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Finite-Time Lyapunov Exponents for SPDEs with Fractional Noise

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Abstract
We estimate the finite-time Lyapunov exponents for a stochastic partial differential equation driven by a fractional Brownian motion (fbm) with Hurst index $H \in (0, 1)$ close to a bifurcation of pitchfork type. We characterize regions depending on the distance from bifurcation, the Hurst parameter of the fbm and the noise strength where finite-time Lyapunov exponents are positive and thus indicate a change of stability. The results on finite-time Lyapunov exponents are novel also for SDEs perturbed by fractional noise.

Keywords Fractional Brownian motion · Finite-time Lyapunov exponents · Amplitude equations · Bifurcations for SPDEs

Mathematics Subject Classification 60H15 · 60H10 · 37H15 · 37H20

1 Introduction

The main goal of this work is to provide a tool for an analysis of a pitchfork-type bifurcation for SPDEs perturbed by fractional noise given by:

$$\begin{cases} \mathrm{d}u = [Au + \nu u + \mathcal{F}(u)]\,\mathrm{d}t + \sigma \mathrm{d}W_t^H. \\ u(0) = u_0. \end{cases} \tag{1.1}$$

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Here A is a linear operator with a one-dimensional kernel, the parameter $\nu \in \mathbb{R}$ shifts the spectrum of A, $\mathcal{F}$ is a stable cubic nonlinearity and $\sigma > 0$ denotes the intensity of the infinite-dimensional noise, which is given by a Hilbert-space-valued fractional Brownian motion $(W^H(t))_{t\in[0,T]}$ with Hurst index $H \in (0, 1)$. The stochastic Allen–Cahn and Swift–Hohenberg equations are covered by this framework. The results obtained in this work provide a novel bifurcation analysis also for stochastic differential equations (SDEs) driven by fractional Brownian motion (fBm). Fractional Brownian motion is a famous example used in order to model memory effects or long-range dependencies. A fBm is a centered stationary Gaussian processes parameterized by the so-called Hurst index/parameter $H \in (0, 1)$. For $H = 1/2$, one recovers the classical Brownian motion. However, for $H \in (1/2, 1)$ and $H \in (0, 1/2)$, fBm exhibits a totally different behavior compared to the Brownian motion. Its increments are no longer independent, but positively correlated for $H > 1/2$ and negatively correlated for $H < 1/2$. Fractional Brownian motion has been used to model a wide range of phenomena such as stock prices and financial markets (Stone 2018), activity of neurons (Richard et al. 2018), dynamics of the nerve growth (Odde et al. 1996) or fluid dynamics (Weiss 2013; Nourdin 2012, Section 2.6).

However, due to its non-Markovianity, dynamical aspects such as bifurcation theory have not been investigated for systems perturbed by a fBm. Here we contribute to this aspect and analyze a classical pitchfork-type bifurcation for equations driven by a fbm using finite-time Lyapunov exponents. In contrast to the Brownian motion (Bedrossian et al. 2022; Blumenthal et al. 2023; Gess and Tsatsoulis 2024; Blömker and Neamţu 2023), to our best knowledge no methods have been developed so far for the computation of (finite-time or asymptotic) Lyapunov exponents. The work (Kuehn et al. 2022) investigates related aspects using early-warning signs to detect changes of stability in a finite-dimensional slow-fast system perturbed by general non-Markovian noise, in particular fractional Brownian motion with Hurst index $H > 1/2$. The analysis of the finite-time Lyapunov exponents for the SDE

$$\mathrm{d}x = (\nu x - x^3)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t \tag{1.2}$$

driven by a Brownian motion around its unique random equilibrium splits into different cases.

For $\nu < 0$, one relies on deterministic stability and shows that FTLE is negative with probability one. This is in other settings well known and was also only briefly sketched in Blömker and Neamţu (2023).

For $\nu > 0$, one essentially relies on the structure of the invariant measure of (1.2), where one needs that this has to be close to zero on a set of positive probability. Together with an estimate of the paths of the Brownian motion on a finite-time horizon, the estimates for the FTLE on a set of positive probability become quite accessible (Callaway et al. 2017).

All these arguments, on which the analysis in Blömker and Neamţu (2023) heavily relies, break down in the context of a fractional Brownian motion. Obviously, the events describing the location of the random equilibrium at the initial time and the finite-time estimates of the paths of the fractional Brownian motion are in general not independent anymore. In order to overcome this issue, we focus on deterministic initial data and expect to obtain positive finite-time Lyapunov exponents for the SDE (1.2) with a fractional Brownian motion. This technical step is justified in Lemma 4.1 which relies on a support theorem for the solution of SDEs perturbed by fractional noise (Hairer and Ohasi 2007). This implies that for any deterministic initial data, the solution of (1.2) is located in a ball around zero with positive probability. This allows us to observe a positive finite-time Lyapunov exponent with positive probability for (1.2) and provides a novel insight in the dynamics of SDEs perturbed by fractional noise.

In order to analyze the SPDE case as in Blömker and Neamţu (2023), we rely on the approximation with amplitude equations (AE) also studied in Blömker and Neamţu (2022); Blömker et al. (2015). These results are valid for the full range of Hurst indices $H \in (0, 1)$ and are applicable if the noise is given by a trace-class fractional Brownian motion. This assumption was crucial in order to control the approximation order, which further reflects the time-scale on which we observe the finite-time Lyapunov exponents. The approach for amplitude equations follows the result of Blömker and Neamţu (2023); nevertheless, we need to adapt the scaling in space time and keep track of the various H-dependent error terms in the approximation via AE and heavily influence the final result on FTLE.

Main Results. Relying on the self-similarity of the fractional Brownian motion

$$W^H(T\varepsilon^{-2}) \overset{\text{law}}{=} W^H(T)\varepsilon^{-2H} \quad \text{and of the derivative} \quad \dot{W}^H(T\varepsilon^{-2}) \overset{\text{law}}{=} \varepsilon^{2-2H}\dot{W}^H(T),$$

the approximation of the SPDE (2.1) via amplitude equations was derived in Blömker and Neamţu (2022). Based on this result and using the theory of finite-time Lyapunov exponents (see Sect. 3.1), we show the following behavior of FTLE from (1.1) near a change of stability. The precise statement is based on the interplay between the distance toward the bifurcation, intensity of the noise and Hurst parameter of the fbm:

I. *Before the bifurcation,* $\nu < 0$. The solutions of (1.1) are stable for all σ with probability one. Therefore, we show in Theorem 6.1 that all FTLEs are negative with probability one.

II. *After the bifurcation, moderate noise strength,* $0 < \sigma \approx \nu^{H+\frac{1}{2}} \ll 1$. Here we have According to Theorem 6.3 instability, namely there is a solution for which the finite-time Lyapunov exponent $\lambda_T > 0$ is positive with positive probability, for times of order $1/\nu^{H+\frac{1}{2}}$. Due to the nature of the approximation result (Theorem 5.3), our result is applicable for times T of order $1/\nu^{H+1/2}$ but for technical reasons not up to 0. This means there is a ν dependent interval, where we can prove that the FTLE is positive and this interval contains values of the type $C/\nu^{H+1/2}$ for some values of the constant $C > 0$.
The proof of this statement relies on the approximation with the amplitude equation $\mathrm{d}b = (b - b^3)\,\mathrm{d}T + \frac{\sigma}{\nu^{\frac{1}{2}+H}}\,\mathrm{d}\beta^H(T)$, as established in Theorem 5.3, where $(\beta^H(T))_{T\geq 0}$ is a fractional Brownian motion. A novel argument in the proof of this statement is to start the amplitude equation in a suitably rescaled deterministic initial data for which the solution of the amplitude equation is located in a ball around zero with positive probability.

III. *After the bifurcation, small noise strength,* $0 < \sigma \ll \nu^{H+\frac{1}{2}} \ll 1$. Similar to case II. we observe in Theorem 6.7 instability using the amplitude equation $\mathrm{d}b = (b - b^3)\,\mathrm{d}T$.
IV. *At the bifurcation,* $0 \leq \nu^{H+\frac{1}{2}} \ll \sigma \ll 1$. Here we have stability as shown in Theorem 6.10, meaning that $\lambda_T < 0$ for all solutions with positive probability. The proof of this statement relies on the approximation with the amplitude equation of the type $\partial_T b = -b^3 \mathrm{d}T + \mathrm{d}\beta^H(T)$, where $(\beta^H(T))_{T\geq 0}$ is a fractional Brownian motion.

This analysis provides novel results for finite-time Lyapunov exponents with fractional noise and also improves the results in Blömker and Neamţu (2023) for the Brownian motion. More precisely in order to prove the statement II., we show that it is not necessary to start the solutions of the SPDE in the rescaled attractor of the SDE but in an arbitrary rescaled small deterministic initial condition. This information allows us to apply a support theorem (Hairer and Ohasi 2007, Proposition 5.8) from which we can further find positive finite-time Lyapunov exponents for SDEs with fractional additive noise and stable cubic nonlinearities with positive probability.

We transfer these results to the finite-time Lyapunov exponents of the SPDE (2.1) approximating it with an amplitude equation as established in Theorem 5.3. A technical step is to quantify the dependence on H of the approximation error between the linearization of the SPDE and the linearization of the amplitude equation around a solution which satisfies the support theorem.

Remark 1.1 Due to the nature of the approximation result (Theorem 5.3), our result is only applicable for positive times T but for technical reasons not up to 0. The main reason is that the error estimate derives a uniform bound of the error to the SDE valid for all times, but as we divide by the time in the Lyapunov exponent, the error estimate is not sufficient for small times. Nevertheless, we conjecture that a careful analysis of the time dependence of the error might close this gap. But this requires completely different methods for the estimates than the one established in the theory of AEs.

Remark 1.2 Another technical novelty and crucial difference to Blömker and Neamţu (2023) arises in the proof of case IV. Here we establish in Lemma 4.2 a lower bound on the probability that the amplitude equation is small for lots of times. This argument does not rely on stationary solutions and Birkhoff's ergodic theorem as for the case of a Brownian motion (Blömker and Neamţu 2023, Lemma 4.2), as this is not available for solutions of SDEs driven by fractional noise. A crucial step in this argument is given by the fact that the density of the amplitude equation perturbed by fractional noise is not concentrated in zero (Besalú et al. 2016).

In conclusion, Lemmas 4.1 and 4.2, on which the main results rely on, have been obtained by totally different methods than those used in Blömker and Neamţu (2023). Moreover, we improve these results by analyzing the dynamics of arbitrary solutions instead of stationary solutions as considered in Blömker and Neamţu (2023), Blumenthal et al. (2023).

In all cases I.–IV. we compute an error term between the two linearizations for $H \in (0, 1)$ obtaining different bounds compared to Blömker and Neamţu (2023) depending on the range of H.

The paper is organized in the following way: Section 2 states all necessary assumptions and establishes the setting we are working in. In Sect. 3, we give a short remark about existence and uniqueness of solutions for our SPDE. The properties of FTLE for the corresponding AE are studied in Sect. 4, while Sect. 5 provides the key approximation result for AE. The final Sect. 6 states the main results in full details and provides their proofs.

2 Setting and Assumptions

We work in the following setting. We let $\mathcal{H}$ stand for a separable Hilbert space and consider the SPDE driven by an Hilbert-space valued fractional Brownian motion $(W^H(t))_{t\in[0,T]}$ with Hurst index $H \in (0, 1)$

$$\begin{cases} \mathrm{d}u = [Au + \nu u + \mathcal{F}(u)]\,\mathrm{d}t + \sigma \mathrm{d}W_t^H \\ u(0) = u_0 \in \mathcal{H}. \end{cases} \tag{2.1}$$

We make the following standard assumptions on the linear operator A and on the cubic nonlinearity $\mathcal{F}$.

Assumption 2.1 (*Differential operator* A) The linear operator A generates a compact analytic semigroup $(e^{tA})_{t\geq 0}$ on $\mathcal{H}$. Moreover, it is symmetric and non-positive and has a one-dimensional kernel which we denote by $\mathcal{N}$. We define the orthogonal projection P_c onto $\mathcal{N}$, set $P_s = \mathrm{Id} - P_c$ and obtain that $\mathcal{H} = \mathcal{N} \oplus \mathcal{S}$, where $\mathcal{S}$ stands for the range of P_s. The semigroup is exponentially stable on $P_s\mathcal{H}$ which means that there exists $\mu > 0$ such that

$$\|e^{tA}P_s\|_{\mathcal{L}(\mathcal{H})} \leq e^{-t\mu}, \quad \text{for all } t \geq 0.$$

We further define the spaces $\mathcal{H}^\alpha = D((1-A)^\alpha)$ for $\alpha \geq 0$ endowed with the norm $\|\cdot\|_\alpha = \|(1-A)^\alpha \cdot\|$ and scalar product $\langle u, v\rangle_\alpha = \langle (1-A)^\alpha u, (1-A)^\alpha v\rangle$ and set $\mathcal{H}^{-\alpha} = (\mathcal{H}^\alpha)^*$ the dual of $\mathcal{H}^\alpha$. It is well-known that $(e^{tA})_{t\geq 0}$ is an analytic semigroup on $\mathcal{H}^\alpha$ for every $\alpha \in \mathbb{R}$. Finally, we have that $\mathcal{N} \subset \mathcal{H}^\alpha$ for all $\alpha > 0$ since $(1-A)^\alpha \mathcal{N} = \mathcal{N}$.

Under our assumptions, we have for some constant $C > 0$ depending on $\alpha > 0$ that $\|A^\alpha P_s u\| \geq C\|P_s u\|$ for all $u \in \mathcal{H}$, which we use frequently.

Assumption 2.2 (*Nonlinearity*) We assume that there exists a Banach space X such that

$$\mathcal{H}^\alpha \subset X \subset \mathcal{H}$$

for $\alpha \in (0, 1/2)$ with continuous and dense embeddings. Moreover, the mapping $\mathcal{F} : X \to X^* \subset \mathcal{H}^{-\alpha}$ is a stable cubic (i.e., trilinear) nonlinearity with

$$\langle \mathcal{F}(u) - \mathcal{F}(v), u - v\rangle \leq -c\|u - v\|_X^4, \quad \text{for } u, v \in X. \tag{2.2}$$

Let us remark that we can allow terms like $C\|u-v\|^2$ on the r.h.s. of (2.2), but we can always modify the linear term to remove these terms.

Assumption 2.3 (*Noise*) We assume that $(W^H(t))_{t\in[0,T]}$ is a trace-class fractional Brownian motion on $\mathcal{H}$. This assumption was made in Blömker and Neamţu (2022), and in particular it implies that the stochastic convolution

$$Z(t) = \int_0^t e^{A(t-s)}\,\mathrm{d}W_s$$

is well-defined and $Z \in C([0,T];\mathcal{H}^\alpha)$ for $\alpha < H$.

Remark 2.4 For $\mathcal{F}$, we can show the following sign condition. For any positive $\delta > 0$, there is a constant $C > 0$ depending on δ such that for all $u, z \in X$

$$\langle \mathcal{F}(u+z), u\rangle \le -c\|u+z\|_X^4 + C\|u+z\|_X^3\|z\| \le -\delta\|u\|_X^4 + C_\delta\|z\|_X^4. \quad (2.3)$$

As $\mathcal{F}$ is trilinear, we readily have that $\mathcal{F}$ is Fréchet-differentiable with

$$D\mathcal{F}(u)[h] = \mathcal{F}(u,u,h) + \mathcal{F}(u,h,u) + \mathcal{F}(h,u,u).$$

Moreover, for $u, h \in X$ we obtain due to (2.3)

$$\langle D\mathcal{F}(u)h, h\rangle = \lim_{t\to 0}\frac{1}{t}\langle \mathcal{F}(u+th) - \mathcal{F}(u), h\rangle \le -\lim_{t\to 0}\frac{1}{t^2}\|th\|^4 = 0. \quad (2.4)$$

Let us remark additionally, that we use an estimate like $D\mathcal{F}_c(b) \le -cb^2$ in our proof that arises from the one-dimensionality of $\mathcal{N}$. In order to remove the condition that $\mathcal{N}$ is one-dimensional, we will need that $\mathcal{F}_c$ is a genuine non-degenerate cubic term with an analogous estimate. We refer to potentials similar to Gess and Tsatsoulis (2024), where for example $\mathcal{F}(b) = -cb|b|^2$ was treated.

In Blömker and Neamţu (2022), we used the following definition of the $\mathcal{O}$ notation.

Definition 2.5 We say that a term $F_\varepsilon = \mathcal{O}(f_\varepsilon)$ if and only if there exist positive ε-independent constants C and ε_0 such that $|F_\varepsilon| \le Cf_\varepsilon$ for all $\varepsilon \in (0, \varepsilon_0]$.

For a random quantity, we write $F_\varepsilon = \mathcal{O}(f_\varepsilon)$ if the above statement holds true on a set with probability going to 1 if $C \to \infty$.

Assumption 2.6 For the stochastic convolution, we have for every small $\kappa > 0$

$$P_s Z = \mathcal{O}(T^\kappa) \qquad \text{and} \qquad P_c Z = P_c W^H = \mathcal{O}(T^H) \quad (2.5)$$

uniformly in T on any interval $[0, T_0]$ in the space X.

Remark 2.7 These bounds were obtained using the scaling properties of the fractional Brownian motion and the factorization method in Blömker and Neamţu (2022, Appendix B). We have for every small $\kappa > 0$

$$\sup_{t\in[0,T_0\varepsilon^{-2}]} \|P_c W^H(t)\| \overset{\text{law}}{=} \varepsilon^{-2H} \sup_{T\in[0,T_0]} \|P_c W^H(T)\| = \mathcal{O}(\varepsilon^{-2H-\kappa})$$

with probability almost 1, whereas, with high probability

$$\sup_{t\in[0,T_0\varepsilon^{-2}]} \|P_s Z(t)\| = \mathcal{O}(\varepsilon^{-\kappa}).$$

3 Existence of Solutions

The existence of solutions of SPDE with additive fractional noise and stable cubic nonlinearities was established in Maslowski and Schmalfuß (2004, Theorem 4.3). In order to obtain some regularity properties of the solution, we briefly sketch an alternative proof similar to the case of the Brownian motion (Blömker and Neamţu 2023) which relies on the Galerkin method and the standard transformation $w = u - Z$ that solves the random PDE

$$\partial_t w = Aw + \nu w + \mathcal{F}(w + Z).$$

For this equation, one can apply classical pathwise existence results, see for example Roger (1997), Tomáš (2013). This is based on (2.3) giving regularity in $L^4(0, T, X)$, together with the compact embedding of X into $\mathcal{H}^{1/2}$ and Aubin-Lions Lemma.

For initial conditions in $\mathcal{H}$ and Z being a continuous stochastic process with values in $\mathcal{H}^\alpha \subset X$, this shows global existence of solutions such that for all $T > 0$

$$u - Z \in L^2(0, T, \mathcal{H}^{1/2}) \cap C^0([0, T], \mathcal{H}) \cap L^4(0, T, X)$$

which also implies some regularity of $\partial_t(u - Z)$ as $A(u - Z) \in L^2(0, T, \mathcal{H}^{-1/2})$ and $\mathcal{F}(u) \in L^{4/3}(0, T, X^*)$.

The pathwise uniqueness of solutions follows immediately from (2.2). For the difference $d = u_1 - u_2$ of two solutions u_1 and u_2 satisfying

$$\partial_t d = Ad + \nu d + \mathcal{F}(u_1) - \mathcal{F}(u_2)$$

we only need the differentiability of the $\mathcal{H}$-norm to conclude

$$\partial_t \|d\|^2 = \langle Ad + \nu d + \mathcal{F}(u_1) - \mathcal{F}(u_2), d\rangle \le \nu\|d\|^2.$$

The differentiability of the norm follows, as we have

$$d = (u_1 - Z) - (u_2 - Z) \in L^2(0, T, \mathcal{H}^{1/2}) \cap L^\infty(0, T, \mathcal{H})$$

by standard parabolic regularity together with $d \in L^4(0,T,X)$ and $\mathcal{F}(u_i) \in L^{4/3}(0,T,X^*)$.

With the arguments sketched above, one can prove the following theorem, which we state without proof.

Theorem 3.1 *Let Assumptions 2.1, 2.2, 2.3 be satisfied. Then, for all initial conditions $u_0 \in \mathcal{H}$ there is a unique (up to global null sets) stochastic process u with continuous paths in $\mathcal{H}$, which is a weak solution of (2.1) and satisfies for all $T > 0$*

$$u - Z \in L^2(0,T,\mathcal{H}^{1/2}) \cap C^0([0,T],\mathcal{H}) \cap L^4(0,T,X).$$

3.1 Finite-Time Lyapunov Exponents

The linearization $D_{u_0}u(t,\omega,u_0)$ of (1.1) around a solution $u(t,\omega,u_0)$ with deterministic initial condition u_0 is defined as the solution $v(t,\omega,u_0,v_0)$ of the linear PDE called also the first variation equation, which due to the additive structure of the noise is given by: see Blumenthal et al. (2023), Blömker and Neamţu (2022)

$$\begin{cases} \mathrm{d}v = [Av + \nu v + D\mathcal{F}(u)v]\,\mathrm{d}t \\ v(0) = v_0. \end{cases} \tag{3.1}$$

Remark 3.2 The Fréchet differentiability of the solution operator $u_0 \mapsto u(t,\omega,u_0)$ follows regarding that $u \in L^2(0,T;\mathcal{H}^{1/2})$ due to Debussche (1998, Lemma 4.4).

For $t > 0$, we denote the random solution operator $U_{u_0}(t) : \mathcal{H} \to \mathcal{H}$ such that $v(t) = U_{u_0}(t)v_0$, where v is a solution of (3.1) given the initial condition $v_0 \in \mathcal{H}$.

Remark 3.3 Note that for any solution $u \in L^4(0,T,X)$ we have $\mathcal{F}(u) \in L^{4/3}(0,T,X^*) \subset L^{4/3}(0,T,\mathcal{H}^{-\alpha})$. We can now use pathwise deterministic theory for linear PDEs. For example, Galerkin methods show that for given $v_0 \in \mathcal{H}$ there is an (up to global null sets) unique stochastic process v with continuous paths in $\mathcal{H}$ and $v \in L^2(0,T;\mathcal{H}^{1/2})$ for all $T > 0$ that solves (3.1).

We define the finite-time Lyapunov exponent as in Blumenthal et al. (2023), Blömker and Neamţu (2022).

Definition 3.4 *(Finite-time Lyapunov exponent).* Let $t > 0$ be fixed. We call a finite-time Lyapunov exponent for a solution u of the SPDE with (random) initial condition u_0

$$\lambda_t(u_0) := \lambda(t,\omega,u_0) = \frac{1}{t}\ln\left(\|U_{u_0}(t)\|_{\mathcal{L}(\mathcal{H})}\right). \tag{3.2}$$

From the definition, it is clear that finite-time Lyapunov exponents measure local expansion rates of nearby solutions. Negative finite-time Lyapunov exponents indicate attraction, whereas positive ones indicate that nearby solutions tend to separate on a finite-time horizon.

Remark 3.5 We can compute $\|U_{u_0}\|_{\mathcal{L}(\mathcal{H})}$ as follows

$$\begin{aligned}\|U_{u_0}(t)\|_{L(\mathcal{H})} &= \sup\{\|v(t)\|/\|v(0)\| \;\; v \text{ solves (3.1) with } v(0) \neq 0\} \\ &= \sup\{\|v(t)\| \;\; v \text{ solves (3.1) with } \|v(0)\| = 1\}.\end{aligned}$$

Remark 3.6 Let us comment on the following.

(1) Both finite-time and asymptotic Lyapunov exponents have not been investigated for S(P)DEs with fractional noise so far.
(2) For technical reasons which will be explained in Sect. 4, we restrict ourselves to deterministic initial data u_0. The independence of u_0 from the fractional Brownian motion helps us.

4 Finite-Time Lyapunov Exponents for Amplitude Equations with Fractional Noise

In this setting, $\sigma > 0$ and $\nu \geq 0$ are fixed quantities that depend on a small parameter ε and we assume the following upper bound:

$$\nu = \mathcal{O}(\varepsilon^2) \text{ and } \sigma = \mathcal{O}(\varepsilon^{2H+1}).$$

Using the cubic nonlinearity and the interplay between ν and σ, we later obtain amplitude equations of two types. In case of $\sigma\nu^{-\frac{1}{2}-H} = \mathcal{O}(1)$, we have

$$\mathrm{d}b = (b + \mathcal{F}_c(b))\,\mathrm{d}T + \frac{\sigma}{\nu^{\frac{1}{2}+H}}\,\mathrm{d}\beta^H_{\nu^H}(T) \tag{4.1}$$

whereas in the case $\nu^{H+1/2} \ll \sigma$

$$\mathrm{d}b = \mathcal{F}_c(b)\,\mathrm{d}T + \mathrm{d}\beta^H_{\sigma^H}(T), \tag{4.2}$$

where $(\beta^H_\gamma(T))_{T\in[0,T_0]}$ is an $\mathcal{N}$-valued fractional Brownian motion rescaled in time by a factor γ, meaning that $\beta^H_\gamma(T) = \gamma^{2H}\beta^H(T\gamma^{-2})$ for some fractional Brownian motion β^H.

Lemma 4.1 (Positive FTLE for (4.1)) *Fix $T_0 > 0$. If $\sigma\nu^{-\frac{1}{2}-H} = \mathcal{O}(1)$, then there is an $\eta > 0$ such that for $|b_0| < \eta$ sufficiently small, then*

$$\mathbb{P}\left(\lambda_T(b_0) \geq \frac{1}{4}\right) > 0 \quad \textit{for all } T \in [0, T_0].$$

Proof Let us think of b_0 being random and introduce the sets

$$A_1 := \{\omega \in \Omega : b_0(\omega) \in (-\eta, \eta)\},$$

$$A_2 := \left\{ \omega \in \Omega : \sup_{t \in [0,T]} \beta^H_{\nu^H}(t) \leq \frac{\eta}{2} \right\}.$$

Here b_0 is the initial data of (4.1), $\eta > 0$ is small and $\beta^H_{\nu^H}$ is just a rescaled fbm. Note that due to the non-Markovianity of the noise the events A_1 and A_2 are in general not independent as in the Brownian case. Therefore, in order to guarantee that $\mathbb{P}(A_1 \cap A_2) > 0$ we consider only deterministic initial conditions b_0 (independent of the fractional Brownian motion) and restrict ourselves to $\omega \in \tilde{\Omega} := A_2$ where the fractional Brownian motion remains small for a finite-time horizon. In this case, we derive for $\omega \in \tilde{\Omega}$ that

$$|b(T)| \leq \left(1 + \frac{\sigma}{\nu^{\frac{1}{2}+H}}\right)\eta e^T < \delta \text{ for all } T \in [0, T_0].$$

This statement follows using the equation (4.1), the stable cubic nonlinearity and the fact that $\omega \in \tilde{\Omega}$. Alternatively one can apply the support theorem (Hairer and Ohasi 2007, Proposition 5.8) which states that for every deterministic initial data and every path of the fbm, the solution of (4.1) will reach a small neighborhood of the origin with positive probability. This information, combined with the fact that the noise remains bounded on the finite-time interval $[0, T]$ which we consider, provides the upper bound on b. Analogously to the case of the Brownian motion, such a bound on the solutions implies the positivity of the FTLEs on the set of positive probability $\tilde{\Omega}$ since

$$\lambda_T(b_0) = \frac{1}{T} \ln \left(\exp \left(T + \int_0^T D\mathcal{F}_c(b(s,\omega))\, \mathrm{d}s \right) \right) \geq \frac{1}{4},$$

choosing $\delta := \frac{1}{2}$. □

For our result, we cannot rely on Birkhoffs ergodic theorem, instead we use a simple argument similar to Blömker (2007, Theorem 3.4) to show that the set of times for which the amplitude equation is small has a small probability. There it was used for $\nu \leq 0$ in order to show pattern formation below the threshold of stability.

Lemma 4.2 (Negative FTLE for (4.2)) *Suppose that* $\nu^{H+1/2} \ll \sigma$, *fix* $T_0 > 0$, *and consider a solution of (4.2). Then, for all* $\theta > 0$ *there is a small positive time* $T_\theta \to 0$ *for* $\theta \to 0$ *such that uniformly for all* $T \in [T_\theta, T_0]$

$$\mathbb{P}\left(\lambda_T(b_0) < -c\theta^2\right) \to 1 \text{ as } \theta \to 0.$$

Corollary 4.3 *Under the assumptions of Lemma 4.2 for a fixed* T, *we have* $\mathbb{P}(\lambda_T(b_0) < 0) = 1$.

For the proof just note that we can choose θ_0 such that $T_\theta < T$ for all $\theta \in (0, \theta_0)$. But now we have

$$\mathbb{P}(\lambda_T(b_0) < 0) \geq \mathbb{P}\left(\lambda_T(b_0) < -c\theta^2\right) \to 1 \quad \text{for } \theta \to 0.$$

Remark 4.4 In the proof we will see that $T_\theta = 2\sqrt{p_\theta}$ which we will choose later by (4.3). Let us remark that we expect $p_\theta \approx c\theta$, as we can approximate the probability by $2p(s)\theta$ where $p(s)$ is the value of the density for $b(s)$ in 0. Thus, we could work out a qualitative bound if we have more knowledge about the density of b.

Proof In order to prove the statement we have to make sure that the solution b does not stay too close to zero for too many times. The linearization of (4.2) around a solution b entails in this case

$$\mathrm{d}\varphi = D\mathcal{F}_c(b)\,\mathrm{d}t.$$

We have to show that there exists a constant $\tilde{c} > 0$ such that

$$\lambda_T(b_0(\omega)) = \frac{1}{T}\ln\exp\left(\int_0^T D\mathcal{F}_c(b(s,\omega))\,\mathrm{d}s\right) < -\tilde{c} < 0 \quad \text{for some } \omega \text{ and } T.$$

Following Blömker (2007, Theorem 3.4), we define the set of times for which the solution of the amplitude equation is small, i.e., for $T_0 > 0$ and $\theta > 0$

$$\mathcal{T}_\theta(T_0) := |\{s \in [0, T_0] : |b(s)| \le \theta\}|$$

and notice that

$$\mathcal{T}_\theta(T_0) = \int_0^{T_0} \mathbb{1}_{\{|b(s)|\le\theta\}}\,\mathrm{d}s.$$

Since $\mathcal{T}_\theta \in [0, T_0]$ a.s. we can bound arbitrary moments of $\mathcal{T}_\theta$. We start by an exponential moment bound. Let $c > 0$ and obtain due to Jensen's inequality

$$\begin{aligned}
\mathbb{E}e^{c\mathcal{T}_\theta(T_0)} &\le \frac{1}{T_0}\int_0^{T_0} \mathbb{E}\exp(cT_0\mathbb{1}_{\{|b(s)|\le\theta\}})\,\mathrm{d}s \\
&= \frac{1}{T_0}\int_0^{T_0} \mathbb{P}(|b(s)| > \theta)\,\mathrm{d}s + \frac{1}{T_0}e^{cT_0}\int_0^{T_0} \mathbb{P}(|b(s)| \le \theta)\,\mathrm{d}s \\
&= 1 + \frac{e^{cT_0}-1}{T_0}\int_0^{T_0} \mathbb{P}(|b(s)| \le \theta)\,\mathrm{d}s.
\end{aligned}$$

Now since $\mathbb{P}(|b(s)| = 0) = 0$ (since the density of the amplitude equation is not concentrated in zero, see for example Besalú et al. (2016, Theorem 1.2, (I))), it follows by dominated convergence that

$$p_\theta := \int_0^{T_0} \mathbb{P}(|b(s)| \le \theta)\,\mathrm{d}s \to 0 \text{ as } \theta \to 0. \tag{4.3}$$

Here we also use that the law of b is independent of θ. Therefore, letting $0 < \gamma_\theta \ll 1$ and applying Markov's inequality for the function $x \to e^{cx} - 1$, $x \geq 0$ we get

$$\mathbb{P}(\mathcal{T}_\theta(T_0) \geq \gamma_\theta) \leq \frac{\mathbb{E}e^{c\mathcal{T}_\theta(T_0)} - 1}{e^{c\gamma_\theta} - 1} \leq \frac{e^{cT_0} - 1}{e^{c\gamma_\theta} - 1} \cdot \frac{p_\theta}{T_0} \to \frac{p_\theta}{\gamma_\theta} \quad \text{for } c \to 0.$$

Setting $\gamma_\theta = \sqrt{p_\theta}$, we obtain that

$$\mathbb{P}(\mathcal{T}_\theta(T_0) \geq \sqrt{p_\theta}) \leq \sqrt{p_\theta}.$$

In conclusion, using $D\mathcal{F}_c(b) = -cb^2$ for $T \in [2\sqrt{p_\theta}, T_0]$

$$\lambda_T(b_0) = \frac{1}{T}\int_0^T D\mathcal{F}_c(b(s,\omega))\,\mathrm{d}s \leq -\frac{c}{T}(T - \mathcal{T}_\theta(T))\theta^2 \leq -\frac{c}{2}\theta^2 < 0$$

on the set of large probability $\Omega_\theta := \{\mathcal{T}_\theta(T_0) < \sqrt{p_\theta}\}$, as we have $\mathbb{P}(\Omega_\theta) \geq 1 - \sqrt{p_\theta}$. □

5 Approximation of SPDEs with Fractional Noise via Amplitude Equations

Here we prove an approximation result for the SPDE (1.1) which is different from the one derived in Blömker and Neamţu (2022). We recall that $T_\varepsilon = T\varepsilon^{-2}$, consider $\nu \geq 0$ and fix $\varepsilon \in (0, \varepsilon_0]$ for some $\varepsilon_0 > 0$ sufficiently small.

Assumption 5.1 We assume that we have the upper bounds $\nu = \mathcal{O}(\varepsilon^2)$ and $\sigma = \mathcal{O}(\varepsilon^{2H+1})$.

Ansatz 5.2 The process b is an $\mathcal{N}$-valued process, which solves the amplitude equation

$$\mathrm{d}b = [\nu\varepsilon^{-2}b + \mathcal{F}_c(b)]\,\mathrm{d}T + \sigma\varepsilon^{-2H-1}\,\mathrm{d}\beta^H(T), \tag{5.1}$$

where $\beta^H(T) = \varepsilon^{2H} P_c W^H(\varepsilon^{-2}T)$ is a rescaled fractional Brownian motion.

Theorem 5.3 *Let u be a solution of the SPDE (1.1) with initial condition $u_0 = \mathcal{O}(\varepsilon)$ in $\mathcal{H}$ such that $P_s u_0 = \mathcal{O}(\varepsilon^{2H+1})$ in $\mathcal{H}$. Further, let b be a solution of(5.1) with $b(0) - \varepsilon^{-1}u_0 = \mathcal{O}(\varepsilon^{2H})$. Then,*

$$u - \varepsilon b(\varepsilon^2\cdot) = \mathcal{O}(\varepsilon^{2H+1-\kappa} + \varepsilon^{2-\kappa}) \quad \textit{on } [0, T_\varepsilon] \textit{ in } \mathcal{H} \textit{ for all small } \kappa > 0. \tag{5.2}$$

Remark 5.4 We notice that for $H > 1/2$ the order of the error term is $\mathcal{O}(\varepsilon^{2-})$, which is given by certain nonlinear terms, whereas for $H < 1/2$ we get $\mathcal{O}(\varepsilon^{2H+1-})$ which arises from bounding the stochastic convolution.

Proof The proof is similar to Blömker and Neamţu (2022, Theorem 4.15) and Blömker and Neamţu (2023, Theorem 5.3). Nevertheless, we provide the main arguments here as well since it is crucial to get the scaling of all the error terms correctly.

First of all we show that if $u_0 = \mathcal{O}(\varepsilon)$ in $\mathcal{H}$, then $u = \mathcal{O}(\varepsilon)$ in $\mathcal{H}$ on $[0, T_\varepsilon]$ regarding that $\sigma Z = \mathcal{O}(\varepsilon)$ on $[0, T_\varepsilon]$ (since $P_s Z = \mathcal{O}(T^\kappa)$ and $P_c Z = \mathcal{O}(T^H)$ on $[0, T_0]$).

Using the standard transformation $\tilde{u} := u - \sigma Z$, we obtain the partial differential equation with random coefficients

$$\partial_t \tilde{u} = A\tilde{u} + \nu(\tilde{u} + \sigma Z) + \mathcal{F}(\tilde{u} + \sigma Z).$$

Due to (2.3) using Young's inequality, we obtain the estimate

$$\begin{aligned}
\frac{1}{2}\partial_t \|\tilde{u}\|^2 &\leq \nu\langle \tilde{u} + \sigma Z, \tilde{u}\rangle + \langle \mathcal{F}(\tilde{u} + \sigma Z), \tilde{u}\rangle \\
&\leq \nu\|\tilde{u}\|^2 + \nu\sigma\langle Z, \tilde{u}\rangle + C\sigma^4\|Z\|_X^4 - \delta\|\tilde{u}\|_X^4 \\
&\leq \frac{1}{2}(\nu\|\tilde{u}\|^2 - \delta\|\tilde{u}\|_X^4) + C(\nu\sigma^2\|Z\|^2 + \sigma^4\|Z\|_X^4) \\
&\leq C(\nu^2 + \nu\sigma^2\|Z\|^2 + \sigma^4\|Z\|_X^4) = \mathcal{O}(\varepsilon^4).
\end{aligned}$$

Here we used that $\sigma Z = \mathcal{O}(\varepsilon)$ on $[0, T_\varepsilon]$ since $\sigma = \mathcal{O}(\varepsilon^{2H+1})$, where $T_\varepsilon = \mathcal{O}(\varepsilon^{-2})$ and that $\nu = \mathcal{O}(\varepsilon^2)$. This completes the proof of the first step. Additionally, we can also conclude that

$$\frac{1}{2}\partial_t \|\tilde{u}\|^2 = -\frac{\delta}{4}\|\tilde{u}\|_X^4 + \mathcal{O}(\varepsilon^4),$$

which gives the $L^4(0, T_\varepsilon, X)$ bound on $\tilde{u}$

$$\int_0^{T_\varepsilon} \|\tilde{u}(t)\|_X^4 \, \mathrm{d}t = \frac{2}{\delta}\|\tilde{u}(0)\|^2 + \mathcal{O}(\varepsilon^4) = \mathcal{O}(\varepsilon^2). \tag{5.3}$$

In particular, we notice that since $\sigma Z = \mathcal{O}(\varepsilon)$ the estimates of $\tilde{u}$ do not depend on H.

Furthermore, following the steps of the proof of Blömker and Neamţu (2023, Theorem 5.3) and regarding that $P_s u_0 = \mathcal{O}(\varepsilon^{2H+1})$ and the properties of the cubic term, we can derive that $u_s := P_s u = \mathcal{O}(\varepsilon^{2H+1})$ in $\mathcal{H}$ on $[0, T_\varepsilon]$ and $\int_0^{T_\varepsilon} \|\tilde{u}_s(t)\|_X^4 \, \mathrm{d}t = \mathcal{O}(\varepsilon^{4H+2})$, consequently $\int_0^T \|\tilde{u}_s(\varepsilon^{-2}t)\|_X^4 \, \mathrm{d}t = \mathcal{O}(\varepsilon^{4H+4})$. For the convenience of the reader, we prove these statements. To this aim, we first use the splitting

$$u = P_c u + P_s u := u_c + u_s,$$

and define $Z_s := P_s Z$. Again we use the standard transformation $\tilde{u} = u - \sigma Z$ so that

$$\tilde{u}_s = u_s - \sigma Z_s = P_s \tilde{u}.$$

Thus, taking the stable projection P_s entails

$$\partial_t \tilde{u}_s = A\tilde{u}_s + \nu(\tilde{u}_s + \sigma Z_s) + P_s\mathcal{F}(u_c + \sigma Z_s + \tilde{u}_s).$$

Assumption 2.1 implies that the quadratic form of A on the unit sphere of $P_s\mathcal{H}$ is bounded from below by a positive constant. Therefore, we further obtain

$$\frac{1}{2}\partial_t \|\tilde{u}_s\|^2 \leq -c\|\tilde{u}_s\|^2 + \nu\|\tilde{u}_s\|^2 + \nu\langle \tilde{u}_s, \sigma Z_s\rangle + \langle \mathcal{F}(u_c + \sigma Z_s + \tilde{u}_s), \tilde{u}_s\rangle.$$

Using (2.3) together with the fact that ε_0 is sufficiently small and thus $\nu = \mathcal{O}(\varepsilon^2)$ is small, we derive the energy estimate

$$\frac{1}{2}\partial_t \|\tilde{u}_s\|^2 \leq -\frac{c}{2}\|\tilde{u}_s\|^2 + C\nu\sigma^2\|Z_s\|^2 + C(\|u_c\|_X^4 + \sigma^4\|Z_s\|_X^4) - \delta\|\tilde{u}_s\|_X^4, \tag{5.4}$$

for two universal constants $c, C > 0$. Hence, via a Gronwall-type estimate, for all $t \leq T_\varepsilon$

$$\|\tilde{u}_s(t)\|^2 \leq \|\tilde{u}_s(0)\|^2 + C\int_0^t e^{-c(t-\tau)}\Big(\nu\sigma^2\|Z_s\|^2 + \|u_c\|_X^4 + \sigma^4\|Z_s\|_X^4\Big)\,\mathrm{d}\tau.$$

We use that all norms are equivalent on $\mathcal{N}$ together with the bounds $\sigma = \mathcal{O}(\varepsilon^{2H+1})$, $\nu = \mathcal{O}(\varepsilon^2)$, $\|Z_s\|_X = \mathcal{O}(T^\kappa)$ and $P_s u_0 = \mathcal{O}(\varepsilon^{2H+1})$ to obtain

$$\|\tilde{u}_s\|^2 = \mathcal{O}(\varepsilon^{4H+2}) \text{ on } [0, T_\varepsilon].$$

Thus,

$$\|u_s\| \leq \|\tilde{u}_s\| + \sigma\|Z_s\| = \mathcal{O}(\varepsilon^{2H+1}) + \mathcal{O}(\varepsilon^{2H+1-\kappa}) \text{ on } [0, T_\varepsilon]$$

which bounds the error on $P_s\mathcal{H}$.

Remark 5.5 We notice that if $H > 1/2$ the order of $\|u_s\|$ is $\mathcal{O}(\varepsilon)$. This follows since $4H + 2 > 4$ and $\|u_c\| \leq \|\tilde{u}_c\| + \sigma\|Z_c\| = \mathcal{O}(\varepsilon)$ is the lower order term appearing in the integral (5.4). We will see in Sect. 6 that the error term appearing in the computation of the FTLEs will always be $\mathcal{O}(\varepsilon)$ if $H > 1/2$.

Moreover, from the previous inequality we can infer bounds on $P_s u$ in X. From (5.4), we also obtain by integration

$$\delta\int_0^t \|\tilde{u}_s\|_X^4\,\mathrm{d}t \leq \|\tilde{u}_s(0)\|^2 + C\int_0^t (\nu\sigma^2\|Z_s\|^2 + \|u_c\|_X^4 + \|Z_s\|_X^4)\,\mathrm{d}t$$

and thus for $H < 1/2$

$$\int_0^{T_\varepsilon} \|\tilde{u}_s(t)\|_X^4 \,\mathrm{d}t = \mathcal{O}(\varepsilon^{4H+2}) \quad \text{or} \quad \int_0^{T_0} \|\tilde{u}_s(\varepsilon^{-2}t)\|_X^4 \,\mathrm{d}t = \mathcal{O}(\varepsilon^{4H+4}).$$

Again if $H > 1/2$ the lowest order term is given by $\|u_c\|_X^4$ which results in

$$\int_0^{T_\varepsilon} \|\tilde{u}_s(t)\|_X^4 \,\mathrm{d}t = \mathcal{O}(\varepsilon^4) \text{ and therefore } \int_0^{T_0} \|\tilde{u}_s(\varepsilon^{-2}t)\|_X^4 \,\mathrm{d}t = \mathcal{O}(\varepsilon^6).$$

Remark 5.6 Here we notice that for a fixed time we get a better estimate for u_s which does not depend on κ, which only comes from taking the supremum. We need then $Z_s(t) = \mathcal{O}(1)$ for a fixed time t which can be proven as in Blömker and Neamţu (2022, Appendix B).

We now sketch the proof for the bound of the error term in $\mathcal{N}$. First note that on $P_s\mathcal{H}$ we obtain

$$\|u_s\| \le \|\tilde{u}_s\| + \sigma\|Z_s\| = \mathcal{O}(\varepsilon^{2H+1}) + \mathcal{O}(\varepsilon^{2H+1-\kappa}) \text{ on } [0, T_\varepsilon],$$

since $u_s = \mathcal{O}(\varepsilon^{2H+1})$. We now show that $\varepsilon^{-1}P_c u(\varepsilon^{-2}\cdot) - b = \mathcal{O}(\varepsilon^{2H})$ on $[0, T_0]$ for our fixed T_0, where $u_c := P_c u$ satisfies the SDE

$$\mathrm{d}u_c = (\nu u_c + \mathcal{F}(u_c + u_s))\mathrm{d}t + \sigma \mathrm{d}W_c^H.$$

We define the error as

$$e := b - \varepsilon^{-1}u_c(\varepsilon^{-2}\cdot)$$

and obtain regarding that $W^H(t\varepsilon^{-2}) = W^H(t)\varepsilon^{-2H}$ in law

$$\partial_t e = \frac{\nu}{\varepsilon^2}e + P_c\mathcal{F}(b) - P_c\mathcal{F}(\varepsilon^{-1}u(\varepsilon^{-2}\cdot)).$$

Taking the inner product with e we further get

$$\frac{1}{2}\partial_t\|e\|^2 = \frac{1}{2}\frac{\nu}{\varepsilon^2}\|e\|^2 + \langle \mathcal{F}_c(b) - \mathcal{F}_c(\varepsilon^{-1}u(\varepsilon^{-2}\cdot))e\rangle. \tag{5.5}$$

We use with the short-hand notation $u^{(\varepsilon)}(\cdot) := \varepsilon^{-1}u(\varepsilon^{-2}\cdot)$ and expand the cubic to derive

$$\begin{aligned}
&\langle \mathcal{F}_c(b) - \mathcal{F}_c(u^{(\varepsilon)}), e\rangle \\
&\quad \le \langle \mathcal{F}_c(b) - \mathcal{F}_c(u_c^{(\varepsilon)}), e\rangle + C\|e\| \cdot (\|u_c^{(\varepsilon)}\|^2\|u_s^{(\varepsilon)}\|_X + \|u_c^{(\varepsilon)}\|\|u_s^{(\varepsilon)}\|_X^2 + \|u_s^{(\varepsilon)}\|_X^3) \\
&\quad \le -\delta\|e\|_X^4 + C\|e\| \cdot (\|u_c^{(\varepsilon)}\|^2\|u_s^{(\varepsilon)}\|_X + \|u_c^{(\varepsilon)}\|\|u_s^{(\varepsilon)}\|_X^2 + \|u_s^{(\varepsilon)}\|_X^3)
\end{aligned}$$

$$\leq -\frac{1}{2}\delta\|e\|_X^4 + C\|u_c^{(\varepsilon)}\|^4 + C\|u_s^{(\varepsilon)}\|_X^4.$$

Thus, from (5.5) we get

$$\frac{1}{2}\partial_T\|e\|^2 \leq \frac{1}{2}\frac{\nu}{\varepsilon^2}\|e\|^2 - \frac{1}{2}\delta\|e\|_X^4 + C\|u_c^{(\varepsilon)}\|^4 + C\|u_s^{(\varepsilon)}\|_X^4.$$

Using a Gronwall-type estimate, we obtain for $H < 1/2$ and for all $T \in [0, T_0]$ with constants depending on T_0

$$\|e(T)\|^2 \leq C\|e(0)\|^2 + C\int_0^T (\|u_c^{(\varepsilon)}\|^4 + C\|u_s^{(\varepsilon)}\|_X^4)\,\mathrm{d}T = \mathcal{O}(\varepsilon^{4H}),$$

using the equivalence of the norms on $\mathcal{N}$, the $L^4(0, T_0, X)$-bound for u_s of order $\mathcal{O}(\varepsilon^{4H+4})$ and the fact that $e(0) = \mathcal{O}(\varepsilon^{2H})$. This means that $\varepsilon e(\cdot)$ is of order $\mathcal{O}(\varepsilon^{2H+1})$ on $[0, T_0]$ as claimed in (5.2). The case $H > 1/2$ leads to an error term of order $\mathcal{O}(\varepsilon)$ using that $\int_0^{T_0} \|\tilde{u}_s(\varepsilon^{-2}t)\|_X^4\,\mathrm{d}t = \mathcal{O}(\varepsilon^6)$. □

Remark 5.7 One can easily show that $b = \mathcal{O}(1)$ using a standard comparison argument for ODEs, see Blömker and Neamţu (2022, Lemma 4.10).

6 Lyapunov Exponents for SPDEs with Fractional Noise

6.1 Case $\nu < 0$: Stability

This is the trivial case where we always have stability meaning that the FTLEs are all negative.

Theorem 6.1 *Let Assumptions 2.1, 2.2, 2.3 hold true and let $\nu < 0$. Furthermore, let u be a solution of (2.1) in the sense of Theorem 3.1 with deterministic initial condition $u_0 \in \mathcal{H}$. Then for all $T > 0$ we have with probability one*

$$\mathbb{P}(\lambda_T(u_0) \leq \nu) = 1.$$

Proof The proof is similar to Blumenthal et al. (2023, Proposition 3.1 a)). We consider a solution v of the linearized problem (3.1) around a solution u of (2.1) with deterministic $\mathcal{H}$-valued initial condition u_0. Recalling that $v \in H^1(0, T, \mathcal{H}^{-1/2}) \cap L^2(0, T, \mathcal{H}^{1/2}) \cap C(0, T, \mathcal{H})$ we obtain using (2.4) the standard energy estimate

$$\frac{1}{2}\partial_t\|v\|^2 = \langle Av, v\rangle + \nu\|v\|^2 + \langle D\mathcal{F}(u)v, v\rangle \leq \nu\|v\|^2.$$

This implies that $\|v(t)\| \leq \|v(0)\|e^{t\nu}$ for all $t > 0$. Due to Remark 3.5, we have for any time $T > 0$

$$\lambda_T(u_0) = \frac{1}{T}\ln(\|U_{u_0}(T)\|_{L(\mathcal{H})}) \leq \nu$$

which finishes the proof. □

Remark 6.2 The statement remains valid if we consider random initial data $u_0(\omega)$, in particular the random fixed point of (2.1) whose existence was established in Maslowski and Schmalfuß (2004).

6.2 Case $1 \gg \sigma \approx \nu^{1/2+H}$: Instability

In this setting we first recall that $\sigma/\nu^{1/2+H}$ and $\nu^{1/2+H}/\sigma$ are both $\mathcal{O}(1)$. Setting $\varepsilon^2 = \nu$ we obtain the amplitude equation

$$\mathrm{d}b = [b + \mathcal{F}_c(b)]\,\mathrm{d}T + \frac{\sigma}{\nu^{\frac{1}{2}+H}}\mathrm{d}\beta_{\nu^H}(T). \tag{6.1}$$

Theorem 6.3 *Let b_0 be an initial data of (4.1) for which the corresponding solution satisfies Lemma 4.1. Furthermore, let λ_T be the finite-time Lyapunov exponent of the SPDE (2.1) with initial data $u_0 = \varepsilon b_0$. For all terminal times $T_0 > 0$ and all probabilities $p \in (0, 1)$, there is a set Ω_p with probability larger than p and a constant $C_p > 0$ such that for $\omega \in \tilde{\Omega} \cap \Omega_p$ we have for all $T \in [0, T_0]$ that*

$$\lambda_{T\nu^{-1}}(\nu^{1/2}b_0) > \begin{cases} \frac{\nu}{4} - C_p\frac{\nu^{1+H}}{T}, & \text{if } H < 1/2 \\ \frac{\nu}{4} - C_p\frac{\nu^{3/2}}{T}, & \text{if } H > 1/2. \end{cases} \tag{6.2}$$

The main ideas are the approximation of the SPDE (2.1) with the amplitude equation (4.1) for $\varepsilon^2 = \nu, \sigma = \mathcal{O}(\varepsilon^{2H+1})$, Lemma 4.1 and the control of the approximation error. We start the SPDE in εb_0 and have that $\varepsilon b_0 = \mathcal{O}(\varepsilon)$ since $b = \mathcal{O}(1)$. In this situation Theorem 5.3 is applicable.

Now we control the approximation error between the linearized SPDE and the linearized ODE. To this aim, we firstly introduce the slow scaling $T = t\varepsilon^2$ and define U via

$$u(t) = \varepsilon U(t\varepsilon^2).$$

Let v be the solution of the linearization of the SPDE around a solution u

$$\partial_t v = Av + \nu v + D\mathcal{F}(u)v.$$

On the slow scale $v(t) = \varepsilon V(t\varepsilon^2)$ we have (using that $D\mathcal{F}$ is quadratic)

$$\partial_T V = \varepsilon^{-2}AV + V + D\mathcal{F}(U)V.$$

Let φ be the solution of the linearization of the amplitude equation around a solution b which satisfies the support theorem

$$\partial_T\varphi = \varphi + D\mathcal{F}_c(b)\varphi.$$

We only consider initial conditions $V(0) = \varphi(0) \in \mathcal{N}$ of order 1 independent of ε.

The first crucial step is the following approximation result.

Theorem 6.4 *Let b_0 be an initial condition for which the corresponding solution satisfies Lemma 4.1. For any probability $p \in (0, 1)$ there is a set Ω_p with probability larger than p such that the error between the linearization of the SPDE (1.1) with initial data $u_0 = \varepsilon b_0$ and of the amplitude equation (6.1) is bounded by $C[\varepsilon + \varepsilon^{2H}]$.*

Proof We show in several steps that the following error bound holds on the set of large probability Ω_p

$$\|V(T) - \varphi(T)\|_{\mathcal{H}} \leq \|P_s V(T) + P_c V(T) - \varphi(T)\| = \mathcal{O}(\varepsilon + \varepsilon^{2H}), \quad T \in [0, T_0]. \tag{6.3}$$

To this aim, we first prove

$$\|P_s V(T)\|_{\mathcal{H}} = \mathcal{O}(\varepsilon) \quad \text{and} \quad \|P_s V\|_{L^2(0,T_0,\mathcal{H}^{1/2})} = \mathcal{O}(\varepsilon^2). \tag{6.4}$$

We first consider V and use standard energy-type estimates to obtain

$$\frac{1}{2}\partial_T \|V\|^2 = \varepsilon^{-2}\langle AV, V\rangle + \|V\|^2 + \langle D\mathcal{F}(U)V, V\rangle \leq \|V\|^2,$$

where we used the non-negativity of A and (2.4). As $V(0) = \mathcal{O}(1)$, this yields a uniform $\mathcal{O}(1)$-bound on V and thus $P_c V$ in $\mathcal{H}$ on $[0, T_0]$ (with constants depending on T_0).

We have (using the short-hand notation $V_s := P_s V$ and $V_c := P_c V$)

$$\begin{aligned}\frac{1}{2}\partial_T \|V_s\|^2 =& \varepsilon^{-2}\langle AV_s, V_s\rangle + \|V_s\|^2 + \langle P_s D\mathcal{F}(U)V, V_s\rangle \\ \leq& -c\varepsilon^{-2}\|V_s\|^2_{\mathcal{H}^{1/2}} + \|V_s\|^2 + \langle P_s D\mathcal{F}(U)V_c, V_s\rangle,\end{aligned} \tag{6.5}$$

where we used the spectral properties of A (Assumption 2.1) and the sign condition on $D\mathcal{F}$ from (2.4).

We now bound the nonlinear term as follows

$$\langle P_s D\mathcal{F}(U)V_c, V_s\rangle \leq C\|U\|_X^2\|V_c\|_X\|V_s\|_{\mathcal{H}^\alpha} \leq C\varepsilon^2\|U\|_X^4\|V_c\|_X^2 + \frac{1}{2}c\varepsilon^{-2}\|V_s\|^2_{\mathcal{H}^\alpha},$$

where we used ε-Young's inequality in the last step. Further, as shown above V is $\mathcal{O}(1)$ in $\mathcal{H}$. Therefore, we obtain that V_c is bounded in X since all norms are equivalent on $\mathcal{N}$. Consequently, we only need a bound on $\int_0^T \|U(S)\|_X^4 \,\mathrm{d}S$, which can be derived from the first step of the approximation result, Theorem 5.3. Namely, using that

$$\int_0^{T_\varepsilon} \|u(t)\|_X^4 \,\mathrm{d}t = \mathcal{O}(\varepsilon^2)$$

we obtain

$$\int_0^{T_0} \|U(S)\|_X^4 \, \mathrm{d}S = \varepsilon^2 \int_0^{T_\varepsilon} \|U(t\varepsilon^2)\|_X^4 \, \mathrm{d}t = \varepsilon^{-2} \int_0^{T_\varepsilon} \|u(t)\|_X^4 \, \mathrm{d}t = \mathcal{O}(1). \quad (6.6)$$

Thus, we can conclude from (6.5) for two different universal constants $c > 0$ and $C > 0$ using that $\|\cdot\|_{\mathcal{H}^\alpha} \le \|\cdot\|_{\mathcal{H}^{1/2}}$

$$\begin{aligned}\frac{1}{2}\partial_T \|V_s\|^2 &\le -c\varepsilon^{-2}\|V_s\|^2_{\mathcal{H}^{1/2}} + \|V_s\|^2_{\mathcal{H}^\alpha} + C\varepsilon^2 \|U\|_X^4 \|V_c\|_X^2 + \frac{1}{2}\varepsilon^{-2} c \|V_s\|^2_{\mathcal{H}^\alpha} \\ &\le -\frac{1}{2} c\varepsilon^{-2}\|V_s\|^2_{\mathcal{H}^{1/2}} + C\varepsilon^2 \|U\|_X^4 \|V_c\|_X^2. \end{aligned} \quad (6.7)$$

Consequently, recalling that $V_s(0) = 0$ via a Gronwall-type estimate we obtain for all $T \in [0, T_0]$ the inequality (with constants depending on T_0)

$$\|V_s(T)\|^2 \le C\varepsilon^2 \int_0^{T_0} \|U(S)\|_X^4 \, \mathrm{d}S \sup_{[0,T_0]} \|V_c\|_X^2 = \mathcal{O}(\varepsilon^2),$$

which means that $\|V_s\|_{\mathcal{H}} = \mathcal{O}(\varepsilon)$, as claimed.

For the second statement in (6.4), we get from (6.7) that

$$c\varepsilon^{-2}\|V_s\|^2_{\mathcal{H}^{1/2}} \le -\frac{1}{2}\partial_T \|V_s\|^2 + C\varepsilon^2 \|U\|_X^4 \|V_c\|_X^2,$$

therefore by integration (recall $V_s(0) = 0$) we derive

$$\int_0^{T_0} \|V_s(S)\|^2_{\mathcal{H}^{1/2}} \, \mathrm{d}S \le -\frac{c\varepsilon^2}{2} \|V_s(T)\|^2 + C\varepsilon^4 \int_0^{T_0} \|U(S)\|_X^4 \, \mathrm{d}S \sup_{[0,T_0]} \|V_c\|_X^2.$$

As $\|V_c\|_X = \mathcal{O}(1)$, $\|V_s\|_{\mathcal{H}} = \mathcal{O}(\varepsilon)$ and $\int_0^{T_0} \|U(S)\|_X^4 \, \mathrm{d}S = \mathcal{O}(1)$ we obtain

$$\|V_s\|_{L^2(0,T_0,\mathcal{H}^{1/2})} = \mathcal{O}(\varepsilon^2).$$

We now focus on the bound for $\|V_c - \varphi\|$. The aim is to show that the error term is of order $\mathcal{O}(\varepsilon^{2H})$. We observe that $V_c - \varphi$ satisfies the equation

$$\partial_T (V_c - \varphi) = V_c - \varphi + (D\mathcal{F}_c(U)V - D\mathcal{F}_c(b)\varphi),$$

so we have to estimate

$$\frac{1}{2}\partial_T \|V_c - \varphi\|^2 = \|V_c - \varphi\|^2 + \langle D\mathcal{F}_c(U)V - D\mathcal{F}_c(b)\varphi, V_c - \varphi \rangle. \quad (6.8)$$

Here, the crucial term contains the nonlinearity

$$\begin{aligned}\langle D\mathcal{F}_c(b)\varphi - P_c D\mathcal{F}(U)V, \varphi - V_c\rangle_{\mathcal{N}} = &-\langle P_c D\mathcal{F}(U)V_s, \varphi - P_c V\rangle_{\mathcal{N}}\\ &+\langle D\mathcal{F}_c(b)\varphi - P_c D\mathcal{F}(b)V_c, \varphi - P_c V\rangle_{\mathcal{N}}\\ &+\langle P_c[D\mathcal{F}(b) - D\mathcal{F}(U)]V_c, \varphi - P_c V\rangle_{\mathcal{N}},\end{aligned}$$

where the bound on $P_s V$ is needed in the space X, but the integral bounds turn out to be sufficient. We also rely on our $\mathcal{O}(1)$-bounds on φ and V_c.

We begin with the first term above which entails

$$\begin{aligned}\langle P_c D\mathcal{F}(U)V_s, \varphi - V_c\rangle_{\mathcal{N}} &\leq C\|U\|_X^2\|V_s\|_X\|\varphi - V_c\|_{\mathcal{N}}\\ &\leq C\|U\|_X^2\|V_s\|_{\mathcal{H}^{1/2}}\|\varphi - V_c\|_{\mathcal{N}}\\ &\leq C\|V_s\|_{\mathcal{H}^{1/2}}^2 + \|U\|_X^4\|\varphi - V_c\|_{\mathcal{N}}^2.\end{aligned}$$

In the last step, we used again Young's inequality.

The second term gives

$$\langle D\mathcal{F}_c(b)(\varphi - V_c), \varphi - V_c\rangle_{\mathcal{N}} \leq C\|b\|_{\mathcal{N}}^2\|\varphi - V_c\|_{\mathcal{N}}^2.$$

For the last one, we use that $P_c D\mathcal{F}$ and $D\mathcal{F}_c$ are the same on $\mathcal{N}$, which can be seen by explicitly using the properties of the cubic $\mathcal{F}$.

$$\begin{aligned}&\langle P_c[D\mathcal{F}(b) - D\mathcal{F}(U)]V_c, \varphi - V_c\rangle_{\mathcal{N}}\\ &\quad\leq C\|b - U\|_X^2\|V_c\|_X\|\varphi - V_c\|_{\mathcal{N}}\\ &\quad\leq C\|b - U_c\|_{\mathcal{N}}^2\|V_c\|_{\mathcal{N}}\|\varphi - V_c\|_{\mathcal{N}} + C\|U_s\|_X^2\|V_c\|_{\mathcal{N}}\|\varphi - V_c\|_{\mathcal{N}}\\ &\quad\leq C\|b - U_c\|_{\mathcal{N}}^4\|V_c\|_{\mathcal{N}}^2 + C\|U_s\|_X^4\|V_c\|_{\mathcal{N}}^2 + C\|\varphi - V_c\|_{\mathcal{N}}^2.\end{aligned}$$

Regarding (6.8) and putting all the estimates together we infer that (with universal constants all denoted by $C > 0$)

$$\begin{aligned}\frac{1}{2}\partial_T\|V_c - \varphi\|^2 &\leq \|V_c - \varphi\|^2 + C\|V_s\|_{\mathcal{H}^{1/2}}^2 + \|U\|_X^4\|\varphi - V_c\|_{\mathcal{N}}^2 + \|b\|_{\mathcal{N}}^2\|\varphi - V_c\|_{\mathcal{N}}^2\\ &\qquad + C\|b - U_c\|_{\mathcal{N}}^4\|V_c\|_{\mathcal{N}}^2 + C\|U_s\|_X^4\|V_c\|_{\mathcal{N}}^2 + C\|\varphi - V_c\|_{\mathcal{N}}^2\\ &\leq C\|V_c - \varphi\|_{\mathcal{N}}^2(1 + \|U\|_X^4 + \|b\|_{\mathcal{N}}^2) + C\|V_s\|_{\mathcal{H}^{1/2}}^2 + C\|b\\ &\qquad - U_c\|_{\mathcal{N}}^4\|V_c\|_{\mathcal{N}}^2 + C\|U_s\|_X^4\|V_c\|_{\mathcal{N}}^2\\ &\leq C\cdot I\cdot\|V_c - \varphi\|_{\mathcal{N}}^2 + C\cdot J,\end{aligned}$$

where we set

$$I := 1 + \|U\|_X^4 + \|b\|_{\mathcal{N}}^2 \quad\text{and}\quad J := \|V_s\|_{\mathcal{H}^{1/2}}^2 + \|b - U_c\|_{\mathcal{N}}^4\|V_c\|_{\mathcal{N}}^2 + \|U_s\|_X^4\|V_c\|_{\mathcal{N}}^2.$$

Using Gronwall's inequality, we get for $T \in [0, T_0]$

$$\|V_c(T) - \varphi(T)\|^2 \leq \Big[\|V_c(0) - \varphi(0)\|^2 + C\int_0^T J(S)\,\mathrm{d}S\Big]\exp\Big(C\int_0^T I(S)\,\mathrm{d}S\Big).$$

We now investigate the order of J. First of all, since we start the SPDE in the rescaled initial condition $u_0 = \varepsilon b_0$ we obtain due to Theorem 5.3 for $H < 1/2$

$$\|b(T) - U_c(T)\|_{\mathcal{H}} = \varepsilon^{-1}\|\varepsilon b - u_c(\varepsilon^{-2}\cdot)\|_{\mathcal{H}} = \mathcal{O}(\varepsilon^{2H})$$

respectively for $H > 1/2$

$$\|b(T) - U_c(T)\|_{\mathcal{H}} = \varepsilon^{-1}\|\varepsilon b - u_c(\varepsilon^{-2}\cdot)\|_{\mathcal{H}} = \mathcal{O}(\varepsilon).$$

Again we use the fact that all norms are equivalent on $\mathcal{N}$. Further, using Theorem 5.3 we know that with $u_s(t) = \varepsilon U_s(t\varepsilon^2)$

$$\int_0^{T_0} \|U_s(T)\|_X^4 \,\mathrm{d}T = \varepsilon^{-2}\int_0^{T_\varepsilon} \|u_s(t)\|_X^4 \,\mathrm{d}t = \mathcal{O}(\varepsilon^{4H}).$$

This term will determine the order of the error $V_c - \varphi$ since $\|b(T) - U_c(T)\|^4 = \mathcal{O}(\varepsilon^{8H})$, which is small only if $H < 1/4$.

Due to the above results, we have pathwise bounds for $\int_0^T J(S)\,\mathrm{d}S$ by $C(\varepsilon^2 + C\varepsilon^{4H})$ on a set of probability going to 1 for $C \to \infty$.

Moreover, we can enlarge this set to have for all $T \in [0, T_0]$

$$\exp\left(C\int_0^T I(S)\,\mathrm{d}S\right) = \exp\left(C\int_0^T (1 + \|U(S)\|_X^4 + \|b(S)\|_{\mathcal{N}}^2 \,\mathrm{d}S\right) \le C.$$

Together with the previous bound this gives another condition for the set Ω_p.

In summary, this entails the following error bound on Ω_p

$$\|\varphi(T) - V_c(T)\|^2 \le C[\varepsilon^2 + \varepsilon^{4H}] \text{ for } T \in [0, T_0].$$

Putting all these deliberations together proves the statement (6.3) on Ω_p, i.e.,

$$\|V(T) - \varphi(T)\|_{\mathcal{H}} \le \|V_s(T)\|_{\mathcal{H}} + \|V_c(T) - \varphi(T)\|_{\mathcal{N}} \le C[\varepsilon + \varepsilon^{2H}], \quad T \in [0, T_0].$$

Here we have to add another condition to Ω_p, as $\|V_s\|_{\mathcal{H}} \le C\varepsilon$ uniformly in T with probability going to 1 if $C \to \infty$. □

Remark 6.5 The previous computation shows that $V_s \in L^2(0, T_0; \mathcal{H}^{1/2})$ has the same order as in the case of a Brownian motion. For Brownian noise, exactly this term determined the order of J, since all the other terms were of higher order, see Blömker and Neamţu (2023). However, here the error between $P_c V$ and φ is now determined by the $L^4(0, T_0; X)$ bound on U_s. The approximation with the AE gives a term of order $\mathcal{O}(\varepsilon^{4H})$ in the estimate $\|V_c(T) - \varphi(T)\|$, which becomes small only for $H < 1/4$.

Using this result, we can proceed with the proof of Theorem 6.3. We first recall the definition of the FTLE for a solution of the SPDE starting in $u_0 = \varepsilon b_0$

$$\lambda_{T\nu^{-1}}(\varepsilon b_0) = \frac{\nu}{T}\ln(\sup\{\|v(T/\nu)\| \;\; \|v(0)\| = 1\})$$

$$= \frac{\nu^{3/2}}{T}\ln(\sup\{\|V(T)\| \ \ \|V(0)\| = \varepsilon^{-1}\})$$
$$= \frac{\nu}{T}\ln(\sup\{\|V(T)\| \ \ \|V(0)\| = 1\}).$$

Using (6.3) for the finite-time Lyapunov exponents of the SPDE, we have on $\tilde{\Omega}\cap\Omega_p$ recalling Lemma 4.1

$$\begin{aligned}\|V(T)\| &\geq \|\varphi(T)\| - \|V(T)-\varphi(T)\| \geq \|\varphi(T)\| - C[\varepsilon+\varepsilon^{2H}]\\ &\geq \exp\{(1-3\delta^2)T\} - C[\varepsilon+\varepsilon^{2H}] > 0,\end{aligned}$$

which is positive if ε_0 is sufficiently small. Here we can choose $\delta = \frac{1}{2}$ as in Lemma 4.1.

To proceed, we use a simple estimate for the logarithm. It is known that there exists a positive constant $c > 0$ such that $\ln(1-x) \geq -cx$ for $0 \leq x \leq \frac{1}{2}$. Therefore as $\varepsilon + \varepsilon^{2H} \ll e^{Tc}$ we have that

$$\begin{aligned}\ln(e^{cT} - [\varepsilon+\varepsilon^{2H}]) &= \ln(e^{cT}(1-[\varepsilon+\varepsilon^{2H}]e^{-cT})) = cT + \ln(1-[\varepsilon+\varepsilon^{2H}]e^{-cT})\\ &\geq cT - C[\varepsilon+\varepsilon^{2H}]e^{-cT} \geq cT - C[\varepsilon+\varepsilon^{2H}].\end{aligned}$$

Thus, we can conclude that on $\tilde{\Omega}\cap\Omega_p$ we can bound

$$\begin{aligned}\lambda_{T\nu^{-1}}(\varepsilon b) &= \frac{\nu}{T}\ln(\sup\{\|V(T)\| \ \ \|V(0)\| = 1\})\\ &\geq \frac{\nu}{T}\ln(\sup\{\|\varphi(T)\| - \|V(T)-\varphi(T)\| : \ \|V(0)\| = 1\})\\ &\geq \frac{\nu}{T}\ln(\sup\{\|\varphi(T)\| - \|V(T)-\varphi(T)\| : \ \|v(0)\| = 1, V(0) = \varphi(0) \in \mathcal{N}\})\\ &\geq \frac{\nu}{T}\ln(\sup\{\|\varphi(T)\| - [\varepsilon+\varepsilon^{2H}] : \ \|V(0)\| = 1, V(0) = \varphi(0) \in \mathcal{N}\})\\ &\geq \frac{\nu}{T}\ln\left(\exp\left\{(T + \int_0^T D\mathcal{F}_c(b(s,\omega))\,\mathrm{d}s\right\} - C[\varepsilon+\varepsilon^{2H}]\right)\\ &\geq \frac{\nu}{T}\ln(\exp\{(1-3\delta^2)T\} - C[\varepsilon+\varepsilon^{2H}]\\ &\geq \nu(1-3\delta^2) - C\frac{\nu[\varepsilon+\varepsilon^{2H}]}{T}.\end{aligned}$$

Choosing, for example, $\delta = 1/2$ as in Lemma 4.1 proves that for $\omega \in \tilde{\Omega}\cap\Omega_p$ and for all $T \in [0, T_0]$ we have

$$\lambda_{T\nu^{-1}}(\varepsilon b_0) > \begin{cases} \frac{\nu}{4} - C_p\frac{\nu\varepsilon^{2H}}{T}, & H < 1/2\\ \frac{\nu}{4} - C_p\frac{\nu\varepsilon}{T}, & H > 1/2.\end{cases}$$

□

Remark 6.6 We notice that the error term between the two linearizations is determined by $\|V_s\|_{\mathcal{H}}$ (which is the same as for the Brownian motion), whereas $V_c - \varphi$ is determined by the $L^4(0, T_0; X)$ bound on U_s which is of order $\mathcal{O}(\varepsilon^{2H})$. If $H < 1/2$ this will become small, whereas for $H > 1/2$ the contribution of V_s dominates.

6.3 Case $1 \gg \nu^{H+\frac{1}{2}} \gg \sigma > 0$: Instability

In this case, the amplitude equation is given by

$$\mathrm{d}b = [b + \mathcal{F}_c(b)]\,\mathrm{d}T. \tag{6.9}$$

Here we consider the solution $b = 0$ of the amplitude equation and let u be the solution of SPDE with $u(0) = u_0 = 0$. Here we simplify the proof by neglecting the small noise $\sigma/\nu^{H+1/2}$ in the approximation. Therefore, we cannot use Theorem 5.3 to approximate the SPDE with (6.9). However, all the bounds provided for u in Theorem 5.3 do not depend on the amplitude equation and are enough for our aims.

As before, let V be the solution of the linearized SPDE

$$\partial_T V = \varepsilon^{-2} A V + V + D\mathcal{F}(U)V$$

and thus

$$\partial_T V_c = V_c + P_c D\mathcal{F}(U)V = V_c + D\mathcal{F}_c(U)(V_c + V_s).$$

The linearization of the amplitude equation around 0 reduces to

$$\partial_T \varphi = \varphi + DF_c(0)\varphi,$$

which gives

$$\partial_T \varphi = \varphi.$$

The main result in this case reads as follows. Recall that T_0 is an arbitrary terminal time and $T \in [0, T_0]$.

Theorem 6.7 *Let λ_T be the finite-time Lyapunov exponent of the SPDE (2.1) with initial data $u_0 = 0$. For all probabilities $p \in (0, 1)$, there is a set Ω_p with probability larger than p and a constant $C_p > 0$ such that for $\omega \in \Omega_p$ we have that*

$$\lambda_{T\nu^{-1}}(0) > \nu - C_p \frac{\nu^{3/2}}{T}, \quad \textit{for all } H \in (0, 1) \textit{ and } T \in [0, T_0].$$

Proof Recall the rescaling $v(T/\nu) = \nu^{1/2} V(T)$. Analogously to the previous case, we have on a set Ω_p that

$$\lambda_{T\nu^{-1}}(0) = \frac{\nu}{T} \ln(\sup\{\|v(T/\nu)\| \;\; \|v(0)\| = 1\})$$

$$
\begin{aligned}
&= \frac{\nu^{3/2}}{T} \ln(\sup\{\|V(T)\| \ \ \|V(0)\| = \nu^{-1/2}\}) \\
&= \frac{\nu}{T} \ln(\sup\{\|V(T)\| \ \ \|V(0)\| = 1\}) \\
&\geq \frac{\nu}{T} \ln(\sup\{\|\varphi(T)\| - \|V(T) - \varphi(T)\| : \ \|V(0)\| = 1\}) \\
&\geq \frac{\nu}{T} \ln(\sup\{\|\varphi(T)\| - \|V(T) - \varphi(T)\| : \ \|v(0)\| = 1, V(0) = \varphi(0) \in \mathcal{N}\}) \\
&\geq \frac{\nu}{T} \ln(\sup\{\|\varphi(T)\| - \varepsilon : \ \|V(0)\| = 1, V(0) = \varphi(0) \in \mathcal{N}\}) \\
&\geq \frac{\nu}{T} \ln(\exp T - \varepsilon) \\
&\geq \nu - C\frac{\nu\varepsilon}{T}.
\end{aligned}
$$

□

Therefore, we get

$$\partial_T(V_c - \varphi) = V_c - \varphi + D\mathcal{F}_c(U)(V_c + V_s),$$

which further leads to

$$
\begin{aligned}
&\frac{1}{2}\partial_T\|V_c - \varphi\|^2 \\
&\quad = \|V_c - \varphi\|^2 + \langle D\mathcal{F}_c(U)(V_c + V_s), V_c - \varphi\rangle \\
&\quad = \|V_c - \varphi\|^2 + \langle D\mathcal{F}_c(U)V_c, V_c - \varphi\rangle + \langle D\mathcal{F}_c(U)V_s, V_c - \varphi\rangle \\
&\quad \leq \|V_c - \varphi\|^2 + c\|U\|_X^2\|V_c\|_{\mathcal{N}}\|V_c - \varphi\|_{\mathcal{N}} + c\|U\|_X^2\|V_s\|_X\|V_c - \varphi\|_{\mathcal{N}} \\
&\quad \leq \|V_c - \varphi\|^2 + c\|U\|_X^2\|V_c\|_{\mathcal{N}}\|V_c - \varphi\|_{\mathcal{N}} + c\|U\|_X^2\|V_s\|_{\mathcal{H}^{1/2}}\|V_c - \varphi\|_{\mathcal{N}} \\
&\quad \leq c\|V_c - \varphi\|^2 + c\|U\|_X^4\|V_c\|_{\mathcal{N}}^2 + c\|V_s\|_{\mathcal{H}^{1/2}}^2 + c\|U\|_X^4\|V_c - \varphi\|_{\mathcal{N}}^2 \\
&\quad \leq c(1 + \|U\|_X^4)\|V_c - \varphi\|_{\mathcal{N}}^2 + c\|U\|_X^4\|V_c\|_{\mathcal{N}}^2 + c\|V_s\|_{\mathcal{H}^{1/2}}^2 \\
&\quad \leq c\|V_c - \varphi\|_{\mathcal{N}}^2 I + cJ,
\end{aligned}
$$

where $I := 1 + \|U\|_X^4$ and $J := \|U\|_X^4\|V_c\|_{\mathcal{N}}^2 + \|V_s\|_{\mathcal{H}^{1/2}}^2$ and c stands for a universal constant which varies from line to line. Again, Gronwall's inequality on $[0, T_0]$ entails

$$\|V_c(T) - \varphi(T)\|^2 \leq c\Big(\|V_c(0) - \varphi(0)\|^2 + c\int_0^T J(S)\,\mathrm{d}S\Big) \cdot \exp\Big(c\int_0^T I(S)\,\mathrm{d}S\Big).$$

This gives $\|V_c(T) - \varphi(T)\| \leq C\nu^{1/2}$ on a set of probability arbitrarily close to 1, when the constant C goes to ∞. In this case, we remark that the order of J does not depend on H since it is determined by $\int_0^T \|U(S)\|^4\,\mathrm{d}S = \mathcal{O}(1)$ and $\|V_s\|_{L^2(0,T_0;\mathcal{H}^{1/2})} = \mathcal{O}(\nu^{1/2})$.

Obviously, the exponent $\int_0^T I(S)\,\mathrm{d}S$ can be bounded by a constant on a set of large probability.

In conclusion, we obtain on $[0, T_0]$ and on a set of probability arbitrarily close to 1

$$\|V(T) - \varphi(T)\|_{\mathcal{H}} \leq \|V_s(T)\|_{\mathcal{H}} + \|V_c(T) - \varphi(T)\|_{\mathcal{H}} \leq C\nu^{1/2}.$$

Remark 6.8 For this argument, we do not need the set $\tilde{\Omega}$ constructed in Lemma 4.1, since we consider the linearization of the amplitude equation around zero. The result obtained here provides a bound of the error term of order $\mathcal{O}(\nu^{1/2})$ independent of the value of $H \in (0, 1)$.

6.4 Case: $\nu = 0, 1 \gg \sigma > 0$: Stability at the Bifurcation Point

At the bifurcation point, we consider $\varepsilon = \sigma^H$. Here the amplitude equation is

$$\mathrm{d}b = P_c\mathcal{F}(b)\,\mathrm{d}T + \mathrm{d}\beta_\varepsilon(T).$$

Therefore, we get

$$\partial_T\varphi = D\mathcal{F}_c(b)\varphi. \tag{6.10}$$

The linearization of the SPDE (2.1) reads now as

$$\partial_t v = Av + D\mathcal{F}(u)v,$$

which means that setting $v(t) = \varepsilon V(t\varepsilon^2)$ we obtain

$$\partial_T V = \varepsilon^{-2}AV + D\mathcal{F}(U)V. \tag{6.11}$$

As in the previous cases we compute the error term between the two linearizations.

Theorem 6.9 *Let b_0 be an initial datum for which the corresponding solution satisfies Lemma 4.2. For all $p \in (0, 1)$ there is a constant C_p and a set Ω_p with probability larger than p such that the approximation order between the linearization of the SPDE (6.11) and of the amplitude equation (6.10) with initial data $u_0 = \varepsilon b_0$ is bounded by $C_p\varepsilon$ if $H > 1/2$ respectively $C_p\varepsilon^{2H}$ if $H < 1/2$ on the set Ω_p.*

Proof Since the linear term containing ν drops out, we compute new energy estimates. To get an $\mathcal{O}(1)$ bound on V, we rely on the energy estimate

$$\frac{1}{2}\partial_T\|V\|^2 \leq \varepsilon^{-2}\langle AV, V\rangle + \langle D\mathcal{F}(U)V, V\rangle,$$

which gives now due to (2.3)

$$\frac{1}{2}\partial_T\|V\|^2 \leq \varepsilon^{-2}\langle AV, V\rangle \leq 0,$$

due to the non-negativity of A. As $V(0) = \mathcal{O}(1)$ this yields a uniform $\mathcal{O}(1)$ bound on V in $\mathcal{H}$ on $[0, T_0]$. Due to the $\mathcal{O}(1)$ bound on V in $\mathcal{H}$, we can also bound V_c in $\mathcal{N}$ in any norm.

For V_s, we obtain as before that $\|V_s(T)\|_{\mathcal{H}} = \mathcal{O}(\varepsilon)$ and that $\|V_s\|_{L^2(0,T_0,\mathcal{H}^{1/2})} = \mathcal{O}(\varepsilon^2)$. This follows by the usual energy estimate regarding Assumption 2.1 and (2.3) combined with the ε-Young inequality. To be more precise, the estimate is based on

$$
\begin{aligned}
\frac{1}{2}\partial_T \|V_s\|^2 &= \varepsilon^{-2}\langle AV_s, V_s\rangle + \langle P_s D\mathcal{F}(U)V, V_s\rangle \\
&\leq -C\varepsilon^{-2}\|V_s\|^2_{\mathcal{H}^{1/2}} + C\|U\|_X^2\|V_c\|_X\|V_s\|_{\mathcal{H}^\alpha} \\
&\leq -C\varepsilon^{-2}\|V_s\|^2_{\mathcal{H}^{1/2}} + C\varepsilon^2\|U\|_X^4\|V_c\|_X^2 + \frac{1}{2}C\varepsilon^{-2}\|V_s\|^2_{\mathcal{H}^\alpha} \\
&\leq -\frac{1}{2}C\varepsilon^{-2}\|V_s\|_{\mathcal{H}^{1/2}} + C\varepsilon^2\|U\|_X^4\|V_c\|_X^2.
\end{aligned}
$$

For V_c and φ, we have

$$
\partial_T V_c = D\mathcal{F}_c(U)V \quad \text{and} \quad \partial_T\varphi = D\mathcal{F}_c(b)\varphi,
$$

leading to

$$
\partial_T(V_c - \varphi) = (D\mathcal{F}_c(U) - D\mathcal{F}_c(b))V + D\mathcal{F}_c(b)(V - \varphi).
$$

For the difference, we estimate as follows. Here c is a universal constant which varies from line to line.

$$
\begin{aligned}
&\frac{1}{2}\partial_T\|V_c - \varphi\|^2 \\
&\quad = \langle (D\mathcal{F}_c(U) - D\mathcal{F}_c(b))V, V_c - \varphi\rangle + \langle D\mathcal{F}_c(b)(V - \varphi), V_c - \varphi\rangle \\
&\quad = \langle (D\mathcal{F}_c(U) - D\mathcal{F}_c(b))V, V_c - \varphi\rangle + \langle D\mathcal{F}_c(b)(V_c - \varphi), V_c - \varphi\rangle \\
&\qquad + \langle D\mathcal{F}_c(b)V_s, V_c - \varphi\rangle \\
&\quad \leq c\|U - b\|_X^2(\|V_s\|_X + \|V_c\|_{\mathcal{N}})\|V_c - \varphi\| + c\|b\|_{\mathcal{N}}^2\|V_c - \varphi\|_{\mathcal{N}}^2 \\
&\qquad + c\|b\|_{\mathcal{N}}^2\|V_s\|_{\mathcal{H}^\alpha}\|V_c - \varphi\| \\
&\quad \leq c\|U_c - b\|_{\mathcal{N}}^2(\|V_s\|_{\mathcal{H}^\alpha} + \|V_c\|_{\mathcal{N}})\|V_c - \varphi\|_{\mathcal{N}} + c\|U_s\|_X^2(\|V_s\|_{\mathcal{H}^\alpha} \\
&\qquad + c\|V_c\|_{\mathcal{N}})\|V_c - \varphi\|_{\mathcal{N}} \\
&\qquad + \|b\|_{\mathcal{N}}^2\|V_c - \varphi\|_{\mathcal{N}}^2 + c\|b\|_{\mathcal{N}}^4\|V_s\|^2_{\mathcal{H}^\alpha} + c\|V_c - \varphi\|_{\mathcal{N}}^2 \\
&\quad \leq c\|U_c - b\|_{\mathcal{N}}^4\|V_c - \varphi\|_{\mathcal{N}}^2 + c\|V_s\|^2_{\mathcal{H}^{1/2}} + c\|U_c - b\|_{\mathcal{N}}^4\|V_c\|_{\mathcal{N}}^2 + c\|V_c - \varphi\|_{\mathcal{N}}^2 \\
&\qquad + c\|U_s\|_X^4\|V_c - \varphi\|_{\mathcal{N}}^2 + c\|U_s\|_X^4\|V_c\|_{\mathcal{N}}^2 + c\|b\|_{\mathcal{N}}^2\|V_c - \varphi\|_{\mathcal{N}}^2 \\
&\qquad + c\|b\|_{\mathcal{N}}^4\|V_s\|^2_{\mathcal{H}^{1/2}}.
\end{aligned}
$$

Thus,

$$
\partial_T\|V_c - \varphi\|^2 \leq cI\|V_c - \varphi\|_{\mathcal{N}}^2 + cJ,
$$

where

$$I := 1 + \|U_c - b\|_{\mathcal{N}}^4 + \|U_s\|_X^4 + \|b\|_{\mathcal{N}}^2$$

and

$$J := \|V_s\|_{\mathcal{H}^{1/2}}^2 + \|U_c - b\|_{\mathcal{N}}^4 \|V_c\|_{\mathcal{N}}^2 + \|U_s\|_X^4 \|V_c\|_{\mathcal{N}}^2 + \|b\|_{\mathcal{N}}^4 \|V_s\|_{\mathcal{H}^{1/2}}^2.$$

Using Gronwall's inequality as before we obtain

$$\|V_c(T) - \varphi(T)\|^2 \leq \Big(\|V_c(0) - \varphi(0)\|^2 + c\int_0^T J(S)\,\mathrm{d}S\Big) \cdot \exp\Big(c\int_0^T I(S)\,\mathrm{d}S\Big).$$

Now we use again the $\mathcal{O}(1)$ bounds on V_c and b and the $\mathcal{O}(\varepsilon^{4H})$-bounds for $\|U_s\|_{L^4(0,T_0,X)}$ and $\mathcal{O}(\varepsilon^2)$ for $\|V_s\|_{L^2(0,T_0,\mathcal{H}^{1/2})}$ together with Theorem 5.3 that yields

$$\|b(T) - U_c(T)\|_{\mathcal{H}} = \varepsilon^{-1}\|\varepsilon b - u_c(\varepsilon^{-2}\cdot)\|_{\mathcal{H}} = \mathcal{O}(\varepsilon^{2H}).$$

In contrast to case 6.2, we only need pathwise bounds on J of order $\mathcal{O}(\varepsilon^2)$, respectively, $\mathcal{O}(\varepsilon^{4H})$ (depending on the range of H) and on b of order $\mathcal{O}(1)$, which hold on a set of probability arbitrarily close to 1. There is no need for b being small.

Moreover, a bound by a constant of the exponent $\int_0^T I(S)\,\mathrm{d}S$ holds as before on some set of probability arbitrarily close to 1.

Thus, we finally conclude that $\|V(T) - \varphi(T)\|_{\mathcal{N}} \leq C[\varepsilon + \varepsilon^{2H}]$ for all $T \in [0, T_0]$ on a set of probability arbitrarily close to 1. Actually its probability goes to 1 if $C \to \infty$. □

Regarding Lemma 4.2, we obtain the following bound on the FTLEs. First of all we recall that for the set $\Omega_\theta = \{\mathcal{T}_\theta(T_0) < \sqrt{p_\theta}\}$ we showed that $\mathbb{P}(\Omega_\theta) \geq 1 - \sqrt{p_\theta}$ where $p_\theta \to 0$ as $\theta \to 0$. Here T_0 is an arbitrary terminal time and $\mathcal{T}_\theta(T_0)$ is the set of times for which the amplitude equation is smaller than θ. Keeping this in mind and recalling that $D\mathcal{F}_c(b) = -cb^2$, we derive the following statement.

Theorem 6.10 *Let b_0 be an initial data of (4.2) for which the corresponding solution satisfies Lemma 4.2. Furthermore, let $\lambda_{\widetilde{T}}$ be the finite-time Lyapunov exponent of the SPDE (2.1) with initial condition $u_0 = \varepsilon b_0$. For all probabilities $p \in (0, 1)$ there exist a set $\tilde{\Omega}_p$ with probability larger than p and constants $C_p, c_p > 0$ and times T_p with $T_p \to 0$ as $p \to 1$, $c_p \to 0$ and $C_p \to \infty$ for $t \to 0$ such that for $\omega \in \tilde{\Omega}_p$ we have for $\widetilde{T} \in [T_p, T_0]$ that*

$$\lambda_{\widetilde{T}\varepsilon^{-1}}(\varepsilon b_0) \leq -c_p\varepsilon + [\varepsilon^2 + \varepsilon^{2H+1}]\frac{e^{c_p\widetilde{T}}}{\widetilde{T}}.$$

Note that the proof relies on Lemma 4.2. Having more knowledge about the density of the solution b of the amplitude we conjecture it should be possible to get an ε-dependent T_p.

Proof Our goal is to find a bound for

$$\begin{aligned}\lambda_{\widetilde{T}\varepsilon^{-1}}(\varepsilon b_0) &= \frac{\varepsilon}{\widetilde{\widetilde{T}}}\ln(\sup\{\|V(\widetilde{T})\| \ \ \|V(0)\| = 1\}) \\ &\leq \frac{\varepsilon}{\widetilde{\widetilde{T}}}\ln\sup(\{\|\varphi(\widetilde{T})\| + \|V(\widetilde{T}) - \varphi(\widetilde{T})\| : \|V(0)\| = 1\}) \\ &\leq \frac{\varepsilon}{\widetilde{\widetilde{T}}}\ln\left(\exp\left\{\int_0^{\widetilde{T}} D\mathcal{F}_c(b(s,\omega))\,\mathrm{d}s\right\} + C_p[\varepsilon + \varepsilon^{2H}]\right).\end{aligned}$$

The upper bound is clear as long as the solution of the amplitude equation (4.2) does not spend too much time in zero. To exclude this possibility, we established in Lemma 4.2 a lower bound on the probability of the set Ω_θ with $\mathbb{P}(\Omega_\theta) \to 1$ if $\theta \to 0$, where

$$\int_0^T D\mathcal{F}_c(b(s,\omega))\,\mathrm{d}s \leq -c_\theta T,$$

for a constant $c_\theta = c\theta^2 \to 0$ if $\theta \to 0$.

This further entails that

$$\lambda_{\widetilde{T}}(b_0) = \frac{1}{\widetilde{\widetilde{T}}}\ln\exp\left(\int_0^{\widetilde{T}} D\mathcal{F}_c(b(s,\omega))\,\mathrm{d}s\right) < -c_\theta < 0$$

on the set Ω_θ.

Regarding this, we easily derive on the set of large probability $\Omega_p \cap \Omega_\theta$ that

$$\begin{aligned}\ln(e^{-c_\theta\widetilde{T}} + C_p[\varepsilon + \varepsilon^{2H}]) &= \ln(e^{-c_\theta\widetilde{T}}(1 + C_p[\varepsilon + \varepsilon^{2H}]e^{c_\theta\widetilde{T}})) \\ &= -c_\theta\widetilde{T} + \ln(1 + C_p[\varepsilon + \varepsilon^{2H}]e^{c_\theta\widetilde{T}}) \\ &\leq -c_\theta\widetilde{T} + C_p[\varepsilon + \varepsilon^{2H}]e^{c_\theta\widetilde{T}}.\end{aligned}$$

This further leads to

$$\lambda_{\widetilde{T}\varepsilon^{-1}}(\varepsilon b_0) \leq -c_\theta\varepsilon + C_p[\varepsilon^2 + \varepsilon^{2H+1}]\frac{e^{c_\theta\widetilde{T}}}{\widetilde{\widetilde{T}}},$$

which proves the statement. □

Remark 6.11 For the Brownian motion, we proved a similar assertion in Blömker and Neamţu (2023) for the stationary solution of the amplitude equation 4.2 using Birkhoff's ergodic theorem. We improve now this result in Lemma 4.2 for an arbitrary solution deriving showing that the probability that the amplitude equation stays close to zero for a lot of times is small. For a higher-dimensional kernel and / or multiplicative noise, this property is expected to hold, see Blömker (2007) for a similar discussion for amplitude equations with Brownian motion.

6.5 Case: $1 \gg \sigma \gg \nu^{H+1/2} > 0$: Stability

This situation can be dealt with similar to Case 6.4 using the amplitude equation

$$\mathrm{d}\tilde{b} = \Big[\frac{\nu^{H+1/2}}{\sigma}\tilde{b} + P_c\mathcal{F}(\tilde{b})\Big]\,\mathrm{d}T + \mathrm{d}\beta_\varepsilon(T), \tag{6.12}$$

and its linearization

$$\partial_T\tilde{\varphi} = \frac{\nu^{H+1/2}}{\sigma}\tilde{\varphi} + D\mathcal{F}_c(\tilde{b})\tilde{\varphi}.$$

Since the difference between (6.12) and (4.2) is of order $\mathcal{O}(\frac{\nu^{H+1/2}}{\sigma})$, the following statement can be obtained analogously to Case 6.5. However, there is a major difference for the error term compared to the previous cases. More precisely, here the order of J will be determined by

$$\|\tilde{b}(T) - U_c(T)\|_{\mathcal{N}} \le C\frac{\nu^{H+1/2}}{\sigma} + \|b(T) - U_c(T)\|_{\mathcal{N}} \le C\frac{\nu^{H+1/2}}{\sigma} + C\varepsilon^{2H},$$

which is in the lowest order $\frac{\nu^{H+1/2}}{\sigma}$, since $\sigma \gg \nu^{H+1/2}$.

Theorem 6.12 *Let b_0 be an initial data of (4.2) for which the corresponding solution satisfies Lemma 4.2. Furthermore let $\lambda_{\widetilde{T}}$ be the finite-time Lyapunov exponent of the SPDE (2.1) with initial data $u_0 = \varepsilon b_0$. For all probabilities $p \in (0, 1)$ there exists a set $\tilde{\Omega}_p$ with probability larger than p, a time $T_p > 0$ with $T_p \to 0$ for $p \to 1$ and positive constants c_p and C_p such that $C_p \to \infty$ and $c_p \to 0$ such that for $\omega \in \tilde{\Omega}_p$ and for all $\widetilde{T} \in [T_p, T_0]$ we have*

$$\lambda_{\widetilde{T}\sigma^{-H}}(\sigma^H b_0) \le \sigma^H\Big(-c_p + \frac{\nu^{H+1/2}}{\sigma}\Big) + \sigma^H\left(\frac{\nu^{H+1/2}}{\sigma}\right)^2 \cdot \frac{e^{c_p T - \frac{\nu^{H+1/2}}{\sigma}}}{\widetilde{T}}.$$

Proof We only give a sketch of the proof, since this is similar to Case 6.5. Regarding the computations in Case 6.4 we infer on a set of probability almost 1 and for $T \in [0, T_0]$ that

$$\|V(T) - \tilde{\varphi}(T)\| \le C\Big(\varepsilon + \Big(\varepsilon^{2H} + \frac{\nu^{H+1/2}}{\sigma}\Big)^2\Big).$$

This follows as before using for $T \in [0, T_0]$ that

$$\|V(T) - \tilde{\varphi}(T)\|_{\mathcal{H}} \le \|V_s(T)\|_{\mathcal{H}} + \|V_c(T) - \varphi(T)\|_{\mathcal{N}}.$$

To estimate the last term, we need a bound on J. As already indicated this is now determined by $\|\tilde{b}(T) - U_c(T)\|^2 = \Big(\varepsilon^{2H} + \frac{\nu^{H+1/2}}{\sigma}\Big)^2$ since this expression becomes small as $\sigma \gg \nu^{H+1/2}$. Note that in Case 6.5 the order of J was in lowest order determined by $\|V_s\|^2_{L^2(0,T;\mathcal{H}^{1/2})} = \mathcal{O}(\varepsilon^2)$ and $U_s \in L^4(0, T_0; X) = \mathcal{O}(\varepsilon^{4H})$ and the other terms were higher order. In conclusion, we now get for $T \in [0, T_0]$ that

$$\|V(T) - \tilde{\varphi}(T)\|_{\mathcal{H}} \leq C\Big(\varepsilon + \Big(\varepsilon^{2H} + \frac{\nu^{H+1/2}}{\sigma}\Big)^2\Big).$$

Therefore, the lower bound for the FTLEs for $\omega \in \tilde{\Omega}_p$ (as in Case 6.5) results in for $\widetilde{T} \in [T_p, T_0]$

$$\begin{aligned}
\lambda_{\widetilde{T}\varepsilon^{-1}}(\varepsilon b_0) &= \frac{\varepsilon}{\widetilde{T}} \ln(\sup\{\|V(\widetilde{T})\| \ \ \|V(0)\| = 1\}) \\
&\leq \frac{\varepsilon}{\widetilde{T}} \ln \sup(\{\|\varphi(\widetilde{T})\| + \|V(\widetilde{T}) - \varphi(\widetilde{T})\| : \|V(0)\| = 1\}) \\
&\leq \frac{\varepsilon}{\widetilde{T}} \ln\Big(\exp\Big\{\frac{\nu^{H+1/2}}{\sigma}\widetilde{T} + \int_0^{\widetilde{T}} D\mathcal{F}_c(b(s,\omega))\,\mathrm{d}s\Big\} + C\Big(\frac{\nu^{H+1/2}}{\sigma}\Big)^2\Big).
\end{aligned}$$

Using Lemma 4.2, this entails for $\omega \in \tilde{\Omega}_p$ that

$$\lambda_{\widetilde{T}\varepsilon^{-1}}(\varepsilon b_0) \leq \varepsilon\Big(-c_p + \frac{\nu^{H+1/2}}{\sigma}\Big) + \varepsilon\Big(\frac{\nu^{H+1/2}}{\sigma}\Big)^2 \frac{e^{c_p - \frac{\nu^{H+1/2}}{\sigma}}}{\widetilde{T}} < 0.$$

This proves the statement regarding that $\varepsilon = \sigma^H$ in this case. □

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