The importance of substitution for elementary algebra

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Summary

Substitution is the process of replacing one term in an algebraic expression with another term. Substitutions can be classified in a simple scheme. Their significance can be justified from a theoretical, pragmatic and empirical point of view. From these findings, suggestions can be derived for promoting the understanding of substitution in the classroom. Digital mathematics tools can also be used to good effect.

Keywords

Algebra; CAS; Substitution; Use of technology

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1. Introduction

A key aim of teaching algebra at lower secondary level is to develop an understanding of equations and the equals sign. Weigand et al. (2022) describe basic concepts of equations that should be developed: The basic operational conception consists of interpreting the equal sign as a request to calculate, i.e. as a "results sign". The basic relational concept, on the other hand, sees an equation as consisting of two terms that have the same value. The basic functional conception sees an equation as a combination of two functions and finally, the basic object conception sees an equation as an independent object that can be operated with according to certain rules. In mathematics didactic research, it has been proven many times that learners from elementary school primarily bring with them the view that the equals sign prompts them to calculate, while the relational view only develops over time (e.g. Borromeo Ferri & Blum, 2011).

The basic relational notion can be further divided into a notion of equality of value and a notion of substitutability (Jones & Pratt, 2012). The idea of equality of value is a static one: the simple equation 10 + 10 = 20 means, for example, in a financial context, that two 10-euro bills have the same value as one 20-euro bill. The concept of substitution, on the other hand, is dynamic, it initiates a process: You can exchange one 20-euro bill for two 10-euro bills - and vice versa!

This article first analyzes the concept of substitution and thus the basic relational concept of equations theoretically. This makes its significance in (elementary) mathematics clear. Subsequently, some empirical findings on the understanding of substitution are presented. Finally, suggestions are made as to how the competence of substitution can be promoted in the classroom.

2. Theory of substitution

Substitution is a process within mathematical formula language in which one part of a term or equation is replaced by another term. [1]There are different ways of writing this in mathematical logic textbooks , I use the following here: $t[a \to b]$ means that in the term t (or in the equation t) every free occurrence of a is replaced by b is re-

[1] Instead of my spelling $t[a \to b]$ Bundy (1983), for example, uses $(t)\{a/b\}$Leary & Kristiansen (2015) write for it t_b^aRautenberg (2010) writes $t\frac{b}{a}$, Beckermann (2014) writes $[t]_a^b$ and Crabbé (2004) finally $t[a \coloneqq b]$.

placed. Free means that the variable is not bound by a quantifier, integral or summation sign. This condition is almost always fulfilled in a school context, but it is important to understand it: For example, in $\exists x: x > 2$ or in $\int x^2 \mathrm{d}x$ simply $x \to 1$ because $\exists 1: 1 > 2$ and $\int 1^2 \mathrm{d}1$ are meaningless.

A distinction can be made between different uses of substitutions:

- Variables are replaced by (numerical) values. Examples:
  $x^2 + 1[x \to 3]$ becomes $3^2 + 1$, i.e. 10
  $x^2 + x \cdot y[x \to 3]$ becomes $9 + 3y$
  $x^2 + x \cdot y[x \to 3, y \to 2]$ becomes 15
  $x^2 = 2 \cdot x[x \to 3]$ becomes false
  $x^2 = 2 \cdot x[x \to 2]$ becomes true
  This form of substitution is also known as assigning values
  to variables. This determines the meaning of terms and equations. For example, two terms are equivalent or two equations are equivalent if they provide the same value or truth value under all assignments of the variables.
  This form of substitution is also required if you want to use a formula, for example from a formulary, to calculate a value by inserting all known values.

- Variables are replaced by terms. Examples:
  $x^2 + 1[x \to a + b]$ becomes $(a + b)^2 + 1$
  $V = \frac{4}{3}\pi r^3[r \to r_0 + \Delta r]$ becomes $V = \frac{4}{3}\pi(r_0 + \Delta r)^3$ As the last example shows, replacing variables with terms is typical when using formulas to derive new relationships. This operation is also very useful when solving systems of equations (substitution method).

- Subterms are replaced by terms or variables. Examples:
  $\frac{1}{x^2+y^2}[x^2 + y^2 \to r^2]$ becomes $\frac{1}{r^2}$
  $x^4 + 3x^2 - 1 = 0[x^2 \to u]$ becomes $u^2 + 3u - 1 = 0$
  This form of substitution is required when solving equations and systems of equations, but also when integrating using the substitution method. It is also relevant when a function defined by a term is applied to another term. For example, if the function f is defined by $f(x) = 2x + 1$then $f(x + 1)$ is the result of the substitution $2x + 1[x \to x + 1]$, therefore $2(x + 1) + 1$.

These three cases of the general principle of substitution already make it clear that this is a very wide-ranging topic, but also one that is rich in aspects and therefore difficult for learners (more on the difficulties can be found in the fourth section).

This is all the more true as, in addition to the application of substitution, finding suitable substitutions is also one of the challenges in this area. For example, if the equation $(x+1)^4+x^2+2x+1=0$ is to be solved, it proves to be extremely helpful to see that the substitution $(x+1)^2 \to u$ changes the equation into $u^2+u=0$ with the solution $u=0 \vee u=-1$ which leads by back substitution to $(x+1)^2=0 \vee (x+1)^2=-1$ and hence very quickly shows that $x=-1$ is the only real solution.

Substitution is also closely linked to the meaning of equations. Thus a a solution of the x containing equation $g$if $g[x \to a]$ results in "true". The above-mentioned substitutional interpretation of the relational understanding of equations means that an equation $a=b$ is interpreted as a possible substitution. This is the substitution principle:

> If $a=b$ holds, then the terms t and $t[a \to b]$ have the same value, and the equations g and $g[a \to b]$ have the same truth value.

This principle goes back to Leibniz, who also made the opposite conclusion: [2]If two expressions can be substituted for each other everywhere without changing the truth values of statements, then they are identical (Künne, 2010).

3. Substitution in computer algebra systems

The importance of substitution for computer science can be seen from the fact that universal calculations are possible through substitution alone (Baader & Nipkow, 1999). Simple consequences of this can already be experienced in elementary school: For example, if you put down 4 blue tiles and give the substitution rule that you can replace one blue tile with 3 red ones, you end up with $4 \cdot 3 = 12$ red tiles. If you start with 14 red tiles and apply the reverse substitution rule, you end up with 4 blue and 2 red tiles. Multiplication and division with remainder are therefore very easy to realize by substitution. A little more work shows that any calculation possible

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[2] This principle that indistinguishability implies identity is largely valid, but not without restrictions: Quantum physics violates it (for example, the particles in a Bose-Einstein condensate are indistinguishable but not identical because their number can be counted), but axiomatic algebra also provides a counterexample: If, for example, you introduce the complex numbers as a body extension $\mathbb{C} \coloneqq \mathbb{R}[X]/(X^2+1)$ the complex unit and its conjugate are indistinguishable, but not identical (they only become distinguishable in a concrete model, e.g. when $\mathbb{C}$ is realized as $\mathbb{R}^2$).

with computers can be carried out by substitution. In fact, even amazingly simple substitution rules can do this (Wolfram, 2002).

Substitutions gain practical importance in computer algebra systems, both for practical work and for understanding them. In the prototype SCAS (Oldenburg, 2017; source code: https://myweb.rz.uni-augsburg.de/~oldenbre/SCAS.zip), you can specify substitution rules in a table and use them to build your own computer algebra system and, to a certain extent, watch how it does its work through continuous substitution. Figure 1 shows a complex application based on a large number of built-in rules (which can all be switched on and off individually). If you delete all rules, the term $0 * x + 1 * y$ is returned unchanged. You can then consider that, for example, the rules $0 * A \rightarrow 0, 1 * A \rightarrow A$ are useful. You need a good 100 rules to cover the majority of school algebra and school analysis.

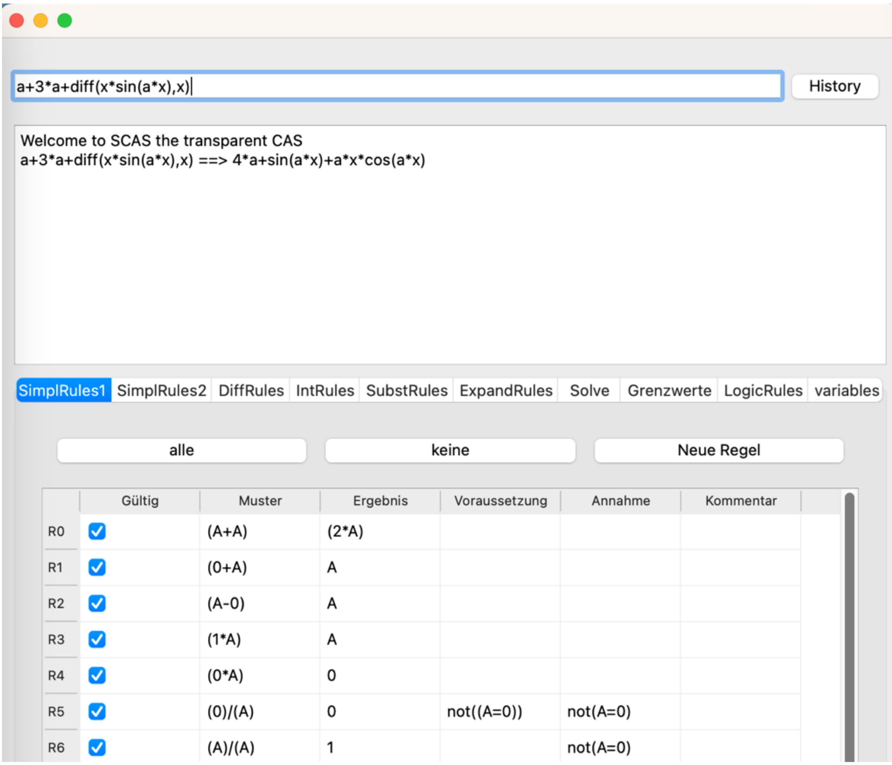

Figure 1: An application of SCAS with many built-in rules

Continued substitution is not only a way of visualizing how a computer algebra system (CAS) works, but one must also apply substitutions systematically when solving problems with such a system. In this context, Drijvers & Gravemeijer (2005) speak of the Solve-Subst scheme, which must be developed for the successful use of CAS. The scheme describes that in many tasks an equation is first solved and the solution found is applied by substitution. The following task shows an example (Fig. 2)

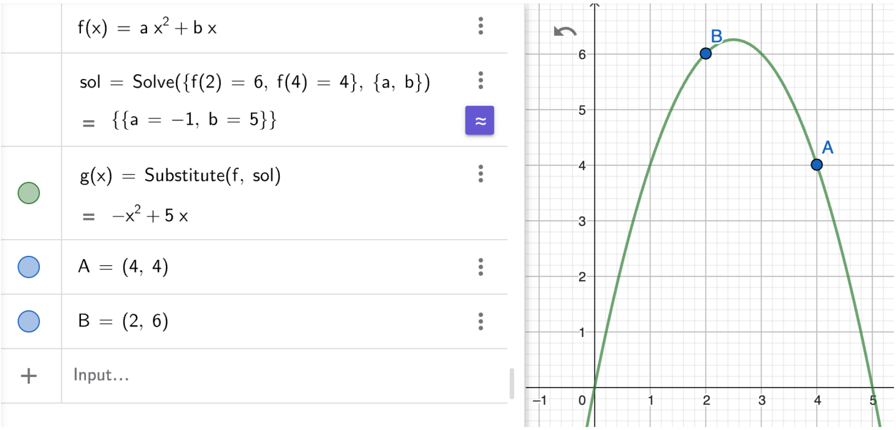

Figure 2: Solving a Task with CAS using the solve-subst-scheme: Find a quadratic function such that the graph passes through (0;0), (2;6), (4;4).

The notation of substitutions is different in the various computer algebra systems. In Mathematica, for example, x^2+y^2 /. {x->1,y->2} gives the result 5, whereas in GeoGebra's CAS calculator the syntax is: Substitute(x^2+y^2,{x=1,y=2}). It is unfortunate that in some systems the equal sign plays the dual role of expressing equality (i.e. establishing a truth value, propositional form) and describing a substitution process, but in our experience this does not lead to difficulties for learners. The use of technology generally broadens the range of meanings of this sign even further than is already traditionally the case (Bardini et al., 2013).

CAS thus offer an environment in which you can control your own substitutions, have substitutions carried out automatically and apply them within complex problem solutions. The above classification of substitutions proves its worth. While the first two forms of substitution are carried out correctly by all computer algebra systems without any problems, the last variant, i.e. the replacement of subterms by other terms, occasionally leads to surprising results. For example, the CAS of GeoGebra returns the following result for Substitute(x^4+x^2,x^2=u) $x^4 + u$. Substitute(x^2+y^2+1,x^2+y^2=u) returns the original term $x^2 + y^2 + 1$ is returned unchanged. Both are due to the fact that substitution is carried out purely syntactically in all common CAS: A substitution $a \to b$ is only used if a is a partial term, i.e. the systems do not apply any transformations in order to, for example x^4 as $(x^2)^2$ so that the substitution $x^2 \to u$ could be applied. Similarly, the CAS would have to convert the term $x^2 + y^2 + 1$ to $(x^2 + y^2) + 1$ in order to $x^2 + y^2 \to u$ to be able to apply it. Therefore, such substitutions, which require a considerably higher computational effort, can only be achieved with special commands, in Maple for example with algsubs(x^2+y^2=u,x^2+y^2+1) or in Mathematica with Simplify[x^2+y^2+1, x^2+y^2==u]. GeoGebra does not have such tools. If such transformations are required, you must convert the equation into a form in which a real partial term is replaced: Substitute(x^2+y^2+1,x^2=u-y^2).

If such phenomena are observed during substitution in class, this can be used as an opportunity to reflect on term structures and operations with terms.

4. The importance of substitution for learning algebra

Malle's didactic classic on elementary algebra (Malle, 1993) already referred in detail to the importance of substitution from a didactic point of view, which in his opinion - as described below - is fed by various sources. On the one hand, substitution helps with the conscious application of rules (Malle, 1993, p. 227ff, 242ff, Weigand et al., 2022, p. 83). For example, if the term $2x \cdot (x^2 + 1)$ is to be multiplied out, substitution plays a role in several ways: The distributive law $A \cdot (B + C) = A \cdot B + A \cdot C$ is applied from left to right, i.e. as a substitution rule $A \cdot (B + C) \to A \cdot B + A \cdot C$to formalize the process of multiplication. To ensure that this rule fits the term at hand, further substitutions must be identified by pattern matching: $\{A \to 2x, B \to x^2, C \to 1\}$. If these are applied to the rule $A \cdot (B + C) \to A \cdot B + A \cdot C$ you get what you want: $2x \cdot (x^2 + 1) \to 2x \cdot x^2 + 2x \cdot 1$.

According to Malle, the further significance of substitution comes from the useful applications, for example to relate initially unrelated variables to each other. The examples of this are discussed in more detail in the next section.

One of the first studies to examine substitution from an empirical perspective was Oldenburg (2010). In a study of 141 students in the eleventh grade of a German grammar school, various sub-dimensions of algebraic skills were assessed. There were tasks on calculating with fractions, simplifying terms, setting up and interpreting terms and equations. Substitution skills were assessed in the test using several items. Some examples:

- The function f is defined by $f(x) = x^3 - 2$. What is then $f(x + 1)$?

- We know that $r = s + t$ and $r + t + s = 30 + 2x$. Determine r.
- We know that $x = 6$ is a solution of $(x + 1)^3 + x = 349$. How can you then find a solution of $(5x + 1)^3 + 5x = 349$?

There were a total of eight such items and the resulting scale has good internal consistency (Cronbach $\alpha = 0.72$). The other scales measured in the test could also be measured reliably. It was shown that the ability to substitute correlates quite highly (r = 0.54) with performance in the rest of the algebra test. Overall, it can be seen that the ability to substitute is a central component of algebraic competence: Implication analyses (SIA, see Gras et al., 2008) also show that the substitution scale implies solving equations, setting up equations and working with functions. This means that in the statistical cross-section, learners who successfully solve substitution tasks are also successful in the other areas (but not necessarily vice versa).

As this was not an intervention study, causality cannot be considered proven, but it does suggest that substitution is a crucial sub-skill in elementary algebra. This result was confirmed in several test repetitions. In a follow-up study with 171 10th grade students, it was further shown that a high level of competence in substitution predicts high algebra performance well, but especially if the ability to set up and interpret formulas is also high (Oldenburg, 2022). This sample also showed a high correlation between overall algebra performance and substitution ability.

Simsek et al. (2019) investigated how the different conceptions of the equal sign (operation sign, equality, substitution) according to Jones et al. (2012) are related to performance in an algebra test. To this end, they examined 57 14-16-year-old adolescents. First, similar to Jones et al. (2012), agreement with statements according to the three conceptions of the equals sign was tested. The rest of the test examined algebraic skills, particularly in dealing with linear equations. This showed that strong agreement with the interpretation of the equal sign as an operation sign correlates significantly negatively with algebra performance (r = -0.34), while the equality interpretation (r = 0.38) and especially the substitution interpretation (r = 0.46) correlate significantly positively with performance. This means that learners who see the role of equation for substitution are better at algebra overall, but it does not yet prove that explicit substitution instruction can improve algebra performance overall - but the suggestion that this is indeed the case is clearly strengthened. Furthermore, the authors found that the correlation of substitution with algebra performance is particularly high for tasks that require a conceptual understanding of algebra, while there is a lower correlation for tasks on pure calculation routines.

It was pointed out above that equations enable substitutions. It is therefore of interest whether learners recognize this meaning. Jones et al. (2012) were able to show that there are substantial differences between English and Chinese schoolchildren aged

11-12 in the interpretation of the equals sign. The children were presented with statements about the equal sign, e.g. "= means that the answer comes after" (operation sign), "= means that both sides have the same value" (equality) and "= means that the one on one side can replace the one on the other side" (substitution). It was found that the sentences on the operation sign were more agreed upon by English children than by Chinese children, the equality interpretation was about the same, but the sentences on substitution were more strongly agreed upon by the Chinese children. These differences do not prove the importance of the substitution concept, but they do show that different teaching can obviously make the interpretation of equations as substitution rules more difficult or easier.

Jones & Pratt (2012) have developed several digital learning environments that enable lower secondary school students (approx. 10 years) to recognize arithmetic relationships between numbers using substitutions. The tasks to be completed on a computer consisted of arithmetic calculations, such as 21 + 9 and a selection of equations, e.g. 21 = 11 + 10, 9 + 10 = 19, 11 + 9 = 20. The students had to choose the equations that made the calculation easier. The authors conclude from their observations that such activities promote relational understanding of the equals sign and can thus promote the transition to symbolic algebra.

It should be noted, however, that this study has also been criticized: Kieran and Martínez-Hernández (2022) claim that sameness and substitutability are exactly the same thing. Now there are also substitutions such as $x \to x + 1$ which are not supposed to express equality at all and which are therefore completely outside the dispute of Jones et al. (2012) and Kieran and Martínez-Hernández (2022). Indeed, as discussed at the end of Section 2, for broad areas of mathematics one can state that substitutability while preserving all (truth) values and equality are synonymous, and thus Kieran and Martínez-Hernández (2022) are right on a mathematical level, but I think they are wrong on a didactic level: Equality is a static concept, substitution is a dynamic process. Tall (2013) calls the connection between process and concept a procept - and that could be exactly what we have here. It would be a didactic mistake to underestimate the importance of processes in the learning process.

As is usual in research, each individual study can only shed light on one aspect, but the synopsis of the results shows a largely consistent picture: being able to see equations (also) as substitution rules and knowing how to carry out substitutions in a meaningful way is closely related to leadership in algebra as a whole.

5. Suggestions for teaching

The scientific evidence on teaching methods that particularly promote substitution is still sparse. Nevertheless, some recommendations can be derived from the existing studies and the logical structure of the subject. These are not meant to imply that there should be a separate teaching unit on substitution - such "knitting without wool" is never recommended - but that the role of substitution in the various algebraic work processes should be made clear. Before this can be done successfully, it is important to develop an understanding of term structures. [3]Only when the recursive structure of terms is clear can substitution be successful. This can be seen, for example, when learners are asked to answer the question of how the area of a rectangle increases when each side is extended by 1 and they have to find the substitution in the formula for the area of a rectangle $A = a \cdot b$ they make the substitution $\{a \to a + 1, b \to b + 1\}$ incorrectly with the result $A = a + 1 \cdot b + 1$. Such errors at the syntactical level can be avoided if the meaning of the formulas is made clear, in this case for example by a sketch of the rectangle and its enlargement. The aim of learning the algebraic formula language, however, means that work is done correctly even without such support. And even if the substitution is carried out correctly, it must be understood that substitutions are not always equivalent transformations of an equation and that the reference of A is different in both equations: In $A = a \cdot b$ is A stands for the content of the small rectangle, in $A = (a + 1) \cdot (b + 1)$ stands for the content of the large rectangle.

Malle has already provided a whole series of tasks to help learners understand the meaning of substitution (Malle, 1993, p. 259ff). He organizes his tasks according to pragmatic goals of what is to be achieved with the substitutions. These are briefly recapitulated here in the formulation of Malle (1993) with some examples of his own:

- Substitution makes it possible to describe facts with different levels of detail in general terms. [4]One example is the equation for the surface area of a cylinder, which can either be given in a rough, but well reflecting the structure of the cylinder as $S = 2D + M$ or in more detail as $S = 2\pi r^2 +$

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[3] Terms have a recursive structure, which means that you can, for example, insert any other term for each variable of a term. This becomes clear when the terms are represented as term trees: A variable is then a leaf of the tree and you can put any other term tree in its place. Another way to emphasize the recursive structure is to describe the syntax in the following form: A term is a number or a variable or a sequence of a term, an operator and another term, etc. In short: Term:=number | variable | (term) | term+term | term - term |...

[4] Here are S, D, M stands for the area of the entire surface, the base circle and the shell.

$2\pi rh$. You can switch between the two variants using the substitutions $\{D \to \pi r^2, M \to 2\pi rh\}$ or $\{\pi r^2 \to D, 2\pi rh \to M\}$.

- Substitution can be used to explain the application of rules (e.g. in the above example of the distributive law).
- Substitution can be used to break down complicated formulas into simple ones or to combine simple forms into more complicated ones. In particular, you can clearly express the quantities on which a complex function term depends. A separate example that cannot be found in Malle: For the lower or upper limit of the 95% confidence interval for estimating the probability of success for a Bernoulli chain with k successes with n trials can be found, for example, in a collection of formulas with the equation $p_{u,o} = \frac{k}{n} \pm \frac{1.96}{\sqrt{n}} \sqrt{\frac{k}{n} \cdot \left(1 - \frac{k}{n}\right)}$. By substituting $\frac{k}{n} \to \hat{p}$ the role of the estimated value can be recognized more clearly and the formula becomes clearer and therefore easier to understand: $p_{u,o} = \hat{p} \pm \frac{1.96}{\sqrt{n}} \sqrt{\hat{p} \cdot (1 - \hat{p})}$.
- You can eliminate variables from a system of formulas by substitution. Here, too, we will use our own example: In a right-angled triangle with the standard designations a, b for the lengths of the cathets and c for the length of the hypotenuse, the height h is drawn above the hypotenuse. This splits the hypotenuse into two sections: $c = p + q$. In addition, the height creates two smaller triangles, which are also right-angled, so that the relationships apply: $a^2 + b^2 = c^2, a^2 = h^2 + p^2, b^2 = h^2 + q^2$. If you eliminate the variables from these four equations a, b, c from these four equations by repeated substitution, it is quite easy to obtain $h^2 = pq$, i.e. the statement of the height theorem.
- By substituting, you can initially relate unrelated quantities to each other. Malle illustrates this with the following interesting task: Give a relationship between the area $A = \pi r^2$ and the circumference $U = 2\pi r$ of a circle. By rearranging the second equation to $r = \frac{U}{2\pi}$ results in a form that can be used in the first equation: $A = \frac{U^2}{4\pi}$.

This 30-year-old list has been repeated here to show that it is still relevant. In particular, it is now possible to see that these substitution functions can also be supported by computer algebra systems in mathematics lessons, and are becoming more important at the same time. A typical algebraic procedure for dealing with a problem situation consists of first introducing variables for all unknown quantities and systematically searching for all relationships between them. New relationships can then be derived from these relationships by substitution and, if necessary, the problem can be solved. This will now be illustrated using simple and complex tasks.

Substitution is particularly helpful when solving a problem step by step. A simple example: The profit of a company is to be calculated. In a first step, you can write down the simple equation $G = E - A$ for the profit calculated from income and expenditure and in the next step for E and A meaningful terms. In a simple case, for example, the revenue could simply be calculated from the number n the number of units sold and the selling price P of a unit: $E \to n \cdot P$.

Step-by-step substitution can also help to proceed in a targeted manner in complex situations. Unlike equations, substitutions express a direction of transformation. This helps, for example, with the correct calculation of quantities with physical units. A non-trivial example: What is the mass of the atmosphere? The weight of the air exerts a pressure of approx. $p = 10000\ \mathrm{Pa}$. The pressure is the force per area $p = \frac{F}{A}$ and the weight of a mass is $F = m \cdot g, g = 9.81\,\frac{\mathrm{N}}{\mathrm{kg}}$. These equations, meaningfully interpreted as substitutions, provide a chain of substitutions: $m = \frac{F}{g} = \frac{pA}{g} = \frac{p \cdot 4\pi r^2}{g} = \frac{100000\ \mathrm{Pa} \cdot 4\pi(6380\ \mathrm{km})^2}{\frac{9.81\ \mathrm{N}}{\mathrm{kg}}} = \frac{\frac{100000\ \mathrm{N}}{\mathrm{m}^2} \cdot 4\pi(6380 \cdot 1000\ \mathrm{m})^2}{\frac{9.81\ \mathrm{N}}{\mathrm{kg}}} = 5.2 \cdot 10^{18}\mathrm{kg}$. Only the last equals sign has been algebraically transformed and calculated, the equals signs in front of it result solely from substitutions. By substituting, you also avoid mental errors when converting the units. It should be noted that this calculation is not only of mathematical interest, but can also be used e.g. as a basis for calculating how much 1ppm is and how much CO2 corresponds to it.

Substitutions invite you to play. The Heron algorithm for approximating the square root of a consists of improving an approximate value x by replacing it with a substitution $x \to \frac{x + \frac{a}{x}}{2}$ substitution. Repeatedly applying this iteration by hand is tedious, but with a CAS it is child's play. In Mathematica, you simply write: H= x->(x+a/x)/2; (x/.H/.H/.H)/.x->1. 1 was inserted at the end as the first approximate value. The result is a term in the variable $a$which approximates the root function very well (it is a so-called Pade approximation). The corresponding solution in Geogebra requires a little more typing (the last line is necessary because a term in a cannot be plotted), but is otherwise easy, as shown in Figure 3.

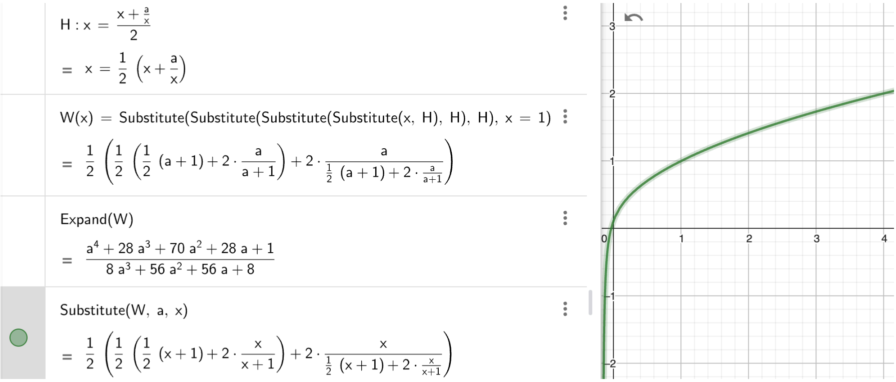

Figure 3: Iterated substitutions that approximate the root function

In the third section, it was pointed out that computer algebra systems are interesting with regard to substitutions. In Oldenburg (2023, and references therein), a curriculum design was proposed that intensively incorporates computer technology.

In the above example of solving the equation $(x+1)^4 + x^2 + 2x + 1 = 0$ above, it became obvious that recognizing common subexpressions is an essential prerequisite for clever substitutions. This aspect is also emphasized by Liang (2021). She argues, for example, that the meaning of the period in $x = 0,\overline{7} = 0{,}777\,...$ can be used productively if you can see that $x = 0,\overline{7} = 0{,}7 + 0{,}0\overline{7} = 0{,}7 + \frac{x}{10}$, therefore $10x = 7 + x$, or $x = \frac{7}{9}$.

But how can learners recognize what suitable substitutions are and what strategies can be used to teach them? As far as I know, there are no systematic didactic considerations on this, even if the role of substitution for structural understanding is discussed (Weigand et al, 2022, p. 103). However, the need to explicate strategies for finding substitutions arises when computers are to be programmed to do so. In this respect, it is not surprising that more detailed considerations on the topic can be found in mathematical computer science. Bundy (1983) lists a large number of strategies in connection with solving equations, many of which are also aimed at substitutions:

- In general, substitution is suitable when a term is contained several times in a larger term.
- Homogenization: to prepare for substitution, it can be useful to resolve calculations. For example, if a term contains $\boldsymbol{x^2}$ and $\boldsymbol{x^4}$ occurs in a term, it

makes sense to $\boldsymbol{x^4 = (x^2)^2}$ can be transformed and substituted. In general, it makes sense to make subterms as similar to each other as possible.

- Weigh up substitution and combination: If the same subterms occur, substitution is always an option, but sometimes also - if possible - combination, which reduces the number of occurrences of the same terms, e.g. : $\boldsymbol{\sin(A) \cdot \cos(A) \rightarrow \frac{1}{2} \cdot \sin(2A)}$.

6. Conclusion

The findings compiled here clearly demonstrate that substitution is a central algebraic operation. It should be reinforced in lessons through suitable exercises and treated as a strategy for solving algebraic problems. Computer algebra systems can be used for this in a variety of ways.

Unfortunately, there are still no scientifically systematically evaluated intervention studies that, in combination with performance measurement, investigate the extent to which targeted promotion of substitution skills increases overall algebra performance. However, this study shows how didactic analyses and empirical findings can work together to provide a coherent and differentiated picture from which recommendations for action can be derived. With regard to substitution, it can be concluded from the consistent results that it is an algebraic sub-skill whose importance is still often underestimated and whose targeted promotion can promote the learning of school algebra.

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