Bulletin of the Iranian Mathematical Society (2025) 51:2
https://doi.org/10.1007/s41980-024-00946-0

ORIGINAL PAPER

Control Semiflows, Chain Controllability, and the Selgrade Decomposition for Linear Delay Systems

Fritz Colonius[1]

Received: 15 August 2024 / Revised: 15 August 2024 / Accepted: 20 October 2024

Abstract
A continuous semiflow is introduced for linear control systems with delays in the states and controls and bounded control range. The state includes the control functions. It is proved that there exists a unique chain control set which corresponds to the chain recurrent set of the semiflow. The semiflow can be lifted to a linear semiflow on an infinite dimensional vector bundle with chain transitive base flow. A decomposition into exponentially separated subbundles is provided by a recent generalization of Selgrade's theorem.

Keywords Delay control system · Chain controllability · Chain transitivity · Selgrade decomposition · Poincaré sphere

Mathematics Subject Classification 93B05 · 93C23 · 37B20 · 34K35

1 Introduction

We will associate a control semiflow to linear systems with delays in the states and controls and study their generalized controllability properties. The considered systems are controlled (retarded) differential delay equations of the form

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{p} A_i x(t-h_i) + B_0 u(t) + \sum_{i=1}^{p} B_i u(t-h_i),\ u \in \mathcal{U},$$
$$x(0) = r,\ x(s) = f(s) \text{ for almost all } -h \le s \le 0 \text{ and } u(t) = 0 \text{ for } t \le 0. \quad (1.1)$$

Communicated by Majid Gazor.

✉ Fritz Colonius
fritz.colonius@uni-a.de

1 Institut für Mathematik, Universität Augsburg, Augsburg, Germany

Here, $A_0, \dots, A_p \in \mathbb{R}^{n\times n}$, $B_0, \dots, B_p \in \mathbb{R}^{n\times m}$, $0 =: h_0 < h_1 < \cdots < h_p =: h$, $r \in \mathbb{R}^n$, $f \in L^2([-h, 0], \mathbb{R}^n)$, and the set $\mathcal{U}$ of admissible control functions is given by

$$\mathcal{U} := \{u \in L^\infty(\mathbb{R}, \mathbb{R}^m) \,|u(t) \in \Omega \text{ for almost all } t \in \mathbb{R}\},$$

for a nonvoid compact and convex set $\Omega \subset \mathbb{R}^m$. The unique solutions $x(t) = \psi(t, r, f, u)$ are absolutely continuous on every interval $[0, T]$, $T > 0$.

A classical topic in control theory is approximate controllability for these systems, where the states at time $t \geq 0$ are $(x(t), x_t)$ with $x_t(s) := x(t+s)$, $s \in [-h, 0]$ in the state space,

$$M_2 := M_2([-h, 0], \mathbb{R}^n) = \mathbb{R}^n \times L^2([-h, 0], \mathbb{R}^n);$$

cf. Manitius [25], Curtain and Zwart [17], Bensoussan, Da Prato, Delfour, and Mitter [4, Chapter 4]. The problem is to determine when for given initial state $(x(0), x_0) = (r, f) \in M_2$ it follows that the reachable set,

$$\{(x(t), x_t) = (\psi(t, r, f, u), \psi_t(\cdot, r, f, u)) \in M_2 \mid t \geq 0 \text{ and } u \in \mathcal{U}\}$$

is dense in M_2. Recently it has found renewed interest by the contribution of Hinrichsen and Oeljeklaus [19]. They show (for systems without control constraints and without delays in the controls) that robustness of approximate controllability with respect to perturbations requires the assumption that $\operatorname{rank}(B_0, A_p B_0, \dots A_p^{n-1} B_0) = n$. We will analyze subsets of the state space where chain controllability holds, which is a weaker version of approximate controllability in infinite time (cf. Definition 5.1) and sometimes may be difficult to distinguish from it in numerical computations. It allows for small jumps in the trajectories and hence it is not a physical notion. In the theory of dynamical systems analogous constructions have been quite successful in order to describe the limit behavior as time tends to infinity for complicated flows; cf., e.g., Robinson [29], Alongi and Nelson [1]. For finite dimensional control systems the monographs Colonius and Kliemann [8] and Kawan [20] contain basic results on chain controllability; cf. also Da Silva and Kawan [14], Ayala, Da Silva, and San Martin [3] and Da Silva [13].

For control system (1.1), we construct a continuous affine semiflow Φ on the infinite dimensional vector bundle $\mathcal{U} \times M_2$ in the form

$$\Phi_t(u, r, f) = ((u(t + \cdot), (x(t), x_t)) \text{ for } t \geq 0, u \in \mathcal{U}, (r, f) \in M_2.$$

Here, $u(t + \cdot)(s) := u(t + s)$, $s \in \mathbb{R}$ is the right shift and $\mathcal{U}$ is endowed with a metric compatible with the weak* topology of $L^\infty(\mathbb{R}, \mathbb{R}^m)$. This generalizes control flows for finite dimensional systems, cf. [8] and [20].

General background on skew product flows (with an emphasis on finite dimensional systems) is provided by Kloeden and Rasmussen [22]. For infinite dimensional systems see Hale [18], Sell and You [31]. Some results on chain recurrence for infinite dimensional linear dynamics in discrete time are presented in Antunez, Mantovani, and Varão [2].

The main results of the present paper are the following. Under an injectivity assumption, Theorem 4.2 shows that Φ is a continuous semiflow on $\mathcal{U} \times M_2$. By Theorem 5.4 there exists a unique maximal subset of chain controllability, i.e., a chain control set E, in M_2. The chain control set corresponds to the unique maximal chain transitive subset of the semiflow Φ; cf. Theorem 5.2. The affine semiflow Φ can be lifted to a linear semiflow Φ^1 on the extended space $\mathcal{U} \times M_2 \times \mathbb{R}$. By a theorem due to Blumenthal and Latushkin [5] the semiflow Φ^1 admits a Selgrade decomposition into exponentially separated subbundles and, equivalently, a Morse decomposition of the induced flow on the projective bundle $\mathcal{U} \times \mathbb{P}(M_2 \times \mathbb{R})$. This construction is related to the Poincaré sphere in the theory of nonlinear differential equations; cf. Remark 6.3. Finally, the special situation is analyzed, where the linear part of the control system is uniformly hyperbolic. This partially generalizes pertinent results of Colonius and Santana [11] and Colonius, Santana, and Viscovini [12] in finite dimensions. Kawan [21] presents a short review of uniformly hyperbolic finite dimensional control systems.

Concerning the construction of the control semiflow it is worth to mention that, for finite dimensional control systems, Desheng Li [23] developed an alternative approach based on differential inclusions, hence avoiding the explicit use of the space $\mathcal{U}$ of control functions. Here continuity of the trajectories with respect to the topology on $\mathcal{U}$ plays no role.

The contents of this paper are as follows: Sect. 2 introduces notation for linear semiflows on infinite dimensional vector bundles and cites a result by Blumenthal and Latushkin on generalized Selgrade decompositions for these systems. In Sect. 3 properties of delay equations and their state space description in M_2 are recalled. Section 4 constructs the control semiflow for injective delay control systems. Section 5 characterizes chain controllable sets by their lifts to chain transitive subsets of the control semiflow. Furthermore, it is shown that there always exists a unique chain control set in M_2. Section 6 extends the affine delay control system to a linear delay control system on the state space $M_2 \times \mathbb{R}$ and applies the generalized Selgrade theorem from Sect. 2. Furthermore, conjugation properties to subsets of the projective bundle are shown. Finally, Sect. 7 considers the special case of uniformly hyperbolic systems. Here the affine delay control system is conjugate to its linear part, and, for the chain control sets in M_2, stronger results can be obtained.

Notation: A semiflow on a metric space X with metric d is a continuous map $\psi : [0, \infty) \times X \to X$ with $\psi(0, x) = x$ and $\psi(t + s, x) = \psi(t, \psi(s, x))$ for $t, s \in [0, \infty)$ and $x \in X$. A subset $X' \subset X$ is forward invariant, if $\psi(t, x) \in X'$ for all $t \geq 0$ and $x \in X'$. For a Banach space Y the space of bounded linear operators on Y is denoted by $\mathcal{L}(Y)$.

2 Semiflows on Banach Bundles

This section presents important properties of linear semiflows on Banach bundles and formulates an infinite dimensional version of Selgrade's theorem due to Blumenthal and Latushkin [5].

Let B be a compact metric space with metric d_B and let Y be a real Banach space with norm $\|\cdot\|$. Let $\theta : \mathbb{R} \times B \to B$ be a continuous flow on B, i.e., $\theta(0, b) = b, \theta(t+s, b) = \theta(t, \theta(s, b))$ for $t, s \in \mathbb{R}$ and $b \in B$. A Banach bundle is given by $\mathcal{V} := B \times Y$. Consider a semiflow of injective linear operators over (B, θ) of the form,

$$\Phi : [0, \infty) \times B \times Y \to B \times Y, \Phi(t, b, y) = (\theta(t, b), \phi(t, b, s)), t \geq 0, b \in B, y \in Y,$$

such that the following hypotheses hold:

(H1) the projection $\pi_B : \mathcal{V} \to B$ satisfies $\pi_B \circ \Phi = \theta$.
(H2) For any $(t, b) \in [0, \infty) \times B$, the map $\Phi(t, b, \cdot) : y \mapsto \Phi(t, b, y)$ is a bounded, injective linear operator of the fibers $\{b\} \times Y \to \{\theta(t, b)\} \times Y$.
(H3) For each fixed $t \geq 0$, the map $b \mapsto \Phi(t, b, \cdot)$ is continuous in the operator norm topology on the space $\mathcal{L}(Y)$ of bounded linear operators on Y.
(H4) The mapping $[0, \infty) \times B : (t, b) \mapsto \Phi(t, b, \cdot)$, is continuous in the strong operator topology on $\mathcal{L}(Y)$, i.e., for all $y \in Y$ one has $\Phi(t_k, b_k, y) \to \Phi(t_0, b_0, y)$ if $(t_k, b_k) \to (t_0, b_0)$.

Where convenient, we will identify the fiber $\{b\} \times Y$ with Y and write $\Phi_t(b, v) = \Phi(t, b, v)$. Note the following consequence of these hypotheses.

Proposition 2.1 *Hypotheses (H1)-(H4) imply that $\Phi : [0, \infty) \times \mathcal{V} \to \mathcal{V}$ is a continuous mapping in the metric $d_{\mathcal{V}}$ on $\mathcal{V}$ given by*

$$d_{\mathcal{V}}((b_1, v_1), (b_2, v_2)) := \max(d_B(b_1, b_2), \|v_1 - v_2\|). \tag{2.1}$$

Proof By (H4) it follows that for every $y \in Y$ and t in a compact interval $I \subset [0, \infty)$ the set $\{\|\Phi_t(b, y)\|, t \in I, b \in B\}$ is bounded. Thus the uniform boundedness principle implies that $\{\|\Phi_t(b, \cdot)\|, t \in I, b \in B\}$ is bounded.

Let $(t_k, b_k, y_k) \to (t_0, b_0, v_0)$ in $[0, \infty) \times B \times Y$ and consider with the metric (2.1) $d_{\mathcal{V}}(\Phi(t_k, b_k, y_k), \Phi(t_0, b_0, y_0))$. Then $d_B(\theta(t_k, b_k), \theta(t_0, b_0)) \to 0$ by continuity of θ. For the second component one estimates

$$\begin{aligned} &\|\phi(t_k, b_k, y_k) - \phi(t_0, b_0, y_0)\| \\ &\quad \leq \|\phi(t_k, b_k, y_k) - \phi(t_k, b_k, y_0)\| + \|\phi(t_k, b_k, y_0) - \phi(t_0, b_0, y_0)\| \\ &\quad \leq \|\Phi(t_k, b_k, \cdot)\| \, ||y_k - y_0|| + \|\phi(t_k, b_k, y_0) - \phi(t_0, b_0, y_0)\|. \end{aligned}$$

Since the factors $\left\|\Phi_{t_k}(b_k, \cdot)\right\|$ remain bounded the first summand converges to 0. The second summand converges to 0 by (H4). □

We write $\mathbb{P}\mathcal{V}$ for the projective bundle $B \times \mathbb{P}Y$. Here $\mathbb{P}Y$ is the projective space of Y defined by $\mathbb{P}Y := (Y \setminus \{0\})/\sim$, where $v \sim w$ for $v, w \in Y \setminus \{0\}$ if $v = \lambda w$ for some

$\lambda \in \mathbb{R}\backslash\{0\}$. The metric on $\mathbb{P}\mathcal{V}$ is defined by

$$d_{\mathbb{P}\mathcal{V}}((b_1, v_1), (b_2, v_2)) := \max\{d_B(b_1, b_2), d_{\mathbb{P}}(v_1, v_2)\} \text{ with}$$
$$d_{\mathbb{P}}(v, w) := \min\left\{\frac{v}{\|v\|} - \frac{w}{\|w\|}, \frac{v}{\|v\|} + \frac{w}{\|w\|}\right\}. \tag{2.2}$$

Since the operators $\Phi_t(b, \cdot)$ are injective by (H2) the linear semiflow descends to the projectivized semiflow $\mathbb{P}\Phi : [0, \infty) \times \mathbb{P}\mathcal{V} \to \mathbb{P}\mathcal{V}$ which is continuous.

Recall the following definition from Blumenthal and Latushkin [5, Definitions 2.3 and 2.7].

Definition 2.2 An asymptotically compact attractor of a semiflow ψ on a metric space X is a compact forward invariant set $A \subset X$ such that for some $\varepsilon > 0$ the following properties hold:

(i) For some $S > 0$ we have that $\overline{\psi([S, \infty) \times B_\varepsilon(A))} \subset B_\varepsilon(A)$ and

$$A = \omega(B_\varepsilon(A)) := \{y \in X \,|\exists t_k \to \infty, \exists x_k \in B_\varepsilon(A) : \psi(t_k, x_k) \to y\};$$

(ii) for any sequence $t_k \to \infty$ and any sequence of points $x_k \in B_\varepsilon(A)$ it follows that $\psi(t_k, x_k)$, $k \in \mathbb{N}$, has a convergent subsequence.

The points which have pre-images will be relevant.

Definition 2.3 For a semiflow ψ on X such that $\psi(t, \cdot)$ is injective for all $t \geq 0$, a point $x \in X$ defines an entire solution, if for all $t > 0$ there is $y \in X$ with $\psi(t, y) = x$.

We slightly abuse notation and write $\psi(-t, x) \in X$ for the pre-image $\psi(t, \cdot)^{-1}(x)$; by injectivity, $\psi(-t, x)$ is a unique element of X when the pre-image exists. When we write $\psi(-t, x)$ we tacitly suppose that this pre-image exists. A set $Y \subset X$ is invariant if $\psi(t, x) \in Y$ for all $t \in \mathbb{R}$.

For $\varepsilon, \tau > 0$ an (ε, τ)-chain from x to y is given by $q \in \mathbb{N}$, $x_0 = x, x_1, \ldots, x_q = y$ in X, and $\tau_0, \ldots, \tau_{q-1} \geq \tau$ with

$$d(\psi(\tau_j, x_j), x_{j+1}) < \varepsilon \text{ for } j = 0, \ldots, q-1.$$

Definition 2.4 (i) A point $x \in X$ is chain recurrent for ψ, if for all $\varepsilon, \tau > 0$ there are (ε, τ)-chains from x to y. The chain recurrent set $\mathcal{R}$ is the set of all chain recurrent points. If $\psi(t, \cdot)$ is injective for all $t \geq 0$, the entire chain recurrent set $\mathcal{R}^\#$ is the set of all chain recurrent points $x \in X$ which define entire solutions in $\mathcal{R}$.

(ii) A nonvoid set $Y \subset X$ is chain transitive if for all $x, y \in Y$ and all $\varepsilon, \tau > 0$ there are (ε, τ)-chains from x to y.

Remark 2.5 For a semiflow with compact state space, it follows that through every point in the chain recurrent set $\mathcal{R}$ there exists an entire solution in $\mathcal{R}$, hence $\mathcal{R}^\# = \mathcal{R}$. This follows as in Bronstein and Kopanskii [6, Section 8]; cf. also Li [23, Proposition 2.5].

For a linear semiflow $\Phi = (\theta, \phi)$ on $\mathcal{V} = B \times Y$ consider two continuously varying, forward invariant subbundles with $\mathcal{V} = \mathcal{E} \oplus \mathcal{F}$ for which $\dim \mathcal{E} < \infty$. The subbundles $\mathcal{E}$ and $\mathcal{F}$ are exponentially separated if there exist constants $K, \gamma > 0$ such that for all fibers $\mathcal{E}_b$ and $\mathcal{F}_b$, $b \in B$,

$$|\Phi_t(b,\cdot)|_{\mathcal{F}_b} \leq K e^{-\gamma t} m(\Phi_t(b,\cdot))_{\mathcal{E}_b} \text{ for all } t > 0,$$

where m is the minimum norm, $m(\Phi_t(b,\cdot))_{\mathcal{E}_b} := \min \{|\phi_t(b,y)|, y \in \mathcal{E}_b\}$.

The following theorem holds by [5, Theorems A and B and Corollary 1.4]. The shorthand $1 \leq i < N+1$ means that if $N = \infty$, then $i \in \mathbb{N}$, and if $N < \infty$, then $1 \leq i \leq N$.

Theorem 2.6 *Assume that Y is a separable Banach space and that B is chain transitive for the base flow θ. Let Φ be a linear semiflow on $\mathcal{V} = B \times Y$ satisfying hypotheses (H1)–(H4) as above. Then there is an at-most countable sequence $\{\mathcal{A}_i\}_{i=0}^N$, $N \in \{0, 1, \ldots\} \cup \{\infty\}$, of subsets of $\mathbb{P}\mathcal{V}$ with $\mathcal{A}_0 = \varnothing$, $\mathcal{A}_i \subset \mathcal{A}_{i+1}$ for $1 \leq i < N$, with the following properties, for any $1 \leq i < N+1$:*

(i) *The set $\mathcal{A}_i$ is an asymptotically compact attractor for $\mathbb{P}\Phi$.*
(ii) *The sequence $\{\mathcal{A}_i\}$ is the finest such collection in the following sense: If $\mathcal{A}$ is any nonempty asymptotically compact attractor for $\mathbb{P}\Phi$, then $\mathcal{A} = \mathcal{A}_i$ for some $1 \leq i < N+1$.*
(iii) *For the finite dimensional subbundles $\mathcal{V}_i^+ = \mathbb{P}^{-1}\mathcal{A}_i$, $i \in \{1, \ldots, N+1\}$, there are subbundles $\mathcal{V}_i^-$ such that $\mathcal{V} = B \times Y = \mathcal{V}_i^+ \oplus \mathcal{V}_i^-$ is an exponentially separated splitting of $\mathcal{V}$.*
(iv) *The subbundles $\mathcal{V}_i := \mathcal{V}_i^+ \cap \mathcal{V}_{i-1}^-$ are finite dimensional, invariant subbundles of $\mathcal{V}$ such that*

$$\mathcal{V}_i^+ = \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_i.$$

(v) *The sets $\mathcal{M}_i = \mathbb{P}\mathcal{V}_i$ are maximal chain transitive for the projectivized flow $\mathbb{P}\Phi$ restricted to $\mathbb{P}\mathcal{V}_i^+$.*

The subbundles $\{\mathcal{V}_i\}_{i=1}^N$ are called the discrete Selgrade decomposition of Φ.

Remark 2.7 For every i with $1 \leq i < N+1$ the sets $\mathcal{M}_j = \mathbb{P}\mathcal{V}_j$, $1 \leq j \leq i$, are the chain recurrent components of the flow $\mathbb{P}\Phi$ restricted to $\mathbb{P}\mathcal{V}_i^+$. Since the $\mathcal{V}_j$ are finite dimensional they are linearly ordered by $\mathcal{M}_i \preceq \mathcal{M}_j$ for $i \leq j$ in the order of Morse sets on compact metric spaces X, cf. Colonius and Kliemann [8, Proposition B.2.8]): $\mathcal{M}_i \preceq \mathcal{M}_j$ if there are $\mathcal{M}_{j_0} = \mathcal{M}_i, \mathcal{M}_{j_1}, \ldots, \mathcal{M}_{j_l} = \mathcal{M}_j$ with $j \leq j_1, \ldots, j_l = i$ and $x_1, \ldots, x_{j_l} \in X$ with $\omega^*(x_k) \subset \mathcal{M}_{j_{k-1}}$ and $\omega(x_k) \subset \mathcal{M}_{j_k}$ for $k = 1, \ldots, l$. Thus the subbundle $\mathcal{V}_1$ is the most unstable subbundle (this is opposite to the numbering in Colonius and Kliemann [8, Theorem 5.1.4]).

3 Delay Equations

In this section we consider linear control systems described by delay equations of the form (1.1). The underlying field is either $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

The solutions $x(t) = \psi(t, r, f, 0)$ of the homogeneous equation

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{p} A_i x(t - h_i) \tag{3.1}$$

satisfy (cf. Curtain and Zwart [17, Theorem 3.3.1])

$$x(t) = e^{A_0 t} r + \sum_{i=1}^{p} \int_0^t e^{A_0(t-s)} A_i x(s - h_i) ds \text{ for } t \geq 0.$$

For $\tau > 0$, there are constants $C_\tau, D_\tau > 0$ such that, for $t \in [0, \tau]$, the following estimates hold:

$$\|x(t)\|^2 \leq C_\tau \left[\|r\|^2 + \|f\|^2_{L^2([-h,0],\mathbb{K}^n)} \right], \tag{3.2}$$

$$\int_0^t \|x(\tau)\|^2 d\tau \leq D_\tau \left[\|r\|^2 + \|f\|^2_{L^2([-h,0],\mathbb{K}^n)} \right]. \tag{3.3}$$

This follows from the proof of [17, Lemma 3.3.3]. Arguing as in [17, Theorem 3.3.1] one finds that the solution $x(t) = \psi(t, r, f, u)$ of the inhomogeneous Eq. (1.1) satisfies

$$\begin{aligned} x(t) = \ & e^{A_0 t} r + \int_0^t e^{A_0(t-s)} \left[\sum_{i=1}^{p} A_i x(s - h_i) \right. \\ & \left. + \sum_{i=0}^{p} B_i u(s - h_i) \right] ds \text{ for } t \geq 0. \end{aligned} \tag{3.4}$$

We also recall the variation-of-parameters formula (cf. Hale [18, Chapter 6, Theorem 2.1 and Corollary 2.1 on pp. 143-145], Delfour [16, Theorem 1.2]). Let $X(t), t \geq 0$, be the $n \times n$-matrix solution of

$$\frac{d}{dt} X(t) = A_0 X(t) + \sum_{i=1}^{p} A_i X(t - h_i) \text{ with } X(t) = 0 \text{ for } t \in [-h, 0), X(0) = I_n.$$

Then the solution of the inhomogeneous Eq. (1.1) with initial value $(r, f) \in \mathbb{R}^n \times L^2([-h, 0], \mathbb{R}^n)$ is given by

$$\begin{aligned} \psi(t, r, f, u) &= \psi(t, r, f, 0) + \int_0^t X(t - s) \sum_{i=0}^{p} B_i u(s - h_i) ds \\ &= \psi(t, r, f, 0) + \psi(t, 0, 0, u). \end{aligned} \tag{3.5}$$

In particular, this shows that the solutions can be split into the homogeneous and inhomogeneous parts.

Next we recall some facts on the description of these systems in the separable Hilbert space $M_2 := M_2([-h,0],\mathbb{K}^n) = \mathbb{K}^n \times L^2([-h,0],\mathbb{K}^n)$; cf. Curtain and Zwart [17, Example 5.1.12]. With the state $y(t) = (x(t), x_t)^\top \in M_2$ the homogeneous Eq. (3.1) can be reformulated as

$$\dot{y}(t) = Ay(t),\, y(0) = y_0 = \begin{pmatrix} r \\ f \end{pmatrix},\ A\begin{pmatrix} r \\ f \end{pmatrix} := \begin{pmatrix} A_0 r + \sum_{i=1}^{p} A_i f(-h_i) \\ \frac{df}{ds}(\cdot) \end{pmatrix}. \tag{3.6}$$

Here A is the infinitesimal generator of a strongly continuous semigroup $T(t), t \geq 0$, and the domain of definition of A is

$$D(A) = \left\{ (r, f)^\top \,\middle|\, f \in W^{1,2}([-h,0],\mathbb{K}^n) \text{ and } f(0) = r \right\}.$$

The inhomogeneous Eq. (1.1) can be reformulated as

$$\dot{y}(t) = Ay(t) + B\left(u(t), u(t-h_1), \ldots, u(t-h_p)\right),\ y(0) = y_0 = (r, f)^\top, \tag{3.7}$$

where

$$B : \mathbb{R}^{m(p+1)} \to M_2,\, B(u_0, u_1, \ldots, u_p) := \left(\sum_{i=0}^{p} B_i u_i, 0\right)^\top,$$

is a bounded linear operator. The mild solution of (3.7) is $y(t) = \varphi(t, y_0, u)$ given by

$$y(t) = T(t)y_0 + \int_0^t T(t-s)B\left(u(s), u(s-h_1), \ldots, u(s-h_p)\right) ds. \tag{3.8}$$

It is related to the solution of (1.1) by $y(t) = (x(t), x_t)^\top$. Note that

$$\varphi(t, y_0, u) = \varphi(t, y_0, 0) + \varphi(t, 0, u) = T(t)y_0 + \varphi(t, 0, u) \in M_2, t \geq 0. \tag{3.9}$$

Since mild solutions are strongly continuous in t the map $t \mapsto \varphi(t, y_0, u)$ is continuous for all (y_0, u) and $\varphi(t, y_0, u) \in D(A)$ for $t \geq h$. This follows since the solutions $x(t)$ of (3.4) are absolutely continuous with derivatives in L^2 on any compact interval.

We note the following spectral properties of linear delay equations referring to Curtain and Zwart [17], Manitius [24], and also to Diekmann, van Gils, Verduyn Lunel, and Walther [15, Chapter 5].

The infinitesimal generator A has a pure point spectrum $\sigma(A)$ consisting of the (countably infinite or finitely many) zeros of $\Delta(s) = \det(sI_n - \sum_{i=0}^{p} A_i e^{-h_i s})$. For every $\mu \in \sigma(A)$ the generalized eigenspace is finite dimensional. The span of the generalized eigenvectors of A is dense in $M_2([-h,0],\mathbb{C}^n)$ if and only if $\det A_p \neq 0$ (cf. [24, Corollary 5.5], [17, Theorem 3.4.4]). The same is true for the span of the real generalized eigenspaces $E(\mu)$, $\mu \in \sigma(A)$, in $M_2([-h,0],\mathbb{R}^n)$.

The linear delay equation is (uniformly) hyperbolic if $\sigma(A) \cap \imath\mathbb{R} = \varnothing$, i.e. if $\Delta(\imath x) \neq 0$ for all $x \in \mathbb{R}$. A consequence of hyperbolicity is the following spectral decomposition into a finite dimensional subspace V^+ and a stable subspace V^-; cf. the discussion in Curtain and Zwart [17, Theorem 8.2.5].

Theorem 3.1 *If the linear semiflow $T(t), t \geq 0$ is hyperbolic, it admits a decomposition $M_2 = V^+ \oplus V^-$ into $T(\cdot)$-invariant closed subspaces V^+ and V^- with* $\dim V^+ < \infty$ *and constants $\alpha, K > 0$ such for all $t \geq 0$*

$$\left\|T(t)y^-\right\| \leq Ke^{-\alpha t}\left\|y^-\right\| \text{ for } y^- \in V^- \text{ and } \left\|T(t)y^+\right\| \geq Ke^{\alpha t}\left\|y^+\right\| \text{ for } y^+ \in V^+.$$

Remark 3.2 Since V^+ is finite dimensional and 0 is not in the spectrum of $T(t)$, it follows that the restriction of $T(t)$ to V^+ is an isomorphism and we can define $T(-t) := T(t)^{-1}, t \geq 0$. The condition above is equivalent to

$$\left\|T(t)z^+\right\| \leq K^{-1}e^{\alpha t}\left\|z^+\right\| \text{ for } t \leq 0 \text{ and } z^+ \in V^+.$$

Indeed, one may write $y^+ = T(-t)z^+$ with $z^+ \in V^+$ and it follows that

$$\left\|z^+\right\| \geq Ke^{\alpha t}\left\|T(-t)z^+\right\| \text{ for } t \geq 0.$$

The following result is a special case of Theorem 2.6.

Theorem 3.3 *Consider the linear semigroup $T(t), t \geq 0$ and assume that $T(t)$ is injective for $t \geq 0$. Order the eigenvalues according to their real parts* $\operatorname{Re}\mu_j$ *in decreasing order and define for $i \in \mathbb{N}$ subspaces of M_2 by*

$$V_i = \bigoplus_{\operatorname{Re}\mu_j = \lambda_i} E(\mu_j),\ V_i^+ = \bigoplus_{\operatorname{Re}\mu_j \geq \lambda_i} E(\mu_j), \text{ and } V_i^- = \overline{\bigoplus_{\operatorname{Re}\mu_j < \lambda_i} E(\mu_j)}.$$

Then $V_i^+ = V_1 \oplus \cdots \oplus V_i$ and it follows that $V_i = V_i^+ \cap V_{i-1}^-$ are finite dimensional, forward invariant subspaces of M_2 and $M_2 = V_i^+ \oplus V_i^-$ is an exponentially separated splitting. Furthermore the following properties hold:

(i) *For any $1 \leq i < N^0 := \infty$ the set $A_i^0 := \mathbb{P}V_i^+$ is an asymptotically compact attractor for the projectivized semiflow $\mathbb{P}T(\cdot)$.*
(ii) *Any nonempty asymptotically compact attractor for $\mathbb{P}T(\cdot)$ coincides with some $A_i^0, i \in \mathbb{N}$.*
(iii) *For every $i \in \mathbb{N}$ the set $\mathcal{M}_i = \mathbb{P}V_i$ is maximal chain transitive for the projectivized flow $\mathbb{P}T(\cdot)$ restricted to $\mathbb{P}V_i^+$.*

Proof The linear semigroup $T(t), t \geq 0$, can be considered as a linear semiflow on a vector bundle with trivial base space. Hypotheses (H1)-(H4) of Theorem 2.6 hold by the definitions and strong continuity of $T(\cdot)$. □

Remark 3.4 For the system without control restriction and without delays in the control, the reachable subspace from the origin is dense in $M_2([-h, 0], \mathbb{C}^n)$ if and only if rank $[\Delta(s), B_0] = n$ for all $s \in \mathbb{C}$ and rank $\left[A_p, B_0\right] = n$ (Curtain and Zwart [17, Theorem 6.3.13]).

We note the following injectivity property of delay equations.

Proposition 3.5 *Consider the linear delay control system (1.1). Then for all $t \geq 0$ the maps $T(t)$ are injective if and only if the matrix A_p is invertible.*

Proof If A_p is not invertible, we may choose $f \in L^2([-h, 0], \mathbb{R}^n)$ with $0 \neq f(s) \in \ker A_p$ for $s \in [-h, -h_{p-1})$ and $f(s) = 0$ for $s \in [-h_{p-1}, 0]$. Then the solutions satisfy $\psi(t, r, f, u) = \psi(t, r, 0, u)$ for $t > 0$ and hence $T(t)$ is not injective.

Conversely, suppose that A_p is invertible. Let $H = h - h_{p-1} = h_p - h_{p-1}$. Since for $t \geq 0$ there are $k \in \{0, 1, \ldots\}$ and $\tau \in [0, H)$ with $t = \tau + kH$, it follows that $T(t) = T(\tau) \circ T(kH)$. This shows that it suffices to prove injectivity of $T(\tau)$ for $\tau \in (0, H]$. We have to prove that $\varphi(\tau, r, f, u) = \varphi(\tau, r', f', u)$ implies $(r, f) = (r', f')$ or, equivalently, that

$$\psi(\tau + s, r, f, 0) = 0, s \in [-h, 0], \text{ implies } r = 0 \text{ and } f = 0. \tag{3.10}$$

Evaluation at $\tau + s = 0$ shows that $r = 0$. In order to show that $f = 0$ in $L^2([-h, 0], \mathbb{R}^n)$, note that, by (3.10), $f(s) = 0$ for $s \in [\tau - h, 0]$ holds. It remains to prove that $f(s) = 0$ for (almost all) $s \in [-h, \tau - h]$. We know that $x(t) = \psi(t, 0, f, 0) = 0$ for $t \in [0, \tau]$, hence it follows that for $t \in [0, \tau]$

$$f(t-h) = x(t-h) = A_p^{-1}\left[\dot{x}(t) - \sum_{i=0}^{p-1} A_i x(t-h_i)\right] = -A_p^{-1}\sum_{i=1}^{p-1} A_i x(t-h_i).$$

Since $\tau \in [0, H] = [0, h - h_{p-1}]$ it follows that $\tau - h \leq -h_{p-1}$. Thus, it holds that. for all $i = 1, \ldots, p-1$ and for $t \geq 0$, that $t - h_i \geq t - h_{p-1} \geq -h_{p-1} \geq \tau - h$ and hence $x(t - h_i) = 0$ for $t - h \in [-h, \tau - h]$. It follows that

$$f(t-h) = x(t-h) = 0 \text{ for } t - h \in [-h, \tau - h],$$

showing that $f(s) = 0$ for $s \in [-h, \tau - h]$. □

Remark 3.6 Injectivity of the operators $T(t)$ is equivalent to the property that the structural operator F of the homogeneous delay differential Eq. (3.1) has a trivial kernel; cf. Manitius [24].

4 The Control Semiflow

This section constructs a continuous semiflow associated with the delay control system (1.1) and analyzes some of its properties.

Endow the set $\mathcal{U}$ of controls with the following metric which is compatible with the weak* topology on $L^\infty(\mathbb{R}, \mathbb{R}^m)$:

$$d_{\mathcal{U}}(u, v) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\mid \int_{\mathbb{R}} (u(t) - v(t))^\top z_k(t) dt \mid}{1 + \mid \int_{\mathbb{R}} (u(t) - v(t))^\top z_k(t) dt \mid}, \tag{4.1}$$

where $\{z_k,\ k \in \mathbb{N}\}$ is a dense subset of $L^1(\mathbb{R},\mathbb{R}^m)$. With this metric, $\mathcal{U}$ is a compact separable metric space and the right shift $\theta : \mathbb{R} \times \mathcal{U} \to \mathcal{U} : \theta_t u := u(t + \cdot)$, $t \in \mathbb{R}$, is a continuous flow that is chain transitive on $\mathcal{U}$; cf. Kawan [20, Proposition 1.9]. The map

$$\Phi : \mathbb{R} \times \mathcal{U} \times M_2 \to \mathcal{U} \times M_2,\ \Phi_t\,(u, y_0) = (\theta_t u, y(t)) = (\theta_t u, \varphi(t, y_0, u))\,, \tag{4.2}$$

satisfies the semigroup properties $\Phi_0\,(u, y_0) = (u, y_0)$ and for $t, s \geq 0$

$$\begin{aligned}\Phi_{t+s}\,(u, y_0) &= (\theta_{t+s} u, \varphi(t + s, y_0, u)) = (\theta_t(\theta_s u), \varphi(t, \varphi(s, y_0, u), u(s + \cdot)) \\ &= \Phi_t \circ \Phi_s\,(u, y_0)\,.\end{aligned}$$

We will show that Φ is a continuous semiflow, called a control semiflow. Observe that Φ is not linear in the second argument, since the maps $y_0 \mapsto \varphi(t, y_0, u)$ are affine; cf. (3.9). The homogeneous part of Φ is a product flow and we denote it by

$$\Phi^0_t(u, y_0) = (\theta_t u, \varphi(t, y_0, 0)) = (\theta_t u, T(t)y_0) \text{ for } t \geq 0, u \in \mathcal{U}, y_0 \in M_2. \tag{4.3}$$

First we show properties of the homogeneous part Φ^0.

Proposition 4.1 *The homogeneous part Φ^0 of the control semiflow $\Phi : [0,\infty) \times \mathcal{U} \times M_2 \to \mathcal{U} \times M_2$ satisfies the following properties:*

(i) *The map $(t, u) \mapsto \Phi^0(t, u, \cdot) : [0,\infty) \times \mathcal{U} \to \mathcal{U} \times \mathcal{L}(M_2)$ is continuous with the strong operator topology on the space $\mathcal{L}(M_2)$ of bounded linear operators on M_2.*

(ii) *For each fixed $t \geq 0$, the map*

$$\mathcal{U} \to \mathcal{U} \times \mathcal{L}(M_2) : u \mapsto \Phi^0_t(u, \cdot) = (\theta_t u, T(t)), u \in \mathcal{U},$$

is continuous with the operator norm topology on $\mathcal{L}(M_2)$.

(iii) *The semiflow Φ^0 is continuous.*

Proof (i) Let $y \in M_2$. We have to show that, for a convergent sequence $(t_k, u^k) \to (t_0, u^0)$, it follows that

$$\Phi^0_{t_k}(u^k, y) = (\theta_{t_k} u^k, T(t_k)y)) \to \Phi^0_{t_0}(u^0, y) = (\theta_{t_0} u^0, T(t_0)y).$$

Convergence in the first component holds by continuity of the right shift θ, and $T(t_k)y$ converges to $T(t^0)y$ by strong continuity of the semigroup $T(t)$, $t \geq 0$.

(ii) Convergence in the first component follows as in (i) and, by (4.3), $\left\|\Phi^0_t(u, \cdot)\right\| = \|T(t)\|$ does not depend on the control u, hence continuity trivially holds.

(iii) Due to assertions (i) and (ii) hypotheses (H1)-(H4) hold for Φ^0, except for injectivity of $\Phi(t, u, \cdot)$. Since Proposition 2.1 does not need the injectivity assumption, this implies that Φ^0 is continuous.

□

The following theorem establishes continuity of the affine control semiflow Φ.

Theorem 4.2 *The control semiflow $\Phi : [0, \infty) \times \mathcal{U} \times M_2 \to \mathcal{U} \times M_2$ defined in (4.2) satisfies the following properties:*

(i) *For every $y \in M_2$, the map $(t, u) \mapsto \Phi(t, u, y) = (\theta_t u, \varphi(t, y, u)) : [0, \infty) \times \mathcal{U} \to \mathcal{U} \times M_2$ is continuous.*

(ii) *Let $t \geq 0$. Then $u^k \to u^0$ in $\mathcal{U}$ implies that*

$$\sup_{\|y\| \leq 1} \left\| \varphi(t, y, u^k) - \varphi(t, y, u^0) \right\| \to 0.$$

(iii) *The semiflow Φ is continuous.*

Proof (i) Let $y \in M_2$. We have to show that for a convergent sequence $(t_k, u^k) \to (t_0, u^0)$ it follows that

$$\Phi_{t_k}(u^k, y) = (\theta_{t_k} u^k, \varphi(t_k, y, u^k)) \to \Phi_{t_0}(u^0, y) = (\theta_{t_0} u^0, \varphi(t_0, y, u^0)).$$

Convergence in the first component holds by continuity of the right shift θ. Concerning the second component, formula (3.8) shows that,

$$\begin{aligned} \varphi(t_k, y, u^k) &= T(t_k)y + \int_0^{t_k} T(t_k - s)B\left(u^k(s), u^k(s - h_1), \ldots, u^k(s - h_p)\right) ds \\ &= \varphi(t_k, y, 0) + \varphi(t_k, 0, u^k). \end{aligned}$$

By Proposition 4.1(i) it follows that

$$(\theta_{t_k} u^k, \varphi(t_k, y, 0)) \to \Phi^0_{t_0}(u^0, y) = (\theta_{t_0} u^0, \varphi(t_0, y, 0)).$$

The other summand in the second component is $\varphi(t_k, 0, u^k) = (x^k(t_k), x^k_{t_k})$, where $x^k(t) = \psi(t, 0, 0, u^k)$, $t \geq -h$ is the solution of

$$\dot{x}^k(t) = A_0 x^k(t) + \sum_{i=1}^{p} A_i x^k(t - h_i) + \sum_{i=0}^{p} B_i u^k(t - h_i), t \geq 0,$$

with initial condition $x^k(t) = 0$ and $u^k(t) = 0$ for $t \in [-h, 0]$.
Let $\tau > 0$. We claim that $\psi(t, 0, 0, u^k) \to \psi(t, 0, 0, u^0)$ uniformly for $t \in [0, \tau]$.
The variation-of-parameters formula (3.5) yields

$$\psi(t, 0, 0, u^k) = \int_0^t X(t - s) \sum_{i=0}^{p} B_i u^k(s - h_i) ds.$$

Since Ω is compact there exists $c_0 > 0$ such that $\left\| \psi(t, 0, 0, u^k) \right\| \le c_0, t \in [0, \tau]$ for all k. Furthermore, for $t_1 < t_2$ in $[0, \tau]$,

$$\left\| \psi(t_2, 0, 0, u^k) - \psi(t_1, 0, 0, u^k) \right\| \le \int_{t_1}^{t_2} \left[\|X(t-s)\| \sum_{i=0}^{p} \|B_i\| \left\| u^k(s-h_i) \right\| \right] ds$$
$$\le (t_2 - t_1)(p+1) \max_{s \in [0,\tau]} \|X(s)\| \max_i \|B_i\| \max_{u \in \Omega} \|u\| .$$

Thus, the functions $x^k(t) = \psi(t, 0, 0, u^k), t \in [0, \tau]$, are equicontinuous and bounded and hence, by the Arzela-Ascoli theorem, a uniformly converging subsequence exists.
Taking into account $u(s) = 0$ for $s \le 0$, weak* convergence of u^k to u^0 implies that, for $t \in [0, \tau]$ and $i = 0, 1, \ldots, p$,

$$\int_0^t X(t-s) B_i u^k(s-h_i) ds$$
$$= \int_{-h_i}^{t-h_i} X(t-s-h_i) B_i u^k(s) ds = \int_0^{\infty} \chi_{[-h_i, t-h_i]}(s) X(t-s-h_i) B_i u^k(s) ds$$
$$\longrightarrow \int_0^{\infty} \chi_{[-h_i, t-h_i]}(s) X(t-s-h_i) B_i u^0(s) ds = \int_0^t X(t-s) B_i u^0(s-h_i) ds.$$

It follows that $x^k(t) = \psi(t, 0, 0, u^k)$ converges to $x^0(t) = \psi(t, 0, 0, u^0)$. By the preceding argument, this convergence is uniform for $t \in [0, \tau]$.
We conclude that $x_{t_k}^k$ converges to $x_{t^0}^0$ in $C([-h, 0], \mathbb{R}^n)$. This implies convergence of $(x^k(t_k), x_{t_k}^k)$ to $(x^0(t^0), x_{t_0}^0)$ in M_2 since the embedding of $C([-h, 0], \mathbb{R}^n)$ into M_2 is continuous.

(ii) Let $t > 0$ and consider $u^k \to u^0$ in $\mathcal{U}$. By formula (3.9) it follows that

$$\sup \left\{ \left\| \varphi(t, y, u^k) - \varphi(t, y, u^0) \right\| \,|\, \|y\| \le 1 \right\} = \left\| \varphi(t, 0, u^k) - \varphi(t, 0, u^0) \right\| .$$

Thus, this does not depend on y, and assertion (i) implies that the right hand side converges to 0 for $k \to \infty$, as claimed.

(iii) For a sequence $(t_k, u^k, y_k) \to (t_0, u^0, y_0)$ in $[0, \infty) \times \mathcal{U} \times M_2$, one obtains that

$$\left\| \varphi(t_k, y_k, u^k) - \varphi(t_0, y_0, u^0) \right\|$$
$$\le \left\| \varphi(t_k, y_k, u^k) - \varphi(t_k, y_0, u^k) \right\| + \left\| \varphi(t_k, y_0, u^k) - \varphi(t_0, y_0, u^0) \right\|$$
$$= \|\varphi(t, y_k, 0) - \varphi(t, y_0, 0)\| + \left\| \varphi(t_k, 0, u^k) - \varphi(t_0, 0, u^0) \right\| .$$

The first summand equals $\|T(t)(y_k - y_0)\| \le \sup_k \|T(t)\| \, \|y_k - y_0\|$ and hence converges to 0. The second summand converges to 0 by (i).

□

The following proposition characterizes injectivity of the semiflow associated with linear delay control systems.

Proposition 4.3 *Consider the linear delay control system (1.1). Then for all $t > 0$ the maps Φ_t on $\mathcal{U} \times M_2$ are injective if and only if the matrix A_p is invertible.*

Proof First we note that for $t > 0$ the map Φ_t is injective if and only if the maps $\Phi_t(u, \cdot), u \in \mathcal{U}$, are injective. In fact, suppose that for all $u \in \mathcal{U}$ the map $\Phi_t(u, \cdot)$ is injective. Then $\Phi_t(u, y) = \Phi_t(u', y')$ implies $u(t + \cdot) = u'(t + \cdot)$, hence $u = u'$ implying $y = y'$. This shows that Φ_t is injective on $\mathcal{U} \times M_2$. The converse holds trivially. Furthermore, the map $\Phi_t(u, \cdot)$ is injective if and only if $\Phi_t(0, \cdot) = T(t)$ is injective. By Proposition 3.5 this holds if and only if $\det A_p \neq 0$. □

Theorem 3.3 on spectral theory of delay equations entails the following consequences for the homogeneous part Φ^0 of Φ.

Proposition 4.4 *Consider the linear flow Φ^0 on $\mathcal{U} \times M_2$ given by (4.3) and assume that* $\det A_p \neq 0$. *Then the sequence $\{\mathcal{A}_i^0\}_{i=0}^N := \{\mathcal{U} \times A_i^0\}_{i=0}^N$ of subsets of $\mathcal{U} \times \mathbb{P}M_2$ has the following properties.*

(i) *For any $i \in \mathbb{N}$ the set $\mathcal{A}_i^0$ is an asymptotically compact attractor for the projectivized semiflow $\mathbb{P}\Phi^0$ on $\mathcal{U} \times \mathbb{P}M_2$.*
(ii) *If $\mathcal{A}^0$ is any nonempty asymptotically compact attractor for $\mathbb{P}\Phi^0$, then $\mathcal{A}^0 = \mathcal{A}_i^0$ for some $i \in \mathbb{N}$.*
(iii) *For $\mathcal{V}_i^{0,+} := \mathbb{P}^{-1}\mathcal{A}_i^0 = \mathcal{U} \times \mathbb{P}^{-1}A_i^0, i \in \mathbb{N}$ the subbundles $\mathcal{V}_i^{0,-} := \mathcal{U} \times V_i^-$ yield an exponentially separated splitting $\mathcal{U} \times M_2 = \mathcal{V}_i^{0,+} \oplus \mathcal{V}_i^{0,-}$.*
(iv) *The subbundles $\mathcal{V}_i^0 := \mathcal{V}_i^{0,+} \cap \mathcal{V}_{i-1}^{0,-} = \mathcal{U} \times \left(V_i^+ \cap V_{i-1}^-\right)$ are finite dimensional, invariant subbundles of $\mathcal{U} \times M_2$ such that*

$$\mathcal{V}_i^{0,+} = \mathcal{V}_1^0 \oplus \cdots \oplus \mathcal{V}_i^0.$$

(v) *The sets $\mathcal{M}_i^0 = \mathbb{P}\mathcal{V}_i^0$ are maximal chain transitive for the projectivized flow $\mathbb{P}\Phi^0$ restricted to $\mathbb{P}\mathcal{V}_i^{0,+}$.*

Proof By Proposition 4.3, the map $\Phi_t^0(u, \cdot)$ on M_2 is injective for every $u \in \mathcal{U}$ since $\det A_p \neq 0$. For the proof of assertion (i), observe that by Theorem 3.3(ii) the sets A_i^0 are asymptotically compact attractors for $\mathbb{P}T(\cdot)$. Hence, for $\varepsilon > 0$ small enough, the neighborhood $B_\varepsilon(A_i^0)$ of A_i^0 satisfies

(a) $\overline{\mathbb{P}T([S, \infty) \times B_\varepsilon(A_i^0))} \subset B_\varepsilon(A_i^0)$ and $A_i^0 = \omega(B_\varepsilon(A_i^0))$;
(b) for any sequence $t_k \to \infty$ and any sequence of points $y_k \in B_\varepsilon(A_i^0)$, it follows that $\mathbb{P}T(t_k)(y_k), k \in \mathbb{N}$ has a convergent subsequence.

It is easily seen that the neighborhood $\mathcal{U} \times B_\varepsilon(A_i^0)$ satisfies the analogous conditions for $\mathbb{P}\Phi^0$ instead of $T(\cdot)$. This proves (i). Assertion (ii) is analogously derived from the corresponding property of the semiflow $T(\cdot)$. Assertions (iii) and (iv) are easily seen using the definitions.

Together we have shown that one obtains a discrete Selgrade decomposition of $\mathbb{P}\Phi^0$ restricted to $\mathcal{U} \times M_2$. Thus Theorem 2.6(v) implies assertion (v). □

5 The Chain Control Set

In this section we define chain control sets and show that they correspond to maximal chain transitive subsets of the control semiflow. Furthermore, we prove that there exists a unique chain control set E in M_2. Throughout the rest of this paper, we assume that the matrix A_p is invertible and hence, by Proposition 4.3, the maps Φ_t are injective on $\mathcal{U} \times M_2$.

As in Sect. 2 we write $\Phi_{-t}(u, y) \in \mathcal{U} \times Y$ for the pre-image $(\Phi_t)^{-1}(u, y)$. Since the shift θ_t is defined for all $t \in \mathbb{R}$, the pre-image exists if and only if the pre-image of y under the map $\varphi(t, \cdot, u)$ denoted by $\varphi(-t, y, u)$ exists. When we write $\Phi_{-t}(u, y)$ or $\varphi(-t, y, u)$ we tacitly suppose that these pre-images exist.

Fix $y, z \in M_2$ and let $\varepsilon, \tau > 0$. A controlled (ε, τ)-chain ζ from y to z is given by $q \in \mathbb{N}$, $y_0 = y, y_1, \ldots, y_q = z$ in M_2, $u_0, \ldots, u_{q-1} \in \mathcal{U}$, and $\tau_0, \ldots, \tau_{q-1} \geq \tau$ with

$$\left\| \varphi(\tau_j, y_j, u_j) - y_{j+1} \right\| < \varepsilon \text{ for } j = 0, \ldots, q-1.$$

Definition 5.1 A nonvoid set $E \subset M_2$ is chain controllable, if for all $y, z \in E$ and all $\varepsilon, \tau > 0$ there are controlled (ε, τ)-chains from y to z. A set $E \subset M_2$ is weakly invariant, if for every $y \in E$ there exists $u \in \mathcal{U}$ such that for all $t \in \mathbb{R}$ one has $\varphi(t, y, u) \in E$. A weakly invariant chain controllable set $E \subset M_2$ is a chain control set if it is maximal (with respect to set inclusion) with these properties.

For finite dimensional systems, the definition of chain control sets above coincides with the standard definition of chain control sets (cf. Colonius and Kliemann [8], Kawan [20]).

Chain control sets can be characterized using the control semiflow Φ on $\mathcal{U} \times M_2$. The following theorem and its proof are generalizations of an analogous result for finite dimensional systems; cf. [8, Theorem 4.3.11]. Recall from Sect. 2 that any invariant set for a semiflow consists of points defining entire solutions.

Theorem 5.2 *Consider the linear delay control system given by (1.1) and assume that the matrix A_p is invertible.*

(i) *If $E \subset M_2$ is a weakly invariant chain controllable set, then the lift*

$$\mathcal{E} := \{(u, y) \in \mathcal{U} \times M_2 \,|\, \forall t \in \mathbb{R} : \varphi(t, y, u) \in E\}$$

is an invariant chain transitive set for the control semiflow Φ and, in particular, $\mathcal{E}$ is contained in the entire chain recurrent set $\mathcal{R}^{\#}$ of Φ.

(ii) *Conversely, let $\mathcal{E} \subset \mathcal{U} \times M_2$ be an invariant chain transitive set for Φ. Then*

$$\pi_{M_2}\mathcal{E} := \{y \in M_2 \,|\, \exists u \in \mathcal{U} : (u, y) \in \mathcal{E}\}$$

is a weakly invariant chain controllable set.

(iii) *For a chain control set E the set $\mathcal{E}$ is a maximal invariant chain transitive set of Φ, and conversely, if $\mathcal{E}$ is a maximal invariant chain transitive set, then $\pi_{M_2}\mathcal{E}$ is a chain control set.*

Proof (i) Let $(u, y), (v, z) \in \mathcal{E}$ and pick $\varepsilon,\ \tau > 0$. Recall the definition of the metric $d_{\mathcal{U}}$ on $\mathcal{U}$ in (4.1) and choose $K \in \mathbb{N}$ large enough such that $\sum_{k=K+1}^{\infty} 2^{-k} < \frac{\varepsilon}{2}$. For $z_1, \ldots, z_K \in L^1(\mathbb{R}, \mathbb{R}^m)$ we may take S large enough such that for all i

$$\int_{\mathbb{R}\setminus[-S,S]} \|z_i(\tau)\|\ d\tau < \frac{\varepsilon}{2 \operatorname{diam} U}.$$

Chain controllability from $\varphi(2\tau, y, u) \in E$ to $\varphi(-\tau, z, v) \in E$ yields the existence of $q \in \mathbb{N}$ and $y_0, \ldots, y_q \in M_2,\ u_0, \ldots, u_{q-1} \in \mathcal{U},\ \tau_0, \ldots, \tau_{q-1} \geq \tau$ with $y_0 = \varphi(2\tau, y, u),\ y_q = \varphi(-\tau, z, v)$, and

$$\left\|\varphi(\tau_j, y_j, u_j) - y_{j+1}\right\| < \varepsilon \text{ for } j = 0, \ldots, q-1.$$

We now construct an (ε, τ)-chain from (u, y) to (v, z) in the following way. Define

$$\begin{aligned} &\tau_{-2} = \tau, \qquad y_{-2} = y, \qquad v_{-2} = u, \\ &\tau_{-1} = \tau,\ y_{-1} = \varphi(\tau, y, u),\ v_{-1}(\tau) = \begin{cases} u(\tau_{-2} + t) & \text{for } t \leq \tau_{-1} \\ u_0(t - \tau_{-1}) & \text{for } t > \tau_{-1} \end{cases} \end{aligned}$$

and let the times $\tau_0, \ldots, \tau_{q-1}$ and the points $y_0, \ldots, y_q$ be as given earlier; furthermore, set $\tau_q = \tau$, $y_{q+1} = z$, $v_{q+1} = v$, and define, for $j = 0, \ldots, q-2$,

$$v_j(t) = \begin{cases} v_{j-1}(\tau_{j-1} + t) & \text{for } t \leq 0 \\ u_j(t) & \text{for } 0 < t < \tau_j \\ u_{j+1}(t - \tau_j) & \text{for } t > \tau_j, \end{cases}$$

$$v_{q-1}(t) = \begin{cases} v_{q-2}(\tau_{q-2} + t) & \text{for } t \leq 0 \\ u_{q-1}(t) & \text{for } 0 < t \leq \tau_{q-1} \\ v(t - \tau_{q-1} - \tau) & \text{for } t > \tau_{q-1}, \end{cases}$$

$$v_q(t) = \begin{cases} v_{q-1}(\tau_{q-1} + t) & \text{for } t \leq 0 \\ v(t - \tau) & \text{for } t > 0. \end{cases}$$

It is easily seen that

$$(v_{-2}, y_{-2}),\ (v_{-1}, y_{-1}), \ldots, (v_{q+1}, y_{q+1}) \text{ and } \tau_{-2},\ \tau_{-1}, \ldots, \tau_q \geq \tau$$

yield an (ε, τ)-chain from (u, y) to (v, z) provided that $d_{\mathcal{U}}(v_j(\tau_j + \cdot), v_{j+1}) < \varepsilon$ for $j = -2, -1, \ldots, q$. By choice of S and K one has for all $w_1,\ w_2 \in \mathcal{U}$ that the distance $d_{\mathcal{U}}(w_1, w_2)$ is bounded by

$$\begin{aligned} \frac{\varepsilon}{2} + \sum_{k=1}^{K} \frac{1}{2^k} \Bigg[&\left\| \int_{\mathbb{R}\setminus[-S,S]} (w_1(t) - w_2(t))^\top z_k(t)\, dt \right\| \\ &+ \left\| \int_{-S}^{S} (w_1(t) - w_2(t))^\top z_k(t)\, dt \right\| \Bigg] \end{aligned}$$

$$< \varepsilon + \max_{k=1,\ldots,K} \int_{-S}^{S} \|w_1(t) - w_2(t)\| \, \|z_k(t)\| \, dt.$$

Hence it suffices to show that for all considered pairs of control functions the integrands vanish. This is immediate from the definition of $v_j,\ j = -2, \ldots, q+1$.

(ii) Let $\mathcal{E}$ be an invariant chain transitive set in $\mathcal{U} \times M_2$. For $y \in \pi_{M_2}\mathcal{E}$ there exists $u \in \mathcal{U}$ such that $\varphi(t, y, u) \in \pi_{M_2}\mathcal{E}$ for all $t \in \mathbb{R}$ by invariance. Now let $y, z \in \pi_M\mathcal{E}$ and choose $\varepsilon, \tau > 0$. Then by chain transitivity of $\mathcal{E}$ we can choose y_j, u_j, τ_j such that the corresponding trajectories satisfy the required conditions.

(iii) It is clear that, for a chain control set E, the set $\mathcal{E}$ is a maximal invariant chain transitive set. Conversely, for a maximal invariant chain transitive set $\mathcal{E}$ the projection $\pi_{M_2}\mathcal{E}$ to M_2 is chain controllable and weakly invariant. Since the maximality property of $\pi_{M_2}\mathcal{E}$ is clear, the assertion follows by (ii).

□

Corollary 5.3 *Assume that the matrix A_p is invertible. Then the following assertions are equivalent:*

(i) *The entire chain recurrent set $\mathcal{R}^{\#}$ of the semiflow Φ is chain transitive.*
(ii) *The set $\mathcal{R}^{\#}$ is the lift $\mathcal{E}$ of a chain control set E.*
(iii) *There is a single chain control set E.*

Proof Suppose that (i) holds. Then, by Theorem 5.2(iii) the projection to M_2 is a chain control set E. By Theorem 5.2(iii) it follows that the lift $\mathcal{E}$ of E is maximal invariant chain transitive set and hence coincides with $\mathcal{R}^{\#}$. This implies (iii) since the lift of any chain control set is contained in $\mathcal{R}^{\#}$. Finally, if E is unique it follows that $\mathcal{R}^{\#}$ coincides with the lift of E. □

The following theorem establishes the announced uniqueness of the chain control set. While this result generalizes the finite dimensional case (Colonius and Santana [11, Theorem 29]), step 3 in the proof is different since here we cannot argue with time reversal. For the convenience of the reader we also write down steps 1 and 2 in the present setting.

Theorem 5.4 *Consider the linear delay control system given by (1.1) and assume that the matrix A_p is invertible. Then there exists a unique chain control set E in M_2.*

Proof First note that for $u \equiv 0$ the origin $0 \in M_2$ is an equilibrium, hence $\{0\}$ is a weakly invariant chain controllable set. Define E as the union of all weakly invariant chain controllable sets containing $\{0\}$. Then E is a weakly invariant chain controllable set and certainly it is maximal with these properties, hence it is a chain control set. It remains to prove uniqueness.

Observe that the trajectories $y(t) = \varphi(t, y_0, u), t \in \mathbb{R}$, of (3.8) satisfy, for $\alpha \in (0, 1)$,

$$\begin{aligned}
\alpha\varphi(t, y_0, u) &= \alpha T(t) y_0 + \alpha \int_0^t T(t-s) B\left(u(s), u(s-h_1), \ldots, u(s-h_p)\right) ds \\
&= T(t)\alpha y_0 + \int_0^t T(t-s) B\left(\alpha u(s), \alpha u(s-h_1), \ldots, \alpha u(s-h_p)\right) ds
\end{aligned}$$

$$= \varphi(t, \alpha y_0, \alpha u). \tag{5.1}$$

Here, $\varphi(\cdot, \alpha y_0, \alpha u)$ is a trajectory of (3.8) since Ω is a convex neighborhood of $0 \in \mathbb{R}^m$ implying that the controls αu are in $\mathcal{U}$.

Suppose that E' is any chain control set and let $y \in E'$. First we will construct controlled (ε, τ)-chains from y to $0 \in E$.

Step 1: There is a controlled (ε, τ)-chain from y to αy for some $\alpha \in (0, 1)$.

For the proof consider a controlled $(\varepsilon/2, \tau)$-chain ζ from y to y given by $y_0 = y, y_1, \ldots, y_q = y$, $u_0, \ldots, u_{q-1} \in \mathcal{U}$, and $\tau_0, \ldots, \tau_{q-1} \geq \tau$ with

$$\|\varphi(\tau_i, y_i, u_i) - y_{i+1}\| < \varepsilon/2 \text{ for } i = 0, \ldots, q-1.$$

Let $\alpha \in (0, 1)$ with $(1-\alpha)\|y\| < \varepsilon/2$. It follows that

$$\left\|\varphi(\tau_{q-1}, y_{q-1}, u_{q-1}) - \alpha y_q\right\| \leq \left\|\varphi(\tau_{q-1}, y_{q-1}, u_{q-1}) - y\right\| + \|y - \alpha y\| < \varepsilon.$$

This defines a controlled (ε, τ)-chain $\zeta^{(1)}$ from y to αy.

Step 2: Replacing y_i by αy_i and u_i by αu_i we get by (5.1)

$$\left\|\varphi(\tau_i, \alpha y_i, \alpha u_i) - \alpha y_{i+1}\right\| = \alpha \left\|\varphi(\tau_i, y_i, u_i) - y_{i+1}\right\| < \varepsilon/2 \text{ for } i = 0, \ldots, q-1,$$
$$\left\|\varphi(\tau_{q-1}, \alpha y_{q-1}, \alpha u_{q-1}) - \alpha^2 y\right\| \leq \left\|\varphi(\tau_{q-1}, \alpha y_{q-1}, \alpha u_{q-1}) - \alpha y\right\| + \left\|\alpha y - \alpha^2 y\right\| < \varepsilon.$$

This defines a controlled (ε, τ)-chain $\zeta^{(2)}$ from αy to $\alpha^2 y$. The concatenation of $\zeta^{(2)}$ and $\zeta^{(1)}$ yields a controlled (ε, τ)-chain $\zeta^{(2)} \circ \zeta^{(1)}$ from y to $\alpha^2 y$.

Repeating this construction, we find that the concatenation $\zeta^{(k)} \circ \cdots \circ \zeta^{(1)}$ is a controlled (ε, τ)-chain from $y \in E'$ to $\alpha^k y$. Since $\alpha^k \to 0$ for $k \to \infty$, we can take $k \in \mathbb{N}$ large enough, such that the last piece of the chain $\zeta^{(k)}$ satisfies

$$\left\|\varphi(\tau_{q-1}, a^k y_{q-1}, a^k u_{q-1})\right\| < \varepsilon.$$

Thus we may take $0 \in E$ as the final point of this controlled (ε, τ)-chain showing that the concatenation $\zeta^{(k)} \circ \cdots \circ \zeta^{(1)}$ defines a controlled (ε, τ)-chain from $y \in E'$ to $0 \in E$.

Step 3: Next we construct controlled chains from 0 to $y \in E'$.

Consider a controlled (ε, τ)-chain from y to y given by $y_0 = y, y_1, \ldots, y_q = y$, $u_0, \ldots, u_{q-1} \in \mathcal{U}$, and $\tau_0, \ldots, \tau_{q-1} \geq \tau$ with

$$\|\varphi(\tau_i, y_i, u_i) - y_{i+1}\| < \varepsilon \text{ for } i = 0, \ldots, q-1.$$

For every $\alpha \in (0, 1)$ formula (5.1) shows that $\alpha y_0 = \alpha y, \alpha y_1, \ldots, \alpha y_q = \alpha y$, $\alpha u_0, \ldots, \alpha u_{q-1} \in \mathcal{U}$, and $\tau_0, \ldots, \tau_{q-1} \geq \tau$ define a controlled $(\alpha\varepsilon, \tau)$- chain from αy to αy with

$$\|\varphi(\tau_i, \alpha y_i, \alpha u_i) - \alpha y_{i+1}\| < \alpha\varepsilon \text{ for } i = 0, \ldots, q-1.$$

Let $\alpha \in (0, \varepsilon)$ be small enough such that $\alpha \|y\| < \varepsilon$. Then we may add a segment $\varphi(t, 0, 0) = 0$, $t \in [0, \tau]$, at the beginning to obtain a controlled $(\alpha\varepsilon, \tau)$-chain from 0 to αy. Furthermore, we find

$$\begin{aligned}\left\|\varphi(\tau_{q-1}, \alpha y_{q-1}, \alpha u_{q-1}) - (\alpha+\varepsilon)y\right\| &\leq \left\|\varphi(\tau_{q-1}, \alpha y_{q-1}, \alpha u_{q-1}) - \alpha y\right\| + \varepsilon \|y\| \\ &\leq \alpha\varepsilon + \varepsilon \|y\| < (\varepsilon + \|y\|)\varepsilon.\end{aligned}$$

Taking $\varepsilon \in (0, 1)$ we have constructed a controlled $((1+\varepsilon)\|y\|, \tau)$-chain $\zeta^{(1)}$ from 0 to $(\alpha + \varepsilon)y$.

Now construct a controlled $(2\varepsilon, \tau)$-chain $\zeta^{(2)}$ from $(\alpha+\varepsilon)y$ to $(2\alpha+\varepsilon)y$. By (5.1),

$$\|\varphi(\tau_i, (\alpha+\varepsilon)y_i, (\alpha+\varepsilon)u_i) - (\alpha+\varepsilon)y_{i+1}\| < (\alpha+\varepsilon)\varepsilon \text{ for } i = 0, \ldots, q-1,$$

and, since $\alpha \|y\| < \varepsilon$,

$$\begin{aligned}&\left\|\varphi(\tau_{q-1}, (\alpha+\varepsilon)y_{q-1}, (\alpha+\varepsilon)u_{q-1}) - (2\alpha+\varepsilon)y\right\| \\ &\quad \leq \left\|\varphi(\tau_{q-1}, (\alpha+\varepsilon)y_{q-1}, (\alpha+\varepsilon)u_{q-1}) - (\alpha+\varepsilon)y\right\| + \alpha \|y\| \\ &\quad < (\alpha+\varepsilon)\varepsilon + \varepsilon.\end{aligned}$$

For $\alpha + \varepsilon < 1$, it follows that $(\alpha + \varepsilon)\varepsilon + \varepsilon \leq 2\varepsilon$ and hence this is a controlled $(2\varepsilon, \tau)$-chain from $(\alpha + \varepsilon)y$ to $(2\alpha + \varepsilon)y$.

Next construct a controlled $(2\varepsilon, \tau)$-chain $\zeta^{(3)}$ starting in $(2\alpha + \varepsilon)y$: By (5.1),

$$\|\varphi(\tau_i, (2\alpha+\varepsilon)y_i, (2\alpha+\varepsilon)u_i) - (2\alpha+\varepsilon)y_{i+1}\| < (2\alpha+\varepsilon)\varepsilon \text{ for } i = 0, \ldots, q-1$$

and

$$\begin{aligned}&\left\|\varphi(\tau_{q-1}, (2\alpha+\varepsilon)y_{q-1}, (2\alpha+\varepsilon)u_{q-1}) - (3\alpha+\varepsilon)y\right\| \\ &\quad \leq \left\|\varphi(\tau_{q-1}, (2\alpha+\varepsilon)y_{q-1}, (2\alpha+\varepsilon)u_{q-1}) - (2\alpha+\varepsilon)y\right\| + \alpha \|y\| \\ &\quad < (2\alpha+\varepsilon)\varepsilon + \varepsilon.\end{aligned}$$

For $2\alpha + \varepsilon < 1$, this defines a controlled $(2\varepsilon, \tau)$-chain from $(2\alpha + \varepsilon)y$ to $(3\alpha + \varepsilon)y$.

As long as $j\alpha+\varepsilon < 1$, we can proceed in this way to obtain controlled $(2\varepsilon, \tau)$-chains $\zeta^{(j+1)}$ from $(j\alpha + \varepsilon)y$ to $((j + 1)\alpha + \varepsilon)y$ satisfying

$$\begin{aligned}&\|\varphi(\tau_i, (j\alpha+\varepsilon)y_i, (j\alpha+\varepsilon)u_i) - (j\alpha+\varepsilon)y_{i+1}\| < (j\alpha+\varepsilon)\varepsilon < \varepsilon \\ &\quad \text{for } i = 0, \ldots, q-1,\end{aligned}$$

with

$$\begin{aligned}&\left\|\varphi(\tau_{q-1}, (j\alpha+\varepsilon)y_{q-1}, (j\alpha+\varepsilon)u_{q-1}) - ((j+1)\alpha+\varepsilon)y\right\| \\ &\quad \leq \left\|\varphi(\tau_{q-1}, (j\alpha+\varepsilon)y_{q-1}, (j\alpha+\varepsilon)u_{q-1}) - (j\alpha+\varepsilon)y\right\| + \alpha \|y\| \\ &\quad \leq (j\alpha+\varepsilon)\varepsilon + \alpha \|y\| < 2\varepsilon.\end{aligned}$$

When for $j = k$ we arrive at $k\alpha + \varepsilon < 1$ and $(k+1)\alpha + \varepsilon \geq 1$, we find

$$\varepsilon > \varepsilon + k\alpha + \varepsilon - 1 > (k+1)\alpha + \varepsilon - 1 \geq 0. \tag{5.2}$$

Thus we get for $i = 0, \ldots, q-1$

$$\|\varphi(\tau_i, (k\alpha+\varepsilon)y_i, (k\alpha+\varepsilon)u_i) - (k\alpha+\varepsilon)y_{i+1}\| < (k\alpha+\varepsilon)\varepsilon < \varepsilon,$$

and, by (5.2),

$$\begin{aligned}
&\left\|\varphi(\tau_{q-1}, (k\alpha+\varepsilon)y_{q-1}, (k\alpha+\varepsilon)u_{q-1}) - y\right\| \\
&\quad\leq \left\|\varphi(\tau_{q-1}, (k\alpha+\varepsilon)y_{q-1}, (k\alpha+\varepsilon)u_{q-1}) - ((k+1)\alpha+\varepsilon)y\right\| \\
&\qquad + \|((k+1)\alpha+\varepsilon)y - y\| \\
&\quad< \left\|\varphi(\tau_{q-1}, (k\alpha+\varepsilon)y_{q-1}, (k\alpha+\varepsilon)u_{q-1}) - (k\alpha+\varepsilon)y\right\| + \varepsilon\,\|y\| \\
&\qquad + \|((k+1)\alpha+\varepsilon-1)y\| \\
&\quad< (k\alpha+\varepsilon)\varepsilon + \varepsilon\,\|y\| + ((k+1)\alpha+\varepsilon-1)\,\|y\| \\
&\quad< \varepsilon + \varepsilon\,\|y\| + \varepsilon\,\|y\| < \varepsilon + 2\varepsilon\,\|y\|\,.
\end{aligned}$$

Thus this defines a controlled $((1+2\,\|y\|)\varepsilon, \tau)$-chain $\zeta^{(k+1)}$ from $(k\alpha+\varepsilon)y$ to y. The concatenation $\zeta^{(k+1)} \circ \zeta^{(k)} \circ \cdots \circ \zeta^{(1)}$ yields a controlled $((1+2\,\|y\|)\varepsilon, \tau)$-chain from 0 to y.

Since $\varepsilon, \tau > 0$ are arbitrary, steps 2 and 3 imply that $y \in E' \cap E$ and hence $E' = E$. □

6 The Linear Lift and the Poincaré Sphere

In this section we lift the affine control semiflow Φ on $\mathcal{U} \times M_2$ to a linear control semiflow Φ^1 on $\mathcal{U} \times M_2^1$ with $M_2^1 := M_2 \times \mathbb{R}$ and obtain a discrete Selgrade decomposition by an application of the generalized Selgrade theorem, Theorem 2.6. Furthermore, conjugation properties of the associated semiflows are derived.

The space M_2^1 becomes a Hilbert space with the scalar product

$$\left\langle (x,\gamma), (x',\gamma') \right\rangle := \langle x.y \rangle_{M_2} + \gamma\gamma' \text{ for } (x,\gamma), (x',\gamma') \in M_2 \times \mathbb{R}.$$

We embed the linear control system (1.1) into a bilinear control system on M_2^1 by introducing an additional state variable x^1. Consider for $t \geq 0$

$$\begin{aligned}
&\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{p} A_i x(t-h_i) + x^1(t) \sum_{i=0}^{p} B_i u(t-h_i),\ u \in \mathcal{U}, \\
&\dot{x}^1(t) = 0, \\
&\quad x(0) = r,\ x(s) = f(s) \text{ for almost all } -h \leq s < 0, \text{ and } x^1(0) = \gamma \in \mathbb{R}.
\end{aligned} \tag{6.1}$$

Denote the solutions of (6.1) by $(x(t), x^1(t)) = \psi^1(t, r, f, \gamma, u) \in \mathbb{R}^{n+1}, t \geq 0$ solving by (3.4)

$$x(t) = e^{A_0 t} r + \int_0^t e^{A_0(t-s)} \left[\sum_{i=1}^{p} A_i x(s - h_i) + \gamma \sum_{i=0}^{p} B_i u(t - h_i) \right] ds,\ x^1(t) = \gamma.$$

In the state space M_2^1 one obtains the bilinear control system

$$\dot{y}(t) = Ay(t) + y^1(s) B\left(u(s), u(s - h_1), \ldots, u(s - h_p)\right), \dot{y}^1(t) = 0, \quad (6.2)$$

with $(y(0), y^1(0)) = (y_0, \gamma) \in M_2^1 = M_2 \times \mathbb{R}$ and controls $u \in \mathcal{U}$.

Equivalently, the control semiflow Φ is lifted to a control semiflow Φ^1 on $\mathcal{U} \times M_2^1$ defined by

$$\begin{aligned}
\Phi_t^1(u, y_0, \gamma) &:= (\theta_t u, \varphi^1(t, y_0, \gamma, u)), \\
\varphi^1(t, y_0, \gamma, u) &:= \left(T(t) y_0 + \int_0^t T(t-s) y^1(s) B\left(u(s), \ldots, u(s - h_p)\right) ds, y^1(t) \right) \\
&= \left(T(t) y_0 + \gamma \int_0^t T(t-s) B\left(u(s), \ldots, u(s - h_p)\right) ds, \gamma \right).
\end{aligned} \quad (6.3)$$

Observe that for $t \geq 0$, $y_0 = (r, f) \in M_2$, $u \in \mathcal{U}$, and $\gamma = 1$ one has

$$\psi^1(t, r, f, 1, u) = (\psi(t, r, f, u), 1) \text{ and } \varphi^1(t, y_0, 1, u) = (\varphi(t, y_0, u), 1).$$

An application of Theorem 2.6 to the linear skew product semiflow Φ^1 defined by (6.3) yields the following discrete Selgrade decomposition of Φ^1.

Theorem 6.1 *For the affine delay control system (1.1), assume that* $\det A_p \neq 0$*, and consider the associated bilinear delay control system (6.1). Then, for the linear control semiflow Φ^1 on $\mathcal{V}^1 = \mathcal{U} \times M_2^1$ defined in (6.3), there is an at-most countable sequence $\{\mathcal{A}_i^1\}_{i=0}^{N^1}$, $N^1 \in \{0, 1, \ldots\} \cup \{\infty\}$, of subsets of $\mathbb{P}\mathcal{V}^1 = \mathcal{U} \times \mathbb{P}M_2^1$ with $\mathcal{A}_0^1 = \varnothing$, $\mathcal{A}_i^1 \subset \mathcal{A}_{i+1}^1$ for all $1 \leq i < N^1$, such that, for $1 \leq i < N^1 + 1$, the following properties hold.*

- (i) *The set $\mathcal{A}_i^1$ is an asymptotically compact attractor for $\mathbb{P}\Phi^1$.*
- (ii) *If $\mathcal{A}^1$ is any nonempty asymptotically compact attractor for $\mathbb{P}\Phi^1$, then it follows that $\mathcal{A}^1 = \mathcal{A}_i^1$ for some $1 \leq i < N^1 + 1$.*
- (iii) *For the finite dimensional subbundles $\mathcal{V}_i^{1,+} = \mathbb{P}^{-1}\mathcal{A}_i^1$ there are subbundles $\mathcal{V}_i^{1,-}$ such that $\mathcal{V}^1 = \mathcal{V}_i^{1,+} \oplus \mathcal{V}_i^{1,-}$ is an exponentially separated splitting.*
- (iv) *The subbundles $\mathcal{V}_i^1 := \mathcal{V}_i^{1,+} \cap \mathcal{V}_{i-1}^{1,-}$ are finite dimensional, invariant subbundles of $\mathcal{V}^1$ such that*

$$\mathcal{V}_i^{1,+} = \mathcal{V}_1^1 \oplus \cdots \oplus \mathcal{V}_i^1.$$

(v) *The sets* $\mathcal{M}_i^1 = \mathbb{P}\mathcal{V}_i^1$ *are maximal chain transitive for the projectivized flow* $\mathbb{P}\Phi^1$ *restricted to* $\mathcal{V}_i^{1,+}$.

Proof We verify the assumptions of Theorem 2.6. The Hilbert space M_2^1 is separable, and the lifted control semiflow Φ^1 on $\mathcal{U} \times M_2^1$ is linear. This is seen as in the finite dimensional case; cf. Colonius and Santana [11]. Theorem 4.2 implies that it is continuous and Hypotheses (H1) is clear by definition.

By Proposition 4.3, invertibility of the matrix A_p is equivalent to injectivity of the maps $\Phi_t(u, \cdot)$. The definition in (6.3) shows that this is also equivalent to injectivity of the maps $\Phi_t^1(u, \cdot)$ since $\Phi_t^1(u, y_0, \gamma) = \Phi_t^1(u, y_0', \gamma')$ holds if and only if $\gamma = \gamma'$ and $\Phi_t(u, y_0) = \Phi_t(u, y_0')$. Thus hypothesis (H2) holds.

We claim that for fixed $t \geq 0$ the map $u \mapsto \Phi_t^1(u, \cdot)$ is continuous in the operator norm, thus (H3) holds. For the proof of the claim, consider for $u^k \to u^0$ in $\mathcal{U}$ the difference in the operator norm on M_2^1. By the definition in (6.3) we obtain

$$\begin{aligned}
&\left\| \Phi_t^1(u^k, y_0, \gamma) - \Phi_t^1(u^0, y_0, \gamma) \right\| \\
&= \sup \left\{ \left\| \varphi^1(t, y_0, \gamma, u^k) - \varphi^1(t, y_0, \gamma, u^0) \right\| \, | \|(y_0, \gamma)\| \leq 1 \right\} \\
&\leq \sup_{|\gamma| \leq 1} |\gamma| \left\| \int_0^t T(t-s)[B\left(u^k(s), \ldots, u^k(s-h_p)\right) - B\left(u^0(s), \ldots, u^0(s-h_p)\right)]ds \right\| \\
&= \left\| \varphi(t, 0, u^k) - \varphi(t, 0, u^0) \right\|.
\end{aligned}$$

By Theorem 4.2(i) the right hand side converges to 0 for $k \to \infty$, and hence the claim follows.

In order to verify hypotheses (H4), let $(t_k, u^k) \to (t_0, u^0)$ and consider $(y_0, \gamma) \in M_2^1$. Then, as for (H3), Theorem 4.2(i) implies $\varphi^1(t_k, u^k, y_0, \gamma) \to \varphi^1(t_0, u^0, y_0, \gamma)$ in M_2^1. Hence the mapping associating to $(t, u) \in [0, \infty) \times \mathcal{U}$ the operator $\Phi_t^1(u, \cdot)$ on M_2^1 is continuous in the strong operator topology showing (H4).

Thus all assumptions of Theorem 2.6 are verified and the assertions follow. □

Next we will analyze the Selgrade bundles $\mathcal{V}_i^1$ in more detail. Define subsets of $M_2^1 = M_2 \times \mathbb{R}$, of the unit sphere $\mathbb{S}M_2^1 = \{(y, \gamma) \in M_2^1 \, | \|(y, \gamma)\| = 1\}$, and of the projective space $\mathbb{P}M_2^1$ by

$$\begin{aligned}
&M_2^{1,0} = M_2 \times \{0\}, \, M_2^{1,1} = M_2 \times (\mathbb{R} \setminus \{0\}), \\
&\mathbb{S}^+ M_2^1 := \left\{ (y, \gamma) \in \mathbb{S}M_2^1 \, | y \in M_2, \gamma > 0 \right\}, \mathbb{S}^0 M_2^1 = \{(y, 0) \in \mathbb{S}M_2^1 \, | y \in M_2\}, \\
&\mathbb{P}M_2^{1,0} = \{\mathbb{P}(y, 0) \in \mathbb{P}M_2^1 \, | y \in M_2\}, \; \mathbb{P}M_2^{1,1} = \{\mathbb{P}(y, \gamma) \in \mathbb{P}M_2^1 \, | y \in M_2, \gamma \neq 0\},
\end{aligned}$$

respectively. One easily sees that the projective space $\mathbb{P}M_2^1 = \overline{\mathbb{P}M_2^{1,1}}$ is the disjoint union of the closed subset $\mathbb{P}M_2^{1,0}$ and the open subset $\mathbb{P}M_2^{1,1}$. Note that $\mathbb{P}M_2^{1,1}$ can be identified with the northern hemisphere $\mathbb{S}^+ M_2^1$ of the sphere $\mathbb{S}M_2^1$, the set $\mathbb{S}^0 M_2^1$ is the equator, and $\mathbb{P}M_2^{1,0}$ is its image in $\mathbb{P}M_2^1$.

Definition 6.2 The Poincaré sphere bundle and the projective Poincaré bundle are $\mathcal{U} \times \mathbb{S}M_2^1$ and $\mathcal{U} \times \mathbb{P}M_2^1$, respectively. The equatorial bundle and the projective equatorial bundle are $\mathcal{U} \times M_2^{1,0}$ and $\mathcal{U} \times \mathbb{P}M_2^{1,0}$, respectively.

Remark 6.3 The construction above is a modification of a classical construction for polynomial ordinary differential equations going back to Poincaré. We consider affine equations and subjoin the additional scalar state variable x^1 in front of the inhomogeneous term, while Poincaré's construction just adds x^1; cf. Perko [27], Cima and Llibre [7]. A consequence (see Proposition 6.4) is that the induced equation on the equatorial bundle is determined by the linear part and the inhomogeneous term vanishes.

A conjugacy of two semiflows ψ and ψ' on metric spaces X and X', respectively, is a homeomorphism $h : X \to X'$ satisfying for all $x \in X$,

$$h(\psi(t, x)) = \psi'(t, h(x)) \text{ for } t \geq 0.$$

Recall that the homogeneous part Φ^0 of the semiflow Φ on $\mathcal{U} \times \mathbb{P}M_2$ (cf. (4.3)) induces a projectivized semiflow $\mathbb{P}\Phi_t^0 = (\theta_t u, \mathbb{P}T(t)), t \geq 0$, on $\mathcal{U} \times \mathbb{P}M_2$.

Proposition 6.4 (i) *The map*

$$h^0 : \mathcal{U} \times M_2 \to \mathcal{U} \times M_2^{1,0}, h^0(u, y) := (u, (y, 0)),$$

and its inverse are uniformly continuous and h^0 conjugates the semiflow Φ^0 on $\mathcal{U} \times M_2$ and the semiflow Φ^1 restricted to the equatorial bundle $\mathcal{U} \times M_2^{1,0}$.

(ii) *The projective map*

$$\mathbb{P}h^0 : \mathcal{U} \times \mathbb{P}M_2 \to \mathcal{U} \times \mathbb{P}M_2^{1,0}, \mathbb{P}h^0(u, \mathbb{P}y) := (u, \mathbb{P}(y, 0)),$$

and its inverse are uniformly continuous and $\mathbb{P}h^0$ conjugates the flow $\mathbb{P}\Phi^0$ on $\mathcal{U} \times \mathbb{P}M_2$ and the flow $\mathbb{P}\Phi^1$ restricted to $\mathcal{U} \times \mathbb{P}M_2^{1,0}$.

(iii) *For $j \leq i$, the maximal invariant chain transitive sets $\mathcal{M}_j^0$ of $\mathbb{P}\Phi^0$ restricted to $\mathcal{V}_i^{0,+}$ are mapped onto the maximal invariant chain transitive sets $\mathcal{M}_j^1 = \mathbb{P}h^0(\mathcal{M}_j^0)$ of $\mathbb{P}\Phi^1$ restricted to $\mathbb{P}h^0(\mathbb{P}\mathcal{V}_i^{0,+})$, and their order is preserved.*

(iv) *The sets $\mathbb{P}^{-1}(\mathbb{P}h^0(\mathcal{M}_j^0)) = \mathcal{V}_j^1$ are finite dimensional subbundles of the subbundle $\mathbb{P}^{-1}\left(\mathbb{P}h^0(\mathcal{V}_i^{0,+})\right) \subset \mathcal{U} \times M_2^{1,0}$.*

Proof (i) The semiflow Φ^1 restricted to $\mathcal{U} \times M_2^{1,0}$ and the semiflow Φ^0 on $\mathcal{U} \times M_2$ satisfy

$$\Phi_t^1(h^0(u, y)) = \Phi_t^1(u, y, 0) = (\theta_t u, T(t)y, 0) = (\Phi_t^0(u, y), 0) = h^0(\Phi_t^0(u, y)).$$

Furthermore, uniform continuity of h^0 and $(h^0)^{-1}$ holds since

$$d((u, y), (v, z)) = d((u, (y, 0)), (v, (z, 0))).$$

Assertion (ii) is a consequence of (i). Since the map $\mathbb{P}h^0$ and its inverse are uniformly continuous they preserve chain transitivity. By Proposition 4.4 the subbundles $\mathcal{V}_i^0$ and the subsets $\mathbb{P}\mathcal{V}_i^0$ are linearly ordered (cf. Remark 2.7) and the sets $\mathcal{M}_i^0 = \mathbb{P}\mathcal{V}_i^0$ are maximal chain transitive sets for $\mathbb{P}\Phi^0$ restricted to $\mathbb{P}\mathcal{V}_i^{0,+}$. This also implies the assertion on the maximal chain transitive sets and the associated subbundles in (iii) and assertion (iv). □

Next we turn to the induced semiflow $\mathbb{P}\Phi^1$ restricted to $\mathcal{U} \times \mathbb{P}M_2^{1,1}$ and note the following lemma.

Lemma 6.5 *Define the map*

$$h^1 : \mathcal{U} \times M_2 \to \mathcal{U} \times \mathbb{P}M_2^{1,1}, h^1(u, y) := (u, \mathbb{P}(y, 1)), (u, y) \in \mathcal{U} \times M_2. \quad (6.4)$$

For every $\varepsilon, \tau > 0$ any (ε, τ)-chain in $\mathcal{U} \times M_2$ is mapped by h^1 onto a $(2\varepsilon, T)$-chain in $\mathcal{U} \times \mathbb{P}M_2^{1,1}$.

Proof It suffices to show that $d((u, y), (u', y')) < \varepsilon$ implies $d(h^1(u, y), h^1(u', y')) < 2\varepsilon$ in $\mathcal{U} \times \mathbb{P}M_2^1$ and this follows by the following estimates of the distances on $\mathbb{P}M_2^1$. According to the definition of the metric in (2.2) on projective space it suffices to estimate

$$\left\| \frac{(y,1)}{\|(y,1)\|} - \frac{(y',1)}{\|(y',1)\|} \right\| = \left(\frac{y}{\|(y,1)\|} - \frac{y'}{\|(y',1)\|}, \frac{1}{\|(y,1)\|} - \frac{1}{\|(y',1)\|} \right).$$

Note that $\|y\| - \|y'\| \le \|y - y'\| < \varepsilon$ and $\|(y,1)\| - \|(y',1)\| \le \|(y - y', 0)\| = \|y - y'\| < \varepsilon$. Hence we find $\delta(\varepsilon)$ with $|\delta(\varepsilon)| < \varepsilon$ such that $\|(y',1)\| = \|(y,1)\| + \delta(\varepsilon)$. The last component satisfies

$$\frac{1}{\|(y,1)\|} - \frac{1}{\|(y',1)\|} = \frac{\|(y',1)\| - \|(y,1)\|}{\|(y,1)\| \, \|(y',1)\|} < \varepsilon$$

and

$$\begin{aligned} \left\| \|(y',1)\| \, y - \|(y,1)\| \, y' \right\| &= \left\| [\|(y,1)\| + \delta(\varepsilon)] \, y - \|(y,1)\| \, y' \right\| \\ &\le \|(y,1)\| \, \|y - y'\| + \delta(\varepsilon) \, \|y\| < \|(y,1)\| \, \varepsilon + \delta(\varepsilon) \, \|y\| . \end{aligned}$$

Hence, the other components satisfy

$$\begin{aligned} \left\| \frac{y}{\|(y,1)\|} - \frac{y'}{\|(y',1)\|} \right\| &= \frac{\left\| \, \|(y',1)\| \, y - \|(y,1)\| \, y' \right\|}{\|(y,1)\| \;\; \|(y',1)\|} \le \frac{(\|(y,1)\| \, \varepsilon + \delta(\varepsilon) \, \|y\|)}{\|(y,1)\| \;\; \|(y',1)\|} \\ &< \varepsilon + |\delta(\varepsilon)| < 2\varepsilon. \end{aligned}$$

This implies the desired estimate. □

The following proposition shows the semiflow Φ on $\mathcal{U} \times M_2$ is conjugate to the semiflow $\mathbb{P}\Phi^1$ restricted to $\mathcal{U} \times \mathbb{P}M_2^{1,1}$. Furthermore, a subset of $\mathcal{U} \times M_2$ is unbounded if and only if the closure of its image in $\mathcal{U} \times \mathbb{P}M_2^1$ intersects the projective equatorial bundle.

Proposition 6.6 (i) *The map h^1 defined in (6.4) is a conjugacy of the semiflows Φ on $\mathcal{U} \times M_2$ and $\mathbb{P}\Phi^1$ restricted to $\mathcal{U} \times \mathbb{P}M_2^{1,1}$,*

$$h^1(\Phi_t(u, y)) = \mathbb{P}\Phi_t^1(u, y, 1) \text{ for } t \geq 0.$$

(ii) *For a subset $C \subset \mathcal{U} \times M_2$ the set $\{y \in M_2 \,|\, (u, y) \in C \text{ for some } u \in \mathcal{U}\}$ is bounded if and only if $\overline{h^1(C)} \cap (\mathcal{U} \times \mathbb{P}M_2^{1,0}) = \varnothing$.*

Proof (i) The proof of Lemma 6.5 shows that h^1 is continuous. The first component of h^1 is the identity on $\mathcal{U}$. Concerning the second component, suppose that $\mathbb{P}(y, 1) = \mathbb{P}(y', 1)$, i.e., $(y, 1) = \lambda(y', 1)$ for some $\lambda \neq 0$. This implies $\lambda = 1$ and hence $y = y'$. Thus h^1 is injective, and it certainly is surjective. It remains to show that the inverse of h^1 is continuous. Suppose that

$$d_{\mathbb{P}}(\mathbb{P}(y_k, 1), \mathbb{P}(y, 1)) = \min\left\{\left\|\frac{(y_k, 1)}{\|(y_k, 1)\|} - \frac{(y, 1)}{\|(y, 1)\|}\right\|,\right.$$
$$\left.\left\|\frac{(y_k, 1)}{\|(y_k, 1)\|} + \frac{(y, 1)}{\|(y, 1)\|}\right\|\right\} \to 0.$$

The second terms cannot converge to 0, since the last component is greater than or equal to $\frac{1}{\|(y,1)\|}$. Hence we know that

$$\left\|\frac{(y_k, 1)}{\|(y_k, 1)\|} - \frac{(y, 1)}{\|(y, 1)\|}\right\| = \left\|\left(\frac{y_k}{\|(y_k, 1)\|} - \frac{y}{\|(y, 1)\|}, \frac{1}{\|(y_k, 1)\|} - \frac{1}{\|(y, 1)\|}\right)\right\| \to 0.$$

The last components converge to 0 implying $\|(y_k, 1)\| \to \|(y, 1)\|$. Since also the other components converge to 0 we conclude that $\|y_k - y\| \to 0$. This shows that h^1 is a homeomorphism. The conjugacy property follows by

$$h^1(\Phi_t(u, y)) = (\theta_t u, \mathbb{P}(\varphi(t, y, u), 1)) = (\theta_t u, \mathbb{P}\varphi^1(t, y, u, 1))$$
$$= \mathbb{P}\Phi_t^1(u, y, 1), t \geq 0.$$

(ii) Consider a sequence $(u^k, y_k), k \in \mathbb{N}$, in C. For the images $h^1(u^k, y_k) = (u^k, \mathbb{P}(y_k, 1))$, the points $\mathbb{P}(y_k, 1)$ are determined by

$$\pm\left(\frac{y_k}{\|(y_k, 1)\|}, \frac{1}{\|(y_k, 1)\|}\right).$$

Then it follows that $\|y_k\| \to \infty$ for $k \to \infty$, is equivalent to the property that the distances of $(u^k, \mathbb{P}(y_k, 1))$ to $\mathcal{U} \times \mathbb{P}M_2^{1,0}$ converge to 0.

□

7 Hyperbolic Semiflows

This section analyzes hyperbolic control semiflows for delay systems. Again we assume throughout that $\det A_p \neq 0$.

Recall that a hyperbolic homogeneous delay equation yields by Theorem 3.1 a spectral decomposition of M_2 into a finite dimensional subspace V^+ and a stable subspace V^-. Since the homogeneous part Φ^0 of the associated control semiflow is the product semiflow $\Phi_t^0(u, y) = (\theta_t u, T(t)y)$, this flow is also hyperbolic with the following decomposition into closed subbundles $\mathcal{U} \times M_2 = \mathcal{V}^- \oplus \mathcal{V}^+$, $\mathcal{V}^- := \mathcal{U} \times V^-$ and $\mathcal{V}^+ := \mathcal{U} \times V^+$. There are constants $\alpha, K > 0$ such that

$$\left\| \Phi_t^0(u, y^-) \right\| = \left\| T(t)y^- \right\| \leq Ke^{-\alpha t} \left\| y^- \right\| \text{ for } t \geq 0 \text{ and } (u, y^-) \in \mathcal{V}^-,$$
$$\left\| \Phi_t^0(u, y^+) \right\| = \left\| T(t)y^+ \right\| \leq Ke^{\alpha t} \left\| y^+ \right\| \text{ for } t \leq 0 \text{ and } (u, y^+) \in \mathcal{V}^+.$$

Since $\dim V^+ < \infty$ the solution map $T(t)$ is an isomorphism on the invariant subspace V^+ and hence, for every $y \in V^+$, there exists an entire solution $\varphi^0(t, y)$, $t \in \mathbb{R}$.

Next we consider the inhomogeneous Eq. (3.7). Let $\pi^\pm : M_2 \to V^\pm$ be the associated projections and define $t \geq 0$

$$\varphi^\pm(t, u, y^\pm) := T(t)y^\pm + \int_0^t T(t-s)\pi^\pm B(u(s), \ldots, u(s-h_p))ds \text{ for } (u, y^\pm) \in \mathcal{U} \times V^\pm,$$

and define associated affine semiflows on $\mathcal{V}^\pm := \mathcal{U} \times V^\pm$ by

$$\Phi_t^\pm(u, y) := \left(\theta_t u, \varphi^\pm(t, u, y)\right) \text{ for } t \geq 0 \text{ and } (u, x^\pm) \in \mathcal{V}^\pm.$$

Our next goal is to prove that for every $u \in \mathcal{U}$ there exists a unique bounded solution of Φ. We start with the stable part.

Lemma 7.1 *Assume that the linear part Φ^0 of the affine semiflow Φ is hyperbolic. Then for every $u \in \mathcal{U}$ there exists a unique entire bounded solution $(\theta_t u, e^-(u, t))$, $t \in \mathbb{R}$ of the affine semiflow Φ^-. It satisfies $e^-(\theta_t u, 0) = e^-(u, t)$ for $t \in \mathbb{R}$, and the map $e^- : \mathcal{U} \times \mathbb{R} \to M_2$ is continuous.*

Proof First we show that the linear semiflow $T(\cdot)$ restricted to V^- has only the trivial entire bounded solution. Any entire bounded solution $\varphi^0(t, y^-)$, $t \in \mathbb{R}$ satisfies, for $t \geq 0$

$$\left\| y^- \right\| = \left\| \varphi^0(0, y^-) \right\| = \left\| \varphi^0(t, \varphi^0(-t, y^-)) \right\| \leq Ke^{-\alpha t} \left\| \varphi^0(-t, y^-) \right\|$$
$$\leq Ke^{-\alpha t} \sup_{s \leq 0} \left\| \varphi^0(s, y^-) \right\|.$$

The right hand side converges to 0 for $t \to \infty$, hence $y^- = 0$. It immediately follows that there is at most a single entire bounded solution for Φ^- since the difference of two bounded entire solutions in V^- is a bounded entire solution in V^- of the linear semiflow. We claim that

$$e^-(u,t) := \int_{-\infty}^{t} T(t-s)\pi^- B(u(s),\ldots,u(s-h_p))ds, t \in \mathbb{R},$$

is the desired solution. The integral exists since $t-s \geq 0$ for $s \in (-\infty, t)$ and for all $u \in \mathcal{U}$ and $s \leq t$

$$\begin{aligned}&\left\|T(t-s)\pi^- B(u(s),\ldots,u(s-h_p))\right\| \leq Ke^{-\alpha(t-s)}\left\|\pi^- B(u(s),\ldots,u(s-h_p))\right\|\\ &\leq Ke^{-\alpha(t-s)}\left\|\pi^-\right\|(p+1)\max_{i=0,\ldots,p}\|B_i\|\max_{u\in\Omega}\|u\|.\end{aligned}$$

This is a solution since for $t_0 \in \mathbb{R}$ and $t \geq t_0 \geq 0$ it satisfies formula (3.8) for the initial value $e^-(u,t_0)$:

$$\begin{aligned}e^-(u,t) &= \int_{-\infty}^{t} T(t-s)\pi^- B(u(s),\ldots,u(s-h_p))ds\\ &= T(t-t_0)\int_{-\infty}^{t_0} T(t_0-s)\pi^- B(u(s),\ldots,u(s-h_p))ds\\ &\quad + \int_{t_0}^{t} T(t-s)\pi^- B(u(s),\ldots,u(s-h_p))ds\\ &= T(t-t_0)e^-(u,t_0)) + \int_{t_0}^{t} T(t-s)\pi^- B(u(s),\ldots,u(s-h_p))ds.\end{aligned}$$

Note that for $t \in \mathbb{R}$

$$\begin{aligned}e^-(\theta_t u, 0) &= \int_{-\infty}^{0} T(-s)\pi^- B(u(t+s),\ldots,u(t+s-h_p))ds\\ &= \int_{-\infty}^{t} T(t-s)\pi^- B(u(s),\ldots,u(s-h_p))ds = e^-(u,t).\end{aligned}$$

In order to prove continuity let $u, u^0 \in \mathcal{U}$ and $t, t_0 \in \mathbb{R}$. Then

$$\begin{aligned}&\left\|e^-(u,t) - e^-(u^0,t_0)\right\|\\ &= \left\|\int_{-\infty}^{t} T(t-s)\pi^- B(u(s),\ldots,u(s-h_p))ds\right.\\ &\quad \left. - \int_{-\infty}^{t_0} T(t_0-s)\pi^- B(u^0(s),\ldots,u^0(s-h_p))ds\right\|\\ &\leq \left\|\int_{\mathbb{R}} \left[\chi_{(-\infty,t]}(s)T(t-s) - \chi_{(-\infty,t_0]}(s)T(t_0-s)\right]\pi^- B(u(s),\ldots,u(s-h_p))ds\right\|\\ &\quad + \left\|\int_{-\infty}^{t_0} T(t_0-s)\pi^- B(u(s)-u^0(s),\ldots,u(s-h_p)-u^0(s-h_p))ds\right\|.\end{aligned}$$

For $(t, u) \to (t_0, u^0)$ the first summand converges to 0 by strong continuity of $T(\cdot)$ and Lebesgue's theorem. The integrand in the second summand is

$$\varphi^-(t_0 - s, 0, u) - \varphi^-(t_0 - s, 0, u^0).$$

For $u \to u^0$ in $\mathcal{U}$, Theorem 4.2(i) implies that this converges to 0, for every $s \in (-\infty, t_0]$. Again Lebesgue's theorem implies that the integral converges to 0. □

An analogous result holds for the unstable part.

Lemma 7.2 *Assume that the linear part Φ^0 of the affine semiflow Φ is hyperbolic. Then for every $u \in \mathcal{U}$ there exists a unique entire bounded solution $(\theta_t u, e^+(u,t))$, $t \in \mathbb{R}$ of the affine semiflow Φ. It satisfies $e^+(\theta_t u, 0) = e^+(u,t)$ for $t \in \mathbb{R}$, and the map $e^+ : \mathcal{U} \times \mathbb{R} \to M_2$ is continuous.*

Proof Let $T(t)y^+ = \varphi^0(t, y^+)$, $t \in \mathbb{R}$, be a bounded solution for $T(t)$ restricted to V^+. Then it follows, for $t \geq 0$,

$$\left\|\varphi^0(t, y^+)\right\| \geq K e^{\alpha t} \left\|y^+\right\| \to \infty \text{ for } k \to \infty.$$

This implies $y^+ = 0$. As above there is at most a single entire bounded solution for Φ^+ since the difference of two bounded entire solutions is a bounded entire solution for the homogeneous semiflow.

Next we show that the entire bounded solution is given by

$$e^+(u,t) = \int_{-\infty}^{t} T(t+s)\pi^+ B(u(s), \ldots, u(s-h_p))ds, \, t \in \mathbb{R}.$$

Observe that the integrand is well defined, since π^+ is a map onto the finite dimensional subspace V^+ and $T(t+s)$ is an isomorphism on V^+. Existence of the integral follows from

$$\|T(t+s)y\| \leq K^{-1} e^{\alpha s} \|T(t)y\| \text{ for } s \leq 0 \text{ and } y \in V^+.$$

The other assertions follow as in the proof of Lemma 7.1. □

A combination of the two previous lemmas establishes the desired unique existence of entire bounded solutions and shows that the affine semiflow is conjugate to its homogeneous part; cf. Colonius and Santana [10, Corollary 1 and Theorem 2.5] for an analogous result in finite dimensions.

Proposition 7.3 *Suppose that the linear part Φ^0 of the affine semiflow Φ is hyperbolic.*

(i) *Then, for every $u \in \mathcal{U}$, there is a unique bounded entire solution given by $(\theta_t u, e(u,t))$, $t \in \mathbb{R}$ for the affine semiflow Φ, the map $e : \mathcal{U} \times \mathbb{R} \to M_2$ is continuous, and $e(\theta_t u, 0) = e(u,t)$ for $t \in \mathbb{R}$.*

(ii) *The affine semiflow* Φ *and its linear part are conjugate by the homeomorphism*

$$H : \mathcal{U} \times M_2 \to \mathcal{U} \times M_2 : H(u, y) := (u, y - e(u, 0)) \text{ for } (u, y) \in \mathcal{U} \times M_2. \quad (7.1)$$

Proof (i) Lemmas 7.1 and 7.2 imply the existence of unique bounded entire solutions $(\theta_t u, e^{\pm}(u, t))$, $t \in \mathbb{R}$. This yields the bounded entire solution for Φ

$$\begin{aligned}(\theta_t u, e(u, t)) &= (\theta_t u, e^+(u, t) + e^-(u, t)) \\ &= (\theta_t u, e^+(u, t)) + (\theta_t u, e^-(u, t)),\ t \in \mathbb{R}.\end{aligned}$$

Since any bounded entire solution for Φ induces bounded entire solutions in $\mathcal{U} \times V^{\pm}$, uniqueness follows. Furthermore, the map $\mathcal{U} \times \mathbb{R} \to \mathcal{U} \times M_2 : (u, t) \mapsto (u, e(u, t))$ is continuous.

(ii) The map H is continuous and bijective with continuous inverse

$$H^{-1}(u, y) := (u, y + e(u, 0)) \text{ for } (u, y) \in \mathcal{U} \times M_2.$$

The conjugation property follows from

$$\begin{aligned}H(\Phi_t(u, y)) &= H(\theta_t u, \varphi(t, u, y)) = (\theta_t u, \varphi(t, u, y) - e(\theta_t u, 0)) \\ &= (\theta_t u, \varphi(t, u, y) - e(u, t)) = (\theta_t u, \varphi(t, u, y) - \varphi(t, u, e(u, 0))) \\ &= (\theta_t u, \varphi^0(t, y - e(u, 0)) = \Phi_t^0(H(u, y)).\end{aligned}$$

□

The following lemma shows that the chain recurrent set of uniformly hyperbolic linear systems is trivial. Antunez, Mantovani, and Varão [2, Corollary 2.11] prove an analogous result for hyperbolic linear operators on Banach spaces.

Lemma 7.4 *Suppose that* Φ^0 *is hyperbolic with decomposition* $\mathcal{V} = \mathcal{U} \times M_2 = \mathcal{V}^+ \oplus \mathcal{V}^-$. *Then the chain recurrent set of* Φ^0 *equals* $\mathcal{U} \times \{0_{M_2}\}$.

Proof It is clear that $\mathcal{U} \times \{0_{M_2}\}$ is contained in the chain recurrent set.

(i) First we show that the chain recurrent set of $\mathcal{V}^-$ equals $\mathcal{U} \times \{0_{M_2}\}$. Suppose, by way of contradiction, that $(u, y) \in \mathcal{V}^-$ with $y \neq 0$ is chain recurrent and consider for $\varepsilon \in (0, 1)$, $\tau > 0$ an (ε, τ)-chain from (u, y) to (u, y) given by

$$\tau_0, \ldots, \tau_{q-1} \geq \tau \text{ and } d(\Phi_{\tau_i}^0(u^i, y_i), (u^{i+1}, y_{i+1})) < \varepsilon \text{ for } i = 0, \ldots, q-1.$$

Let $\tau > 0$ such that $\beta := Ke^{-\alpha\tau} < 1$. Then $\left\| y_{i+1} - \varphi(\tau_i, u^i, y_i) \right\| < \varepsilon$ implies

$$\begin{aligned}\|y\| = \left\| y_q \right\| &\leq \left\| \varphi(\tau_{q-1}, u^{q-1}, y_{q-1}) \right\| + \varepsilon \leq \beta \left\| y_{q-1} \right\| + \varepsilon \\ &\leq \beta^2 \left\| \varphi(\tau_{q-2}, u^{q-2}, y_{q-2}) \right\| + \beta\varepsilon + \varepsilon\end{aligned}$$

$$\leq \beta^q \|y\| + \beta^{q-2}\varepsilon^{q-2} + \cdots + \beta\varepsilon + \varepsilon$$
$$< \beta^q \|y\| + (1-\beta\varepsilon)^{-1} - 1 + \varepsilon.$$

Since $(1-\beta\varepsilon)^{-1} - 1 + \varepsilon \to 0$ for $\varepsilon \to 0$ we may take $\varepsilon > 0$ small enough such that for any $q \geq 1$,

$$(1-\beta^q)\|y\| > (1-\beta\varepsilon)^{-1} - 1 + \varepsilon,$$

and hence $\|y\| > \beta^q \|y\| + (1-\beta\varepsilon)^{-1} - 1 + \varepsilon$. This contradiction shows that, for $\varepsilon > 0$ small enough and $\tau > 0$ large enough, there are no (ε, τ)-chains from (u, y) to (u, y).

(ii) Next we show that the chain recurrent set of $\mathcal{V}^+$ equals $\mathcal{U} \times \{0\}$. Suppose that $(u, y) \in \mathcal{V}^-$ with $y \neq 0$ is chain recurrent and consider for $\varepsilon \in (0, 1)$, $\tau > 0$ an (ε, τ)-chain from (u, y) to (u, y). Let $\tau > 0$ such that $\beta := Ke^{\alpha\tau} > 1$. Similarly as in (i) we compute

$$\begin{aligned}\|y\| = \|y_q\| &\geq \left\|y_q - \varphi(\tau_{q-1}, u^{q-1}, y_{q-1}) + \varphi(\tau_{q-1}, u^{q-1}, y_{q-1})\right\| \\ &\geq \left\|\varphi(\tau_{q-1}, u^{q-1}, y_{q-1})\right\| - \left\|y_q - \varphi(\tau_{q-1}, u^{q-1}, y_{q-1})\right\| \\ &> \beta\left\|y_{q-1}\right\| - \varepsilon \geq \beta^2\left\|y_{q-2}\right\| - \beta\varepsilon - \varepsilon \\ &\geq \beta^q \|y\| - (\beta\varepsilon)^{q-2} - (\beta\varepsilon)^{q-3} - \cdots - \beta\varepsilon - \varepsilon.\end{aligned}$$

Let $\varepsilon > 0$ be small enough such that $\beta\varepsilon < 1$. Then it follows that

$$\|y\| > \beta^q \|y\| - (1-\beta\varepsilon)^{-1} + 1 - \varepsilon.$$

For $\varepsilon > 0$ small enough this contradicts $\beta > 1$ and hence, for $\varepsilon > 0$ small enough and $\tau > 0$ large enough, there is no (ε, τ)-chain from (u, y) to (u, y) if $y \neq 0$.

(iii) Any (ε, τ)-chain in $\mathcal{U} \times \mathbb{R}^n = \mathcal{V}^+ \oplus \mathcal{V}^-$ projects to (ε, τ) chains in $\mathcal{V}^+$ and $\mathcal{V}^-$. Thus (i) and (ii) imply the assertion.

□

The following result characterizes the chain recurrent set for hyperbolic control semiflows.

Theorem 7.5 *Consider the affine control semiflow Φ on $\mathcal{U} \times M_2$ defined in (4.2) associated with the delay system (1.1). Suppose that* $\det A_p \neq 0$ *and that the linear part Φ^0 of Φ is hyperbolic. Then, for the linear semiflow Φ^0, the entire chain recurrent set is $\mathcal{R}^{\#}(\Phi^0) = \mathcal{U} \times \{0_{M_2}\}$ and, for the affine semiflow Φ*

$$\mathcal{R}^{\#}(\Phi) = H(\mathcal{R}^{\#}(\Phi^0)) = \{(u, -e(u, 0)) \,|u \in \mathcal{U}\},$$

where H is the homeomorphism defined in (7.1) and $e(u, t) \in M_2, t \in \mathbb{R}$ is the unique entire bounded solution of (3.6). The entire chain recurrent set $\mathcal{R}^{\#}(\Phi)$ is compact, invariant, and chain transitive.

Proof Lemma 7.4 shows that the entire chain recurrent set of Φ^0 is $\mathcal{R}^{\#}(\Phi^0) = \mathcal{U} \times \{0_{M_2}\}$. By Proposition 7.3 the set

$$H(\mathcal{R}^{\#}(\Phi^0)) = H(\mathcal{U} \times \{0_{M_2}) = \{(u, -e(u, 0)) \,|u \in \mathcal{U}\}.$$

is compact using that $\mathcal{U}$ is compact and $e(\cdot, 0)$ is continuous.

The set $\{(u, -e(u, 0)) \,|u \in \mathcal{U}\}$ is invariant since, by Proposition 7.3(i),

$$\Phi_t(u, -e(u, 0)) = (\theta_t u, -e(u, t)) = (\theta_t u, -e(\theta_t u, 0)), t \in \mathbb{R}.$$

The map H is uniformly continuous: In fact, for $\varepsilon > 0$ it follows by compactness of $\mathcal{U}$ and continuity of $e(\cdot, 0)$ that there is $\delta(\varepsilon) \in (0, \varepsilon/2)$ such that $d(u, u') < \delta(\varepsilon)$ and $\left\| y - y' \right\| < \delta(\varepsilon)$ implies

$$\begin{aligned} \left\| y - e(u, 0) - \left(y' - e(u', 0)\right) \right\| &\le \left\| y - y' \right\| \\ &+ \left\| e(u, 0) - e(u', 0) \right\| < \delta(\varepsilon) + \varepsilon/2 < \varepsilon. \end{aligned}$$

Hence $d(u, y), (u', y')) < \delta(\varepsilon)$ implies $d(H(u, y), H(u', y')) < \varepsilon$. Analogously one proves that the inverse of H given by $H^{-1}(u, y) = (u, y + e(u, 0))$ is uniformly continuous.

Let $\varepsilon, \tau > 0$ and consider $H(u, 0), H(u', 0) \in H(\mathcal{R}^{\#}(\Phi^0))$ with $u, u' \in \mathcal{U}$. By chain transitivity of $\mathcal{U}$ there is a $(\delta(\varepsilon), \tau)$-chain in $\mathcal{U} \times \{0_{M_2}\}$ from $(u, 0)$ to $(u', 0)$. Then H maps this chain onto an (ε, τ)-chain from $H(u, 0)$ to $H(u', 0)$. Since $\varepsilon, \tau > 0$ are arbitrary, this proves that $H(\mathcal{R}^{\#}(\Phi^0))$ is chain transitive and certainly this set is invariant and consists of points defining entire solutions.

It remains to prove that $H(\mathcal{R}^{\#}(\Phi^0))$ is the entire chain recurrent set of Φ. Let $\varepsilon > 0$. By uniform continuity of H^{-1} there is $\delta'(\varepsilon) > 0$ such that $d(u, y), (u', y')) < \delta'(\varepsilon)$ implies $d(H^{-1}(u, y), H^{-1}(u', y')) < \varepsilon$. For any chain recurrent point (u, y) of Φ and $\tau > 0$ there is a $(\delta'(\varepsilon), \tau)$-chain from (u, y) to (u, y). This is mapped by H^{-1} to an (ε, τ)-chain of Φ from $H^{-1}(u, y)$ to $H^{-1}(u, y)$. This proves that $H^{-1}(u, y) \in \mathcal{R}^{\#}(\Phi^0)$ and hence $(u, y) = H(H^{-1}(u, y)) \in H(\mathcal{R}^{\#}(\Phi^0))$. □

Next we use the linear lift to describe the image of the entire chain recurrent set.

Theorem 7.6 *Consider the delay control system (1.1) and suppose that* $\det A_p \neq 0$. *Assume, for the associated affine control semiflow* Φ *on* $\mathcal{U} \times M_2$ *defined in (4.2), that the linear part* Φ^0 *is hyperbolic.*

(i) *Then the lift* Φ^1 *on* $\mathcal{U} \times M_2^1$ *defined in (6.3) possesses an invariant one dimensional subbundle* $\mathcal{V}_c^1$ *of* $\mathcal{U} \times M_2^1$ *defined by*

$$\mathcal{V}_c^1 = \{(u, -re(u, 0), r) \in \mathcal{U} \times M_2 \times \mathbb{R} \,|u \in \mathcal{U}, r \in \mathbb{R}\}. \tag{7.2}$$

(ii) *The projection* $\mathcal{M}_c^1 = \mathbb{P}\mathcal{V}_c^1$ *to* $\mathcal{U} \times \mathbb{P}M_2^1$ *is a compact subset of* $\mathcal{U} \times \mathbb{P}M_2^{1,1}$ *and coincides with the image of the entire chain recurrent set of* Φ*, i.e.,*

$$\mathcal{M}_c^1 = \left\{(u, \mathbb{P}(x, 1)) \in \mathcal{U} \times \mathbb{P}M_2^1 \,\middle|\, (u, x) \in \mathcal{R}^{\#}(\Phi) \right\}. \tag{7.3}$$

Proof (i) Denote by $\mathcal{V}_*^1$ the right hand side of (7.2). For every $u \in \mathcal{U}$ the fiber $\{(u, -re(u,0), r), r \in \mathbb{R}\}$, is one-dimensional and $\mathcal{V}_*^1$ is closed. In fact, suppose that a sequence $(u^k, -r_k e(u^k, 0), r_k), k \in \mathbb{N}$ in this set converges to $(u, x, r) \in \mathcal{U} \times M_2 \times \mathbb{R}$. Then $u^k \to u$ and $r_k \to r$ and, by continuity of $e(\cdot, 0)$, it follows that $r_k e(u^k, 0) \to re(u, 0)$. This shows that $(u, x, r) = (u, -re(u, 0), r) \in \mathcal{V}_*^1$. According to Blumenthal and Latushkin [5, Lemma 3.8] it follows that $\mathcal{V}_c^1 = \mathcal{V}_*^1$ is a one dimensional subbundle of $\mathcal{U} \times M_2 \times \mathbb{R}$. This subbundle is invariant, since by Proposition 7.3

$$\Phi_t^1(u, -re(u,0), r) = (\theta_t u, -re(u,t), r) = (\theta_t u, -re(\theta_t u, 0), r) \text{ for } t \in \mathbb{R}.$$

(ii) The equality in (7.3) follows from the definitions and Theorem 7.5. This also implies that the set $\mathcal{R}^\#(\Phi)$ is compact and hence the set $\{(u, -e(u,0), 1) \in \mathcal{U} \times M_2 \times \mathbb{R} \,|\, u \in \mathcal{U}\}$ is also compact. It follows that the right hand side of (7.3) is compact. Finally. Proposition 6.6(ii) shows that $\mathcal{M}_c^1 \subset \mathcal{U} \times M_2^{1,1}$.

□

This result implies the following consequences for chain control sets; cf. Colonius and Santana [11, Theorem 35] for the finite dimensional case.

Corollary 7.7 *Consider the delay system (1.1) and suppose that* $\det A_p \neq 0$ *and that the linear part* Φ^0 *of the semiflow* Φ *is hyperbolic. Then the chain control set* E *of system (1.1) is compact, its lift* $\mathcal{E}$ *to* $\mathcal{U} \times M_2$ *coincides with the entire chain recurrent set of the control semiflow* Φ*, i.e.,* $\mathcal{E} = \mathcal{R}^\#(\Phi)$*, and for every* $u \in \mathcal{U}$ *there is a unique element* $x \in E$ *with* $\psi(t, x, u) \in E$ *for all* $t \in \mathbb{R}$*. Furthermore, the image of* E *in* $\mathbb{P}M_2^1$ *satisfies*

$$\{\mathbb{P}(x,1) \,|\, x \in E\} = \{\mathbb{P}(x,1) \,\Big|\, \exists u \in \mathcal{U} : (u, \mathbb{P}(x,1)) \in \mathcal{M}_c^1 = \mathbb{P}\mathcal{V}_c^1\}.$$

Proof By Theorem 7.5, the chain recurrent set $\mathcal{R}(\Phi^\#)$ is compact, invariant, and chain transitive. Hence Theorem 5.2(iii) implies that it is the lift of a chain control set, i.e., by Theorem 5.4 it is the lift of the unique chain control set E. Then the second assertion follows by Theorem 7.6(ii). □

Funding Open Access funding enabled and organized by Projekt DEAL.

References

1. Alongi, J.M., Nelson, G.S.: Recurrence and Topology, Grad. Stud. Math. Vol. 85, Am. Math. Soc., Providence, RI, (2007)
2. Antunez, M.B., Mantovani, G.E., Varão, R.: Chain recurrence and positive shadowing in linear dynamics. J. Math. Anal. Appl. **506**(1), 125622 (2022)
3. Ayala, V., Da Silva, A., San Martin, L.A.B.: *Control systems on flag manifolds and their chain control sets*, Discrete Cont. Dyn. Syst., 37(5), (2017), pp. 2301-2313. https://doi.org/10.3934/dcds.2017101.
4. Bensoussan, A., Da Prato, G., Delfour, M.C., Mitter, S.K.: *Representation and Control of Infinite Dimensional Systems.* Systems and Control: Foundations and Applications, 2nd edn., Birkhäuser Inc., Boston, (2007)
5. Blumenthal, A., Latushkin, Yu.: The Selgrade decomposition for linear semiflows on Banach spaces. J. Dyn. Diff. Eqs. **31**(3), 1427–1456 (2019)
6. Bronstein, I.U., Kopanskii, A.. Ya..: Chain recurrence in dynamical systems without uniqueness, Nonlinear Analysis. Theory Methods Appl. **12**(2), 147–154 (1988)
7. Cima, A., Llibre, J.: Bounded polynomial vector fields. Trans. Am. Math. Soc. **318**(2), 557–579 (1990)
8. Colonius, F., Kliemann, W.: The Dynamics of Control. Birkhäuser, Boston (2000)
9. Colonius, F., Kliemann, W.: *Dynamical Systems and Linear Algebra,* Grad. Stud. Math. Vol. 156, Providence, RI, (2014)
10. Colonius, F., Santana, A.J.: Topological conjugacy for affine-linear flows and control systems. Commun. Pure Appl. Anal. **10**(3), 847–857 (2011)
11. Colonius, F., Santana, A.J.: Chain recurrence and Selgrade's theorem for affine flows. J. Dynam. Differential Equations (2024). https://doi.org/10.1007/s10884-024-10351-0, published online 4 March
12. Colonius, F., Santana, A.J., Viscovini, E.C.: Chain controllability of linear control systems. SIAM J. Control Optim. **62**(4), 2387–2411 (2024). https://doi.org/10.1137/23M162634
13. Da Silva, A.:, *The chain control set of a linear control system,* preprint, arXiv:2306.12936v1 [math.OC], (2023)
14. Da Silva, A., Kawan, C., Da Silva, A.: Hyperbolic chain control sets on flag manifolds. J. Dyn. Control Syst. **22**, 725–745 (2016). https://doi.org/10.1007/s10883-015-9308-1
15. Diekmann, O., van Gils, S.A., Verduyn Lunel, S.M., Walther, H.-O.: *Delay Equations. Functional, Complex, and Nonlinear Analysis*. Applied Mathematical Sciences, vol. 110 (Springer, New York) (1995)
16. Delfour, M.C.: State theory of linear hereditary differential systems. J. Math. Anal. Appl. **60**, 8–35 (1977)
17. Curtain, R., Zwart, H.: *Introduction to Infinite-Dimensional Systems Theory*, Springer-Verlag (2020)
18. Hale, J.: *Theory of Functional Differential Equations*, Springer-Verlag (1977)
19. Hinrichsen, D., Oeljeklaus, E.: *Are delay-differential systems generically controllable?*, Math. Control, Signals, Syst. https://doi.org/10.1007/s00498-022-00329-y
20. Kawan, C.: *Invariance Entropy for Deterministic Control Systems*, Lecture Notes in Math. Vol 2089, Springer Cham, (2013)
21. Kawan, C.: Uniformly hyperbolic control theory. Ann. Rev. Control **44**, 89–96 (2017)
22. Kloeden, P.E., Rasmussen, M.: Nonautonomous Dynamical Systems. Amer. Math, Soc (2011)
23. Li, Desheng: Morse decompositions for general dynamical systems and differential inclusions with applications to control systems. SIAM J. Control Opt. **46**(1), 35–60 (2007)
24. Manitius, A.: Completeness and F-completeness of eigenfunctions associated with retarded functional differential equations. J. Diff. Eqs. **35**, 1–29 (1980)
25. Manitius, A.: Necessary and sufficient conditions of approximate controllability for general linear retarded systems. SIAM J Control Opt. **19**(4), 516–532 (1981)
26. Pazy, A.: Semigroups of Linear Operators and Applications to Partial Differential Equations. Springer, New York (1983)
27. Perko, L.: *Differential Equations and Dynamical Systems,* Springer, 3rd ed., Cham, (2001)
28. Poincare, H., Memoire sur les courbes definies par une equation differentielle, J. Mathematiques, 7,: pp. 375–422; Ouevre Tome I. Gauthier-Villar, Paris **1928**, 3–84 (1881)
29. Robinson, C.: *Dynamical Systems: Stability, Symbolic Dynamics, and Chaos* Taylor & Francis Inc., 2nd ed., (1998)
30. Selgrade, J.: Isolated invariant sets for flows on vector bundles. Trans. Am. Math. Soc. **203**, 259–390 (1975)

31. Sell, G.R., You, Y.: Dynamics of Evolutionary Equations. Applied Mathematical Sciences, vol. 143. Springer, New York (2002)