n_B(n_B+1)}{n_B(n_B+1)+1}$. The obtained networks consist of monochromatic cliques (of a size equal to the number of agents of the minority type), connected by a matching of bichromatic edges. All agents of the majority type are connected to all other agents of their type, but whether non-curious agents of the majority type have further edges depends on the threshold.

The construction for more agent types exploits the same idea. The structure of the subnetwork induced by the agents in $V_B \cup V_T$ for any type $T \in \mathcal{T}$ with $T \neq B$ is essentially the same. However, dependent on α, agents from larger communities might have an incentive to maintain further bichromatic edges, captured by the multiple thresholds. Interestingly, when $k = 2$, the stable networks constructed in the proof of Proposition 4.3 give an almost full characterization of stable networks for an edge costs smaller than but close to 1.

Theorem 4.5 *Consider the ICF-NCG with parameter α and $k = 2$ agent types. Let $\frac{n_R}{n_R+1} < \alpha < 1$ and assume that G is pairwise stable. Then, the blue agents are fully intra-connected, the bichromatic edges form a matching of size n_B, and curious red agents are connected to all other red agents.*

Proof Let $\frac{n_R}{n_R+1} < \alpha < 1$ and assume that G is a pairwise stable network in the ICF-NCG with cost parameter α. By Proposition 4.1(4.1), the diameter of G is bounded by 2 and there exists a curious type of agents. By Proposition 4.1(4.1), the curious type of agents forms a clique C and the curious agents of the other type form a clique as well.

Assume towards a contradiction that the bichromatic edges do not form a matching. Assume that there is an agent $x \in C$ that maintains bichromatic edges with two different agents y and z. We will show that agent y has an incentive to sever the edge xy. For this, consider the network $G' = G - xy$. First, the distance cost of agent y decreases by at most 1. Indeed, since all agents of the type of x are still curious in G' and since agent y forms edges to all curious agents of her type, the distance to all these agents is 2 in G' and 1 to agents other than x to which a bichromatic edge exists in G. Also, since agent y is connected to all curious agents of her type, the shortest paths

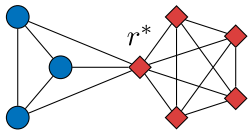

Fig. 5 Pairwise stable network for $\frac{n_B}{n_B+1} \leq \alpha \leq \frac{n_R}{n_R+1}$

to agents of her own type in G cannot use x and still exist after severing the edge xy. Now, the neighborhood cost decreases by

$$\alpha\left(1+\frac{1}{\mathrm{f}_G(y)+1}\right) \geq \alpha\left(1+\frac{1}{n_R}\right) > 1.$$

Hence, no agent in the clique C maintains more than one bichromatic edge.

Next, assume that two agents $w, x \in C$ maintain a bichromatic edge to the same agent y. It is quickly checked that severing the edge xy increases the distance cost by 1 for agent y and her neighborhood cost decreases by more than 1, as above.

Together, the bichromatic edges form a matching. Hence, only a minority type can be a curious type and we can conclude that the blue agents are fully intra-connected and that the matching of bichromatic edges is of size n_B. It remains to show that all curious red agents maintain edges with non-curious red agents. Assume that agent y is a curious red agent forming a bichromatic edge to the blue agent x and that there is no edge to a non-curious red agent z, i.e., the edge yz is not present in G. But then, $\mathrm{d}_G(x, z) \geq 3$, contradicting Proposition 4.1(4.1). □

Example 4.6 The characterization encountered in Theorem 4.5 does not cover the whole parameter range of Proposition 4.3. In fact, it does not hold for $\frac{n_B}{n_B+1} \leq \alpha \leq \frac{n_R}{n_R+1}$, and further pairwise stable networks exist. Assume that $n_R \geq 2$ and let $r^* \in V_R$. Consider the network $G = (V, E)$, where

$$E = \{\{v, w\} : v, w \in V_R\} \cup \{\{v, w\} : v, w \in V_B\} \cup \{\{v, r^*\} : v \in V_B\},$$

i.e., the network is fully intra-connected and there is a special agent r^* to which all blue agents are connected. The structure of this network is depicted in Fig. 5. It is straightforward to check that the network is pairwise stable.

Moreover, recall that Proposition 4.1(4.1) implies full intra-connectivity for $\alpha < \frac{6}{7}$. If this is not the case, i.e., for $\frac{6}{7} \leq \alpha \leq \frac{n_R}{n_R+1}$ (which implies $n_R \geq 6$, i.e., a sufficiently large number of agents), then there exist even pairwise stable networks where most agents of one type have exactly one neighbor (recall Example 4.2). However, it is necessarily the case that the agents of the other type are fully intra-connected. ◁

Until now, we set our focus on the existence of pairwise stable networks. In the remainder of the section, we want to consider the segregation of pairwise stable networks. Recall that our segregation measures have the range [0, 1], where a value of 1 means total segregation. We can apply Theorem 4.5 to obtain very high segregation, asymptotically approaching 1, for $\frac{n_R}{n_R+1} < \alpha < 1$. The corollary follows from a direct computation based on the characterization of Theorem 4.5.

Corollary 4.7 *Consider the ICF-NCG with parameter α and $k = 2$ agent types. Let $\frac{n_R}{n_R+1} < \alpha < 1$ and assume that G is pairwise stable. Then, $GS(G) \geq 1 - \frac{1}{n_R}$ and $LS(G) \geq 1 - \frac{2}{n}$.*

Proof Let $\frac{n_R}{n_R+1} < \alpha < 1$ and assume that $G = (V, E)$ is a pairwise stable network for an ICF-NCG with cost parameter α.

We start with computing the global segregation. By Theorem 4.5, there are n_B bichromatic edges. Additionally,

$$|E| \geq n_B + 2\binom{n_B}{2} + n_B(n_R - n_B) = n_B n_R.$$

Hence,

$$GS = \frac{|E| - n_B}{|E|} \geq 1 - \frac{1}{n_R}.$$

For the local segregation, we need to compute the quantity $\frac{\mathrm{f}_G(u)}{\deg_G(u)}$ for every agent u. We can apply the characterization of Theorem 4.5 again to find

$$\frac{\mathrm{f}_G(u)}{\deg_G(u)} = \begin{cases} \frac{n_B-1}{n_B} & \text{if } u \text{ blue,} \\ \frac{n_R-1}{n_R} & \text{if } u \text{ red and curious,} \\ 1 & \text{otherwise.} \end{cases}$$

Consequently,

$$\begin{aligned} LS(G) &= \frac{1}{n}\left(n_B\frac{n_B-1}{n_B} + n_B\frac{n_R-1}{n_R} + (n_R - n_B)\right) \\ &= \frac{1}{n}\left(n - 1 - \frac{n_B}{n_R}\right) \geq 1 - \frac{2}{n} \ . \end{aligned}$$

□

We know that segregation is low for sufficiently low parameter α, where cliques are (uniquely) pairwise stable. Then, there is a transition at $\alpha = \frac{n_R}{n_R+1}$, where segregation is provably high regardless of further parameters like the distribution of agents into types. Once, the cost parameter increases to $\alpha \geq 1$, the picture becomes less clear. Stars yield very high or very low segregation, i.e., the segregation measures attain values close to the extreme values of 0 and 1.

Proposition 4.8 *Consider the ICF-NCG with parameter $\alpha \geq 1$. Then, for every $n \geq 2$, there exist pairwise stable networks G and G' with n nodes such that $GS(G) = LS(G) = 1$ and $GS(G') = LS(G') = \frac{1}{n-1}$.*

Proof Note that in the considered parameter range, the star $\mathbf{S}_n$ is pairwise stable according to Proposition 4.1(4.1). If there are only agents of one type, then $G = \mathbf{S}_n$ fulfills $GS(G), LS(G) = 1$. On the other hand, if there are two blue agents and $n - 2$ red agents, consider $G' = \mathbf{S}_n$ where the center agent is blue. Then $GS(G')$, $LS(G') = \frac{1}{n-1}$. □

The networks in the previous proposition have the drawback that we need to fix the exact numbers of agents of each type to obtain the desired segregation. By contrast, for $\alpha \geq \frac{4}{3}$, the double star is always highly segregated.

Proposition 4.9 *Consider the ICF-NCG with $\alpha \geq \frac{4}{3}$. Then, the double star $\mathbf{DS}_n$ is a pairwise stable network with $GS(\mathbf{DS}_n) = 1 - \frac{1}{n-1}$ and $LS(\mathbf{DS}_n) \geq 1 - \frac{2}{n}$.*

Proof By Proposition 4.1(4.1), we already know that the double star is pairwise stable. It remains to compute the segregation measures for the double star.

First,

$$GS(\mathbf{DS}_n) = \frac{n_B - 1 + n_R - 1}{n - 1} = 1 - \frac{1}{n-1}.$$

Second,

$$\begin{aligned} LS(\mathbf{DS}_n) &= \frac{1}{n}\left(n_B - 1 + n_R - 1 + \frac{n_B - 1}{n_B} + \frac{n_R - 1}{n_R}\right) \\ &= 1 - \frac{1}{n}\left(\frac{1}{n_B} + \frac{1}{n_R}\right) \geq 1 - \frac{2}{n}. \end{aligned}$$

□

5 Decreasing effort of integration

We consider the DEI-NCG. We start by collecting some results determining simple stable networks for sufficiently small and large values of α, respectively. Recall that we implicitly assume the restriction to two agent types when considering the networks $\mathbf{DS}_n$ and $\mathbf{DSX}_n$. All other statements hold for an arbitrary number of agent types.

Proposition 5.1 *For the DEI-NCG the following holds:*

1. *If $\alpha < \frac{1}{2}$, then $\mathbf{K}_n$ is the unique pairwise stable network.*
2. *If $\alpha < 1$, then every pairwise stable network is fully intra-connected.*
3. *If $\alpha < 1$, then every pairwise stable network G satisfies* $\mathrm{diam}(G) \leq 2$.
4. *The network $\mathbf{K}_n$ is pairwise stable if $\alpha \leq \frac{n-n_R}{n-n_R+1}$.*
5. *If $\alpha \geq 1$, then $\mathbf{S}_n$ and $\mathbf{DS}_n$ are pairwise stable networks.*
6. *If $\alpha \geq \frac{4}{3}$, then $\mathbf{DSX}_n$ is a pairwise stable network.*

As with Proposition 4.1, all bounds for α in Proposition 5.1 are tight:

1. If $\alpha \geq \frac{1}{2}$, then the star with red center and one blue and two red agents is pairwise stable because the red leaf agent has no incentive to create a first bichromatic edge for a cost of 2α. We discuss another interesting class of networks with a variable number of agents in Example 5.2.
2. If $\alpha \geq 1$, then the star (or double star) is pairwise stable and not fully intra-connected.
3. If $\alpha \geq 1$, then the double star is pairwise stable and has a diameter of 3.
4. If $\alpha > \frac{n-n_R}{n-n_R+1}$, the every red agent in a complete network has an incentive to sever a bichromatic edge.

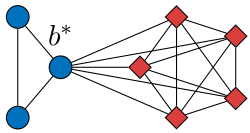

Fig. 6 Pairwise stable network for $\frac{1}{2} \leq \alpha \leq \frac{n_R}{n_R+1}$

5. If $\alpha < 1$, then all leaf agents in stars and double stars have an incentive to create a joint edge.
6. If $\alpha < \frac{4}{3}$, then two leaves of different types have an incentive to create a bichromatic edge. Interestingly, this determines the threshold for the pairwise stability of $\mathbf{DSX}_n$ because every agent in $\mathbf{DSX}_n$ is curious. By contrast, there is only an incentive to create monochromatic edges for $\alpha < 1$.

Proposition 5.1(5.1) and Proposition 5.1(5.1) imply that, for $\alpha < 1$, every pairwise stable network consists of two monochromatic cliques and one type of agents is curious. Still, there are highly segregated pairwise stable networks. Also, the upper bound in Proposition 5.1(5.1) is equal to $\frac{n_B}{n_B+1}$ in the case of two agent types.

An interesting difference between the ICF-NCG and the DEI-NCG is that the agents determining the pairwise stability of the complete network change. In the ICF-NCG, the agents of the minority type have the least friends and therefore the largest gain when severing a bichromatic edge. In contrast, in the DEI-NCG the agents of the majority type have the highest marginal gain from severing a bichromatic edge because they sponsor the smallest number of bichromatic edges. This is reflected in the respective thresholds for the pairwise stability of the complete network in Propositions 4.1 and 5.1. Finally, we provide an example for $\alpha \geq \frac{1}{2}$ but close to $\frac{1}{2}$ with a variable number of agents, in which the complete network is not the unique stable network. We consider the case $k = 2$, but similar examples also exist for more than two types.

Example 5.2 Assume $k = 2$ and $\frac{1}{2} \leq \alpha \leq \frac{n_R}{n_R+1}$. Recall that n_R is the size of the majority type of agents. In particular, this covers the case $\alpha \leq \frac{n_B}{n_B+1} = \frac{n-n_R}{n-n_R+1}$. Assume that $n_B \geq 2$ and let b^* be some fixed blue agent, i.e., an agent from the minority type. Consider the network $G = (V, E)$ with

$$E = \{vw : v, w \in R\} \cup \{vw : v, w \in B\} \cup \{vb^* : v \in R\},$$

i.e., the network is fully intra-connected and there is a special blue agent b^* to which all red agents are connected. There are no further bichromatic edges. For an illustration of the network, see Fig. 6.

We prove pairwise stability of the network. First, no agent can sever a monochromatic edge. Red agents cannot sever the bichromatic edge, because this decreases the distance to every blue agent by 1. The blue agent b^* cannot sever a bichromatic edge, because this increases her cost by $1 - \alpha\frac{n_R+1}{n_R} \geq 0$. Also, further bichromatic edges cannot be added since their cost is more than 1 for a blue agent while decreasing the distance cost only by 1. ◁

In the previous example, it was still possible to simultaneously have full intra-connectivity while there are agents entertaining several bichromatic edges. This is not possible anymore if we further increase α.

Lemma 5.3 *Let $k = 2$ in the DEI-NCG. Consider a fully intra-connected and pairwise stable network G.*

1. *If $\alpha > \frac{n_B}{n_B+1}$, then every red agent in G entertains at most one bichromatic edge.*
2. *If $\alpha > \frac{n_R}{n_R+1}$, then every agent in G entertains at most one bichromatic edge.*

As a consequence, we can even characterize all pairwise stable networks for $\frac{n_R}{n_R+1} < \alpha < 1$ and $k = 2$.

Theorem 5.4 *Let $k = 2$ in the DEI-NCG. Assume that $\frac{n_R}{n_R+1} < \alpha < 1$ and consider a network G. Then, network G is pairwise stable if and only if it is fully intra-connected and its bichromatic edges form a matching covering V_B.*

Proof Clearly, if $k = 2$ and $n_R = 1$, then the unique stable network consists of a neighboring blue and red agent. Hence, the assertion is true. Thus, we may assume that $n_R \geq 2$.

Let $\frac{n_R}{n_R+1} < \alpha < 1$ and assume first that G is a pairwise stable network. By Proposition 5.1(5.1), the network is fully intra-connected. By Lemma 5.3, the bichromatic edges form a matching. Finally, by Proposition 5.1(5.1), one type of the agents must be curious, and therefore the matching covers the minority type of agents.

Conversely, assume that G is a fully intra-connected network such that the bichromatic edges form a matching covering one type of agents. Then, no edge can be severed because monochromatic edges only decrease the neighborhood cost by $\alpha < 1$ while increasing the distance cost by 1. Also, bichromatic edges decrease the neighborhood cost by $2\alpha < 2$ while increasing the distance cost by 2. Finally, it is impossible to create another bichromatic edge. This edge would be the second bichromatic edge incident to its endpoint from the minority type of agents. This agent would only decrease her distance cost by 1 while increasing her neighborhood cost by $\frac{3}{2}\alpha \geq \frac{3}{2}\frac{n_R}{n_R+1} \geq 1$, where we use $n_R \geq 2$ in the last step. $\square$

The second part of the above proof shows that the networks characterized in the theorem are even stable for $\frac{2}{3} \leq \alpha < 1$. Putting together Proposition 5.1, Example 5.2, and Theorem 5.4, we have proven the existence of pairwise stable networks for almost every DEI-NCG if $k = 2$ (except a limit case when $n_B = 1$). By generalizing the encountered networks, we can show the existence of stable networks for an arbitrary number of types in the next theorem. The generalization of the network in Example 5.2 is straightforward, maintaining the property that there exists one specific agent entertaining all bichromatic edges. However, the generalization of the network in Theorem 5.4 is a bit disguised. We define the network by providing an efficient algorithm. This algorithm initially considers a fully intra-connected network and adds edges by having agents create bichromatic edges via specific better responses. In the special case of $k = 2$, this results precisely in the matchings encountered in Theorem 5.4.

Proposition 5.5 *In the DEI-NCG, there exists a pairwise stable network for every $\frac{n-n_R}{n-n_R+1} < \alpha < 1$.*

Proof Suppose that $\mathcal{T} = \{T_1, \dots, T_k\}$ with $n_{T_1} \leq \cdots \leq n_{T_k}$ and, for each $1 \leq j \leq k$, $V_{T_j} = \{t_j^1, \dots, t_j^{n_{T_j}}\}$. We will construct pairwise stable networks for this parameter range.

First, we will generalize the network of Example 5.2 to an arbitrary number of agent types. Let $j^* = \min(\{1 \leq j \leq k : n_{T_j} \geq 2\} \cup \{k\})$, i.e., the index of the smallest type of size at least 2 or the index of the last type if there exists exactly one agent per type. Consider the network $G = (V, E)$ with edge set defined by

- $\{t_j^i, t_j^l\} \in E$ for $1 \leq j \leq k$, $1 \leq i < l \leq n_{T_j}$,
- $\{t_{j^*}^1, t_j^i\} \in E$ for $1 \leq j \leq k$, $j \neq j^*$, $1 \leq i \leq n_{T_j}$, and
- no further edges are in E.

We now provide conditions, under which the network G is pairwise stable.

Lemma 5.6 *The network G is pairwise stable if*

(i) $j^* = k$ *and* $\frac{2}{3} \leq \alpha \leq 1$,
(ii) $k = 2$, $j^* = k$, $n_{T_k} \geq 2$ *and* $\frac{1}{2} \leq \alpha \leq 1$, *or*
(iii) $\frac{2}{3} \leq \alpha \leq \frac{n - n_{T_{j^*}}}{n - n_{T_{j^*}} + 1}$.

Proof *(i)* Assume that $j^* = k$ and $\frac{2}{3} \leq \alpha \leq 1$. Then, no monochromatic edge can be severed because of $\alpha \leq 1$. Since $j^* = k$, bichromatic edges cannot be severed due to connectivity. Also, and creating an edge costs $\frac{3}{2}\alpha \geq 1$ for an agent of type different to k while it decreases her distance cost by exactly 1.

(ii) Next, consider the case that $k = 2$, $j^* = k$, $n_{T_k} \geq 2$ and $\frac{1}{2} \leq \alpha \leq 1$. Then, again, no monochromatic edge can be severed because of $\alpha \leq 1$. The unique bichromatic edge cannot be severed as this would disconnect the network. Also, adding another bichromatic edge must include a non-curious red agent. This agent would increase her neighborhood cost by $2\alpha \geq 1$ while only decreasing her distance cost by 1.

(iii) Now, assume that $\frac{2}{3} \leq \alpha \leq \frac{n - n_{T_{j^*}}}{n - n_{T_{j^*}} + 1}$. Again, monochromatic edges cannot be severed as $\alpha < 1$. Further, bichromatic edges incident to an agent t_j^1 for $1 \leq j \leq j^* - 1$ cannot be severed as this would disconnect the network. Next, agent $t_{j^*}^1$ cannot sever another bichromatic edge, because this would increase her cost by $1 - \alpha \frac{n - n_{T_{j^*}} + 1}{n - n_{T_{j^*}}} \geq 0$. Also, for $j^* < j \leq k$ and $1 \leq i \leq n_{T_j}$, agent t_j^i cannot sever $\{t_{j^*}^1, t_j^i\}$, because this increases the distance to at least $n_{T_{j^*}} \geq 2$ agents (in T_{j^*}) by 1 while decreasing the neighborhood cost by 2.
It remains to consider the creation of edges. Every agent in $V \setminus V_{T_{j^*}}$ entertains exactly one bichromatic edge. Creating a second bichromatic edge costs $\frac{3}{2}\alpha \geq 1$ while it decreases the distance cost by exactly 1. Together, the network is pairwise stable. □

Second, we generalize the network from Theorem 5.4. To this end, we design an algorithm that constructs pairwise stable networks. In the special case of two agent types, it yields the networks encountered in Theorem 5.4. Note that this must specifically hold for the parameter range where the uniqueness of the theorem applies.

Algorithm 1: Determination of Edge Set for Network G'

Input: Set of agents V.
Output: Edge set E'.

$E' \leftarrow \{\{t_j^i, t_j^l\} \colon 1 \leq j \leq k, 1 \leq i < l \leq n_{T_j}\}$;
while *there exist* $u, v \in V$ *with* $\mathrm{d}_{(V,E')}(u, v) \geq 3$ **do**
  $E' \leftarrow E' \cup \{uv\}$;
return E'

Therefore, consider the network $G' = (V, E')$ where the edge set E' is computed according to Algorithm 1.

The algorithm starts with the fully intra-connected network without any bichromatic edges. Then, bichromatic edges are added whenever the distance between two agents is too large. Clearly, this algorithm has to terminate by returning E' after at most $\binom{n}{2}$ executions of the while loop.

Lemma 5.7 *The following properties are valid.*

- *The diameter of G' satisfies* $\mathrm{diam}(G') \leq 2$.
- *Every triangle*[2] *in G' consists of monochromatic edges only.*
- *Every agent is incident to at most $k-1$ bichromatic edges in G'.*

Proof The first property is immediate from the definition of the while loop. We prove the second property by contradiction. Assume that network G' contains a triangle containing agents u, v, and w of at least two different types. Assume that uv is the last edge that was added by the algorithm. At this point uw and vw were already present, so $\mathrm{d}_{(V,E')}(u, v) \leq 2$, which is a contradiction to adding uv.

For the third property, we observe that every agent can add at most one bichromatic edge to an agent of each fixed type. Once this edge is added, the distance to all agents of this type is at most 2 due to the intra-connectivity of the network. As there are at most $k-1$ other types, the assertion follows. $\square$

It is easy to deduce the pairwise stability of network G'.

Lemma 5.8 *The network G' is pairwise stable for* $\frac{k}{k+1} \leq \alpha \leq 1$.

Proof As in previous networks, monochromatic edges cannot be severed because of $\alpha \leq 1$. Now, consider a bichromatic edge uv. Then, $\mathrm{d}_{G-uv}(u, v) \geq 3$. Indeed, if $\mathrm{d}_{G-uv}(u, v) = 2$, then edge uv is part of a triangle, contradicting the second statement in Lemma 5.7. Hence, severing the edge uv increases the distance cost for uv by at least 2 while saving a neighborhood cost of at most 2.

It remains to consider the creation of edges. As the network is fully intra-connected, only bichromatic edges can be created. Hence, consider the creation of a bichromatic edge uv. Its creation decreases the distance cost for agent u by exactly 1. Indeed, as $\mathrm{diam}(G') \leq 2$, the distance to agent v is decreased by exactly 1, and the distance to other agents is no shorter. On the other hand, as agent u is incident to at most $k-1$

[2] A triangle is defined as a complete subnetwork induced by three vertices.

bichromatic edges, the creation of uv costs at least $\alpha\left(1+\frac{1}{k}\right) \geq 1$. Hence, the total cost for agent u cannot have decreased. □

To conclude the proof, we want to argue that we can cover the whole parameter range of α. First, we cover the range until $\alpha = \frac{2}{3}$. According to Proposition 5.1(5.1), this is covered by $\mathbf{K}_n$ if $n - n_{T_k} \geq 2$. In particular, this is the case if $k \geq 3$ or $n_{T_1} \geq 2$. If $k = 2$ and $n_{T_1} = 1$, we can apply case *(ii)* of Lemma 5.6 if $n_{T_k} \geq 2$. If $k = 2$ and $n_{T_k} = 1$, then the network consisting of two agents of different types, connected by an edge, is pairwise stable.

Finally, consider the parameter range $\frac{2}{3} \leq \alpha \leq 1$. If $j^* = k$, then case *(i)* of Lemma 5.6 applies. Otherwise, $j^* < k$, and therefore $n - n_{T_{j^*}} \geq k$. This implies that $\frac{n-n_{T_{j^*}}}{n-n_{T_{j^*}}+1} \geq \frac{k}{k+1}$, and the parameter range is covered by case *(iii)* of Lemma 5.6 and Lemma 5.8. □

Combining Propositions 5.1 and 5.5, we also obtain the existence of stable networks in the DEI-NCG.

Theorem 5.9 *In the DEI-NCG pairwise stable networks always exist.*

Finally, we want to consider the segregation of pairwise stable networks in the DEI-NCG. Clearly, segregation only depends on the networks, not on the type of NCG. Hence, based on our investigation of ICF-NCGs, we already know that that cliques provide low segregation for small α and stars provide high or low segregation for higher α, but require a specific distribution of agents into types. Independently of this distribution, double stars provide high segregation and it is clear that $GS(\mathbf{DSX}_n) = LS(\mathbf{DSX}_n) = 0$. Finally, for an intermediate range of α, high segregation, approaching the extreme value of 1 for both segregation measures, is inevitable.

Corollary 5.10 *Let $k = 2$ and $\frac{n_R}{n_R+1} < \alpha < 1$. Then, every pairwise stable network G in the DEI-NCG with parameter α satisfies $GS(G) \geq 1 - \frac{2}{n}$ and $LS(G) \geq 1 - \frac{2}{n}$.*

Proof Consider a network $G = (V, E)$ satisfying the assumptions of the corollary. We start with the global segregation measure. According to Theorem 5.4, there are n_B bichromatic edges and a total of

$$n_B + \frac{n_B(n_B-1)}{2} + \frac{n_R(n_R-1)}{2} \geq n_B + \frac{n_B(n_B-1)}{2} + \frac{n_B(n_R-1)}{2} = \frac{n_B n}{2}$$

edges. Hence,

$$GS(G) = \frac{|E| - n_B}{|E|} = 1 - \frac{n_B}{|E|} \geq 1 - \frac{n_B}{n_B n/2} = 1 - \frac{2}{n}.$$

Using the characterization in Theorem 5.4 once again, the computation of the local segregation measure is identical as in the proof of Corollary 4.7. □

6 Simulations

While our theoretical results indicate a clear structure of stable networks for $\alpha \leq 1$, there is a broad range of possibilities for larger α. Therefore, we support our theoretical findings for $\alpha > 1$ by a detailed experimental analysis. To this end, we simulate best response dynamics. These are a simple dynamic process based on distributed and strategic edge creation and deletion over time, incentivized by agents optimizing their individual cost functions.

The dynamics start with sparse initial networks (spanning trees or grids) and distribute agents of two equally-sized types such that the segregation of the initial network is either very low or very high. In each step, a single agent is activated uniformly at random and can either create an edge (if the other endpoint of the edge also profits from this) or delete an edge. Recall that among the available creations and deletions, the best responses are the options with the highest cost decrease for the activated agent.

We start with highlighting that despite the existence of stable networks proven in Sects. 4 and 5, such dynamics are not guaranteed to converge. This means that instances exist where an infinite sequence of best responses is possible, i.e., best response dynamics might never converge to a pairwise stable network. This can be shown by providing a *best response cycle*, i.e., a cyclic sequence of networks where neighboring networks in the sequence only differ by a best response of some agent. For the classical bilateral network creation game by Corbo and Parkes (2005), a similar observation was already made by Kawald and Lenzner (2013) for a variant where additionally bilateral edge-swaps are allowed. As this construction cannot be directly transferred to our setting, we provide exemplary best response cycles for both the ICF-NCG and the DEI-NCG with two types. This indicates that best response dynamics have no convergence guarantee.

Example 6.1 Consider the sequence of networks depicted in Fig. 7. The instance encompasses 10 agents from two types. The network G_1 evolves from G_0 by having agent a sever the edge ab. Then, to obtain G_2, edge gb is created. Further, network G_3 evolves by deleting the edge fa from network G_2. Finally, if the edge fg is created in network G_3, we obtain a network isomorphic to G_0 where a and g have swapped their roles.

It can be shown that each step is a best response for the agent highlighted in green, where we assume that $9 < \alpha < 10$ for the DEI-NCG and $\frac{108}{13} < \alpha < \frac{60}{7}$ for the ICF-NCG. $\triangleleft$

Despite not having a convergence guarantee, the dynamics for all our randomly generated instances always converged. However, as shown in Fig. 8, this can take a long time. To circumvent long convergence times, we only demand 1.01-approximate pairwise stability in our experiments, as defined below. Moreover, the long convergence times motivate us to additionally study an *add-only* variant of the model, where agents can only create edges—this trivially guarantees convergence and a speed-up in the involved computations. In the case of add-only dynamics, we speak of *best additions* instead of best responses. Such add-only dynamics are particularly natural when modeling social networks, as confirmed by the observation that many real-world social networks get denser over time (Leskovec et al. 2005). In both variants of the dynamics,

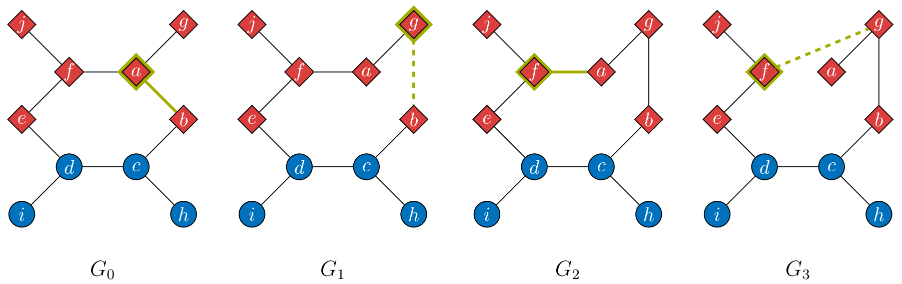

Fig. 7 Sequence of networks with two agent types. Severed and created edges are indicated in green, where the latter are dashed. We obtain a best response cycle for the DEI-NCG when $9 < \alpha < 10$ and for the ICF-NCG when $\frac{108}{13} < \alpha < \frac{60}{7}$

if no improvement is possible, then the active agent's strategy remains unchanged and the agent is marked as *content*. The convergence process is iterated until eventually all agents are found to be content and, hence, the network is (approximately) pairwise stable. Finally, we measure the segregation strength in the obtained (approximately) stable networks.

6.1 Detailed experimental setup

For our simulations, we first generated an initial network and an initial agent-type distribution. Then agents are activated at random and compute a best response. Recall that an edge addition requires the consent of the other endpoint of the edge, i.e., this agent also has to strictly decrease her cost by adding the edge. This sequential activation process is then run until no agent has a better response and a pairwise stable network is found. We now discuss the details of this setup.

General setup Our experiments regarding the obtained segregation strength considered 1000 agents partitioned into two types with 500 agents each. For each run we chose

- a random spanning tree or a grid as initial network,
- an integrated or perfectly segregated initial agent distribution,
- if best responses or best additions are performed,
- if the segregation strength is measured via the local segregation measure *LS* or via the global segregation measure *GS*, and
- the value of α in 19 steps between 1 and 255.

In total this yielded $2^4 \cdot 19 = 304$ different configurations and for every configuration we simulated 50 runs, yielding a total number of 15200 considered networks.

We also ran additional experiments to investigate the convergence dynamics in more detail. For these simulations we used a similar setup but only 500 agents partitioned into two types of 250 agents each. We considered $\alpha \in \{1, 2, 3, 4, 5, 10, 15\}$ and for each configuration we sampled 20 runs, yielding a total of 1120 experiments, where we tracked the exact convergence trajectory, i.e., which agent performed which move, discriminating between the addition of monochromatic and of bichromatic edges and the deletion of edges.

Generating the initial networks We considered random spanning trees and grids as initial networks. For our experiments on the segregation strength we used grids of size 20×50, for evaluating the dynamics we used grids of size 20×25. Moreover, we sampled the random spanning trees by the following scheme: starting from a single node, add nodes one by one, and each new arriving node is attached by an edge to one of the existing nodes chosen uniformly at random.

Generating the initial agent distribution We focus on two cases: perfectly segregated and integrated initial states. An integrated initial state is sampled by a uniformly random type assignment to each node. To generate a perfectly segregated spanning tree, we generate two one-type spanning trees of 500 nodes (or 250 nodes) and join them by connecting the initial nodes of each tree. A perfectly segregated grid is sampled by assigning one type to all 500 nodes (250 nodes) in the first ten rows and the other type to the rest.

Random activation of the agents We start with marking all nodes as potentially discontent, i.e., as willing to improve. In each step of the algorithm, one agent is chosen from the set of the potentially discontent nodes uniformly at random. This active agent is searching for a best response. If no better response is possible, the agent is marked as content. If the agent has a better response, the new strategy is applied to the network, and all agents move back to being potentially discontent to be ready to become activated again. The algorithm stops when the last agent is marked as content.

Convergence criteria Fig. 8 shows a representative timeline of the local segregation of the obtained networks in each step of the best response dynamics starting from a random tree with a random initial color distribution. We observe that the segregation value quickly reaches a high value and remains in the interval [0.8, 1] until the end of the execution of the dynamics. It illustrates the need for relaxation of the solution concept to avoid long calculations. Therefore, our experimental study of the best response dynamics uses 1.01-approximate pairwise stable states as solution concept. We say that a network is a 1.01-*approximate pairwise stable* if no agent can improve her cost by more than a factor of 1.01. The approximation factor is chosen empirically to minimize the convergence time and the approximation gap.

Note that for the add-only dynamics, the process naturally stops at the latest when a complete network is reached. Hence, the computation time is rather low compared to the best response dynamics and we could consider exact pairwise stable networks. However, as we will see, in some cases this leads to trajectories where many edge additions happen that only yield a negligible cost decrease, which then leads to a rather high number of edge additions compared to focusing on 1.01-approximate best responses.

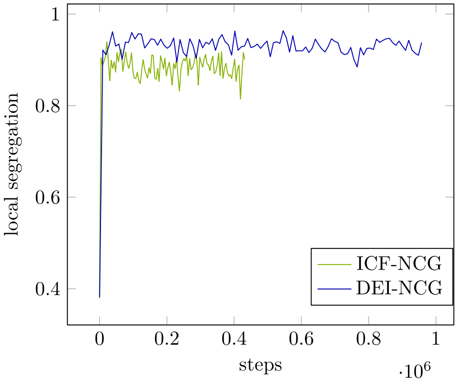

Fig. 8 Trajectory of the local segregation of a network obtained by the best response dynamics for $n = 50$, $\alpha = 15$ starting from a tree with random color distribution in the ICF-NCG and DEI-NCG. The x-axis displays the number of steps taken in the best response dynamics

Visualization of our results In our experiments regarding the qualitative behavior of the dynamics we display the number of discontent agents over time or we show the number of performed moves per agent, sorted decreasingly.

The results of our experiments on the segregation strength are visualized via box-and-whiskers plots of the local and global segregation for the networks obtained by the best response dynamics for $n = 1000$ over 50 runs. Lower and upper whiskers are the minimal and maximal local segregation values over 50 runs of the algorithm. The middle lines are the median values, while the bottom and top of the boxes represent the first and the third quartiles.

We refer the reader to Appendix B for a detailed visualization of the results of our simulations. In the remainder of this section, we restrict attention to representative visualizations for the ICF-NCG. The behavior for the DEI-NCG is very similar, and important differences are already discussed here.

6.2 Qualitative observations regarding the dynamics

Before our main experiment that concerns segregation in the obtained networks, we want to get a feeling about the general behavior of the dynamics. For this, we ran the smaller experiment with 500 agents for which we created more extensive data.

Number of discontent agents First, we want to get an understanding of how far we are from pairwise stability during the execution of the dynamics. Figure 9 displays exemplary time lines for the number of discontent agents, i.e., agents that would want to sever an edge or, together with another agent, create an edge. Initially, essentially all agents are discontent, while most of the agents stay discontent for about half of the dynamics. Then, the number of discontent agents starts to decrease more and more rapidly. Consistent with the intuition that it is less beneficial to create edges when they are more costly, the number of deviations decreases for an increasing cost parameter.

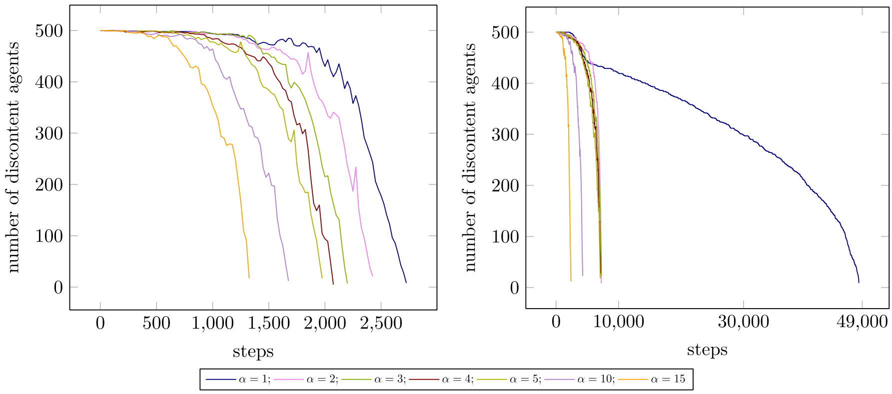

Fig. 9 Time line of the number of discontent agents averaged over 20 runs when starting from a random grid in the ICF-NCG with $n = 500$ agents partitioned into two types of 250 agents each. Left: 1.01-approximate dynamics. Right: Add-only dynamics

Even though the displayed figure assumes a random grid as the initial network, the behavior is very similar for all other considered initial configurations.

Interestingly, while the add-only variant is faster than the exact best response dynamics, it is significantly slower than the approximate dynamics. In fact, when analyzing the approximate dynamics, we observed that it never deletes an edge. This can be explained by the fact that our initial networks are very sparse. Hence, in the simulations, the approximate dynamics is rather a refinement of the add-only than of the exact improvement dynamics. The long run-time of the add-only dynamics is due to a large number of edge additions that each yield only minuscule cost improvements.

Another interesting observation is that the add-only dynamics take significantly longer for $\alpha = 1$. Interestingly, the same behavior cannot be observed for the DEI-NCG, where the case $\alpha = 1$ is not an outlier (see Figs. 17, 18, 19, 20 in the appendix). This observation might rely on a qualitative difference between the two games. In the ICF-NCG, creating a *monochromatic* edge affects the cost for all edges created so far. Hence, creating such an edge can reduce both the neighborhood and the distance cost. In contrast, in the DEI-NCG, creating any edge only decreases the neighborhood cost. This causes a stronger bias towards creating monochromatic edges in the ICF-NCG compared to the DEI-NCG.

Once we reach a network containing a moderately large number of edges, buying further edges hugely depends on the edge cost. If α is around 1, then creating a monochromatic edge in the ICF-NCG costs less than 1 for any agent having a small number of enemies (see also the cost update formulas in Lemma A.1), whereas the cost for creating any edge is at least 1 in the DEI-NCG. Because the distance cost is decreased by at least 1 whenever an edge is created, this causes monochromatic edges to be build inevitably in the ICF-NCG. Hence, agents having a few enemies become content only when they have created all possible monochromatic edges (with other such agents). This also explains that, after initially making some agents content quickly (the ones with only one or no enemies), the slope of making agents content flattens again. Due to the uniformly random activation of agents, it now takes much

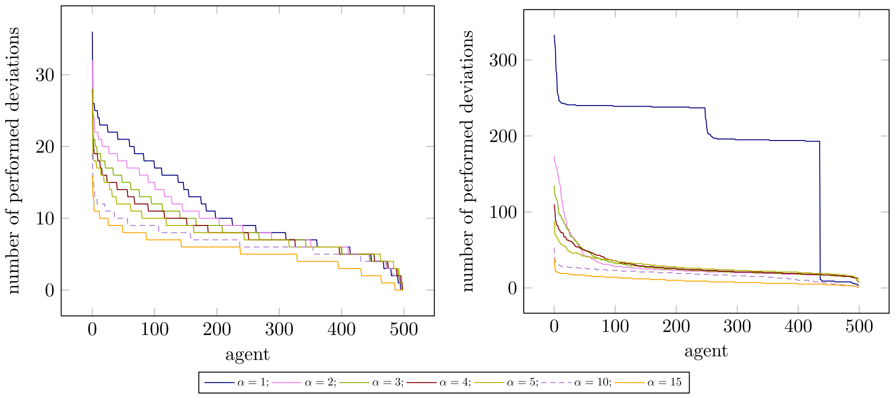

Fig. 10 Distribution of the number of deviations averaged over 20 runs when starting from a random grid in the ICF-NCG with $n = 500$ agents partitioned into two types of 250 agents each. Agents are relabeled and sorted decreasingly by their number of performed deviations, e.g., agent 1 has performed the most deviations while agent 500 had the least deviations. Left: 1.01-approximate dynamics. Right: Add-only dynamics

longer to make agents content. In contrast, once α is larger, e.g., for $\alpha \geq 2$, there is no necessity to build monochromatic edges, even for agents with a moderate number of enemies and we do not observe this behavior.

Distribution of agent deviations Next, we want to explore the distributions of active agents performing a deviation. Figure 10 displays this for the ICF-NCG starting from a random grid. We see that it is quite skewed with some agents performing significantly more deviations than others. Again, the picture differs for the add-only dynamics with $\alpha = 1$. There are two large plateaus of agents performing approximately the same number of deviations. When analyzing the specific edges created by these agents, it becomes apparent that they essentially create the same set of edges. More specifically, the two plateaus correspond to clusters of the two types of agents that form the same monochromatic edges. A close look reveals that the plateaus encompass about the same number of agents as the number of edges that these agents form. The first, larger cluster, contains most of the agents of one type whereas the other cluster only contains about 200 agents (i.e., 80 % of the other type). This visualizes the creation of monochromatic edges in the ICF-NCG as observed when discussing the number of discontent agents over time above. Again, this behavior can be observed independent of the starting configuration (in particular, regardless of whether it is integrated or segregated) while it does not happen for the DEI-NCG or for larger cost parameters.

6.3 Analysis of the segregation strength

We now discuss the findings of our main experiment aiming at understanding the segregation of networks. Figure 11 and Fig. 12 consider the dynamics for the ICF-NCG under the 1.01-approximate version of the dynamics and for its add-only version, respectively.

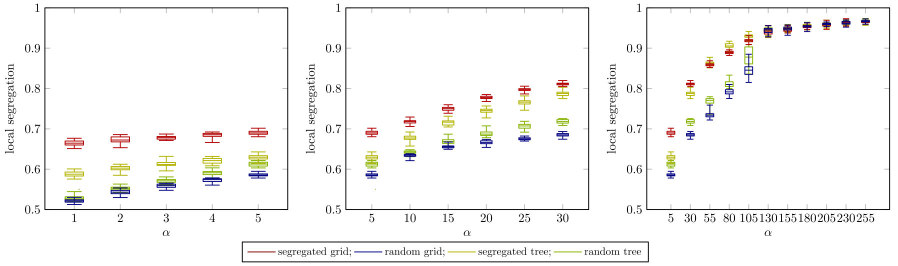

Fig. 11 Local segregation of 1.01-approximate stable networks in the ICF-NCG obtained by iterative best responses for $n = 1000$ over 50 runs starting on a random spanning tree or a grid as initial graph and having a uniformly random or already strongly segregated initial distribution of the agent types. Note that a uniformly random initial type distribution yields very low segregation. E.g., “segregated tree” is the case where the initial graph is a random spanning tree and the initial type distribution of the agents is strongly segregated

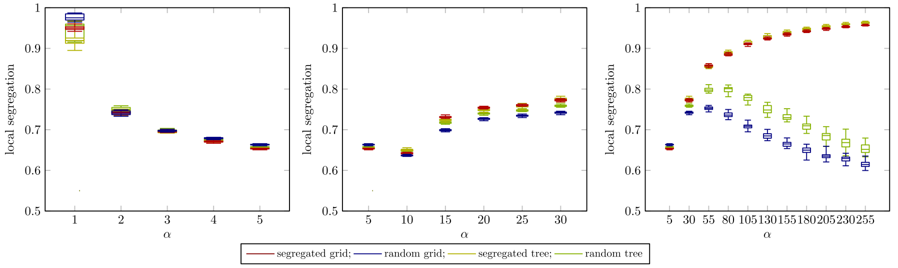

Fig. 12 Local segregation of pairwise stable networks in the ICF-NCG obtained by iterative best additions for $n = 1000$ over 50 runs starting on a random spanning tree or a grid as initial graph and having a uniformly random or already strongly segregated initial distribution of the agent types

Again, we see a qualitative difference between the ICF-NCG and DEI-NCG for the add-only variant for small α. For $\alpha = 1$, the segregation in the ICF-NCG is almost 1, and decreasing until a value of $\alpha = 10$ (compare Fig. 12 with Figs. 30 and 31 in the appendix). This is due to the bias towards creating monochromatic edges as observed in Sect. 6.2. While this effect is very strong for $\alpha = 1$, it is still present for moderately small α.

Once we reach a value of $\alpha = 10$, the results for the ICF-NCG and DEI-NCG are qualitatively the same. From this point on, the games generally behave similarly. The experiments indicate that the segregation strength is proportional to α, with low segregation for low α. For the DEI-NCG, this is true for the whole range of α, despite the theoretical necessity of high segregation for α close to 1.[3] The qualitative behavior of the segregation strength does not depend on the segregation measure, i.e., we obtain similar figures for the global segregation measure, see the appendix for details.

Moreover, except for high α, the initial agent distribution significantly influences the segregation strength, with higher observed segregation strength when starting on

[3] The provably high segregation for $\alpha < 1$ close to 1 is not contradicting the experimental results for the DEI-NCG. Just before we reach a cost parameter of α, we hit the sweet spot where buying monochromatic edges is desirable while buying bichromatic edges is not.

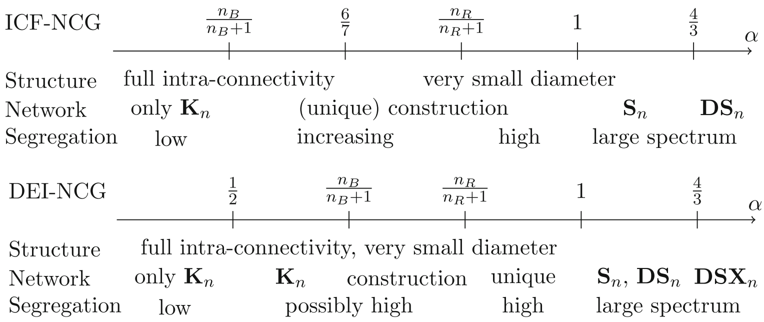

Fig. 13 Overview of our theoretical results. We display structural properties of pairwise stable networks, explicit pairwise stable networks and findings about the segregation of pairwise stable networks. The two models behave surprisingly similar

already segregated initial states. The structure of the initial network seems less important for the qualitative behavior. Interestingly, the add-only version displays a similar behavior for low α, but the behavior changes drastically for moderately high α. Instead of high segregation, we find that initially integrated networks converge to only moderately segregated states, whereas this is not true for initially segregated networks, suggesting an escape route from segregation.

7 Conclusion

We have investigated two network creation games that consider heterogeneous edge creation of agents acting according to homophily. Our main goal was to analyze segregation within reasonable networks measured by pairwise stability. Our theoretical results are summarized in Fig. 13.

Clearly, stable networks are highly integrated for a very small edge cost, when agents can afford to buy all available edges. Once our cost parameter reaches the sweet spot where agents need to balance neighborhood and distance cost, there is provably high segregation, following from characterizations of stable networks. For slightly higher edge cost, our theoretical results cannot give a clear tendency of the segregation strength. In principle, both low and high segregation can be achieved by stable networks. Therefore, we performed an average-case analysis by running extensive simulation experiments. These experiments provide general tendencies about segregation contrasting the large theoretical spectrum for $\alpha \geq 1$. Most importantly, except for a high cost parameter α, we consistently observe lower obtained segregation under integrated initial conditions. While this difference seems to vanish for high α when edges can also be deleted, in the add-only setting we even see a drastically increasing difference in the obtained segregation strength for high edge price α. This yields a possible escape route from segregation: by a high initial investment in integrated initial states and by incentivizing agents to keep their established connections, permanent integration might be reached.

Even though our two game models feature two seemingly orthogonal perspectives based on a direct and an indirect consideration of homophily, their qualitative behavior is surprisingly similar. This holds both for our theoretical analysis and our simulations except for a parameter range close to $\alpha = 1$. There, the creation of *monochromatic* edges can cause a decrease of the neighborhood cost for already present edges in addition to a decrease in distance cost for the cost function in the ICF-NCG. By contrast, the creation of edges in the DEI-NCG only decreases the neighborhood cost. We observe a creation of a large number of monochromatic edges in the ICF-NCG, leading to highly segregated networks. In summary, we find a fairly robust segregation strength across both models, except for a small sensitive range for the cost parameter.

There are several exciting avenues for future work. First, network creation games can be considered for a broad range of cost functions and studying models with different cost functions might lead to new discoveries. A natural idea would, for instance, be to consider the average or maximum distance instead of the sum of distances as a distance cost. However, considering the average distance merely scales the distance cost by a factor of n, which does lead to an equivalent model when scaling α accordingly. By contrast, considering the maximum distance leads to a qualitatively different network creation game. Demaine et al. (2012) show that this change yields a different price of anarchy, i.e., the worst-case ratio of the total cost in stable and cost-minimal networks, in the single-type model by Corbo and Parkes (2005). It would be interesting to get a better understanding of the nature of pairwise stable networks in our settings and even in the base model by Corbo and Parkes (2005).

A second direction for further research is a deeper investigation of best (or better) response dynamics. While best response cycles are possible, it would be interesting to see, whether dynamics possibly (or necessarily) converge under certain conditions. Moreover, our simulations assume sparse initial networks. This makes sense in our context as our focus is on understanding the process of creating a network. Still, this has the effect that edge deletions may be negligible for the execution of dynamics (as in our approximate dynamics). Moreover, the obtained networks might depend heavily on the first created edges, which may affect the obtained segregation. To complement our simulations, one could perform an analysis where initial networks are parameterized by the initial average degree, or where initial networks follow the structure of networks observed in real-world social networks.

Appendix

A Missing proofs

In this appendix, we provide missing proofs.

A.1 Increasing comfort among friends

For the analysis of pairwise stability, we frequently have to compute an agent's cost change after creating or severing one edge. To clarify the calculations, we gather the respective formulae in a lemma.

Lemma A.1 *Consider a network $G = (V, E)$ and an agent $u \in V$ in the ICF-NCG. Consider an agent $v \in V_{\mathcal{T}(u)}$ of the same type and an agent $w \in V \setminus V_{\mathcal{T}(u)}$ of a different type. Then, the following statements hold:*

1. $a_{G+uv}(u) - a_G(u) = \alpha\left(1 + \frac{\mathrm{f}_G(u)-\deg_G(u)+1}{(\mathrm{f}_G(u)+1)(\mathrm{f}_G(u)+2)}\right)$ *if $uv \notin E$ (creation of a monochromatic edge),*
2. $a_{G-uv}(u) - a_G(u) = -\alpha\left(1 + \frac{\mathrm{f}_G(u)-\deg_G(u)+1}{(\mathrm{f}_G(u)+1)\mathrm{f}_G(u)}\right)$ *if $uv \in E$ (deletion of a monochromatic edge),*
3. $a_{G+uw}(u) - a_G(u) = \alpha\left(1 + \frac{1}{\mathrm{f}_G(u)+1}\right)$ *if $uw \notin E$ (creation of a bichromatic edge), and*
4. $a_{G-uw}(u) - a_G(u) = -\alpha\left(1 + \frac{1}{\mathrm{f}_G(u)+1}\right)$ *if $uw \in E$ (deletion of a bichromatic edge).*

Proof We perform the calculations for each case accordingly. Let G' be the network after the respective edge creation or deletion.

1. Creation of a monochromatic edge: $a_{G'}(u) - a_G(u) = (\deg_G(u) + 1) \cdot \alpha\left(1 + \frac{1}{\mathrm{f}_G(u)+2}\right) - \deg_G(u) \cdot \alpha\left(1 + \frac{1}{\mathrm{f}_G(u)+1}\right) = \alpha\left(1 + \frac{\mathrm{f}_G(u)-\deg_G(u)+1}{(\mathrm{f}_G(u)+1)(\mathrm{f}_G(u)+2)}\right)$.
2. Deletion of a monochromatic edge: $a_{G'}(u) - a_G(u) = (\deg_G(u) - 1) \cdot \alpha\left(1 + \frac{1}{\mathrm{f}_G(u)}\right) - \deg_G(u) \cdot \alpha\left(1 + \frac{1}{\mathrm{f}_G(u)+1}\right) = -\alpha\left(1 + \frac{\mathrm{f}_G(u)-\deg_G(u)+1}{(\mathrm{f}_G(u)+1)\mathrm{f}_G(u)}\right)$.
3. Creation of a bichromatic edge: $a_{G'}(u)-a_G(u) = (\deg_G(u)+1)\cdot\alpha\left(1 + \frac{1}{\mathrm{f}_G(u)+1}\right) - \deg_G(u) \cdot \alpha\left(1 + \frac{1}{\mathrm{f}_G(u)+1}\right) = \alpha\left(1 + \frac{1}{\mathrm{f}_G(u)+1}\right)$.
4. Deletion of a bichromatic edge: $a_{G'}(u)-a_G(u) = (\deg_G(u)-1)\cdot\alpha\left(1 + \frac{1}{\mathrm{f}_G(u)+1}\right) - \deg_G(u) \cdot \alpha\left(1 + \frac{1}{\mathrm{f}_G(u)+1}\right) = -\alpha\left(1 + \frac{1}{\mathrm{f}_G(u)+1}\right)$. □

Next, we provide proofs for the collected statements about ICF-NCGs concerning structural properties of pairwise stable networks and simple pairwise stable networks.

Proposition 4.1 *For the ICF-NCG the following hold:*

1. *If $\alpha < \frac{6}{7}$, then every pairwise stable network is fully intra-connected.*

2. *If* $\alpha < \frac{4}{3}$*, then* $\mathrm{diam}(G) \leq 2$ *for every pairwise stable network* G*. In particular, network* G *contains a curious type.*
3. *Let* $\alpha < 1$*, let* G *be a pairwise stable network, and let* $C \subseteq V$ *such that every agent in* C *is curious and* $C \subseteq V_T$*, for some type* $T \in \mathcal{T}$*. Then, network* $G[C]$ *is a clique. In particular, every curious type of agents is fully intra-connected.*
4. *If* $\alpha \leq \frac{n_B}{n_B+1}$*, then the complete network* $\mathbf{K}_n$ *is pairwise stable. Moreover for* $\alpha < \min\{\frac{6}{7}, \frac{n_B}{n_B+1}\}$*, the network* $\mathbf{K}_n$ *is the unique pairwise stable network.*
5. *If* $\alpha \geq 1$*, then the star* $\mathbf{S}_n$ *is pairwise stable.*
6. *If* $\alpha \geq \frac{4}{3}$*, then the double star* $\mathbf{DS}_n$ *is pairwise stable.*

Proof We prove the statements one after another.

1. Let $\alpha < \frac{6}{7}$. Assume that a network $G = (V, E)$ is given that is not fully intra-connected. Let $u, v \in V$ be agents of the same type with $uv \notin E$. Define $G' = G + uv$. We will show that $c_{G'}(u) - c_G(u) < 0$ (the computation for v is identical). We can assume that $\deg_G(u) \geq 1$, because otherwise agent u's cost would be infinite and adding uv would be beneficial. We compute the difference in the neighborhood cost, using Lemma A.1 in the first equality.

$$\begin{aligned} a_{G'}(u) - a_G(u) &= \alpha\left(1 + \frac{\mathrm{f}_G(u) - \deg_G(u) + 1}{(\mathrm{f}_G(u)+1)(\mathrm{f}_G(u)+2)}\right) \\ &= \alpha\left(\frac{\mathrm{f}_G(u)+3}{\mathrm{f}_G(u)+2} - \deg_G(u)\frac{1}{(\mathrm{f}_G(u)+2)(\mathrm{f}_G(u)+1)}\right) \\ &\leq \alpha\left(\frac{\mathrm{f}_G(u)+3}{\mathrm{f}_G(u)+2} - \frac{1}{(\mathrm{f}_G(u)+2)(\mathrm{f}_G(u)+1)}\right). \end{aligned}$$

   Now, consider the function $f\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$, $f(x) = \frac{x+3}{x+2} - \frac{1}{(x+2)(x+1)}$. This function attains its maximum for $x = \sqrt{2}$ and is monotonically increasing for $0 \leq x \leq \sqrt{2}$ and monotonically decreasing for $x \geq \sqrt{2}$. Moreover, $f(1) = f(2) = \frac{7}{6}$. Hence, the maximum attained by integer values is $\frac{7}{6}$. We conclude that $a_{G'}(u) - a_G(u) \leq \frac{7}{6}\alpha < 1$. Since $\mathrm{d}_{G'}(u) - \mathrm{d}_G(u) \leq -1$, we obtain $c_{G'}(u) - c_G(u) < 0$. Hence, creation of the edge uv is beneficial for u.
2. Let $\alpha < \frac{4}{3}$ and consider a pairwise stable network G. In particular, G is connected. Assume that there are agents v and w of distance at least 3. We will show that $G' = G + vw$ is better for both of these agents, contradicting the pairwise stability of G.
   The same computations as in the proof of the first property show that the neighborhood cost increases by at most $\frac{7}{6}\alpha$ if vw is monochromatic. On the other hand, if vw is bichromatic, then the neighborhood cost increases by at most $\frac{3}{2}\alpha$. Since the distance cost decreases by at least 2, we conclude that $c_{G'}(x) - c_G(x) < 0$ for $\alpha < \frac{4}{3}$ and $x \in \{v, w\}$.
   The curiosity of one agent type follows from the fact that two agents from different types, which are both not curious, must have distance at least 3.

3. Let $\alpha < 1$ and assume that u is a curious agent of a network $G = (V, E)$. Consider an agent v of the same type such that $uv \notin E$ and let $G' = G + uv$. Then,

$$\begin{aligned}
&a_{G'}(u) - a_G(u) \\
&= \alpha \left(\frac{\mathrm{f}_G(u) + 3}{\mathrm{f}_G(u) + 2} - \frac{\deg_G(u)}{(\mathrm{f}_G(u) + 2)(\mathrm{f}_G(u) + 1)} \right) \\
&\leq \alpha \left(\frac{\mathrm{f}_G(u) + 3}{\mathrm{f}_G(u) + 2} - \frac{\mathrm{f}_G(u) + 1}{(\mathrm{f}_G(u) + 2)(\mathrm{f}_G(u) + 1)} \right) = \alpha < 1.
\end{aligned}$$

The first equality is derived by the same computations as in the proof of the first property. Consequently, $c_{G'}(u) - c_G(u) < 0$. Hence, if v and w are both curious agents of the same type, then the edge vw must be present in any pairwise stable network.

4. We start to show that $\mathbf{K}_n$ is pairwise stable for $\alpha \leq \frac{n_B}{n_B+1}$.
To this end, we show that no edge can be deleted by one of its endpoints. Consider a pair of agents $u, v \in V$. If they are of the same type, then severing the edge uv by u decreases her cost by

$$\begin{aligned}
&c_{G-uv}(u) - c_G(u) = -\alpha \left(1 + \frac{\mathrm{f}_G(u) - \deg_G(u) + 1}{(\mathrm{f}_G(u) + 1)\mathrm{f}_G(u)} \right) \\
&= -\alpha \left(1 + \frac{\mathrm{f}_G(u) + 2 - n}{\mathrm{f}_G(u)(\mathrm{f}_G(u) + 1)} \right) + 1 \\
&\geq -\frac{n_B}{n_B + 1} \cdot \frac{n^2 - n + 1}{(n - 1)n} + 1 \\
&\geq -\frac{n}{n + 1} \cdot \frac{n^2 - n + 1}{(n - 1)n} + 1 \geq 0.
\end{aligned}$$

Hence, no agent can improve her strategy by severing an edge to an agent of the same color.
If u and v have different colors, the cost decrease is

$$\begin{aligned}
&c_{G-uv}(u) - c_G(u) = -\alpha \left(1 + \frac{1}{\mathrm{f}_G(u) + 1} \right) + 1 \\
&\geq -\alpha \left(1 + \frac{1}{n_B} \right) + 1 \geq -\frac{n_B}{n_B + 1} \cdot \frac{n_B + 1}{n_B} + 1 = 0.
\end{aligned}$$

Therefore, there is no better response for any agent in the network, which implies that $\mathbf{K}_n$ is pairwise stable.
For the uniqueness, consider any pairwise stable network $G = (V, E)$ and assume that $\alpha < \min\{\frac{6}{7}, \frac{n_B}{n_B+1}\}$. Note that G is fully intra-connected according to Proposition 4.1(4.1). Assume for contradiction that there are two agents $u, v \in V$ with $uv \notin E$ which have a different type.

Then, creating the edge uv increases the neighborhood cost for each involved agent by at most

$$\alpha\left(1+\frac{1}{\mathrm{f}_G(u)+1}\right) \leq \alpha\left(1+\frac{1}{n_B}\right) < 1,$$

while it decreases the distance to at least one node, a contradiction. Hence, $uv \in E$, which implies that G is a clique.

5. Let $\alpha \geq 1$. Consider a star graph $\mathbf{S}_n$ with central node c. To show that $\mathbf{S}_n$ is pairwise stable, we need to prove that no two leaves can jointly create an edge. Consider two leafs u and v. There can be a few possible situations. The first two cases cover the case that c and one of u and v are of the same color, say $u \in V_{\mathcal{T}(c)}$. If $v \in V_{\mathcal{T}(c)}$, then creating uv causes an increase in neighborhood cost of $a_{\mathbf{S}_n+uv}(u) - a_{\mathbf{S}_n}(u) = \alpha\left(1+\frac{1}{6}\right) = \frac{7}{6}\alpha \geq 1$, while the distance cost is only decreased by 1. Hence, for $\alpha \geq 1$, creating the edge uv is not beneficial for u. If v has a different color, then $a_{\mathbf{S}_n+uv}(u) - a_{\mathbf{S}_n}(u) = \frac{3}{2}\alpha$, and u would again prevent the creation of uv.
It remains that u and v both have a different color from c. If $v \in V_{\mathcal{T}(u)}$, then creating the edge uv increases the neighborhood cost by α and decreases the distance cost by 1 for both u and v. Thus, since $\alpha \geq 1$, this is not beneficial.
If all three nodes u, v, and c have different colors, then the creation of the edge uv increases the neighborhood cost of u by $2\alpha \geq 2$ and decreases her distance cost by only 1.
Therefore, no pair of nodes can create an edge to improve their cost. Clearly, also no edge can be unilaterally deleted. The assertion follows.
6. Let $\alpha \geq \frac{4}{3}$. Consider the double star $\mathbf{DS}_n$ and let c_B and c_R be the blue and red star center, respectively. Note that no agent can sever an edge, because this would disconnect the network.
Also, no edge between a star center and a leaf node can be created, because it is not profitable for the center node. Indeed, consider a pair of nodes $v \in V_R$ and the central node c_B. Adding the edge $c_B v$ improves the distance to only one node for the agent c_B, while the neighborhood cost increases by

$$\begin{aligned} & a_{\mathbf{DS}_n+c_B v}(c_B) - a_{\mathbf{DS}_n}(c_B) \\ & = \alpha\left((\deg_{\mathbf{DS}_n}(c_B)+1)\left(1+\frac{1}{n_B}\right) - \deg_{\mathbf{DS}_n}(c_B)\cdot\left(1+\frac{1}{n_B}\right)\right) = \alpha\left(\frac{n_B+1}{n_B}\right) \geq 1. \end{aligned}$$

Hence, the edge $c_B v$ will be rejected by the agent c_B. Analogously, a new edge between the center node c_R and a node $v \in V_B$ is not profitable for the center node c_R, because it increases the neighborhood cost by $\alpha\left(1+\frac{1}{n_R}\right) \geq 1$ and decreases the distance cost by 1.
Next, consider the case of creating a bichromatic edge between two leave nodes. Then, the distance cost is decreased by 2, while the neighborhood cost is increased by $\frac{3}{2}\alpha \geq 2$.
Finally, consider the creation of an edge between two nodes u, v of the same type, say type R. The new edge improves the distance cost by 1 for both agents

but increases the neighborhood cost by $\alpha\left(2\cdot\left(1+\frac{1}{2+1}\right)-1-\frac{1}{2}\right)=\frac{7\alpha}{6}\geq 1$. Hence, $\mathbf{DS}_n$ is pairwise stable for any $\alpha\geq\frac{4}{3}$. □

A.2 Decreasing effort of integration

The update formulas for the DEI-NCG are much easier then the formulas for the ICF-NCG and can be directly obtained from the definition of $a_G^{DEI}(u)$. For reference, we collect them in a lemma analogous to Lemma A.1.

Lemma A.2 *Consider a network $G=(V,E)$ and an agent $u\in V$ in the DEI-NCG. Consider an agent $v\in V_{\mathcal{T}(u)}$ of the same type and an agent $w\in V\setminus V_{\mathcal{T}(u)}$ of a different type. Then, the following statements hold:*

1. $a_{G+uv}(u)-a_G(u)=\alpha$ *if $uv\notin E$ (creation of a monochromatic edge),*
2. $a_{G-uv}(u)-a_G(u)=-\alpha$ *if $uv\in E$ (deletion of a monochromatic edge),*
3. $a_{G+uw}(u)-a_G(u)=\alpha\left(1+\frac{1}{\mathrm{e}_G(u)+1}\right)$ *if $uw\notin E$ (creation of a bichromatic edge), and*
4. $a_{G-uw}(u)-a_G(u)=-\alpha\left(1+\frac{1}{\mathrm{e}_G(u)}\right)$ *if $uw\in E$ (deletion of a bichromatic edge).*

We now provide the proofs of the statements collected in Proposition 5.1.

Proposition 5.1 *For the DEI-NCG the following holds:*
1. *If $\alpha<\frac{1}{2}$, then $\mathbf{K}_n$ is the unique pairwise stable network.*
2. *If $\alpha<1$, then every pairwise stable network is fully intra-connected.*
3. *If $\alpha<1$, then every pairwise stable network G satisfies* $\mathrm{diam}(G)\leq 2$.
4. *The network $\mathbf{K}_n$ is pairwise stable if $\alpha\leq\frac{n-n_R}{n-n_R+1}$.*
5. *If $\alpha\geq 1$, then $\mathbf{S}_n$ and $\mathbf{DS}_n$ are pairwise stable networks.*
6. *If $\alpha\geq\frac{4}{3}$, then $\mathbf{DSX}_n$ is a pairwise stable network.*

Proof We prove the statements one by one.

1. If some edge is not present, it has cost at most $2\alpha<1$ and creating it decreases the distance cost by at least 1.
2. Creating a monochromatic edge has cost $\alpha<1$ and decreases the distance cost by at least 1.
3. Let $\alpha<1$. Assume that there are agents $u,v\in V$ with $\mathrm{d}_G(u,v)\geq 3$. Then, creating uv increases the neighborhood cost by at most $2\alpha<2$, while decreasing the distance cost by at least 2 for each of its endpoints. Hence, G is not pairwise stable.
4. Clearly, no monochromatic edge can be severed. Now, consider a bichromatic edge uv. Then, severing uv increases the total cost for v by $1-\alpha\left(1+\frac{1}{n-n_{\mathcal{T}(v)}}\right)\geq 1-\alpha\left(1+\frac{1}{n-n_R}\right)\geq 1-\frac{n-n_R}{n-n_R+1}\left(1+\frac{1}{n-n_R}\right)=0$. Hence, also bichromatic edges cannot be severed.
5. No edge can be severed, because these networks are trees. Due to the sufficiently large distance cost, no agent favors to create an edge if this only improves the distance cost by 1. Hence, $\mathbf{S}_n$ is stable, the two centers of $\mathbf{DS}_n$ will not agree to build

further edges, and leaves of $\mathbf{DS}_n$ will not agree to create further monochromatic edges. Finally, the cost for creating an edge between two leaves of different types is $2\alpha \geq 2$ which does not make up for a distance improvement of 2.

6. As for $\mathbf{DS}_n$, no edges can be severed, and the centers will not benefit from creating further edges. Also, leaves have no incentive to create monochromatic edges. Finally, the cost for a bichromatic edge between leaves of different types is $\frac{3}{2}\alpha \geq 2$, but creating such an edge yields only a distance improvement of 2. □

Lemma 5.3 *Let $k = 2$ in the DEI-NCG. Consider a fully intra-connected and pairwise stable network G.*

1. *If $\alpha > \frac{n_B}{n_B+1}$, then every red agent in G entertains at most one bichromatic edge.*
2. *If $\alpha > \frac{n_R}{n_R+1}$, then every agent in G entertains at most one bichromatic edge.*

Proof The proof of both statements follows from a unified approach. Let $G = (V, E)$ be a fully intra-connected and pairwise stable network. Let $u \in V$. By full intra-connectivity, severing one of several bichromatic edges incident to u, increases the distance cost of u by exactly 1 while decreasing the neighborhood cost by $\Delta = \alpha\frac{\mathrm{e}_G(u)+1}{\mathrm{e}_G(u)}$. If $\alpha > \frac{n_R}{n_R+1}$, then $\Delta > 1$ and severing a bichromatic edge is beneficial for u. This proves the second statement. If even $\alpha > \frac{n_B}{n_B+1}$ and u is an agent of the majority type, then $\mathrm{e}_G(u) \leq n_B$, and $\Delta \geq \alpha\frac{n_B+1}{n_B} > 1$. □

B Detailed experimental results

In this section we provide the detailed results of our experimental analysis complementing Sect. 6. We first present the results for our experiments that shed light on the general behavior of dynamics. Then we discuss additional experimental results regarding the obtained segregation strength.

B.1 Additional experiments regarding the number of discontent agents

We provide additional simulation results on the number of discontent agents during the convergence process for the ICF-NCG and the DEI-NCG.

Number of discontent agents in the ICF-NCG In addition to the plot in Fig. 9 in the main body of our paper, we present in Figs. 14, 15, 16 the corresponding plots for the number of deviations when the process is started on a spanning tree with a uniformly random initial agent placement, and on grids or spanning trees with a segregated initial agent placement.

We find that in all cases the observed behavior is very similar to our observations from Fig. 9. We consistently see that the case $\alpha = 1$ has a specific behavior for the add-only dynamics. This is no surprise since the same arguments as discussed in Sect. 6.2 apply. Overall, we see that the initial network and agent placement have no significant impact on the observed number of discontent agents.

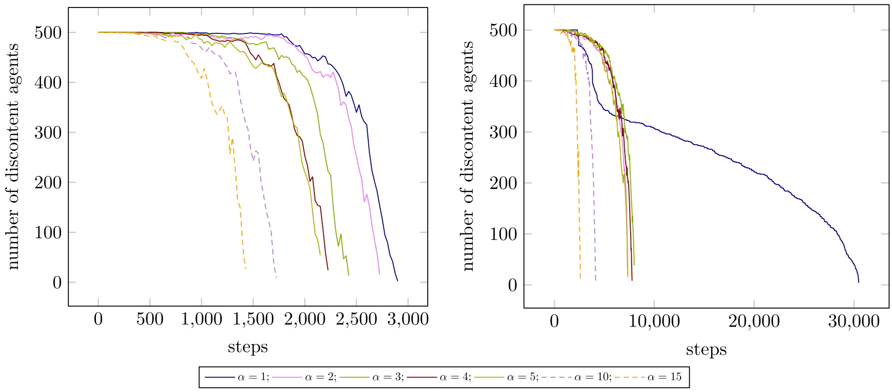

Fig. 14 Time line of the number of discontent agents when starting from a random tree in the ICF-NCG. Left: 1.01-approximate dynamics. Right: Add-only dynamics

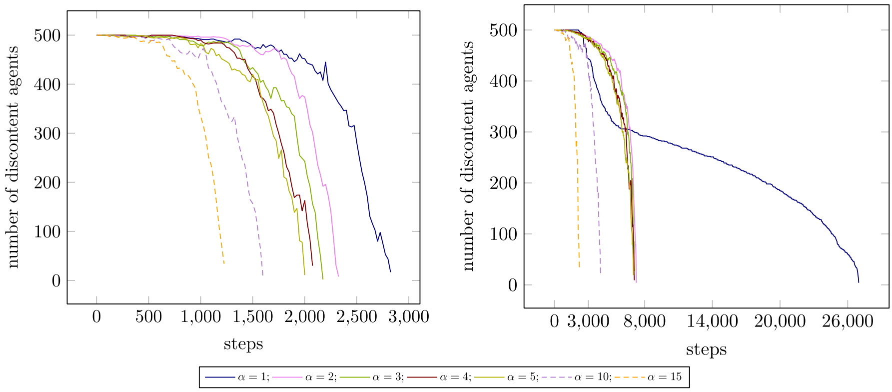

Fig. 15 Time line of the number of discontent agents when starting from a segregated grid in the ICF-NCG. Left: 1.01-approximate dynamics. Right: Add-only dynamics

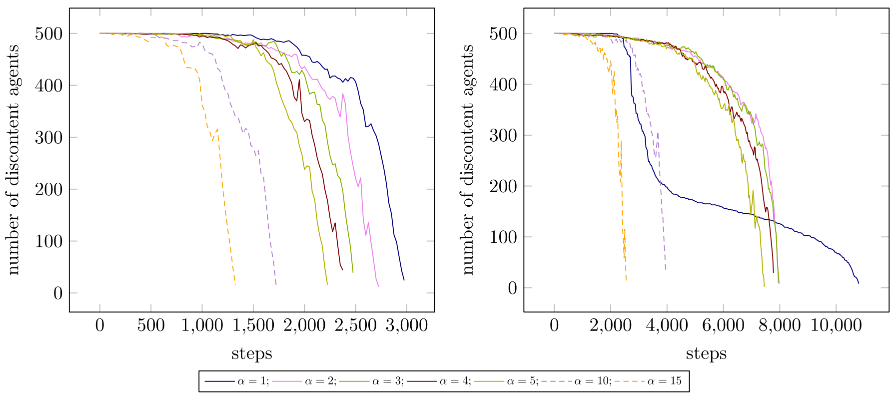

Fig. 16 Time line of the number of discontent agents when starting from a segregated tree in the ICF-NCG. Left: 1.01-approximate dynamics. Right: Add-only dynamics

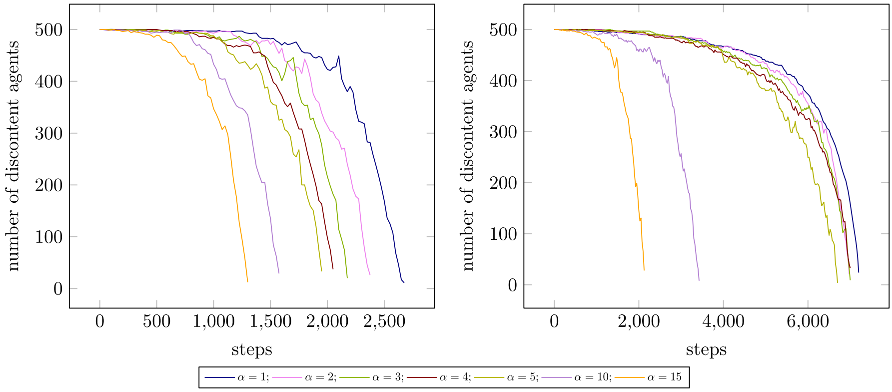

Fig. 17 Time line of the number of discontent agents when starting from a random grid in the DEI-NCG. Left: 1.01-approximate dynamics. Right: Add-only dynamics

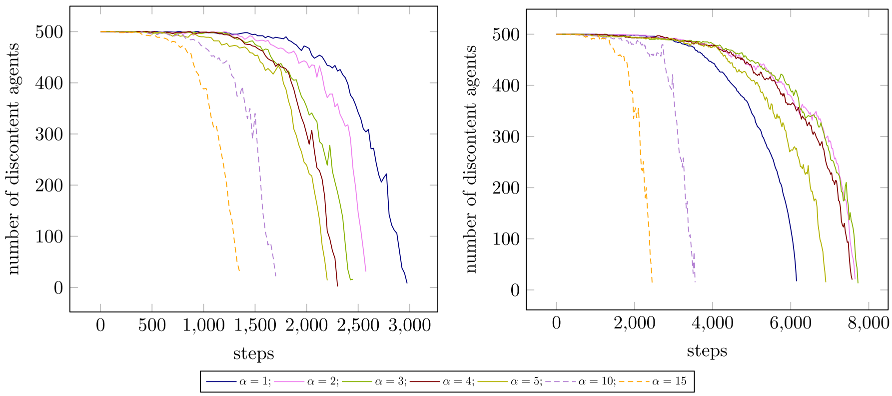

Fig. 18 Time line of the number of discontent agents when starting from a random tree in the DEI-NCG. Left: 1.01-approximate dynamics. Right: Add-only dynamics

Number of discontent agents in the DEI-NCG For the DEI-NCG we show the plots regarding the number of discontent agents in Figs. 17, 18, 19, 20. We find that all settings behave very similarly and in particular, we do not find the exceptional behavior for the case $\alpha = 1$. As discussed in Sect. 6.2, this relies on the fact that the increase in neighborhood cost cannot be below α, so there is no necessity to create monochromatic edges as in the ICF-NCG.

B.2 Additional experiments regarding the number of deviations per agent

Next, we consider the distribution of deviations among agents. In addition to the plots in Fig. 10, we present the corresponding plots for all other initial settings for both the ICF-NCG and the DEI-NCG.

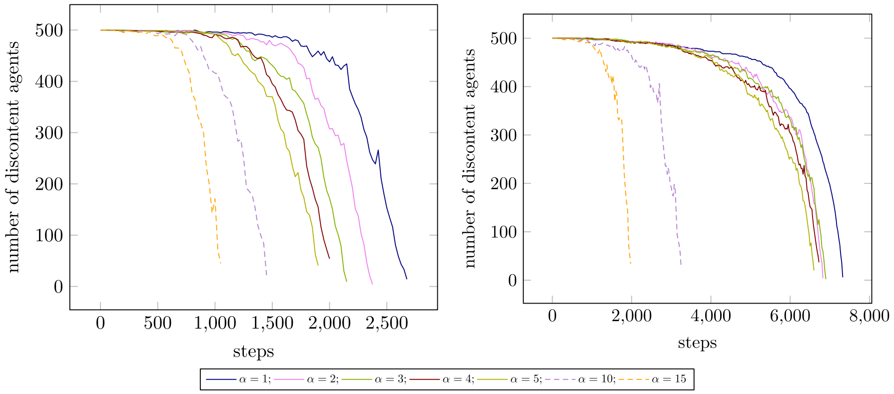

Fig. 19 Time line of the number of discontent agents when starting from a segregated grid in the DEI-NCG. Left: 1.01-approximate dynamics. Right: Add-only dynamics

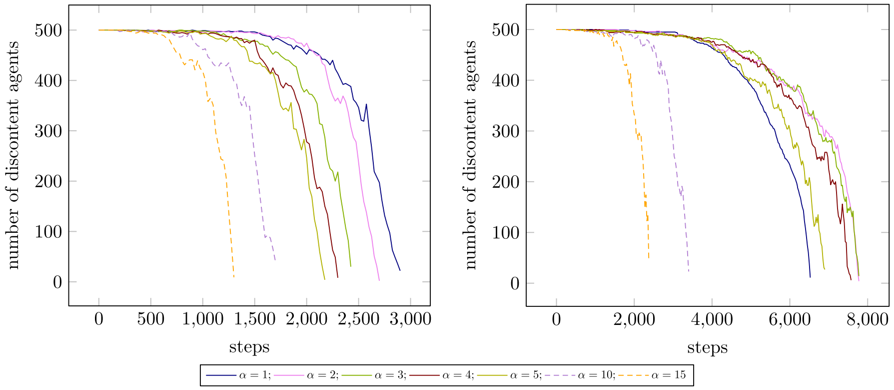

Fig. 20 Time line of the number of discontent agents when starting from a segregated tree in the DEI-NCG. Left: 1.01-approximate dynamics. Right: Add-only dynamics

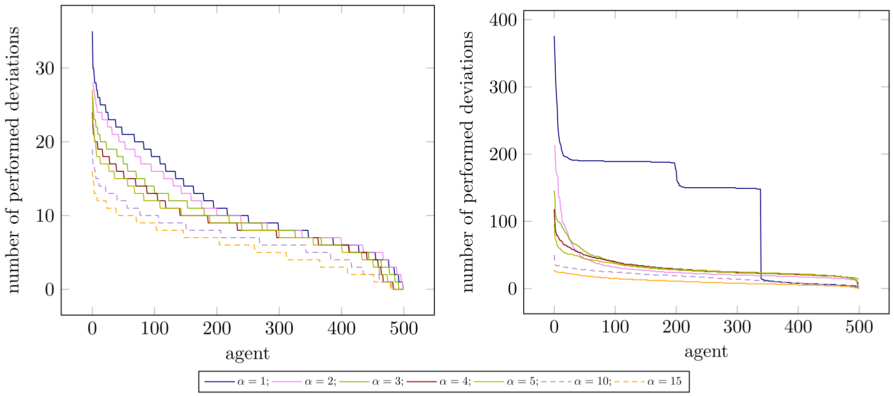

Fig. 21 Distribution of the number of deviations when starting from a random tree in the ICF-NCG. Left: 1.01-approximate dynamics. Right: Add-only dynamics

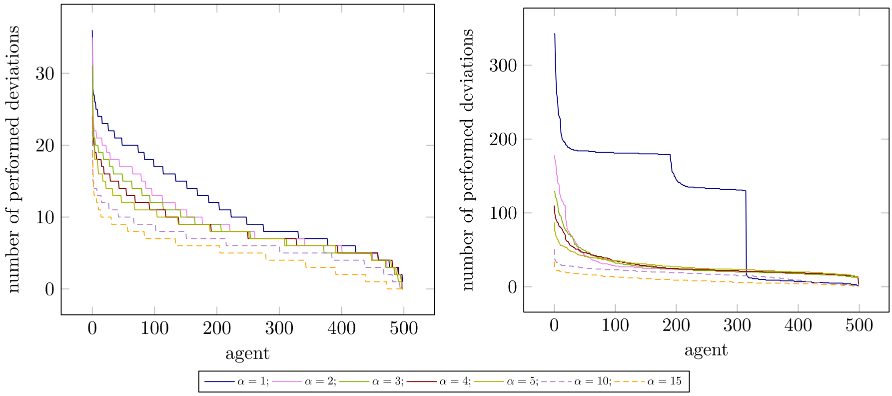

Fig. 22 Distribution of the number of deviations when starting from a segregated grid in the ICF-NCG. Left: 1.01-approximate dynamics. Right: Add-only dynamics

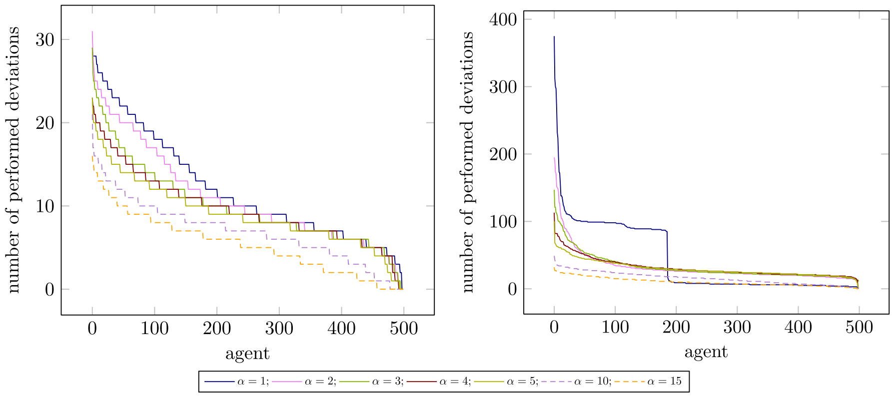

Fig. 23 Distribution of the number of deviations when starting from a segregated tree in the ICF-NCG. Left: 1.01-approximate dynamics. Right: Add-only dynamics

Deviation distribution in the ICF-NCG We present the results for the ICF-NCG in Figs. 21 to 23. We find a very consistent behavior to what we already observed in Fig. 10. For $\alpha = 1$, the size of the plateaus changes depending on the initial setting. Recall that the plateaus are caused by the necessity of creating monochromatic edges by agents that have at least a small number of enemies at some point. The number of such agents changes. While it concerns many agents for a uniformly random initial distribution, there are far less agents sponsoring bichromatic edges in segregated networks. This is most extreme for the case of segregated trees, which only have a single bichromatic edge initially.

Deviation distribution in the DEI-NCG The results on the distribution of performed agent deviations for the DEI-NCG are shown in Figs. 24 to 27. We find a very consistent behavior in all settings. In particular, we do not observe plateaus for the case $\alpha = 1$. Thus, for the DEI-NCG the exact initial setting only has a very low impact on the observed convergence process.

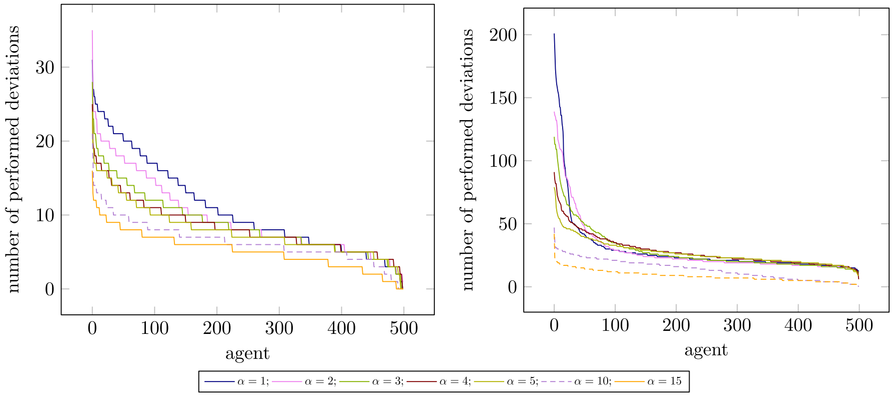

Fig. 24 Distribution of the number of deviations when starting from a random grid in the DEI-NCG. Left: 1.01-approximate dynamics. Right: Add-only dynamics

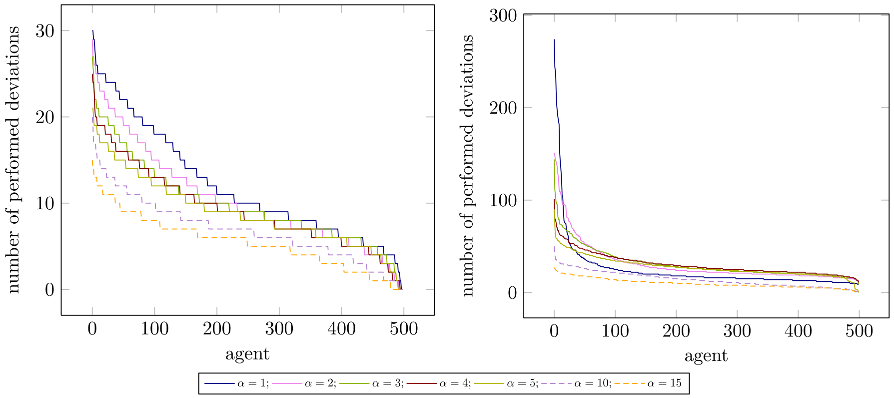

Fig. 25 Distribution of the number of deviations when starting from a random tree in the DEI-NCG. Left: 1.01-approximate dynamics. Right: Add-only dynamics

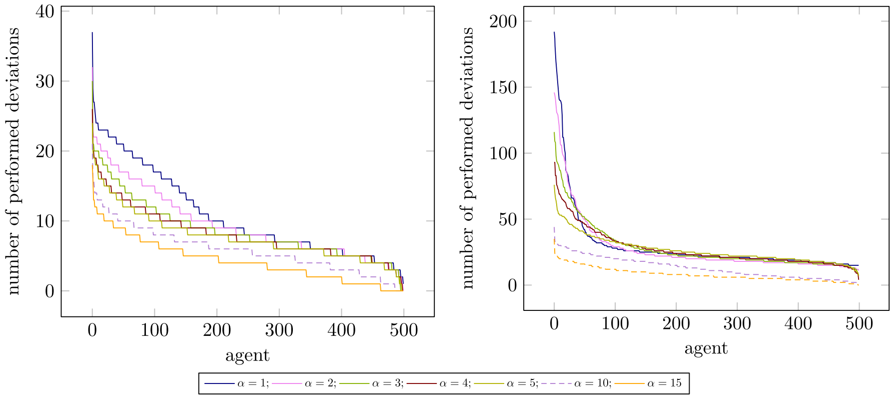

Fig. 26 Distribution of the number of deviations when starting from a segregated grid in the DEI-NCG. Left: 1.01-approximate dynamics. Right: Add-only dynamics

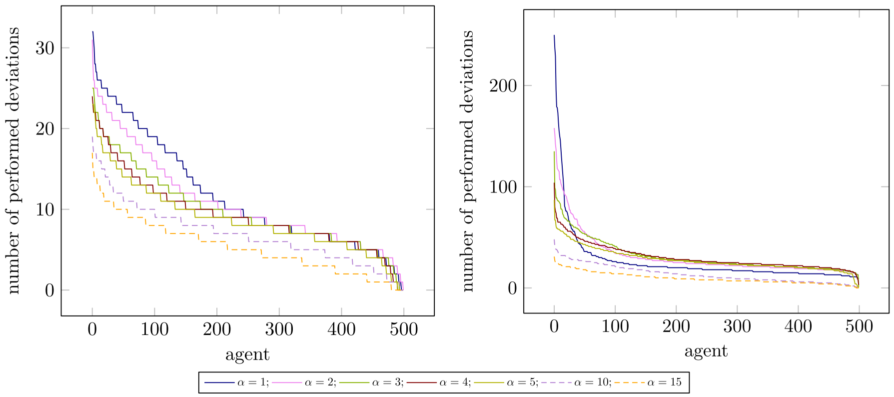

Fig. 27 Distribution of the number of deviations when starting from a segregated tree in the DEI-NCG. Left: 1.01-approximate dynamics. Right: Add-only dynamics

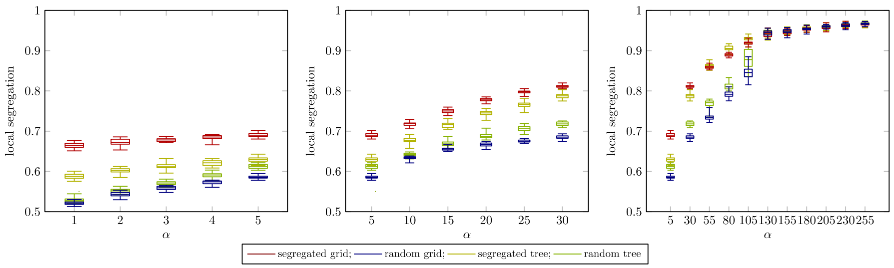

Fig. 28 Local segregation of 1.01-approximate pairwise stable networks in the ICF-NCG obtained by the best response dynamics for $n = 1000$ over 50 runs starting from initially integrated or initially segregated random spanning trees or grids

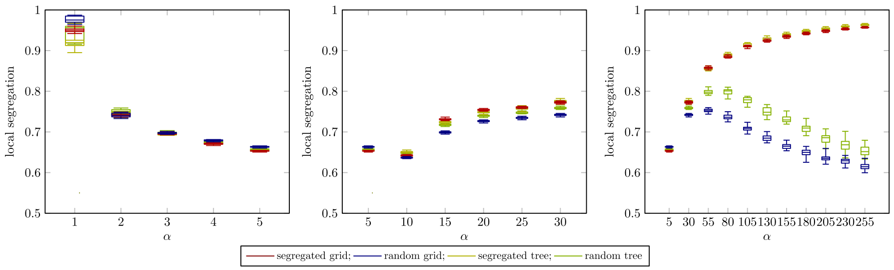

Fig. 29 Local segregation of pairwise stable networks in the add-only ICF-NCG obtained by the best response dynamics for $n = 1000$ over 50 runs starting from initially integrated or initially segregated random spanning trees or grids

B.3 Additional experiments regarding the local segregation measure

We provide additional simulation results for the local segregation measure for the ICF-NCG and the DEI-NCG.

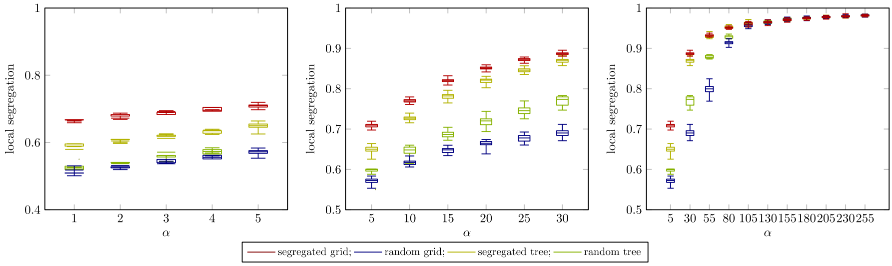

Fig. 30 Local segregation of 1.01-approximate pairwise stable networks in the DEI-NCG obtained by the best response dynamics for $n = 1000$ over 50 runs starting from initially integrated or initially segregated random spanning trees or grids

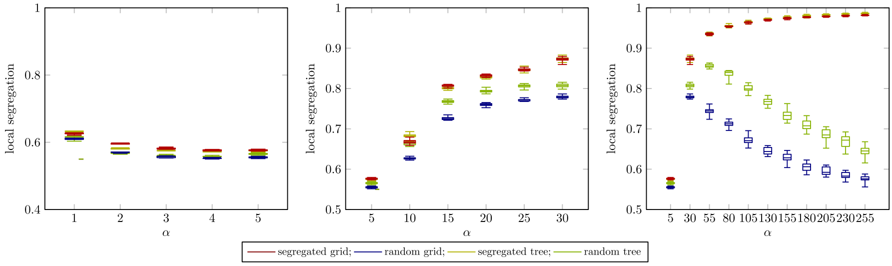

Fig. 31 Local segregation of pairwise stable networks in the add-only DEI-NCG obtained by the best response dynamics for $n = 1000$ over 50 runs starting from initially integrated or initially segregated random spanning trees or grids

Local segregation strength in the ICF-NCG For the sake of comparison, we include the results for the local segregation measure for the ICF-NCG again. Figures 28 and 29 are identical to the respective figures in Sect. 6.

They show that high segregation of stable networks can be avoided by a lower cost of the connections ($\alpha < 30$) and if started from an initially integrated state. Moreover, as shown in Fig. 29, this even holds for high connection cost if the add-only dynamics starts with an initially integrated network.

Local segregation strength in the DEI-NCG Compared to the results in Figs. 28 and 29, the following results in Figs. 30 and 31 for the DEI-NCG clearly show the difference in segregation strength for small α as discussed in Sect. 6.3. For larger α, the structure of the networks is highly similar. In particular, the tendency of decreasing segregation in case of the add-only version of the dynamics with integrated initial networks is observed for both games.

B.4 Experiments regarding the global segregation measure

We now consider the global segregation measure. We illustrate the dependence of the global segregation measure on the parameter α and the initial state in both the DEI-NCG and ICF-NCG. The observations are similar as for the local segregation measure, highlighting the robustness of our results.

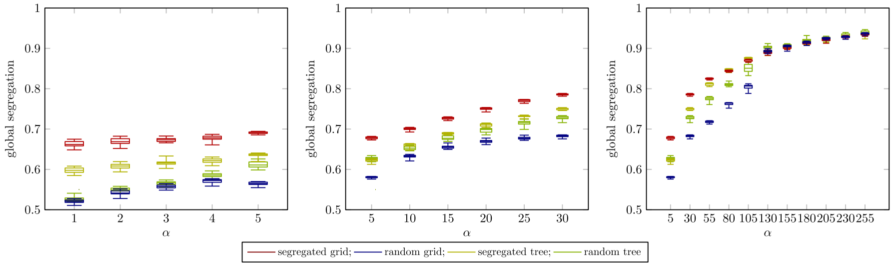

Fig. 32 Global segregation of 1.01-approximate pairwise stable networks in the ICF-NCG obtained by the best response dynamics for $n = 1000$ over 50 runs starting from initially integrated or initially segregated random spanning trees or grids

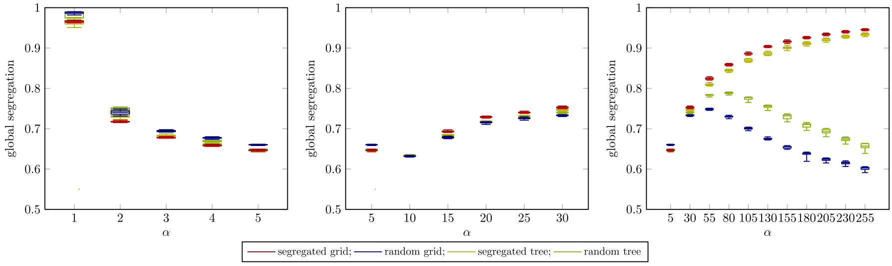

Fig. 33 Global segregation of pairwise stable networks in the add-only ICF-NCG obtained by the best response dynamics for $n = 1000$ over 50 runs starting from initially integrated or initially segregated random spanning trees or grids

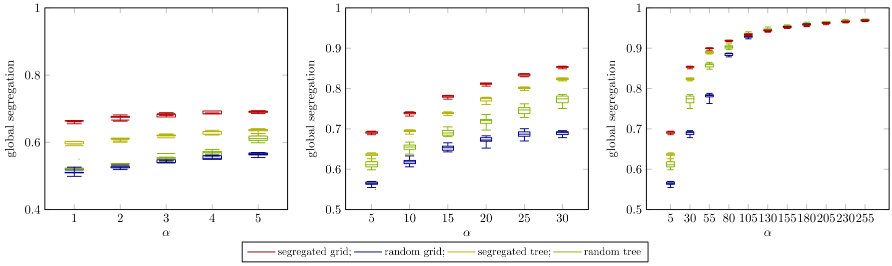

Fig. 34 Global segregation of 1.01-approximate pairwise stable networks in the DEI-NCG obtained by the best response dynamics for $n = 1000$ over 50 runs starting from initially integrated or initially segregated random spanning trees or grids

Global segregation strength in the ICF-NCG The results for the global segregation measure for 1.01-approximate networks in the ICF-NCG and pairwise stable networks in the add-only ICF-NCG are presented in Figs. 32 and 33.

Global segregation strength in the DEI-NCG The results for the global segregation measure for 1.01-approximate pairwise stable networks in the DEI-NCG and pairwise stable networks in the add-only DEI-NCG are presented in Figs. 34 and 35. Also these results are in line with the corresponding results for the local segregation measure.

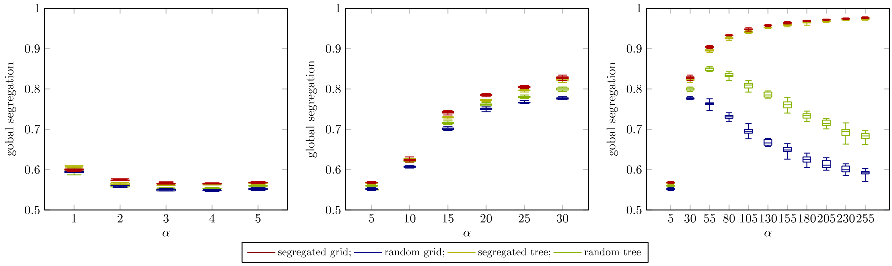

Fig. 35 Global segregation of pairwise stable networks in the add-only DEI-NCG obtained by the best response dynamics for $n = 1000$ over 50 runs starting from initially integrated or initially segregated random spanning trees or grids

Acknowledgements A preliminary version of this article appeared in the Proceedings of the 31th International Joint Conference on Artificial Intelligence (2021). We would like to thank the anonymous reviewers from IJCAI and SCW for their helpful feedback.

Funding This work was supported in part by the Deutsche Forschungsgemeinschaft (German Research Foundation) under grants BR 2312/11-2, BR 2312/12-1, and LE 3445/3-1, and by the AI Programme of The Alan Turing Institute.

Data availability The source code for the simulations is publicly available at https://github.com/melnan/HomophilicNCG.

Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

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