Research Article Open Access

Kevin Jakob* and Matthias Fischer

Quantifying the impact of different copulas in a generalized CreditRisk$^+$ framework

An empirical study

Abstract: Without any doubt, credit risk is one of the most important risk types in the classical banking industry. Consequently, banks are required by supervisory audits to allocate economic capital to cover unexpected future credit losses. Typically, the amount of economical capital is determined with a credit portfolio model, e.g. using the popular CreditRisk$^+$ framework (1997) or one of its recent generalizations (e.g. [8] or [15]). Relying on specific distributional assumptions, the credit loss distribution of the CreditRisk$^+$ class can be determined analytically and in real time. With respect to the current regulatory requirements (see, e.g. [4, p. 9-16] or [2]), banks are also required to quantify how sensitive their models (and the resulting risk figures) are if fundamental assumptions are modified. Against this background, we focus on the impact of different dependence structures (between the counterparties of the bank's portfolio) within a (generalized) CreditRisk$^+$ framework which can be represented in terms of copulas. Concretely, we present some results on the unknown (implicit) copula of generalized CreditRisk$^+$ models and quantify the effect of the choice of the copula (between economic sectors) on the risk figures for a hypothetical loan portfolio and a variety of parametric copulas.

Keywords: copula; credit risk; model risk; quantitative finance; CreditRisk+; capital requirements

MSC: 91G40, 62H86

***Corresponding Author: Kevin Jakob:** Department of Economics, University of Augsburg, E-mail: Kevin.Jakob@Student.Uni-Augsburg.de
Matthias Fischer: Department of Statistics and Econometrics, University of Nuremberg, E-mail: Matthias.Fischer@fau.de

1 Introduction

Financial institutions are allowed to use their own internal models in order to manage different types of risk, such as market, liquidity, credit or operational risk, on an economic (in contrast to a regulatory) basis. In this context, banks are requested by national supervisors to quantify the accompanied amount of model risk.

> If methods and processes, underlying assumptions, parameters or influent data are rather complex, a qualitative and quantitative validation of these components as well as the useability of the resulting risk figures is necessary (see [2]).

Regarding credit risk, portfolio models always assume a specific dependency structure between economic sectors and obligors. Mathematically, dependency can be described with the help of copula functions, originally introduced by [33]. Recently, several authors such as [20], [11], [6] or [24] have addressed the topic of copulas in credit portfolio models. However, they have always concentrated on only one copula class. By contrast, we take a wide range of copulas into account, analyze which fits best to default data and how risk figures change for several copulas. Because in practical applications the number of counterparties is simply too high to model their dependencies directly, we focus on the dependency structure between sector variables. In this regard the article can be seen as an extension of [6], who exchanged the copula of the compound gamma model[1] with a Gaussian copula and a t-copula. Our analysis incorporates representatives from the elliptical, the generalized hyperbolic and the Archimedean copula classes together with two copulas, implicitly defined by the corresponding portfolio model.

The outline of this article is as follows: Section 2 gives a brief introduction to copulas and defines some notation used throughout the article. In the third section we shortly introduce the CreditRisk$^+$model together with two model enhancements covering flexible sector dependencies. Here we also analyze the implicit copulas induced by sector models in order to compare them with other dependency models. In section 4 the credit portfolio under consideration and the default data used for parameter estimation are described. Afterwards we analyze the change in risk figures concerning different copulas. The article concludes with a summary.

2 Copulas

The notion of copulas dates back to [33]. Technically, a copula C is a multivariate distribution function on the d-dimensional unit hypercube $\boldsymbol{I}^d$ with uniform margins. With the help of copulas one can separate the problem of finding a suitable multivariate distribution function into two parts. In a first step, one can concentrate on the univariate margins independently from each other. In a second step one analyzes the dependency structure regardless of the marginal distribution. The justification for this procedure is given by Sklar's theorem.

Theorem 1. *Let F denote a d-dimensional distribution function on $\mathbb{R}^d$ with univariate margins F_i. Furthermore, let $Im(F_i)$ denote the image of F_i. Then, a unique copula function $C : \times_{i=1}^d Im(F_i) \to \boldsymbol{I}$ exists such that*

$$F(\boldsymbol{x}) = C(F_1(x_1), \ldots, F_d(x_d)) \quad \forall \boldsymbol{x} \in \mathbb{R}^d.$$

It should be noted that, especially if all F_i are continuous on $\boldsymbol{I}$, the copula is unique on $\boldsymbol{I}^d$. All information about the dependency structure are contained in the copula.

In this article we will use Sklar's theorem in reverse. In order to analyze the impact of the copula on the risk figures of the CreditRisk$^+$model, we exchange the copula function and create new multivariate distribution functions with the same margins as in the original case.

Our analysis includes the following copulas:

- **Independence copula** defined as $C(\boldsymbol{u}) := \prod_{i=1}^d u_i$. The copula is the one of d stochastically independent random variables U_i.
- **Elliptical copulas** are implicitly given by the class of elliptical distributions, see [7]. Famous representatives are the normal or Gaussian copula
$$C(\boldsymbol{u}) = \boldsymbol{\Phi}_\Sigma\left(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d)\right)$$
and the t-copula
$$C(\boldsymbol{u}) = \boldsymbol{t}_{\Sigma,\nu}\left(t_\nu^{-1}(u_1), \ldots, t_\nu^{-1}(u_d)\right).$$
Here, $\boldsymbol{\Phi}_\Sigma$ / $\boldsymbol{t}_{\Sigma,\nu}$ denotes the multivariate normal / t- distribution with zero mean, dispersion matrix Σ (and ν degrees of freedom). Φ^{-1} and t_ν^{-1} are the corresponding quantile functions.
- **Generalized hyperbolic copulas** (ghyp) arise implicitly from the corresponding multivariate distribution first studied by [3]. In general, ghyp copulas are not elliptically symmetric. Under certain parameter restrictions, ghyp copulas belong to the elliptical copula class. Therefore, the class of ghyp copulas also contains the normal and the t-copula as limiting cases. A detailed parametrization of the ghyp copula is given in the appendix.

1 The compound gamma model is an extension of the ordinary CreditRisk$^+$model, incorporating sector dependencies, see section 3

- **Archimedean copulas** (AC) are defined by a d-monotone[2] generator function $\psi : [0, \infty) \to \boldsymbol{I}$ with $\psi(0) = 1$ and $\lim_{x\to\infty} \psi(x) = 0$. In contrast to the former classes, these copulas have an explicit representation. For dimension d, an AC is defined by:

$$C_\psi(\boldsymbol{u}) := \psi\left(\sum_{i=1}^{d} \psi^{-1}(u_i)\right), \quad \boldsymbol{u} \in \boldsymbol{I}^d. \tag{1}$$

- **Hierarchical Archimedean** copulas (HAC) are an extension of the Archimedean class where one can combine different generator functions. For example, consider a generator function ψ_θ with parameter $\theta \in \Theta$, the ordinary AC is given by

$$C_{\psi_\theta}(\boldsymbol{u}) = \psi_\theta\left(\psi_\theta^{-1}(u_1) + \left(\psi_\theta^{-1} \circ \psi_\theta\right)\left(\sum_{i=2}^{d} \psi_\theta^{-1}(u_i)\right)\right) = C_{\psi_\theta}\left(u_1, \underbrace{C_{\psi_\theta}\left(\boldsymbol{u}_{\{2,\dots,d\}}\right)}_{(\#)}\right).$$

Under suitable conditions[3] one can exchange the inner AC denoted by (#) with another AC arising from a different generator function. This nesting procedure can be repeated until only two variables are left. The resulting copula is called a fully nested HAC (left panel of figure 1). Another nesting procedure could be to exclude not only one variable at each nesting level, but two or more, which again can be grouped via an AC. The resulting copula is called a partially nested HAC (right panel of figure 1). The technical definition of a general HAC depends on a rather complex notation. Since, for our purposes, the general definition is not necessary, we refer to [32].

In case of the Frank, Clayton or Gumbel copula, one can alternate the parameter values θ for different nesting levels. Thus, one can create hierarchical structures with stronger dependencies at the ground level (e.g. between industry groups of specific countries) and weaker dependencies at the top level (e.g. between countries themselves).

Figure (1) shows two possible trees of a five dimensional HAC where the generator function is of the Clayton type, i.e. $\psi_\theta^{\mathrm{Cl}}(x) := (\theta x + 1)^{-1/\theta}$ for $\theta \geq 0$. For a detailed discussion of HAC we refer to [32] and [29] as well as [23] and [16], where the sufficiency of the nesting condition has been proven and first examples were presented.

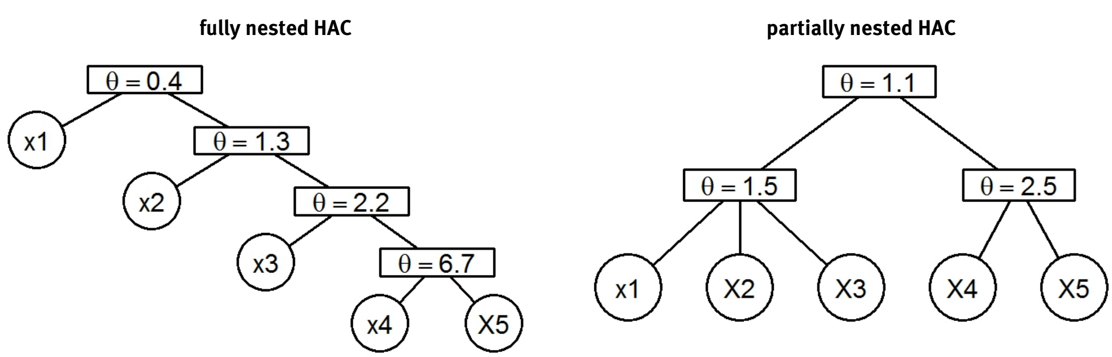


Fig. 1. Fully and partially nested HAC with a Clayton generator function.

Please note that the partially nested structure is just a special case of the fully nested one.

2 A function f is d-monotone on D iff $(-1)^k f^{(k)}(x) \geq 0, \quad k = 0, \dots, d-2$ for all x in the interior of D and $(-1)^{d-2} f^{(d-2)}$ is a decreasing and convex function on D.

3 The including generator functions ψ_θ as well as the derivatives of $\psi_\theta^{-1} \circ \psi_{\tilde{\theta}}$, where $\psi_{\tilde{\theta}}$ denotes a different generator function, have to be completely monotone. A function f is called completely monotone iff $(-1)^k f^{(k)}(x) \geq 0, \ \forall k \in \mathbb{N}$

3 The CreditRisk$^+$ model and extensions

In this section, after a short primer on credit risk in general, where we explain the reasons for using credit portfolio models in practice, the CreditRisk$^+$model is introduced. At the end we discuss the topic of sector dependencies and present two extensions of the basic model to incorporate correlated sectors.

3.1 A short primer on credit risk in general

In the banking industry, credit portfolio models are used to quantify the risk arising from a portfolio of obligors within a fixed time horizon (e.g. one year). Thinking of the overall portfolio loss, one distinguishes between expected and unexpected losses. The former one corresponds to the expectation value of the portfolio loss (e.g. caused by defaults). Since this component is not affected by relationships between counterparties, it can be calculated separately for each obligor. Typically, the expected loss is incorporated within the credit pricing process. By contrast, the unexpected loss corresponds to the possible loss exceeding the expected one. With the help of so called economic capital, financial institutions measure the amount of capital needed to cover unexpected losses. In order to quantify the necessary amount, the so called value at risk approach is used. For a given level $\alpha \in [0, 1]$ the corresponding value at risk (VaR_α) is defined as the α–quantile of the portfolio loss distribution. The economic capital is defined as

$$\mathrm{EC}_\alpha := \mathrm{VaR}_\alpha - \mathrm{EL},$$

where EL denotes the expected loss (over one year) of the portfolio.

The major task of credit portfolio models is to calculate the distribution function of the portfolio loss, such that one can extract the VaR_α and other characteristic values we define later.

In general, credit risk consists of two different parts:

- **default risk** arising from the loss caused by counterparties' defaults and
- **migration risk** occurring if the creditworthiness of counterparties decreases.

Please note that within this article we only concentrate on the default risk component. For a more detailed introduction to credit risk we refer to [24].

3.2 The basic CreditRisk$^+$ model

The CreditRisk$^+$ portfolio model was introduced by the Financial Products division of Credit Suisse in 1997. A detailed description of the model is given in [35]. It belongs to the class of mixture models. This model type is characterized by the assumption that defaults are conditionally independent given the state of the economy or a specific sector. As a consequence of certain distributional assumptions the pdf of the portfolio loss distribution can be calculated analytically. Therefore, it enjoys great popularity in practice.

Consider a portfolio of M counterparties $CP_1, \ldots, CP_M$. Let $\tilde{p}_i \in [0, 1]$ denote the probability of default (PD) for obligor i. The overall portfolio loss is defined by

$$\tilde{L} = \sum_{i=1}^{M} \mathrm{ead}_i \cdot \mathrm{lgd}_i \cdot \tilde{D}_i$$

with a default indicator $\tilde{D}_i \sim \mathrm{Bernoulli}(\tilde{p}_i)$. The variables $\mathrm{ead}_i > 0$ and $\mathrm{lgd}_i \in [0, 1]$ represent the exposure at default and the loss given default, respectively. The standard model assumes that ead_i and lgd_i are deterministic and independent from each other as well as from $\tilde{D}_i$.

In a first step, the exposures are discretized with respect to a common loss unit $L_0 > 0$. With the help of the discretization, the model becomes numerically tractable. Afterwards, the original potential loss $\mathrm{ead}_i \cdot \mathrm{lgd}_i$ and the PD $\tilde{p}_i$ are then replaced by

$$\nu_i := \max\left\{\left\lceil \frac{\mathrm{ead}_i \cdot \mathrm{lgd}_i}{L_0} \right\rceil, 1\right\} \quad \text{and} \quad p_i := \frac{\mathrm{ead}_i \cdot \mathrm{lgd}_i \cdot \tilde{p}_i}{\nu_i \cdot L_0}$$

respectively. Here $\lceil x \rfloor$ denotes the nearest integer to x. Since the original PD $\tilde{p}_i$ is replaced by p_i the discretization of the potential loss does not affect the expected loss of the portfolio, because:

$$\mathbb{E}\left(\tilde{L}\right) = \sum_{i=1}^{M} \text{ead}_i \cdot \text{lgd}_i \cdot \tilde{p}_i = \sum_{i=1}^{M} v_i \cdot L_0 \cdot p_i = \mathbb{E}\left(L\right),$$

with $L := \sum_{i=1}^{M} v_i \cdot L_0 \cdot D_i$ representing the discretized portfolio loss and $D_i \sim \text{Bernoulli}\left(p_i\right)$. In a second step, the Bernoulli distribution is replaced by a Poisson distribution. This replacement ensures, that the density function of the overall portfolio loss can be calculated analytically. The intensity of defaults depends on the modified PD p_i and some sector variables $S_k \sim \Gamma\left(\theta_k, \delta_k\right)$ representing the state of the economy or specific sectors.[4] Every counterparty is mapped onto one or more of K sectors via sector weights $w_{i,k} \in [0, 1]$ for $i = 1, .., M$ and $k = 1, \dots, K$. The idiosyncratic risk is represented by

$$w_{i,0} := 1 - \sum_{k=1}^{K} w_{i,k} \geq 0.$$

The default intensity of CP_i is defined by

$$\lambda_i := p_i \left(w_{i,0} + \sum_{k=1}^{K} w_{i,k} S_k \right). \tag{2}$$

In the basic model, the sector variables S_k are assumed to be independent from each other. Extensions to correlated sectors will be discussed in the following section. In combination with the sector weights $w_{i,k}$ the default intensities of two counterparties CP_i and CP_j sharing at least one sector, [5] are correlated with

$$\text{Cor}\left(\lambda_i, \lambda_j\right) = \frac{\sum_{k=1}^{K} w_{i,k} w_{j,k} \sigma_k^2}{\sqrt{\left(\sum_{k=1}^{K} w_{i,k}^2 \sigma_k^2\right)\left(\sum_{k=1}^{K} w_{j,k}^2 \sigma_k^2\right)}},$$

where σ_k^2 denotes the variance of S_k. In order to ensure an unchanged expected loss, $\mathbb{E}\left(S_k\right) = 1$ has to be assured, which means that $\delta_k = \frac{1}{\theta_k}$ for all sectors $k = 1, \dots, K$. As a second condition for the parametrization of the sector distribution one can estimate the variances σ_k^2 either from historical default data or use one out of several approximation formulas given in [14].

With the help of the probability generating function (pgf) of the Poisson distribution and the independence of sector variables and default events, the pgf of the total portfolio loss is given by

$$G_L(t) = \int_0^{\infty} \prod_{i=1}^{M} G_{L_i|\mathbf{S}}(t) \cdot \prod_{k=1}^{K} f_{S_k}(s)\, ds \tag{3}$$

with $G_{L_i|\mathbf{S}}(t) = 1 + p_i\left(t^{v_i} - 1\right)$. The pgf can be evaluated with the help of a nested evaluation algorithm. A detailed derivation of the pgf as well as a numerically stable algorithm to calculate the (discrete) density of L from its pgf is given in [14].

3.3 Modeling dependent sectors

Modeling dependencies between several counterparties in a credit portfolio is a crucial issue. Typically, credit portfolios consist of thousands of obligors. Modeling the dependencies directly between counterparties is

4 Throughout this article $\Gamma(a, b)$ denotes the Gamma distribution with shape parameter $a > 0$ and scale parameter $b > 0$. The mean is given by ab.

5 CP_i and CP_j share sector k iff $w_{i,k} w_{j,k} > 0$.

not manageable because the dimension of the copula or multivariate distribution function is just too high. Instead, the counterparties are categorized by industry, country and other attributes. Then each group is mapped onto a sector specific variable affecting the default probability (PD) of the group members. The dependency between sector variables translates into a dependency between counterparties. Since dependencies between counterparties have a major impact on the portfolio loss distribution, the way they are modeled directly influences the amount of economic capital which must be provided by financial institutions to cover unexpected losses.

Modeling the dependencies between sectors rather then counterparties reduces the dimension of the copula dramatically. In our empirical analysis (section 4) we work with a ten dimensional copula. In the following we introduce two extensions of the basic CreditRisk$^+$model, in order to overcome the assumption of independent sectors.

3.3.1 The Common-Background-Vector model

In practice, economic sectors are not independent. Therefore, [8] proposed the so called common-background-vector (CBV) model, which is a generalization of the model from [15]. The main idea is to replace each sector variable by a linear combination of L independent gamma distributed variables $\hat{S}_\ell \sim \Gamma\left(\hat{\theta}_\ell, 1\right)$ and an independent sector specific variable $S_k \sim \Gamma(\theta_k, \delta_k)$. I.e. we define

$$\overline{S}_k := S_k + \sum_{\ell=1}^{L} y_{k,\ell}\hat{S}_\ell \tag{4}$$

with non-negative weights $y_{k,\ell}$ for $k = 1, \dots, K$ and $\ell = 1, \dots, L$. The vector $\hat{\boldsymbol{S}} := \left(\hat{S}_1, \dots, \hat{S}_L\right)^T$ is called common-background-vector. $\hat{\boldsymbol{S}}$ is equal for all sectors k. How much the background factor $\hat{S}_l$ influences sector k is determined by $y_{k,l}$. Defining additional counterparty sector weights $w_{i,K+\ell} := \sum_{k=1}^{K} w_{i,k}y_{k,\ell}$ for $\ell = 1, \dots, L$ and plugging (4) into (2) we can rewrite the formula for the default intensity as

$$\begin{aligned}\lambda_i &= p_i\left(w_{i,0} + \sum_{k=1}^{K} w_{i,k}\overline{S}_k\right) \\ &= p_i\left(w_{i,0} + \sum_{k=1}^{K} w_{i,k}S_k + \sum_{\ell=1}^{L} w_{i,K+\ell}\hat{S}_\ell\right).\end{aligned}$$

This setting is equal to the basic model with $K + L$ independent sectors. Therefore, the pgf can be expressed analytically as in case of the basic model.

The variance covariance (VCV) of the original sectors $\overline{S}_k$ can be derived as

$$\operatorname{Var}\left(\overline{S}_k\right) = \theta_k\delta_k^2 + \sum_{\ell=1}^{L} y_{k,\ell}^2\hat{\theta}_\ell \quad k = 1, \dots, K \tag{5}$$

$$\operatorname{Cov}\left(\overline{S}_i, \overline{S}_j\right) = \sum_{\ell=1}^{L} y_{i,\ell}y_{j,\ell}\hat{\theta}_\ell \qquad i \neq j. \tag{6}$$

The model parameters $y_{k,\ell}$, δ_k, θ_k and $\hat{\theta}_\ell$ should be chosen such that the theoretical VCV structure given by equations (5) and (6) meets an empirical one. This can be achieved by solving a high dimensional optimization problem with $2K + L(1 + K)$ variables. The dimension of the optimization problem should not be mixed up with the dimension K of the corresponding copula (equation 7). In order to guarantee that $\mathbb{E}\left(\overline{S}_k\right) = 1$ we have to restrict the parameter space such that $\theta_k\delta_k + \sum_{\ell=1}^{L} y_{k,\ell}\hat{\theta}_k = 1$ for all sectors $k = 1, \dots, K$.

The multivariate distribution of the sector variables is determined by a linear combination given in equation (4). Writing this in a more general way with vectors $\boldsymbol{X} \in \mathbb{R}^d$ and $\boldsymbol{Y} \in \mathbb{R}^K$, where each single $X_{i=1,\dots,d}$ is

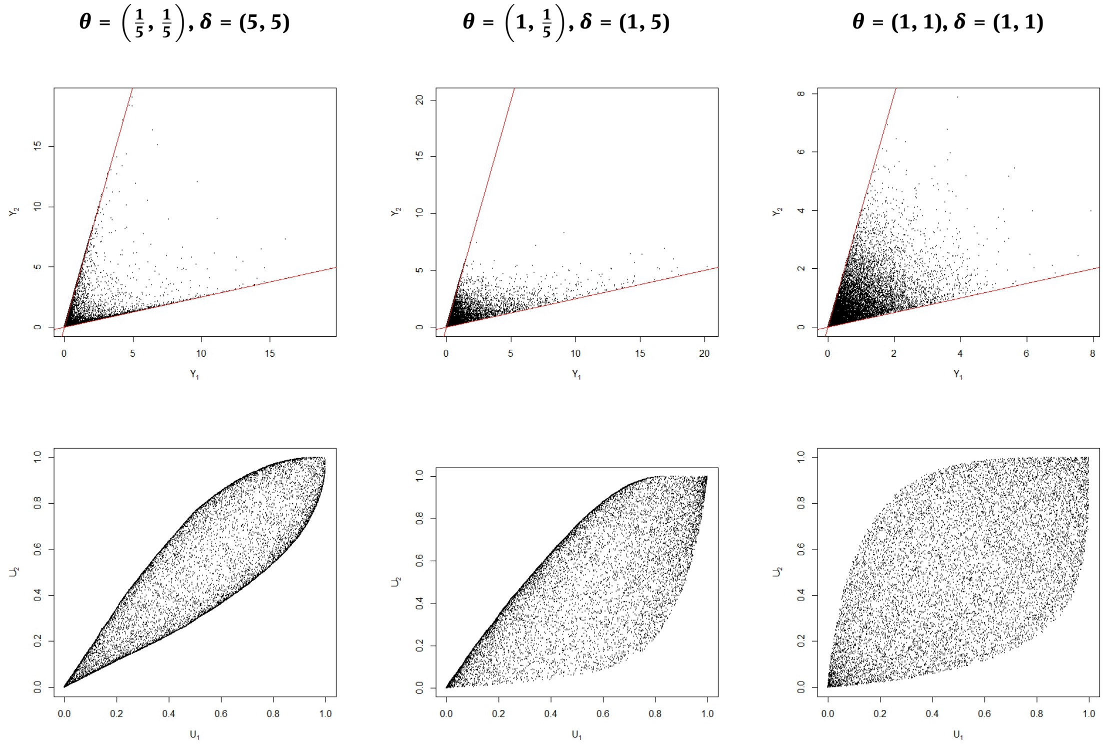


Fig. 2. 10,000 realizations of the multivariate distribution (first row) and the implicit copula (second row) defined by equation (7).

independently gamma distributed with an individual shape and scale parameter and a matrix $A \in \mathbb{R}^{K \times d}$, we have

$$\boldsymbol{Y} = A\boldsymbol{X}. \tag{7}$$

The univariate distribution of the single Y_k has already been studied by [26]. The copula corresponds to a multi-factor copula, investigated by [28], with gamma distributed factors. Figure (2) shows 10,000 realizations of the bivariate distribution (first row) and the implicit copula (second row) for the bivariate case with $A = \begin{pmatrix} 0.2 & 0.8 \\ 0.8 & 0.2 \end{pmatrix}$. The shape and scale parameters for the gamma distributed variables X_1 and X_2 are given in the header.

As the plot in the left-hand column shows, the support for the bivariate copula function is not the whole unit square. Instead we have a concave and a convex zero curve. In the bivariate distribution above, these curves occur as linear bounds. For illustrative issues we have added the red lines representing them. One can show that the slope of these lines is given by the minimum and the maximum of the ratios between the matrix entries in the several columns of A. In particular the following theorem holds.

Theorem 2. *Given a vector $\boldsymbol{X}$ of non-negative random variables X_i with arbitrary copula and a matrix $A \in \mathbb{R}^{K \times d}$ with entries $a_{j,i}$ for some $K, d \in \mathbb{N}_{>0}$, then for the vector $\boldsymbol{Y} = A\boldsymbol{X}$ it holds:*

$$\min_{i=1,\ldots,d} \left\{ \frac{a_{k,i}}{a_{\ell,i}} \right\} \leq \frac{Y_k}{Y_\ell} \leq \max_{i=1,\ldots,d} \left\{ \frac{a_{k,i}}{a_{\ell,i}} \right\} \qquad \mathbb{P} - a.s.,$$

if $\mathbb{P}(Y_\ell = 0) = 0$.

Proof. The proof is given in the appendix. □

Because of the nonlinear probability transformation, for the copula the linear bounds of the bivariate distribution transform into concave and convex curves. One can also observe that the number of realizations in the area around the zero curves grows with the variance of the corresponding X_i. In general, this copula belongs to none of the copula classes presented in section 2. Because of its shape and the bounded support it is neither elliptical nor generalized hyperbolic. The copula is also not an Archimedean one because those are unable to form concave zero curves (see [27] Theorem 4.3.2).

For some parameter setting, the scatter plots and especially the zero curves look very similar to those of a two parameter extreme value copula class named BC_2, introduced by [22]. Also the coefficients of lower and upper tail dependence (see section 3) of these two copula classes are similar. Since the density and distribution function of the copula given by (7) cannot be stated explicitly, we leave the question in which cases these two copula classes coincide as an open one for further research.

In the special case of the CBV model, equation (7) takes the form

$$\overline{\boldsymbol{S}} = I_{K\times K}\boldsymbol{S} + \Gamma\hat{\boldsymbol{S}}$$

where $\Gamma \in \mathbb{R}^{K\times L}$ with entries $y_{k,\ell}$. So the first columns of matrix A contain the K dimensional identity matrix. Therefore the lower and upper bounds of Theorem (2) disappear because they are given by 0 and ∞.

3.3.2 The Multi Compound Gamma model

Another approach to model correlated sectors within a CreditRisk$^+$ framework goes back to [13]. The basis of the (multi) compound gamma (MCG) is a mixture approach as in the CreditRisk$^+$model itself. In the standard model, the default distribution (Poisson) is mixed with a Gamma distribution or a linear combination of several independent ones. In the MCG model the gamma distribution of the sectors again is mixed with one or more gamma distributions. In more detail, again L background variables $\hat{S}_\ell \sim \Gamma\left(\hat{\sigma}_\ell^{-2}, \hat{\sigma}_\ell^2\right)$ are introduced. The shape parameter θ_k of the original sector variables is assumed to follow a linear combination of the background variables with weights $\alpha_{k,\ell} > 0$. The scale parameter is fixed. Summarizing this we have:

$$\lambda_i = p_i\left(w_{i,0} + \sum_{k=1}^{K} w_{i,k}S_k^\star\right)$$

with

$$S_k^\star \Big| \hat{\boldsymbol{S}} \sim \Gamma\left(\sum_{\ell=1}^{L}\alpha_{k,\ell}\hat{S}_\ell, \beta_k\right) \quad \text{and} \quad \hat{S}_\ell \sim \Gamma\left(\hat{\sigma}_\ell^{-2}, \hat{\sigma}_\ell^2\right).$$

Again to ensure that $\mathbb{E}\left(S_k^\star\right) = 1$ we have the additional parameter condition $\beta_k = \left(\sum_{\ell=1}^{L}\alpha_{k,\ell}\right)^{-1}$. The VCV structure of the MCG model is given by:

$$\operatorname{Var}\left(S_k^\star\right) = \beta_k + \beta_k^2\sum_{\ell=1}^{L}\alpha_{k,\ell}^2\hat{\sigma}_\ell^2 \tag{8}$$

$$\operatorname{Cov}\left(S_i^\star, S_j^\star\right) = \sum_{\ell=1}^{L}\beta_i\beta_j\alpha_{i,\ell}\alpha_{j,\ell}\hat{\sigma}_\ell^2. \tag{9}$$

The pgf of the portfolio loss L reads as

$$G_L(t) = \exp\left(P_0(t) - \sum_{\ell=1}^{L}\frac{1}{\hat{\sigma}_\ell^2}\log\left[1 + \hat{\sigma}_\ell^2\sum_{k=1}^{K}\log\left(1 - \delta_k P_k(t)\right)\right]\right)$$

with $P_k(t) := \sum_{i=1}^{M} w_{i,k}p_i\left(t^{\nu_i} - 1\right)$. Again, the pgf can be calculated with the help of a nested evaluation algorithm.

Having a set of CBV- or MCG-parameters, one can easily switch between the two models with the help of the following lemmas.

Lemma 3. *For $k = 1, \dots, K$ and $\ell = 1, \dots, L$ let β_k, $\hat{\sigma}_\ell$ and $\alpha_{k,\ell}$ denote the parameters of a MCG model. The identical VCV structure is generated by a CBV model with fixed parameter $\hat{\theta}_\ell \equiv \hat{\theta}$ where $0 < \hat{\theta} < \left(\sum_{\ell=1}^{L} \beta_k \alpha_{k,\ell} \hat{\sigma}_\ell\right)$ holds for all $k = 1, .., K$ and*

$$\begin{aligned}
\theta_k &= \left(1 - \sum_{\ell=1}^{L} \sqrt{\hat{\theta}_\ell} \beta_k \alpha_{k,\ell} \hat{\sigma}_\ell\right)^2 \beta_k^{-1}, \\
\delta_k &= \beta_k \left(\sum_{\ell=1}^{L} \sqrt{\hat{\theta}_\ell} \beta_k \alpha_{k,\ell} \hat{\sigma}_\ell\right)^{-1}, \\
y_{k,\ell} &= \beta_k \alpha_{k,\ell} \hat{\sigma}_\ell \hat{\theta}_\ell^{-1/2}.
\end{aligned}$$

Proof. A simple algebraical calculation shows that the equations (5) and (6) with the parameters stated in the lemma are equivalent with (8) and (9) which proves the lemma. □

Lemma 4. *Let δ_k, θ_k, $\hat{\theta}_\ell$, $y_{k,\ell}$ denote the parametrization of a CBV model. The identical VCV structure is asymptotically generated by an MCG setup with K sectors and $\hat{L} = L + 1$ background sectors and the following parameters*

$$\beta_k = \delta_k^2 \theta_k, \quad \hat{\sigma}_\ell \equiv \hat{\sigma} = \max_{k=1,\dots,K} \sum_{\ell=1}^{L} y_{k,\ell} \sqrt{\hat{\theta}_\ell}, \quad \hat{\sigma}_{L+1} = \epsilon,$$

$$\alpha_{k,\ell} = \frac{y_{k,\ell}\sqrt{\hat{\theta}_\ell}}{\beta_k \hat{\sigma}}, \quad \alpha_{k,L+1} = \frac{1 - \sum_{\ell=1}^{L} \alpha_{k,\ell} \beta_k}{\beta_k}.$$

It holds:

$$\lim_{\epsilon \to 0} \Sigma_{MCG} = \Sigma_{CBV}.$$

Proof. Again, simply plugging in the stated parametrization into (8) and (9) and taking the limit leads to the formulas (5) and (6). □

In both models, finding a suitable parametrization for an empirical VCV structure - especially for a higher number of sectors - is crucial. Hence, the last two lemmas are very helpful because now one has to find the parametrization for just one model. Furthermore, we do not have to consider any effects of unequal parametrization caused by different algorithms when we switch from one model to another.

4 Empirical analysis

4.1 Portfolio and data

The underlying portfolio consists of 5000 counterparties. For reasons of simplicity, we assume a constant loss given default (LGD) of one for each counterparty. Since we only want to concentrate on the relative changes of the economic capital for different copulas under consideration, this assumption will not restrict our results. Assuming counterparty specific constant LGDs (not necessarily 1) is equal to a scaling of the corresponding exposure, resulting in a shift / reduction of the value at risk. The same applies to the case of stochastic LGDs. For those who are interessted in the effect of stochastic LGDs, we refer to [14, section 7].

Each counterparty is mapped onto a single sector out of the ten sectors. To keep the analysis manageable, we do not distinguish between different regions or countries. The distribution of potential losses (PL) and counterparties (CP) across sectors is shown in table 1.

For parameter estimation a data pool with more than 30,000 corporates around the world is used. Based on a Merton model (see [25]), the data represent the one-year PD of exchange traded corporates between 2003 and 2013. The PDs are estimated monthly over the last ten years and then aggregated on sector level via the

Table 1. Economic sectors.

#	Sector	# CP	% of PL
1	Basic materials	16	1.7
2	Communication	5	2.5
3	Cyclical consumer goods	4631	19.5
4	Non-cyclical consumer goods	15	1.5
5	Diversified companies	28	3
6	Energy	10	4.3
7	Finance	146	45.9
8	Industry	75	11.1
9	Technology	19	1.8
10	Utilities	55	8.7

median. Finally, in order to take time dependencies into account, we have fitted a univariate auto-regressive process to every sector time series and use the residuals for the maximum likelihood estimation instead.

4.2 Copulas under consideration

In order to estimate the unknown parameters for the copulas from section 2, we have used the maximum likelihood approach. For this purpose, we used the R-packages "copula" by [17], "ghyp" from [21] and the "HAC" package from [30]. The implicit copula of the CBV model was estimated with the help of an optimization algorithm, minimizing the L_2–distance between the empirical and the theoretical VCV matrix. For the parameters of the MCG model we used lemma 4. A short overview of the estimation results is given over the next few pages.

If we exchange the independence copula of the basic CreditRisk$^+$ model with some arbitrary one, in general the model can no longer be solved analytically.[6] Therefore, we simulate the sector variables $S_1, .., S_k$ with a specified copula C, calculate the individual default intensity λ_i via (2) and restrict the analytical calculation to the first product in (3). Thus we just have to calculate the distribution of the portfolio loss caused by M independent obligors. Simulating the vector $\boldsymbol{S} = (S_1, .., S_k)^T$ of sector variables N times and averaging each single exposure band over all N pdfs gives us a final estimation for the overall pdf. This idea has already been mentioned in [6].

4.2.1 Tail dependence

During the following analysis we will also focus on the index of tail dependence. In contrast to dependency measures such as the linear correlation coefficient, Kendall's τ or Spearman's ρ, which measure dependency on an overall level, the index of tail dependence measures dependency only in extreme situations. Following [27], the coefficients of upper and lower tail dependence are defined for a pair (X_1, X_2) of random variables by

$$\lambda_U := \lim_{u \nearrow 1} \mathbb{P}\left[X_2 \geq F_2^{-1}(u) \,|\, X_1 \geq F_1^{-1}(u)\right] \quad \text{and}$$
$$\lambda_L := \lim_{u \searrow 0} \mathbb{P}\left[X_2 \leq F_2^{-1}(u) \,|\, X_1 \leq F_1^{-1}(u)\right],$$

where F_i and F_i^{-1} denotes the marginal distributions and the quantile functions, respectively.

In the context of the financial crisis, the Gaussian copula was blamed for failing to model economic dependencies correctly because it admits no tail dependencies. This means that, especially during the crisis, counterparties or economic sectors tend to behave independently from each other rather than dependently.

6 As [10] shows, an analytical solution is also possible in the special case of a multivariate tempered α-stable distribution.

Therefore, the amount of economic capital required to cover the loss in such situations, estimated with the help of the Gaussian copula, may be not sufficient. Interestingly, this disadvantage of the Gaussian copula has already been pointed out by several mathematicians years before the financial crisis started (see [34]).

4.2.2 Gaussian vs. Student-t copula

For reasons of clarity, we focus our discussion of the estimation results on just three of the ten sectors, namely the industrial sector, the financial sector and the sector of cyclical consumer goods, which cover around 75% of the total exposure.

Table 2 shows the estimated entries of the dispersion matrix as well as the index of tail dependence. Please note that, in contrast to the Student-t copula, the Gaussian copula admits no upper nor lower tail dependence. In the case of the Student-t copula, the coefficients of upper and lower tail dependence are equal because of the elliptical symmetry. For the t-copula we estimated 3.786 degrees of freedom. The corresponding copula functions together with the empirical observations are illustrated in figure 3.

Table 2. Estimated coefficient of tail dependence ($\hat{\lambda}$) and entries of the dispersion matrix ($\hat{\sigma}_{i,j}$) for Gaussian (upper triangle matrix) and t-copula (lower triangle matrix)

industry	$\hat{\sigma}_{7,8} = 0.78$ $\hat{\lambda} = 0$	$\hat{\sigma}_{3,8} = 0.94$ $\hat{\lambda} = 0$
$\hat{\sigma}_{7,8} = 0.78$ $\hat{\lambda} = 0.48$	**finance**	$\hat{\sigma}_{3,7} = 0.77$ $\hat{\lambda} = 0$
$\hat{\sigma}_{3,8} = 0.93$ $\hat{\lambda} = 0.69$	$\hat{\sigma}_{3,7} = 0.78$ $\hat{\lambda} = 0.47$	**cyclical consumer goods**

The strongest dependency occurs between the industrial sector and the sector of cyclical consumer goods. Here, we observe the highest σ as well as the highest coefficient of tail dependence for the t-copula. Furthermore, the copula functions between these two sectors concentrate more mass on the main diagonal compared to the others.

Table 3 summarizes the log-likelihood values of the Gaussian- and t copula as well as those from the estimated ghyp copulas, presented in the next section. The log-likelihood of the Gaussian copula is around 634 whereas the t-copula has a value of approximately 728. Based on a likelihood ratio test, we can conclude that the t-copula fits the data significantly better than the Gaussian one.

4.2.3 Generalized hyperbolic copulas

The class of generalized hyperbolic copulas contains the class of elliptical copulas if no asymmetries are taken into account. Therefore, as in the case of the t- and the Gaussian copula, we would expect a considerably higher log-likelihood than in the former cases. For the asymmetric ghyp copula and the underlying data set we obtain a log-likelihood value of 13566. Again, a likelihood ratio test indicates that the ghyp copula fits the data significantly better than the t-copula and, of course, better than the Gaussian copula. Besides this, we also considered a symmetric ghyp copula, where we restrict $\boldsymbol{y} = (0, \dots, 0)^T \in \mathbb{R}^d$. In this case, the log-likelihood is still very much higher compared to the case of the Gaussian and the t-copula, i.e. the log-likelihood equals 8848. Performing likelihood ratio tests leads us to two results. On the one hand, the symmetric ghyp copula fits the data significantly better than the Gaussian and the t-copula. On the other hand, the asymmetric ghyp copula also has a better fit compared to the symmetric one. Hence, we can state that the asymmetry in our empirical data is significant and therefore should not be neglected.

In the asymmetric case, all components of $\hat{\boldsymbol{y}}$ are strictly positive. Therefore, the multivariate distribution and the copula are skewed towards higher values. Since higher values correspond to higher default rates,

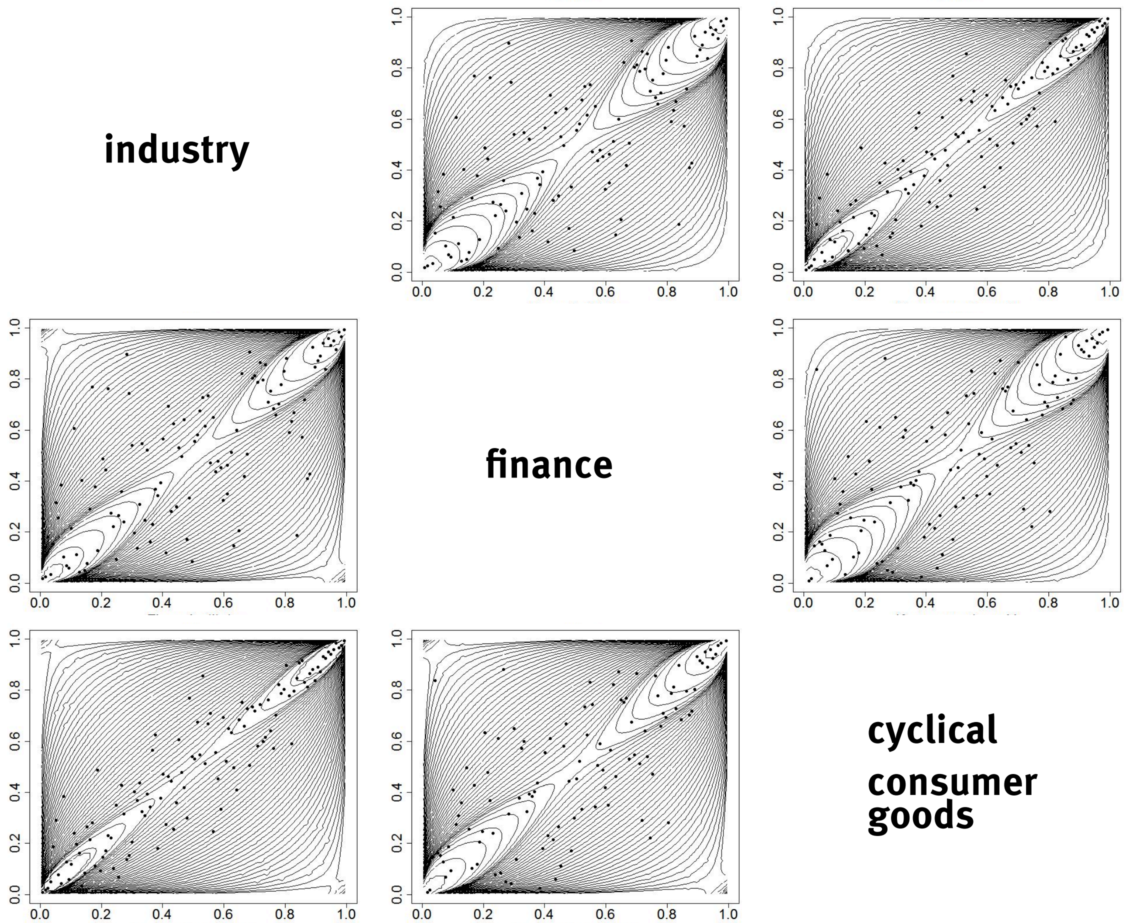


Fig. 3. Fitted Gaussian and t-copula together with empirical observations.

this supports the economic intuition of a higher dependency between corporates when economic conditions become worse. However, as figure 4 shows, the optical differences to the elliptical copulas (lower triangle matrix) are not as high as expected. Nonetheless, in all three bivariate copulas of the upper triangle matrix of figure 4 there is slightly more probability mass concentrated in the upper right corner than in the lower left one.

4.2.4 (Hierarchical) Archimedean copulas

Another flexible class of copulas is the Archimedean one. Since the ghyp copula has already indicated that a positive skewed distribution is needed to fit the data, we only consider the Gumbel copula for our analysis. In fact, after fitting other Archimedean copulas (Clayton and Frank) and the adoption of a goodness of fit test as discussed by [12], we can clearly reject them. Besides the ordinary Gumbel copula, we also take a hierarchical

Table 3. Rounded log-likelihood values for elliptical and ghyp copulas.

copula	log-likelihood
Gaussian	634
t	728
symmetric ghyp	8848
asymmetric ghyp	13566

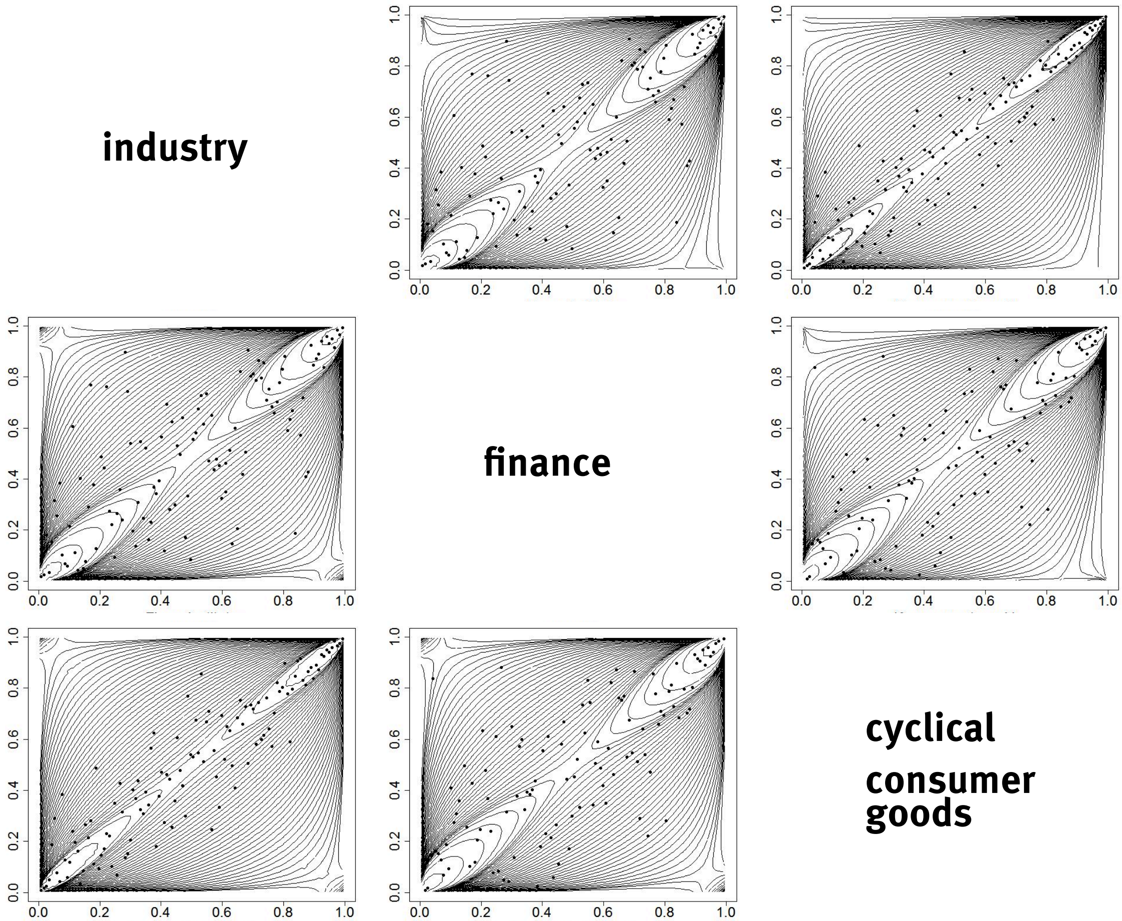
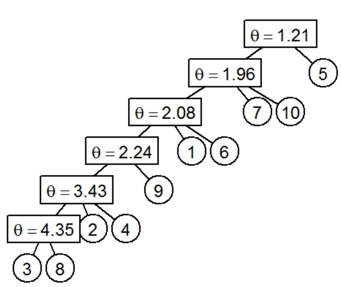


Fig. 4. Symmetric (lower triangle matrix) and asymmetric ghyp copula (upper triangle matrix) together with empirical observations.

construction into account. To estimate the parameters and the nesting structure we used the package "HAC" by [30]. The package uses a stepwise maximum likelihood approach[7]. Since the estimation of a HAC is not in the scope of this paper, please refer to the mentioned article for further details on the estimation process. The estimated hierarchical structure is illustrated in figure 5.



1	basic materials
2	communication
3	cyclical consumer goods
4	non-cyclical consumer goods
5	diversified companies
6	energy
7	finance
8	industry
9	technology
10	utilities

Fig. 5. Hierarchical Archimedean copula estimated from default data.

7 The function *estimate.copula()* with *method=ML* was used.

The Gumbel copula is positively ordered, which means that a higher parameter value implies a higher dependency between variables. Therefore, we have the strongest dependency on the lowest level between cyclical consumer goods and the industrial sector. Indicated by the lowest parameter value, diversified companies have the weakest dependency to all other sectors. The parameter range reaches from $\hat{\theta}$=1.21 to $\hat{\theta}$ = 4.35, which corresponds to values of Kendall's τ between 0.17 and 0.77. Calculating the empirical values of Kendall's τ by means of the default data gives us a similar interval.

If we consider just an ordinary Gumbel copula instead of a hierarchical one, we obtain $\hat{\theta}$ = 1.836. Since all of the bivariate copulas of a multivariate AC are identical,[8] choosing the copula parameter is always a tradeoff between stronger and weaker dependencies. The "average" value for $\hat{\theta}$ corresponds to τ = 0.46.

Again, figure 6 shows the bivariate copulas for sectors 3, 7 and 8 of the ordinary multivariate Archimedean (upper triangle matrix) and the hierarchical Archimedean copula (lower triangle matrix). In the ordinary case, all the bivariate copulas coincide. In case of the industrial and the financial sector or the financial sector and the one of cyclical consumer goods, the ordinary and the HAC are quite close $\left(\hat{\theta}_{\text{HAC}} = 1.96 \text{ vs. } \hat{\theta}_{\text{ord.}} = 1.836\right)$. But in the upper right and lower left case (industry vs. cycl. consumer goods) the choice of a common θ is a major disadvantage, because the dependency measured by the HAC (τ = 0.77) and the observations $\left(\hat{\tau} = 0.76\right)$ are higher than the ordinary Gumbel copula suggests (τ = 0.46).

Because of the specific generator function, the copula function is not as symmetric as the Gaussian or the t-copula. Comparing the Gumbel copula to a symmetric Archimedean one (Frank) and one with inverted asymmetry (Clayton) by means of goodness of fit tests shows that the Gumbel copula is the most suitable one for our data. The coefficients for implied upper tail dependency are summarized in table 4. Comparing these values to those of table 2 we can state that the HAC in all three cases generates a higher upper tail dependence than the t-copula does. The coefficient of lower tail dependency of the ordinary and the hierarchical copula is zero, by definition.

Table 4. Coefficients of upper tail dependency of ordinary Gumbel copula (upper triangle matrix) and HAC (lower triangle matrix).

industry	$\hat{\lambda}_U$ **= 0.54**	$\hat{\lambda}_U$ **= 0.54**
$\hat{\lambda}_U$ **= 0.58**	**finance**	$\hat{\lambda}_U$ **= 0.54**
$\hat{\lambda}_U$ **= 0.83**	$\hat{\lambda}_U$ **= 0.58**	**cycl. consumer goods**

4.2.5 The CBV and MCG copula

In a final step we also analyze the implicit copulas of the CBV and the MCG model for the given data. Since we do not have an analytical representation of the copula densities, we have estimated them based on 10^6 simulated values for each pair of variables. For the CBV model we chose five background variables. For the calibration of the MCG model, we used lemma 4 with $\epsilon = 10^{-6}$. In the upper triangle matrix of figure 7 we plotted the bivariate copulas of the CBV model and in the lower triangle matrix those of the MCG model.

Please note that, in contrast to the former copulas, the parametrization of the CBV and the MCG model was executed on the basis of the empirical VCV structure rather than the likelihood of the observations. In general, the copulas of the two models are fairly similar. The copulas of the MCG model show a slightly higher concentration of probability mass around the ridge of the density function (black curve) whereas the implicit copula of the CBV model is wider. What differentiates both copulas from their competitors (e.g. Gaussian or t-copula) is the large amount of asymmetry. As in the case of the Gumbel copula, mutual higher realizations are more likely than mutual lower ones. However, the magnitude of this skewness is much greater. In addition,

8 For every Archimedean copula it holds: $C_\psi(u_i, u_j, 1, \ldots, 1) = \psi\left(\psi^{-1}(u_i) + \psi^{-1}(u_j) + 0 + \ldots + 0\right) = C_\psi(u_i, u_j)$

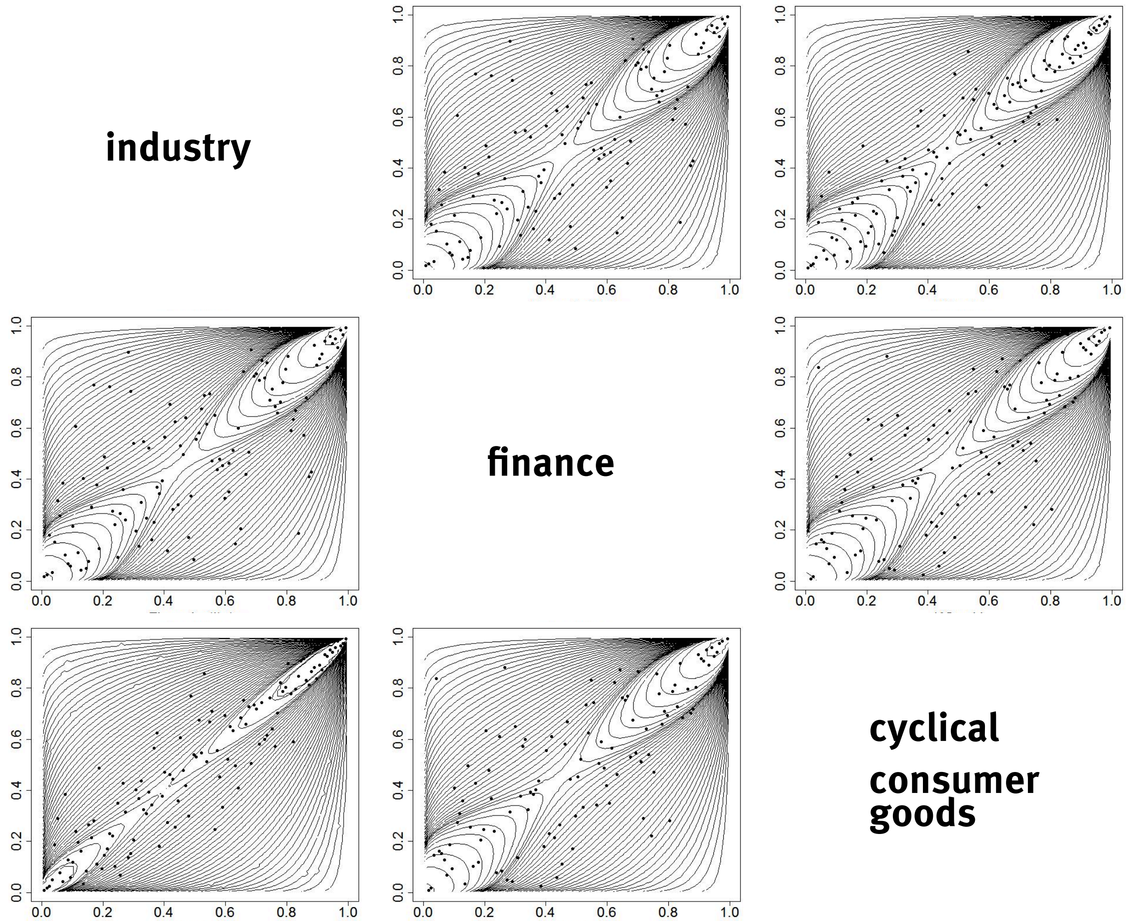


Fig. 6. Estimated bivariate (hierarchical) Archimedean Gumbel copulas (upper triangle matrix: ordinary Gumbel copula, lower triangle matrix: hierarchical Archimedean Gumbel copula).

the copulas of both models are not symmetric with respect to the main diagonal. Since the estimation was done on the basis of the VCV structure, which in general does not cause asymmetry, this is quite an interesting observation.

Because of the significant asymmetries, we decided to numerically estimate the coefficients of lower and upper tail dependence with the help of methods discussed by [5] and [9]. The results, based on 10^7 simulations, are stated in table 5. We used three different estimators for upper and lower tail dependence. However, the results are fairly equal. So we state only one value for lower and one for upper tail dependency.

Irrespective of the model (CBV or MCG) the estimated upper tail dependence in all cases is considerably higher than the lower tail dependence[9]. Whereas for the MCG model λ_L is clearly positive, the copula of the CBV model seems to be lower tail independent. Besides the significant difference between λ_L and λ_U, the level of the upper tail dependence of all pictured copulas is also remarkably high.

9 Please note, that Proposition 3 in [28] cannot be applied to the the factor copula of the CBV model because the gamma distribution has no positive tail index.

Table 5. Estimator for coefficients of upper and lower tail dependence for CBV (upper triangle matrix) and MCG model (lower triangle matrix).

industry	$\hat{\lambda}_U = 0.91$ $\hat{\lambda}_L = 0.01$	$\hat{\lambda}_U = 0.96$ $\hat{\lambda}_L = 0.05$
$\hat{\lambda}_U = 0.86$ $\hat{\lambda}_L = 0.11$	finance	$\hat{\lambda}_U = 0.93$ $\hat{\lambda}_L = 0.02$
$\hat{\lambda}_U = 0.93$ $\hat{\lambda}_L = 0.38$	$\hat{\lambda}_U = 0.9$ $\hat{\lambda}_L = 0.13$	cycl. consumer goods

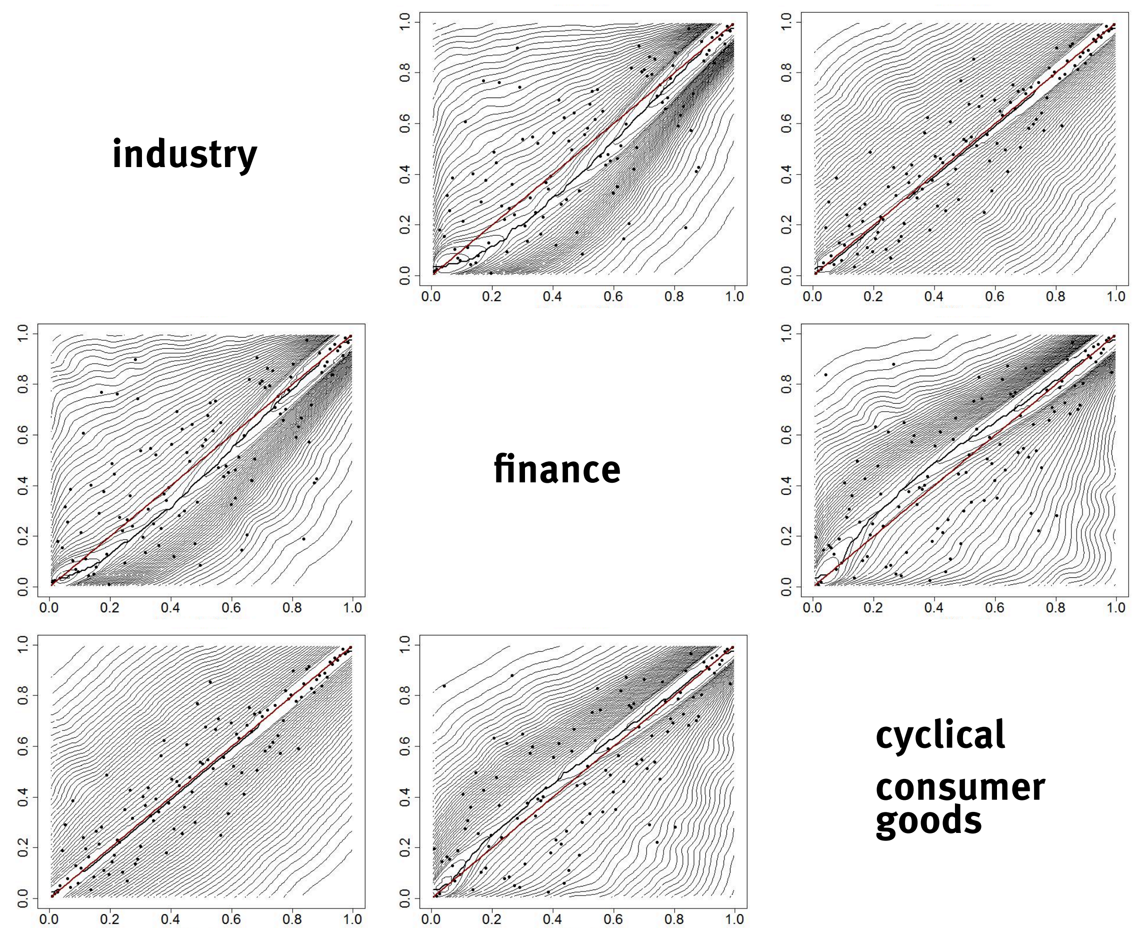


Fig. 7. Estimated bivariate copulas of the CBV (upper triangle matrix) and the MCG model (lower triangle matrix) together with empirical observations.

4.3 Impact on risk figures

After analyzing the estimation results, now we discuss the impact of the several dependency structures and, in a first step, also of the marginal sector distributions on the risk figures within a CreditRisk$^+$framework. As already mentioned in section 3, banks use internal models and the concept of economic capital to allocate equity capital necessary to cover unexpected losses. Against this background, the variations in risk figures can be interpreted as the model risk arising from the choice of a specific dependency structure.

Besides the economic capital, we also state the values for expected shortfall. The expected shortfall on level α, denoted by ES_α, is defined as the conditional expectation value of the portfolio loss, once the corresponding value at risk VaR_α is exceeded. It is an alternative risk measure to VaR fulfilling the property of subadditivity, i.e. for two portfolios A and B it holds: $\mathrm{ES}_\alpha(A) + \mathrm{ES}_\alpha(B) \geq \mathrm{ES}_\alpha(A + B)$, see [1].

4.3.1 Marginal sector distributions

Originally, the distribution of each single sector of the CreditRisk+ model was assumed to be gamma. In order to handle correlated sectors, the distribution changed by construction to a compound gamma distribution (MCG) or a linear combination of independent gamma distributions (CBV), which in general is not gamma any more. Hence, before analyzing the impact of the copula, we have to quantify the effect caused by a change from the model specific margins to gamma distributed ones.

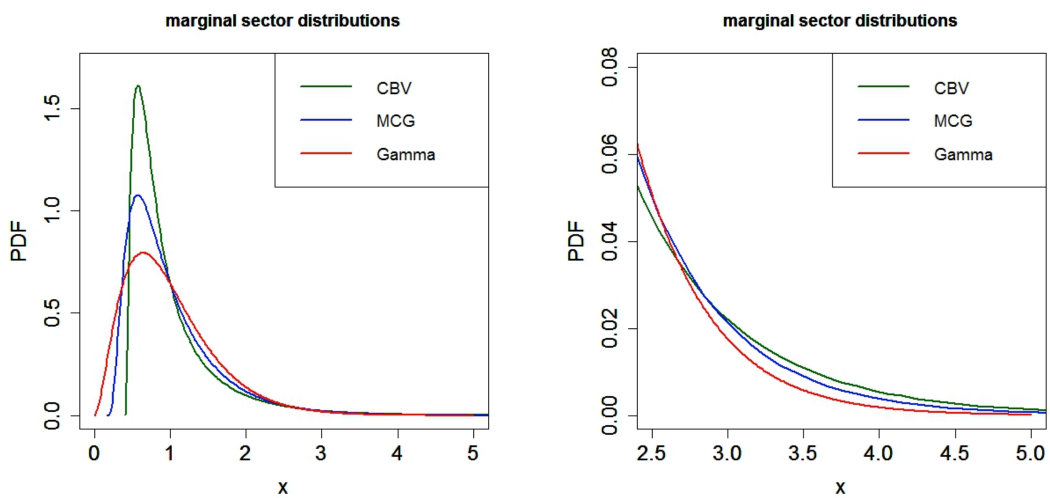


Fig. 8. Densities of sector variable 3 in the CBV (green), the MCG (blue), and the original (red) CreditRisk+ model. For other sectors plots are very similar.

Exemplarily, figure 8 shows the pdf of sector variable 3 in the CBV (green), and the MCG model (blue) as well as the pdf of an ordinary gamma distributed random variable (red) with equal mean and variance. The marginal sector distributions of the CBV and the MCG model are more heavily tailed than an ordinary gamma distribution. Hence, we could expect a remarkable decrease in risk figures when we switch the marginal sector distribution back to an ordinary gamma distribution. The effect will be stronger for the CBV than for the MCG model.

Table 6. Impact of marginal sector distribution on risk figures.

Copula	Margins	EL	SD	EC_{90}	EC_{99}	$EC_{99,9}$	ES_{90}	ES_{99}
CBV	comb. Γ	4344	3402	4256	12496	21686	12148	20777
CBV	Γ	4346	3400	4544	11364	18224	11857	18366
ratio		1	1	0.937	1.100	1.190	1.024	1.131
MCG	comp. Γ	4336	3404	4434	11884	19844	11985	19301
MCG	Γ	4356	3413	4564	11374	18204	11881	18377
ratio		0.995	0.997	0.972	1.045	1.090	1.009	1.05

Table 6 summarizes different risk figures,[10] describing the portfolio loss distribution of the CBV and the MCG model with different marginal sector distributions. In both models, the loss distributions of the CBV/MCG models are more heavily tailed compared to the models with gamma distributed sectors. On a 90% loss level

10 EC_α and ES_α denote the economic capital and the expected shortfall (or conditional value at risk) respectively.

alone, the values for economic capital are lower in the case of the combined Γ distribution than in the case of an ordinary Γ distribution. The markups for higher quantiles are in the range of 4.5% to 19% depending on the model and quantile level. Naturally, the impact is stronger for higher quantiles. The original CBV model also accounts for higher risk numbers compared to the original MCG model. This is consistent with former studies (e.g. [8]) and the observation of more heavily tailed sector distributions from figure 8.

4.3.2 Copulas

Finally we analyze how the loss distribution and hence the risk figures change under copula assumptions. The results are summarized in table 7. In order to facilitate comparison, we state the values in percent of the corresponding CBV value. The right tail area of the loss distributions is illustrated in figure 9.

At first we note that all alternative copulas (independence, Gaussian, t-, ghyp, AC, and HAC) imply a lower risk than the copulas of the CBV and the MCG model. The pdfs of the portfolio loss of the CBV and the MCG model are fairly equal, as figure 9 shows. The risk figures of the model with independent sectors show that 25% of the portfolio loss standard deviation and nearly one third of the required economic capital (99.9% level) are due to sector dependencies. The risk reduction effect grows with the considered risk level.

Since the Gaussian copula is elliptically symmetric and admits no tail dependence, it produces the second lowest risk. In the tail area above the 99% loss quantile, the pdf is strictly dominated by all others. On average the values for economic capital are 5% to 9% lower compared to the CBV model with gamma distributed margins. Using a symmetric ghyp copula instead of a Gaussian one, the results are similar. Switching to a t-copula, the positive tail dependence causes an increase in risk of approximately 2% on the highest level. The asymmetric ghyp, Gumbel and hierarchical Archimedean copula imply the highest risk among the alternative copulas under consideration. In contrast to the Gaussian and t-copula, they generate an asymmetric dependency structure. Since the realization of multiple high default rates, in those cases, is more likely than the realization of multiple lower ones, high losses are also more likely. The resulting loss distribution is more heavily tailed than those of the elliptical models. Since the pdfs of the asymmetric ghyp, Gumbel and HAC model are very close, we have plotted just the density of the model with an asymmetric ghyp copula. Although having no tail dependency, the risk arising from this copula is fairly equal to that arising from the Archimedean copulas, which admits a positive upper tail dependence. Therefore, we can conclude that, the risk arising from an asymmetric dependency structure is higher than the risk implied by a copula with positive tail dependence for this data set.

The highest risk is observed for the copulas underlying the MCG and the CBV model. This is reasonable because they have the highest upper tail dependence and, in the case of the CBV copula, no lower tail dependence. Furthermore, the probability mass of these copulas is more concentrated around the main diagonal as figure 7 shows.

Table 7. Risk figures for different copulas.

copula	margins	EL	SD	EC_{90}	EC_{99}	$EC_{99,9}$	ES_{90}	$ES_{99,9}$
CBV	Γ	1.000	1.000	1.000	1.000	1.000	1.000	1.000
MCG	Γ	1.002	1.004	1.004	1.001	0.999	100.2	0.998
independence	Γ	1,001	0,750	0,739	0,704	0,688	0,820	0,763
Gaussian	Γ	1.000	0.947	0.947	0.925	0.918	0.958	0.939
sym. ghyp	Γ	0.999	0.943	0.937	0.927	0.933	0.955	0.953
t	Γ	1.000	0.949	0.943	0.934	0.941	0.960	0.960
asym. ghyp	Γ	1.000	0.953	0.942	0.949	0.960	0.966	0.974
Gumbel	Γ	0.999	0.939	0.924	0.944	0.962	0.959	0.975
HAC	Γ	1.000	0.945	0.934	0.948	0.957	0.963	0.967

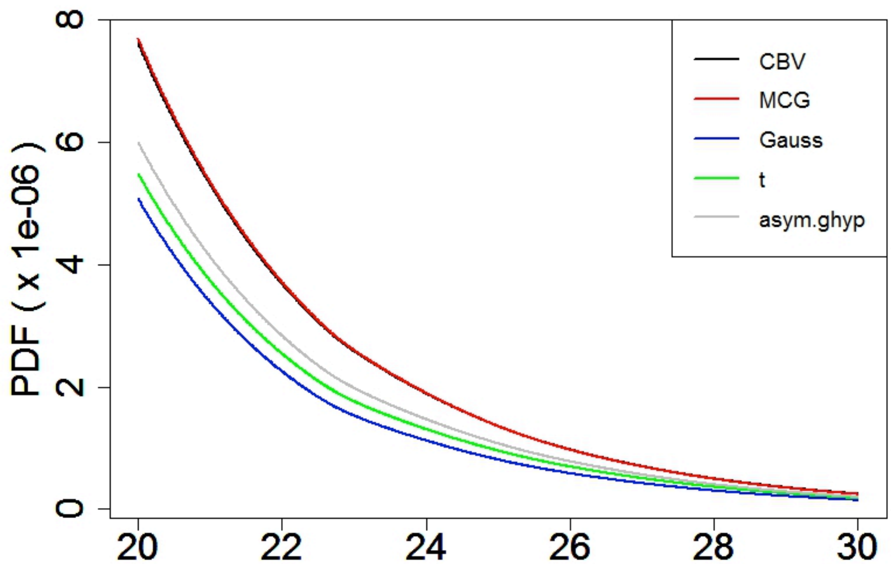


Fig. 9. Pdf (right tail) of portfolio loss distribution for several copulas.

5 Conclusions

In a credit portfolio model such as CreditRisk+, the choice of a particular copula noticeably affects the risk figures. Of course, using a Gaussian copula implies the lowest risk. Switching to a skewed copula with or without positive tail dependence increases the risk. Here the impact of an asymmetric dependency structure is stronger than the impact of a positive tail dependence. However, the implied copulas of the CBV and the MCG model account for the highest risk. Therefore, from a copula point of view, the MCG and the CBV model are rather conservative within the framework of CreditRisk+. With the help of likelihood ratio tests, we showed that for our data, the magnitude of asymmetry is significant. Different estimators suggest that these copulas admit a high upper tail dependence, while the lower one is considerably smaller. The difference between the copulas of the CBV and the MCG model is negligible.

Aside from the copula, the marginal distributions of the sector variables also considerably affect the risk figures. For the CreditRisk+ model, this effect is even stronger than the effect of the copula. Hence, the difference between the CBV and the MCG model mainly arises from the marginal sector distribution.

Since modeling the dependency directly on counterparty level is not manageable, an interesting question is what the resulting copula on counterparty level would be after the translation of the sector copula via the link function. But this is a more general topic which we leave open for further research.

A Proof of Theorem 2

Theorem. *Let $\boldsymbol{X}$ be a vector of real random variables X_i with arbitrary copula and $\mathbb{P}(X_i \geq 0) = 1$ for all $i = 1, \dots, d$. Furthermore let $a_{j,i}$ denote the elements of a matrix $A \in \mathbb{R}^{K \times d}$ for some $K,\ d \in \mathbb{N}_{>0}$. Then the ratio of any two components of the vector $\boldsymbol{Y} = A\boldsymbol{X}$ is bounded below and above. If $\mathbb{P}(Y_\ell = 0) = 0$ it holds:*

$$\min_{i=1,\dots,d} \left\{ \frac{a_{k,i}}{a_{\ell,i}} \right\} \leq \frac{Y_k}{Y_\ell} \leq \max_{i=1,\dots,d} \left\{ \frac{a_{k,i}}{a_{\ell,i}} \right\} \qquad \mathbb{P} - a.s.$$

Proof. Without loss of generality we set $k = 1$ and $\ell = 2$. At first we concentrate on the bivariate case of $\boldsymbol{X}$, so $d = 2$. For two realizations (x_1, x_2) of (X_1, X_2) set $q = \frac{a_{1,1}x_1+a_{1,2}x_2}{a_{2,1}x_1+a_{2,2}x_2}$. Taking derivatives yields:

$$\frac{\partial q}{\partial x_1} = \frac{x_2\left(a_{1,1}a_{2,2} - a_{2,1}a_{1,2}\right)}{\left(a_{2,1}x_1 + a_{2,2}x_2\right)^2} \quad \text{and} \quad \frac{\partial q}{\partial x_2} = -\frac{x_1\left(a_{1,1}a_{2,2} - a_{2,1}a_{1,2}\right)}{\left(a_{2,1}x_1 + a_{2,2}x_2\right)^2}.$$

Now we can distinguish between three cases according to the sign of $(a_{1,1}a_{2,2} - a_{2,1}a_{1,2})$.

– $(a_{1,1}a_{2,2} - a_{2,1}a_{1,2}) > 0$:
  q is increasing with x_1 for all $x_2 > 0$ and decreasing with x_2 for all $x_1 > 0$. Therefore we get:
  $$\frac{a_{1,2}}{a_{2,2}} = q(0, x_2) < q(x_1, x_2) < q(x_1, 0) = \frac{a_{1,1}}{a_{2,1}}.$$
– $(a_{1,1}a_{2,2} - a_{2,1}a_{1,2}) < 0$:
  We have the opposite monotonicity of q compared to case 1. Therefore we get:
  $$\frac{a_{1,1}}{a_{2,1}} = q(x_1, 0) < q(x_1, x_2) < q(0, x_2) = \frac{a_{1,2}}{a_{2,2}}$$
– $(a_{1,1}a_{2,2} - a_{2,1}a_{1,2}) = 0$:
  q is constant with $q(x_1, x_2) = \frac{a_{1,1}}{a_{2,1}} = \frac{a_{1,2}}{a_{2,2}}$.

Analogously, for arbitrary $d \in \mathbb{N}_{>0}$ we have:

$$\frac{\partial q(\boldsymbol{x})}{\partial x_i} = \frac{\sum_{k\neq i} x_k\left(a_{1,i}a_{2,k} - a_{2,i}a_{1,k}\right)}{\left(\sum_{k=1}^{d} a_{2,k}x_k\right)^2}.$$

So for every x_i the ratio $q(\boldsymbol{x})$ is strictly monotone. The direction depends on the values of x_k, $k \neq i$. The set of the possible bounds is given by

$$\{q(0, \dots, 0, x_i, 0, \dots, 0)\}_{i=1,..,d} = \left\{\frac{a_{1,i}}{a_{2,i}}\right\}_{i=1,..,d}.$$

The min and max of this set are the lower and upper bound stated in the theorem.

□

B Generalized hyperbolic distributions

The family of generalized hyperbolic (ghyp) distributions was originally introduced by [3]. A real valued d-dimensional random variable $\boldsymbol{X}$ follows a ghyp distribution iff it admits the following stochastic representation

$$\boldsymbol{X} \overset{d}{=} \boldsymbol{\mu} + W\boldsymbol{y} + \sqrt{W}A\boldsymbol{Y}$$

with $\boldsymbol{\mu}, \boldsymbol{y} \in \mathbb{R}^d$, $A \in \mathbb{R}^{d\times d}$, $\boldsymbol{Y} \sim \mathcal{N}(\boldsymbol{0}, I_d)$ and $W \sim \text{GIG}(\lambda, \chi, \psi)$, where $\mathcal{N}$ denotes the multivariate normal distribution and GIG the generalized inverse Gaussian distribution with parameters:

$$\begin{aligned} &\lambda \in \mathbb{R}, \quad \chi > 0, \quad \psi > 0 \\ \text{or } &\lambda > 0, \quad \chi = 0, \quad \psi > 0 \;\; (\Gamma \text{ distribution}) \\ \text{or } &\lambda < 0, \quad \chi > 0, \quad \psi = 0 \;\; (\text{inv. } \Gamma - \text{distribution}). \end{aligned}$$

The ghyp family contains a lot of special cases e.g. normal, (skewed) t, variance gamma or the normalized inverse Gaussian distribution. For more information on this topic as well as the GIG distribution we refer to [31]. The ghyp family also possesses several different representations. For other parametrizations and ways of switching between them, we refer to [21]. These authors also created the R-package “ghyp”.

Finally, the ghyp copula is given via the multivariate distribution F and the quantile functions of the margins F_i, i.e.

$$C(\boldsymbol{u}) = F\left(F_1^{-1}(u_1), \dots, F_d^{-1}(u_d)\right).$$

References

[1] Artzner, P., Delbaen, F., Eber, J.-M., and Heath, D. (1999). Coherent measures of risk. *Math. Finance*, 9:203-228.
[2] BaFin. (2012). Erläuterung zu den MaRisk in der Fassung vom 14.12.2012, Dec 2012.
[3] Barndorff-Nielsen, O. E. (1977). Exponentially decreasing distributions for the logarithm of particle size. *Proc. R. Soc. A*, 353:401-419.
[4] Board of Governors of the Federal Reserve System. (2011). Supervisory guidance on model risk management. Letter 11-7. http://www.federalreserve.gov
[5] Dobric, J. and Schmid, F. (2005). Nonparametric estimation of the lower tail dependence in bivariate copulas. *J. Appl. Stat.*, 32:387-407.
[6] Ebmeyer, D., Klaas, R., and Quell, P. (2006). The role of copulas in the CreditRisk+ framework. In *Copulas*. Risk Books London.
[7] Fang, K.-T., Kotz, S., and Wang, K. (1990). *Symmetric Multivariate and Related Distributions*. Chapman & Hall/CRC London.
[8] Fischer, M. and Dietz, C. (2011/12). Modeling sector correlations with CreditRisk+: The common background vector model. *The Journal of Credit Risk*, **7**:23-43.
[9] Fischer, M. and Dörflinger, M. (2010). A note on a non-parametric tail dependence estimator. *Far East J. Theor. Stat.*, 32:1-5.
[10] Fischer, M. and Mertel, A. (2012). Quantifying model risk within a CreditRisk+ framework. *The Journal of Risk Model Validation*, 6:47-76.
[11] Frey, R., McNeil, A.J., and Nyfeler, M.A. (2001). Copulas and credit models. *RISK*, October: 111-114.
[12] Genest, C., Remillard, B., and Beaudoin, D. (2009). Goodness-of-t tests for copulas: A review and a power study. *Insurance Math. Econom.*, 44:199-213.
[13] Giese, G. (2003). Enhancing CreditRisk+. *RISK*, 16:73-77.
[14] Gundlach, M. and Lehrbass, F. (2003). *CreditRisk+ in the Banking Industry*. Springer- Verlag Berlin Heidelberg.
[15] Han, C. and Kang, J. (2008). An extended CreditRisk+ framework for portfolio credit risk management. *The Journal of Credit Risk*, 4:63-80.
[16] Hering, C., Hofert, M., Mai, J., and Scherer, M. (2010). Constructing hierarchical Archimedean copulas with Lévy subordinators. *J. Multivariate Anal.*, 101(6):1428-1433.
[17] Hofert, M., Kojadinovic, I., Mächler, M., and Yan, J. (2012). *copula: Multivariate Dependence with Copulas*, R package version 0.999-5 edition. URL: http://CRAN. R-project.org/package=copula.
[18] Jaworski, P., Durante, F., Härdle, W., and Rychlik, T. (2010). *Copula Theory and Its Applications.* Springer-Verlag Berlin Heidelberg.
[19] Joe, H. (1997). *Multivariate Models and Dependence Concepts.* Chapman & Hall/CRC London.
[20] Li, D.X. (2000). On default correlation: A copula function approach. *Journal of Fixed Income*, 9:43-54.
[21] Luethi, D. and Breymann, W. (2011). *ghyp: A package on the generalized hyperbolic distribution and its special cases*. URL: http://CRAN.R-project.org/package= ghyp.
[22] Mai, J.F. and Scherer, M. (2009). Bivariate extreme-value copula with discrete pickands dependence measure. *Extremes*, 14:311-324.
[23] McNeil, A.J. (2008). Sampling nested Archimedean copulas. *J. Stat. Comput. Simul.*, 78:567-581.
[24] McNeil, A.J., Frey, R., and Embrechts, P. (2005). *Quantitative Risk Management*. Princeton University Press.
[25] Merton, R.C. (1973). On the pricing of corporate debt: The risk structure of interest rates. *Journal of Finance*, 29:449-470.
[26] Moschopoulos, P.G. (1985). The distribution of the sum of independendent gamma random variables. *Ann. Inst. Statist. Math.*, 37:541-544.
[27] Nelsen, R.B. (2006). *An Introduction to Copulas*. Springer New York.
[28] Oh, D.H. and Patton, A.J. (2012). Modelling dependence in high dimension with factor copulas. *Manuscript, Duke University*. URL: http://public.econ.duke.edu/~ap172/Oh_Patton_MV_factor_copula_6dec12.pdf
[29] Okhrin, O. and Ristig, A. (2012). *Hierarchical Archimedean Copulae: The HAC Package*. Humbold Universität Berlin. URL: http://cran.r-project.org/web/ packages/HAC/index.html.
[30] Okhrin, O., Okhrin, Y., and Schmid, W. (2013). Properties of hierarchical Archimedean copulas. *Statistics & Risk Modeling*, 30:21-54.
[31] Paolella, M.S. (2007). *Intermediate Probability: A Computational Approach*. John Wiley & Sons Chichester.
[32] Savu, C. and Trede, M. (2010). Hierarchical Archimedean Copulas. *Quant. Finance*, 10:295-304.
[33] Sklar, A. (1959). Fonctions de répartition á n dimensions et leurs marges. *Publ. Inst. Stat. Univ. Paris*, 8:229-231.
[34] Szpiro, G. (2009). Eine falsch angewendete Formel und ihre Folgen. *Neue Züricher Zeitung*, 18 März.
[35] Wilde, T. (1997). *CreditRisk+ A Credit Risk Management Framework*. Working Paper, Credit Suisse First Boston.

Received November 4, 2013; accepted January 23, 2014.