Universität
Augsburg
University

DOCTORAL THESIS

Exceptional Lefschetz collections on orthogonal Grassmannians $OGr(3, 2n+1)$

Author:
Max BRIEST, M. Sc.

Supervisor:
Prof. Dr. Maxim SMIRNOV

A thesis submitted in fulfillment of the requirements for the degree of

Doctor of natural sciences

in the reseach group

Algebra and number theory, Institute of Mathematics

Augsburg
November 2024

First examiner : Prof. Dr. Maxim Smirnov
Second examiner : Assist. Prof. Pieter Belmans
Date of oral examination : April 29th, 2025

Abstract

For the orthogonal Grassmannian $X = \mathrm{OGr}(3, N)$ of 3-dimensional subspaces in a symplectic vector space of dimension $N = 2n + 1$, we construct an exceptional Lefschetz collection in its bounded derived category of coherent sheaves $\mathbf{D}^{\mathrm{b}}(X)$. For the case $n = 4$ or equivalently $N = 9$, we present an exceptional Lefschetz collection of maximal expected length.

"Young man, in mathematics you don't understand things. You just get used to them."

John von Neumann

To my family.

Contents

Acknowledgements

First, I express my deepest gratitude to my first supervisor Maxim Smirnov for his outstanding patience and guidance. Further, I thank Pieter Belmans and Marco Hien for their support in the past months.

This thesis benefited from various discussions, a lot of feedback and plenty suggestions. Hence, I want to thank my colleagues; in particular, Alexander Kuznetsov and Marco Rampazzo.

This work was supported by the Deutsche Forschungsgemeinschaft (German Research Foundation) in course of the project *Lefschetz-Folgen auf homogenen Räumen und Quantenkohomologie* (No. 448537907).

Finally, I thank my wife for non-mathematical help as well as my children for any diversion.

Chapter 1

Introduction

1.1 Motivation

For a smooth projective variety X, its associated bounded derived category of coherent sheaves, namely

$$\mathbf{D}^{\mathrm{b}}(X),$$

is a highly interesting homological invariant. I refer you to [5, 8, 23] and references therein to gain an overview to this topic and the several possible perspectives. Let us just sketch two of the latter and hence catch a glimpse: First, the bounded derived category of a projective variety provides data about the underlying base space. For instance, under suitable conditions as in [6], we can reconstruct X from $\mathbf{D}^{\mathrm{b}}(X)$. Second, the bounded derived category of a projective variety can carry its own geometric structure. For instance, in [19, 24], we can assign notions such as dimension or orientation to the entity $\mathbf{D}^{\mathrm{b}}(X)$ or suitable components.

In [9, 10], Dubrovin conjectures a relation between the bounded derived category of coherent sheaves over X and its (big) quantum cohomology: The first of three parts proposes that $\mathbf{D}^{\mathrm{b}}(X)$ permits a full exceptional collection if and only if $\mathrm{BQH}^{\bullet}(X)$ is generically semisimple. Emanating from this, in [26, Conjecture 1.3.], Kuznetsov and Smirnov refine the test area for smooth Fano varities of Picard rank r and with index m by an analogy between the μ_m-action on the (canonical) quantum spectrum $\mathrm{QS}_X := \mathrm{Spec}(\mathrm{QH}^{\bullet}(X) \otimes_{\mathbb{Q}[q_1,\cdots,q_r]} \mathbb{C})$ on the one side and the twist by $\mathcal{O}_X(1)$ on Lefschetz exceptional collections in $\mathbf{D}^{\mathrm{b}}(X)$ on the other side. The advantage of this approach relies on the fact that the small quantum cohomology $\mathrm{QH}^{\bullet}(X)$ is much easier to compute in contrast to the big one $\mathrm{BQH}^{\bullet}(X)$. In this thesis, we construct exceptional Lefschetz collections such that their patterns might be in accordance with the prediction arising from the corresponding (small) quantum cohomologies.

1.2 State of the art

Given an arbitrary homogeneous variety $X = \mathrm{G}/\mathrm{P}$, we can break it down into an iterated fibration such that each fibre is of the form $\mathrm{G}^{(i)}/\mathrm{P}^{(i)}$ where $\mathrm{G}^{(i)}$ is semi-simple and $P^{(i)} \subseteq G^{(i)}$ is a maximal parabolic subgroup. Due to Orlov's projectivization formula in [29] we can establish a full exceptional collection on $\mathbf{D}^{\mathrm{b}}(G/P)$ whenever we have for each fibre such collection on $\mathbf{D}^{\mathrm{b}}(G^{(i)}/P^{(i)})$. Thus, we are left to consider minimal homogeneous varieties $X = G/P$ – that is, the algebraic group G is simple and the corresponding parabolic subgroup $P \subseteq G$ is maximal.

There is a general algorithm in [25] arising from a representation-theoretic study of G to construct an exceptional collection of maximal expected length (Kuznetsov-Polishchuk collection): First, we take specific subsets in the cone of L-dominant weights, the so-called exceptional blocks B. Then we form the exceptional subcollections $\{\mathcal{U}^{\lambda}\}_{\lambda \in B}$ in the G-equivariant bounded derived category $\mathbf{D}^{\mathrm{b}}_{G}(G/P)$ of coherent sheaves and next we set up the associated right dual exceptional subcollections $\{\mathcal{E}^{\lambda}\}_{\lambda \in B}$ in $\mathbf{D}^{\mathrm{b}}_{G}(G/P)$. Finally, dropping off the G-equivariant structure yields (miraculously) exceptional subcollections in $\mathbf{D}^{\mathrm{b}}(G/P)$. The desired exceptional collection of maximal expected length is obtained by merging the previous subcollections $\{\mathcal{E}^{\lambda}\}_{\lambda \in B}$. Unfortunately, this collection has not yet been proven to be full unless for a few sporadic cases.

Besides the (systematic) approach we sketched before, in the past the extensive research on homogeneous varieties $X = G/P$ has come up with various subfamilies where it is possible to construct an exceptional collections which are even full. If we confine ourselves to a subfamily of homogeneous varieties with specific geometric and/or representation-theoretic properties, this can provide us with suitable conditions to prove fullness. The website [2] serves as a periodic table of (generalized) Grassmannians.

A_n: In [1], Beilinson constructed his famous pioneering minimal full exceptional Lefschetz collection $\mathcal{O}, \mathcal{O}(1), \cdots, \mathcal{O}(n)$ on the projective space $\mathbb{P}^n$.
In [18], Kapranov established a full exceptional collection on the Grassmannian $\mathrm{Gr}(k, n+1)$. These collections are far from being minimal if one implements a Lefschetz structure. Later, in [13], Fonarev presented two exceptional Lefschetz collections: the first one is proven to be full but only almost minimal, while the second one is conjectured to be full and minimal.

B_n: In case of the quadric space Q^{2n-1}, let me also refer you to [18].
In [20], Kuznetsov implemented a collection on $\mathrm{OGr}(2, 2n+1)$. Its pattern seems to extend the aforementioned Kapranov's collection on $Q^{2n-1} = \mathrm{OGr}(1, 2n+1)$ in some sense. In this thesis, we are going to partially advance in this direction as we will consider the cases $\mathrm{OGr}(3, 2n+1)$.
If $k = n$, then $\mathrm{OGr}(n, 2n+1)$ is isomorphic to the spinor $\frac{1}{2}n(n+1)$-fold $\mathrm{OGr}_{+}(n+1, 2n+2)$. For the cases $n \in \{3, 4, 5\}$, let me refer you to the corresponding cases appearing in type D_{n+1}.

C_n: The spaces $\mathrm{SGr}(1, 2n) = \mathbb{P}^{2n-1}$ are covered by Beilinson's collection. For $\mathrm{SGr}(2, 2n)$, we proceed analogously as before in the type B_n: See again [20], where Kuznetsov

extends the collection of the aforementioned case on $\mathrm{SGr}(1,2n)$ to $\mathrm{SGr}(2,2n)$. Furthermore, the following sporadic cases are covered: For $\mathrm{SGr}(3,8)$ let me refer you to [14], and for $\mathrm{SGr}(3,10)$ to [28].
If $k = n$, i.e. the Lagrangian Grassmannian $\mathrm{LGr}(n,2n)$, Fonarev elaborated a full exceptional collection for the Lagrangian Grassmannians in general in [12].

D_n: For the quadric space Q^{2n-2}, let me refer you to Kapranov's collection in [18], as before.
The orthogonal Grassmannian $\mathrm{OGr}(2,6)$ is isomorphic to the projective space $\mathbb{P}^3$ and therefore let me refer you to Beilionson's collection in type A_3. For higher n, let me refer you again to Kuznetsov's collection in [26].
If $k = n$, then we consider the spinor $\frac{1}{2}(n-1)n$-fold $\mathrm{OGr}_+(n,2n)$. If $k = n = 3$, i.e. the spinor 3-fold $\mathrm{OGr}_+(3,6)$ is isomorphic to the projective space $\mathbb{P}^3$, we fall back to Beilinson's collection. If $k = n = 4$, i.e. the spinor 6-fold $\mathrm{OGr}_+(4,8)$ is isomorphic to the quadric 6-fold Q^6, we need to recall the corresponding collection. If $k = n = 5$, i.e. the spinor 10-fold $\mathrm{OGr}_+(5,10)$, let me refer you to [22]. Furthermore, in [27], Moschetti and Rampazzo linked this collection with the one arising from the general algorithm of [25] via suitable mutations.
If $k = n = 6$, i.e. the spinor 15-fold $\mathrm{OGr}_+(6,12)$, let me refer you to [4].

E_6: If $k = 1$, i.e. the Cayley plane $\mathbb{OP}^2$, or if $k = 6$, i.e. the dual Cayley plane $\mathbb{OP}^{2,\vee}$, in [11], Faenzi and Manivel constructed a full strongly exceptional collection.

F_4: In [30], Smirnov came up with a full rectangular Lefschetz collection on the adjoint Grassmannian of type F_4, namely on $\mathrm{F}_4/\mathrm{P}_1$.
In [3], Belmans, Kuznetsov and Smirnov provided a full exceptional collection on the coadjoint Grassmannian of type F_4, namely on $\mathrm{F}_4/\mathrm{P}_4$. Indeed, they restricted the Faenzi–Manivel collection on the Cayley plane to a suitable hyperplane section.

G_2: This space $\mathrm{G}_2/\mathrm{P}_1$ can be realized as quadric 5-fold Q^5 and therefore let me refer you to Kapranov's collection in [18].
Kuznetsov covered the sporadic case $\mathrm{G}_2/\mathrm{P}_2$ in [22].

1.3 Results

First, we initialize our setting for $X = \mathrm{OGr}(3,2n+1)$ in chapter 2 and prepare auxiliary calculations in chapter 3.
Thereafter, we construct exceptional collections in the bounded derived category $\mathbf{D}^{\mathrm{b}}(X)$: the tautological subcollection in chapter 4 (Propositions 4.1.2, 4.2.2, 4.3.4) and the spinor one in chapter 5 (Proposition 5.0.10). Then, we merge the above subcollections in chapter 6 (Proposition 6.1.2 or 6.2.2 respectively).
In chapter 7, we focus on the case $X = \mathrm{OGr}(3,9)$ and develop an exceptional collection of maximal expected length for it (Proposition 7.2.1).

Finally, in chapter 8, we present an outlook on open issues, namely the fullness of our collection on $X = \mathrm{OGr}(3, 9)$ as well as its residual part.

Chapter 2

Preliminaries

In the following, we introduce the required notions and recall the necessary foundations. For further details, refer to [17, 15].

2.1 Notation

We denote by e_i the i^{th} standard basis vector.

We will write intervals as the following:

$$\begin{aligned}
\mathbb{Z}_{[a,b]} &= [a,b] &&= \{\, x \in \mathbb{Z} \,:\, a \le x \le b \,\} \\
\mathbb{Q}_{[a,b]} & &&= \{\, x \in \mathbb{Q} \,:\, a \le x \le b \,\} \\
\mathbb{R}_{[a,b]} & &&= \{\, x \in \mathbb{R} \,:\, a \le x \le b \,\}
\end{aligned}$$

2.2 Setting

We fix an algebraically closed field $\mathbf{K}$ of characteristic zero, some integer $n \geq 4$ and $N := 2n+1$. Let V be a N-dimensional vector space over $\mathbf{K}$ equipped with a non-degenerate, symmetric bilinear form $\langle -,- \rangle$. Let $X = \mathrm{OGr}(3,V)$ be the space parameterizing 3-dimensional isotropic subspaces in V. It is well-known that X is a Fano variety and that its Picard group $\mathrm{Pic}(X)$ is torsion free and of rank 1. We compute the following invariants of X straightforward:

$$\begin{aligned}
d &:= \dim(X) = 6n-12, \\
w_{max} &:= \mathrm{indx}(X) = 2n-3, \text{ and} \\
l_{max} &:= \mathrm{rk}\,\mathrm{K}_0(X) = \frac{4}{3}(n-2)(n-1)n \\
&\qquad\qquad\qquad\;\; = \frac{4}{3}n^3 - 4n^2 + \frac{8}{3}n.
\end{aligned} \tag{2.1}$$

2.3 Bounded derived category

Our main object of interest is the bounded derived category of coherent sheaves on X, i.e.

$$\mathbf{D}^{\mathrm{b}}(X) = \mathbf{D}^{\mathrm{b}}(\mathrm{Coh}(X)). \tag{2.2}$$

Given an object $\mathcal{E}$ from $\mathbf{D}^{\mathrm{b}}(X)$, the functor

$$\mathbf{D}^{\mathrm{b}}(pt) := \mathbf{D}^{\mathrm{b}}(\mathrm{Vec}/\mathbf{K}) \to \mathbf{D}^{\mathrm{b}}(X) \quad \text{via} \quad V^\bullet \mapsto V^\bullet \otimes \mathcal{E} \tag{2.3}$$

is fully faithful if and only if $\mathcal{E}$ is exceptional. This means, its automorphism group contains only scalar multiplication and there are no higher self-extensions; or in formulas

$$\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(X)}(\mathcal{E},\mathcal{E}) = \mathrm{Ext}^\bullet(\mathcal{E},\mathcal{E}) = \mathbf{K}[0]. \tag{2.4}$$

We say an object $\mathcal{E}_2$ is right orthogonal to another one $\mathcal{E}_1$ or likewise $\mathcal{E}_1$ is left orthogonal to $\mathcal{E}_2$ if there are no morphisms from $\mathcal{E}_2$ to $\mathcal{E}_1$, i.e.

$$\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(X)}(\mathcal{E}_2,\mathcal{E}_1) = \mathrm{Ext}^\bullet(\mathcal{E}_2,\mathcal{E}_1) = 0. \tag{2.5}$$

In this case, we write briefly $\mathcal{E}_2 \perp \mathcal{E}_1$. We explicitly mention that this relation is not symmetric. A collection of l many objects $\mathcal{E}_1, \cdots, \mathcal{E}_l$ from $\mathbf{D}^{\mathrm{b}}(X)$ is called exceptional if

1. any object $\mathcal{E}_i$ is exceptional, and
2. for any pair $i < j$ of indices, $\mathcal{E}_j$ is right orthogonal to $\mathcal{E}_i$.

In this case, the full triangulated subcategory $\mathcal{D} = \langle \mathcal{E}_1, \cdots, \mathcal{E}_l \rangle \subseteq \mathbf{D}^{\mathrm{b}}(X)$ admits a semi-orthogonal decomposition. We call this collection full if $\mathcal{D}$ coincides with $\mathbf{D}^{\mathrm{b}}(X)$. Due to the Euler pairing $\chi : \mathbf{D}^{\mathrm{b}}(X) \times \mathbf{D}^{\mathrm{b}}(X) \to \mathbb{Z}$ via

$$(\mathcal{E}'', \mathcal{E}') \mapsto \sum_i (-1)^i \dim(\,\mathrm{Ext}^i(\mathcal{E}'', \mathcal{E}')\,), \tag{2.6}$$

we see that l_{max} is an upper bound for the length l of any exceptional collection, and moreover reaching it is a necessary (but not sufficient) criterion for fullness – see [16, Proposition 3.5.] for further details. We equip such a collection with a Lefschetz structure by establishing the following data; see [21]:

Twist Let τ be an automorphism on $\mathbf{D}^{\mathrm{b}}(X)$. In later sections, it will be tensoring with an ample generator $\mathcal{O}_X(1)$ of the Picard group. Hence, we already write $\mathcal{E}(m)$ for $\tau^m \mathcal{E}$.

Support partition Let $h_1 \geq h_2 \geq \cdots \geq h_w \geq 0$ be a descending sequence of integers summing up to l.

It is $\mathcal{E}_i = \mathcal{E}_y(x-1)$ whenever $i = \sum_{c=1}^{x-1} h_c + y$ for $x \in [1, w]$ and $y \in [1, h_x]$. This means, we arrange the objects of the collection in the following grid:

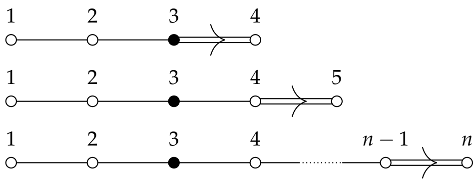
$$\begin{pmatrix} \vdots & \vdots & \\ \mathcal{E}_2 & \mathcal{E}_2(1) & \cdots \\ \mathcal{E}_1 & \mathcal{E}_1(1) & \cdots \end{pmatrix}. \tag{2.7}$$

The columns are called blocks and the x^{th} one has height h_x. The rows are called orbits and the y^{th} one has width/length $w_y = \max\{x : y \leq h_x\}$. We mention for Fano varieties that its Fano index is an upper bound for the length of the orbits by Serre duality.

Remark 2.3.1. In the following, we label the objects of a collection with the index i if we want to emphasize the total ordering (without the data of a Lefschetz structure); and otherwise, we label them with the indices x, y if we want to highlight the 2D grid (arising from some Lefschetz structure).

2.4 Homogeneous variety

We can see X as realisation of the quotient G/P where G is the universal covering (hence simply connected) of the simple algebraic group $\mathrm{SO}(V)$ of Dynkin type B_n and P is the maximal parabolic subgroup $\mathrm{P}_3 \subseteq \mathrm{G}$ associated to the third node in the Dynkin diagram



if $n = 4$,

if $n = 5$, or (2.8)

if $n \geq 6$.

In detail, we fix a maximal torus T and construct in a first step the corresponding *negative* Borel subgroup B_- by adding all negative roots of G to the maximal torus T. Then we construct the parabolic subgroup $\mathrm{P} = \mathrm{P}_3$ in a second step by extending B_- along all *positive* roots except the third one. Hence, we have inclusions

$$\mathrm{T} \subseteq \mathrm{B}_- \subseteq \mathrm{P} \subseteq \mathrm{G}. \tag{2.9}$$

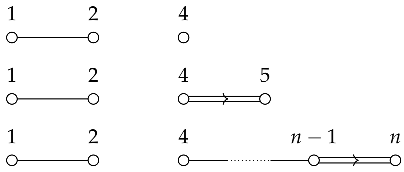
Similar, we obtain the *positive* Borel subgroup B_+ from the maximal torus T by involving the positive roots.

Since the parabolic subgroup P is a semi-direct product of its unipotent radical U_P acting on its quotient, it admits the splitting short exact sequence

$$0 \to \mathrm{U}_\mathrm{P} \to \mathrm{P} \to \mathrm{L} \to 0. \tag{2.10}$$

Accordingly, this gives us the Levi subgroup $L := P/U_P$ of the parabolic subgroup P containing the maximal torus T. It is a reductive group and its semi-simple part is of Dynkin type

$$\begin{array}{lll} A_2 \times A_1 & \text{– i.e.} & \text{if } n = 4, \\ A_2 \times B_2 & \text{– i.e.} & \text{if } n = 5, \text{ or} \\ A_2 \times B_{n-3} & \text{– i.e.} & \text{if } n \geq 6. \end{array} \tag{2.11}$$



The intersection $B_- \cap L$ is a negative Borel subgroup in the Levi subgroup L and analogously $B_+ \cap L$ a positive one.

We equip $\mathbb{Q}^n$ with the standard scalar product

$$v_1 \cdot v_2 = \sum_{j=1}^{n} v_{1,j} v_{2,j} \tag{2.12}$$

where $v_i = (v_{i,1}, \cdots, v_{i,n}) \in \mathbb{Q}^n$ for $i \in \{1, 2\}$ and we generate the root lattice $\mathbf{Q}_G$ of the algebraic group G as a lattice by the simple roots

$$\alpha_i = e_i - e_{i+1} \text{ for } i \in \{1, \cdots, n-1\} \quad \text{and} \quad \alpha_n = e_n. \tag{2.13}$$

The root lattice $\mathbf{Q}_L \subseteq \mathbf{Q}_G$ of the Levi subgroup L consists of the linear combinations of α_1 and α_2 as well as $\alpha_4, \ldots, \alpha_n$.

The weight lattice $\mathbf{P}_G$ of G is generated in $\mathbb{Q}^n$ by the fundamental weights

$$\omega_i = \sum_{j=1}^{i} e_j \text{ for } i \in \{1, \cdots, n-1\} \quad \text{and} \quad \omega_n = \frac{1}{2} \sum_{j=1}^{n} e_j. \tag{2.14}$$

This means, a tuple $\lambda = (\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n) \in \mathbb{Q}^n$ describes a weight of G if and only if

$$\lambda_i - \lambda_{i+1} \in \mathbb{Z} \text{ for } i \in \{1, \cdots, n-1\} \quad \text{and} \quad \lambda_n \in \frac{1}{2}\mathbb{Z}. \tag{2.15}$$

In this context, let us reformulate that the root lattice $\mathbf{Q}_L$ of the Levi subgroup L consists of those roots which are orthogonal to the third fundamental weight ω_3 with respect to the scalar product (2.12). We denote by $\mathbf{P}_G^+$ the cone of dominant weights with respect to B_+. Concretely, a weight $\lambda = (\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n)$ is G-dominant if and only if

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n \geq 0. \tag{2.16}$$

Obviously from (2.14), the sum of the fundamental weights is given by

$$\rho_G = \sum_{i=1}^{n} \omega_i = (n - \frac{1}{2}, n - \frac{3}{2}, n - \frac{5}{2}, \cdots, \frac{1}{2}). \tag{2.17}$$

Thanks to the embedding $L \subseteq G$, we identify the weight lattice $\mathbf{P}_L$ of the Levi subgroup L with the weight lattice $\mathbf{P}_G$. Consequently, $\mathbf{P}_L^+$ is the cone of dominant weights with respect to $B_+ \cap L$ and we have

$$\mathbf{P}_G^+ \subseteq \mathbf{P}_L^+. \tag{2.18}$$

A weight $\lambda = (\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n)$ is L-dominant if and only if

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \quad \text{and} \quad \lambda_4 \geq \cdots \geq \lambda_n \geq 0. \tag{2.19}$$

The Weyl group W_G of G is generated by the simple reflections

$$\begin{aligned}
(\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n) &\xmapsto{w_1} (\lambda_2, \lambda_1, \lambda_3, \cdots, \lambda_n), \\
(\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n) &\xmapsto{w_2} (\lambda_1, \lambda_3, \lambda_2, \cdots, \lambda_n), \\
&\vdots \\
(\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_4) &\xmapsto{w_{n-1}} (\lambda_1, \lambda_2, \cdots, \lambda_n, \lambda_n - 1), \text{ and} \\
(\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_4) &\xmapsto{w_n} (\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_{n-1}, -\lambda_n).
\end{aligned} \tag{2.20}$$

Let $\ell_G : W_G \to \mathbb{Z}$ be the length function on the Weyl group W_G, i.e. if $w = w_{i_1} \cdots w_{i_n}$ is the minimal representation by simple reflections, then we have $\ell_G w = n$. The Weyl group W_L of L is generated by w_1 and w_2 as well as w_4, ..., w_n. The presentations (2.20) imply that the longest elements act as

$$(\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n) \xmapsto{w_{G,\infty}} (-\lambda_1, -\lambda_2, -\lambda_3, -\lambda_4, \cdots, -\lambda_n) \tag{2.21}$$

and

$$(\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n) \xmapsto{w_{L,\infty}} (\lambda_3, \lambda_2, \lambda_1, -\lambda_4, \cdots, -\lambda_n) \tag{2.22}$$

respectively. Obviously, we have $W_L \subseteq W_G$. Let $\overline{W}$ be the set of minimal length representatives of the quotient W_G / W_L, i.e.

$$\overline{W} = \{ \, w_{i_1} \cdots w_{i_n} \text{ representative of minimal length } : \; i_n = 3 \, \} \tag{2.23}$$

A weight $\lambda = (\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n)$ is called G-singular if it is invariant under some element of the Weyl group W_G. This means conceptionally that it is orthogonal to some root of the algebraic group G and therefore lies on one of the so-called Weyl chambers. Concrete for our case, we notice that λ is G-singular if and only

$$|\lambda_i| = |\lambda_j| \text{ for two distinct entries of } \lambda \quad \text{or} \quad \lambda_i = 0 \text{ for at least one entry of } \lambda. \tag{2.24}$$

Otherwise, we say that such a weight λ is G-regular. This means, we find a unique element w in the Weyl group W_G such that the entries of $\lambda' = w\lambda$ are proper descending.

The Weyl group W_G acts on the weight lattice $\mathbf{P}_G$ via the dot action

$$w \cdot \lambda := w(\lambda + \rho_G) - \rho_G. \tag{2.25}$$

2.5 Equivariant vector bundles

The foundation of our machinery is the natural equivalence between the following tensor abelian categories: on one side the category of G-equivariant coherent sheaves on the homogeneous variety $X = \mathrm{G}/\mathrm{P}$ and on the other side the category of representations of the parabolic subgroup P, namely

$$\mathrm{Coh}_{\mathrm{G}}(\mathrm{G}/\mathrm{P}) \cong \mathrm{Rep}(\mathrm{P}) \quad \text{via} \quad F \mapsto F_{[P]} \tag{2.26}$$

where $F_{[P]}$ is the fiber of F at the point $[P] \in X = \mathrm{G}/\mathrm{P}$. In particular, tensor products and duals are preserved. We explicitly mention that $\mathrm{Rep}(\mathrm{L})$ is the semi-simple part of the category $\mathrm{Rep}(\mathrm{P})$ and describe in the following how to construct G-equivariant vector bundles from L-representations. Given a L-dominant weight λ, i.e. $\lambda \in \mathbf{P}_{\mathrm{L}}^{+}$, we take the irreducible L-representation $\mathrm{V}_{\mathrm{L}}^{\lambda}$ with highest weight λ and extend it via the projection $\mathrm{P} \to \mathrm{L}$ of (2.10) to a P-representation $\mathrm{V}_{\mathrm{P}}^{\lambda}$. Therefore, moving from the right side of (2.26) to the left, we obtain a G-equivariant vector bundle $\mathcal{U}^{\lambda}$. This means that we manage the tensor multiplication as well as the dualisation of G-equivariant vector bundles by the manipulations of the corresponding L-representations.

Proposition 2.5.1. *1. It is*

$$\mathcal{U}^{\lambda\vee} = \mathcal{U}^{-w_{\mathrm{L},\infty}\lambda} \tag{2.27}$$

2. It is

$$\mathrm{V}_{\mathrm{L}}^{\lambda_1} \otimes \mathrm{V}_{\mathrm{L}}^{\lambda_2} = \bigoplus_{\mu} M_{\mu}^{(\lambda_1,\lambda_2)} \otimes \mathrm{V}_{\mathrm{L}}^{\mu} \quad \Rightarrow \quad \mathcal{U}^{\lambda_1} \otimes \mathcal{U}^{\lambda_2} = \bigoplus_{\mu} M_{\mu}^{(\lambda_1,\lambda_2)} \otimes \mathcal{U}^{\mu} \tag{2.28}$$

where $M_{\mu}^{(\lambda_1,\lambda_2)} = \mathrm{Hom}(\mathrm{V}_{\mathrm{L}}^{\mu}, \mathrm{V}_{\mathrm{L}}^{\lambda_1} \otimes \mathrm{V}_{\mathrm{L}}^{\lambda_2})$.

Proof. Cf. [25, Formula (8)]. □

We denote by $\mathcal{O}(1) := \mathcal{U}^{\omega_3}$ the ample generator of the Picard group $\mathrm{Pic}(X)$. In particular, we write

$$\mathcal{U}^{\lambda}(t) := \mathcal{U}^{\lambda + t\omega_3}. \tag{2.29}$$

Finally, we illustrate how we compute the cohomology of G-equivariant vector bundles and therefore see the basics for later Ext-computations.

Proposition 2.5.2 (Borel–Weil–Bott)**.** *Let λ be from $\mathbf{P}_{\mathrm{L}}^{+}$ and let $\mathcal{U}^{\lambda}$ be the corresponding G-equivariant vector bundle. Then, we compute*

$$\mathrm{H}^{\bullet}(\mathrm{X},\mathcal{U}^{\lambda}) = \begin{cases} \mathrm{V}_{\mathrm{G}}^{w\cdot\lambda}[-\ell_{\mathrm{G}}(w)] & , \textit{if } \lambda+\rho_{\mathrm{G}} \textit{ is G-regular} \\ 0 & , \textit{if } \lambda+\rho_{\mathrm{G}} \textit{ is G-singular} \end{cases} \tag{2.30}$$

where, in the first case, w denotes the unique element from the Weyl group W_{G} mapping $\lambda+\rho_{\mathrm{G}}$ to $\mathbf{P}_{\mathrm{G}}^{+}$ and $w\cdot\lambda$ is the dot action as in (2.25).

Proof. Cf. [25, Theorem 2.15.] and the reference [7, Theorem IV'] therein. □

2.6 Tautological vector bundles

We have the short exact sequence

$$0 \to \mathcal{U} \to V \otimes \mathcal{O} \to \mathcal{Q} \to 0. \tag{2.31}$$

build up from the following vector bundles:

The trivial vector bundle: $\mathcal{V} = V \otimes \mathcal{O}$ is the vector bundle of rank $N = 2n+1$ with fiber V.

The tautological subbundle: $\mathcal{U}$ is the subbundle of isotropic subspaces. It has rank 3 and determinant $\mathcal{O}(-1)$.

The tautological quotient bundle: $\mathcal{Q} = \mathcal{V}/\mathcal{U}$ is the corresponding quotient bundle of rank $N-3 = 2n-2$. It has determinant $\mathcal{O}(1)$.

Dualising (2.31) and implementing the isomorphism $V = V^{\vee}$ arising by the mapping $v \mapsto \langle v, -\rangle$ yields the morphism

$$V \otimes \mathcal{O} \cong V^{\vee} \otimes \mathcal{O} \to \mathcal{U}^{\vee}. \tag{2.32}$$

Applying the snake lemma shows that its kernel is isomorphic to the dual of the quotient bundle $\mathcal{Q}$, i.e.

$$0 \to \mathcal{Q}^{\vee} \to V \otimes \mathcal{O} \to \mathcal{U}^{\vee} \to 0. \tag{2.33}$$

We have the inclusion $\mathcal{U} \subseteq \mathcal{Q}^{\vee} \subseteq V \otimes \mathcal{O}$ of vector bundles and consequently the short exact sequence

$$0 \to \mathcal{U} \to \mathcal{Q}^{\vee} \to \mathcal{Q}^{\vee}/\mathcal{U} \to 0. \tag{2.34}$$

In fact, locally for a point $U \in X$, we have the zero mapping

$$U \hookrightarrow V \cong V^{\vee} \twoheadrightarrow U^{\vee} \quad \text{via} \quad u \mapsto \langle u, -\rangle|_U = 0 \tag{2.35}$$

since the bilinear form $\langle -, - \rangle$ vanishes on the isotropic subspace $U \subseteq V$ by definition. So, we deduce the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{U} \arrow[r] \arrow[d] & V \otimes \mathcal{O} \arrow[r] \arrow[d, equal, "\wr"] & \mathcal{Q} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{Q}^\vee \arrow[r] & V^\vee \otimes \mathcal{O} \arrow[r] & \mathcal{U}^\vee \arrow[r] & 0 \end{tikzcd} \tag{2.36}$$

for the zero morphism (2.35) and see the desired embedding. Moreover, snaking shows us that the quotient $\mathcal{Q}^\vee/\mathcal{U}$ is self-dual.

The dual of the tautological subbundle $\mathcal{U}^\vee$ has highest weight ω_1 as the trivial vector bundle $\mathcal{V}$ has the following weights:

$$\left.\begin{array}{ccc} \omega_1 & -\omega_3 + 2\omega_4 & \omega_2 - \omega_3 \\ -\omega_1 + \omega_2 & 0 & \omega_1 - \omega_2 \\ -\omega_2 + \omega_3 & \omega_3 - 2\omega_4 & -\omega_1 \end{array}\right\} \text{ if } n = 4 \tag{2.37}$$

$$\left.\begin{array}{ccc} \omega_1 & -\omega_3 + \omega_4 & \omega_2 - \omega_3 \\ -\omega_1 + \omega_2 & -\omega_4 + 2\omega_5 & \omega_1 - \omega_2 \\ -\omega_2 + \omega_3 & 0 & -\omega_1 \\ & \omega_4 - 2\omega_5 & \\ & \omega_3 - \omega_4 & \end{array}\right\} \text{ if } n = 5 \tag{2.38}$$

$$\left.\begin{array}{ccc} \omega_1 & -\omega_3 + \omega_4 & \omega_2 - \omega_3 \\ -\omega_1 + \omega_2 & \vdots & \omega_1 - \omega_2 \\ -\omega_2 + \omega_3 & -\omega_{n-2} - \omega_{n-1} & -\omega_1 \\ & -\omega_{n-1} - 2\omega_n & \\ & 0 & \\ & \omega_{n-1} - 2\omega_n & \\ & \omega_{n-2} - \omega_{n-1} & \\ & \vdots & \\ & \omega_3 - \omega_4 & \end{array}\right\} \text{ if } n \geq 6 \tag{2.39}$$

Lemma 2.6.1. *We have*

$$\wedge^i \mathcal{Q} = (\wedge^{2n-2-i} \mathcal{Q})^\vee(1) \tag{2.40}$$

Proof. We apply the determinant to the short exact sequence (2.31) and therefore see

$$\mathcal{O} = \det(\mathcal{V}) = \det(\mathcal{U}) \cdot \det(\mathcal{Q}). \tag{2.41}$$

Since $\mathcal{Q}$ is a vector bundle of rank $2n-2$ and $\det(\mathcal{U}) = \mathcal{O}(-1)$, we conclude

$$\wedge^{2n-2} \mathcal{Q} = \det(\mathcal{Q}) = \mathcal{O}(1). \tag{2.42}$$

For $i \in \{0, \cdots, 2n-2\}$, the claimed isomorphism follows from the perfect pairing

$$\wedge^i \mathcal{Q} \times \wedge^{2n-2-i} \mathcal{Q} \to \wedge^{2n-2} \mathcal{Q} = \mathcal{O}(1). \tag{2.43}$$

□

2.7 Spinor bundle

We write the spinor bundle as

$$\mathcal{S} := \mathcal{U}^{\omega_n}. \tag{2.44}$$

It has rank 2^{n-3} and determinant $\mathcal{O}(2^{n-4})$ – see [20, Corollary 6.5.].

First, we present a filtration of $\mathrm{S} \otimes \mathcal{O}$.

Proposition 2.7.1.

$$\begin{gathered}
\mathcal{F}_4^{(1)} = 0 \\
0 \to \mathcal{F}_4^{(1)} \to \mathcal{F}_3^{(1)} \to \mathcal{U}^{\omega_n}(-1) \to 0 \\
0 \to \mathcal{F}_3^{(1)} \to \mathcal{F}_2^{(1)} \to \mathcal{U}^{\omega_1+\omega_n}(-1) \to 0 \\
0 \to \mathcal{F}_2^{(1)} \to \mathcal{F}_1^{(1)} \to \mathcal{U}^{\omega_2+\omega_n}(-1) \to 0 \\
0 \to \mathcal{F}_1^{(1)} \to \mathcal{F}_0^{(1)} \to \mathcal{U}^{\omega_n} \to 0 \\
\mathcal{F}_0^{(1)} = \mathrm{S} \otimes \mathcal{O}
\end{gathered} \tag{2.45}$$

Proof. We apply [20, Proposition 6.3.] and check

$$\mathcal{F}_i^{(1)} / \mathcal{F}_{i+1}^{(1)} \cong \mathcal{S} \otimes \wedge^i \mathcal{U} \tag{2.46}$$

for $i \in \{0, \cdots, 3\}$. □

Secondly, we give useful description of $\mathrm{S} \otimes \mathcal{S}$.

Proposition 2.7.2.

$$\begin{gathered}
\mathcal{F}_n^{(2)} = 0 \\
0 \to \wedge^{2n-2} \mathcal{Q} \to \mathcal{F}_{n-1}^{(2)} \to \mathcal{F}_n^{(2)} \to 0 \\
0 \to \wedge^{2n-4} \mathcal{Q} \to \mathcal{F}_{n-2}^{(2)} \to \mathcal{F}_{n-1}^{(2)} \to 0 \\
\vdots \\
0 \to \wedge^{2} \mathcal{Q} \to \mathcal{F}_{1}^{(2)} \to \mathcal{F}_{2}^{(2)} \to 0 \\
0 \to \mathcal{O} \to \mathcal{F}_{0}^{(2)} \to \mathcal{F}_{1}^{(2)} \to 0 \\
\mathcal{F}_0^{(2)} = \mathrm{S} \otimes \mathcal{S}
\end{gathered} \tag{2.47}$$

Proof. We apply [20, Proposition 6.7.] and dualise the induced short exact sequences. □

2.8 Equivariant extensions

Irreducible vector bundles are our building blocks. However, there are several more ways to construct new objects than just summing up directly. Given two objects $\mathcal{E}'$ and $\mathcal{E}''$, the n^{th} Ext-space $\mathrm{Ext}^n(\mathcal{E}', \mathcal{E}'')$ classifies the degree n extensions of $\mathcal{E}'$ by $\mathcal{E}''$, i.e. exact sequences of the form

$$0 \to \mathcal{E}'' \to \mathcal{E}_1 \to \cdots \to \mathcal{E}_n \to \mathcal{E}' \to 0. \tag{2.48}$$

Any simple summand $\mathbf{K}$ in $\mathrm{Ext}^n(\mathcal{E}', \mathcal{E}'')$ describes a family of G-equivariant extensions of degree n. In detail, the zero belongs to the direct sum while any non-zero induces a non-splitting extension. In the last case, its representatives coincide up to rescaling. Mostly, we will be interested in G-equivariant extension of degree $n = 1$.

Let $0 \to \mathcal{E}'' \to \mathcal{E} \to \mathcal{E}' \to 0$ be a short exact sequence defining an extension $\mathcal{E}$, let $\mathcal{F}$ be a further object.

Lemma 2.8.1.

$$\mathcal{F} \textit{ is right orthogonal to } \mathcal{E}'' \quad \Rightarrow \quad \mathrm{Ext}^\bullet(\mathcal{F}, \mathcal{E}) = \mathrm{Ext}^\bullet(\mathcal{F}, \mathcal{E}') \tag{2.49}$$

$$\mathcal{E}'' \textit{ is right orthogonal to } \mathcal{F} \quad \Rightarrow \quad \mathrm{Ext}^\bullet(\mathcal{E}, \mathcal{F}) = \mathrm{Ext}^\bullet(\mathcal{E}', \mathcal{F}) \tag{2.50}$$

$$\mathcal{F} \textit{ is right orthogonal to } \mathcal{E} \quad \Rightarrow \quad \mathrm{Ext}^\bullet(\mathcal{F}, \mathcal{E}') = \mathrm{Ext}^{\bullet+1}(\mathcal{F}, \mathcal{E}'') \tag{2.51}$$

$$\mathcal{E} \textit{ is right orthogonal to } \mathcal{F} \quad \Rightarrow \quad \mathrm{Ext}^\bullet(\mathcal{E}'', \mathcal{F}) = \mathrm{Ext}^{\bullet+1}(\mathcal{E}', \mathcal{F}) \tag{2.52}$$

$$\mathcal{F} \textit{ is right orthogonal to } \mathcal{E}' \quad \Rightarrow \quad \mathrm{Ext}^\bullet(\mathcal{F}, \mathcal{E}) = \mathrm{Ext}^\bullet(\mathcal{F}, \mathcal{E}'') \tag{2.53}$$

$$\mathcal{E}' \textit{ is right orthogonal to } \mathcal{F} \quad \Rightarrow \quad \mathrm{Ext}^\bullet(\mathcal{E}, \mathcal{F}) = \mathrm{Ext}^\bullet(\mathcal{E}'', \mathcal{F}) \tag{2.54}$$

Proof. **(2.49), (2.51), and (2.53)** We apply $\mathrm{Hom}(\mathcal{F}, -)$ to the short exact sequence defining $\mathcal{E}$ and induce the long exact sequence

$$\cdots \to \mathrm{Ext}^i(\mathcal{F}, \mathcal{E}'') \to \mathrm{Ext}^i(\mathcal{F}, \mathcal{E}) \to \mathrm{Ext}^i(\mathcal{F}, \mathcal{E}') \to \cdots. \tag{2.55}$$

The vanishing of one out of the three Ext-spaces gives us an isomorphism between the remaining two ones.

(2.50), (2.52), and (2.54) We apply $\mathrm{Hom}(-, \mathcal{F})$ to the short exact sequence defining $\mathcal{E}$ and argue similar as before with the long exact sequence

$$\cdots \to \mathrm{Ext}^i(\mathcal{E}', \mathcal{F}) \to \mathrm{Ext}^i(\mathcal{E}, \mathcal{F}) \to \mathrm{Ext}^i(\mathcal{E}'', \mathcal{F}) \to \cdots. \tag{2.56}$$

□

Corollary 2.8.2. *If $\mathcal{F}$ is right orthogonal to two out of the three objects $\mathcal{E}'$, $\mathcal{E}''$, and $\mathcal{E}$, then it is also right orthogonal to the third one. Likewise, if two out of the three objects $\mathcal{E}'$, $\mathcal{E}''$, and $\mathcal{E}$ are right orthogonal to $\mathcal{F}$, then the third also does.*

Chapter 3

Calculations

3.1 Dualisations

First, we describe the dual of an irreducible G-equivariant vector bundle explicitly.

Lemma 3.1.1.

$$\begin{aligned} &\mathcal{U}^{c_1\omega_1+c_2\omega_2+c_4\omega_4+\cdots+c_n\omega_n}(t)^\vee \\ &\quad = \mathcal{U}^{c_2\omega_1+c_1\omega_2+c_4\omega_4+\cdots+c_n\omega_n}(-t - \sum_{i=1}^{2} c_i - 2\sum_{i=4}^{n-1} c_i - c_n) \end{aligned} \tag{3.1}$$

Proof. We combine (2.22) with (2.27) and apply it to the weight

$$\begin{aligned} &c_1\omega_1 + c_2\omega_2 + t\omega_3 + c_4\omega_4 + \cdots + c_n\omega_n \\ &\quad = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \cdots, \lambda_n) \end{aligned} \tag{3.2}$$

where $\lambda_i = \sum_{j=i}^{n-1} c_j + \frac{1}{2}c_n$ and $c_3 = t$. Accordingly, the claimed statement follows from

$$\begin{aligned} &-w_{\mathrm{L},\infty}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \cdots, \lambda_n) \\ &\quad = (-\lambda_3, -\lambda_2, -\lambda_1, \lambda_4, \cdots, \lambda_n). \end{aligned} \tag{3.3}$$

□

3.2 Tensor product decomposition

Next, we recall the computations of tensor products, exterior powers, and symmetric powers given in [30, Section 2.3.]. Therefore, we introduce the following two closed algebraic subgroups of the Levi subgroup L:

The semi-simple part $\mathrm{ss}(\mathrm{L})$ It is a connected simply connected semi-simple algebraic group with Dynkin diagram (2.11). Its corresponding weight lattice $\mathbf{P}_{\mathrm{ss}(\mathrm{L})}$ is the quotient of the map

$$\mathbf{P}_\mathrm{G} \xrightarrow{\pi_{\mathrm{ss}(\mathrm{L})}} \mathbf{P}_{\mathrm{ss}(\mathrm{L})} := \mathbf{P}_\mathrm{G}/\mathbb{Z}\omega_3. \tag{3.4}$$

Since the fundamental weights ω_1, ω_2, ω_3, ω_4, ..., and ω_n form a basis of the weight lattice $\mathbf{P}_\mathrm{G}$, the projection in (3.4) is determined by

$$\omega_i \mapsto \begin{cases} \bar{\omega}_i = \omega_i + \mathbb{Z}\omega_3 & \text{, if } i \neq 3 \\ 0 & \text{, if } i = 3. \end{cases} \tag{3.5}$$

Obviously, we have a lifting from $\mathbf{P}_{\mathrm{ss(L)}}$ to $\mathbf{P}_\mathrm{G}$ via

$$\sum_i c_i \bar{\omega}_i \mapsto \sum_{i \neq 3} c_i \omega_i. \tag{3.6}$$

We observe $\mathbf{P}_{\mathrm{ss(L)}} \otimes \mathbb{Q} = \sum_{\alpha \in \mathbf{Q}_\mathrm{L}} \mathbb{Q}\alpha$. Let $\lambda \in \mathbf{P}_\mathrm{L}^+$ be a weight and consider the corresponding irreducible L-representation $\mathrm{V}_\mathrm{L}^\lambda$. Then restriction to the semisimple part yields the irreducible ss(L)-representation

$$\mathrm{V}_{\mathrm{ss(L)}}^{\bar{\lambda}} = \mathrm{Res}_{\mathrm{ss(L)}}^{\mathrm{L}}(\mathrm{V}_\mathrm{L}^\lambda) \tag{3.7}$$

where $\bar{\lambda} = \pi_{\mathrm{ss(L)}}\lambda$.

The center $\mathrm{Z(L)}$ Because we excluded only a single node, namely the third one, it is isomorphic to the multiplicative group $\mathbb{G}_m = \Bbbk^*$ (1-dimensional torus) and its weight lattice $\mathbf{P}_{\mathrm{Z(L)}}$ arise by

$$\mathbf{P}_\mathrm{G} \xrightarrow{\pi_{\mathrm{Z(L)}}} \mathbf{P}_{\mathrm{Z(L)}} := \mathbf{P}_\mathrm{G}/(\sum_{\alpha \in \mathbf{Q}_\mathrm{L}} \mathbb{Q}\alpha \cap \mathbf{P}_\mathrm{G}). \tag{3.8}$$

Similar as before, we note $\mathbf{P}_{\mathrm{Z(L)}} \otimes \mathbb{Q} = \mathbb{Q}\omega_3$. For a weight $\lambda \in \mathbf{P}_\mathrm{L}^+$ and accordingly for an irreducible L-representation $\mathrm{V}_\mathrm{L}^\lambda$, we have the restriction to the irreducible Z(L)-representation

$$\mathrm{V}_{\mathrm{Z(L)}}^{\bar{\lambda}} = \mathrm{Res}_{\mathrm{Z(L)}}^{\mathrm{L}}(\mathrm{V}_\mathrm{L}^\lambda) \tag{3.9}$$

where $\bar{\lambda} = \pi_{\mathrm{Z(L)}}\lambda$.

The group law of the Levi subgroup L yields a short exact sequence of algebraic groups

$$0 \to \mathrm{ss(L)} \cap \mathrm{Z(L)} \to \mathrm{ss(L)} \times \mathrm{Z(L)} \to \mathrm{L} \to 0 \tag{3.10}$$

where the kernel is finite. The map $(\pi_{\mathrm{ss(L)}}, \pi_{\mathrm{Z(L)}}) : \mathbf{P}_\mathrm{G} \to \mathbf{P}_{\mathrm{ss(L)}} \oplus \mathbf{P}_{\mathrm{Z(L)}}$ is an embedding and the weight lattice $\mathbf{P}_\mathrm{G}$ is a subgroup of finite index in $\mathbf{P}_{\mathrm{ss(L)}} \oplus \mathbf{P}_{\mathrm{Z(L)}}$. Tensoring with $\mathbb{Q}$ yields the isomorphism $\mathbf{P}_\mathrm{G} \otimes \mathbb{Q} = (\mathbf{P}_{\mathrm{ss(L)}} \otimes \mathbb{Q}) \oplus (\mathbf{P}_{\mathrm{Z(L)}} \otimes \mathbb{Q})$.

Lemma 3.2.1. *Let λ or λ_1 and λ_2 be weights from $\mathbf{P}_\mathrm{L}^+$ and let us denote the corresponding* L*-representations by $\mathrm{V}_\mathrm{L}^\lambda$ or $\mathrm{V}_\mathrm{L}^{\lambda_1}$ and $\mathrm{V}_\mathrm{L}^{\lambda_2}$ respectively. We denote by $\mathrm{V}_{\mathrm{ss(L)}}^{\bar{\lambda}}$ or $\mathrm{V}_{\mathrm{ss(L)}}^{\bar{\lambda}_1}$ and $\mathrm{V}_{\mathrm{ss(L)}}^{\bar{\lambda}_2}$ the restrictions as defined above in* (3.7).

1. *It is*

$$\begin{aligned} V^{\bar{\lambda}_1}_{ss(L)} \otimes V^{\bar{\lambda}_2}_{ss(L)} &= \bigoplus_{\bar{\mu}} M^{(\bar{\lambda}_1,\bar{\lambda}_2)}_{\bar{\mu}} \otimes V^{\bar{\mu}}_{ss(L)} \\ \Rightarrow \qquad V^{\lambda_1}_{L} \otimes V^{\lambda_2}_{L} &= \bigoplus_{\bar{\mu}} M^{(\bar{\lambda}_1,\bar{\lambda}_2)}_{\bar{\mu}} \otimes V^{\mu}_{L}(m) \end{aligned} \tag{3.11}$$

where $M^{(\bar{\lambda}_1,\bar{\lambda}_2)}_{\bar{\mu}} = \mathrm{Hom}(V^{\bar{\mu}}_{ss(L)}, V^{\bar{\lambda}_1}_{ss(L)} \otimes V^{\bar{\lambda}_2}_{ss(L)})$.

2. *It is*

$$\begin{aligned} \wedge^p V^{\bar{\lambda}}_{ss(L)} &= \bigoplus_{\bar{\mu}} M^{(\bar{\lambda})}_{\bar{\mu}} \otimes V^{\bar{\mu}}_{ss(L)} \\ \Rightarrow \qquad \wedge^p V^{\lambda}_{L} &= \bigoplus_{\bar{\mu}} M^{(\bar{\lambda})}_{\bar{\mu}} \otimes V^{\mu}_{L}(m) \end{aligned} \tag{3.12}$$

where $M^{(\bar{\lambda})}_{\bar{\mu}} = \mathrm{Hom}(V^{\bar{\mu}}_{ss(L)}, \wedge^p V^{\bar{\lambda}}_{ss(L)})$.

3. *It is*

$$\begin{aligned} S^p V^{\bar{\lambda}}_{ss(L)} &= \bigoplus_{\bar{\mu}} M^{(\bar{\lambda})}_{\bar{\mu}} \otimes V^{\bar{\mu}}_{ss(L)} \\ \Rightarrow \qquad S^p V^{\lambda}_{L} &= \bigoplus_{\bar{\mu}} M^{(\bar{\lambda})}_{\bar{\mu}} \otimes V^{\mu}_{L}(m) \end{aligned} \tag{3.13}$$

where $M^{(\bar{\lambda})}_{\bar{\mu}} = \mathrm{Hom}(V^{\bar{\mu}}_{ss(L)}, S^p V^{\bar{\lambda}}_{ss(L)})$.

μ and m are constructed explicitly in the following proof.

Proof. We only discuss the case (1) of tensor multiplication. The two other cases, namely (2) on the exterior power and (3) on the symmetric power respectively, proceed analogously. Given two weights λ_1 and λ_2 in terms of fundamental weights, we do a base change by

$$\begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 & \cdots & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 & \cdots & 0 \\ \frac{1}{3} & \frac{2}{3} & 1 & a_{3,4} & \cdots & a_{3,n} \\ 0 & 0 & 0 & b_{4,4} & \cdots & b_{4,n} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & b_{n,4} & \cdots & b_{n,n} \end{pmatrix} \tag{3.14}$$

where

$$a_{3,i} = \begin{cases} 1 & \text{, if } i \in [4, n-1] \\ \frac{1}{2} & \text{, if } i = n \end{cases} \quad \text{and} \quad b_{i,j} = \begin{cases} j-3 & \text{, if } j \in [4, n-1],\ j \leq i \\ i-3 & \text{, if } j \in [4, n-1],\ i < j \\ \frac{1}{2}i - \frac{3}{2} & \text{, if } j = n \end{cases} \tag{3.15}$$

to replace all fundamental weights by simple roots except the third one (mixed basis). Hence, we write them as linear combinations

$$\lambda_1 = a_{\lambda_1,1}\alpha_1 + a_{\lambda_1,2}\alpha_2 + c_{\lambda_1,3}\omega_3 + a_{\lambda_1,4}\alpha_4 + \cdots + a_{\lambda_1,n}\alpha_n \tag{3.16}$$

and

$$\lambda_2 = a_{\lambda_2,1}\alpha_1 + a_{\lambda_2,2}\alpha_2 + c_{\lambda_2,3}\omega_3 + +a_{\lambda_2,4}\alpha_4 + \cdots + a_{\lambda_2,n}\alpha_n. \tag{3.17}$$

The projections to the semi-simple part along (3.5) are given by

$$\bar{\lambda}_1 = a_{\lambda_1,1}\bar{\alpha}_1 + a_{\lambda_1,2}\bar{\alpha}_2 + a_{\lambda_1,4}\bar{\alpha}_4 + \cdots + a_{\lambda_1,n}\bar{\alpha}_n \tag{3.18}$$

and

$$\bar{\lambda}_2 = a_{\lambda_2,1}\bar{\alpha}_1 + a_{\lambda_2,2}\bar{\alpha}_2 + a_{\lambda_2,4}\bar{\alpha}_4 + \cdots + a_{\lambda_2,n}\bar{\alpha}_n. \tag{3.19}$$

Now, we compute $V^{\bar{\lambda}_1}_{\mathrm{ss(L)}} \otimes V^{\bar{\lambda}_2}_{\mathrm{ss(L)}}$ via the Littlewood–Richardson rule. For any $\bar{\mu}$ appearing in the decomposition, we take a lift μ by (3.6) and do the inverse base change as before. This means, we change from the mixed basis over to the one consisting of fundamental weights and write

$$\mu = c_{\mu,1}\omega_1 + c_{\mu,2}\omega_2 + c_{\mu,3}\omega_3 + c_{\mu,4}\omega_4 + \cdots + c_{\mu,n}\omega_n. \tag{3.20}$$

Finally, we sum up $m = c_{\lambda_1,3} + c_{\lambda_2,3}$. □

Corollary 3.2.2. *It is*

$$\wedge^p \mathcal{U}^\vee = \begin{cases} \mathcal{O} & \text{, if } p = 0 \\ \mathcal{U}^{\omega_1} & \text{, if } p = 1 \\ \mathcal{U}^{\omega_2} & \text{, if } p = 2 \\ \mathcal{U}^{\omega_3} = \mathcal{O}(1) & \text{, if } p = 3 \\ 0 & \text{, if } 4 \leq p \end{cases} \tag{3.21}$$

as well as

$$\mathrm{Sym}^p \mathcal{U}^\vee = \mathcal{U}^{p\omega_1}. \tag{3.22}$$

If we have irreducible G-equivariant vector bundles supported over different parts of the Levi part – i.e. in our case one highest weight is a linear combination of ω_1 and ω_2 while the other one is a linear combination of ω_4, ..., ω_n – then the tensor product is an irreducible G-equivariant vector bundles where the highest weight is just the sum.

Lemma 3.2.3.

$$\mathcal{U}^{c_1\omega_1 + c_2\omega_2} \otimes \mathcal{U}^{c_4\omega_4 + \cdots + c_n\omega_n} = \mathcal{U}^{c_2\omega_1 + c_1\omega_2 + c_4\omega_4 + \cdots + c_n\omega_n} \tag{3.23}$$

Proof. If we multiply two irreducible ss(L)-representations with the highest weights $c_1\bar{\omega}_1 + c_2\bar{\omega}_2$ and $c_4\bar{\omega}_4 + \cdots + c_n\bar{\omega}_n$, respectively, then this gives a single irreducible ss(L)-representation with highest weight equal to the sum. Hence, we apply first 3.2.1.(1) and then Proposition 2.5.1.(2) to deduce the desired statement. □

3.3 Some landmarks in the jungle of Ext-computations

In the later chapters 4 and 5, we are going to compute a lot of Ext-spaces between irreducible G-equivariant vector bundles. Therefore we present two very concrete approaches:

1. For the tautological subcollection $C^{(\mathcal{U})}$ in chapter 4, we compute the cohomology of irreducible summands $\mathcal{U}^{\mu}$ which appear in the direct sum decomposition of tensor products of the form $\mathcal{U}^{\mu'\vee} \otimes \mathcal{U}^{\mu''}$. If we need to check the vanishings, we show that the appropriate highest weights λ are G-singular.

2. In the context of the spinor subcollection $C^{(\mathcal{S})}$ in chapter 5, we need to check (partial) vanishings of Ext-spaces of the form

$$\begin{aligned} &\mathrm{Ext}^\bullet(\mathcal{U}^{\mu'} \otimes \mathcal{S}, \mathcal{U}^{\mu''}), \\ &\mathrm{Ext}^\bullet(\mathcal{U}^{\mu'}, \mathcal{U}^{\mu''} \otimes \mathcal{S}), \text{ or} \\ &\mathrm{Ext}^\bullet(\mathcal{U}^{\mu'} \otimes \mathcal{S}, \mathcal{U}^{\mu''} \otimes \mathcal{S}) \end{aligned}$$

   where $\mathcal{U}^{\mu'}$ and $\mathcal{U}^{\mu''}$ are from the previously mentioned collection $C^{(\mathcal{U})}$ in chapter 4. Hence, we deduce the desired vanishing from the one of $\mathrm{Ext}^\bullet(\mathcal{U}^{\mu'}, \mathcal{U}^{\mu''})$ which has been checked before.

Throughout this section, let $\mathcal{U}^{\mu'}$ and $\mathcal{U}^{\mu''}$ be two irreducible G-equivariant vector bundles such that their highest weights μ' and μ'' are supported over the first Levi part, namely

$$\mu' = (\mu'_1, \mu'_2, \mu'_3, 0 \cdots, 0) \quad \text{and} \quad \mu'' = (\mu''_1, \mu''_2, \mu''_3, 0 \cdots, 0) \tag{3.24}$$

respectively. We denote by μ the highest weights corresponding to the irreducible summands $\mathcal{U}^{\mu}$ appearing in the direct sum decomposition of the tensor product $\mathcal{U}^{\mu'\vee} \otimes \mathcal{U}^{\mu''}$.

Auxiliary statements where the highest weights are supported over the first Levi part.

Lemma 3.3.1. *Let $\mathcal{U}^{\mu'}$, $\mathcal{U}^{\mu''}$, and $\mathcal{U}^{\mu}$ as introduced above. Then it is the following facts:*

$$\mu_1 \geq \mu_2 \geq \mu_3 \text{ in } \mathbb{Z}, \tag{3.25}$$

$$\mu_i = -\mu'_{4-i} + \sum_{j=1}^{3} t_{i,j}\mu''_j \text{with } t_{i,j} \in \mathbb{R}_{[0,1]} \text{ as well as } \sum_{j=1}^{3} t_{i,j} = 1 \text{ for any } i \in [1,3], \tag{3.26}$$

$$\mu_1 + \mu_2 + \mu_3 = -(\mu'_1 + \mu'_2 + \mu'_3) + (\mu''_1 + \mu''_2 + \mu''_3) \tag{3.27}$$

and

$$\mu_4 = \cdots = \mu_n = 0. \tag{3.28}$$

Proof. Abstract description of μ We deduce two aspects from [25, Equation (8) and Lemma 2.9]: First, μ need to be a L-dominant weight. This means, the difference $\mu_i - \mu_{i+1}$ lies in $\mathbb{Z}_{\geq 0}$ whenever the index i is from $[1,2]$ or from $[4, n-1]$ and μ_n is an element in $\frac{1}{2}\mathbb{Z}_{\geq 0}$. Second, it lies in the convex hull of $\{ -w_{\mathrm{L},\infty}\mu' + w\mu'' \ : \ w \in \mathrm{W_L} \}$ and consequently it can be written as

$$\begin{aligned} \mu &= \sum_{w \in \mathrm{W_L}} t_w \cdot (-w_{\mathrm{L},\infty}\mu' + w\mu'') \\ &= -w_{\mathrm{L},\infty}\mu' + \sum_{w \in \mathrm{W_L}} t_w w \mu'' \end{aligned} \tag{3.29}$$

where the coefficients t_w lie in $\mathbb{R}_{[0,1]}$ for any $w \in \mathrm{W_L}$ and they sum up to 1. In particular, let us observe by (3.29) that μ is the sum of $-w_{\mathrm{L},\infty}\mu'$ with an element from the convex hull of $\{ w\mu'' \ : \ w \in \mathrm{W_L} \}$.

Concretising the description of μ. Since Weyl elements $w \in \mathrm{W_L}$ act on the weights μ' and μ'' respectively by permutation within the first three entries as well as permutation and sign-alternation within the remaining entries (recall (2.20)), we can rewrite

$$-w_{\mathrm{L},\infty}\mu' = (-\mu'_3, -\mu'_2, -\mu'_1, 0, \cdots, 0) \tag{3.30}$$

as in (2.22) and

$$w\mu'' = (\mu''_{w_{1\mathrm{st}}^{-1}(1)}, \mu''_{w_{1\mathrm{st}}^{-1}(2)}, \mu''_{w_{1\mathrm{st}}^{-1}(3)}, 0, \cdots, 0). \tag{3.31}$$

$w_{1^{\mathrm{st}}}$ is a permutation on $[1,3]$ which one obtains from w after restriction to the first Levi part. By assumption on the support of the weights μ' and μ'' respectively, there happens nothing on the second Levi part. This means, we have

$$\begin{aligned} \mu = {} & (-\mu'_3, -\mu'_2, -\mu'_1, 0, \cdots, 0) \\ & + \sum_{t_{w_{1\mathrm{st}}} \in S_3} t_{w_{1\mathrm{st}}} (\mu''_{w_{1\mathrm{st}}^{-1}(1)}, \mu''_{w_{1\mathrm{st}}^{-1}(2)}, \mu''_{w_{1\mathrm{st}}^{-1}(3)}, 0, \cdots, 0) \end{aligned} \tag{3.32}$$

where $t_{w_{1\text{st}}}$ is the sum of all coefficients t_w such that $w \in \mathrm{W_L}$ restricts over the first Levi part to $w_{1^{\text{st}}}$.

Simplifying the description of μ any more. We write the partial sum in (3.32) running over $t_{w_{1\text{st}}} \in S_3$ as

$$\begin{aligned}
&\sum_{t_{w_{1\text{st}}} \in S_3} t_{w_{1\text{st}}} (\mu''_{w^{-1}_{1\text{st}}(1)}, \mu''_{w^{-1}_{1\text{st}}(2)}, \mu''_{w^{-1}_{1\text{st}}(3)}, 0, \cdots, 0) \\
&\quad = \sum_{t_{w_{1\text{st}}} \in S_3} t_{w_{1\text{st}}} \sum_i^3 \mu''_{w^{-1}_{1\text{st}}(i)} \mathrm{e}_i \\
&\quad = \sum_i^3 \Big(\sum_{t_{w_{1\text{st}}} \in S_3} t_{w_{1\text{st}}} \mu''_{w^{-1}_{1\text{st}}(i)} \Big) \mathrm{e}_i \\
&\quad = \sum_i^3 (t_{i,1}\mu''_1 + t_{i,2}\mu''_2 + t_{i,3}\mu''_3) \mathrm{e}_i
\end{aligned} \tag{3.33}$$

where $t_{i,j}$ is the sum of all $t_{w_{1\text{st}}}$ such that $w_{1^{\text{st}}} \in S_3$ maps j to i. In particular, the i^{th} entry of μ is the sum of $-\mu'_{4-i}$ with an element from the convex hull of $\{ \mu''_1, \mu''_2, \mu''_3 \}$. Moreover, we compute

$$\begin{aligned}
\mu_1 + \mu_2 + \mu_3 &= -\mu'_3 + t_{1,1}\mu''_1 + t_{1,2}\mu''_2 + t_{1,3}\mu''_3 \\
&\qquad - \mu'_2 + t_{2,1}\mu''_1 + t_{2,2}\mu''_2 + t_{2,3}\mu''_3 \\
&\qquad - \mu'_1 + t_{3,1}\mu''_1 + t_{3,2}\mu''_2 + t_{3,3}\mu''_3 && (3.34) \\
&= -(\mu'_1 + \mu'_2 + \mu'_3) \\
&\qquad + \sum_{i=1}^3 \underbrace{(\sum_{j=1}^3 t_{i,j})}_{=1} \mu''_i && (3.35) \\
&= -(\mu'_1 + \mu'_2 + \mu'_3) + (\mu''_1 + \mu''_2 + \mu''_3) && (3.36)
\end{aligned}$$

Indeed, given some i, we see

$$\sum_{j=1}^3 t_{i,j} = \sum_{j=1}^3 \sum_{\substack{w_{1\text{st}} \in S_3 \\ w_{1\text{st}}(j)=i}} t_{w_{1\text{st}}} = 1 \tag{3.37}$$

as each $w_{1^{\text{st}}}$ appears exactly once. □

The above introduced description of μ can be summarized by handy inequalities on the entries μ_i.

Corollary 3.3.2. *Let $\mathcal{U}^{\mu'}$, $\mathcal{U}^{\mu''}$, and $\mathcal{U}^{\mu}$ as introduced above. Then we can estimate for the first three entries of the highest weight μ:*

$$-\mu'_{4-i} + \mu''_3 \leq \mu_i \leq \min\{-\mu'_{4-i} + \mu''_1, -\mu'_3 + \mu''_i\} \tag{3.38}$$

for $i \in [1,3]$.

Proof. We consider (3.32) and recall $\mu''_1 \geq \mu''_2 \geq \mu''_3$. The i^{th} entry of μ is the sum of $-\mu'_{4-i}$ with an element from the convex hull of $\{\mu''_1, \mu''_2, \mu''_3\}$. For the upper bound $-\mu'_3 + \mu''_i$ we refer to [13, Lemma 2.10.]. □

The cohomology of $\mathcal{U}^\mu$ vanishes by Proposition 2.5.2 if the weight $\mu + \rho_G$ is G-singular. Therefore, let us write out explicitly.

Lemma 3.3.3. *$\mu + \rho_G$ is G-singular if and only if one of the following conditions is satisfied (exclusively for a given index i):*

$$\mu_i \in [-n+i, -4+i] \qquad \text{where } i \in [1,3] \tag{3.39}$$
$$\mu_i + \mu_j = -2n - 1 + i + j \qquad \text{where } i < j \in [1,3] \tag{3.40}$$
$$\mu_i \in [-2n+3+i, -n-1+i] \qquad \text{where } i \in [1,3] \tag{3.41}$$

Proof. As the i^{th} entry of $\mu + \rho_G$ takes the value $\mu_i + n + \frac{1}{2} - i$, it is G-singular if and only if one of the following conditions is satisfied:

1. $\mu_i + n + \frac{1}{2} - i = 0$ or equivalently $-\mu_i - n + i = \frac{1}{2}$. However, this case will never happen, as the left-hand side of the second equation is from $\mathbb{Z}$ while its right-hand side is from $\frac{1}{2}\mathbb{Z}$.

2. $\mu_i + n + \frac{1}{2} - i = \mu_j + n + \frac{1}{2} - j$ or equivalently $\mu_i - \mu_j = i - j$ for two distinct indices i and j from $[1, n]$. Without loss of generality, we assume $i < j$. If i and j are both from $[1, 3]$, then we have $i - j < 0$ and it is $\mu_i \geq \mu_j$ or equivalently $0 \leq \mu_i - \mu_j$ by inequalities of Lemma 3.3.1. If i and j are from $[4, n]$, then we see $\mu_i - \mu_j = 0 < i - j$ by the vanishings of μ_4, ..., μ_n in Lemma 3.3.1. Accordingly, we are left with $i \in [1,3]$ and $j \in [4, n]$. The current case, namely the i^{th} entry of $\mu + \rho_G$ coincides with its j^{th} entry, can only appear if and only if $\mu_i \in [-n+i, -4+i]$ where $i \in [1,3]$.

3. $\mu_i + n + \frac{1}{2} - i = -\mu_j - n - \frac{1}{2} + j$ or equivalently $\mu_i + \mu_j = -2n - 1 + i + j$ for two distinct indices i and j from $[1, n]$. In detail, the current case is given if one of the following equations holds: $\mu_i + \mu_j = -2n - 1 + i + j$ where $i < j$ in $[1,3]$ or $\mu_i \in [-2n+3+i, -n-1+i]$ where $i \in [1,3]$.

□

Auxiliary statements where the spinor bundle $\mathcal{S}$ appears in one component.

Lemma 3.3.4. *Let $\mathcal{U}^{\mu'}$ and $\mathcal{U}^{\mu''}$ be as introduced above.*

1. *Let $\mathcal{U}^\mu$ be an irreducible summand from the direct sum decomposition of the tensor product $\mathcal{U}^{\mu'\vee} \otimes \mathcal{U}^{\mu''}$. We assume the following technical condition: If μ contains two entries $\mu_i \geq \mu_j$ with $i < j \in [1,3]$ such that it is the equation $\mu_i + \mu_j = -2n - 1 + i + j$, then we can check at least one of the following statements:*

- *$\mu_i - \mu_j \leq 2n - 5 + i - j$,*
- *there is some i^* satisfying conditions (3.39) or (3.41) respectively, or*
- *$\mu + \omega_n + \rho_G$ is G-singular.*

If $\mathcal{U}^{\mu'}$ is right orthogonal to $\mathcal{U}^{\mu''}$, then $\mathcal{U}^{\mu'}$ is also right orthogonal to $\mathcal{U}^{\mu''} \otimes \mathcal{S}$.

2. *Let $\mathcal{U}^{\mu}$ be an irreducible summand from the direct sum decomposition of the tensor product $\mathcal{U}^{\mu'}(1)^{\vee} \otimes \mathcal{U}^{\mu''}$ and we assume the same technical condition as before.*
If $\mathcal{U}^{\mu'}(1)$ is right orthogonal to $\mathcal{U}^{\mu''}$, then $\mathcal{U}^{\mu'} \otimes \mathcal{S}$ is right orthogonal to $\mathcal{U}^{\mu''}$.

Remark 3.3.5. In many cases we can show already $\mu_i - \mu_j \leq 2n - 5 - i - j$ without the assumption $\mu_i + \mu_j = -2n - 1 + i + j$. Hence, we only mention this requirement if it is necessary for sure.

Proof. 1^{st} *statement.* Due to assumption $0 = \mathrm{Ext}^{\bullet}(\mathcal{U}^{\mu'}, \mathcal{U}^{\mu''}) = \mathrm{H}^{\bullet}(X, \mathcal{U}^{\mu'\vee} \otimes \mathcal{U}^{\mu''})$, any irreducible summand $\mathcal{U}^{\mu}$ appearing in the direct sum decomposition of the tensor product $\mathcal{U}^{\mu'\vee} \otimes \mathcal{U}^{\mu''}$ needs to have vanishing cohomology by Proposition (2.5.2). So, the weight $\mu + \rho_G$ need to be G-singular. This means that the entries of $\mu + \rho_G$ satisfy at least one of the following conditions (see Lemma 3.3.3) and we conclude in each case that $\mu + \omega_n + \rho_G$ is G-singular likewise (see 2.24):

1. $\mu_i \in [-n+i, -4+i]$ where $i \in [1,3]$ – cf. (3.39): We have $\mu_i = -j + i$ for some $j \in [4, n]$. Hence, the i^{th} entry and the j^{th} entry of $\mu + \omega_n + \rho_G$ are the same, as we have $(\mu_i + \frac{1}{2}) + n + \frac{1}{2} - i = \frac{1}{2} + n + \frac{1}{2} - j$.
2. $\mu_i + \mu_j = -2n - 1 + i + j$ where $i < j \in [1,3]$ – cf. (3.40): It is $\mu_j \leq \mu_i$ in $\mathbb{Z}$ by Lemma 3.3.1. So, we deduce from $\mu_j - j < \mu_i - i = -\mu_j - 2n - 1 + j$ the inequalities $\mu_j < -n - \frac{1}{2} + j$ as well as $-n - \frac{1}{2} + i < \mu_i$. As μ_i and μ_j lie in $\mathbb{Z}$, we even have $\mu_j \leq -n - 1 + j$ and $-n + i \leq \mu_i$. Now, let us consider various subcases with respect to μ_j and μ_i respectively:
   (a) $\mu_j \in [-2n + 2 + j, -n - 1 + j]$ or equivalently $(\mu_j + \frac{1}{2}) + n + \frac{1}{2} - j \in [-n + 3, 0]$: $\mu + \omega_n + \rho_G$ is G-singular as its j^{th} entry lies in the range from $-n+3$ to -1, is zero, or lies in the range from 1 to $n - 3$ (see (2.24)). In fact, we recall from Lemma 3.3.1 that the entries μ_4 to μ_n are zero and let us mention that consequently the 4^{th} entry of $\mu + \omega_n + \rho_G$ is $\frac{1}{2} + n + \frac{1}{2} - 4 = n - 3$, ..., and the n^{th} one is $\frac{1}{2} + n + \frac{1}{2} - n = 1$.
   (b) $\mu_i \in [-n + i, -4 + i]$ or equivalently $(\mu_i + \frac{1}{2}) + n + \frac{1}{2} - i \in [1, n-3]$: We argue as in the previous subcase and see likewise that the weight $\mu + \omega_n + \rho_G$ is G-singular.
   (c) $\mu_j \leq -2n + 1 + j$ and $-3 + i \leq \mu_i$: As we have $2n - 4 + i - j \leq \mu_i - \mu_j$, we refer to our assumed technical condition: If we are not already done with $\mu + \omega_n + \rho_G$, then we either have a contradiction to the estimation $\mu_i - \mu_j \leq 2n - 5 + i - j$ or repeat the proof with respect to i^*.

3. $\mu_i \in [-2n+3+i, -n-1+i]$ where $i \in [1,3]$ – cf. (3.41): We have $\mu_i = -2n-1+i+j$ for some $j \in [4,n]$ and accordingly see $(\mu_i + \frac{1}{2}) + n + \frac{1}{2} - i = (-2n-1+i+j)+n+1-i = -n+j$. If $j \in [4, n-1]$, then the i^{th} and the $(j+1)^{\text{th}}$ entry of $\mu + \omega_n + \rho_G$ coincide up to a sign, namely $\mu_{j+1} + \frac{1}{2} + n + \frac{1}{2} - (j+1) = n - j$. Otherwise if j is n, then the i^{th} entry of $\mu + \omega_n + \rho_G$ is zero.

We summarize all this by the following statement: If $\mathcal{U}^{\mu'\vee} \otimes \mathcal{U}^{\mu''}$ decomposes as the direct sum $\bigoplus_\mu \mathcal{U}^\mu$ where all irreducible summands $\mathcal{U}^\mu$ have no cohomology, then $\mathcal{U}^{\mu'\vee} \otimes \mathcal{U}^{\mu''} \otimes \mathcal{S}$ decomposes likewise as $\bigoplus_\mu \mathcal{U}^\mu \otimes \mathcal{S} = \bigoplus_\mu \mathcal{U}^{\mu+\omega_n}$ and again without any cohomology. Finally, we see for the first statement

$$\begin{aligned} \operatorname{Ext}^\bullet(\mathcal{U}^{\mu'}, \mathcal{U}^{\mu''} \otimes \mathcal{S}) &= \mathrm{H}^\bullet(X, \mathcal{U}^{\mu'\vee} \otimes \mathcal{U}^{\mu''} \otimes \mathcal{S}) \\ &= \bigoplus_\mu \mathrm{H}^\bullet(X, \mathcal{U}^\mu \otimes \mathcal{S}) \\ &= \bigoplus_\mu \mathrm{H}^\bullet(X, \mathcal{U}^{\mu+\omega_n}) \\ &= 0 \end{aligned} \tag{3.42}$$

2^{nd} *statement.* As we have $\mathcal{S}^\vee = \mathcal{S}(-1)$ due to Lemma 3.1.1, we start with

$$\begin{aligned} \operatorname{Ext}^\bullet(\mathcal{U}^{\mu'} \otimes \mathcal{S}, \mathcal{U}^{\mu''}) &= \operatorname{Ext}^\bullet(\mathcal{U}^{\mu'}(1) \otimes \mathcal{S}^\vee, \mathcal{U}^{\mu''}) \\ &= \operatorname{Ext}^\bullet(\mathcal{U}^{\mu'}(1), \mathcal{U}^{\mu''} \otimes \mathcal{S}) \end{aligned} \tag{3.43}$$

and then our assumption on $\operatorname{Ext}^\bullet(\mathcal{U}^{\mu'}(1), \mathcal{U}^{\mu''}) = 0$ implies $\operatorname{Ext}^\bullet(\mathcal{U}^{\mu'}(1), \mathcal{U}^{\mu''} \otimes \mathcal{S}) = 0$ by the previous statement. □

Lemma 3.3.6. *Let $\mathcal{U}^{\mu'}$ and $\mathcal{U}^{\mu''}$ be as in the previous Lemma 3.3.1.*

1. *If $-(\mu'_1 + \mu'_2 + \mu'_3) + (\mu''_1 + \mu''_2 + \mu''_3) < 0$ and $-\mu'_3 + \mu''_i \leq -3 + i$ for some index $i \in [1,3]$, then the Ext-space $\operatorname{Ext}^p(\mathcal{U}^{\mu'}, \mathcal{U}^{\mu''} \otimes \mathcal{S})$ vanishes at least for any $p \in [0, n-4]$.*

2. *If $-(\mu'_1 + \mu'_2 + \mu'_3) + (\mu''_1 + \mu''_2 + \mu''_3) < 3$ and $-\mu'_3 + \mu''_i \leq -2 + i$ for some index $i \in [1,3]$, then the Ext-space $\operatorname{Ext}^p(\mathcal{U}^{\mu'} \otimes \mathcal{S}, \mathcal{U}^{\mu''})$ vanishes at least for any $p \in [0, n-4]$.*

Proof. 1^{st} *statement.* As we want to compute partially the Ext-space $\operatorname{Ext}^\bullet(\mathcal{U}^{\mu'}, \mathcal{U}^{\mu''} \otimes \mathcal{S})$, we start with

$$\begin{aligned} \operatorname{Ext}^\bullet(\mathcal{U}^{\mu'}, \mathcal{U}^{\mu''} \otimes \mathcal{S}) &= \mathrm{H}^\bullet(X, \mathcal{U}^{\mu'\vee} \otimes \mathcal{U}^{\mu''} \otimes \mathcal{S}) \\ &= \bigoplus_\mu \mathrm{H}^\bullet(X, \mathcal{U}^\mu \otimes \mathcal{S}) \\ &= \bigoplus_\mu \mathrm{H}^\bullet(X, \mathcal{U}^{\mu+\omega_n}) \end{aligned} \tag{3.44}$$

where the index μ runs through the set of highest weights μ such that their corresponding vector bundles $\mathcal{U}^\mu$ appear as irreducible summands in the direct sum decomposition of the tensor product $\mathcal{U}^{\mu'\vee} \otimes \mathcal{U}^{\mu''}$. Due to Lemma 3.3.1, we have $\mu_1 \geq \mu_2 \geq \mu_3$ in $\mathbb{Z}$ and the remaining entries vanish, namely from $\mu_4 = \cdots = \mu_n = 0$. Moreover, we estimate the i^{th} entry μ_i for $i \in [1,3]$ as $\mu_i \leq -\mu'_3 + \mu''_i$ by Corollary 3.3.2. Next, we distinguish the following cases with respect to $\mu_i \leq -\mu'_3 + \mu''_i \leq -3 + i$:

1. $\mu_i = -3 + i$:
   (a) $i \in \{1,2\}$: It is $\mu_3 \leq \mu_i = -3 + i \leq -1$ and therefore we consider the cases with respect to $\mu_3 \leq -1$.
   (b) $i = 3$: As we have $\mu_1 \geq \mu_2 \geq \mu_3 = -3 + 3 = 0$, we observe $\mu_1 + \mu_2 + \mu_3 \geq 0$. However this contradicts to the fact $\mu_1 + \mu_2 + \mu_3 = -(\mu'_1 + \mu'_2 + \mu'_3) + (\mu''_1 + \mu''_2 + \mu''_3) < 0$.
2. $\mu_i \in [-2n + 2 + i, -4 + i]$ or equivalently $(\mu_i + \frac{1}{2}) + n + \frac{1}{2} - i \in [-n + 3, n - 3]$: $(\mu_i + \frac{1}{2}) + n + \frac{1}{2} - i$ coincides with $\frac{1}{2} + n + \frac{1}{2} - j$ where j is from $[4,n]$, is zero, or equals $-\frac{1}{2} - n - \frac{1}{2} + j$ where j is from $[4,n]$. Thus, $\mu + \omega_n + \rho_{\mathrm{G}}$ is G-singular by (2.24).
3. $\mu_i \leq -2n + 1 + i$ or equivalently $(\mu_i + \frac{1}{2}) + n + \frac{1}{2} - i \leq -n + 2$: The i^{th} entry of $\mu + \omega_n + \rho_{\mathrm{G}}$ is smaller than the 4^{th} one which is $n - 3$, ..., and the n^{th} one which is 1. Hence, the unique element $w \in \mathrm{W_G}$ such that $w(\mu + \omega_n + \rho_{\mathrm{G}})$ lies in $\mathbf{P}^+_{\mathrm{G}}$ has at least length $n - 3 \leq \ell_{\mathrm{G}}(w)$.

We summarize the previous cases: Either $\mathcal{U}^{\mu+\omega_n}$ has no cohomology at all or otherwise at least in degree $n - 3$. Thus, the Ext-space $\mathrm{Ext}^p(\mathcal{U}^{\mu'}, \mathcal{U}^{\mu'} \otimes \mathcal{S})$ in (3.44) vanishes for any $p < n - 3$.

2^{nd} *statement.* We have $\mathcal{S}^\vee = \mathcal{S}(-1)$ due to Lemma 3.1.1 and therefore we write

$$\begin{aligned}\mathrm{Ext}^\bullet(\mathcal{U}^{\mu'} \otimes \mathcal{S}, \mathcal{U}^{\mu''}) &= \mathrm{Ext}^\bullet(\mathcal{U}^{\mu'}(1) \otimes \mathcal{S}(-1), \mathcal{U}^{\mu''}) \\ &= \mathrm{Ext}^\bullet(\mathcal{U}^{\mu'}(1) \otimes \mathcal{S}^\vee, \mathcal{U}^{\mu''}) \\ &= \mathrm{Ext}^\bullet(\mathcal{U}^{\mu'}(1), \mathcal{U}^{\mu''} \otimes \mathcal{S}).\end{aligned} \tag{3.45}$$

Due to our assumption, we deduce $\mathrm{Ext}^\bullet(\mathcal{U}^{\mu'}(1), \mathcal{U}^{\mu''} \otimes \mathcal{S}) = 0$. □

Auxiliary statements where the spinor bundle $\mathcal{S}$ appears in both components. Given two vector bundles $\mathcal{U}^{\mu'}$ and $\mathcal{U}^{\mu''}$ as introduced in the beginning of this section. Then we observe the following: First

$$\begin{aligned}\mathrm{Ext}^\bullet(\mathcal{U}^{\mu'} \otimes \mathcal{S}, \mathcal{U}^{\mu''} \otimes \mathcal{S}) &= \mathrm{H}^\bullet(\mathrm{X}, (\mathcal{U}^{\mu'} \otimes \mathcal{S})^\vee \otimes \mathcal{U}^{\mu''} \otimes \mathcal{S}) \\ &= \mathrm{H}^\bullet(\mathrm{X}, \mathcal{U}^{\mu'\vee} \otimes \mathcal{U}^{\mu''} \otimes \mathcal{S}(-1) \otimes \mathcal{S}).\end{aligned} \tag{3.46}$$

Second, since we have $\mathcal{S}(-1) \otimes \mathcal{S} = \bigoplus_{l=3}^{n} \mathcal{U}^{\nu_l}$ where $\mathcal{U}^{\nu_l}$ is the irreducible G-equivariant vector bundle with highest weight $\nu_l = \sum_{i=4}^{l} \mathrm{e}_i$, it is

$$\mathrm{Ext}^\bullet(\mathcal{U}^{\mu'} \otimes \mathcal{S}, \mathcal{U}^{\mu''} \otimes \mathcal{S}) = \bigoplus_{l=3}^{n} \mathrm{Ext}^\bullet(\mathcal{U}^{\mu'}, \mathcal{U}^{\mu''+\nu_l}). \tag{3.47}$$

This means that the highest weights ν_l are supported over the second Levi part and are of the form $\nu_l = (0,0,0,1,\cdots,1,0,\cdots,0)$. For this purpose, let l be an integer from $[3,n]$ and we consider those Ext-spaces where the second component is tensored with those $\mathcal{U}^{\nu_l}$.

Lemma 3.3.7. *Let $\mathcal{U}^{\mu'}, \mathcal{U}^{\mu''}, l, \mathcal{U}^{\nu_l}$, and $\mathcal{U}^{\mu}$ as introduced above.*
We assume the following two technical conditions:

1. *If μ contains an entry $\mu_i = -l + i$ where $i \in [1,3]$, then we can check that $\mu + \nu_l + \rho_\mathrm{G}$ is* G*-singular.*

2. *If μ contains an entry $\mu_i = 2n - 1 + i + l$ where $i \in [1,3]$, then we can check likewise that $\mu + \nu_l + \rho_\mathrm{G}$ is* G*-singular.*

If $\mathcal{U}^{\mu'}$ is right orthogonal to $\mathcal{U}^{\mu''}$, then $\mathcal{U}^{\mu'}$ is also right orthogonal to $\mathcal{U}^{\mu''} \otimes \mathcal{U}^{\nu_l}$.

Proof. The right orthogonal relation between $\mathcal{U}^{\mu'}$ and $\mathcal{U}^{\mu''}$ – i.e. the vanishing $0 = \mathrm{Ext}^\bullet(\mathcal{U}^{\mu'}, \mathcal{U}^{\mu''}) = \mathrm{H}^\bullet(X, \mathcal{U}^{\mu'\vee} \otimes \mathcal{U}^{\mu''})$ – implies that any irreducible summand $\mathcal{U}^{\mu}$ appearing in the direct sum decomposition of the tensor product $\mathcal{U}^{\mu'\vee} \otimes \mathcal{U}^{\mu''}$ must have vanishing cohomology by Proposition (2.5.2). Consequently, the corresponding weight $\mu + \rho_\mathrm{G}$ need to be G-singular. This means that the entries of $\mu + \rho_\mathrm{G}$ satisfy at least one of the following conditions (see Lemma 3.3.3) and we conclude in each case that $\mu + \nu_l + \rho_\mathrm{G}$ is G-singular likewise (see 2.24):

1. $\mu_i \in [-n+i, -4+i]$ where $i \in [1,3]$ – cf. (3.39): We have $\mu_i = -j + i$ for some $j \in [4,n]$. So, the i^{th} entry of $\mu + \nu_l + \rho_\mathrm{G}$ is calculated as $(-j+i) + 0 + n + \frac{1}{2} - i = n + \frac{1}{2} - j$. If $j \in [4, l-1]$, then the i^{th} entry of $\mu + \nu_l + \rho_\mathrm{G}$ coincides with the $j+1^{\text{th}}$ one, as we have $1 + n + \frac{1}{2} - (j+1) = n + \frac{1}{2} - j$. If $j = l$, we apply our first assumed technical condition. Finally, if $j \in [l+1, n]$, then the i^{th} entry of $\mu + \nu_l + \rho_\mathrm{G}$ is clearly equal to the j^{th} one.

2. $\mu_i + \mu_j = -2n - 1 + i + j$ where $i < j \in [1,3]$ – cf. (3.40): As we have $\mu_i + n + \frac{1}{2} - i = -(\mu_j + n + \frac{1}{2} - j)$, the i^{th} and the j^{th} entry of $\mu + \nu_l + \rho_\mathrm{G}$ coincide up to a sign. In fact, both the i^{th} and the j^{th} entry of ν_l are zero.

3. $\mu_i \in [-2n+3+i, -n-1+i]$ where $i \in [1,3]$ – cf. (3.41): We have $\mu_i = -2n - 1 + i + j$ for some $j \in [4,n]$ and therefore see $\mu_i + 0 + n + \frac{1}{2} - i = (-2n-1+i+j) + n + \frac{1}{2} - i = -n - \frac{1}{2} + j$. If $j \in [4, l-1]$, then the i^{th} entry of $\mu + \nu_l + \rho_\mathrm{G}$ coincides with the $j+1^{\text{th}}$ one up to a sign. as we have $1 + n + \frac{1}{2} - (j+1) = n + \frac{1}{2} - j$. If $j = l$, we apply our second assumed technical condition

and see that $\mu + \nu_l + \rho_G$ is G-singular. Finally if $j \in [l+1, n]$, then the i^{th} entry of $\mu + \nu_l + \rho_G$ is equal to the j^{th} one up to a sign.

□

Lemma 3.3.8. *Let $\mathcal{U}^{\mu'}, \mathcal{U}^{\mu''}, l, \mathcal{U}^{\nu_l}$, and $\mathcal{U}^{\mu}$ as introduced above.*
If $-\mu'_3 + \mu''_i < -3 + i$ for some index $i \in [1,3]$, then the Ext*-space* $\operatorname{Ext}^p(\mathcal{U}^{\mu'}, \mathcal{U}^{\mu''} \otimes \mathcal{U}^{\nu_l})$ *vanishes for any*

$$p \in \begin{cases} [0, 2l-5] & , \textit{if } -(\mu'_1 + \mu'_2 + \mu'_3) + (\mu''_1 + \mu''_2 + \mu''_3) \leq -l + 2 \\ [0, l-4] & , \textit{else} \end{cases}. \tag{3.48}$$

Proof. First, we want to compute partially the Ext-space

$$\operatorname{Ext}^\bullet(\mathcal{U}^{\mu'}, \mathcal{U}^{\mu''} \otimes \mathcal{U}^{\nu_l}) = \mathrm{H}^\bullet(\mathrm{X}, \mathcal{U}^{\mu' \vee} \otimes \mathcal{U}^{\mu''} \otimes \mathcal{U}^{\nu_l}) = \mathrm{H}^\bullet(\mathrm{X}, \bigoplus_{\mu} \mathcal{U}^{\mu} \otimes \mathcal{U}^{\nu}) \tag{3.49}$$

where μ is the highest weight of those vector bundles $\mathcal{U}^{\mu}$ appearing as irreducible summands in the direct sum decomposition of the tensor product $\mathcal{U}^{\mu' \vee} \otimes \mathcal{U}^{\mu''}$. In addition, we have the usual constraints on μ_i as before (see Lemma 3.3.1 and Corollary 3.3.2). Next, we consider the weight $\mu + \nu_l + \rho_G$. Its entries are given by

$$\mu_i + \nu_{l,i} + n + \frac{1}{2} - i = \begin{cases} \mu_i + n + \frac{1}{2} - i & , \text{if } i \in [1,3] \\ n + \frac{3}{2} - i & , \text{if } i \in [4, l] \\ n + \frac{1}{2} - i & , \text{if } i \in [l+1, n] \end{cases}. \tag{3.50}$$

We distinguish the following cases with respect to $\mu_i \leq -\mu'_3 + \mu''_i \leq -3 + i$:

1. $\mu_i \in [-l+1+i, -3+i]$ or equivalently $\mu_i + n + \frac{1}{2} - i \in [n + \frac{3}{2} - l, n - \frac{5}{2}]$: The i^{th} entry of $\mu + \nu_l + \rho_G$ coincides with the j^{th} one where j is from $[4, l]$. Hence, $\mu + \nu_l + \rho_G$ is G-singular by (2.24).

2. $\mu_i = -l + i$ or equivalently $\mu_i + n + \frac{1}{2} - i = n + \frac{1}{2} - l$: Either $\mu + \nu_l + \rho_G$ is G-singular or it is G-regular. In the later case, there is a unique element $w \in \mathrm{W}_G$ mapping $\mu + \nu_l + \rho_G$ to $\mathbf{P}_G^+$ and we estimate its length as following:

   (a) $i = 1$: We have $n + \frac{3}{2} - l - 1 = \mu_1 + n - \frac{1}{2} \geq \mu_2 + n - \frac{3}{2} \geq \mu_3 + n - \frac{5}{2}$ and accordingly $\ell_G(w) \geq 3(l-1)$ as we need to permute at least the first three entries of $\mu + \nu_l + \rho_G$ to $\mathbf{P}_G^+$ at behind the 4^{th}, ..., and the l^{th} one.

   (b) $i = 2$: Then $\mu_1 + n - \frac{1}{2} \geq n + \frac{3}{2} - l - 1 = \mu_2 + n - \frac{3}{2} \geq \mu_3 + n - \frac{5}{2}$: We conclude $\ell_G(w) \geq 2(l-2)$ analogously as in the previous subcase.

   (c) $i = 3$: In general, it is $\ell_G(w) \geq l - 3$. However, if we have $\mu_1 + \mu_2 + \mu_3 = -(\mu'_1 + \mu'_2 + \mu'_3) + (\mu''_1 + \mu''_2 + \mu''_3) \leq -l + 2$, then this case can not occur. Indeed, it is $\mu_1 \geq \mu_2 \geq \mu_3 = -l + 3$. If μ_1 would lie in the range

$[-l+3,-2]$ or μ_2 in the range $[-l+3,-1]$, then $\mu+\nu_l+\rho_G$ would be G-singular. Hence, we need to have $\mu_1 \geq \mu_2 \geq 0$ and therefore we compute $\mu_1+\mu_2+\mu_3 \geq -l+3$ which contradicts to our previous assumption.

We summarize the previous cases: Either $\mathcal{U}^{\mu+\nu_l}$ has no cohomology at all or otherwise at least in degree

$$\begin{cases} 2l-4 & \text{, if } \mu_1+\mu_2+\mu_3 \leq -l+1 \\ l-3 & \text{, else} \end{cases} . \tag{3.51}$$

$\square$

Chapter 4

The tautological subcollection

General construction. The starting block $C_0^{(\mathcal{U})}$ of the tautological subcollection $C^{(\mathcal{U})}$ consists of irreducible G-equivariant vector bundles $\mathcal{U}^\lambda$ with highest weights λ which are supported over the first component of the Levi part and which satisfy specific constraints. In detail, we start with weights $\lambda = (\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n) \in \mathbf{P}_G$ such that $\lambda_1 \geq \lambda_2 \geq \lambda_3 = 0$ and $\lambda_4 = \cdots \lambda_n = 0$. We order these weights lexicographically. Then we require additional conditions to form the following to building blocks $C_0^{(\mathcal{U})} = C_0^{(\mathcal{U},1)} \cup C_0^{(\mathcal{U},2)}$.

4.1 The first part

Construction. For the starting block $C_0^{(\mathcal{U},1)}$ of the first part $C^{(\mathcal{U},1)}$ of the tautological subcollection $C^{(\mathcal{U})}$, we take the subset of those weights λ where we have $n-3 \geq \lambda_1$. Then $C_0^{(\mathcal{U},1)}$ is the set $\{\mathcal{U}^\lambda\}$ with the induced order. Each row has length $w_{max} = 2n-3$. This means, the collection $C^{(\mathcal{U},1)}$ has support partition $(h_0, \cdots, h_{2n-4})$ with

$$h_x = \frac{1}{2}n^2 - \frac{3}{2}n + 1 \tag{4.1}$$

for any $x \in [0, 2n-4]$. Accordingly, the collection $C^{(\mathcal{U},1)}$ consists of $n^3 - \frac{9}{2}n^2 + \frac{13}{2}n - 3$ objects.

Example 4.1.1. For $\mathrm{OGr}(3,V)$ with $n=7$, we write completely

$$C^{(\mathcal{U},1)} = \begin{pmatrix} \mathcal{U}^{4\omega_2} & \mathcal{U}^{4\omega_2}(1) & \cdots & \mathcal{U}^{4\omega_2}(10) \\ \mathcal{U}^{\omega_1+3\omega_2} & \mathcal{U}^{\omega_1+3\omega_2}(1) & \cdots & \mathcal{U}^{\omega_1+3\omega_2}(10) \\ \mathcal{U}^{2\omega_1+2\omega_2} & \mathcal{U}^{2\omega_1+2\omega_2}(1) & \cdots & \mathcal{U}^{2\omega_1+2\omega_2}(10) \\ \mathcal{U}^{3\omega_1+\omega_2} & \mathcal{U}^{3\omega_1+\omega_2}(1) & \cdots & \mathcal{U}^{3\omega_1+\omega_2}(10) \\ \mathcal{U}^{4\omega_1} & \mathcal{U}^{4\omega_1}(1) & \cdots & \mathcal{U}^{4\omega_1}(10) \\ \mathcal{U}^{3\omega_2} & \mathcal{U}^{3\omega_2}(1) & \cdots & \mathcal{U}^{3\omega_2}(10) \\ \mathcal{U}^{\omega_1+2\omega_2} & \mathcal{U}^{\omega_1+2\omega_2}(1) & \cdots & \mathcal{U}^{\omega_1+2\omega_2}(10) \\ \mathcal{U}^{2\omega_1+\omega_2} & \mathcal{U}^{2\omega_1+\omega_2}(1) & \cdots & \mathcal{U}^{2\omega_1+\omega_2}(10) \\ \mathcal{U}^{3\omega_1} & \mathcal{U}^{3\omega_1}(1) & \cdots & \mathcal{U}^{3\omega_1}(10) \\ \mathcal{U}^{2\omega_2} & \mathcal{U}^{2\omega_2}(1) & \cdots & \mathcal{U}^{2\omega_2}(10) \\ \mathcal{U}^{\omega_1+\omega_2} & \mathcal{U}^{\omega_1+\omega_2}(1) & \cdots & \mathcal{U}^{\omega_1+\omega_2}(10) \\ \mathcal{U}^{2\omega_1} & \mathcal{U}^{2\omega_1}(1) & \cdots & \mathcal{U}^{2\omega_1}(10) \\ \mathcal{U}^{\omega_2} & \mathcal{U}^{\omega_2}(1) & \cdots & \mathcal{U}^{\omega_2}(10) \\ \mathcal{U}^{\omega_1} & \mathcal{U}^{\omega_1}(1) & \cdots & \mathcal{U}^{\omega_1}(10) \\ \mathcal{O}_X & \mathcal{O}_X(1) & \cdots & \mathcal{O}_X(10) \end{pmatrix}. \tag{4.2}$$

Consequently, the Lefschetz structure is determined by the starting block, i.e. the first column of (4.2), and the support partition $(15,15,15,15,15,15,15,15,15,15,15)$.

Let $\mathcal{U}^\lambda$ be an element from the starting block $C_0^{(\mathcal{U})}$ and we write its highest weight either as $\lambda = (\lambda_1, \lambda_2, 0, 0, \cdots, 0)$ or $\lambda = c_1\omega_1 + c_2\omega_2$. Then $\mathcal{U}^\lambda$ has rank

$$\begin{aligned} \mathrm{rk}(\mathcal{U}^\lambda) &= \frac{1}{2}(\lambda_1-\lambda_2+1)(\lambda_2+1)(\lambda_1+2) \\ &= \frac{1}{2}(c_1+1)(c_2+1)(c_1+c_2+2) \end{aligned} \tag{4.3}$$

due to [15, Example 10.23.] and we conjecture it has determinant $\det(\mathcal{U}^\lambda) = \mathcal{O}(t)$ with

$$\begin{aligned} t &= \frac{1}{6}(\lambda_1-\lambda_2+1)(\lambda_1+\lambda_2)(\lambda_2+1)(\lambda_1+2) \\ &= \frac{1}{6}(c_1+1)(c_1+2c_2)(c_2+1)(c_1+c_2+2). \end{aligned} \tag{4.4}$$

Proving exceptionality. Throughout this paragraph, let $\mathcal{E}_{i'}^{(\mathcal{U},1)} = \mathcal{U}^{\lambda'}(x')$ as well as $\mathcal{E}_{i''}^{(\mathcal{U},1)} = \mathcal{U}^{\lambda''}(x'')$ be two objects from the first part $C^{(\mathcal{U},1)}$ of the tautological subcollection $C^{(\mathcal{U})}$ such that $i'' \leq i'$ – i.e. we have either the case $i' = i''$ which is

$$x' = x'' \text{ and } \lambda' = \lambda'' \tag{4.5}$$

or the case $i'' < i'$ which means

$$x' = x'' \text{ and } \lambda'' < \lambda' \text{ lexicographically, or } x'' < x'. \tag{4.6}$$

We write their highest weights as $\mu' = \lambda' + x'\omega_3$ and $\mu'' = \lambda'' + x''\omega_3$ respectively. Furthermore, let $\mathcal{U}^\mu$ be an irreducible component in the direct sum decomposition of the tensor product

$$\mathcal{U}^{\mu'\vee} \otimes \mathcal{U}^{\mu''} = \mathcal{U}^{-w_{\mathrm{L},\infty}\mu'} \otimes \mathcal{U}^{\mu''} \tag{4.7}$$

Proposition 4.1.2. *The first part $C^{(\mathcal{U},1)}$ of the tautological subcollection is exceptional.*

Proof. To compute the Ext-space from $\mathcal{E}_{i'}^{(\mathcal{U},1)}$ into $\mathcal{E}_{i''}^{(\mathcal{U},1)}$, we write

$$\begin{aligned} \mathrm{Ext}^\bullet(\mathcal{E}_{i'}^{(\mathcal{U},1)}, \mathcal{E}_{i''}^{(\mathcal{U},1)}) &= \mathrm{H}^\bullet(X, \mathcal{U}^{-w_{\mathrm{L},\infty}\mu'} \otimes \mathcal{U}^{\mu''}) \\ &= \bigoplus_\mu M_\mu^{(-w_{\mathrm{L},\infty}\mu',\mu'')} \otimes \mathrm{H}^\bullet(X, \mathcal{U}^\mu). \end{aligned} \tag{4.8}$$

Let us recall from (2.28) that $M_\mu^{(-w_{\mathrm{L},\infty}\mu',\mu'')}$ is the multiplicity of $\mathrm{V}_\mathrm{L}^\mu$ in the direct sum decomposition of the tensor product $\mathrm{V}_\mathrm{L}^{w_{\mathrm{L},\infty}\mu'} \otimes \mathrm{V}_\mathrm{L}^{\mu''}$. We apply Lemma 4.1.3 to describe the weights μ appearing in the tensor product (4.7) and to see $M_\mu^{(-w_{\mathrm{L},\infty}\mu',\mu'')} = \Bbbk$ if μ is trivial. In Lemma 4.1.4 we compute the cohomology of the components $\mathcal{U}^\mu$ which is $\Bbbk$ in degree 0 if μ is trivial and vanishes otherwise. Consequently, the case (4.5) covers the exceptionality of our objects $\mathcal{U}^\lambda(x)$ in the first part $C^{(\mathcal{U},1)}$ of the tautological subcollection $C^{(\mathcal{U})}$ as well as the case (4.6) ensures the right orthogonal relations. □

Lemma 4.1.3. *We describe the weights μ appearing in the direct sum decomposition of the tensor product* (4.7) *with respect to the above cases:*

1. *Case* (4.5)*: It is either trivial and the corresponding vector bundle $\mathcal{O}$ has multiplicity* 1*, or (exclusively) the entries satisfy the inequalities*

$$\max\{\mu_2, 1\} \le \mu_1 \le n-3, \tag{4.9}$$
$$\max\{\mu_3, \lceil -\tfrac{1}{2}n + \tfrac{3}{2} \rceil\} \le \mu_2 \le \min\{\mu_1, \lfloor \tfrac{1}{2}n - \tfrac{3}{2} \rfloor\}, \tag{4.10}$$
$$-n+3 \le \mu_3 \le \min\{\mu_2, -1\}, \tag{4.11}$$
$$\mu_i = 0 \qquad \textit{for } i \in [4,n], \tag{4.12}$$

and

$$2 \le \mu_1 - \mu_3 \le 2n-6. \tag{4.13}$$

2. *Case* (4.6)*: We have the inequalities*

$$\max\{\mu_2, -2n+4\} \le \mu_1 \le n-3, \tag{4.14}$$
$$\max\{\mu_3, -3n+7\} \le \mu_2 \le \min\{\mu_1, n-4\}, \tag{4.15}$$
$$-3n+7 \le \mu_3 \le \min\{\mu_2, -1\}, \tag{4.16}$$
$$\mu_i = 0 \qquad \textit{for } i \in [4,n], \tag{4.17}$$

and

$$0 \leq \mu_1 - \mu_3 \leq 2n - 6. \tag{4.18}$$

Proof. First general description of μ. We recall from the Lemma 3.3.1 and Corollary 3.3.2 the following:

$$\mu_3 \leq \mu_2 \leq \mu_1, \tag{4.19}$$

$$-x' + x'' = -\mu_3' + \mu_3'' \leq \mu_1 \leq -\mu_3' + \mu_1'' = \lambda_1'' - x' + x'', \tag{4.20}$$

$$\begin{aligned} -\lambda_2' - x' + x'' = -\mu_2' + \mu_3'' \leq \mu_2 &\leq \min\{-\mu_2' + \mu_1'', -\mu_3' + \mu_2''\} \\ &= \min\left\{ \begin{array}{l} -\lambda_2' + \lambda_1'' - x' + x'' \\ \lambda_2'' - x' + x'' \end{array} \right\}, \end{aligned} \tag{4.21}$$

$$\begin{aligned} -\lambda_1' - x' + x'' = -\mu_1' + \mu_3'' \leq \mu_3 &\leq \min\{-\mu_1' + \mu_1'', -\mu_3' + \mu_3''\} \\ &= \min\left\{ \begin{array}{l} -\lambda_1' + \lambda_1'' - x' + x'' \\ -x' + x'' \end{array} \right\}, \end{aligned} \tag{4.22}$$

$$\mu_i = 0 \qquad \text{for } i \in [4, n]. \tag{4.23}$$

Due to the above bounds of μ_i and $-\mu_j$, we observe $0 \leq \mu_i - \mu_j \leq (-\mu_{4-i}' + \mu_1'') + (\mu_{4-j}' - \mu_3'') = (\mu_{4-j}' - \mu_{4-i}') + (\mu_1'' - \mu_3'') = \lambda_{4-j}' - \lambda_{4-i}' + \lambda_1'' \leq 2n - 6$.

Case (4.5). As we have $x' = x''$ and $\lambda' = \lambda''$, this gives us clearly $\mu' = \mu''$.

If μ is trivial, we compute

$$\begin{aligned} M_0^{-w_{L,\infty}\mu',\mu'} &= \dim(\mathrm{Hom}(\mathbf{K}, \mathrm{V}_{\mathrm{L}}^{\mu'\vee} \otimes \mathrm{V}_{\mathrm{L}}^{\mu'})) \\ &= \dim(\mathrm{Hom}(\mathrm{V}_{\mathrm{L}}^{\mu'}, \mathrm{V}_{\mathrm{L}}^{\mu'})) = 1. \end{aligned} \tag{4.24}$$

Indeed, as $\mathrm{V}_{\mathrm{L}}^{\mu'}$ is irreducible, we have $\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathrm{X})}(\mathcal{U}^{\mu'}, \mathcal{U}^{\mu'})^{\mathrm{G}} = \mathrm{Hom}_{\mathrm{Rep}(\mathrm{L})}(\mathrm{V}_{\mathrm{L}}^{\mu'}, \mathrm{V}_{\mathrm{L}}^{\mu'}) = \mathbf{K}$ by Schur's Lemma.

Otherwise, we assume that μ is non-trivial and check the following inequalities:

(4.9)**:** We estimate $0 \leq \mu_1 \leq \lambda_1'' \leq n - 3$. If we assume that μ_1 equals zero, then we have both $0 = \mu_1 \geq \mu_2 \geq \mu_3$ as well as $\mu_1 + \mu_2 + \mu_3 = 0$ and consequently we see $\mu_2 = \mu_3 = 0$. However, this contradicts with our assumption that μ is non-trivial. Thus, μ_1 needs to lie in the range from 1 to $n - 3$.

(4.11)**:** As we have $-n + 3 \leq \mu_3 \leq 0$, μ_3 lies between $-n + 3$ and 0. If we assume that μ_3 equals zero, then we have $\mu_1 \geq \mu_2 \geq \mu_3 = 0$ as well as $\mu_1 + \mu_2 + \mu_3 = 0$ and therefore it is also $\mu_1 = \mu_2 = 0$. However, this contradicts to the assumption that μ is non-trivial. So, we can tighten the upper bound to -1.

(4.10): We refer to the first step introducing a general description of μ.

If we assume $\mu_2 < -\frac{1}{2}n + \frac{3}{2}$, then we have

$$\begin{aligned} 0 &= -(\mu_1' + \mu_2' + \mu_3') + (\mu_1'' + \mu_2'' + \mu_3'') \\ &= \mu_1 + \mu_2 + \mu_3 \\ &< \mu_1 + 2 \cdot (-\frac{1}{2}n + \frac{3}{2}) \\ &= \mu_1 - n + 3 \end{aligned} \tag{4.25}$$

and accordingly $n - 3 < \mu_1$. However, this contradicts (4.9). So, we see $-\frac{1}{2}n + \frac{3}{2} \leq \mu_2$. Since μ_2 is an integer we even get $\lceil -\frac{1}{2}n + \frac{3}{2} \rceil \leq \mu_2$.

We proceed analogously to show $\mu_2 \leq \lfloor \frac{1}{2}n - \frac{3}{2} \rfloor$.

(4.12): Certainly, this is the above vanishing of the entries of μ supported over the second Levi part.

(4.13): We can improve the lower bound up to 2 since we have $1 \leq \mu_1$ by (4.9) as well as $1 \leq -\mu_3$ by (4.11).

Case (4.6). We mention the fact $-x' + x'' \in [-2n + 4, 0]$ and check the following inequalities:

(4.14): It is $-2n + 4 \leq -x' + x'' \leq \mu_1 \leq -x' + x'' + \lambda_1'' \leq n - 3$.

(4.15): It is $-3n + 7 \leq -\lambda_2' - x' + x'' \leq \mu_2 \leq -x' + x'' \leq 0$.

If $x' = x''$ and $\lambda'' < \lambda'$ lexicographically, then we need to have either $\lambda_1'' = \lambda_1' = n - 3$ and $0 \leq \lambda_2' < \lambda_2'$ or it is $\lambda_1'' < \lambda_1' \leq n - 3$ and $0 \leq \lambda_2'$. In both cases, we conclude $-\lambda_2' + \lambda_1'' \leq n - 4$. Otherwise, if $x'' < x'$ or equivalently $-x' + x'' < 0$, then it is $-x' + x'' - \lambda_2' + \lambda_1'' \leq n - 4$ as we have $0 \leq \lambda_2'$ and $\lambda_1'' \leq n - 3$ by construction.

(4.16): It is $-3n + 7 \leq -\lambda_1' - x' + x'' \leq \mu_3 \leq -x' + x'' - \lambda_1' + \lambda_1'' \leq 0$. Now, let us show that we can tighten the upper bound to -1 in any subcase of (4.6).

$x' = x''$ **and** $\lambda'' < \lambda'$ **lexicographically:** We have $-x' + x'' = 0$ and hence we estimate immediately $\mu_3 \leq -\mu_1' + \mu_1'' = -\lambda_1' + \lambda_1''$. If $\lambda_1'' < \lambda_1'$, we have clearly $\mu_3 \leq -1$. If $\lambda_1'' = \lambda_1'$ and $\lambda_2'' < \lambda_2''$, then $\mu_1 + \mu_2 + \mu_3 = -(\mu_1' + \mu_2' + \mu_3') + (\mu_1'' + \mu_2'' + \mu_3'') = -\lambda_2' + \lambda_2'' \leq -1$. Thus, μ_3 need also to be negative; otherwise we would have a contradiction by $0 \leq \mu_1 + \mu_2 + \mu_3$.

$x'' < x'$**:** We write $\mu_3 \leq -x' + x'' \leq -1$.

(4.17): Similar as before in the proof of (4.12), we refer to the vanishing of the entries μ_4, ..., and μ_n.

(4.18): See above at the proof of (4.13).

$\square$

Lemma 4.1.4. *We compute*

$$\mathrm{H}^\bullet(X,\mathcal{U}^\mu) = \begin{cases} \mathbf{K}[0] & \text{, if } \mu = 0 \\ 0 & \text{, else} \end{cases}. \tag{4.26}$$

Proof. *Case* (4.5).

μ is trivial: We take the identity $w = id$ and therefore see $w \cdot \mu = w(\mu + \rho_G) - \rho_G = 0$. Hence, we obtain the claimed result by Proposition 2.5.2.

μ is not trivial: μ_3 need to be in the range from $-n+3$ to -1 by inequality (4.11). Hence, $\mu + \rho_G$ is G-singular by condition (3.39) where $i = 3$.

Case (4.6). We distinguish with respect to μ_3 which ranges from $-3n+7$ to -1 by (4.16).

$\mu_3 \in [-2n+6, -1]$: We apply conditions (3.39) or (3.41) where $i = 3$ and deduce that $\mu + \rho_G$ is G-singular.

$\mu_3 = -2n+5$: It follows $-2n+5 \leq \mu_1 \leq -1$ from (4.18) and accordingly μ_2 is between $-2n+5$ and -1 by (4.15). If $\mu_2 = -1$, we compute $\mu_2 + \mu_3 = -2n+4 = -2n-1+(2+3)$ which is precisely condition (3.40). Otherwise, if $\mu_2 \in [-2n+5, -2]$, we apply conditions (3.39) or (3.41) where $i = 2$.

$\mu_3 = -2n+4$: Similar as before, it follows $-2n+4 \leq \mu_2 \leq \mu_1 \leq -2$ from (4.15) as well as (4.18). If $\mu_1 = -2$, then μ_2 is in $[-2n+4, -2]$. Hence, if μ_2 lies even in $[-2n+5, -2]$, we refer to the computation of the previous subcase; and if $\mu_2 = -2n+4$, then we consider $\mu_1 + \mu_2 = -2n+2 = -2n-1+(1+2)$ which is condition (3.40). Otherwise, if $\mu_1 \in [-2n+4, -3]$, apply conditions (3.39) or (3.41) where $i = 1$.

$\mu_3 \in [-3n+7, -2n+3]$: We have $-2n+4 \leq \mu_1$ by (4.14) as well as $\mu_1 \leq \mu_3 + 2n - 6$ by (4.18). Hence, μ_1 lies in the range from $-2n+4$ to $\mu_3 + 2n - 6 \leq -3$. We apply conditions (3.39) or (3.41) where $i = 1$ again as in the previous subcase.

□

4.2 The second part

Construction. The starting block $C_0^{(\mathcal{U},2)}$ of the second part $C^{(\mathcal{U},2)}$ of the tautological subcollection $C^{(\mathcal{U})}$ is made up of those weights λ where we have $\lfloor \frac{3}{2}n - \frac{9}{2} \rfloor \geq \lambda_1 \geq n-2$ as well as $\lceil \frac{1}{2}n - \frac{3}{2} \rceil \geq \lambda_2 \geq -n+3+\lambda_1$. Then $C_0^{(\mathcal{U},2)}$ is the set $\{\mathcal{U}^\lambda\}$ with the induced order. Each row has length $w_{max} = 2n-3$. This means, the collection $C^{(\mathcal{U},2)}$ has support partition $(h_0, \cdots, h_{2n-4})$ with

$$h_x = \frac{1}{8}n^2 - \frac{3}{8}n + \frac{1}{8}n \cdot (-1)^n - \frac{5}{16} - \frac{11}{16} \cdot (-1)^n \tag{4.27}$$

for any $x \in [0, 2n-4]$. In fact, h_x is the sum of all terms $\lceil \frac{3}{2}n - \frac{7}{2} \rceil - \lambda_1$ while λ_1 runs from $n-2$ to $\lfloor \frac{3}{2}n - \frac{9}{2} \rfloor$. Accordingly, the collection $C^{(\mathcal{U},2)}$ consists of

$$\frac{1}{4}n^3 - \frac{9}{8}n^2 + \frac{1}{4}n^2 \cdot (-1)^n + \frac{1}{2}n - \frac{7}{4}n \cdot (-1)^n + \frac{15}{16} + \frac{33}{16} \cdot (-1)^n \tag{4.28}$$

objects.

Example 4.2.1. For $\mathrm{OGr}(3, V)$ with $n = 7$, we write completely

$$C^{(\mathcal{U},2)} = \begin{pmatrix} \mathcal{U}^{4\omega_1+2\omega_2} & \mathcal{U}^{4\omega_1+2\omega_2}(1) & \cdots & \mathcal{U}^{4\omega_1+2\omega_2}(10) \\ \mathcal{U}^{3\omega_1+2\omega_2} & \mathcal{U}^{3\omega_1+2\omega_2}(1) & \cdots & \mathcal{U}^{3\omega_1+2\omega_2}(10) \\ \mathcal{U}^{4\omega_1+\omega_2} & \mathcal{U}^{4\omega_1+\omega_2}(1) & \cdots & \mathcal{U}^{4\omega_1+\omega_2}(10) \end{pmatrix}. \tag{4.29}$$

Consequently, the Lefschetz structure is determined by the starting block, i.e. the first column of (4.29), and the support partition $(3,3,3,3,3,3,3,3,3,3,3)$.

For the rank of $\mathcal{U}^\lambda$ and its determinant, we refer to equations (4.3) and (4.4) respectively.

Proving exceptionality. We proceed similarly to the first part $C^{(\mathcal{U},1)}$ of the tautological subcollection $C^{(\mathcal{U})}$ in previous section 4.1. Let $\mathcal{E}_{i'}^{(\mathcal{U},2)} = \mathcal{U}^{\lambda'}(x')$ as well as $\mathcal{E}_{i''}^{(\mathcal{U},2)} = \mathcal{U}^{\lambda''}(x'')$ be two objects from the second part $C^{(\mathcal{U},2)}$ of the tautological subcollection $C^{(\mathcal{U})}$ such that $i'' \le i'$. Again, we distinguish the following two cases: Either $i' = i''$ which is

$$x' = x'' \text{ and } \lambda' = \lambda'' \tag{4.30}$$

or $i'' < i'$ which means

$$x' = x'' \text{ and } \lambda'' < \lambda' \text{ lexicographically}, \text{ or } x'' < x'. \tag{4.31}$$

We set $\mu' = \lambda' + x'\omega_3$ and $\mu'' = \lambda'' + x''\omega_3$ respectively. Let $\mathcal{U}^\mu$ be an irreducible component in the direct sum decomposition of the tensor product

$$\mathcal{U}^{\mu'\vee} \otimes \mathcal{U}^{\mu''} = \mathcal{U}^{-w_{\mathrm{L},\infty}\mu'} \otimes \mathcal{U}^{\mu''} \tag{4.32}$$

Proposition 4.2.2. *The second part $C^{(\mathcal{U},2)}$ of the tautological subcollection is exceptional.*

Proof. We start with the direct sum decomposition

$$\begin{aligned} \mathrm{Ext}^\bullet(\mathcal{E}_{i'}^{(\mathcal{U},2)}, \mathcal{E}_{i''}^{(\mathcal{U},2)}) &= \mathrm{H}^\bullet(X, \mathcal{U}^{-w_{\mathrm{L},\infty}\mu'} \otimes \mathcal{U}^{\mu''}) \\ &= \bigoplus_\mu M_\mu^{(-w_{\mathrm{L},\infty}\mu', \mu'')} \otimes \mathrm{H}^\bullet(X, \mathcal{U}^\mu). \end{aligned} \tag{4.33}$$

In a first step, we apply the Lemma 4.2.3 to describe the weights μ appearing in the tensor product (4.32); and then in a second step, we deduce from Lemma 4.2.4 the desired statement. The case (4.30) shows that the objects $\mathcal{U}^{\lambda}(x)$ in the collection $C^{(\mathcal{U},2)}$ are exceptional, and the case (4.31) ensures the right orthogonal relations. □

Lemma 4.2.3. *We describe the weights μ appearing in the direct sum decomposition of the tensor product (4.32) with respect to the above cases:*

1. *Case (4.30): It is either trivial and the corresponding vector bundle $\mathcal{O}$ has multiplicity* 1*, or (exclusively) the entries satisfy the inequalities*

$$\max\{\mu_2, 1\} \leq \mu_1 \leq \lfloor \frac{3}{2}n - \frac{9}{2} \rfloor, \tag{4.34}$$

$$\max\{\mu_3, \lfloor -\frac{1}{2}n + \frac{3}{2} \rfloor\} \leq \mu_2 \leq \min\{\mu_1, \lceil \frac{1}{2}n - \frac{3}{2} \rceil\}, \tag{4.35}$$

$$\lceil -\frac{3}{2}n + \frac{9}{2} \rceil \leq \mu_3 \leq \min\{\mu_2, -1\}, \tag{4.36}$$

$$\mu_i = 0 \qquad \textit{for } i \in [4, n], \tag{4.37}$$

$$0 \leq \mu_1 - \mu_2 \leq 2n - 6, \tag{4.38}$$

$$2 \leq \mu_1 - \mu_3 \leq 2 \cdot \lfloor \frac{3}{2}n - \frac{9}{2} \rfloor, \tag{4.39}$$

and

$$0 \leq \mu_2 - \mu_3 \leq 2n - 6. \tag{4.40}$$

2. *Case (4.31): We have the inequalities*

$$\max\{\mu_2, -2n + 4\} \leq \mu_1 \leq \lfloor \frac{3}{2}n - \frac{9}{2} \rfloor, \tag{4.41}$$

$$\max\{\mu_3, \lfloor -\frac{5}{2}n + \frac{11}{2} \rfloor\} \leq \mu_2 \leq \min\{\mu_1, \lceil \frac{1}{2}n - \frac{3}{2} \rceil\}, \tag{4.42}$$

$$\lceil -\frac{7}{2}n + \frac{17}{2} \rceil \leq \mu_3 \leq \max\{\mu_2, -1\}, \tag{4.43}$$

$$\mu_i = 0 \qquad \textit{for } i \in [4, n], \tag{4.44}$$

$$0 \leq \mu_1 - \mu_2 \leq 2n - 6, \tag{4.45}$$

$$0 \leq \mu_1 - \mu_3 \leq 2 \cdot \lfloor \frac{3}{2}n - \frac{9}{2} \rfloor, \tag{4.46}$$

and

$$0 \leq \mu_2 - \mu_3 \leq 2n - 6. \tag{4.47}$$

Proof. We argue analogously as in the proof of Lemma 4.1.3.

First, we deduce from the Lemma 3.3.1 and Corollary 3.3.2 general inequalities for the entries μ_1, μ_2, and μ_3 as well as the vanishing of μ_4, ..., and μ_n. Due to the bounds of μ_1 and $-\mu_3$, we observe $0 \leq \mu_1 - \mu_3 \leq \lambda'_1 + \lambda''_1 \leq 2 \cdot \lfloor \frac{3}{2}n - \frac{9}{2} \rfloor$.

Second, we distinguish the two cases (4.30) and (4.31): If we assume $x' = x''$ as well as $\lambda' = \lambda''$ – i.e. $\mu' = \mu''$, then μ is either trivial and therefore $M_0^{-w_{L,\infty}\mu',\mu'}$ computes as 1 by Schur's Lemma, or it is non-trivial and we check the inequalities (4.34)-(4.40). Otherwise, if $x' = x''$ and $\lambda'' < \lambda'$ lexicographically or $x'' < x'$, then we check (4.41)-(4.47). □

Lemma 4.2.4. *We compute*

$$\mathrm{H}^{\bullet}(X, \mathcal{U}^{\mu}) = \begin{cases} \mathbf{K}[0] & , \textit{ if } \mu = 0 \\ 0 & , \textit{ else} \end{cases}. \tag{4.48}$$

Proof. *Case* (4.30).

μ is trivial: We take the identity $w = id$ and therefore see $w \cdot \mu = w(\mu + \rho_G) - \rho_G = 0$. Hence, we obtain the claimed result by Proposition 2.5.2.

μ is not trivial: μ_3 need to be in the range from $\lceil -\frac{3}{2}n + \frac{9}{2} \rceil$ to -1 by inequality (4.36).

$\mu_3 \in [-n+3, -1]$**:** $\mu + \rho_G$ is G-singular by condition (3.39) where $i = 3$.

$\mu_3 \in [\lceil -\frac{3}{2}n + \frac{9}{2} \rceil, -n+2]$**:** We mention the fact $-2n + 6 \leq -\frac{3}{2}n + \frac{9}{2}$ for $n \geq 3$. Hence, $\mu + \rho_G$ is G-singular by condition (3.41) where $i = 3$.

Case (4.31). We distinguish with respect to μ_3 which ranges from $\lceil -\frac{7}{2}n + \frac{17}{2} \rceil$ to -1 by (4.43).

$\mu_3 \in [-2n+6, -1]$**:** We apply conditions (3.39) or (3.41) where $i = 3$ and deduce that $\mu + \rho_G$ is G-singular.

$\mu_3 = -2n + 5$**:** It follows $-2n + 5 \leq \mu_2 \leq -1$ from (4.47).

$\mu_2 = -1$**:** $\mu_2 + \mu_3 = -2n + 4 = -2n - 1 + (2 + 3)$ is precisely condition (3.40).

$\mu_2 \in [-2n+5, -2]$**:** We apply the conditions (3.39) or (3.41) where $i = 2$.

$\mu_3 = -2n + 4$**:** Similar to before, it follows $-2n + 4 \leq \mu_2 \leq -2$ from (4.47).

$\mu_2 \in [-2n+5, -2]$ **:** This is either condition (3.39) or (3.41) where $i = 2$.

$\mu_2 = -2n + 4$ **:** We observe $-2n + 4 \leq \mu_1 \leq -2$ from (4.45).

$\mu_1 = -2$**:** $\mu_1 + \mu_2 = -2n + 2 = -2n - 1 + (1 + 2)$ is also condition (3.40).

$\mu_1 \in [-2n+4, -3]$**:** We set $i = 1$ and apply (3.39) or (3.41) respectively.

$\mu_3 \in [-3n+7, -2n+3]$**:** We have $-3n + 7 \leq \lfloor -\frac{5}{2}n + \frac{11}{2} \rfloor \leq \mu_2 \leq \lfloor -3$ due to (4.47).

$\mu_2 \in [-2n+4, -3]$**:** We argue analogously as in the previous case.

$\mu_2 \leq -2n + 3$**:** We have $\mu_2 < -2n + 4 \leq \mu_1 \leq -3$ because of (4.45). This means as before, we apply (3.39) or (3.41) respectively for $i = 1$.

$\mu_3 \in [\lceil -\frac{7}{2}n + \frac{17}{2} \rceil, -3n+6]$: The inequality (4.46) implies $\mu_3 < -2n+4 \leq \mu_1 \leq 2\lfloor \frac{3}{2}n - \frac{9}{2} \rfloor + \mu_3 \leq -3$. Hence, we have $\mu_1 \in [-2n+4, -3]$ and therefore proceed as in previous cases.

$\square$

4.3 Merging the parts

Construction. Finally, we form the tautological subcollection $C^{(\mathcal{U})}$: Its starting block is the union $C_0^{(\mathcal{U})} = C_0^{(\mathcal{U},1)} \cup C_0^{(\mathcal{U},2)}$ where we remain the lexicographical ordering, and its support partition arises from summing up (4.1) and (4.27).

We present a few auxiliary lemmas for later computations. For this purpose, let λ be a highest weight of an objects appearing in the tautological subcollection $C^{(\mathcal{U})}$.

Lemma 4.3.1. $\lambda_1 + \lambda_2 \leq 2n - 6$

Proof. For the first part $C^{(\mathcal{U},1)}$ of the tautological subcollection (cf. section 4.1), the statement is obvious. For the second part $C^{(\mathcal{U},2)}$ (cf. section 4.1), we compute $\lambda_1 + \lambda_2 \leq \lfloor \frac{3}{2}n - \frac{9}{2} \rfloor + \lceil \frac{1}{2}n - \frac{3}{2} \rceil \leq 2n - 6$. $\square$

Lemma 4.3.2. $\lambda_2 \leq n - 3$

Proof. For the first part $C^{(\mathcal{U},1)}$ of the tautological subcollection (cf. section 4.1), the statement is obvious. For the second part $C^{(\mathcal{U},2)}$ (cf. section 4.1), we observe compute $\lambda_2 \leq \lceil \frac{1}{2}n - \frac{3}{2} \rceil \leq n - 3$ or equivalently $0 \leq \lfloor \frac{1}{2}n - \frac{3}{2} \rfloor$. $\square$

Lemma 4.3.3. *If $\lambda_2 = n - 3$, then we need to have $\lambda_1 = n - 3$ and accordingly $\lambda = (n-3)\omega_2$.*

Proof. If λ appears in the first part $C^{(\mathcal{U},1)}$ of the tautological subcollection (cf. section 4.1), then we have $n - 3 = \lambda_2 \leq \lambda_1 \leq n - 3$. Otherwise, if λ appears in the second part $C^{(\mathcal{U},2)}$ (cf. section 4.2), then it needs to satisfy $n - 3 = \lambda_2 \leq \lceil \frac{1}{2}n - \frac{3}{2} \rceil$ or equivalently $\lfloor \frac{1}{2}n - \frac{3}{2} \rfloor \leq 0$. However, this fact implies $n \leq 4$ which yields a contradiction as the corresponding second part $C^{(\mathcal{U},2)}$ is empty. $\square$

Proving exceptionality. In the following we show that the two parts $C^{(\mathcal{U},1)}$ and $C^{(\mathcal{U},2)}$ can be combined. We continue with the notation introduced in the previous sections.

Proposition 4.3.4. *The tautological subcollection $C^{(\mathcal{U})}$ is exceptional.*

Proof. Thanks to Propositions 4.1.2 as well as 4.2.2, we are left to check the following right orthogonal relations given by Lemma 4.3.6 as well as 4.3.8. $\square$

Lemma 4.3.5. *Let $\mathcal{U}^{\lambda'}$ be from the starting block $C_0^{(\mathcal{U},2)}$, x' ranges in $[0, 2n-4]$, and let $\mathcal{U}^{\lambda''}$ be from the the starting block $C_0^{(\mathcal{U},1)}$. We describe the weights μ appearing in the direct sum decomposition of the tensor product $\mathcal{U}^{\mu'\vee} \otimes \mathcal{U}^{\mu''}$ where $\mu' = \lambda' + x'\omega_3$ and $\mu'' = \lambda''$:*

$$\max\{\mu_2, -2n+4\} \leq \mu_1 \leq n-3, \tag{4.49}$$
$$\max\{\mu_3, \lfloor -\frac{5}{2}n + \frac{11}{2} \rfloor\} \leq \mu_2 \leq \min\{\mu_1, n-3\}, \tag{4.50}$$
$$\lceil -\frac{7}{2}n + \frac{17}{2} \rceil \leq \mu_3 \leq \min\{\mu_2, -1\}, \tag{4.51}$$
$$\mu_i = 0 \qquad \textit{for } i \in [4, n], \tag{4.52}$$
$$0 \leq \mu_1 - \mu_2 \leq \lceil \frac{3}{2}n - \frac{9}{2} \rceil, \tag{4.53}$$
$$0 \leq \mu_1 - \mu_3 \leq \lfloor \frac{5}{2}n - \frac{15}{2} \rfloor, \tag{4.54}$$

and

$$0 \leq \mu_2 - \mu_3 \leq 2n - 6. \tag{4.55}$$

Proof. We aruge analogously as before in the proofs of Lemmas 4.1.3 or 4.2.3. □

Lemma 4.3.6. *$\mathcal{U}^{\lambda'}(x')$ is right orthogonal to $\mathcal{U}^{\lambda''}$ whenever $\mathcal{U}^{\lambda'}$ is from the starting block $C_0^{(\mathcal{U},2)}$, x' ranges in $[0, 2n-4]$, and $\mathcal{U}^{\lambda''}$ is from $C_0^{(\mathcal{U},1)}$.*

Proof. Let $\mathcal{U}^{\mu}$ be an irreducible summand that appears in the direct sum decomposition of the tensor product $\mathcal{U}^{\mu'\vee} \otimes \mathcal{U}^{\mu''}$ with $\mu' = \lambda' + x'\omega_3$ and $\mu'' = \lambda''$. We observe $\lceil -\frac{7}{2}n + \frac{17}{2} \rceil \leq \mu_3 \leq -1$ by (4.51):

$\mu_3 \in [-2n+6, -1]$: We apply the conditions (3.39) or (3.41) respectively where $i = 3$.

$\mu_3 = -2n+5$: Due to (4.55), we have $-2n+5 \leq \mu_2 \leq -1$.

- **$\mu_2 = -1$:** We observe $\mu_2 + \mu_3 = -2n+4 = -2n-1+(2+3)$ and hence refer to condition (3.40).
- **$\mu_2 \in [-2n+5, -2]$:** We apply the conditions (3.39) or (3.41) respectively where $i = 2$.

$\mu_3 \in [\lceil -\frac{7}{2}n + \frac{17}{2} \rceil, -2n+4]$: We recall (4.50) for the lower bound and (4.55) for the upper one. Thus, we see $\lfloor -\frac{5}{2}n + \frac{11}{2} \rfloor \leq \mu_2 \leq -2$.

- **$\mu_2 \in [-2n+5, -2]$:** As before, we apply condition (3.39) or (3.41) respectively where $i = 2$.
- **$\mu_2 = -2n+4$:** We compute $-2n+4 \leq \mu_1 \leq \lceil -\frac{1}{2}n - \frac{1}{2} \rceil$ by (4.53). For $n \geq 3$, we have $\mu_1 \leq -2$.
  - **$\mu_1 = -2$:** It is $\mu_1 + \mu_2 = -2n+2 = -2n-1+(1+2)$ and hence the condition (3.40) holds.

$\mu_1 \in [-2n+4, -3]$: We apply condition (3.39) or (3.41) respectively where $i = 1$.

$\mu_2 \in [\lfloor -\frac{5}{2}n + \frac{11}{2} \rfloor, -2n+3]$: We compute $\mu_2 \leq -2n+3 < -2n+4 \leq \mu_1 \leq \lceil -\frac{1}{2}n - \frac{3}{2} \rceil$ by (4.53). For $n \geq 3$, we have $\mu_1 \leq -3$. Hence, we proceed as before.

□

Lemma 4.3.7. *Let $\mathcal{U}^{\lambda'}$ be from the starting block $C_0^{(\mathcal{U},1)}$, x' ranges in $[1, 2n-4]$, and let $\mathcal{U}^{\lambda''}$ be from the the starting block $C_0^{(\mathcal{U},2)}$. We describe the weights μ appearing in the direct sum decomposition of the tensor product $\mathcal{U}^{\mu' \vee} \otimes \mathcal{U}^{\mu''}$ where $\mu' = \lambda' + x'\omega_3$ and $\mu'' = \lambda''$:*

$$\max\{\mu_2, -2n+4\} \leq \mu_1 \leq \lfloor \frac{3}{2}n - \frac{11}{2} \rfloor, \tag{4.56}$$

$$\max\{\mu_3, -3n+7\} \leq \mu_2 \leq \min\{\mu_1, \lceil \frac{1}{2}n - \frac{5}{2} \rceil\}, \tag{4.57}$$

$$-3n+7 \leq \mu_3 \leq \min\{\mu_2, -1\}, \tag{4.58}$$

$$\mu_i = 0 \qquad \text{for } i \in [4, n], \tag{4.59}$$

$$0 \leq \mu_1 - \mu_2 \leq \lfloor \frac{5}{2}n - \frac{15}{2} \rfloor, \tag{4.60}$$

$$0 \leq \mu_1 - \mu_3 \leq \lfloor \frac{3}{2}n - \frac{9}{2} \rfloor, \tag{4.61}$$

and

$$0 \leq \mu_2 - \mu_3 \leq \lceil \frac{1}{2}n - \frac{3}{2} \rceil. \tag{4.62}$$

Proof. Again we refer to the analogous proofs, namely Lemmas 4.1.3, 4.2.3, or 4.3.5. □

Lemma 4.3.8. *$\mathcal{U}^{\lambda'}(x')$ is right orthogonal to $\mathcal{U}^{\lambda''}$ whenever $\mathcal{U}^{\lambda'}$ is from the starting block $C_0^{(\mathcal{U},1)}$, x' ranges in $[1, 2n-4]$ and $\mathcal{U}^{\lambda''}$ is from the starting block $C_0^{(\mathcal{U},2)}$.*

Proof. We set $\mu' = \lambda' + x'\omega_3$ and $\mu'' = \lambda''$. Let $\mathcal{U}^{\mu}$ be an irreducible summand in the direct sum decomposition of the tensor product $\mathcal{U}^{\mu' \vee} \otimes \mathcal{U}^{\mu''}$. We have $\mu_3 \in [-3n+7, -1]$ by (4.58). Then we do the same case distinction as in the proof of Lemma 4.3.6.

□

Chapter 5

The spinor subcollection

Construction of the subcollection. The starting block $C_0^{(\mathcal{S})}$ of the spinor subcollection $C^{(\mathcal{S})}$ consists of $n-1$ objects, namely the spinor bundle

$$\mathcal{S}^{(0)} = \mathcal{U}^{\omega_n} \tag{5.1}$$

as well as $n-2$ non-splitting G-equivariant extensions

$$\begin{aligned}
0 \to \mathcal{U}^{\omega_n} \to &\mathcal{S}^{(1)} \to \mathcal{U}^{\omega_1+\omega_n} \to 0,\\
&\vdots\\
0 \to \mathcal{U}^{(n-4)\omega_1+\omega_n} \to &\mathcal{S}^{(n-3)} \to \mathcal{U}^{(n-3)\omega_1+\omega_n} \to 0, \text{ and}\\
0 \to \mathcal{U}^{(n-3)\omega_1+\omega_n} \to &\mathcal{S}^{(n-2)} \to \mathcal{U}^{(n-2)\omega_1+\omega_n} \to 0.
\end{aligned} \tag{5.2}$$

The rows of objects $\mathcal{S}^{(0)}, \cdots, \mathcal{S}^{(n-3)}$ have length $w_{max} = 2n-3$ and the last one belonging to the object $\mathcal{S}^{(n-2)}$ has length $n-2$. This means, the collection $C^{(\mathcal{S})}$ has support partition $(h_0, \cdots, h_{2n-4})$ where

$$h_x = \begin{cases} n-1 & , \text{ if } x \in [0, n-3] \\ n-2 & , \text{ if } x \in [n-2, 2n-4] \end{cases}. \tag{5.3}$$

All in all, we establish the following Lefschetz collection $C^{(\mathcal{S})}$ on $\mathbf{D}^{\mathrm{b}}(X)$ consisting of $2n^2-6n+4$ objects:

$$\begin{pmatrix}
\mathcal{S}^{(n-2)} & \mathcal{S}^{(n-2)}(1) & \cdots & \mathcal{S}^{(n-2)}(n-3) & & & \\
\mathcal{S}^{(n-3)} & \mathcal{S}^{(n-3)}(1) & \cdots & \mathcal{S}^{(n-3)}(n-3) & \mathcal{S}^{(n-3)}(n-2) & \cdots & \mathcal{S}^{(n-3)}(2n-4) \\
\vdots & \vdots & & \vdots & \vdots & & \vdots \\
\mathcal{S}^{(0)} & \mathcal{S}^{(0)}(1) & \cdots & \mathcal{S}^{(0)}(n-3) & \mathcal{S}^{(0)}(n-2) & \cdots & \mathcal{S}^{(0)}(2n-4)
\end{pmatrix} \tag{5.4}$$

Example 5.0.1. For $\mathrm{OGr}(3, V)$ with $n = 7$, we have entirely

$$C^{(\mathcal{S})} = \begin{pmatrix} \mathcal{S}^{(5)} & \mathcal{S}^{(5)}(1) & \cdots & \mathcal{S}^{(5)}(4) & & & \\ \mathcal{S}^{(4)} & \mathcal{S}^{(4)}(1) & \cdots & \mathcal{S}^{(4)}(4) & \mathcal{S}^{(4)}(5) & \cdots & \mathcal{S}^{(4)}(10) \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \mathcal{S}^{(0)} & \mathcal{S}^{(0)}(1) & \cdots & \mathcal{S}^{(0)}(4) & \mathcal{S}^{(0)}(5) & \cdots & \mathcal{S}^{(0)}(10) \end{pmatrix} \tag{5.5}$$

Again, the Lefschetz structure is determined by the starting block, i.e. the first column of (5.5), and the support partition $(6, 6, 6, 6, 6, 5, 5, 5, 5, 5, 5)$.

Given $y \in [0, n-2]$, we conjecture that the object $\mathcal{S}^{(y)}$ has rank

$$\mathrm{rk}(\, \mathcal{S}^{(y)}\,) = 2^{n-3} \cdot (y+1)^2 \tag{5.6}$$

and that it has determinant $\det(\, \mathcal{S}^{(y)}\,) = \mathcal{O}(t)$ with

$$t = 2^{n-4} \cdot \frac{1}{3} n(2n^2 + 1). \tag{5.7}$$

Construction of the objects $\mathcal{S}^{(y)}$. Initially, we claim that the objects $\mathcal{S}^{(y)}(x)$ of the spinor subcollection $C^{(\mathcal{S})}$ are well-defined. Therefore, we focus on the objects $\mathcal{S}^{(y)}$ of the starting block $C_0^{(\mathcal{S})}$. The statement is clear for the case $y = 0$ as the first object $\mathcal{S}^{(0)}$ is by construction (5.1) the spinor bundle $\mathcal{S}$; for the higher cases $y > 0$, we give the following lemma.

Lemma 5.0.2. *Let y be an integer greater than* 0*. Then there is non-splitting* G*-equivariant extension*

$$0 \to \mathcal{U}^{(y-1)\omega_1+\omega_n} \to \mathcal{S}^{(y)} \to \mathcal{U}^{y\omega_1+\omega_n} \to 0 \tag{5.8}$$

which is unique up to rescaling.

Proof. Our approach. We show that the first Ext-space $\mathrm{Ext}^1(\mathcal{U}^{y\omega_1+\omega_n}, \mathcal{U}^{(y-1)\omega_1+\omega_n})$ contains exactly one $\Bbbk$ as summand. Hence, our desired extension $\mathcal{S}^{(y)}$ is induced by a non-negative value from this summand.

Identify the suitable summand. As we have

$$\mathrm{Ext}^\bullet(\mathcal{U}^{y\omega_1+\omega_n}, \mathcal{U}^{(y-1)\omega_1+\omega_n}) = \mathrm{H}^\bullet(X, \mathcal{U}^{y\omega_1+\omega_n \vee} \otimes \mathcal{U}^{(y-1)\omega_1+\omega_n}), \tag{5.9}$$

we decompose the following tensor product into a direct sum of irreducible G-equivariant vector bundles:

$$
\begin{aligned}
\mathcal{U}^{y\omega_1+\omega_n\vee} \otimes \mathcal{U}^{(y-1)\omega_1+\omega_n} &= \mathcal{U}^{y\omega_2+\omega_n}(-y-1) \otimes \mathcal{U}^{(y-1)\omega_1+\omega_n} \\
&= \mathcal{O}(-y-1) \\
&\quad \otimes \mathcal{U}^{\omega_n} \otimes \mathcal{U}^{\omega_n} \\
&\quad \otimes \mathcal{U}^{y\omega_2} \otimes \mathcal{U}^{(y-1)\omega_1} \\
&= \mathcal{O}(-y-1) \\
&\quad \otimes (\mathcal{U}^{\omega_3} \oplus \cdots \oplus \mathcal{U}^{\omega_{n-1}} \oplus \mathcal{U}^{2\omega_n}) \\
&\quad \otimes (\mathcal{U}^{\omega_2}(y-1) \oplus \cdots)
\end{aligned}
\tag{5.10}
$$

If $n = 4$, we observe that $\mathcal{U}^{\omega_2+2\omega_4}(-2)$ is a summand in the above tensor product; otherwise if $5 \leq n$, it is for $\mathcal{U}^{\omega_2+\omega_4}(-2)$. The corresponding highest weight is written as

$$
\lambda = (0,0,-1,1) \quad \text{or} \quad \lambda = (0,0,-1,1,0,\cdots,0) \tag{5.11}
$$

respectively.

Compute cohomology partially. We apply Proposition 2.5.2 to compute the cohomology $\mathrm{H}^\bullet(X,\mathcal{U}^\lambda) = \Bbbk[-1]$ for the summand $\mathcal{U}^\lambda$ of the previous step. Indeed, we recall (2.17) as well as (5.11) and we consider the sum $\lambda + \rho_{\mathrm{G}}$, namely

$$
\lambda + \rho_{\mathrm{G}} = \begin{cases} (\frac{7}{2}, \frac{5}{2}, \frac{1}{2}, \frac{3}{2}) & , \text{ if } n = 4 \\ (n-\frac{1}{2}, n-\frac{3}{2}, n-\frac{7}{2}, n-\frac{5}{2}, n-\frac{9}{2}, \cdots, \frac{1}{2}) & , \text{ if } 5 \leq n \end{cases}. \tag{5.12}
$$

In both cases, $\lambda + \rho_{\mathrm{G}}$ is G-regular and the third simple reflection w_3 is the unique element of the Weyl group W_{G} mapping $\lambda + \rho_{\mathrm{G}}$ to the cone of dominant weights $\mathbf{P}^+_{\mathrm{G}}$. In particular, we have $\ell_{\mathrm{G}}(w_3) = 1$ as well as $w_3 \cdot \lambda = w_3(\lambda + \rho_{\mathrm{G}}) - \rho_{\mathrm{G}} = 0$.

Uniqueness of the summand $\Bbbk$ *in degree* 1. Let λ be the highest weight of a summand $\mathcal{U}^\lambda$ appearing in the direct sum decomposition of the above tensor prodcut $\mathcal{U}^{y\omega_1+\omega_n\vee} \otimes \mathcal{U}^{(y-1)\omega_1+\omega_n}$. Hence, it is L-dominant. Now, we assume $\mathrm{H}^\bullet(X,\mathcal{U}^\lambda) = \Bbbk[-1]$ which means precisely $w_i \cdot \lambda = w_i(\lambda+\rho_{\mathrm{G}}) - \rho_{\mathrm{G}} = 0$ for a simple reflection w_i. Consequently, we need to compare the entries of $\lambda + \rho_{\mathrm{G}}$ and ρ_{G}. Let us explicitly mention that the j -th entry of ρ_{G} can be written as $n + \frac{1}{2} - j$. If $i \in \{1,\cdots,n-1\}$, we compute $\lambda_i = -1$, $\lambda_{i+1} = 1$, and $\lambda_j = 0$ for all other entries where $j \in [1,n] \setminus \{i,i+1\}$. Otherwise, if $i = n$, then we obtain $\lambda_j = 0$ for all $j \in [1,n-1]$ and $\lambda_n = -1$. We conclude that i needs to be 3 because for any other case we see that λ could not be L-dominant. Hence, λ is exactly of the form (5.11). □

Resolution of the objects $\mathcal{S}^{(y)}$ for $y \geq 1$. Our next aim is to characterize the extensions $\mathcal{S}^{(y)}$ for later computations by a handy resolution.

Proposition 5.0.3. *For $y \in \{1, \cdots, n-2\}$, the object $\mathcal{S}^{(y)}$ is resolved by the following exact sequence*

$$\begin{aligned} 0 \to \mathcal{S}^{(y)} \to &\mathrm{S} \otimes \mathcal{U}^{(y-1)\omega_1}(1) \to \mathrm{S} \otimes \mathcal{U}^{(y-2)\omega_1+\omega_2}(1) \to \\ &\cdots \to \mathrm{S} \otimes \mathcal{U}^{(y-1)\omega_2}(1) \to \mathcal{S}^{(y)\vee}(y+1) \to 0. \end{aligned} \tag{5.13}$$

Proof. First, we construct the first part of the exact sequence (5.13) by Lemma 5.0.5, the middle pieces by Lemma 5.0.7, and the last one by Lemma 5.0.9. Then we glue all the short exact sequences to obtain the desired resolution. □

Example 5.0.4. We consider the case $\mathrm{OGr}(3, V)$ with $n = 7$. Then we have the following resolutions for the objects $\mathcal{S}^{(y)}$:

$$\begin{aligned} &0 \to \mathcal{S}^{(1)} \to \mathrm{S} \otimes \mathcal{O}(1) \to \mathcal{S}^{(1)\vee}(2) \to 0 \\ &0 \to \mathcal{S}^{(2)} \to \mathrm{S} \otimes \mathcal{U}^{\omega_1}(1) \to \mathrm{S} \otimes \mathcal{U}^{\omega_2}(1) \to \mathcal{S}^{(2)\vee}(3) \to 0 \\ &\qquad\qquad \vdots \\ &0 \to \mathcal{S}^{(5)} \to \mathrm{S} \otimes \mathcal{U}^{4\omega_1}(1) \to \mathrm{S} \otimes \mathcal{U}^{3\omega_1+\omega_2}(1) \to \\ &\qquad\qquad \cdots \to \mathrm{S} \otimes \mathcal{U}^{4\omega_2}(1) \to \mathcal{S}^{(5)\vee}(6) \to 0 \end{aligned} \tag{5.14}$$

Before we start with the proof of the above exact sequence (5.13), let us introduce a bunch of objects as well as labellings. Recall that $\mathcal{F}_i^{(1)}$ is the i^{th} component appearing in the filtration (2.45) of Proposition 2.7.1 and let us write the factors in (5.13) as

$$F_{y,i} = \mathcal{U}^{(y-1-i)\omega_1+i\omega_2}(1) \tag{5.15}$$

where $i \in [0, y-1]$ – i.e. $F_{y,i}$ is the irreducible G-equivariant vector bundle with highest weight $(y, i+1, 1, 0, \cdots, 0)$. We consider the short exact sequences appearing in (2.45) and tensor it with $F_{y,i}$:

$$\mathcal{F}_{y,i,0} = \mathrm{S} \otimes F_{y,i}, \tag{5.16}$$

$$0 \to \mathcal{F}_{y,i,1} \to \mathcal{F}_{y,i,0} \to \mathcal{U}^{\omega_n} \otimes F_{y,i} \to 0, \tag{5.17}$$

$$0 \to \mathcal{F}_{y,i,2} \to \mathcal{F}_{y,i,1} \to \mathcal{U}^{\omega_2+\omega_n}(-1) \otimes F_{y,i} \to 0, \tag{5.18}$$

$$0 \to \mathcal{F}_{y,i,3} \to \mathcal{F}_{y,i,2} \to \mathcal{U}^{\omega_1+\omega_n}(-1) \otimes F_{y,i} \to 0, \tag{5.19}$$

and

$$\mathcal{F}_{y,i,3} = \mathcal{U}^{\omega_n}(-1) \otimes F_{y,i}. \tag{5.20}$$

Then, we decompose the tensor products appearing in the above short exact sequences (5.17)-(5.20), namely

$$\mathcal{U}^{\omega_n} \otimes F_{y,i} = \mathcal{U}^{(y-1-i)\omega_1+i\omega_2+\omega_n}(1) =: \mathcal{M}_{y,i,0}, \tag{5.21}$$

$$\mathcal{U}^{\omega_2+\omega_n}(-1) \otimes F_{y,i} = \mathcal{M}_{y,i,1,1} \oplus \mathcal{M}_{y,i,1,2} =: \mathcal{M}_{y,i,1}, \tag{5.22}$$

where

$$\mathcal{M}_{y,i,1,1} = \begin{Bmatrix} 0 & \text{, if } i=0 \\ \mathcal{U}^{(y-i)\omega_1+i\omega_2+\omega_n}(1) & \text{, if } i\geq 1 \end{Bmatrix}$$

$$\mathcal{M}_{y,i,1,2} = \mathcal{U}^{(y-1-i)\omega_1+(i+1)\omega_2+\omega_n} \oplus \begin{Bmatrix} \mathcal{U}^{(y-2-i)\omega_1+i\omega_2+\omega_n}(1) & \text{, if } i\leq y-2 \\ 0 & \text{, if } i=y-1 \end{Bmatrix},$$

$$\mathcal{U}^{\omega_1+\omega_n}(-1) \otimes F_{y,i} = \mathcal{M}_{y,i,2,1} \oplus \mathcal{M}_{y,i,2,2} =: \mathcal{M}_{y,i,2}, \tag{5.23}$$

where

$$\mathcal{M}_{y,i,2,1} = \mathcal{U}^{(y-i)\omega_1+i\omega_2+\omega_n} \oplus \begin{Bmatrix} 0 & \text{, if } i=0 \\ \mathcal{U}^{(y-1-i)\omega_1+(i-1)\omega_2+\omega_n}(1) & \text{, if } i\geq 1 \end{Bmatrix}$$

$$\mathcal{M}_{y,i,2,2} = \begin{Bmatrix} \mathcal{U}^{(y-2-i)\omega_1+(i+1)\omega_2+\omega_n} & \text{, if } i\leq y-2 \\ 0 & \text{, if } i=y-1 \end{Bmatrix},$$

and

$$\mathcal{U}^{\omega_n}(-1) \otimes F_{y,i} = \mathcal{U}^{(y-1-i)\omega_1+i\omega_2+\omega_n} =: \mathcal{M}_{y,i,3}. \tag{5.24}$$

Let us explicitly mention the following interweaving identities:

$$\mathcal{M}_{y,i,0} = \mathcal{M}_{y,i,3}(1) \tag{5.25}$$

and

$$\mathcal{M}_{y,i,1,2} = \mathcal{M}_{y,i+1,2,1} \text{ whenever } (y,i)=(1,0) \text{ or } 1\leq i\leq y-2. \tag{5.26}$$

We construct the resolution (2.45) of $\mathcal{S}^{(y)}$ inductively. For this purpose, let us claim that for any $i \in [0, y-1]$ it exists short exact sequences

$$0 \to C_{y,i-1,0} \to \mathrm{S} \otimes F_{y,i} \to C_{y,i,0} \to 0 \tag{5.27}$$

such that the first object $C_{y,-1,0}$ is precisely $\mathcal{S}^{(y)}$ and later $C_{y,i,0}$'s are characterized by

$$0 \to C_{y,i,1} \to C_{y,i,0} \to \mathcal{M}_{y,i,0} \to 0, \tag{5.28}$$
$$0 \to C_{y,i,2} \to C_{y,i,1} \to \mathcal{M}_{y,i,1,2} \to 0, \tag{5.29}$$

and

$$C_{y,i,2} = \mathcal{M}_{y,i,2,2}. \tag{5.30}$$

Furthermore, the short exact sequence (5.28) is a non-splitting G-equivariant extension for all cases and the short exact sequence (5.29) is so whenever it is $i \leq y-2$.

Now, let us start with the base case, namely the first part from the left of the exact sequence (5.13).

Lemma 5.0.5. *For $y \geq 1$, we have the short exact sequence*

$$0 \to \mathcal{S}^{(y)} \to \mathrm{S} \otimes F_{y,0} \to C_{y,0,0} \to 0 \tag{5.31}$$

with the following components

$$0 \to C_{y,0,1} \to C_{y,0,0} \to \underbrace{\mathcal{U}^{(y-1)\omega_1+\omega_n}(1)}_{=\mathcal{M}_{y,0,0}} \to 0, \tag{5.32}$$

$$0 \to C_{y,0,2} \to C_{y,0,1} \to \underbrace{\begin{matrix} \mathcal{U}^{(y-1)\omega_1+\omega_2+\omega_n} \\ \oplus \begin{Bmatrix} 0 & , \textit{ if } y=1 \\ \mathcal{U}^{(y-2)\omega_1+\omega_n}(1) & , \textit{ if } y \geq 2 \end{Bmatrix} \end{matrix}}_{=\mathcal{M}_{y,0,1,2}} \to 0, \tag{5.33}$$

and

$$C_{y,0,2} = \underbrace{\begin{cases} 0 & , \textit{ if } y=1 \\ \mathcal{U}^{(y-2)\omega_1+\omega_2+\omega_n} & , \textit{ if } y \geq 2 \end{cases}}_{=\mathcal{M}_{y,0,2,2}}. \tag{5.34}$$

The object $C_{y,0,0}$ is a non-splitting G*-equivariant extension for all $y \geq 1$; and $C_{y,0,1}$ is so for $y \geq 2$ respectively.*

Proof. Filtration on $\mathrm{S} \otimes F_{y,0}$. We recall (5.16)-(5.20) for the case $i=0$:

$$\mathcal{F}_{y,0,0} = \mathrm{S} \otimes F_{y,0}, \tag{5.35}$$
$$0 \to \mathcal{F}_{y,0,1} \to \mathcal{F}_{y,0,0} \to \underbrace{\mathcal{U}^{(y-1)\omega_1+\omega_n}(1)}_{=\mathcal{M}_{y,0,0}} \to 0, \tag{5.36}$$
$$0 \to \mathcal{F}_{y,0,2} \to \mathcal{F}_{y,0,1} \to \underbrace{\mathcal{M}_{y,0,1,1} \oplus \mathcal{M}_{y,0,1,2}}_{=\mathcal{M}_{y,0,1}} \to 0 \tag{5.37}$$

where

$$
\begin{aligned}
\mathcal{M}_{y,0,1,1} &= 0 \\
\mathcal{M}_{y,0,1,2} &= \mathcal{U}^{(y-1)\omega_1+\omega_2+\omega_n} \\
&\oplus \begin{Bmatrix} 0 & \text{, if } y=1 \\ \mathcal{U}^{(y-2)\omega_1+\omega_n}(1) & \text{, if } y \geq 2 \end{Bmatrix},
\end{aligned}
$$

$$
0 \to \mathcal{F}_{y,0,3} \to \mathcal{F}_{y,0,2} \to \underbrace{\mathcal{M}_{y,0,2,1} \oplus \mathcal{M}_{y,0,2,2}}_{=\mathcal{M}_{y,0,2}} \to 0 \tag{5.38}
$$

where

$$
\begin{aligned}
\mathcal{M}_{y,0,2,1} &= \mathcal{U}^{y\omega_1+\omega_n} \\
\mathcal{M}_{y,0,2,2} &= \begin{Bmatrix} 0 & \text{, if } y=1 \\ \mathcal{U}^{(y-2)\omega_1+\omega_2+\omega_n} & \text{, if } y \geq 2 \end{Bmatrix},
\end{aligned}
$$

and

$$
\mathcal{F}_{y,0,3} = \underbrace{\mathcal{U}^{(y-1)\omega_1+\omega_n}}_{=\mathcal{M}_{y,0,3}}. \tag{5.39}
$$

Embed $\mathcal{S}^{(y)}$ into $\mathcal{F}_{y,0,2}$ and construct $C_{y,0,2}$. We compose the projection of (5.38) with the one onto $\mathcal{M}_{y,0,2,2}$, namely

$$
\mathcal{F}_{y,0,2} \twoheadrightarrow \underbrace{\mathcal{M}_{y,0,2,1} \oplus \mathcal{M}_{y,0,2,2}}_{=\mathcal{M}_{y,0,2}} \twoheadrightarrow \mathcal{M}_{y,0,2,2}. \tag{5.40}
$$

Snaking yields the short exact sequences

$$
0 \to \widetilde{\mathcal{S}^{(y)}} \to \mathcal{F}_{y,0,2} \to \mathcal{M}_{y,0,2,2} \to 0 \tag{5.41}
$$

and

$$
0 \to \underbrace{\mathcal{U}^{(y-1)\omega_1+\omega_n}}_{=\mathcal{M}_{y,0,3}} \to \widetilde{\mathcal{S}^{(y)}} \to \underbrace{\mathcal{U}^{y\omega_1+\omega_n}}_{=\mathcal{M}_{y,0,2,1}} \to 0. \tag{5.42}
$$

We claim that $\widetilde{\mathcal{S}^{(y)}}$ is isomorphic to the object $\mathcal{S}^{(y)}$. In (5.41), the object $\widetilde{\mathcal{S}^{(y)}}$ is constructed as kernel of a G-equivariant morphism (5.40). Thus, it need to be G-equivariant as well. If we show that the extension $\widetilde{\mathcal{S}^{(y)}}$ does also not split, then it needs to coincide with $\mathcal{S}^{(y)}$ up to rescaling. In fact, both objects are an extension of $\mathcal{U}^{y\omega_1+\omega_n}$ by $\mathcal{U}^{(y-1)\omega_1+\omega_n}$ as we see from (5.8) and (5.42). Hence, the non-splitting of $\widetilde{\mathcal{S}^{(y)}}$ implies the stated isomorphism from the uniqueness proved in previous Lemma 5.0.2. So, let us assume the opposite, namely that $\widetilde{\mathcal{S}^{(y)}}$ splits – i.e. it is a direct sum. Then, we construct an G-equivariant morphism

$$\mathcal{U}^{y\omega_1+\omega_n} \hookrightarrow \widetilde{\mathcal{S}^{(y)}} \hookrightarrow \mathcal{F}_{y,0,2} \hookrightarrow \mathcal{F}_{y,0,1} \hookrightarrow \mathrm{S} \otimes F_{y,0} \tag{5.43}$$

where the first embedding comes from the assumed splitting, the second one from (5.41), and the remaining ones from (5.36) as well as (5.37). However, this contradicts the fact that there are no non-trivial morphisms from $\mathcal{U}^{y\omega_1+\omega_n}$ to $F_{y,0}$. Indeed, we will see this in (5.53) of Lemma 5.0.6 hereinafter. We finish this step by setting $C_{y,0,2} = \mathcal{M}_{y,0,2,2}$ to see (5.34).

Embed $\mathcal{S}^{(y)}$ into $\mathcal{F}_{y,0,1}$ and construct $C_{y,0,1}$. We compose the embedding of (5.37) with the one of (5.41), namely

$$\mathcal{S}^{(y)} \hookrightarrow \mathcal{F}_{y,0,2} \hookrightarrow \mathcal{F}_{y,0,1}. \tag{5.44}$$

Analogously as before, snaking gives us the short exact sequences

$$0 \to \mathcal{S}^{(y)} \to \mathcal{F}_{y,0,1} \to C_{y,0,1} \to 0 \tag{5.45}$$

and

$$0 \to C_{y,0,2} \to C_{y,0,1} \to \underbrace{\mathcal{M}_{y,0,1,1} \oplus \mathcal{M}_{y,0,1,2}}_{=\mathcal{M}_{y,0,1}} \to 0. \tag{5.46}$$

The last one is precisely (5.33) as the summand $\mathcal{M}_{y,0,1,1}$ vanishes.
If $y \geq 2$, then the G-equivariant extension $C_{y,0,1}$ does not split. Otherwise, if we assume the opposite, then we have a non-trivial surjection

$$\mathcal{F}_{y,0,1} \twoheadrightarrow C_{y,0,1} \twoheadrightarrow C_{y,0,2} = \mathcal{U}^{(y-2)\omega_1+\omega_2+\omega_n}. \tag{5.47}$$

However, this contradicts the fact that the Hom-space $\mathrm{Hom}(\mathcal{F}_{y,0,1}, C_{y,0,2})$ vanishes. In fact, we apply $\mathrm{Hom}(-, C_{y,0,2})$ to the short exact sequence (5.36) and obtain the long exact sequence

$$\cdots \to \mathrm{Ext}^p(\mathcal{M}_{y,0,0}, C_{y,0,2}) \to \mathrm{Ext}^p(\mathrm{S} \otimes F_{y,0}, C_{y,0,2}) \to \mathrm{Ext}^p(\mathcal{F}_{y,0,1}, C_{y,0,2}) \to \cdots. \tag{5.48}$$

Then, we observe the vanishing of the Hom-space $\mathrm{Hom}(\mathrm{S} \otimes F_{y,0}, C_{y,0,2})$ by the later computation (5.54) in Lemma 5.0.6 as well as the vanishing of the first Ext-space $\mathrm{Ext}^1(\mathcal{M}_{y,0,0}, C_{y,0,2})$ by (5.55).

Embed $\mathcal{S}^{(y)}$ into $\mathcal{F}_{y,0,0} = \mathrm{S} \otimes F_{y,0}$ and construct $C_{y,0,0}$. We compose the embedding of (5.36) with the one of (5.45), namely

$$\mathcal{S}^{(y)} \hookrightarrow \mathcal{F}_{y,0,1} \hookrightarrow \mathrm{S} \otimes F_{y,0}. \tag{5.49}$$

We snake a third time and obtain the short exact sequences

$$0 \to \mathcal{S}^{(y)} \to \mathrm{S} \otimes F_{y,0} \to C_{y,0,0} \to 0 \tag{5.50}$$

and

$$0 \to C_{y,0,1} \to C_{y,0,0} \to \mathcal{M}_{y,0,0} \to 0. \tag{5.51}$$

These are the desired sequences (5.31) and (5.32) respectively.
We claim that the G-equivariant extension $C_{y,0,0}$ does not split. If we assume the opposite – i.e. it is a direct sum, then we have a non-trivial surjection

$$S \otimes F_{y,0} \twoheadrightarrow C_{y,0,0} \twoheadrightarrow C_{y,0,1} \twoheadrightarrow \mathcal{M}_{y,0,1} = \mathcal{M}_{y,0,1,2} \twoheadrightarrow \mathcal{U}^{(y-1)\omega_1+\omega_2+\omega_n}. \tag{5.52}$$

The first comes from the projection in (5.50), the second from the assumed splitting, the third one from (5.46), the last one from the projection onto the summand $\mathcal{U}^{(y-1)\omega_1+\omega_2+\omega_n}$. However, this contradicts the fact that there are no non-trivial morphisms from $F_{y,0}$ to $\mathcal{U}^{(y-1)\omega_1+\omega_2+\omega_n}$. In fact, we refer to (5.56) in the subsequent Lemma 5.0.6. □

Lemma 5.0.6. *We show the following vanishings:*

$$\mathrm{Hom}(\mathcal{U}^{y\omega_1+\omega_n}, F_{y,0}) = 0 \tag{5.53}$$
$$\mathrm{Hom}(F_{y,0}, C_{y,0,2}) = 0 \tag{5.54}$$
$$\mathrm{Ext}^1(\mathcal{M}_{y,0,0}, C_{y,0,2}) = 0 \tag{5.55}$$
$$\mathrm{Hom}(F_{y,0}, \mathcal{U}^{(y-1)\omega_1+\omega_2+\omega_n}) = 0 \tag{5.56}$$

Proof. (5.53)**:** We apply Lemma 3.3.6 with $\mu' = y\omega_1$, $\mu'' = (y-1)\omega_1 + \omega_3$ and $i = 3$. In fact, we check $-(\mu'_1 + \mu'_2 + \mu'_3) + (\mu''_1 + \mu''_2 + \mu''_3) = 2$ and $-\mu'_3 + \mu''_3 = 1$.

(5.54)**:** We recall $C_{y,0,2}$ from (5.30): If y is one, then $C_{y,0,2}$ is by construction zero and therefore the claimed statement follows obviously. Otherwise, if we assume $y \geq 2$, then we have $C_{y,0,2} = \mathcal{U}^{(y-2)\omega_1+\omega_2+\omega_n}$. We apply Lemma 3.3.6 with $\mu' = (y-1)\omega_1 + \omega_3$, $\mu'' = (y-2)\omega_1 + \omega_2$ and $i = 3$. For this purpose, we check $-(\mu'_1 + \mu'_2 + \mu'_3) + (\mu''_1 + \mu''_2 + \mu''_3) = -2$ and $-\mu'_3 + \mu''_3 = -1$.

(5.55)**:** We recall $\mathcal{M}_{y,0,0}$ is the irreducible vector bundle $\mathcal{U}^{(y-1)\omega_1+\omega_n}(1)$. Similar as in the step before, the statement is obvious for $y = 1$. Hence, we assume $y \geq 2$ for the remaining part and therefore have $C_{y,0,2} = \mathcal{U}^{(y-2)\omega_1+\omega_2+\omega_n}$. We start with the following preparation where $\mu' = (y-1)\omega_1 + \omega_3$ and $\mu'' = (y-2)\omega_1 + \omega_2$ (see (3.47)):

$$\mathrm{Ext}^\bullet(\mathcal{M}_{y,0,0}, C_{y,0,2}) = \mathrm{Ext}^\bullet(\mathcal{U}^{\mu'+\omega_n}, \mathcal{U}^{\mu''+\omega_n}) = \cdots = \bigoplus_{l=3}^{n} \mathrm{Ext}^\bullet(\mathcal{U}^{\mu'}, \mathcal{U}^{\mu''+\nu_l}) \tag{5.57}$$

with $\nu_l = \sum_{i=4}^{l} e_i$. We apply Lemma 3.3.8 to compute the vanishing of the Ext-spaces $\mathrm{Ext}^p(\mathcal{U}^{\mu'}, \mathcal{U}^{\mu''} \otimes \mathcal{U}^{\nu_l})$ for $p \in [0,1]$ and $l \in [3,n]$. Indeed, it is $-\mu'_3 + \mu''_3 = -1 < 0$ as well as $-(\mu'_1 + \mu'_2 + \mu'_3) + (\mu''_1 + \mu''_2 + \mu''_3) = -2 \leq -l + 2$ for $l \in [3,4]$.

(5.56): We apply Lemma 3.3.6 a third time with $\mu' = (y-1)\omega_1 + \omega_3$, $\mu'' = (y-1)\omega_1 + \omega_2$ and $i = 3$. Clearly, it is $-(\mu'_1 + \mu'_2 + \mu'_3) + (\mu''_1 + \mu''_2 + \mu''_3) = -1$ and $-\mu'_3 + \mu''_3 = -1$.

□

Next, we do the induction step from the case i to $i+1$.

Lemma 5.0.7. *For $y \geq 2$ and $i \in [0, y-2]$, we have the short exact sequence*

$$0 \to C_{y,i,0} \to S \otimes F_{y,i+1} \to C_{y,i+1,0} \to 0 \tag{5.58}$$

with the following components

$$0 \to C_{y,i+1,1} \to C_{y,i+1,0} \to \underbrace{\mathcal{U}^{(y-2-i)\omega_1+(i+1)\omega_2+\omega_n}(1)}_{=\mathcal{M}_{y,i+1,0}} \to 0, \tag{5.59}$$

$$0 \to C_{y,i+1,2} \to C_{y,i+1,1} \to \underbrace{\begin{matrix} \mathcal{U}^{(y-2-i)\omega_1+(i+2)\omega_2+\omega_n} \\ \oplus \begin{Bmatrix} \mathcal{U}^{(y-3-i)\omega_1+(i+1)\omega_2+\omega_n}(1) & , \text{if } i \leq y-3 \\ 0 & , \text{if } i = y-2 \end{Bmatrix} \end{matrix}}_{=\mathcal{M}_{y,i+1,1,2}} \to 0, \tag{5.60}$$

and

$$C_{y,i+1,2} = \underbrace{\begin{Bmatrix} \mathcal{U}^{(y-3-i)\omega_1+(i+2)\omega_2+\omega_n} & , \text{if } i \leq y-3 \\ 0 & , \text{if } i = y-2 \end{Bmatrix}}_{=\mathcal{M}_{y,i+1,2,2}}. \tag{5.61}$$

The object $C_{y,i+1,0}$ is a non-splitting G-equivariant extension for all $y \geq i+2$; and $C_{y,i+1,1}$ is so for $y \geq i+3$ respectively.

Proof. Filtration on $S \otimes F_{y,i+1}$. We start similar as in the base case. For this purpose, we consider (5.16)-(5.20) for the case $i+1$:

$$\mathcal{F}_{y,i+1,0} = S \otimes F_{y,i+1}, \tag{5.62}$$

$$0 \to \mathcal{F}_{y,i+1,1} \to \mathcal{F}_{y,i+1,0} \to \underbrace{\mathcal{U}^{(y-2-i)\omega_1+(i+1)\omega_2+\omega_n}(1)}_{=\mathcal{M}_{y,i+1,0}} \to 0, \tag{5.63}$$

$$0 \to \mathcal{F}_{y,i+1,2} \to \mathcal{F}_{y,i+1,1} \to \underbrace{\mathcal{M}_{y,i+1,1,1} \oplus \mathcal{M}_{y,i+1,1,2}}_{=\mathcal{M}_{y,i+1,1}} \to 0 \tag{5.64}$$

where

$$\mathcal{M}_{y,i+1,1,1} = \mathcal{U}^{(y-1-i)\omega_1+(i+1)\omega_2+\omega_n}(1)$$
$$\mathcal{M}_{y,i+1,1,2} = \mathcal{U}^{(y-2-i)\omega_1+(i+2)\omega_2+\omega_n}$$

$$\oplus \begin{Bmatrix} \mathcal{U}^{(y-3-i)\omega_1+(i+1)\omega_2+\omega_n}(1) & \text{, if } i \leq y-3 \\ 0 & \text{, if } i = y-2 \end{Bmatrix},$$

$$0 \to \mathcal{F}_{y,i+1,3} \to \mathcal{F}_{y,i+1,2} \to \underbrace{\mathcal{M}_{y,i+1,2,1} \oplus \mathcal{M}_{y,i+1,2,2}}_{=\mathcal{M}_{y,i+1,2}} \to 0 \tag{5.65}$$

where

$$\begin{aligned} \mathcal{M}_{y,i+1,2,1} &= \mathcal{U}^{(y-1-i)\omega_1+(i+1)\omega_2+\omega_n} \\ &\oplus \mathcal{U}^{(y-2-i)\omega_1+i\omega_2+\omega_n}(1) \\ \mathcal{M}_{y,i+1,2,2} &= \begin{Bmatrix} \mathcal{U}^{(y-3-i)\omega_1+(i+2)\omega_2+\omega_n} & \text{, if } i \leq y-3 \\ 0 & \text{, if } i = y-2 \end{Bmatrix}, \end{aligned}$$

and

$$\mathcal{F}_{y,i+1,3} = \underbrace{\mathcal{U}^{(y-2-i)\omega_1+(i+1)\omega_2+\omega_n}}_{=\mathcal{M}_{y,i+1,3}}. \tag{5.66}$$

Induction hypothesis. Let us assume that the statement holds for the previous case i. This means that we have (5.27)-(5.30). We define $C_{y,i-1,0}$ recursively for $i > 0$; and if $i = 0$, we set $C_{y,-1,0} = \mathcal{S}^{(y)}$ as in the previous Lemma 5.0.5.

$C_{y,i,2}$ coincides with $\mathcal{F}_{y,i+1,3}$. Due to our assumption $i \leq y-2$, we conclude the identity if we compare (5.66) with (5.30).

Embed $C_{y,i,1}$ into $\mathcal{F}_{y,i+1,2}$. We combine the projection of (5.65) with the one onto the summand $\mathcal{M}_{y,i+1,2,2}$ and consider the G-equivariant projection

$$\mathcal{F}_{y,i+1,2} \twoheadrightarrow \mathcal{M}_{y,i+1,2} \twoheadrightarrow \mathcal{M}_{y,i+1,2,2}. \tag{5.67}$$

Snaking yields the short exact sequences

$$0 \to \widetilde{C_{y,i,1}} \to \mathcal{F}_{y,i+1,2} \to \mathcal{M}_{y,i+1,2,2} \to 0 \tag{5.68}$$

and

$$0 \to C_{y,i,2} \to \widetilde{C_{y,i,1}} \to \mathcal{M}_{y,i+1,2,1} \to 0. \tag{5.69}$$

In the remaining part of this step, let us show that $\widetilde{C_{y,i,1}}$ is already $C_{y,i,1}$. So, (5.68) gives the desired embedding. The object $\widetilde{C_{y,i,1}}$ is constructed in (5.68) as kernel of the above projection (5.67) and consequently a G-equivariant object. If we assume that the short exact sequence (5.69) characterising $\widetilde{C_{y,i,1}}$ splits, i.e. $\widetilde{C_{y,i,1}}$ is a direct sum of its components, then we have the following G-equivariant embedding

$$\mathcal{U}^{(y-1-i)\omega_1+(i+1)\omega_2+\omega_n} \hookrightarrow \mathcal{M}_{y,i+1,2,1} \hookrightarrow \widetilde{C_{y,i,1}} \hookrightarrow \mathcal{F}_{y,2,2} \hookrightarrow \mathcal{F}_{y,2,1} \hookrightarrow \mathrm{S} \otimes F_{y,i+1}. \tag{5.70}$$

However, this contradicts to the fact that there are no morphisms from the component $\mathcal{U}^{(y-1-i)\omega_1+(i+1)\omega_2+\omega_n}$ to $F_{y,i+1}$ thanks to the later computation (5.95) in Lemma 5.0.8. Hence, we see that $\widetilde{C_{y,i,1}}$ does not split and accordingly it needs to coincide with $C_{y,i,1}$ up to a scalar. In fact, we recall that $C_{y,i,1}$ is by our induction hypothesis a non-splitting G-equivariant extension.

Embed $C_{y,i,1}$ into $\mathcal{F}_{y,i+1,1}$. We compose the embedding of (5.64) with (5.68) and obtain the G-equivariant embedding

$$C_{y,i,1} \hookrightarrow \mathcal{F}_{y,i+1,2} \hookrightarrow \mathcal{F}_{y,i+1,1}. \tag{5.71}$$

Snaking a further time yields the short exact sequences

$$0 \to C_{y,i,1} \to \mathcal{F}_{y,i+1,1} \to \mathcal{N}_1 \to 0 \tag{5.72}$$

and

$$0 \to \mathcal{M}_{y,i+1,2,2} \to \mathcal{N}_1 \to \mathcal{M}_{y,i+1,1} \to 0. \tag{5.73}$$

Do the technical transition and construct $C_{y,i+1,2}$. We take (5.73) and consider the G-equivariant projection onto the component $\mathcal{M}_{y,i+1,1,2}$, namely

$$\mathcal{N}_1 \twoheadrightarrow \mathcal{M}_{y,i+1,1} \twoheadrightarrow \mathcal{M}_{y,i+1,1,2}. \tag{5.74}$$

Then snaking yields the short exact sequences

$$0 \to \mathcal{N}_2 \to \mathcal{N}_1 \to \mathcal{M}_{y,i+1,1,2} \to 0 \tag{5.75}$$

and

$$0 \to \mathcal{M}_{y,i+1,2,2} \to \mathcal{N}_2 \to \mathcal{M}_{y,i+1,1,1} \to 0. \tag{5.76}$$

Let us observe that the later sequence (5.76) splits and therefore $\mathcal{N}_2$ needs to be the direct sum of its components. If $i \leq y-3$, then we compute the first Ext-space

$$\operatorname{Ext}^1(\underbrace{\mathcal{U}^{(y-1-i)\omega_1+(i+1)\omega_2+\omega_n}(1)}_{=\mathcal{M}_{y,i+1,1,1}}, \underbrace{\mathcal{U}^{(y-3-i)\omega_1+(i+2)\omega_2+\omega_n}}_{=\mathcal{M}_{y,i+1,2,2}}) = 0 \tag{5.77}$$

Indeed, in the case where $i = y-2$, it is obvious as $\mathcal{M}_{y,i+1,2,2}$ vanishes; otherwise see the later computation (5.96) in Lemma 5.0.8. We set

$$C_{y,i+1,2} := \underbrace{\begin{cases} 0 & \text{, if } i = y-2 \\ \mathcal{U}^{(y-3-i)\omega_1+(i+2)\omega_2+\omega_n} & \text{, if } i \leq y-3 \end{cases}}_{=\mathcal{M}_{y,i+1,2,2}}. \tag{5.78}$$

We combine (5.72) with (5.75) and therefore obtain the G-equivariant projection

$$\mathcal{F}_{y,i+1,1} \twoheadrightarrow \mathcal{N}_1 \twoheadrightarrow \mathcal{M}_{y,i+1,1,2}. \tag{5.79}$$

We deduce from snaking the short exact sequences

$$0 \to \mathcal{N}_3 \to \mathcal{F}_{y,i+1,1} \to \mathcal{M}_{y,i+1,1,2} \to 0 \tag{5.80}$$

and

$$0 \to C_{y,i,1} \to \mathcal{N}_3 \to \mathcal{N}_2 \to 0. \tag{5.81}$$

Embed $C_{y,i,0}$ into $\mathcal{F}_{y,i+1,1}$ and construct $C_{y,i+1,1}$. We take the short exact sequence (5.81) and the projection in (5.76) onto $\mathcal{M}_{y,i+1,2,2}$ coming from the splitting to obtain the G-equivariant projection

$$\mathcal{N}_3 \twoheadrightarrow \mathcal{N}_2 \twoheadrightarrow C_{y,i+1,2}. \tag{5.82}$$

This gives us by snaking the short exact sequences

$$0 \to \widetilde{C_{y,i,0}} \to \mathcal{N}_3 \to C_{y,i+1,2} \to 0 \tag{5.83}$$

and

$$0 \to C_{y,i,1} \to \widetilde{C_{y,i,0}} \to \mathcal{M}_{y,i+1,1,1} \to 0. \tag{5.84}$$

Now, let us show that $\widetilde{C_{y,i,0}}$ is already $C_{y,i,0}$. The object $\widetilde{C_{y,i,0}}$ is constructed in (5.83) as kernel of the above projection (5.82) and consequently a G-equivariant object. If we assume that the short exact sequence (5.84) characterising $\widetilde{C_{y,i,0}}$ splits, i.e. $\widetilde{C_{y,i,0}}$ is a direct sum of its components, then we have the following G-equivariant embedding

$$\underbrace{\mathcal{U}^{(y-1-i)\omega_1+(i+1)\omega_2+\omega_n}(1)}_{=\mathcal{M}_{y,i+1,1,1}} \hookrightarrow \widetilde{C_{y,i,0}} \hookrightarrow \mathcal{N}_3 \hookrightarrow \mathcal{F}_{y,i+1,1} \hookrightarrow \mathrm{S} \otimes F_{y,i+1} \tag{5.85}$$

However, this contradicts to the fact that there are no morphisms from the component $\mathcal{M}_{y,i+1,1,1}$ to $F_{y,i+1}$. Indeed, we refer to the subsequent computation (5.97) in Lemma 5.0.8. Hence, we see that $\widetilde{C_{y,i,0}}$ does not split and accordingly it needs to coincide with $C_{y,i,0}$ up to a scalar. In fact, we recall that $C_{y,i,0}$ is by our induction hypothesis a non-splitting G-equivariant extension.
Next, we construct $C_{y,i+1,1}$. For this purpose, we combine (5.80) with (5.83) and obtain the G-equivariant embedding

$$C_{y,i,0} \hookrightarrow \mathcal{N}_3 \hookrightarrow \mathcal{F}_{y,i+1,1}. \tag{5.86}$$

We apply the snake lemma and write the short exact sequences

$$0 \to C_{y,i,0} \to \mathcal{F}_{y,i+1,1} \to C_{y,i+1,1} \to 0 \tag{5.87}$$

and

$$0 \to C_{y,i+1,2} \to C_{y,i+1,1} \to \mathcal{M}_{y,i+1,1,2} \to 0. \tag{5.88}$$

We claim that $C_{y,i+1,1}$ is a non-splitting G-equivariant extension. It is a G-equivariant object since it is by construction of (5.87) the cokernel of a G-equivariant morphism.

Let i be smaller than or equal to $y-3$. If we assume that (5.88) splits, then we have the projection

$$\mathcal{F}_{y,i+1,1} \twoheadrightarrow C_{y,i+1,1} \twoheadrightarrow \underbrace{\mathcal{U}^{(y-3-i)\omega_1+(i+2)\omega_2+\omega_n}}_{=C_{y,i+1,2}}. \tag{5.89}$$

However this contradicts to the fact that there are no morphisms from $\mathcal{F}_{y,i+1,1}$ to $C_{y,i+1,2}$. In fact, we apply $\mathrm{Hom}(-, C_{y,i+1,2})$ to the short exact sequence (5.63) and obtain the long exact sequence

$$\cdots \to \underbrace{\mathrm{Hom}(\mathrm{S} \otimes F_{y,i+1}, C_{y,i+1,2})}_{=0 \text{ by } (5.98)} \to \mathrm{Hom}(\mathcal{F}_{y,i+1,1}, C_{y,i+1,2}) \to \underbrace{\mathrm{Ext}^1(\mathcal{M}_{y,i+1,0}, C_{y,i+1,2})}_{=0 \text{ by } (5.99)} \to \cdots. \tag{5.90}$$

Embed $C_{y,i,0}$ into $\mathcal{F}_{y,i+1,0}$ and define $C_{y,i+1,0}$. We combine (5.63) and (5.87) to obtain the G-equivariant embedding

$$C_{y,i,0} \hookrightarrow \mathcal{F}_{y,i+1,1} \hookrightarrow \mathrm{S} \otimes F_{y,i+1}. \tag{5.91}$$

A last snaking yields the short exact sequences

$$0 \to C_{y,i,0} \to \mathrm{S} \otimes F_{y,i+1} \to C_{y,i+1,0} \to 0 \tag{5.92}$$

and

$$0 \to C_{y,i+1,1} \to C_{y,i+1,0} \to \mathcal{M}_{y,i+1,0} \to 0. \tag{5.93}$$

By construction of (5.92), $C_{y,i+1,0}$ is a G-equivariant object. If we assume that (5.93) splits, we have the projection

$$\mathrm{S} \otimes F_{y,i+1} \twoheadrightarrow C_{y,i+1,0} \twoheadrightarrow C_{y,i+1,1} \twoheadrightarrow \mathcal{M}_{y,i+1,1,2} \to \mathcal{U}^{(y-2-i)\omega_1+(i+2)\omega_2+\omega_n}. \tag{5.94}$$

However, this contradicts to the fact that there are no morphisms from $F_{y,i+1}$ to $\mathcal{U}^{(y-2-i)\omega_1+(i+2)\omega_2+\omega_n}$ due to the computation (5.100) below in Lemma 5.0.8.

□

Lemma 5.0.8. *We show the following vanishings:*

$$\mathrm{Hom}(\mathcal{U}^{(y-1-i)\omega_1+(i+1)\omega_2+\omega_n}, F_{y,i+1}) = 0 \tag{5.95}$$
$$\mathrm{Ext}^1(\mathcal{M}_{y,i+1,1,1}, \mathcal{M}_{y,i+1,2,2}) = 0 \tag{5.96}$$
$$\mathrm{Hom}(\mathcal{M}_{y,i+1,1,1}, F_{y,i+1}) = 0 \tag{5.97}$$
$$\mathrm{Hom}(F_{y,i+1}, C_{y,i+1,2}) = 0 \tag{5.98}$$
$$\mathrm{Ext}^1(\mathcal{M}_{y,i+1,0}, C_{y,i+1,2}) = 0 \tag{5.99}$$
$$\mathrm{Hom}(F_{y,i+1}, \mathcal{U}^{(y-2-i)\omega_1+(i+2)\omega_2+\omega_n}) = 0 \tag{5.100}$$

Proof. We proceed analogously as in the proof of Lemma 5.0.6.

(5.95): We check $-(\mu'_1+\mu'_2+\mu'_3)+(\mu''_1+\mu''_2+\mu''_3)=2$ and $-\mu'_3+\mu''_3=1$. and therefore apply Lemma 3.3.6 with $\mu'=(y-1-i)\omega_1+(i+1)\omega_2$, $\mu''=(y-2-i)\omega_1+(i+1)\omega_2+\omega_3$ and $i=3$.

(5.96): We recall $\mathcal{M}_{y,i+1,1,1}$ is the irreducible vector bundle $\mathcal{U}^{\mu'+\omega_n}$ with $\mu'=(y-i-1)\omega_1+(i+1)\omega_2+\omega_3$ and $\mathcal{M}_{y,i+1,2,2}$ is the irreducible vector bundle $\mathcal{U}^{\mu''+\omega_n}$ with $\mu''=(y-3-i)\omega_1+(i+1)\omega_2$. Then we observe the following similar as for (5.55) in the proof of Lemma 5.0.6 (see (3.47)):

$$\operatorname{Ext}^\bullet(\mathcal{M}_{y,i+1,1,1},\mathcal{M}_{y,i+1,2,2})=\operatorname{Ext}^\bullet(\mathcal{U}^{\mu'+\omega_n},\mathcal{U}^{\mu''+\omega_n})=\cdots=\bigoplus_{l=3}^{n}\operatorname{Ext}^\bullet(\mathcal{U}^{\mu'},\mathcal{U}^{\mu''+\nu_l}) \tag{5.101}$$

with $\nu_l=\sum_{i=4}^{l}e_i$. We apply Lemma 3.3.8 to compute the vanishing of the Ext-spaces $\operatorname{Ext}^p(\mathcal{U}^{\mu'},\mathcal{U}^{\mu''}\otimes\mathcal{U}^{\nu_l})$ for $p\in[0,1]$ and $l\in[3,n]$. Indeed, it is $-\mu'_3+\mu''_3=-1<0$ as well as $-(\mu'_1+\mu'_2+\mu'_3)+(\mu''_1+\mu''_2+\mu''_3)=-5\leq -l+2$ for $l\in[3,4]$.

(5.97): We recall $\mathcal{M}_{y,i+1,1,1}=\mathcal{U}^{(y-i-1)\omega_1+(i+1)\omega_2+\omega_n}(1)$ from (5.22) and apply again Lemma 3.3.6 with $\mu'=(y-1-i)\omega_1+(i+1)\omega_2+\omega_3$, $\mu''=(y-2-i)\omega_1+(i+1)\omega_2+\omega_3$ and $i=3$. It is $-(\mu'_1+\mu'_2+\mu'_3)+(\mu''_1+\mu''_2+\mu''_3)=-1$ and $-\mu'_3+\mu''_3=0$.

(5.98): We recall $C_{y,i+1,2}$ from (5.30): If $i+1=y$, then $C_{y,i+1,2}$ is zero and the desired vanishing follows immediately. Otherwise, if $i+1\leq y-1$, then $C_{y,i+1,2}=\mathcal{U}^{(y-3-i)\omega_1+(i+2)\omega_2+\omega_n}$. We apply as before Lemma 3.3.6 with $\mu'=(y-2-i)\omega_1+(i+1)\omega_2+\omega_3$, $\mu''=(y-3-i)\omega_1+(i+2)\omega_2$ and $i=3$. We check $-(\mu'_1+\mu'_2+\mu'_3)+(\mu''_1+\mu''_2+\mu''_3)=-2$ and $-\mu'_3+\mu''_3=-1$.

(5.99): We recall $\mathcal{M}_{y,i+1,0}$ is the irreducible vector bundle $\mathcal{U}^{\mu'+\omega_n}$ with $\mu'=(y-i-2)\omega_1+(i+1)\omega_2+\omega_3$ and $C_{y,i+1,0}$ is the irreducible vector bundle $\mathcal{U}^{\mu''+\omega_n}$ with $\mu''=(y-3-i)\omega_1+(i+2)\omega_2$. Then we observe the following similar as before (see (3.47)):

$$\operatorname{Ext}^\bullet(\mathcal{M}_{y,i+1,0},C_{y,i+1,2})=\operatorname{Ext}^\bullet(\mathcal{U}^{\mu'+\omega_n},\mathcal{U}^{\mu''+\omega_n})=\cdots=\bigoplus_{l=3}^{n}\operatorname{Ext}^\bullet(\mathcal{U}^{\mu'},\mathcal{U}^{\mu''+\nu_l}) \tag{5.102}$$

with $\nu_l=\sum_{i=4}^{l}e_i$. We apply Lemma 3.3.8 to compute the vanishing of the Ext-spaces $\operatorname{Ext}^p(\mathcal{U}^{\mu'},\mathcal{U}^{\mu''}\otimes\mathcal{U}^{\nu_l})$ for $p\in[0,1]$ and $l\in[3,n]$. In fact, it is $-\mu'_3+\mu''_3=-1<0$ as well as $-(\mu'_1+\mu'_2+\mu'_3)+(\mu''_1+\mu''_2+\mu''_3)=-2\leq -l+2$ for $l\in[3,4]$.

(5.100): We apply Lemma 3.3.6 at last time. Hence, we take $\mu'=(y-2-i)\omega_1+(i+1)\omega_2+\omega_3$, $\mu''=(y-2-i)\omega_1+(i+2)\omega_2$ as well as $i=3$, and we check $-(\mu'_1+\mu'_2+\mu'_3)+(\mu''_1+\mu''_2+\mu''_3)=-1$ and $-\mu'_3+\mu''_3=-1$.

□

Finally, we check the last part of the resolution 5.13.

Lemma 5.0.9. *For $y \geq 1$, we have the isomorphism*

$$C_{y,y-1,0} \cong \mathcal{S}^{(y)\vee}(y+1). \tag{5.103}$$

Proof. We construct the object $C_{y,y-1,0}$ through the previous lemma 5.0.7 and therefore obtain the following G-equivariant extensions

$$0 \to C_{y,y-1,1} \to C_{y,y-1,0} \to \mathcal{M}_{y,y-1,0} \to 0, \tag{5.104}$$
$$0 \to C_{y,y-1,2} \to C_{y,y-1,1} \to \mathcal{M}_{y,y-1,1,2} \to 0, \tag{5.105}$$

and

$$C_{y,y-1,2} = \mathcal{M}_{y,y-1,2,2}. \tag{5.106}$$

Since we have

$$\mathcal{M}_{y,y-1,0} = \mathcal{U}^{(y-1)\omega_2+\omega_n}(1), \tag{5.107}$$
$$\mathcal{M}_{y,y-1,1,2} = \mathcal{U}^{y\omega_2+\omega_n}, \tag{5.108}$$

and

$$\mathcal{M}_{y,y-1,2,2} = 0, \tag{5.109}$$

we deduce that $C_{y,y-1,0}$ needs to be non-splitting G-equivariant extension

$$0 \to \mathcal{U}^{y\omega_2+\omega_n} \to C_{y,y-1,0} \to \mathcal{U}^{(y-1)\omega_2+\omega_n}(1) \to 0. \tag{5.110}$$

Thus, it coincides with $\mathcal{S}^{(y)\vee}(y+1)$ up to rescaling. □

Proving exceptionality. Throughout this paragraph, let $\mathcal{S}^{(i')} = \mathcal{S}^{(y')}(x')$ and $\mathcal{S}^{(i'')} = \mathcal{S}^{(y'')}(x'')$ be two objects from the spinor subcollection such that $i'' \leq i'$ – i.e. we have either the case $i' = i''$ which is

$$x' = x'' \text{ and } y' = y'' \tag{5.111}$$

or the case $i'' < i'$ which means

$$x' = x'' \text{ and } y'' < y', \text{ or } x'' < x'. \tag{5.112}$$

We recall the defining short exact sequence (5.2) to write

$$0 \to \mathcal{U}^{(y'-1)\omega_1+\omega_n}(x') \to \mathcal{S}^{(y')}(x') \to \mathcal{U}^{y'\omega_1+\omega_n}(x') \to 0 \tag{5.113}$$

and

$$0 \to \mathcal{U}^{(y''-1)\omega_1+\omega_n}(x'') \to \mathcal{S}^{(y'')}(x'') \to \mathcal{U}^{y''\omega_1+\omega_n}(x'') \to 0. \tag{5.114}$$

Proposition 5.0.10. *The spinor subcollection $C^{(\mathcal{S})}$ is exceptional.*

Proof. Case (5.111). We check that the objects $\mathcal{S}^{(i')} = \mathcal{S}^{(y')}(x')$ in the spinor subcollection $C^{(\mathcal{S})}$ are exceptional. We write $\mathcal{E}' = \mathcal{S}^{(y')}$ in the following. If $y' = 0$ – i.e. $\mathcal{S}^{(0)}$ is the spinor bundle $\mathcal{S}$, we refer to [20, Proposition 6.8.] before we start decomposing $\mathcal{S}^\vee \otimes \mathcal{S}$ and fall back on Proposition 2.5.2 with respect to the appearing summands. Otherwise, if $y' \in [1, n-2]$, we apply the Hom-functor $\mathrm{Hom}(-, \mathcal{E}')$ to the resolution (5.13) for y' given in Proposition 5.0.3. Accordingly, we obtain families of long exact sequences:

$$\begin{aligned}
\cdots \to \mathrm{Ext}^p(\mathcal{E}', \mathcal{E}') \to \mathrm{Ext}^p(\mathrm{S} \otimes F_{y',0}, \mathcal{E}') \to \mathrm{Ext}^p(C_{y',0,0}, \mathcal{E}') \to \cdots \\
\cdots \to \mathrm{Ext}^p(C_{y',0,0}, \mathcal{E}') \to \mathrm{Ext}^p(\mathrm{S} \otimes F_{y',1}, \mathcal{E}') \to \mathrm{Ext}^p(C_{y',1,0}, \mathcal{E}') \to \cdots \\
\vdots \quad \\
\cdots \to \mathrm{Ext}^p(C_{y',y'-1,0}, \mathcal{E}') \to \mathrm{Ext}^p(\mathrm{S} \otimes F_{y',y'-1}, \mathcal{E}') \to \mathrm{Ext}^p(\mathcal{F}', \mathcal{E}') \to \cdots
\end{aligned} \tag{5.115}$$

where $\mathcal{F}' = \mathcal{S}^{(y')\vee}(y'+1)$. As the Ext-spaces $\mathrm{Ext}^\bullet(F_{y',i}, \mathcal{E}')$ for $i \in [0, y'-1]$ vanish by the later computation (5.122) of Lemma 5.0.11, we deduce the isomorphisms

$$\begin{aligned}
\mathrm{Ext}^p(\mathcal{E}', \mathcal{E}') \cong \mathrm{Ext}^{p-1}(C_{y',0,0}, \mathcal{E}') \cong \mathrm{Ext}^{p-2}(C_{y',1,0}, \mathcal{E}') \cong \cdots \\
\cong \mathrm{Ext}^{p-y'}(C_{y',y'-1,0}, \mathcal{E}') \cong \mathrm{Ext}^{p-y'-1}(\mathcal{F}', \mathcal{E}').
\end{aligned} \tag{5.116}$$

The last Ext-space computes as

$$\mathrm{Ext}^p(\mathcal{F}', \mathcal{E}') \cong \mathrm{Ext}^p(\mathcal{S}^{(y')\vee}(y'+1), \mathcal{S}^{(y')}) = \begin{cases} \mathbf{K} & , \text{ if } p = y' \\ 0 & , \text{ else} \end{cases}. \tag{5.117}$$

thanks to the computation (5.125) in Lemma 5.0.12.

Case (5.112). We check that the object $\mathcal{S}^{(i')} = \mathcal{S}^{(y')}(x')$ is right orthogonal to the object $\mathcal{S}^{(i'')} = \mathcal{S}^{(y'')}(x'')$. Let us write $\mathcal{E}' = \mathcal{S}^{(y')}$ as well as $\mathcal{E}'' = \mathcal{S}^{(y'')}$.

1. If we are in the subcase $x' = x''$ and $y'' < y'$, then we argue analogously as in the case before. First we apply $\mathrm{Hom}(-, \mathcal{E}'')$ to the resolution (5.13) for y' and then we deduce the vanishings

$$\mathrm{Ext}^p(\mathcal{E}', \mathcal{E}'') \cong \mathrm{Ext}^{p-y'-1}(\mathcal{F}', \mathcal{E}'') = 0 \tag{5.118}$$

where $\mathcal{F}' = \mathcal{S}^{(y')\vee}(y'+1)$. In fact, we refer to the subsequent computations (5.122) of Lemma 5.0.11 and (5.125) of Lemma 5.0.12.

2. Let us assume the subcase $x'' < x'$.

First, if y'' equals zero, we consider the Ext-space $\mathrm{Ext}^\bullet(\mathcal{E}'(x'), \mathcal{S}(x''))$ and therefore skip the following long exact sequence. Otherwise, if y'' is from $[1, n-2]$, we apply the Hom-functor $\mathrm{Hom}(\mathcal{E}'(x'), -)$ to (5.114):

$$\begin{aligned} \cdots \to \mathrm{Ext}^p(\mathcal{E}'(x'), \mathcal{U}^{(y''-1)\omega_1+\omega_n}(x'')) \to \\ \mathrm{Ext}^p(\mathcal{E}'(x'), \mathcal{E}''(x'')) \to \\ \mathrm{Ext}^p(\mathcal{E}'(x'), \mathcal{U}^{y''\omega_1+\omega_n}(x'')) \to \cdots \end{aligned} \tag{5.119}$$

Second, if y' equals zero, we consider the outer Ext-spaces. Notice that the one on the left hand side does only appear if $1 \leq y''$. Accordingly, we skip (maybe a second time) the following long exact sequence. Otherwise, if y' is from $[1, n-2]$, we apply the Hom-functors $\mathrm{Hom}(-, \mathcal{U}^{(y''-1)\omega_1+\omega_n}(x''))$ and $\mathrm{Hom}(-, \mathcal{U}^{y''\omega_1+\omega_n}(x''))$ to (5.113):

$$\begin{aligned} \cdots \to \mathrm{Ext}^p(\mathcal{U}^{y'\omega_1+\omega_n}(x'), \mathcal{U}^{(y''-1)\omega_1+\omega_n}(x'')) \to \\ \mathrm{Ext}^p(\mathcal{E}'(x'), \mathcal{U}^{(y''-1)\omega_1+\omega_n}(x'')) \to \\ \mathrm{Ext}^p(\mathcal{U}^{(y'-1)\omega_1+\omega_n}(x'), \mathcal{U}^{(y''-1)\omega_1+\omega_n}(x'')) \to \cdots \end{aligned} \tag{5.120}$$

as well as

$$\begin{aligned} \cdots \to \mathrm{Ext}^p(\mathcal{U}^{y'\omega_1+\omega_n}(x'), \mathcal{U}^{y''\omega_1+\omega_n}(x'')) \to \\ \mathrm{Ext}^p(\mathcal{E}'(x'), \mathcal{U}^{y''\omega_1+\omega_n}(x'')) \to \\ \mathrm{Ext}^p(\mathcal{U}^{(y'-1)\omega_1+\omega_n}(x'), \mathcal{U}^{y''\omega_1+\omega_n}(x'')) \to \cdots \end{aligned} \tag{5.121}$$

Again, let us mention that the first one does only appear if $1 \leq y''$. Finally, we refer to the subsequent computations (5.135) of Lemma 5.0.14.

□

Lemma 5.0.11. *Recall the irreducible G-equivariant vector bundle $F_{y,i} = \mathcal{U}^{(y-1-i)\omega_1+i\omega_2}(1)$ with $i \in [0, y-1]$ from (5.15). If $y' \in [0, n-2]$ and $y'' \in [0, y']$, then we have the vanishing*

$$\mathrm{Ext}^\bullet(F_{y',i}, \mathcal{S}^{(y'')}) = 0 \tag{5.122}$$

Proof. Preparations if $y'' \in [1, n-2]$. If y'' equals zero, then we drop this step and proceed immediately to the next one. Otherwise, if $y'' \in [1, n-2]$, then we apply the Hom-functor $\mathrm{Hom}(F_{y',i}, -)$ to the short exact sequence (5.114) that defines $\mathcal{S}^{(y'')}$ with $x'' = 0$. This gives us the long exact sequences of Ext-spaces, namely

$$\cdots \to \mathrm{Ext}^p(F_{y',i}, \mathcal{U}^{(y''-1)\omega_1+\omega_n}) \to \mathrm{Ext}^p(F_{y',i}, \mathcal{S}^{(y'')}) \to \mathrm{Ext}^p(F_{y',i}, \mathcal{U}^{y''\omega_1+\omega_n}) \to \cdots \tag{5.123}$$

By the subsequent computation, we see that the outer Ext-spaces vanish, and therefore also the middle term as desired.

$F_{y',i}$ is right orthogonal to $\mathcal{U}^{y''\omega_1+\omega_n}$ whenever $y' \in [1, n-2]$ and $y'' \in [0, y']$. First, we check that $F_{y',i} = \mathcal{U}^{(y'-1-i)\omega_1+i\omega_2}(1)$ is right orthogonal to $\mathcal{U}^{y''\omega_1}$ in any mentioned cases of y' and y'':

1. $y' \in [1, n-3]$: We observe that $F_{y',i} = \mathcal{U}^{(y'-1-i)\omega_1+i\omega_2}(1)$ is right orthogonal to $\mathcal{U}^{y''\omega_1}$ due to Proposition 4.1.2. Let us mention that it is $\mu_i - \mu_j \leq \mu_1 - \mu_3 \leq -\mu_3' + \mu_1'' + \mu_1' - \mu_3'' = y'' + y' - 1 \leq 2n-7 \leq 2n-5+i-j$ for any $i < j \in [1, 3]$.

2. $y' = n-2$: Let $\mathcal{U}^{\mu}$ be an irreducible summand in the direct sum decomposition of the tensor product $\mathcal{U}^{(n-3-i)\omega_1+i\omega_2+\omega_3 \vee} \otimes \mathcal{U}^{y''\omega_1}$. Then we estimate the entries of μ thanks to Lemma 3.3.1 and Corollary 3.3.2:

$$\begin{aligned} -1 \leq \mu_1 &\leq y''-1 \\ -i-1 \leq \mu_2 &\leq -1 \\ -n+2 \leq \mu_3 &\leq -1 \\ \mu_i &= 0 \qquad \text{for } i \in [4, n] \end{aligned} \tag{5.124}$$

If we have $\mu_3 + n + \frac{1}{2} - 3 \in [\frac{1}{2}, n - \frac{7}{2}]$ or equivalently $\mu_3 \in [-n+3, -1]$, then the third entry of $\mu + \rho_G$ coincides with the j^{th} ones where j is from $[4, n]$. Otherwise, if $\mu_3 = -n+2$ or equivalently $\mu_3 + n - \frac{5}{2} = -\frac{1}{2}$, then the third entry equals the n^{th} one up to a sign. In both cases, we refer to (2.24) and deduce that $\mu + \rho_G$ is G-singular.

Let us check that the difference $\mu_i - \mu_j$ can be estimated by $2n-5+i-j$ whenever $\mu_i + \mu_j = -2n-1+i+j$ for $i < j \in [1, 3]$. In fact, due to $4 \leq n$ we have $-2n+2+j \leq -2n+5 \leq -n+2$ and therefore $-2n+2+j \leq \mu_3 \leq \mu_j$. So, it follows $\mu_i - \mu_j = -2\mu_j - 2n - 1 + i + j \leq 2n-5+i-j$.

Next, Lemma 3.3.4 implies that $F_{y',i}$ is even right orthogonal to $\mathcal{U}^{y\omega_1} \otimes \mathcal{S} = \mathcal{U}^{y\omega_1+\omega_n}$. □

Lemma 5.0.12. *We compute*

$$\mathrm{Ext}^\bullet(\mathcal{S}^{(y')\vee}(y'+1), \mathcal{S}^{(y'')}) = \begin{cases} \mathbf{K}[-y'] & , \text{ if } y' = y'' \\ 0 & , \text{ if } y'' < y' \end{cases} \tag{5.125}$$

Proof. $y' = 0$. In this case, we need to have $y'' = 0$ likewise. We have $\mathcal{S}^{(0)\vee}(1) = \mathcal{S}^\vee(1) = \mathcal{S}$ as the spinor bundle $\mathcal{S}$ is self-dual up to a $\mathcal{O}(1)$-twist (see (3.1) or [20, Proposition 6.6.]) and it is also exceptional (see [20, Proposition 6.8.]). All in all, this gives us $\mathrm{Ext}^\bullet(\mathcal{S}^\vee(1), \mathcal{S}) = \mathbf{K}[0]$.

$y' \in [1, n-2]$. First we dualize and twist the short exact sequence (5.2) defining the object $\mathcal{S}^{(y')}$:

$$0 \to \underbrace{\mathcal{U}^{y'\omega_1+\omega_n \vee}(y'+1)}_{=\mathcal{U}^{y'\omega_2+\omega_n}} \to \mathcal{S}^{(y')\vee}(y'+1) \to \underbrace{\mathcal{U}^{(y'-1)\omega_1+\omega_n \vee}(y'+1)}_{=\mathcal{U}^{(y'-1)\omega_2+\omega_n}(1)} \to 0. \tag{5.126}$$

Then we apply the Hom-functor $\mathrm{Hom}(-, \mathcal{S}^{(y'')})$ and obtain the following:

$$\begin{aligned} \cdots \to \mathrm{Ext}^p(\mathcal{U}^{(y'-1)\omega_2+\omega_n}(1), \mathcal{S}^{(y'')}) \to \\ \mathrm{Ext}^p(\mathcal{S}^{(y')\vee}(y'+1), \mathcal{S}^{(y'')}) \to \\ \mathrm{Ext}^p(\mathcal{U}^{y'\omega_2+\omega_n}, \mathcal{S}^{(y'')}) \to \cdots \end{aligned} \tag{5.127}$$

Thanks to the subsequent steps, we compute the outer terms in (5.127) and therefore deduce the claimed statement of the Lemma. In fact, if y'' equals zero, then we skip the remaining part of this step and argue immediately to the corresponding statements as it is $\mathcal{S}^{(0)} = \mathcal{S}$. Otherwise, if $y'' \in [1, y']$, we apply the Hom-functors $\mathrm{Hom}(\mathcal{U}^{(y'-1)\omega_2+\omega_n}(1), -)$ or $\mathrm{Hom}(\mathcal{U}^{y'\omega_2+\omega_n}, -)$ respectively to (5.114) with $x'' = 0$. This gives us

$$\begin{aligned} \cdots \to \mathrm{Ext}^p(\mathcal{U}^{(y'-1)\omega_2+\omega_n}(1), \mathcal{U}^{(y''-1)\omega_1+\omega_n}) \to \\ \mathrm{Ext}^p(\mathcal{U}^{(y'-1)\omega_2+\omega_n}(1), \mathcal{S}^{(y'')}) \to \\ \mathrm{Ext}^p(\mathcal{U}^{(y'-1)\omega_2+\omega_n}(1), \mathcal{U}^{y''\omega_1+\omega_n}) \to \cdots \end{aligned} \tag{5.128}$$

as well as

$$\begin{aligned} \cdots \to \mathrm{Ext}^p(\mathcal{U}^{y'\omega_2+\omega_n}, \mathcal{U}^{(y''-1)\omega_1+\omega_n}) \to \\ \mathrm{Ext}^p(\mathcal{U}^{y'\omega_2+\omega_n}, \mathcal{S}^{(y'')}) \to \\ \mathrm{Ext}^p(\mathcal{U}^{y'\omega_2+\omega_n}, \mathcal{U}^{y''\omega_1+\omega_n}) \to \cdots. \end{aligned} \tag{5.129}$$

Left term in (5.127). Let y'' be from $[0, y']$. We claim that $\mathcal{U}^{(y'-1)\omega_2+\omega_n}(1)$ is right orthogonal to $\mathcal{U}^{y''\omega_1+\omega_n}$ – i.e.

$$\mathrm{Ext}^\bullet(\mathcal{U}^{(y'-1)\omega_2+\omega_n}(1), \mathcal{U}^{y''\omega_1+\omega_n}) = 0. \tag{5.130}$$

Due to (3.47) we consider the subspaces $\mathrm{Ext}^\bullet(\mathcal{U}^{\mu'}, \mathcal{U}^{\mu''+\nu_l})$ where $\mu' = (y', y', 1, 0, \cdots, 0)$, $\mu'' = (y'', 0, 0, 0, \cdots, 0)$, $\nu_l = \sum_{i=4}^{l} \mathrm{e}_i$, and $l \in [3, n]$ and we intend to apply Lemma 3.3.7 for each possible l. For this purpose, let us show the following aspects:

- $\mathcal{U}^{\mu'}$ is right orthogonal to $\mathcal{U}^{\mu''}$. If $y'' \in [0, n-3]$, then we refer to Proposition 4.3.4; and if $y'' = y' = n-2$ – i.e. $\mathrm{Ext}^\bullet(\mathcal{U}^{(n-3)\omega_2}(1), \mathcal{U}^{(n-2)\omega_1})$, then we refer to the later computation (5.133) in Lemma 5.0.13.
- Let μ be a highest weight of an irreducible summand $\mathcal{U}^{\mu}$ appearing in the direct sum decomposition of $\mathcal{U}^{\mu'\vee} \otimes \mathcal{U}^{\mu''}$. If μ contains an entry $\mu_i = -l + i$ where $i \in [1, 3]$, then we can check that $\mu + \nu_l + \rho_{\mathrm{G}}$ is G-singular. In fact, if $l = 3$ and

hence $\nu_l = 0$, $\mu + \nu_l + \rho_G = \mu + \rho_G$ is G-singular as we checked previously the corresponding right orthogonal relation and therefore $\mathcal{U}^{\mu}$ has no cohomology. Otherwise, if $4 \leq l$, we distinguish the following cases with respect to i:

1. $i = 3$ – i.e. $\mu_3 = -l + 3$: We have $-l + 3 = \mu_3 \leq \mu_2 \leq -1$ and therefore $\mu_2 \in [-l + 3, -1]$. This gives us $\mu_2 + n - \frac{3}{2} \in [n + \frac{3}{2} - l, n - \frac{5}{2}]$. So, the second entry of $\mu + \nu_l + \rho_G$ coincides with the j^{th} one where j is from $[4, l]$.

2. $i = 2$ – i.e. $\mu_2 = -l + 2$: If $\mu_3 = -n + 2$, then the third entry of $\mu + \nu_l + \rho_G$ is calculated as $-\frac{1}{2}$ and therefore it coincides up to a sign with the n^{th} one if $l < n$ or with the second one if $l = n$. Otherwise, if $\mu_3 \in [-n + 3, -l + 2]$ or equivalently $\mu_3 + n - \frac{5}{2} \in [\frac{1}{2}, n - \frac{1}{2} - l]$, then the third entry of $\mu + \nu_l + \rho_G$ equals the j^{th} one where j is from $[l + 1, n]$.

3. $i = 1$ – i.e. $\mu_1 = -l + 1$: This case can not occur as $-1 \leq \mu_1 = -l + 1$ and accordingly $l \leq 2$ contradicts to the fact $4 \leq l$.

- Let $\mathcal{U}^{\mu}$ be as before. If μ contains an entry $\mu_i = 2n - 1 + i + l$ where $i \in [1, 3]$, then we can check likewise that $\mu + \nu_l + \rho_G$ is G-singular. However, this case can never occur as $\mu_i = 2n - 1 + i + l$ contradicts to the fact $\mu_3 \leq \mu_2 \leq \mu_1 \leq n - 3$.

Right term in (5.127). Let y'' be from $[0, y']$. We claim that $\mathcal{U}^{y'\omega_2 + \omega_n}$ is right orthogonal to $\mathcal{U}^{y''\omega_1 + \omega_n}$ if $y'' < y'$ and $\operatorname{Ext}^\bullet(\mathcal{U}^{y'\omega_2 + \omega_n}, \mathcal{U}^{y'\omega_1 + \omega_n})$ has only a trivial component in degree y' if $y'' = y'$ – i.e.

$$\operatorname{Ext}^p(\mathcal{U}^{y'\omega_2 + \omega_n}, \mathcal{U}^{y''\omega_1 + \omega_n}) = \begin{cases} 0 & , \text{ if } y'' < y' \\ \mathbf{K}[-y'] & , \text{ if } y'' = y' \end{cases}. \tag{5.131}$$

Due to (3.47) we consider the subspaces $\operatorname{Ext}^\bullet(\mathcal{U}^{\mu'}, \mathcal{U}^{\mu'' + \nu_l})$ where $\mu' = (y', y', 0, 0, \cdots, 0)$, $\mu'' = (y'', 0, 0, 0, \cdots, 0)$, $\nu_l = \sum_{i=4}^{l} \mathrm{e}_i$, and $l \in [3, n]$ and we are going to show

$$\operatorname{Ext}^\bullet(\mathcal{U}^{\mu'}, \mathcal{U}^{\mu'' + \nu_l}) = \begin{cases} 0 & , \text{ if } y'' < y' \\ 0 & , \text{ if } y'' = y' \leq n - 3 \text{ and } l \neq y' + 3 \\ \mathbf{K}[-y'] & , \text{ if } y'' = y' \leq n - 3 \text{ and } l = y' + 3 \\ 0 & , \text{ if } y'' = y' = n - 2 \text{ and } l \neq n \\ \mathbf{K}[-y'] & , \text{ if } y'' = y' = n - 2 \text{ and } l = n \end{cases}. \tag{5.132}$$

For this purpose, let us consider the five cases separately:

1. $y'' < y'$: The claimed statement follows from Lemma 3.3.7 as we check the necessary assumptions:

   - $\mathcal{U}^{\mu'}$ is right orthogonal to $\mathcal{U}^{\mu''}$. If $y' \in [1, n - 3]$, then we refer to Proposition 4.3.4; otherwise, if $y' = n - 2$, then we refer to the subsequent computation (5.134) in Lemma 5.0.13.

- Let μ be a highest weight of an irreducible summand $\mathcal{U}^{\mu}$ appearing in the direct sum decomposition of $\mathcal{U}^{\mu'\vee} \otimes \mathcal{U}^{\mu''}$. If μ contains an entry $\mu_i = -l + i$ where $i \in [1,3]$, then we can check that $\mu + \nu_l + \rho_G$ is G-singular. Indeed, if $l = 3$ and hence $\nu_l = 0$, we have $\mu + \nu_l + \rho_G = \mu + \rho_G$. Due to the previous right orthogonal relation the irreducible summand $\mathcal{U}^{\mu}$ has no cohomology and therefore $\mu + \rho_G$ is G-singular. Otherwise, if $4 \leq l$, we distinguish the following cases with respect to i:
  (a) $i = 3$ – i.e. $\mu_3 = -l + 3$: If $\mu_2 = 0$, then we have $\mu_1 + 0 + (-l + 3) = -2y' + y'' < -y'$ or equivalently $\mu_1 < l - 3 - y'$. However this contradicts the fact $-y' \leq \mu_3 = -l + 3$ or equivalently $l - 3 - y' \leq 0$ as we need to have $0 \leq \mu_1$ by Corollary 3.3.2. Otherwise, if $\mu_2 \in [-l + 3, -1]$ or equivalently $\mu_2 + n - \frac{3}{2} \in [n + \frac{3}{2} - l, n - \frac{5}{2}]$, then the second entry of $\mu + \nu_l + \rho_G$ coincides with the j^{th} one where j is from $[4, l]$.
  (b) $i = 2$ – i.e. $\mu_2 = -l + 2$: If $\mu_3 = -n + 2$, then the third entry of $\mu + \nu_l + \rho_G$, namely $-\frac{1}{2}$, equals up to a sign with the n^{th} one if $l < n$ or with the second one if $l = n$. Otherwise, if $\mu_3 \in [-n + 3, -l + 2]$ or equivalently $\mu_3 + n - \frac{5}{2} \in [\frac{1}{2}, n - \frac{1}{2} - l]$, then the third entry of $\mu + \nu_l + \rho_G$ coincide with the j^{th} one where j is from $[l + 1, n]$.
  (c) $i = 1$ – i.e. $\mu_1 = -l + 1$: This case can not occur as $\mu_1 = -l + 1 < 0$ contradicts to the fact $0 \leq \mu_1$ coming from Corollary 3.3.2.
- Let $\mathcal{U}^{\mu}$ be as before. If μ contains an entry $\mu_i = 2n - 1 + i + l$ where $i \in [1,3]$, then we can check likewise that $\mu + \nu_l + \rho_G$ is G-singular. However, this case can not occur as we always have $\mu_3 \leq \mu_2 \leq \mu_1 \leq y'' < y' \leq n - 2$.

2. $y'' = y' \leq n - 3$ and $l \neq y' + 3$: Again, we apply Lemma 3.3.7:
   - $\mathcal{U}^{\mu'}$ is right orthogonal to $\mathcal{U}^{\mu''}$. Again, we refer to Proposition 4.3.4.
   - Let μ be a highest weight of an irreducible summand $\mathcal{U}^{\mu}$ appearing in the direct sum decomposition of $\mathcal{U}^{\mu'\vee} \otimes \mathcal{U}^{\mu''}$. If μ contains an entry $\mu_i = -l + i$ where $i \in [1,3]$, then we can check that $\mu + \nu_l + \rho_G$ is G-singular. In fact, we follow the arguments both for $l = 3$ and $4 \leq l$ as in the previous case $y'' < y'$ straightforward. Let us just mention the case where $\mu_3 = -l + 3$ and $\mu_2 = 0$, then we have $\mu_1 + 0 + (-l + 3) = -y'$ or equivalently $\mu_1 = l - 3 - y' = l - (y' + 3)$. Hence, we deduce $\mu_1 \neq 0$ from our current assumption $l \neq y' + 3$ and therefore $1 \leq \mu_1$ by Corollary 3.3.2. However, this contradicts to the fact $-y' \leq \mu_3 = -l + 3$ or equivalently $l - 3 - y' \leq 0$.
   - Let $\mathcal{U}^{\mu}$ be as before. If μ contains an entry $\mu_i = 2n - 1 + i + l$ where $i \in [1,3]$, then we can check likewise that $\mu + \nu_l + \rho_G$ is G-singular. Thanks to $\mu_3 \leq \mu_2 \leq \mu_1 \leq y'' = y' \leq n - 3$ there is nothing to check.

3. $y'' = y' \leq n-3$ and $l = y'+3$: Let $\mathcal{U}^{\mu}$ be an irreducible summand appearing in the direct sum decomposition of $\mathcal{U}^{\mu'^\vee} \otimes \mathcal{U}^{\mu''}$. We compute the cohomology of $\mathcal{U}^{\mu+\nu_{y'+3}}$. First, recall Corollary 3.3.2 to estimate the ranges of entries of μ:

   (a) $\mu_3 \in [-y'+1, 0]$ or equivalently $\mu_3 + n - \frac{5}{2} \in [n - \frac{3}{2} - y', n - \frac{5}{2}]$: The third entry of $\mu + \nu_{y'+3} + \rho_G$ coincides with the i^{th} one where i is from $[4, y'+3]$.

   (b) $\mu_3 = -y'$: It is $\mu_1 + \mu_2 + \mu_3 = -y'$ and therefore in this subcase even $\mu_2 = -\mu_1$. If μ_1 is from $[1, y']$, then μ_2 need to be from $[-y', -1]$ and hence $\mu_2 + n - \frac{3}{2}$ lies in $[n - \frac{3}{2} - y', n - \frac{5}{2}]$. So, the second entry of $\mu + \nu_{y'+3} + \rho_G$ is equal to the i^{th} one where i is from $[4, y'+3]$. Otherwise, if μ_1 is zero and equivalently μ_2 likewise, then the entries of $\mu + \nu_{y'+3} + \rho_G$ are of the form

   $$(\underset{1^{\text{st}}}{n-\frac{1}{2}}, \underset{2^{\text{nd}}}{n-\frac{3}{2}}, \underset{3^{\text{rd}}}{n-\frac{5}{2}-y'}, \underset{4^{\text{th}}}{n-\frac{5}{2}}, \cdots, \underset{(y'+3)^{\text{th}}}{n-\frac{3}{2}-y'}, \underset{(y'+4)^{\text{th}}}{n-\frac{7}{2}-y'}, \cdots, \underset{n^{\text{th}}}{\frac{1}{2}}).$$

   This means, the unique Weyl element $w \in \mathrm{W_G}$ mapping $\mu + \nu_{y'+3} + \rho_G$ to the dominant cone $\mathbf{P}_G^+$ is given by the composition of simple reflections from the third entry to the $(y'+2)^{\text{th}}$, namely $w = w_{y'+2} \circ \cdots \circ w_3$. Certainly, w has length y' and it maps $\mu + \nu_{y'+3} + \rho_G$ to ρ_G. By Proposition 2.5.2 we compute the desired cohomology.

4. $y'' = y' = n-2$ and $l \neq n$: Again, we apply Lemma 3.3.7:

   - $\mathcal{U}^{\mu'}$ is right orthogonal to $\mathcal{U}^{\mu''}$. We refer to the subsequent computation (5.134) in Lemma 5.0.13.
   - Let μ be a highest weight of an irreducible summand $\mathcal{U}^{\mu}$ appearing in the direct sum decomposition of $\mathcal{U}^{\mu'^\vee} \otimes \mathcal{U}^{\mu''}$. If μ contains an entry $\mu_i = -l + i$ where $i \in [1,3]$, then we can check that $\mu + \nu_l + \rho_G$ is G-singular. Indeed, we proceed similar as before in the first case $y'' < y'$.
   - Let $\mathcal{U}^{\mu}$ be as before. If μ contains an entry $\mu_i = 2n - 1 + i + l$ where $i \in [1,3]$, then we can check likewise that $\mu + \nu_l + \rho_G$ is G-singular. We mention $\mu_3 \leq \mu_2 \leq \mu_1 \leq y'' = y' = n-2$ and thus finish this subcase.

5. $y'' = y' = n-2$ and $l = n$: Similar as before, let $\mathcal{U}^{\mu}$ be an irreducible summand appearing in the direct sum decomposition of $\mathcal{U}^{\mu'^\vee} \otimes \mathcal{U}^{\mu''}$ and we compute the cohomology of $\mathcal{U}^{\mu+\nu_n}$. By Corollary 3.3.2 we estimate the lower and upper bounds on the entries of μ:

   (a) $\mu_3 \in [-n+3, 0]$: Due to $-n+3 \leq \mu_3 \leq \mu_2 \leq 0$ we have $\mu_2 \in [-n+3, 0]$. If $\mu_2 = 0$, then it follows from $\mu_1 + \mu_2 + \mu_3 = -n+2$ the inequality $\mu_1 = -n+2-\mu_3 \leq -1$. However this contradicts to the fact $0 \leq \mu_1$. Otherwise, if μ_2 is in $[-n+3, -1]$ or equivalently $\mu_2 + n - \frac{3}{2} \in [\frac{3}{2}, n - \frac{5}{2}]$, then the second entry of $\mu + \nu_n + \rho_G$ is equal to the i^{th} one where i is from $[4, n]$.

$\mu_3 = -n+2$: It is $\mu_1 + \mu_2 + \mu_3 = -n+2$ and consequently even $\mu_2 = -\mu_1$. If $\mu_1 = n-2$ and equivalently $\mu_2 = -n+2$, then the second and third entry of $\mu + \nu_n + \rho_G$ coincide up to a sign, namely $\mu_2 + n - \frac{3}{2} = \frac{1}{2} = -\mu_3 - n + \frac{5}{2}$. If μ_1 is from $[1, n-3]$ and thus μ_2 from $[-n+3, -1]$, then $\mu_2 + n - \frac{3}{2}$ need to be an element of $[\frac{3}{2}, n - \frac{5}{2}]$. In this case, the second entroy of $\mu + \nu_n + \rho_G$ equals the i^{th} one where i is from $[4, n]$. Finally, if μ_1 vanishes and μ_2 is zero likewise, then the entries of $\mu + \nu_n + \rho_G$ are of the form

$$(\underset{1^{\text{st}}}{n - \frac{1}{2}}, \underset{2^{\text{nd}}}{n - \frac{3}{2}}, \underset{3^{\text{rd}}}{-\frac{1}{2}}, \underset{4^{\text{th}}}{n - \frac{5}{2}}, \cdots, \underset{n^{\text{th}}}{\frac{3}{2}}).$$

The unique Weyl element $w \in W_G$ mapping $\mu + \nu_n + \rho_G$ to the dominant cone $\mathbf{P}_G^+$ is given by the composition of simple reflections from the third entry to the n^{th}, namely $w = w_n \circ \cdots \circ w_3$. The first $n-3$ reflections mutate the third entry to the last positon and then the last reflection switches the sign. Certainly, w has length $n-2$ and it maps $\mu + \nu_n + \rho_G$ to ρ_G. By Proposition 2.5.2 we compute again the desired cohomology.

□

Lemma 5.0.13.

$$\operatorname{Ext}^\bullet(\mathcal{U}^{(n-3)\omega_2}(1), \mathcal{U}^{(n-2)\omega_1}) = 0 \tag{5.133}$$

$$\operatorname{Ext}^\bullet(\mathcal{U}^{(n-2)\omega_2}, \mathcal{U}^{y''\omega_1}) = 0 \qquad y'' \in [0, n-2] \tag{5.134}$$

Proof. (5.133)**:** Let $\mathcal{U}^\mu$ be an irreducible summand in the direct sum decomposition of the tensor product $\mathcal{U}^{\mu'\vee} \otimes \mathcal{U}^{\mu''}$ where $\mu' = (n-2, n-2, 1, 0, \cdots, 0)$ and $\mu'' = (n-2, 0, 0, 0, \cdots, 0)$. The third entry of μ can be estimate from below by $-n+2$ and from above by -1 by Corollary 3.3.2 – i.e. $\mu_3 \in [-n+2, -1]$. If $\mu_3 \in [-n+3, -1]$, then we apply condition (3.39) for $i = 3$; and if $\mu_3 = -n+2$, then we refer to (3.41) for $i = 3$.

(5.134)**:** Let $\mathcal{U}^\mu$ be an irreducible summand in the direct sum decomposition of the tensor product $\mathcal{U}^{\mu'\vee} \otimes \mathcal{U}^{\mu''}$ where $\mu' = (n-2, n-2, 0, 0, \cdots, 0)$ and $\mu'' = (y'', 0, 0, 0, \cdots, 0)$. The third entry of μ ranges from $-n+2$ to 0. If $\mu_3 = 0$, then we have $\mu_1 + \mu_2 + \mu_3 = -2n + 4 + y''$ and hence $\mu_1 + \mu_2 = -2n + 4 + y'' \leq -n + 2$. However this contradicts to $0 \leq \mu_1 + \mu_2$ as we have $0 = \mu_3 \leq \mu_2 \leq \mu_1$. Otherwise, if $\mu_3 \in [-n+3, -1]$, we apply condition (3.39) for $i = 3$; or if $\mu_3 = -n+2$, then condition (3.41) for $i = 3$.

□

Lemma 5.0.14. *Let $x'' < x'$ be in $[0, 2n-4]$, let y' be from*

$$\begin{cases} [0, n-2] & , \text{ if } x' \in [1, n-3] \\ [0, n-3] & , \text{ if } x' \in [n-2, 2n-4] \end{cases},$$

and let y'' be from

$$\begin{cases} [0, n-2] & , \text{ if } x'' \in [0, n-3] \\ [0, n-3] & , \text{ if } x'' \in [n-2, 2n-5] \end{cases}.$$

Then we have the vanishing

$$\mathrm{Ext}^\bullet(\mathcal{U}^{y'\omega_1+\omega_n}(x'), \mathcal{U}^{y''\omega_1+\omega_n}(x'')) = 0 \tag{5.135}$$

Proof. We recall (3.47) and then write $\mu' = (y'+x', x', x', 0, \cdots, 0)$, $\mu'' = (y''+x'', x'', x'', 0, \cdots, 0)$, and $\nu_l = \sum_{i=4}^{l} \mathrm{e}_i$ with $l \in [3, n]$. Let μ be a highest weight of an irreducible summand $\mathcal{U}^\mu$ appearing in the direct sum decomposition of $\mathcal{U}^{\mu'^\vee} \otimes \mathcal{U}^{\mu''}$. Now, we check the necessary conditions of Lemma 3.3.7 to show the desired right orthogonal relation.

- If y' and y'' are both from $[0, n-3]$, then $\mathcal{U}^{\mu'} = \mathcal{U}^{y'\omega_1}(x')$ is right orthogonal to $\mathcal{U}^{\mu''} = \mathcal{U}^{y''\omega_1}(x'')$ by Proposition 4.3.4; otherwise, if $y' = n-2$ or $y'' = n-2$, then we refer to the subsequent computation (5.136) in Lemma 5.0.15.
- If μ contains an entry $\mu_i = -l+i$ where $i \in [1, 3]$, then we can check that $\mu + \nu_l + \rho_G$ is G-singular. In fact, if $l = 3$ and hence $\nu_l = 0$, we have $\mu + \nu_l + \rho_G = \mu + \rho_G$ which G-singular as any irreducible summand $\mathcal{U}^\mu$ is without cohomology by the previous right orthogonal relation. Otherwise, if $4 \le l$, we distinguish the following cases with respect to i:
  1. $i = 3$ – i.e. $\mu_3 = -l+3$: We have $-l+3 \le \mu_3 \le \mu_2 \le -x'+x'' \le -1$ and therefore $\mu_2 \in [-l+3, -1]$ or equivalently $\mu_2 + n - \frac{3}{2} \in [n + \frac{3}{2} - l, n - \frac{5}{2}]$. So, the second entry of $\mu + \nu_l + \rho_G$ coincides with the j^{th} one where j is from $[4, l]$.
  2. $i = 2$ – i.e. $\mu_2 = -l+2$: It is $-x'+x'' = -\mu_2' + \mu_3'' \le \mu_2 \le -\mu_3' + \mu_2'' = -x'+x''$ by Corollary 3.3.2. Thus, we deduce $-l+2 = \mu_2 = -x'+x''$. Furthermore, this gives us $-2n+4 \le -n-l+4 \le \mu_3 \le \mu_2 = -l+2$, i.e. μ_3 lies in the integer interval $[-2n+4, -l+2]$.
     (a) $\mu_3 \in [-n+3, -l+2]$ or equivalently $\mu_3 + n - \frac{5}{2} \in [\frac{1}{2}, n - \frac{1}{2} - l]$: We compare the third entry of $\mu + \nu_l + \rho_G$ to the j^{th} one where j is from $[l+1, n]$.
     (b) $\mu_3 \in [-2n+3+l, -n+2]$ or equivalently $\mu_3 + n - \frac{5}{2} \in [-n + \frac{1}{2} + l, -\frac{1}{2}]$: The third entry of of $\mu + \nu_l + \rho_G$ is up to a sign the j^{th} one where j is from $[l+1, n]$.

(c) $\mu_3 = -2n+2+l$ or equivalently $\mu_3+n-\frac{5}{2} = -n-\frac{1}{2}+l$: The third entry of $\mu+\nu_l+\rho_G$ is up to a sign the second one which is $\mu_2+n-\frac{3}{2} = n+\frac{1}{2}-l$.

(d) $\mu_3 \in [-2n+5, -2n+1+l]$ or equivalently $\mu_3+n-\frac{5}{2} \in [-n+\frac{5}{2}, -n-\frac{3}{2}+l]$: The third entry of $\mu+\nu_l+\rho_G$ coincides up to a sign the j^{th} one for some j from $[4,l]$. In fact, it is $\mu_j+1+n+\frac{1}{2}-j = n+\frac{3}{2}-j$.

(e) $\mu_3 = -2n+4$: We start with $\mu_1+(-l+2)+(-2n+4) = -y'+y''+3(-x'+x'') = -y'+y''+3(-l+2) = -y'+y''-3l+6$ or equivalently $\mu_1 = -y'+y''+2n-2l$. Then we consider $-y'+y''+2n-2l = \mu_1 \leq -l+2+y''$ and therefore $2n-l-2 \leq y' \leq n-2$. So, this case can only occur if $l=n$ and accordingly $y'=n-2$ as well as $x' \in [1, n-3]$. As we have $x'' = x'-n+2 \leq -1$ this contradicts to the fact $0 \leq x''$.

3. $i=1$ – i.e. $\mu_1 = -l+1$: Due to $-2n+4 \leq -x'+x'' = \mu_2 \leq \mu_1 \leq -l+1$, we have $\mu_2 \in [-2n+4, -l+1]$.

   (a) $\mu_2 \in [-n+2, -l+1]$ or equivalently $\mu_2+n-\frac{3}{2} \in [\frac{1}{2}, n-\frac{1}{2}-l]$: We compare the second entry of $\mu+\nu_l+\rho_G$ to the j^{th} one where j is from $[l+1, n]$.

   (b) $\mu_2 \in [-2n+2+l, -n+1]$ or equivalently $\mu_2+n-\frac{3}{2} \in [-n+\frac{1}{2}+l, -\frac{1}{2}]$: The second entry of $\mu+\nu_l+\rho_G$ equals up to a sign to the j^{th} one where j is from $[l+1, n]$.

   (c) $\mu_2 = -2n+1+l$: We compute $\mu_2+n-\frac{3}{2} = -n-\frac{1}{2}+l = -(\mu_1+n-\frac{1}{2})$.

   (d) $\mu_2 \in [-2n+4, -2n+l]$ or equivalently $\mu_2+n-\frac{3}{2} \in [-n+\frac{5}{2}, -n-\frac{3}{2}+l]$: The second entry of $\mu+\nu_l+\rho_G$ coincides up to a sign with the j^{th} one where j is from $[4,l]$.

- If μ contains an entry $\mu_i = 2n-1+i+l$ where $i \in [1,3]$, then we can check likewise that $\mu+\nu_l+\rho_G$ is G-singular. However, this case can not occur as we always have $\mu_3 \leq \mu_2 \leq \mu_1 \leq -x'+x''+y'' < n-2$.

□

Lemma 5.0.15. *Let $x'' < x'$ be from $[0, 2n-4]$ and let y' as well as y'' from $[0, n-2]$ such that y', y'' or both are equal to $n-2$.*

$$\operatorname{Ext}^\bullet(\mathcal{U}^{y'\omega_1}(x'), \mathcal{U}^{y''\omega_1}(x'')) = 0 \tag{5.136}$$

Proof. (5.136)**:** Let $\mathcal{U}^\mu$ be an irreducible summand in the direct sum decomposition of the tensor product $\mathcal{U}^{\mu'^\vee} \otimes \mathcal{U}^{\mu''}$ where $\mu' = (y'+x', x', x', 0, \cdots, 0)$ and $\mu'' = (y''+x'', x'', x'', 0, \cdots, 0)$. For the second entry of μ we observe $-x'+x'' \leq$

$\mu_2 \leq -x' + x''$ by Corollary 3.3.2 – i.e. $\mu_2 = -x' + x'' \in [-2n+4, -1]$. We distinguish with respect to μ_2:

1. $\mu_2 = -1$: We have $-n+1 \leq \mu_3 \leq \mu_2 = -1$ – i.e. $\mu_3 \in [-n+1, -1]$. If $\mu_3 \in [-n+3, -1]$, then we refer to condition (3.39) for $i = 3$. Given $\mu_3 = -n+2$, we apply condition (3.41) for $i = 3$. If $\mu_3 = -n+1$, we need to distinguish if either $n = 4$ or $5 \leq n$. In the first subcase, we compare the second entry of $\mu + \rho_G$, namely $-1 + 4 + \frac{1}{2} - 2 = \frac{3}{2}$, to the third one, namely $-3 + 4 + \frac{1}{2} - 3 = -\frac{3}{2}$ – see also condition (3.40) for $i = 2 < j = 3 \in [1, 3]$. In the second subcase, we refer to condition (3.41) as before.

2. $\mu_2 \in [-n+2, -2]$: We apply condition (3.39) for $i = 2$.

3. $\mu_2 \in [-2n+5, -n+1]$: We refer to condition (3.41) for $i = 2$.

4. $\mu_2 = -2n+4$: It is $-2n+4 = \mu_2 \leq \mu_1 \leq -n+2$ – i.e. $\mu_1 \in [-2n+4, -n+2]$. If $\mu_1 = -n+2$ and $n = 4$, we compare the first and the second entry of $\mu + \rho_G$ – i.e. condition (3.40) for $i = 1 < j = 2 \in [1, 3]$. For the case where $\mu_1 = -n+2$ and $5 \leq n$ or where $\mu_1 = -n+1$, we apply condition (3.39) for $i = 1$. For $\mu_1 = -n+1$, we apply condition (3.39) for $i = 1$ likewise. Finally, if $\mu_1 \in [-2n+4, -n]$, then we refer to condition (3.41) for $i = 1$.

$\square$

Chapter 6

Merging the subcollections

6.1 Consecutive composition

We merge our two collections by concatenating the tautological one $C^{(\mathcal{U})}$ with the spinor one $C^{(\mathcal{S})}$. This means concretely, the starting block $C^{(\mathrm{Con})}$ is the ordered set $C_0^{(\mathcal{U})} \cup C_0^{(\mathcal{S})}$ and summing up the support partitions (4.1), (4.27), and (5.3) yields $(h_0, \cdots, h_{2n-4})$ with

$$h_x = \frac{5}{8}n^2 - \frac{7}{8}n + \frac{1}{8}n \cdot (-1)^n + \left\{ \begin{array}{ll} -\frac{5}{16} & \text{, if } x \in [0, n-3] \\ -\frac{21}{16} & \text{, if } x \in [n-2, 2n-4] \end{array} \right\} - \frac{11}{16} \cdot (-1)^n. \quad (6.1)$$

Consequently, we count

$$\frac{5}{4}n^3 - \frac{29}{8}n^2 + \frac{1}{4}n^2 \cdot (-1)^n + n - \frac{7}{4}n \cdot (-1)^n + \frac{31}{16} + \frac{33}{16} \cdot (-1)^n \quad (6.2)$$

objects for the total collection $C^{(\mathrm{Con})}$. Comparing this amount to the expected length l_{max} in (2.1), we have a difference of the form

$$\begin{aligned} &\frac{1}{12}n^3 - \frac{3}{8}n^2 - \frac{1}{4}n^2 \cdot (-1)^n + \frac{5}{3}n + \frac{7}{4}n \cdot (-1)^n - \frac{31}{16} - \frac{33}{16} \cdot (-1)^n \\ &= \left(\frac{1}{24}n^2 - \frac{1}{8}n - \frac{1}{8}n \cdot (-1)^n + \frac{31}{48} + \frac{11}{16} \cdot (-1)^n \right) \left(2n - 3 \right). \end{aligned} \quad (6.3)$$

Example 6.1.1. For $\mathrm{OGr}(3, V)$ with $n = 7$, we combine (4.2) and (5.5) such that we obtain

$$C^{(\mathrm{Con})} = \begin{pmatrix} \mathcal{S}^{(5)} & \mathcal{S}^{(5)}(1) & \cdots & \mathcal{S}^{(5)}(4) & & & \\ \mathcal{S}^{(4)} & \mathcal{S}^{(4)}(1) & \cdots & \mathcal{S}^{(4)}(4) & \mathcal{S}^{(4)}(5) & \cdots & \mathcal{S}^{(4)}(10) \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \mathcal{S}^{(0)} & \mathcal{S}^{(0)}(1) & \cdots & \mathcal{S}^{(0)}(4) & \mathcal{S}^{(0)}(5) & \cdots & \mathcal{S}^{(0)}(10) \\ \mathcal{U}^{4\omega_1+2\omega_2} & \mathcal{U}^{4\omega_1+2\omega_2}(1) & \cdots & \mathcal{U}^{4\omega_1+2\omega_2}(4) & \mathcal{U}^{4\omega_1+2\omega_2}(5) & \cdots & \mathcal{U}^{4\omega_1+2\omega_2}(10) \\ \mathcal{U}^{3\omega_1+2\omega_2} & \mathcal{U}^{3\omega_1+2\omega_2}(1) & \cdots & \mathcal{U}^{3\omega_1+2\omega_2}(4) & \mathcal{U}^{3\omega_1+2\omega_2}(5) & \cdots & \mathcal{U}^{3\omega_1+2\omega_2}(10) \\ \mathcal{U}^{4\omega_1+\omega_2} & \mathcal{U}^{4\omega_1+\omega_2}(1) & \cdots & \mathcal{U}^{4\omega_1+\omega_2}(4) & \mathcal{U}^{4\omega_1+\omega_2}(5) & \cdots & \mathcal{U}^{4\omega_1+\omega_2}(10) \\ \mathcal{U}^{4\omega_2} & \mathcal{U}^{4\omega_2}(1) & \cdots & \mathcal{U}^{4\omega_2}(4) & \mathcal{U}^{4\omega_2}(5) & \cdots & \mathcal{U}^{4\omega_2}(10) \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \mathcal{O} & \mathcal{O}(1) & \cdots & \mathcal{O}(4) & \mathcal{O}(5) & \cdots & \mathcal{O}(10) \end{pmatrix} \quad (6.4)$$

This means, the support partition is $(24,24,24,24,24,23,23,23,23,23,23)$.

Proposition 6.1.2. *The collection $C^{(Con)}$ is exceptional.*

Proof. First of all. We refer to Proposition 4.1.2 for the semi-orthogonal relations of the tautological subcollection $C^{(\mathcal{U})}$ and to Proposition 5.0.10 for the semi-orthogonal relations of the spinor subcollection $C^{(\mathcal{S})}$. Hence, we are left to show the necessary relations among these two subcollections. For this purpose, we fix throughout this proof the following: Let $\mathcal{U}^\lambda$ be an object from the starting block $C_0^{(\mathcal{U})}$ of the tautological subcollection $C^{(\mathcal{U})}$ (see section 4) and let $\mathcal{S}^{(y)}$ be an object from the starting block $C_0^{(\mathcal{S})}$ of the spinor subcollection $C^{(\mathcal{S})}$ (see section 5).

$\mathcal{S}^{(y)}(x)$ is right orthogonal to $\mathcal{U}^\lambda$ for suitable x and y. If $y=0$ – i.e. we consider $\mathcal{S}^{(0)}$, we skip the next intermediate step and argue immediately to the later claim. Otherwise, for $y \in [1,n-2]$ we apply $\mathrm{Hom}(-,\mathcal{U}^\lambda)$ to a $\mathcal{O}(x)$-twisted version of short exact sequence (5.8) defining $\mathcal{S}^{(y)}(x)$ and obtain the long exact sequence of Ext-spaces:

$$\begin{aligned}\cdots \to \mathrm{Ext}^\bullet(\mathcal{U}^{y\omega_1+\omega_n}(x),\mathcal{U}^\lambda) \to \mathrm{Ext}^\bullet(\mathcal{S}^{(y)}(x),\mathcal{U}^\lambda) \to \\ \mathrm{Ext}^\bullet(\mathcal{U}^{(y-1)\omega_1+\omega_n}(x),\mathcal{U}^\lambda) \to \cdots\end{aligned} \tag{6.5}$$

As the outer Ext-spaces vanish by the following computations, we see the desired right orthogonal relation.

We claim that $\mathcal{U}^{y\omega_1+\omega_n}(x)$ is right orthogonal to $\mathcal{U}^\lambda$ and therefore refer to Lemma 6.1.3.

$\mathcal{U}^\lambda(x)$ is right orthogonal to $\mathcal{S}^{(y)}$ for suitable x and y. Similarly as before, if $y=0$ – i.e. the object $\mathcal{S}^{(0)}$, then we skip the following intermediate step. Otherwise, if $y \in [1,n-2]$, the desired vanishing follows from the long exact sequence of Ext-spaces:

$$\begin{aligned}\cdots \to \mathrm{Ext}^\bullet(\mathcal{U}^\lambda(x),\mathcal{U}^{(y-1)\omega_1+\omega_n}) \to \mathrm{Ext}^\bullet(\mathcal{U}^\lambda(x),\mathcal{S}^{(y)}) \to \\ \mathrm{Ext}^\bullet(\mathcal{U}^\lambda(x),\mathcal{U}^{y\omega_1+\omega_n}) \to \cdots\end{aligned} \tag{6.6}$$

We claim that $\mathcal{U}^\lambda(x)$ is right orthogonal to $\mathcal{U}^{y\omega_1+\omega_n}$ and accordingly refer to the later Lemma 6.1.6. □

Lemma 6.1.3. *The vector bundle $\mathcal{U}^{y\omega_1+\omega_n}(x)$ is right orthogonal to the vector bundle $\mathcal{U}^\lambda$ whenever $(x,y) \in [0,2n-4] \times [0,n-3] \cup [0,n-3] \times \{n-2\}$ and λ is highest weight of a vector bundle appearing in the starting block $C_0^{(\mathcal{U})}$ of the tautological subcollection $C^{(\mathcal{U})}$.*

Proof. **$x \in [0,2n-5]$ and $y \in [0,n-3]$:** We refer to Propositions 4.3.4 and observe the vanishing of the Ext-space from $\mathcal{U}^{y\omega_1}(x+1)$ to $\mathcal{U}^\lambda$. Hence, we apply Lemma 3.3.4. Its technical assumption is covered by the following considerations.

$i = 1$ **and** $j = 2$**:** We estimate $\mu_1 - \mu_2 \le \lambda_1 \le \lfloor \frac{3}{2}n - \frac{9}{2} \rfloor \le 2n - 6 = 2n - 5 + i - j$.

$i = 2$ **and** $j = 3$**:** We compute $\mu_2 - \mu_3 \le \lambda_2 + y \le 2n - 6 = 2n - 5 + i - j$.

$i = 1$ **and** $j = 3$**:** We start with $\mu_1 - \mu_3 \le \lambda_1 + n - 2 \le \lfloor \frac{5}{2}n - \frac{15}{2} \rceil$ and we can often estimate $\mu_1 - \mu_3 \le 2n - 7 = 2n - 5 + i - j$.
However, let us give a proof for this case which works always: We show that μ_{i^*} ranges between $-2n + 5 = -2n + 3 + i^*$ and $-2 = -4 + i^*$ for $i^* = 2$. For this purpose, we assume $\mu_1 + \mu_3 = -2n - 1 + i + j = -2n + 3$. If $-1 \le \mu_2$, then we have $2n - 5 \le \mu_1 + \mu_2 + 2n - 3 = \mu_2 - \mu_3$ and this contradicts with the previous observations in the case $i = 2, j = 3$. Otherwise, if $\mu_2 \le -2n + 4$, then $2n - 5 \le -\mu_2 - \mu_3 - 2n + 3 = \mu_1 - \mu_2$ contradicts with the previous computation of the case $i = 1, j = 2$.

$x = 2n - 4$ **and** $y \in [0, n - 3]$**:** We distinguish with respect to the form of λ.

$\lambda \notin \{0, \omega_1, \cdots, y\omega_1\}$**:** This means, we have $1 \le \lambda_2$ if $\lambda_1 \in [0, y]$.
We refer to the vanishing statement of the later computation (6.7) in Lemma (6.1.4). Hence, we apply Lemma 3.3.4 and therefore check the necessary technical condition analogously to the previous case.

$\lambda \in \{0, \omega_1, \cdots, y\omega_1\}$**:** It is $y \ge \lambda_1 \ge \lambda_2 = 0$.
We refer to the vanishing statement of the later computation (6.8) in Lemma (6.1.4).

$x \in [0, n - 3]$ **and** $y = n - 2$**:** We consider the following subcases.

$x \ne n - 4$ **or** $\lambda \ne (n - 3)\omega_2$**:** If $\lambda \ne (n - 3)\omega_2$, then we need to have $\lambda_2 \le n - 4$. Indeed, either λ is a highest weight appearing in the starting block $C_0^{(\mathcal{U},1)}$ of the first part $C^{(\mathcal{U},1)}$ of the tautological subcollection $C^{(\mathcal{U})}$ and then it follows from $n - 3 \ge \lambda_1 \ge \lambda_2$; or λ appears in the starting block $C_0^{(\mathcal{U},2)}$ of the second part $C^{(\mathcal{U},2)}$ such that $\lambda_2 \le \lceil \frac{1}{2}n - \frac{3}{2} \rceil \le n - 4$ as $5 \le n$. (Recall that the second part $C^{(\mathcal{U},2)}$ is empty if $n \le 4$.)
We refer to the later computation (6.9) in Lemma 6.1.4, namely that $\mathcal{U}^{(n-2)\omega_1}(x + 1)$ is right orthogonal to $\mathcal{U}^{\lambda}$. Thus, we apply Lemma 3.3.4 and convince ourselves that the technical assumption holds. For this purpose, let us write $\mu' = (n - 1 + x, x + 1, x + 1, 0, \cdots, 0)$ and $\mu'' = (\lambda_1, \lambda_2, 0, 0, \cdots, 0)$.

$i = 1$ **and** $j = 2$**:** It is $\mu_1 - \mu_2 \le \lambda_1 \le \lfloor \frac{3}{2}n - \frac{9}{2} \rfloor \le 2n - 6 = 2n - 5 + i - j$.

$i = 2$ **and** $j = 3$**:** We consider various cases with respect to x:

$x \in [0, n - 5]$**:** We compute $-2n + 6 \le -n + 1 - x \le \mu_3 \le -x - 2 \le -1$. This means, the third entry μ_3 lives in the interval $[-2n + 6, -1]$ and therefore μ satisfies the conditions (3.39) or (3.41) respectively with $i^* = 3$.

$x = n-4$: We need to have $\lambda \neq (n-3)\omega_2$ due to the above assumption. Consequently, it is $\lambda_2 \leq n-4$ and therefore it follows immediately $\mu_2 - \mu_3 \leq \lambda_2 + n - 2 \leq 2n-6 = 2n-5+i-j$.

$x = n-3$: First, we observe that $-2n+4 \leq \mu_3 \leq -n+2 \leq \mu_2 \leq -1$ as well as $\mu_1 + \mu_2 + \mu_3 \leq -2n+2$ (since $\lambda_1 + \lambda_2 \leq 2n-6$). Second, if we assume $\mu_2 + \mu_3 = -2n-1+i+j = -2n+4$, then we sharpen the upper bound to $\mu_2 \leq \mu_1 \leq -2$. Accordingly, we estimate $\mu_2 - \mu_3 \leq 2n-6 = 2n-5+i-j$ as desired.

$i = 1$ and $j = 3$: We start with $\mu_1 - \mu_3 \leq \lambda_1 + n - 2 \leq \lfloor \frac{5}{2}n - \frac{13}{2} \rceil$ and we can often estimate $\mu_1 - \mu_3 \leq 2n-7 = 2n-5+i-j$.
Nevertheless, let us present a proper proof.

$x \in [0, n-5]$: We proceed the same way as in the corresponding subcase of $(i,j) = (2,3)$.

$x = n-4$: The third entry μ_3 ranges between $-2n+5$ and $-x-1 \leq -1$. For the case $\mu_3 \in [-2n+6, -1]$ we argue similar as before. If $\mu_3 = -2n+5$, then we have $-n+3 \leq \mu_2 \leq \mu_1 = -2n+3-\mu_1 = -2$ and therefore μ satisfies the condition (3.39) with $i^* = 2$.

$x = n-3$: Again we start with $\mu_1 + \mu_2 + \mu_3 \leq -2n+2$. If we assume $\mu_1 + \mu_3 = -2n-1+i+j = -2n+3$, then we deduce $-n+2 \leq \mu_2 \leq -1$.

$\mu_2 = -1$: We have $-1 = \mu_2 \leq \mu_1 = -2n+3-\mu_3 \leq -1$ and accordingly $\mu_3 = -2n+4$. This means, we have $\mu = (-1,-1,-2n+4,0,\cdots,0)$ and accordingly $\mu + \omega_n + \rho_G = (n-1, n-2, -n+2, n-3, \cdots, 1)$ is G-singular as its second and third entry coincide up to sign.

$\mu_2 \in [-n+2, -2]$: μ satisfies the condition (3.39) with $i^* = 2$.

$x = n-4$ and $\lambda = (n-3)\omega_2$: It is $\lambda_1 = \lambda_2 = n-3$. We refer to the later computation (6.10) presented in Lemma 6.1.4.

□

Lemma 6.1.4. *Let $\mathcal{U}^\lambda$ be an object from the starting block $C_0^{(\mathcal{U})}$ of the tautological subcollection $C^{(\mathcal{U})}$. We show the following vanishings:*

$$\mathrm{Ext}^\bullet(\mathcal{U}^{y\omega_1}(2n-3), \mathcal{U}^\lambda) = 0 \qquad \lambda \notin \{0, \omega_1, \cdots, y\omega_1\} \tag{6.7}$$
$$\mathrm{Ext}^\bullet(\mathcal{U}^{y\omega_1+\omega_n}(2n-4), \mathcal{U}^\lambda) = 0 \qquad \lambda \in \{0, \omega_1, \cdots, y\omega_1\} \tag{6.8}$$
$$\mathrm{Ext}^\bullet(\mathcal{U}^{(n-2)\omega_1}(x+1), \mathcal{U}^\lambda) = 0 \qquad x \neq n-4 \text{ or } \lambda \neq (n-3)\omega_2 \tag{6.9}$$
$$\mathrm{Ext}^\bullet(\mathcal{U}^{(n-2)\omega_1+\omega_n}(n-4), \mathcal{U}^{(n-3)\omega_2}) = 0 \tag{6.10}$$

Remark 6.1.5. One can compute $\mathrm{Ext}^\bullet(\mathcal{U}^{y\omega_1}(2n-3), \mathcal{U}^{\lambda_1\omega_1}) = \mathrm{V}_{\mathrm{G}}^{(y-\lambda_1)\omega_1}[-d]$ whenever $\lambda_1 \leq y$ in $[0, n-3]$ as well as $\mathrm{Ext}^\bullet(\mathcal{U}^{(n-2)\omega_1}(n-4), \mathcal{U}^{(n-3)\omega_2}) = \mathbf{K}[-2n+5]$. Hence, we can not show the desired vanishing of Ext-spaces presented in (6.8) and (6.10) respectively as we did several times before via Lemma 3.3.4. However, we need to check these Ext-spaces explicitly.

Proof. **(6.7):** Let μ' be the weight $(y+2n-3, 2n-3, 2n-3, 0, \cdots, 0)$ and let μ'' be the weight $(\lambda_1, \lambda_2, 0, 0, \cdots, 0)$ with $1 \leq \lambda_2$ if $\lambda_1 \in [0, y]$. Given an irreducible summand $\mathcal{U}^\mu$ from the direct sum decomposition of the tensor product $\mathcal{U}^{\mu'\vee} \otimes \mathcal{U}^{\mu''}$, we estimate for the second entry by Corollary 3.3.2 the range $-2n+3 \leq \mu_2 \leq -2n+3+\lambda_2 \leq -n$.

$\mu_2 \in [-2n+5, -n]$: μ is satisfies condition (3.41) with $i = 2$. Thus, the corresponding summand $\mathcal{U}^\mu$ does not have any cohomology by Proposition 2.5.2.

$\mu_2 \in [-2n+3, -2n+4]$: We compute $\mu_1 - \mu_2 \leq \lambda_1$ and hence obtain $-2n+3 \leq \mu_2 \leq \mu_1 \leq \lambda_1 - 2n + 4 \leq \lfloor -\frac{1}{2}n - \frac{1}{2} \rfloor \leq -3$.

$\mu_1 \in [-2n+4, -3]$: $\mu + \rho_{\mathrm{G}}$ is G-singular by conditions (3.39) or (3.41) respectively with $i = 1$. Again, we can state that the corresponding summand $\mathcal{U}^\mu$ has vanishing cohomology by Proposition 2.5.2.

$\mu_1 = -2n+3$: This means, we have $-4n+6+\mu_3 = \mu_1+\mu_2+\mu_3 = -y - 6n+9+\lambda_1+\lambda_2$ and accordingly deduce $\mu_3 = -y+\lambda_1+\lambda_2-2n+3$. Therefore we first write $\mu = (-2n+3, -2n+3, -y+\lambda_1+\lambda_2-2n+3, 0, \cdots, 0)$ and then likewise $\mu + \rho_{\mathrm{G}} = (-n+\frac{5}{2}, -n+\frac{3}{2}, -y+\lambda_1+\lambda_2-n+\frac{1}{2}, n-\frac{7}{2}, \cdots, \frac{1}{2})$. Thus, we are left to check that the third entry of $\mu + \rho_{\mathrm{G}}$ coincides with one of the other entries up to a sign, namely

$$-y+\lambda_1+\lambda_2-n+\frac{1}{2} \in \Big\{-n+\frac{3}{2}, -n+\frac{5}{2}, -n+\frac{7}{2}, \cdots, -\frac{1}{2}, \\ \frac{1}{2}, \cdots, n-\frac{7}{2}, n-\frac{5}{2}, n-\frac{3}{2}\Big\}$$

or equivalently

$$-y+\lambda_1+\lambda_2 \in \{1, 2, 3, \cdots, n-1, \\ n, \cdots, 2n-4, 2n-3, 2n-2\}$$

$\lambda_1 \in [y+1, \lfloor \frac{3}{2}n - \frac{9}{2} \rfloor]$: In this case, λ_2 lies in the interval $[0, n-3]$ by construction of λ in chapter 4. Thus, we have $1 \leq -y+\lambda_1 \leq -y+\lambda_1+\lambda_2 \leq \lambda_1+\lambda_2 \leq 2n-6$.

$\lambda_1 \in [0, y]$: λ_2 needs to be in the interval $[1, \lambda_1]$ by the initial assumption of this case. So, we have $\mu_3 = -y - 2n + 3 + \lambda_1 < \mu_3 = -y - 2n + 3 + \lambda_1 + \lambda_2$ which contradicts to the assumption $\mu_3 \leq -y - 2n + 3 + \lambda_1$ arising by Corollary 3.3.2.

(6.8): We rewrite the tensor product $\mathcal{U}^{y\omega_1+\omega_n}(2n-4)^\vee \otimes \mathcal{U}^\lambda = \mathcal{U}^{y\omega_1}(2n-3)^\vee \otimes \mathcal{U}^\lambda \otimes \mathcal{U}^{\omega_n}$ due to Lemma 3.2.3. Hence, we need to check that any weight $\mu + \omega_n + \rho_G$ is G-singular whenever μ is a highest weight of an irreducible summand appearing in the direct sum decomposition of the tensor product $\mathcal{U}^{y\omega_1}(2n-3)^\vee \otimes \mathcal{U}^\lambda$. For this purpose, let μ' be $(y + 2n - 3, 2n - 3, 2n - 3, 0, \cdots, 0)$ and let μ'' be $(\lambda_1, 0, 0, 0, \cdots, 0)$. Corollary 3.3.2 implies that any highest weight μ appearing in the tensor product $\mathcal{U}^{\mu'\vee} \otimes \mathcal{U}^{\mu''}$ need to be of the form $(\mu_1, -2n + 3, \mu_3, 0, \cdots, 0)$ where $\mu_1 \in [-2n + 3, -2n + 3 + \lambda_1]$ and $\mu_3 \in [-y - 2n + 3, -2n + 3]$. Given such a weight μ, then $\mu + \omega_n + \rho_G$ needs to be of the form $(\mu_1 + n, -n + 2, \mu_3 + n - 2, n - 3, \cdots, 1)$. We consider the first entry of $\mu + \omega_n + \rho_G$ and observe that it lies in the interval $[-n + 3, -n + 3 + \lambda_1] \subseteq [-n + 2, n - 2]$. This means in any case, $\mu + \omega_n + \rho_G$ is G-singular as its first entry is zero, coincides with the second entry or one of the last $n - 3$ entries up to a sign.

(6.9): Let μ' be the weight $(n - 1 + x, x + 1, x + 1, 0, \cdots, 0)$ and let μ'' be the weight $(\lambda_1, \lambda_2, 0, 0, \cdots, 0)$.

$x \in [0, n - 5]$: It is $-2n + 4 \leq -n + 1 - x \leq \mu_3 \leq -x - 1 \leq -1$. This means, μ satisfies conditions (3.39) or (3.41) respectively with $i = 3$ and thus the corresponding summand $\mathcal{U}^\mu$ has no cohomology by Proposition 2.5.2.

$x = n - 4$: Due to our assumption we need to have $\lambda \neq (n - 3)\omega_3$ and accordingly it is $\lambda_2 \leq n - 4$. (Recall the argument given in the case where $x \neq n - 4$ or $\lambda \neq (n - 3)\omega_2$ of the proof of Lemma 6.1.3.) We rewrite μ' as the weight $(2n - 5, n - 3, n - 3, 0, \cdots, 0)$ and keep μ'' as the weight $(\lambda_1, \lambda_2, 0, 0, \cdots, 0)$. Then Corollary 3.3.2 gives us the estimation $-n + 3 \leq \mu_2 \leq -x - 1 + \lambda_2 \leq -1$.

$\mu_2 = -1$: We mention $-2n + 5 \leq \mu_3 \leq \mu_2 = -1$.

$\mu_3 \in [-2n + 6, -1]$: μ satisfies conditions (3.39) or (3.41) respectively with $i = 3$ and the vanishing cohomology of $\mathcal{U}^\mu$ follows by Proposition 2.5.2.

$\mu_3 = -2n + 5$: μ satisfies condition 3.40 for $i = 2$ and $j = 3$. In fact, it is $\mu_2 + \mu_3 = -2n + 4 = -2n - 1 + 2 + 3$. Hence, we apply Proposition 2.5.2.

$\mu_2 \in [-n + 3, -2]$: μ satisfies condition (3.39) with $i = 2$ and Proposition 2.5.2 gives rise to the vanishing of the cohomology of the corresponding summand $\mathcal{U}^\mu$.

$x = n-3$: Let μ' be the weight $(2n-4, n-2, n-2, 0, \cdots, 0)$ and let μ'' be as before the weight $(\lambda_1, \lambda_2, 0, 0, \cdots, 0)$. It is $-n+2 \leq \mu_2 \leq -n+2+\lambda_2 \leq -1$.

$\mu_2 = -1$: We observe $-2n+4 \leq \mu_3 \leq \mu_2 = -1$.

$\mu_3 \in [-2n+5, -1]$: We argue analgously as in the previous case where $x = n-4$, $\mu_2 = -1$, and μ_3 as assumed.

$\mu_3 = -2n+4$: On the one hand we have $\mu_1 + \mu_2 + \mu_3 = -4n+8+\lambda_1+\lambda_2$ and on the other hand $\mu_2 + \mu_3 = -2n+3$. Combining this two facts yields $-1 \leq \mu_2 \leq \mu_1 = -2n+5+\lambda_1+\lambda_2 \leq -1$. This means, it is $\mu_1 = -1$ and we see immediately $\mu_1 + \mu_3 = -2n+3 = -2n-1+1+3$. So, μ satisfies condition 3.40 for $i=1$ and $j=3$. Proposition 2.5.2 provides vanishing cohomology for $\mathcal{U}^{\mu}$.

$\mu_2 \in [-n+2, -2]$: We argue analogously as in the corresponding subcase of $x = n-4$.

(6.10): We proceed similar as in the previous computation for (6.8). It is $\mathcal{U}^{(n-2)\omega_1+\omega_n}(n-4)^{\vee} \otimes \mathcal{U}^{(n-3)\omega_2} = \mathcal{U}^{(n-2)\omega_1}(n-3)^{\vee} \otimes \mathcal{U}^{(n-3)\omega_2} \otimes \mathcal{U}^{\omega_n}$ by Lemma 3.2.3. First, let $\mathcal{U}^{\mu}$ be an irreducible summand from the direct sum decomposition of the tensor product $\mathcal{U}^{\mu'\vee} \otimes \mathcal{U}^{\mu''}$ where $\mu' = (2n-5, n-3, n-3, 0, \cdots, 0)$ and $\mu'' = (n-3, n-3, 0, 0, \cdots, 0)$. We apply Corollary 3.3.2 to see that its third entry μ_3 ranges between $-2n+5$ and $-n+3$. Now, we consider the weight $\mu + \omega_n + \rho_G$: It needs to be of the form $(\mu_1+n, \mu_2+n-1, \mu_3+n-2, n-3, \cdots, 1)$. Thus, its third entry lives in the interval $[-n+3, 1] \subseteq [-n+3, n-3]$. So, $\mu + \omega_n + \rho_G$ is G-singular as its third entry is zero or coincides with one of the last $n-3$ entries up to a sign.

□

Lemma 6.1.6. *The vector bundle $\mathcal{U}^{\lambda}(x)$ is right orthogonal to the vector bundle $\mathcal{U}^{y\omega_1+\omega_n}$ whenever $(x,y) \in [1, 2n-4] \times [0, n-3] \cup [1, 2n-4] \times \{n-2\}$ and λ is the highest weight of a vector bundle appearing in the starting block $C_0^{(\mathcal{U})}$ of the tautological subcollection $C^{(\mathcal{U})}$.*

Proof. $x \in [1, 2n-4]$ **and** $y \in [0, n-3]$: We refer to Propositions 4.3.4 and observe the vanishing of the Ext-space from $\mathcal{U}^{\lambda}(x)$ to $\mathcal{U}^{y\omega_1}$. Then we proceed similarly as before. This means that we apply Lemma 3.3.4. For the technical assumption, we consider the differences $\mu_i - \mu_j$ and argue analogously as before in the proof of Lemma 6.1.3.

$i=1$ **and** $j=2$: It is $\mu_1 - \mu_2 \leq \lambda_2 + y \leq 2n-6 \leq 2n-5+i-j$.

$i=2$ **and** $j=3$: We deduce $\mu_2 - \mu_3 \leq \lambda_1 \leq \lceil \frac{3}{2}n - \frac{9}{2} \rceil \leq 2n-6 = 2n-5+i-j$.

$i = 1$ and $j = 3$: We start with $\mu_1 - \mu_3 \leq \lambda_1 + y \leq \lfloor \frac{5}{2}n - \frac{15}{2} \rfloor$ and we can often estimate $\mu_1 - \mu_3 \leq 2n - 7 = 2n - 5 + i - j$ but unfortunately not always. However, we show similar as before that $\mu_{i^*} \in [-2n + 3 + i^*, -4 + i^*]$ where $i^* = 2$. In fact, if we assume $\mu_1 + \mu_3 = -2n + 3$, then both cases $-1 \leq \mu_2$ and $\mu_2 \leq -2n + 4$ lead to contradictions with respect to the differences $\mu_i - \mu_j$ where $(i, j) = (2, 3)$ or $(i, j) = (1, 2)$ respectively.

$x \in [1, 2n - 4]$ and $y = n - 2$: We distinguish with respect to x and λ.

$x \neq n$ or $\lambda \neq (n - 3)\omega_2$: $\mathcal{U}^\lambda(x)$ is right orthogonal to $\mathcal{U}^{(n-2)\omega_1}$ by the later computation (6.11) of Lemma 6.1.7. Thus we are left to check the technical assumption of Lemma 3.3.4.

$i = 1$ and $j = 2$: It is $\mu_1 - \mu_2 \leq \lambda_2 + n - 2$.

$\lambda_2 = n - 3$: Due to Lemma 4.3.3 we have $\lambda = (n - 3)\omega_2$ and in particular $x \neq n$. This means, we are working with the weights $\mu' = (n - 3 + x, n - 3 + x, x, 0, \cdots, 0)$ as well as $\mu'' = (n - 2, 0, 0, 0, \cdots, 0)$. So, $\mu_1 + \mu_2 + \mu_3 = -n + 4 - 3x$.
We assume $\mu_1 + \mu_2 = -2n + 2$ and therefore deduce $\mu_3 = n + 2 - 3x$. Corollary 3.3.2 gives us $-n + 3 - x \leq \mu_3 = n + 2 - 3x \leq -x$. Rewriting yields $\frac{1}{2}n + 1 \leq x \leq n - \frac{1}{2}$. As x is an integer we see that it must lie in the interval $[\lceil \frac{1}{2}n \rceil + 1, n - 1] \subseteq [3, n - 1]$. Consequently μ_3 ranges from $-2n + 4$ to -3.

$\mu_3 \in [-2n + 6, -3]$: We have μ_{i^*} in the interval $[-2n + 3 + i^*, -4 + i^*]$ for $i^* = 3$.

$\mu_3 \leq -2n + 5$: It is $x = n - 1$ because of $-n + 4 - 3x = \mu_1 + \mu_2 + \mu_3 \leq -4n + 7$ and hence $-2n + 5 = \mu_3 \leq \mu_2 \leq -n + 1$. This means, we have $\mu_{i^*} \in [-2n + 3 + i^*, -4 + i^*]$ for $i^* = 2$.

$\lambda_2 \in [0, n - 4]$: We see $\mu_1 - \mu_2 \leq 2n - 6 \leq 2n - 5 + i - j$.

$i = 2$ and $j = 3$: We argue the same way as in the corresponding case of $x \in [1, 2n - 4]$ and $y \in [0, n - 3]$.

$i = 1$ and $j = 3$: The estimation $\mu_1 - \mu_3 \leq \lambda_1 + n - 2 \leq \lfloor \frac{5}{2}n - \frac{13}{2} \rfloor$ does not work to deduce $\mu_1 - \mu_3 \leq 2n - 7 = 2n - 5 + i - j$.
However, we follow the same arguements as in the corresponding case of $x \in [1, 2n - 4]$ and $y \in [0, n - 3]$.

$x = n$ and $\lambda = (n - 3)\omega_2$: We refer to the later computation (6.12) given in Lemma 6.1.7.

□

Lemma 6.1.7. *Let $\mathcal{U}^\lambda$ be an object from the starting block $C_0^{(\mathcal{U})}$ of the tautological subcollection $C^{(\mathcal{U})}$. We show the following vanishings:*

$$\operatorname{Ext}^\bullet(\mathcal{U}^\lambda(x), \mathcal{U}^{(n-2)\omega_1}) = 0 \qquad x \neq n \text{ or } \lambda \neq (n-3)\omega_2 \tag{6.11}$$

$$\operatorname{Ext}^\bullet(\mathcal{U}^{(n-3)\omega_2}(n), \mathcal{U}^{(n-2)\omega_1+\omega_n}) = 0 \tag{6.12}$$

Remark 6.1.8. We claim $\operatorname{Ext}^\bullet(\mathcal{U}^{(n-3)\omega_2}(n), \mathcal{U}^{(n-2)\omega_1}) = \mathbf{K}[-4n+7]$ without proof. Nevertheless, we can not deduce desired vanishing of the Ext-spaces stated in (6.12) just by applying the Lemma 3.3.4. Therefore, we are going to check this Ext-space explicitly analogously as before.

Proof. (6.11): $x \in [n+1, 2n-4]$: Corollary 3.3.2 implies for the first entry of μ: $-2n+4 \leq -x \leq \mu_1 \leq -x+n-2 \leq -3$. Hence, μ satisfies conditions (3.39) or (3.41) respectively with $i = 1$.

$x = n$: We need to have $\lambda \neq (n-3)\omega_2$ and therefore see $\lambda_2 \leq n-4$. (Recall the argument given in the case where $x \neq n-4$ or $\lambda \neq (n-3)\omega_2$ of the proof of Lemma 6.1.3.) It is $-2n+4 \leq -\lambda_2 - n \leq \mu_2 \leq -n$

$\mu_2 \in [-2n+5, -n]$: We need condition (3.41) with $i = 2$.

$\mu_2 = -2n+4$: The first entry μ_1 is in the interval $[-n, -2]$.

$\mu_1 = -2$: We have $\mu_1 + \mu_2 = -2n+2 = -2n-1+1+2$ and consequently apply condition (3.40) with $i = 1$ and $j = 2$.

$\mu_1 \in [-n, -3]$: We refer to conditions (3.39) or (3.41) respectively with $i = 1$.

$x = n-1$: We distinguish with respect to $\lambda_2 \in [0, n-3]$.

$\lambda_2 = n-3$: Recall Lemma 4.3.3 and we state that in this case λ needs to be $(n-3)\omega_2$.
Let μ' be the weight $(2n-4, 2n-4, n-1, 0, \cdots, 0)$ and let μ' be the weight $(n-2, 0, 0, 0, \cdots, 0)$. It is $-2n+4 \leq \mu_2 \leq -n+1$.

$\mu_2 \in [-2n+5, -n+1]$: We refer to condition (3.41) with $i = 2$.

$\mu_2 = -2n+4$: Due to $\mu_1 + \mu_2 + \mu_3 = -4n+7$ we see immediately $\mu_1 + \mu_3 = -2n+3 = -2n-1+1+3$. So, μ satisfies condition (3.40) with $i = 1$ and $j = 3$.

$\lambda_2 \in [0, n-4]$: We observe $-2n+5 \leq -\lambda_2 - x \leq \mu_2 \leq -x \leq -n+1$ by Corollary 3.3.2 and accordingly μ satisfies condition (3.41) with $i = 2$.

$x \in [\lceil \frac{1}{2}n - \frac{1}{2} \rceil, n-2]$: Similar as in the previous computation for $x = n-1$ and $\lambda_2 \in [0, n-4]$, we observe $-2n+5 \leq -\lambda_2 - x \leq \mu_2 \leq -x \leq -\lceil \frac{1}{2}n -$

$\frac{1}{2}\rceil \leq -2$ by Corollary 3.3.2 and accordingly μ satisfies conditions (3.39) or (3.41) respectively with $i = 2$.

$x \in [1, \lceil \frac{1}{2}n - \frac{3}{2} \rceil]$**:** It follows $-2n + 6 = -\lfloor \frac{3}{2}n - \frac{9}{2} \rfloor - \lceil \frac{1}{2}n - \frac{3}{2} \rceil \leq -\lambda_1 - x \leq \mu_1 \leq -x \leq -1$ from Corollary 3.3.2. Thus, μ satisfies conditions (3.39) or (3.41) respectively with $i = 3$.

(6.12)**:** We proceed similar as in the previous computations for (6.8) or (6.10) respectively. Lemma 3.2.3 allows us to write $\mathcal{U}^{(n-3)\omega_2}(n)^\vee \otimes \mathcal{U}^{(n-2)\omega_1+\omega_n} = \mathcal{U}^{(n-3)\omega_2}(n)^\vee \otimes \mathcal{U}^{(n-2)\omega_1} \otimes \mathcal{U}^{\omega_n}$. We start with an irreducible summand $\mathcal{U}^\mu$ from the direct sum decomposition of the tensor product $\mathcal{U}^{\mu'\vee} \otimes \mathcal{U}^{\mu''}$ where $\mu' = (2n-3, 2n-3, n, 0, \cdots, 0)$ and $\mu'' = (n-2, 0, 0, 0, \cdots, 0)$ and then we apply Corollary 3.3.2 to conclude for the entries of μ: $\mu_1 \in [-n, -2]$, $\mu_2 \in [-2n+3, -n]$, and $\mu_3 \in [-2n+3, -n]$.

$\mu_2 \in [-2n+4, -n]$**:** We see $\mu_2 + n - 1 \in [-n+3, -1]$ and therefore the second entry of $\mu + \omega_n + \rho_G$ coincides with one of the last $n-3$ entries up to a sign.

$\mu_2 = -2n+3$**:** It is $-2n+3 \leq \mu_3 \leq \mu_2 = -2n+3$ and hence μ_1 needs to be -2 as $\mu_1 + \mu_2 + \mu_3 = -4n+4$. So, μ is the weight $(-2, -2n+3, -2n+3, 0, \cdots, 0)$ and $\mu + \omega_n + \rho_G$ is certainly the weight $(n-2, -n+2, -n+1, n-3, \cdots, 1)$ which is G-singular as its first and second entry coincide up to a sign.

□

6.2 Alternating composition

A second pattern arises when we intertwine the subcollection: Let $C_{0,y}^{(\mathrm{Alt})}$ be the ordered set $\{ \mathcal{U}^\lambda \in C_0^{(\mathcal{U})} : \lambda_1 = y \} \cup \{ \mathcal{S}^{(y)} \}$. Certainly, the subset of the left hand side inherits the lexicographical ordering induced from the tautological part. Then the starting block of $C^{(\mathrm{Alt})}$ is the ordered set $C_0^{(\mathrm{Alt})} = C_{0,0}^{(\mathrm{Alt})} \cup C_{0,1}^{(\mathrm{Alt})} \cup \cdots \cup C_{0,n-2}^{(\mathrm{Alt})}$. The corresponding support partition is again (6.1) for sure.

Example 6.2.1. For $\mathrm{OGr}(3,V)$ with $n=7$, we combine again (4.2) and (5.5) such that we obtain

$$
C^{(\mathrm{Alt})} = \begin{pmatrix}
\mathcal{U}^{4\omega_1+2\omega_2} & \mathcal{U}^{4\omega_1+2\omega_2}(1) & \cdots & \mathcal{U}^{4\omega_1+2\omega_2}(4) & \mathcal{U}^{4\omega_1+2\omega_2}(5) & \cdots & \mathcal{U}^{4\omega_1+2\omega_2}(10) \\
 & & & & & & \\
\mathcal{S}^{(5)} & \mathcal{S}^{(5)}(1) & \cdots & \mathcal{S}^{(5)}(4) & & & \\
\mathcal{U}^{3\omega_1+2\omega_2} & \mathcal{U}^{3\omega_1+2\omega_2}(1) & \cdots & \mathcal{U}^{3\omega_1+2\omega_2}(4) & \mathcal{U}^{3\omega_1+2\omega_2}(5) & \cdots & \mathcal{U}^{3\omega_1+2\omega_2}(10) \\
\mathcal{U}^{4\omega_1+\omega_2} & \mathcal{U}^{4\omega_1+\omega_2}(1) & \cdots & \mathcal{U}^{4\omega_1+\omega_2}(4) & \mathcal{U}^{4\omega_1+\omega_2}(5) & \cdots & \mathcal{U}^{4\omega_1+\omega_2}(10) \\
 & & & & & & \\
\mathcal{S}^{(4)} & \mathcal{S}^{(1)}(4) & \cdots & \mathcal{S}^{(4)}(4) & \mathcal{S}^{(4)}(5) & \cdots & \mathcal{S}^{(4)}(10) \\
\mathcal{U}^{4\omega_2} & \mathcal{U}^{4\omega_2}(1) & \cdots & \mathcal{U}^{4\omega_2}(4) & \mathcal{U}^{4\omega_2}(5) & \cdots & \mathcal{U}^{4\omega_2}(10) \\
\vdots & \vdots & & \vdots & \vdots & & \vdots \\
\mathcal{U}^{3\omega_1+\omega_2} & \mathcal{U}^{3\omega_1+\omega_2}(1) & \cdots & \mathcal{U}^{3\omega_1+\omega_2}(4) & \mathcal{U}^{3\omega_1+\omega_2}(5) & \cdots & \mathcal{U}^{3\omega_1+\omega_2}(10) \\
\mathcal{U}^{4\omega_1} & \mathcal{U}^{4\omega_1}(1) & \cdots & \mathcal{U}^{4\omega_1}(4) & \mathcal{U}^{4\omega_1}(5) & \cdots & \mathcal{U}^{4\omega_1}(10) \\
 & & & & & & \\
\vdots & \vdots & & \vdots & \vdots & & \vdots \\
 & & & & & & \\
\mathcal{S}^{(1)} & \mathcal{S}^{(1)}(1) & \cdots & \mathcal{S}^{(1)}(4) & \mathcal{S}^{(1)}(5) & \cdots & \mathcal{S}^{(1)}(10) \\
\mathcal{U}^{\omega_2} & \mathcal{U}^{\omega_2}(1) & \cdots & \mathcal{U}^{\omega_2}(4) & \mathcal{U}^{\omega_2}(5) & \cdots & \mathcal{U}^{\omega_2}(10) \\
\mathcal{U}^{\omega_1} & \mathcal{U}^{\omega_1}(1) & \cdots & \mathcal{U}^{\omega_1}(4) & \mathcal{U}^{\omega_1}(5) & \cdots & \mathcal{U}^{\omega_1}(10) \\
 & & & & & & \\
\mathcal{S}^{(0)} & \mathcal{S}^{(0)}(1) & \cdots & \mathcal{S}^{(0)}(4) & \mathcal{S}^{(0)}(5) & \cdots & \mathcal{S}^{(0)}(10) \\
\mathcal{O} & \mathcal{O}(1) & \cdots & \mathcal{O}(4) & \mathcal{O}(5) & \cdots & \mathcal{O}(10)
\end{pmatrix} \tag{6.13}
$$

Proposition 6.2.2. *The collection $C^{(Alt)}$ is exceptional.*

Proof. We start with the consecutive collection $C^{(\mathrm{Con})}$ which is exceptional by Proposition 6.1.2. Let us show that the object $\mathcal{S}^{(y)}$ is even orthogonal to the objects $\mathcal{U}^{\lambda}$ where $\lambda_1 > y$ and therefore the objects $\mathcal{S}^{(y)}(x)$ can be mutated easily to the appropriate positions: The vanishing $\mathrm{Ext}^\bullet(\mathcal{S}^{(y)}, \mathcal{U}^{\lambda}) = 0$ is clear from the right orthogonal relations of $C^{(\mathrm{Con})}$ and the vanishings $\mathrm{Ext}^\bullet(\mathcal{U}^{\lambda}, \mathcal{S}^{(y)}) = 0$ need to be checked in the following.

If we consider $\mathcal{S}^{(0)}$, then skip the following intermediate step; otherwise, if $y \in [1, n-2]$, we apply the Hom-fucntor $\mathrm{Hom}(\mathcal{U}^{\lambda}, -)$ to the short exact (5.8) defining $\mathcal{S}^{(y)}$:

$$
\cdots \to \mathrm{Ext}^\bullet(\mathcal{U}^{\lambda}, \mathcal{U}^{(y-1)\omega_1+\omega_n}) \to \mathrm{Ext}^\bullet(\mathcal{U}^{\lambda}, \mathcal{S}^{(y)}) \to \mathrm{Ext}^\bullet(\mathcal{U}^{\lambda}, \mathcal{U}^{y\omega_1+\omega_n}) \to \cdots \tag{6.14}
$$

As the outer Ext-spaces vanishes by the subsequent computations, it follows also for the middle one. In fact, we claim that $\mathcal{U}^{\lambda}$ is right orthogonal to $\mathcal{U}^{y\omega_1+\omega_n}$ with $\lambda_1 > y$ by Lemma 6.2.3). □

Lemma 6.2.3. *The vector bundle $\mathcal{U}^{\lambda}$ is right orthogonal to the vector bundle $\mathcal{U}^{y\omega_1+\omega_n}$ whenever $y \in [0, n-2]$ and λ is the highest weight of a vector bundle appearing in the starting block $C_0^{(\mathcal{U})}$ of the tautological subcollection $C^{(\mathcal{U})}$ such that $\lambda_1 > y$.*

Proof. If $y \in [0, n-3]$, then we refer to Propositions 4.3.4; and otherwise if $y = n-2$, then we refer to the later computation (6.15) in Lemma 6.2.4. In both cases, we observe the vanishing of the Ext-space from $\mathcal{U}^{\lambda}$ to $\mathcal{U}^{y\omega_1}$. Hence, we apply Lemma 3.3.4. The technical assumption is covered by the following observations.

$i = 1$ **and** $j = 2$**:** We start with $\mu_1 - \mu_2 \le y + \lambda_2$.

$y \in [0, n-3]$**:** As we have $y \le n-3$ and $\lambda_2 \le n-3$ likewise, it is certainly $\mu_1 - \mu_2 \le 2n - 6 = 2n + 5 + i - j$.

$y = n-2$**:** Due to $\lambda_1 > y = n-2$, we have $-2n-6 \le -\lambda_1 \le \mu_3 \le -1$. Hence, there is some $\mu_{i^*} \in [-2n+3+i^*, -4+i^*]$ for $i^* = 3$.

$i = 2$ **and** $j = 3$**:** It is $\mu_2 - \mu_3 \le \lambda_1 \le \lfloor \frac{3}{2}n - \frac{9}{2} \rfloor \le 2n - 6 = 2n - 5 + i - j$.

$i = 1$ **and** $j = 3$**:** We have $-\lambda_2 \le \mu_2 \le 0$. However, if we assume $\mu_1 + \mu_3 = -2n - 1 + i + j = -2n + 3$ this leads to a contradiction. It is $\mu_2 - 2n + 3 = \mu_1 + \mu_2 + \mu_3 = -\lambda_1 - \lambda_2 + y$ or equivalently $\mu_2 = 2n - 3 - \lambda_1 - \lambda_2 + y$. Then, $0 < 3 + y \le \mu_2$ since $\lambda_1 + \lambda_2 \le 2n - 6$.

□

Lemma 6.2.4. *Let $\mathcal{U}^{\lambda}$ be an object from the starting block $C_0^{(\mathcal{U})}$ of the tautological subcollection $C^{(\mathcal{U})}$ such that $\lambda_1 > n-2$. We show the following vanishings:*

$$\mathrm{Ext}^{\bullet}(\mathcal{U}^{\lambda}, \mathcal{U}^{(n-2)\omega_1}) = 0 \tag{6.15}$$

Proof. Let μ' be the weight $(\lambda_1, \lambda_2, 0, 0, \cdots, 0)$ as well as μ'' the weight $(n-2, 0, 0, 0, \cdots, 0)$. Given an irreducible summand $\mathcal{U}^{\mu}$ in the direct sum decomposition of the tensor product $\mathcal{U}^{\mu'\vee} \otimes \mathcal{U}^{\mu''}$, then the third entry μ_3 of its highest weight μ can be estimated as $-2n + 6 \le -\lfloor \frac{3}{2}n - \frac{9}{2} \rfloor \le -\lambda_1 \le \mu_3 \le -\lambda_1 + n - 2 \le -1$. Thus, μ is G-singular by conditions (3.39) or (3.41) respectively with $i = 3$. The summand $\mathcal{U}^{\mu}$ has trivial cohomology by Proposition 2.5.2. □

Chapter 7

An exceptional collection on $OGr(3,9)$ of maximal expected length

We fix $n = 4$ and consequently $N = 2n + 1 = 9$. Hence, $X = \mathrm{OGr}(3, V_9)$ parameterizes 3-dimensional isotropic subspaces in a 9-dimensional vector space V_9. The invariants (2.1) are computed as

$$\begin{aligned} d &:= \dim(X) = 12, \\ w_{max} &:= \mathrm{indx}(X) = 5, \text{ and} \\ l_{max} &:= \mathrm{rk}\,\mathrm{K}_0(X) = 32. \end{aligned} \tag{7.1}$$

The bounded derived category $\mathbf{D}^{\mathrm{b}}(\mathrm{OGr}(3, V_9))$ admits two exceptional collections consisting of 27 objects (cf. the previous chapter 6.1):

$$C^{(\mathrm{Con})} = \begin{pmatrix} \mathcal{S}^{(2)} & \mathcal{S}^{(2)}(1) & & & \\ \mathcal{S}^{(1)} & \mathcal{S}^{(1)}(1) & \mathcal{S}^{(1)}(2) & \mathcal{S}^{(1)}(3) & \mathcal{S}^{(1)}(4) \\ \mathcal{S}^{(0)} & \mathcal{S}^{(0)}(1) & \mathcal{S}^{(0)}(2) & \mathcal{S}^{(0)}(3) & \mathcal{S}^{(0)}(4) \\ \wedge^2\mathcal{U}^\vee & (\wedge^2\mathcal{U}^\vee)(1) & (\wedge^2\mathcal{U}^\vee)(2) & (\wedge^2\mathcal{U}^\vee)(3) & (\wedge^2\mathcal{U}^\vee)(4) \\ \mathcal{U}^\vee & \mathcal{U}^\vee(1) & \mathcal{U}^\vee(2) & \mathcal{U}^\vee(3) & \mathcal{U}^\vee(4) \\ \mathcal{O} & \mathcal{O}(1) & \mathcal{O}(2) & \mathcal{O}(3) & \mathcal{O}(4) \end{pmatrix} \tag{7.2}$$

as well as

$$C^{(\mathrm{Alt})} = \begin{pmatrix} \mathcal{S}^{(2)} & \mathcal{S}^{(2)}(1) & & & \\ \mathcal{S}^{(1)} & \mathcal{S}^{(1)}(1) & \mathcal{S}^{(1)}(2) & \mathcal{S}^{(1)}(3) & \mathcal{S}^{(1)}(4) \\ \wedge^2\mathcal{U}^\vee & (\wedge^2\mathcal{U}^\vee)(1) & (\wedge^2\mathcal{U}^\vee)(2) & (\wedge^2\mathcal{U}^\vee)(3) & (\wedge^2\mathcal{U}^\vee)(4) \\ \mathcal{U}^\vee & \mathcal{U}^\vee(1) & \mathcal{U}^\vee(2) & \mathcal{U}^\vee(3) & \mathcal{U}^\vee(4) \\ \mathcal{S}^{(0)} & \mathcal{S}^{(0)}(1) & \mathcal{S}^{(0)}(2) & \mathcal{S}^{(0)}(3) & \mathcal{S}^{(0)}(4) \\ \mathcal{O} & \mathcal{O}(1) & \mathcal{O}(2) & \mathcal{O}(3) & \mathcal{O}(4) \end{pmatrix} \tag{7.3}$$

From a numerical point of view we are left to construct a further exceptional orbit consisting of 5 objects.

7.1 The missing link

In the following, we will construct step-by-step objects $\mathcal{M}^{(i)}$. We start with the filtration on $\mathrm{S} \otimes \mathcal{S}$ as presented in Proposition 2.7.2. In later sections, we will then work with the last object $\mathcal{M}^{(3)}$ and therefore abbreviate this just by $\mathcal{M}$.

Step 0 We consider the quotient $\mathcal{Q}^\vee/\mathcal{U}$ in (2.34): It is the irreducible G-equivariant vector bundle $(\mathrm{Sym}^2\mathcal{S})(-1)$ – i.e. it has highest weight $-\omega_3 + 2\omega_4$. This object is self-dual, of rank 3 and has trivial determinate. Furthermore, we observe the fact $\mathcal{S}^{\otimes 2} = \mathcal{Q}^\vee/\mathcal{U}(1) \oplus \mathcal{O}(1)$.

We combine Lemma 2.6.1 with Proposition 2.7.2 and deduce the short exact sequences

$$\begin{aligned} 0 \to (\wedge^2\mathcal{Q}^\vee)(1) \to \mathcal{F}_2^{(2)} \to \mathcal{O}(1) \to 0 \\ 0 \to \wedge^2\mathcal{Q} \to \mathcal{F}_1^{(2)} \to \mathcal{F}_2^{(2)} \to 0 \\ 0 \to \mathcal{O} \to \mathrm{S} \otimes \mathcal{S} \to \mathcal{F}_1^{(2)} \to 0. \end{aligned} \tag{7.4}$$

Furthermore, we recall the exact sequence

$$0 \to \mathrm{Sym}^2\mathcal{U} \to V \otimes \mathcal{U} \to \wedge^2 V \otimes \mathcal{O} \to \wedge^2\mathcal{Q} \to 0 \tag{7.5}$$

which arise if we apply $\wedge^2$ to the tautological sequence (2.31).

Lemma 7.1.1. *Let x be in $[0,4]$ and let p be in $[0,2]$.*

$$\mathrm{Ext}^\bullet((\mathrm{Sym}^p\mathcal{U}^\vee)(x), \wedge^2\mathcal{Q}) = \begin{cases} \wedge^2 V[0] & \text{, } \textit{if } x = p = 0 \\ \mathbf{K}[-1] & \text{, } \textit{if } x = 0,\ p = 1 \\ V[-2] & \text{, } \textit{if } x = 0,\ p = 2 \\ \mathbf{K}[-3] & \text{, } \textit{if } x = 1,\ p = 2 \\ 0 & \text{, } \textit{else} \end{cases} \tag{7.6}$$

Proof. First, we compute by Proposition 2.5.2 the Ext-spaces from $(\mathrm{Sym}^p\mathcal{U}^\vee)(x) = \mathcal{U}^{p\omega_1}(x)$ to $\mathrm{Sym}^q\mathcal{U} = \mathcal{U}^{q\omega_2}(-q)$ for $p, q \in [0,2]$:

$$\mathrm{Ext}^\bullet((\mathrm{Sym}^p\mathcal{U}^\vee)(x), \mathrm{Sym}^q\mathcal{U}) = \begin{cases} \mathbf{K}[0] & \text{, if } x = p = q = 0 \\ \mathbf{K}[-3] & \text{, if } x = 0,\ p = 1,\ q = 2 \\ \mathbf{K}[-3] & \text{, if } x = 0,\ p = 2,\ q = 1 \\ \mathbf{K}[-5] & \text{, if } x = 1,\ p = q = 2 \\ 0 & \text{, else} \end{cases}. \tag{7.7}$$

Then we apply $\mathrm{Hom}((\mathrm{Sym}^p\mathcal{U}^\vee)(x), -)$ to the exact sequence (7.5) and obtain the following two long exact sequences of Ext-spaces:

$$\cdots \to \mathrm{Ext}^i((\mathrm{Sym}^p\mathcal{U}^\vee)(x), \mathrm{Sym}^2\mathcal{U}) \to \mathrm{Ext}^i((\mathrm{Sym}^p\mathcal{U}^\vee)(x), V\otimes\mathcal{U}) \to \\ \mathrm{Ext}^i((\mathrm{Sym}^p\mathcal{U}^\vee)(x), K) \to \cdots \tag{7.8}$$

and

$$\cdots \to \mathrm{Ext}^i((\mathrm{Sym}^p\mathcal{U}^\vee)(x), K) \to \mathrm{Ext}^i((\mathrm{Sym}^p\mathcal{U}^\vee)(x), \wedge^2 V\otimes\mathcal{O}) \to \\ \mathrm{Ext}^i((\mathrm{Sym}^p\mathcal{U}^\vee)(x), \wedge^2\mathcal{Q}) \to \cdots \tag{7.9}$$

The object K is the cokernel of $\mathrm{Sym}^p\mathcal{U} \to V\otimes\mathcal{U}$ and likewise the kernel of $\wedge^2 V\otimes\mathcal{O} \to \wedge^2\mathcal{Q}$.

Second, we distinguish the following cases:

$x = p = 0$: Since $\mathcal{O}$ is right orthogonal to both $\mathrm{Sym}^2\mathcal{U}$ and $\mathcal{U}$, it follows that it is also right orthogonal to K. Hence we have the isomorphisms

$$\mathrm{Ext}^i(\mathcal{O}, \wedge^2 V\otimes\mathcal{O}) = \mathrm{Ext}^i(\mathcal{O}, \wedge^2\mathcal{Q}) \tag{7.10}$$

and accordingly $\mathrm{Ext}^\bullet(\mathcal{O}, \wedge^2\mathcal{Q}) = \wedge^2 V[0]$.

$x = 0$, $p = 1$: We have the isomorphisms

$$\begin{aligned} \mathrm{Ext}^i(\mathcal{U}^\vee, \wedge^2\mathcal{Q}) &= \mathrm{Ext}^{i+1}(\mathcal{U}^\vee, K) \\ &= \mathrm{Ext}^{i+2}(\mathcal{U}^\vee, \mathrm{Sym}^2\mathcal{U}) \end{aligned} \tag{7.11}$$

since $\mathcal{U}^\vee$ is right orthogonal to both $\mathcal{U}$ and $\mathcal{O}$.

$x = 0$, $p = 2$: We have the isomorphisms

$$\begin{aligned} \mathrm{Ext}^i(\mathrm{Sym}^2\mathcal{U}^\vee, \wedge^2\mathcal{Q}) &= \mathrm{Ext}^{i+1}(\mathrm{Sym}^2\mathcal{U}^\vee, K) \\ &= \mathrm{Ext}^{i+1}(\mathrm{Sym}^2\mathcal{U}^\vee, V\otimes\mathcal{U}) \end{aligned} \tag{7.12}$$

since $\mathrm{Sym}^2\mathcal{U}^\vee$ is right orthogonal to both $\mathrm{Sym}^2\mathcal{U}$ and $\mathcal{O}$.

$x = 1$, $p = 2$: We have the isomorphisms

$$\begin{aligned} \mathrm{Ext}^i((\mathrm{Sym}^2\mathcal{U}^\vee)(1), \wedge^2\mathcal{Q}) &= \mathrm{Ext}^{i+1}((\mathrm{Sym}^2\mathcal{U}^\vee)(1), K) \\ &= \mathrm{Ext}^{i+2}((\mathrm{Sym}^2\mathcal{U}^\vee)(1), \mathrm{Sym}^2\mathcal{U}) \end{aligned} \tag{7.13}$$

since $(\mathrm{Sym}^2\mathcal{U}^\vee)(1)$ is right orthogonal to $\mathcal{U}$ as well as $\mathcal{O}$.

x and p otherwise: The Ext-space $\mathrm{Ext}^\bullet((\mathrm{Sym}^p\mathcal{U}^\vee)(x), \wedge^2\mathcal{Q})$ vanishes since $(\mathrm{Sym}^p\mathcal{U}^\vee)(x)$ is right orthogonal to all three components $\mathrm{Sym}^q\mathcal{U}$ where $q \in [0,2]$.

□

Lemma 7.1.2. *Let x be in $[0,4]$ and let q be in $[0,2]$.*

$$\mathrm{Ext}^\bullet((\wedge^2\mathcal{Q})(x),\mathrm{Sym}^q\mathcal{U}^\vee)=\begin{cases}\mathrm{Sym}^qV[-1] & ,\ \textit{if } x=0\\ \mathbf{K}[-9] & ,\ \textit{if } x=4,\ q=2\\ 0 & ,\ \textit{else}\end{cases}\tag{7.14}$$

Proof. $x=0$**:** We apply the functor $\wedge^2$ to the dualized version of the sequence (2.34). This yields the exact sequence

$$0\to\underbrace{\mathrm{Sym}^2(\mathcal{Q}^\vee/\mathcal{U})}_{=(\mathrm{Sym}^4\mathcal{S})(-2)\oplus\mathcal{O}}\to\mathcal{Q}\otimes(\mathcal{Q}^\vee/\mathcal{U})\to\wedge^2\mathcal{Q}\to\wedge^2\mathcal{U}^\vee\to 0.\tag{7.15}$$

Then we compute the Ext-spaces from the three remaining terms into $\mathrm{Sym}^q\mathcal{U}^\vee$:

$\mathrm{Ext}^\bullet(\wedge^2\mathcal{U}^\vee,\mathrm{Sym}^q\mathcal{U}^\vee)=0$**:** We refer to Proposition 4.1.2 to see that $\wedge^2\mathcal{U}^\vee$ is right orthogonal to $\mathcal{O}$ or $\mathcal{U}^\vee$. We refer to Proposition 2.5.2 to cover the case involving $\mathrm{Sym}^2\mathcal{U}^\vee$.

$\mathrm{Ext}^\bullet(\mathcal{Q}\otimes(\mathcal{Q}^\vee/\mathcal{U}),\mathrm{Sym}^q\mathcal{U}^\vee)=0$**:** We tensor $(\mathcal{Q}^\vee/\mathcal{U})$ to the tautological sequence (2.31), namely

$$0\to\underbrace{\mathcal{U}\otimes(\mathcal{Q}^\vee/\mathcal{U})}_{=\mathcal{U}^{\omega_2+2\omega_4}(-2)}\to V\otimes\underbrace{(\mathcal{Q}^\vee/\mathcal{U})}_{=\mathcal{U}^{2\omega_4}(-1)}\to\mathcal{Q}\otimes(\mathcal{Q}^\vee/\mathcal{U})\to 0,\tag{7.16}$$

and check by Proposition 2.5.2 that both $\mathcal{U}^{2\omega_4}(-1)$ and $\mathcal{U}^{\omega_2+2\omega_4}(-2)$ are right orthogonal to $\mathrm{Sym}^q\mathcal{U}^\vee$. Hence, we see the desired right orthogonal relation.

$\mathrm{Ext}^\bullet((\mathrm{Sym}^4\mathcal{S})(-2),\mathrm{Sym}^q\mathcal{U}^\vee)=0$**:** The vector bundle $(\mathrm{Sym}^4\mathcal{S})(-2)$ has highest weight $-2\omega_3+4\omega_4$ and it is right orthogonal to $\mathrm{Sym}^q\mathcal{U}^\vee$ by Proposition 2.5.2.

$\mathrm{Ext}^\bullet(\mathcal{O},\mathrm{Sym}^q\mathcal{U}^\vee)=\mathrm{Sym}^qV[0]$**:** We recall Proposition 2.5.2.

Finally, we apply $\mathrm{Hom}(-,\mathrm{Sym}^q\mathcal{U}^\vee)$ to exact sequence (7.15) and obtain two long exact sequences of Ext-spaces. It follows precisely $\mathrm{Ext}^\bullet(\wedge^2\mathcal{Q},\mathcal{O})=\mathbf{K}[-1]$ due to the above relations.

$x\in[1,4]$**:** First, we compute by Proposition 2.5.2 the Ext-spaces from $(\mathrm{Sym}^p\mathcal{U})(x)=\mathcal{U}^{p\omega_2}(-p+x)$ to $\mathrm{Sym}^q\mathcal{U}^\vee=\mathcal{U}^{q\omega_1}$ for $p,q\in[0,2]$:

$$\mathrm{Ext}^\bullet((\mathrm{Sym}^p\mathcal{U})(x),\mathrm{Sym}^q\mathcal{U}^\vee)=\begin{cases}\mathbf{K}[-7] & ,\ \text{if } x=4,\ p=q=2\\ 0 & ,\ \text{else}\end{cases}.\tag{7.17}$$

Then we apply $\mathrm{Hom}(-, \mathrm{Sym}^q \mathcal{U}^\vee)$ to the $\mathcal{O}(x)$-twisted version of the exact sequence (7.5) and obtain the following two long exact sequences of Ext-spaces:

$$\cdots \to \mathrm{Ext}^i((\wedge^2 \mathcal{Q})(x), \mathrm{Sym}^q \mathcal{U}^\vee) \to \mathrm{Ext}^i(\wedge^2 V \otimes \mathcal{O}(x), \mathrm{Sym}^q \mathcal{U}^\vee) \to \\ \mathrm{Ext}^i(K, \mathrm{Sym}^q \mathcal{U}^\vee) \to \cdots \tag{7.18}$$

and

$$\cdots \to \mathrm{Ext}^i(K, \mathrm{Sym}^q \mathcal{U}^\vee) \to \mathrm{Ext}^i(V \otimes \mathcal{U}(x), \mathrm{Sym}^q \mathcal{U}^\vee) \to \\ \mathrm{Ext}^i((\mathrm{Sym}^2 \mathcal{U})(x), \mathrm{Sym}^q \mathcal{U}^\vee) \to \cdots \tag{7.19}$$

Second, we distinguish the following cases:

$x = 4$, $q = 2$: We have the isomorphisms

$$\begin{aligned} \mathrm{Ext}^i((\wedge^2 \mathcal{Q})(4), \mathrm{Sym}^2 \mathcal{U}^\vee) &= \mathrm{Ext}^{i-1}(K, \mathrm{Sym}^2 \mathcal{U}^\vee) \\ &= \mathrm{Ext}^{i-2}((\mathrm{Sym}^2 \mathcal{U})(4), \mathrm{Sym}^2 \mathcal{U}^\vee) \end{aligned} \tag{7.20}$$

since both components $\mathcal{O}(4)$ and $\mathcal{U}(4)$ are right orthogonal to $\mathrm{Sym}^2 \mathcal{U}^\vee$.

x and q otherwise: The Ext-space $\mathrm{Ext}^\bullet((\wedge^2 \mathcal{Q})(x), \mathrm{Sym}^p \mathcal{U}^\vee)$ vanishes since each of the components $(\mathrm{Sym}^p \mathcal{U})(x)$ where $p \in [0, 2]$ is right orthogonal to $\mathrm{Sym}^q \mathcal{U}^\vee$.

□

Lemma 7.1.3. *Let x be in $[0, 4]$ and let p be in $[0, 2]$.*

$$\mathrm{Ext}^\bullet((\wedge^p \mathcal{U}^\vee)(x), \wedge^2 \mathcal{Q}) = \begin{cases} \wedge^2 V[0] & , \textit{ if } x = p = 0 \\ \mathbf{K}[-1] & , \textit{ if } x = 0,\ p = 1 \\ \mathbf{K}[-10] \oplus V[-11] & , \textit{ if } x = 4,\ p = 2 \\ 0 & , \textit{ else} \end{cases} \tag{7.21}$$

Proof. First, we compute by Proposition 2.5.2 the Ext-spaces from $(\wedge^p \mathcal{U}^\vee)(x)$ to $\mathrm{Sym}^q \mathcal{U} = \mathcal{U}^{q\omega_2}(-q)$ for $p, q \in [0, 2]$:

$$\mathrm{Ext}^\bullet((\wedge^p \mathcal{U}^\vee)(x), \mathrm{Sym}^q \mathcal{U}) = \begin{cases} \mathbf{K}[0] & , \text{ if } x = p = q = 0 \\ \mathbf{K}[-3] & , \text{ if } x = 0,\ p = 1,\ q = 2 \\ \mathbf{K}[-12] & , \text{ if } x = 4,\ p = 2,\ q = 1\,. \\ \mathbf{K}[-12] & , \text{ if } x = 4,\ p = q = 2 \\ 0 & , \text{ else} \end{cases} \tag{7.22}$$

Let us mention that $\wedge^p\mathcal{U}^\vee = \mathcal{U}^{\omega_p}$ if $p \in \{1,2\}$. Then we analogously as in the proof of the previous Lemma 7.1.1. This yields the two long exact sequences of Ext-spaces:

$$\cdots \to \mathrm{Ext}^i((\wedge^p\mathcal{U}^\vee)(x), \mathrm{Sym}^2\mathcal{U}) \to \mathrm{Ext}^i((\wedge^p\mathcal{U}^\vee)(x), V\otimes\mathcal{U}) \to \\ \mathrm{Ext}^i((\wedge^p\mathcal{U}^\vee)(x), K) \to \cdots \tag{7.23}$$

and

$$\cdots \to \mathrm{Ext}^i((\wedge^p\mathcal{U}^\vee)(x), K) \to \mathrm{Ext}^i((\wedge^p\mathcal{U}^\vee)(x), \wedge^2 V\otimes\mathcal{O}) \to \\ \mathrm{Ext}^i((\wedge^p\mathcal{U}^\vee)(x), \wedge^2\mathcal{Q}) \to \cdots \tag{7.24}$$

Second, we distinguish the following cases:

$x = 0$ and $p \in \{0,1\}$: We refer to arguments in the corresponding case appearing in the proof of Lemma 7.1.1.

$x = 4$, $p = 2$: It is $\mathrm{Ext}^\bullet((\wedge^2\mathcal{U}^\vee)(4), K) = \mathbf{K}[-11] \oplus V[-12]$. Since $(\wedge^2\mathcal{U}^\vee)(4)$ is right orthogonal to $\mathcal{O}$, it follows the appropriate result.

x and p otherwise: The Ext-space $\mathrm{Ext}^\bullet((\wedge^p\mathcal{U}^\vee)(x), \wedge^2\mathcal{Q})$ vanishes since $(\wedge^p\mathcal{U}^\vee)(x)$ is right orthogonal to all three components $\mathrm{Sym}^q\mathcal{U}$ where $q \in [0,2]$.

□

Lemma 7.1.4. *Let x be in $[0,4]$.*

$$\mathrm{Ext}^\bullet(\mathcal{S}(x), \wedge^2\mathcal{Q}) = 0 \tag{7.25}$$

Proof. The spinor bundle $\mathcal{S} = \mathcal{U}^{\omega_4}$ and its higher twists are right orthogonal to each component $\mathrm{Sym}^p\mathcal{U} = \mathcal{U}^{p\omega_2}(-p)$ where $p \in [0,2]$ which appear in (7.5). □

Lemma 7.1.5. *Let x be in $[0,4]$.*

$$\mathrm{Ext}^\bullet((\wedge^2\mathcal{Q})(x), \mathcal{S}) = 0 \tag{7.26}$$

Proof. We apply $\wedge^4$ to the tautological sequence (2.31) and recall Lemma 2.6.1 for the last term, namely

$$0 \to \mathrm{Sym}^4\mathcal{U} \to V\otimes\mathrm{Sym}^3\mathcal{U} \to \wedge^2 V\otimes\mathrm{Sym}^2\mathcal{U} \to \\ \wedge^3 V\otimes\mathcal{U} \to \wedge^4 V\otimes\mathcal{O} \to \underbrace{\wedge^4\mathcal{Q}}_{=(\wedge^2\mathcal{Q}^\vee)(1)} \to 0. \tag{7.27}$$

Since $\mathcal{S}(x)$ is right orthogonal to any term $\mathrm{Sym}^p\mathcal{U} = \mathcal{U}^{p\omega_2}(-p)$, we deduce the claimed vanishing $\mathrm{Ext}^\bullet((\wedge^2\mathcal{Q})(x), \mathcal{S}) = \mathrm{Ext}^\bullet(\mathcal{S}(x), (\wedge^2\mathcal{Q}^\vee)(1)) = 0$. □

Lemma 7.1.6.

$$\mathrm{Ext}^\bullet((\wedge^2\mathcal{Q}^\vee)(1), \wedge^2\mathcal{Q}) = \mathbf{K}[-1] \tag{7.28}$$

Proof. We dualize (7.5) and twist it by $\mathcal{O}(1)$:

$$0 \to (\wedge^2\mathcal{Q}^\vee)(1) \to \wedge^2 V \otimes \mathcal{O}(1) \to V \otimes \mathcal{U}^\vee(1) \to (\mathrm{Sym}^2\mathcal{U}^\vee)(1) \to 0. \tag{7.29}$$

Then we apply $\mathrm{Hom}(-, \wedge^2\mathcal{Q})$ and recall the previous computations of Lemma 7.1.1 with $x = 1$ and $p \in [0,2]$. Consequently, it follows that $\mathrm{Ext}^i((\wedge^2\mathcal{Q}^\vee)(1), \wedge^2\mathcal{Q})$ is isomorphic to $\mathrm{Ext}^{i+2}((\mathrm{Sym}^2\mathcal{U}^\vee)(1), \wedge^2\mathcal{Q})$. □

Lemma 7.1.7. *Let x be in $[1,4]$.*

$$\mathrm{Ext}^\bullet((\wedge^2\mathcal{Q})(x), \wedge^2\mathcal{Q}) = 0 \tag{7.30}$$

Proof. First, we check that the Ext-space $\mathrm{Ext}^\bullet((\wedge^2\mathcal{Q})(x), \mathrm{Sym}^q\mathcal{U})$ vanishes for any $q \in [0,2]$. Indeed, each component $(\mathrm{Sym}^p\mathcal{U})(x) = \mathcal{U}^{p\omega_2}(-p+x)$ where $p \in [0,2]$ appearing in a $\mathcal{O}(x)$-twisted version of (7.5) is right orthogonal to $\mathrm{Sym}^q\mathcal{U}$.

Next, we apply $\mathrm{Hom}((\wedge^2\mathcal{Q})(x), -)$ to (7.5) and deduce the desired vanishing. □

Step 1 We refer to (7.4) and say that the object $\mathcal{M}^{(1)} := \mathcal{F}_2^{(2)\vee}(1)$ is defined by the short exact sequence

$$0 \to \mathcal{O} \to \mathcal{M}^{(1)} \to \wedge^2\mathcal{Q} \to 0. \tag{7.31}$$

Lemma 7.1.8. *The object $\mathcal{M}^{(1)}$ is a non-splitting* G*-equivariant extension of $\wedge^2\mathcal{Q}$ by $\mathcal{O}$.*

Proof. First, we compute the Ext-space from $\wedge^2\mathcal{Q}$ to $\mathcal{O}$: see $\mathrm{Ext}^\bullet(\wedge^2\mathcal{Q}, \mathcal{O}) = \mathbf{K}[-1]$ by the computation in Lemma 7.1.2 where $x = 0$ and $q = 0$.

Second, we check that (7.31) is not-splitting. Therefore, we assume the opposite – i.e. there is G-equivariant embedding of $\wedge^2\mathcal{Q}^\vee$ into $\mathcal{M}^{(1)}$. Thus, concatenating with the appropriate embeddings of the dualized versions of (7.4) yields a non-trivial morphism

$$\wedge^2\mathcal{Q}^\vee \hookrightarrow \mathcal{M}^{(1)} := \mathcal{F}_2^{(2)\vee}(1) \hookrightarrow \mathcal{F}_1^{(2)\vee}(1) \hookrightarrow \mathrm{S} \otimes \mathcal{S}.$$

However, this contradicts the fact that $\wedge^2\mathcal{Q}^\vee$ is right orthogonal to $\mathcal{S}$ – see the computation in Lemma 7.1.5 where $x = 0$. □

Lemma 7.1.9. *The object $\mathcal{M}^{(1)}$ is resolved by the short exact sequence*

$$0 \to \mathcal{M}^{(1)} \to \mathrm{S} \otimes \mathcal{S} \to \mathcal{M}^{(1)\vee}(1) \to 0. \tag{7.32}$$

Proof. We concatenate the surjection onto $\mathcal{F}_1^{(2)}$ with the one onto $\mathcal{M}^{(1)\vee}(1) = \mathcal{F}_2^{(2)}$ which are both from (7.4). Thus, we obtain a G-equivariant surjection

$$\mathrm{S} \otimes \mathcal{S} \twoheadrightarrow \mathcal{F}_1^{(2)} \twoheadrightarrow \mathcal{F}_2^{(2)} = \mathcal{M}^{(1)\vee}(1). \tag{7.33}$$

Then, snaking gives us the two short exact sequences

$$0 \to K \to \mathrm{S} \otimes \mathcal{S} \to \mathcal{M}^{(1)\vee}(1) \to 0 \tag{7.34}$$

and

$$0 \to \mathcal{O} \to K \to \wedge^2 \mathcal{Q} \to 0. \tag{7.35}$$

We observe that the kernel K is likewise as the object $\mathcal{M}^{(1)}$ a G-equivariant extension of $\wedge^2\mathcal{Q}$ by $\mathcal{O}$ – we compare (7.31) with (7.35).

Now, we are left to show that both objects are isomorphic. Indeed, K does not split and accordingly it coincides with $\mathcal{M}^{(1)}$ up to a scalar. Otherwise, we would have a G-equivariant embedding

$$\wedge^2 \mathcal{Q} \hookrightarrow K \hookrightarrow \mathrm{S} \otimes \mathcal{S} \tag{7.36}$$

contradicting the fact $\mathrm{Ext}^\bullet(\wedge^2\mathcal{Q}, \mathcal{S}) = 0$ as we computed in the previous Lemma 7.1.5 for $x = 0$. ☐

Lemma 7.1.10. 1. *The object $\mathcal{M}^{(1)}$ is exceptional.*

2. *The twists $\mathcal{M}^{(1)}(x)$ are right orthogonal $\mathcal{M}^{(1)}$ whenever $x \in [1,4]$ – i.e. the object $\mathcal{M}^{(1)}$ has an exceptional orbit of length* 5.

Proof. **7.1.10.(1):** We apply $\mathrm{Hom}(-, \mathcal{M}^{(1)})$ to the resolution (7.32):

$$\begin{aligned} \cdots \to \mathrm{Ext}^i(\mathcal{M}^{(1)\vee}(1), \mathcal{M}^{(1)}) \to \mathrm{Ext}^i(\mathrm{S} \otimes \mathcal{S}, \mathcal{M}^{(1)}) \to \\ \mathrm{Ext}^i(\mathcal{M}^{(1)}, \mathcal{M}^{(1)}) \to \cdots \end{aligned} \tag{7.37}$$

The middle term vanishes by computation (7.40) in the following Lemma 7.1.11 and consequently we have isomorphisms

$$\mathrm{Ext}^i(\mathcal{M}^{(1)}, \mathcal{M}^{(1)}) = \mathrm{Ext}^{i+1}(\mathcal{M}^{(1)\vee}(1), \mathcal{M}^{(1)}). \tag{7.38}$$

Finally we refer to the computation (7.41) in Lemma 7.1.11.

7.1.10.(2): We apply $\mathrm{Hom}(-, \mathcal{M}^{(1)})$ to the $\mathcal{O}(x)$-twisted version of the short exact sequence (7.31) defining $\mathcal{M}^{(1)}$:

$$\begin{aligned} \cdots \to \mathrm{Ext}^i((\wedge^2\mathcal{Q})(x), \mathcal{M}^{(1)}) \to \mathrm{Ext}^i(\mathcal{M}^{(1)}(x), \mathcal{M}^{(1)}) \to \\ \mathrm{Ext}^i(\mathcal{O}(x), \mathcal{M}^{(1)}) \to \cdots. \end{aligned} \tag{7.39}$$

By the later computations (7.42) and (7.43) in the following Lemma 7.1.11 we observe the vanishing of the outer Ext-spaces. Hence, we also deduce the vanishing of the desired Ext-space in the middle.

□

Lemma 7.1.11.

$$\operatorname{Ext}^\bullet(\mathcal{S}(x), \mathcal{M}^{(1)}) = 0 \quad , \textit{ if } x \in [0,4] \tag{7.40}$$
$$\operatorname{Ext}^\bullet(\mathcal{M}^{(1)\vee}(1), \mathcal{M}^{(1)}) = \mathbf{K}[-1] \tag{7.41}$$
$$\operatorname{Ext}^\bullet(\mathcal{O}(x), \mathcal{M}^{(1)}) = 0 \quad , \textit{ if } x \in [1,4] \tag{7.42}$$
$$\operatorname{Ext}^\bullet((\wedge^2\mathcal{Q})(x), \mathcal{M}^{(1)}) = 0 \quad , \textit{ if } x \in [1,4] \tag{7.43}$$

Proof. (7.40)**:** The object $\mathcal{S}(x)$ is right orthogonal to the component $\mathcal{O}$ by Proposition 6.1.2 and it is also to the component $\wedge^2\mathcal{Q}$ by computation in Lemma 7.1.4 where $x = 0$.

(7.41)**:** First, we apply $\operatorname{Hom}(-, \mathcal{M}^{(1)})$ to the short exact sequence defining $\mathcal{M}^{(1)\vee}(1) = \mathcal{F}_2^{(2)}$ in (7.4), namely

$$\begin{aligned} \cdots \to \operatorname{Ext}^i(\mathcal{O}(1), \mathcal{M}^{(1)}) \to \operatorname{Ext}^i(\mathcal{M}^{(1)\vee}(1), \mathcal{M}^{(1)}) \to \\ \operatorname{Ext}^i((\wedge^2\mathcal{Q}^\vee)(1), \mathcal{M}^{(1)}) \to \cdots, \end{aligned} \tag{7.44}$$

and see the isomorphisms

$$\operatorname{Ext}^i(\mathcal{M}^{(1)\vee}(1), \mathcal{M}^{(1)}) = \operatorname{Ext}^i((\wedge^2\mathcal{Q}^\vee)(1), \mathcal{M}^{(1)}) \tag{7.45}$$

since $\mathcal{O}(1)$ is right orthogonal to $\mathcal{M}^{(1)}$ by the later computation (7.42).

Second, we apply $\operatorname{Hom}((\wedge^2\mathcal{Q}^\vee)(1), -)$ to the short exact sequence (7.31), namely

$$\begin{aligned} \cdots \to \operatorname{Ext}^i((\wedge^2\mathcal{Q}^\vee)(1), \mathcal{O}) \to \operatorname{Ext}^i((\wedge^2\mathcal{Q}^\vee)(1), \mathcal{M}^{(1)}) \to \\ \operatorname{Ext}^i((\wedge^2\mathcal{Q}^\vee)(1), \wedge^2\mathcal{Q}) \to \cdots, \end{aligned} \tag{7.46}$$

and deduce the isomorphisms

$$\operatorname{Ext}^i((\wedge^2\mathcal{Q}^\vee)(1), \mathcal{M}^{(1)}) = \operatorname{Ext}^i((\wedge^2\mathcal{Q}^\vee)(1), \wedge^2\mathcal{Q}) \tag{7.47}$$

since $(\wedge^2\mathcal{Q}^\vee)(1)$ is right orthogonal to $\mathcal{O}$. In fact, $\operatorname{Ext}^\bullet((\wedge^2\mathcal{Q}^\vee)(1), \mathcal{O})$ is isomorphic to $\operatorname{Ext}^\bullet(\mathcal{O}(1), \wedge^2\mathcal{Q})$. Thus, we refer again to the computation in Lemma 7.1.1 with $x = 1$ and $p = 0$.

Third, we finish with a reference to Lemma 7.1.6.

(7.42)**:** We observe that $\mathcal{O}(x)$ is right orthogonal to both components of $\mathcal{M}^{(1)}$: For $\operatorname{Ext}^\bullet(\mathcal{O}(x), \mathcal{O}) = 0$ we refer to Propositon 4.1.2 and for $\operatorname{Ext}^\bullet(\mathcal{O}(x), \wedge^2\mathcal{Q}) = 0$ we refer to the computation in Lemma 7.1.1 where $x = 0$ and $p = 0$.

(7.43): The object $(\wedge^2\mathcal{Q})(x)$ is right orthogonal to both components of $\mathcal{M}^{(1)}$: We refer to the computation in Lemma 7.1.2 with $q = 0$ and to Lemma 7.1.7.

□

Remark 7.1.12. The object $\mathcal{M}^{(1)}$ can already be used to extend our consecutive collection from chapter 6 in the third row. This means that the bounded derived category $\mathbf{D}^{\mathrm{b}}(\mathrm{OGr}(3,9))$ admits an exceptional collection of maximal expected length with starting block

$$\left(\mathcal{O}\,,\, \mathcal{U}^\vee\,,\, \mathcal{M}^{(1)}\,,\, \wedge^2\mathcal{U}^\vee\,,\, \mathcal{S}^{(0)}\,,\, \mathcal{S}^{(1)}\,,\, \mathcal{S}^{(2)} \right).$$

The object $\mathcal{M}^{(1)\vee}(1)$ does an analogous job for the alternating collection from chapter 6. It induces a starting block of the form

$$\left(\mathcal{O}\,,\, \mathcal{S}^{(0)}\,,\, \mathcal{U}^\vee\,,\, \mathcal{M}^{(1)\vee}(1)\,,\, \wedge^2\mathcal{U}^\vee\,,\, \mathcal{S}^{(1)}\,,\, \mathcal{S}^{(2)} \right).$$

Step 2

Lemma 7.1.13. *The object $\wedge^2\mathcal{U}^\vee$ is right orthogonal to $\mathcal{M}^{(1)}$ – i.e. the pair $(\mathcal{M}^{(1)}, \wedge^2\mathcal{U}^\vee)$ is exceptional.*

Proof. We show that $\wedge^2\mathcal{U}^\vee$ is right orthogonal to both components appearing in (7.31). For $\mathrm{Ext}^\bullet(\wedge^2\mathcal{U}^\vee, \mathcal{O}) = 0$ we refer to Proposition 4.1.2 and for $\mathrm{Ext}^\bullet(\wedge^2\mathcal{U}^\vee, \wedge^2\mathcal{Q}) = 0$ we refer to the computation in Lemma 7.1.3 with $x = 0$ and $p = 2$. □

Lemma 7.1.14. *The right mutation of $\mathcal{M}^{(1)}$ through $\wedge^2\mathcal{U}^\vee$ is described by the short exact sequence*

$$0 \to \mathcal{M}^{(2)} \to \mathcal{M}^{(1)} \to \wedge^2\mathcal{U}^\vee \to 0 \tag{7.48}$$

such that $\mathcal{M}^{(2)}$ is characterized as the non-splitting G*-equivariant extension defined by the short exact sequence*

$$0 \to \mathcal{O} \oplus (\mathcal{Q}^\vee/\mathcal{U}) \to \mathcal{M}^{(2)} \to \mathcal{U}^\vee \otimes (\mathcal{Q}^\vee/\mathcal{U}) \to 0. \tag{7.49}$$

Proof. First, we apply $\wedge^2$ to the dualized version of the short exact sequence (2.34):

$$0 \to \mathrm{Sym}^2(\mathcal{Q}^\vee/\mathcal{U}) \to \mathcal{Q} \otimes (\mathcal{Q}^\vee/\mathcal{U}) \to \wedge^2\mathcal{Q} \to \wedge^2\mathcal{U}^\vee \to 0. \tag{7.50}$$

Breaking up this into short exact sequenes gives rise to the non-splitting G-equivariant extension K: The left side is

$$0 \to \mathrm{Sym}^2(\mathcal{Q}^\vee/\mathcal{U}) \to \mathcal{Q} \otimes (\mathcal{Q}^\vee/\mathcal{U}) \to K \to 0, \tag{7.51}$$

the right side is

$$0 \to K \to \wedge^2\mathcal{Q} \to \wedge^2\mathcal{U}^\vee \to 0, \tag{7.52}$$

and the kernel K appearing in the middle is precisely of the form

$$0 \to \mathcal{Q}^\vee/\mathcal{U} \to K \to \mathcal{U}^\vee \otimes (\mathcal{Q}^\vee/\mathcal{U}) \to 0. \tag{7.53}$$

Indeed, $\wedge^2\mathcal{Q}$ has semi-simplification

$$\underbrace{\wedge^2\mathcal{U}^\vee}_{=\mathcal{U}^{\omega_2}} \oplus \underbrace{\mathcal{U}^\vee \otimes (\mathcal{Q}^\vee/\mathcal{U})}_{=\mathcal{U}^{\omega_1+2\omega_4}(-1)} \oplus \underbrace{\mathcal{Q}^\vee/\mathcal{U}}_{=\mathcal{U}^{2\omega_4}(-1)} \tag{7.54}$$

and one can compute easily the following Ext-spaces $\mathrm{Ext}^\bullet(\mathcal{U}^\vee \otimes (\mathcal{Q}^\vee/\mathcal{U}), \mathcal{Q}^\vee/\mathcal{U}) = \mathbf{K}[-1]$ as well as $\mathrm{Ext}^\bullet(\mathcal{Q}^\vee/\mathcal{U}, \mathcal{U}^\vee \otimes (\mathcal{Q}^\vee/\mathcal{U})) = 0$. If we assume that (7.53) splits, then there a G-equivariant surjection

$$V \otimes (\mathcal{Q}^\vee/\mathcal{U}) \twoheadrightarrow \mathcal{Q} \otimes (\mathcal{Q}^\vee/\mathcal{U}) \twoheadrightarrow K \twoheadrightarrow \mathcal{Q}^\vee/\mathcal{U}. \tag{7.55}$$

However, there are no non-trivial G-equivariant morphisms from $V \otimes (\mathcal{Q}^\vee/\mathcal{U})$ to $\mathcal{Q}^\vee/\mathcal{U}$ since we compute $\mathrm{Ext}^\bullet(V \otimes (\mathcal{Q}^\vee/\mathcal{U}), (\mathcal{Q}^\vee/\mathcal{U})) = V[0]$.

Second, we concatenate the surjection $\mathcal{M}^{(1)} \twoheadrightarrow \wedge^2\mathcal{Q}$ in (7.31) with the surjection $\wedge^2\mathcal{Q} \twoheadrightarrow \wedge^2\mathcal{U}^\vee$ in (7.52). Snaking yields the two short exact sequences

$$0 \to \mathcal{M}^{(2)} \to \mathcal{M}^{(1)} \to \wedge^2\mathcal{U}^\vee \to 0 \tag{7.56}$$

and

$$0 \to \mathcal{O} \to \mathcal{M}^{(2)} \to K \to 0. \tag{7.57}$$

Thanks to the fact $\mathrm{Ext}^\bullet(\mathcal{M}^{(1)}, \wedge^2\mathcal{U}^\vee) = \mathbf{K}[0]$ (cf. following Lemma 7.1.15), the short exact sequence (7.52) induces the mutation triangle defining the object $\mathcal{M}^{(2)}$ as right mutation of $\mathcal{M}^{(1)}$ through $\wedge^2\mathcal{U}^\vee$.

Third, we combine the surjection of (7.53) with the one of (7.57) and obtain by the snake lemma the desired short exact sequence (7.49). We explicitly mention that the kernel of $\mathcal{M}^{(2)} \twoheadrightarrow \mathcal{U}^\vee \otimes (\mathcal{Q}^\vee/\mathcal{U})$ is the direct sum $\mathcal{O} \oplus (\mathcal{Q}^\vee/\mathcal{U})$ as both Ext-spaces from one component to the other one vanishes. □

Lemma 7.1.15.

$$\mathrm{Ext}^\bullet(\mathcal{M}^{(1)}, \wedge^2\mathcal{U}^\vee) = \mathbf{K}[0] \tag{7.58}$$

Proof. First, we apply $\mathrm{Hom}(-, \wedge^2\mathcal{U}^\vee)$ to resolution (7.32) and obtain the long exact sequence of Ext-spaces:

$$\cdots \to \mathrm{Ext}^i(\mathcal{M}^{(1)\vee}(1), \wedge^2\mathcal{U}^\vee) \to \mathrm{Ext}^i(\mathrm{S} \otimes \mathcal{S}, \wedge^2\mathcal{U}^\vee) \to \mathrm{Ext}^i(\mathcal{M}^{(1)}, \wedge^2\mathcal{U}^\vee) \to \cdots. \tag{7.59}$$

Since $\mathcal{S}$ is right orthogonal to $\wedge^2\mathcal{U}^\vee$ by Proposition 6.2.2, we have the isomorphism

$$\mathrm{Ext}^i(\mathcal{M}^{(1)}, \wedge^2\mathcal{U}^\vee) = \mathrm{Ext}^{i+1}(\mathcal{M}^{(1)\vee}(1), \wedge^2\mathcal{U}^\vee). \tag{7.60}$$

Second, we observe the isomorphisms

$$\mathrm{Ext}^i(\mathcal{M}^{(1)\vee}(1), \wedge^2\mathcal{U}^\vee) = \mathrm{Ext}^i(\mathcal{U}^\vee, \mathcal{M}^{(1)}) \tag{7.61}$$

by dualisation and we are left to show $\mathrm{Ext}^i(\mathcal{U}^\vee, \mathcal{M}^{(1)}) = \mathbf{K}[-1]$. In fact, the object $\mathcal{U}^\vee$ is right orthogonal to the component $\mathcal{O}$ by Propositon 4.1.2. This induces the isomorphisms

$$\mathrm{Ext}^i(\mathcal{U}^\vee, \mathcal{M}^{(1)}) = \mathrm{Ext}^i(\mathcal{U}^\vee, \wedge^2\mathcal{Q}) \tag{7.62}$$

and we refer to the computation in Lemma 7.1.1 with $x = 0$ and $p = 1$. □

Lemma 7.1.16.

1. *The object $\mathcal{M}^{(2)}$ is exceptional.*
2. *The twists $\mathcal{M}^{(2)}(x)$ are right orthogonal $\mathcal{M}^{(2)}$ whenever $x \in [1,4]$ – i.e. the object $\mathcal{M}^{(2)}$ has an exceptional orbit of length 5.*

Proof. Mutating $\mathcal{M}^{(1)}$ through $\wedge^2\mathcal{U}^\vee$ preserves the exceptional orbit we checked before in Lemma 7.1.10. □

Remark 7.1.17. Assuming the exceptional collection mentioned in the previous Remark 7.1.12, then one can modify this to an exceptional collection with the starting block

$$\Big(\mathcal{O} \, , \, \mathcal{U}^\vee \, , \, \wedge^2\mathcal{U}^\vee \, , \, \mathcal{M}^{(2)} \, , \, \mathcal{S}^{(0)} \, , \, \mathcal{S}^{(1)} \, , \, \mathcal{S}^{(2)} \Big) \, .$$

Furthermore, let us state that we could generalize this pattern to higher orthogonal Grassmannians $\mathrm{OGr}(3, 2n+1)$ with $n = 5, 6, 7, \cdots$. This means, we could construct non-splitting G-equivariant extensions of the form

$$0 \to \mathcal{U}^{\omega_{n-1}}(-1) \oplus \mathcal{U}^{2\omega_n}(-1) \to \mathcal{M}^{(2)} \to \mathcal{U}^{\omega_1 + 2\omega_n}(-1) \to 0 \tag{7.63}$$

and check at least numerically that these objects are exceptional.

Step 3

Lemma 7.1.18. *The object $\mathcal{S}^{(0)}$ is right orthogonal to $\mathcal{M}^{(2)}$ – i.e. the pair $(\mathcal{M}^{(2)}, \mathcal{S}^{(0)})$ is exceptional.*

Proof. We apply $\mathrm{Hom}(\mathcal{S}^{(0)}, -)$ to the short exact sequence (7.48) defining $\mathcal{M}^{(2)}$. The object $\mathcal{S}^{(0)}$ is right orthogonal to $\mathcal{M}^{(1)}$ by Lemma 7.40 and likewise right orthogonal to $\wedge^2\mathcal{U}^\vee$ by Proposition 6.1.2. □

Lemma 7.1.19. *The right mutation of $\mathcal{M}^{(2)}$ through $\mathcal{S}^{(0)}$ is described by the short exact sequence*

$$0 \to \mathcal{M}^{(2)} \to \mathrm{S} \otimes \mathcal{S}^{(0)} \to \mathcal{M}^{(3)} \to 0 \tag{7.64}$$

such that $\mathcal{M}^{(3)}$ is characterized as the non-splitting G*-equivariant extension defined by the short exact sequence*

$$0 \to \wedge^2\mathcal{U}^\vee \to \mathcal{M}^{(3)} \to \mathcal{M}^{(1)\vee}(1) \to 0. \tag{7.65}$$

Proof. We concatenate the embedding $\mathcal{M}^{(2)} \hookrightarrow \mathcal{M}^{(1)}$ in (7.48) with the embedding $\mathcal{M}^{(1)} \hookrightarrow \mathrm{S} \otimes \mathcal{S}$ in (7.1.9). Then snaking yields both (7.64) and (7.65).

It is $\mathrm{Ext}^\bullet(\mathcal{M}^{(1)\vee}(1), \wedge^2\mathcal{U}^\vee) = \mathbf{K}[-1]$ as we computed earlier in the proof of Lemma 7.1.15.

If we assume that (7.65) splits, then we have a G-equivariant surjection

$$\mathrm{S} \otimes \mathcal{S}^{(0)} \twoheadrightarrow \mathcal{M}^{(3)} \twoheadrightarrow \wedge^2\mathcal{U}^\vee \tag{7.66}$$

if we combine the splitting with the surjection in (7.64). However, this contradicts the fact that $\mathcal{S}^{(0)}$ is right orthogonal to $\wedge^2\mathcal{U}^\vee$ by Proposition 6.1.2.

As we have $\mathrm{Ext}^\bullet(\mathcal{M}^{(2)}, \mathcal{S}^{(0)}) = \mathrm{S}[0]$ (cf. following Lemma 7.1.20), the short exact sequence (7.64) induces the mutation triangle defining the object $\mathcal{M}^{(3)}$ as right mutation of $\mathcal{M}^{(2)}$ through $\mathcal{S}^{(0)}$. □

Lemma 7.1.20.

$$\mathrm{Ext}^\bullet(\mathcal{M}^{(2)}, \mathcal{S}^{(0)}) = \mathrm{S}[0] \tag{7.67}$$

Proof. Since $\wedge^2\mathcal{U}^\vee$ is right orthogonal to $\mathcal{S}^{(0)}$, the short exact sequence (7.48) yields the isomorphisms

$$\mathrm{Ext}^i(\mathcal{M}^{(2)}, \mathcal{S}^{(0)}) = \mathrm{Ext}^i(\mathcal{M}^{(1)}, \mathcal{S}^{(0)}). \tag{7.68}$$

Moreover, $\wedge^2\mathcal{Q}$ is right orthogonal to $\mathcal{S}^{(0)}$ by the computation in Lemma 7.1.5 where $x = 0$. Therefore, it follows from (7.31) the isomorphisms

$$\mathrm{Ext}^i(\mathcal{M}^{(1)}, \mathcal{S}^{(0)}) = \mathrm{Ext}^i(\mathcal{O}, \mathcal{S}^{(0)}). \tag{7.69}$$

Due to Proposition 2.5.2 we compute $\mathrm{Ext}^\bullet(\mathcal{O}, \mathcal{S}^{(0)}) = \mathrm{S}[0]$. □

Lemma 7.1.21.

1. *The object $\mathcal{M}^{(3)}$ is exceptional.*
2. *The twists $\mathcal{M}^{(3)}(x)$ are right orthogonal $\mathcal{M}^{(3)}$ whenever $x \in [1, 4]$ – i.e. the object $\mathcal{M}^{(3)}$ has an exceptional orbit of length* 5.

Proof. Mutating $\mathcal{M}^{(2)}$ through $\mathcal{S}^{(0)}$ preserves the the exceptional orbit we checked before in Lemma 7.1.16. □

7.2 The collection

We set $\mathcal{M} = \mathcal{M}^{(3)}$ as constructed in Lemma 7.1.19 and recall the necessary construction by the exact sequences

$$0 \to \wedge^2\mathcal{U}^\vee \to \mathcal{M} \to \mathcal{F}_2^{(2)} \to 0, \tag{7.70}$$

$$0 \to \underbrace{(\wedge^2\mathcal{Q}^\vee)(1)}_{=\wedge^4\mathcal{Q}} \to \mathcal{F}_2^{(2)} \to \mathcal{O}(1) \to 0, \tag{7.71}$$

and

$$0 \to (\wedge^2\mathcal{Q}^\vee)(1) \to \wedge^2 V \otimes \mathcal{O}(1) \to V \otimes \mathcal{U}^\vee(1) \to (\mathrm{Sym}^2\mathcal{U}^\vee)(1) \to 0. \tag{7.72}$$

Moreover, we can resolve the component $\mathcal{F}_2^{(2)}$ appearing in (7.70) by

$$0 \to \mathcal{F}_2^{(2)\vee}(1) \to \mathrm{S} \otimes \mathcal{S} \to \mathcal{F}_2^{(2)} \to 0, \tag{7.73}$$

$$0 \to \mathcal{O} \to \mathcal{F}_2^{(2)\vee}(1) \to \wedge^2\mathcal{Q} \to 0, \tag{7.74}$$

and

$$0 \to \mathrm{Sym}^2\mathcal{U} \to V \otimes \mathcal{U} \to \wedge^2 V \otimes \mathcal{O} \to \wedge^2\mathcal{Q} \to 0. \tag{7.75}$$

The bounded derived category $\mathbf{D}^{\mathrm{b}}(\mathrm{OGr}(3, V_9))$ admits two exceptional collections consisting of maximal expected length, namely

$$\begin{pmatrix} \mathcal{M} & \mathcal{M}(1) & \mathcal{M}(2) & \mathcal{M}(3) & \mathcal{M}(4) \\ \mathcal{S}^{(2)} & \mathcal{S}^{(2)}(1) & & & \\ \mathcal{S}^{(1)} & \mathcal{S}^{(1)}(1) & \mathcal{S}^{(1)}(2) & \mathcal{S}^{(1)}(3) & \mathcal{S}^{(1)}(4) \\ \mathcal{S}^{(0)} & \mathcal{S}^{(0)}(1) & \mathcal{S}^{(0)}(2) & \mathcal{S}^{(0)}(3) & \mathcal{S}^{(0)}(4) \\ \wedge^2\mathcal{U}^\vee & (\wedge^2\mathcal{U}^\vee)(1) & (\wedge^2\mathcal{U}^\vee)(2) & (\wedge^2\mathcal{U}^\vee)(3) & (\wedge^2\mathcal{U}^\vee)(4) \\ \mathcal{U}^\vee & \mathcal{U}^\vee(1) & \mathcal{U}^\vee(2) & \mathcal{U}^\vee(3) & \mathcal{U}^\vee(4) \\ \mathcal{O} & \mathcal{O}(1) & \mathcal{O}(2) & \mathcal{O}(3) & \mathcal{O}(4) \end{pmatrix} \tag{7.76}$$

and

$$\begin{pmatrix} \mathcal{M} & \mathcal{M}(1) & \mathcal{M}(2) & \mathcal{M}(3) & \mathcal{M}(4) \\ \mathcal{S}^{(2)} & \mathcal{S}^{(2)}(1) & & & \\ \mathcal{S}^{(1)} & \mathcal{S}^{(1)}(1) & \mathcal{S}^{(1)}(2) & \mathcal{S}^{(1)}(3) & \mathcal{S}^{(1)}(4) \\ \wedge^2\mathcal{U}^\vee & (\wedge^2\mathcal{U}^\vee)(1) & (\wedge^2\mathcal{U}^\vee)(2) & (\wedge^2\mathcal{U}^\vee)(3) & (\wedge^2\mathcal{U}^\vee)(4) \\ \mathcal{U}^\vee & \mathcal{U}^\vee(1) & \mathcal{U}^\vee(2) & \mathcal{U}^\vee(3) & \mathcal{U}^\vee(4) \\ \mathcal{S}^{(0)} & \mathcal{S}^{(0)}(1) & \mathcal{S}^{(0)}(2) & \mathcal{S}^{(0)}(3) & \mathcal{S}^{(0)}(4) \\ \mathcal{O} & \mathcal{O}(1) & \mathcal{O}(2) & \mathcal{O}(3) & \mathcal{O}(4) \end{pmatrix} \tag{7.77}$$

respectively. Moreover, the object $\mathcal{M}$ is even orthogonal to $\mathcal{S}^{(1)}$ as well as $\mathcal{S}^{(2)}$.

Proving exceptionality.

Proposition 7.2.1. *Both the consecutive collection (7.2) and the alternating one (7.3) can be extended by the exceptional orbit* $(\mathcal{M}, \cdots, \mathcal{M}(4))$ *after the fourth, fifth or sixth row respectively.*

Proof. We refer to the following three Lemmas 7.2.2, 7.2.3, and 7.2.4. □

Lemma 7.2.2. *Let* $\mathcal{E}''$ *be an element from the starting block of the consecutive collection (7.2) – i.e.* $\mathcal{E}''$ *is from* $(\mathcal{O}, \mathcal{U}^\vee, \wedge^2\mathcal{U}^\vee, \mathcal{S}^{(0)}, \mathcal{S}^{(1)}, \mathcal{S}^{(2)})$ *– and let* x *be from* $[0,4]$*. Then* $\mathcal{M}(x)$ *is right orthogonal to* $\mathcal{E}''$*.*

Proof. $x \in \{0,1\}$**:** $x = 0$ **and** $\mathcal{E}'' = \wedge^2\mathcal{U}^\vee$**:** We combine Proposition 6.1.2 with the previous section 7.1. This means that the triple $(\mathcal{M}^{(1)}, \wedge^2\mathcal{U}^\vee, \mathcal{S}^{(0)})$ is exceptional. Furthermore, $\mathcal{M} = \mathcal{M}^{(3)}$ is the right-mutation of $\mathcal{M}^{(1)}$ through $(\wedge^2\mathcal{U}^\vee, \mathcal{S}^{(0)})$. In particular, $\mathcal{M}$ is right orthogonal to $\wedge^2\mathcal{U}^\vee$.

Otherwise: We apply $\mathrm{Hom}(-, \mathcal{E}'')$ to $\mathcal{O}(x)$-twisted versions of (7.71) and (7.72) respectively. The components $(\mathrm{Sym}^p\mathcal{U}^\vee)(1+x)$ where $p \in [0,2]$ are right orthogonal to $\mathcal{E}''$. In fact, for $p \in \{0,1\}$, we refer to Proposition 6.1.2; and for $p = 2$, we check by Proposition 2.5.2 that $(\mathrm{Sym}^2\mathcal{U}^\vee)(1+x) = \mathcal{U}^{2\omega_1}(1+x)$ is right orthogonal to each irreducible component of $\mathcal{E}''$. So, we see that $(\wedge^2\mathcal{Q})(1+x)$ is right orthogonal to $\mathcal{E}''$ and the same holds also for $\mathcal{F}_2^{(2)}$.

Next, we move on to (7.70) and recall that $(\wedge^2\mathcal{U}^\vee)(x)$ is right orthogonal to $\mathcal{E}''$ by Proposition 6.1.2. Accordingly, it follows the desired right orthogonal relation from $\mathcal{M}(x)$ to $\mathcal{E}''$.

$x \in [2,4]$**:** First, we point out that $(\wedge^2\mathcal{Q})(x)$ is right orthogonal to $\mathcal{E}''$ due to (7.75). In fact, $(\mathrm{Sym}^p\mathcal{U})(x)$ is right orthogonal to $\mathcal{E}''$ as we can either refer back to Proposition 6.1.2 or we compute by Proposition 2.5.2 that $(\mathrm{Sym}^2\mathcal{U})(x) = \mathcal{U}^{2\omega_2}(-2+x)$ is right orthogonal to each irreducible component of $\mathcal{E}''$.

Second, $\mathcal{F}_2^{(2)\vee}(1+x)$ is right orthogonal to $\mathcal{E}''$ by (7.74).

Third, we see that $\mathcal{S}(x)$ is right orthogonal to $\mathcal{E}''$ by Proposition 6.1.2 and we deduce by (7.73) the similar statement for $\mathcal{F}_2^{(2)}(x)$.

Finally, (7.70) yields that $\mathcal{M}(x)$ is right orthogonal to $\mathcal{E}''$ since each of its components is right orthogonal to $\mathcal{E}''$.

□

Lemma 7.2.3. *Let* $\mathcal{E}'(x)$ *be an object from the consecutive collection (7.2) such that* $\mathcal{E}'$ *is an element of the starting block* $(\mathcal{O}, \mathcal{U}^\vee, \wedge^2\mathcal{U}^\vee, \mathcal{S}^{(0)}, \mathcal{S}^{(1)}, \mathcal{S}^{(2)})$ *and* x *is from* $[1,4]$*. Then* $\mathcal{E}'(x)$ *is right orthogonal to* $\mathcal{M}$*.*

Proof. $x = 1$ **and** $\mathcal{E}' \in \{\mathcal{O}, \mathcal{U}^\vee\}$**:** First, we consider (7.74) and (7.75) and we show that $\mathcal{E}'(1)$ is right orthogonal to $\mathrm{Sym}^q\mathcal{U} = \mathcal{U}^{q\omega_2}(-q)$ where $q \in [0,2]$. In fact, for $q \in \{0,1\}$, we recall the fact $\mathcal{U} = (\wedge^2\mathcal{U}^\vee)(-1)$ and refer to Proposition 4.1.2; and for $q = 2$, we check by Proposition 2.5.2. Thus, $\mathcal{E}'(1)$ is right orthogonal to $\wedge^2\mathcal{Q}$ and likewise it is also right orthogonal to $\mathcal{F}_2^{(2)\vee}(1)$.

Second, we consider (7.73) and we recall that $\mathcal{E}'(1)$ is right orthogonal to $\mathcal{S}$ by Proposition 6.1.2. So, it follows that $\mathcal{E}'(1)$ is right orthogonal to $\mathcal{F}_2^{(2)}$.

Third, we work with (7.70) and we recall that $\mathcal{E}'(1)$ is right orthogonal to $\wedge^2\mathcal{U}^\vee$ by Proposition 4.1.2. This yields finally that $\mathcal{E}'(1)$ is also right orthogonal to $\mathcal{M}$ as desired.

Otherwise: The object $\mathcal{E}'(x)$ appears in the consecutive collection (7.2) after $\mathcal{U}^\vee(1)$ – i.e. we have either $x = 1$ and $\mathcal{E}' \in \{\wedge^2\mathcal{U}^\vee, \mathcal{S}^{(0)}, \mathcal{S}^{(1)}, \mathcal{S}^{(2)}\}$ or it is $x \in [2,4]$ and $\mathcal{E}' \in \{\mathcal{O}, \mathcal{U}^\vee, \wedge^2\mathcal{U}^\vee, \mathcal{S}^{(0)}, \mathcal{S}^{(1)}\}$.

First, we start with (7.71) and (7.72). We observe that $\mathcal{E}'(x)$ is right orthogonal to $(\mathrm{Sym}^q\mathcal{U}^\vee)(1)$ where $q \in [0,2]$. In fact, for $q \in \{0,1\}$, we refer to Proposition 6.1.2; and for $q = 2$, we check by Proposition 2.5.2 that any irreducible component of $\mathcal{E}'(x)$ is right orthogonal to $(\mathrm{Sym}^2\mathcal{U}^\vee)(1) = \mathcal{U}^{2\omega_1}(1)$. Hence, we deduce that $\mathcal{E}'(1)$ is right orthogonal to $(\wedge^2\mathcal{Q}^\vee)(1)$ and likewise $\mathcal{F}_2^{(2)}$.

Second, let us mention that $\mathcal{E}'(x)$ is also right orthogonal to $\wedge^2\mathcal{U}^\vee$ by Proposition 6.1.2.

All in all, we see that $\mathcal{E}'(x)$ is right orthogonal to any component of $\mathcal{M}$ and therefore it is also right orthogonal to $\mathcal{M}$ itself.

□

Lemma 7.2.4. *The two objects $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$ of the spinor subcollection are each even orthogonal to $\mathcal{M}$.*

Proof. We already showed in the previous Lemma 7.2.2 that $\mathcal{M}$ is right orthogonal to the objects $\mathcal{S}^{(y)}$ where $y \in \{1,2\}$. Hence, we are left to show the opposite direction, namely the objects $\mathcal{S}^{(y)}$ where $y \in \{1,2\}$ is right orthogonal to $\mathcal{M}$.

First, we apply $\mathrm{Hom}(\mathcal{S}^{(y)}, -)$ to (7.74) as well as (7.75). The object $\mathcal{S}^{(y)}$ is right orthogonal to $\mathrm{Sym}^q\mathcal{U}$ since we have for $q \in \{0,1\}$ Proposition 6.1.2 and the fact $\mathcal{U} = (\wedge^2\mathcal{U}^\vee)(-1)$ as well as for $q = 2$ Proposition 2.5.2 and the fact $\mathrm{Sym}^2\mathcal{U} = \mathcal{U}^{2\omega_2}(-1)$. So, $\mathcal{S}^{(y)}$ is right orthogonal to $\wedge^2\mathcal{Q}$ and also to $\mathcal{F}_2^{(2)\vee}(1)$.

Second, we focus to (7.73) and recall that $\mathcal{S}^{(y)}$ where $y \in \{1,2\}$ is right orthogonal to $\mathcal{S}$ by Proposition 5.0.10. Therefore, $\mathcal{S}^{(y)}$ is likewise right orthogonal to $\mathcal{F}_2^{(2)}$.

Finally, we mention that $\mathcal{S}^{(y)}$ is right orthogonal to $\wedge^2\mathcal{U}^\vee$ by Proposition 6.1.2 and thus it follows from (7.70) that $\mathcal{S}^{(y)}$ is right orthogonal to $\mathcal{M}$. □

Chapter 8

Outlook

We present an overview of our remaining open problems concerning the family of orthogonal Grassmannians $\mathrm{OGr}(3, V_{2n+1})$ where V_{2n+1} is a $2n+1$-dimensional vector space. First, we take up the case $\mathrm{OGr}(3, V_9)$ which has been considered in the previous chapter 7 explicitly. Then, we shortly outline some computational results for the cases $\mathrm{OGr}(3, V_{2n+1})$ with higher n.

8.1 Fullness for $OGr(3,9)$

The bounded derived category $\mathbf{D}^{\mathrm{b}}(X)$ where $X = \mathrm{OGr}(3, V_9)$ admits two exceptional collections (7.76) and (7.77). Let $\mathcal{D}$ be their full triangulated subcategory. A priori, we have a semi-orthogonal decomposition $\mathbf{D}^{\mathrm{b}}(X) = \langle \mathcal{D}^{\perp}, \mathcal{D} \rangle$ where $\mathcal{D}^{\perp}$ is the right orthogonal of $\mathcal{D}$ in $\mathbf{D}^{\mathrm{b}}(X)$, namely

$$\mathcal{D}^{\perp} = \left\{ \mathcal{F} \in \mathbf{D}^{\mathrm{b}}(X) \ : \ \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(X)}(\mathcal{E}, \mathcal{F}) = 0 \text{ for any } \mathcal{E} \in \mathcal{D} \right\}. \tag{8.1}$$

It is an open question whether $\mathcal{D}^{\perp}$ vanishes and consequently whether we have the identity $\mathcal{D} = \mathbf{D}^{\mathrm{b}}(X)$. Nevertheless, let us present our most advanced approach which relies on the embedding

$$Y = \mathrm{OGr}(3, V_7) = \mathrm{B}_3/\mathrm{P}_3 \quad \subseteq \quad X = \mathrm{OGr}(3, V_9) = \mathrm{B}_4/\mathrm{P}_3 \tag{8.2}$$

and proceeds analogously as in [20].

Set-up of the embedding. Let v_1 and v_2 be two arbitrary but fixed vectors from V_9 such that the span $\langle v_1, v_2 \rangle \subseteq V_9$ is a 2-dimensional subspace and the quantity $\langle v_1, v_2 \rangle$ does not vanish. We introduce the associated section $\phi : V_9^{\oplus 2} \to \mathbf{K}^{\oplus 2}$ defined via $\phi = (\langle v_1, - \rangle, \langle v_2, - \rangle)$. Write δ for the diagonal morphism $V_9 \to V_9^{\oplus 2}$ via $v \mapsto (v, v)$. Then we obtain the following zero loci:

- The orthogonal complement of $\langle v_1, v_2 \rangle$ in V_9, namely

$$V_7 := \langle v_1, v_2 \rangle^{\perp} := \{ \, v \in V \ : \ (\phi \circ \delta)(v) = 0 \, \}.$$

It is a 7-dimensional vector space equipped with a non-degenerate, symmetric bilinear form $\langle -,- \rangle |_{V_7^{\oplus 2}}$.

- The space

$$Y := \{ U \in X \;:\; (\phi \circ \delta)(u) \text{ for any } u \in U \} = \{ U \in X \;:\; U \subseteq V_7 \}$$

is the orthogonal Grassmannian $\mathrm{OGr}(3, V_7)$. We have the tautological short exact sequence

$$0 \to \mathcal{U}_Y \to V_7 \otimes \mathcal{O}_Y \to \mathcal{Q}_Y \to 0 \tag{8.3}$$

on Y and observe that the tautological quotient bundle $\mathcal{Q}_Y$ is a non-splitting G_Y-equivariant extension of $\mathcal{U}_Y{}^\vee$ by $\mathcal{O}_Y$. Furthermore, there is the Koszul resolution

$$0 \to K_6 \to \cdots \to K_1 \to K_0 \to \iota_{\phi *}\mathcal{O}_Y \to 0 \tag{8.4}$$

where K_i is the bundle $\wedge^i(\mathcal{U}^{\oplus 2})$ and ι_ϕ the embedding $Y \hookrightarrow X$.

We deduce from Kapranov's collection on the quadric 6-fold Q^6 which is isomorphic to $Y = \mathrm{OGr}(3, V_7)$ the full exceptional Lefschetz collection

$$\begin{pmatrix} \mathcal{Q}_Y & \mathcal{Q}_Y(1) & & & \\ \mathcal{O}_Y & \mathcal{O}_Y(1) & \mathcal{O}_Y(2) & \cdots & \mathcal{O}(5) \end{pmatrix}. \tag{8.5}$$

The crucial open issue. Now, we present the following lifting from $\mathbf{D}^{\mathrm{b}}(Y)$ to $\mathbf{D}^{\mathrm{b}}(X)$:

$$\begin{array}{cccl} \mathbf{D}^{\mathrm{b}}(X) & \xrightarrow{\iota_\phi{}^*} & \mathbf{D}^{\mathrm{b}}(Y) & \\ \mathcal{O}_X(t) & \mapsto & \mathcal{O}_Y(2t) & \text{for } t \in [0,2] \\ \mathcal{S}(t) & \mapsto & \mathcal{O}_Y(2t+1)^{\oplus 2} & \text{for } t \in [0,2] \\ \mathcal{U}_X{}^\vee & \mapsto & \mathcal{U}_Y{}^\vee & \\ \mathcal{U}_X{}^\vee \otimes \mathcal{S} & \mapsto & \mathcal{U}^\vee(1)^{\oplus 2} & \end{array} \tag{8.6}$$

We mention that we can find every irreducible component (maybe as a summand) that appears in the collection (8.5) on the right side, and every lift on the left side lies in the subcategory $\mathcal{D}$.

Let us assume that we could check the containment:

Assumption 8.1.1. *$\mathcal{F}''' \otimes \mathcal{F}'' \otimes \mathcal{F}' \in \mathcal{D}$ where $\mathcal{F}'$ is from the left side of (8.6) and where $\mathcal{F}''$ and $\mathcal{F}'''$ are of the form $\wedge^i \mathcal{U}_X{}^\vee$.*

In other words, it means $\mathcal{F}' \otimes \mathcal{F}'' \otimes \mathcal{F}''' \in \mathcal{D}$ whenever

$$\mathcal{F}' \in \{ \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2), \mathcal{S}, \mathcal{S}(1), \mathcal{S}(2), \mathcal{U}_x{}^\vee, \mathcal{U}_X{}^\vee \otimes \mathcal{S} \}$$

and

$$\mathcal{F}'', \mathcal{F}''' \in \{ 0, \mathcal{O}_X, \mathcal{U}_X{}^\vee, \wedge^2\mathcal{U}_X{}^\vee, \mathcal{O}_X(1) \}.$$

Then, it follows clearly $K_i{}^\vee \otimes \mathcal{F}' = \bigoplus_{j=0}^{i} \wedge^j \mathcal{U}^\vee \otimes \wedge^{i-j} \mathcal{U}^\vee \otimes \mathcal{F}' \in \mathcal{D}$.

Remark 8.1.2. We could show many of the necessary containments $\mathcal{F}''' \otimes \mathcal{F}'' \otimes \mathcal{F}' \in \mathcal{D}$. However, we are still left with a handful of open issues. In fact, it boils down to prove that the following vector bundles lie in $\mathcal{D}$:

$$\mathcal{O}_X(m) \text{ for } m \in [0,4] \tag{8.7}$$
$$\mathcal{S}_X(m) \text{ for } m \in [0,4] \tag{8.8}$$
$$\mathcal{U}_X{}^\vee(m) \text{ for } m \in [0,3] \tag{8.9}$$
$$(\wedge^2\mathcal{U}_X{}^\vee)(m) \text{ for } m \in [0,3] \tag{8.10}$$
$$\mathcal{S} \otimes \mathcal{U}_X{}^\vee(m) \text{ for } m \in [0,3] \tag{8.11}$$
$$\mathcal{S}_X \otimes (\wedge^2\mathcal{U}_X{}^\vee)(m) \text{ for } m \in [0,3] \tag{8.12}$$
$$(\mathrm{Sym}^2\mathcal{U}_X{}^\vee)(m) \text{ for } m \in [0,2] \tag{8.13}$$
$$(\Sigma^{2,1}\mathcal{U}_X{}^\vee)(m) \text{ for } m \in [0,2] \tag{8.14}$$
$$(\Sigma^{2,2}\mathcal{U}_X{}^\vee)(m) \text{ for } m \in [0,2] \tag{8.15}$$
$$\mathcal{S}_X \otimes (\mathrm{Sym}^2\mathcal{U}_X{}^\vee)(m) \text{ for } m \in [0,2] \tag{8.16}$$
$$\mathcal{S}_X \otimes (\Sigma^{2,1}\mathcal{U}_X{}^\vee)(m) \text{ for } m \in [0,2] \tag{8.17}$$
$$\mathcal{S}_X \otimes (\Sigma^{2,2}\mathcal{U}_X{}^\vee)(m) \text{ for } m \in [0,2] \tag{8.18}$$
$$\mathrm{Sym}^3\mathcal{U}_X{}^\vee \tag{8.19}$$
$$\Sigma^{3,1}\mathcal{U}_X{}^\vee \tag{8.20}$$
$$\Sigma^{3,2}\mathcal{U}_X{}^\vee \tag{8.21}$$
$$\mathcal{S}_X \otimes \mathrm{Sym}^3\mathcal{U}_X{}^\vee \tag{8.22}$$
$$\mathcal{S}_X \otimes \Sigma^{3,1}\mathcal{U}_X{}^\vee \tag{8.23}$$
$$\mathcal{S}_X \otimes \Sigma^{3,2}\mathcal{U}_X{}^\vee \tag{8.24}$$

We are left to prove the cases $\mathrm{Sym}^2\mathcal{U}_X{}^\vee$ which is the case $m = 0$ in (8.13), $\mathcal{S}_X \otimes (\mathrm{Sym}^2\mathcal{U}_X{}^\vee)(2)$ which is the case $m = 2$ in (8.16), $(\mathrm{Sym}^3\mathcal{U}_X{}^\vee)(m)$ for $m \in [0,2]$ which is the case (8.19) if $m = 0$, $(\Sigma^{3,1}\mathcal{U}_X{}^\vee)(1)$, and $\mathcal{S}_X \otimes \mathrm{Sym}^3\mathcal{U}_X{}^\vee$ which is the case (8.22).

Final conclusion. Let $\mathcal{R}$ be an object from the right orthogonal complement $\mathcal{D}^\perp$; that is, $\mathrm{Hom}_{\mathbf{D}^b(X)}(\mathcal{E},\mathcal{R}) = 0$ for any generator appearing in our collection (7.76) or (7.77) respectively.

Lemma 8.1.3. *Assuming 8.1.1, the restriction $\iota_\phi{}^*\mathcal{R}$ is right orthogonal to any $\iota_\phi{}^*\mathcal{F}$ as introduced on the right side of* (8.6).

Proof. Let $\mathcal{F}$ be some object from the set $\{\ \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2), \mathcal{S}, \mathcal{S}(1), \mathcal{S}(2), \mathcal{U}_x{}^\vee, \mathcal{U}_X{}^\vee \otimes \mathcal{S}\ \}$. Tensoring $\mathcal{F}^\vee \otimes \mathcal{R}$ to the resolution (8.4) yields the complex

$$\begin{aligned} 0 \to K_6 \otimes \mathcal{F}^\vee \otimes \mathcal{R} \to \cdots \to K_1 \otimes \mathcal{F}^\vee \otimes \mathcal{R} \to \\ K_0 \otimes \mathcal{F}^\vee \otimes \mathcal{R} \to \iota_{\phi*}(\mathcal{O}_Y) \otimes \mathcal{F}^\vee \otimes \mathcal{R} \to 0. \end{aligned} \tag{8.25}$$

We claim that $\iota_{\phi *}(\mathcal{O}_Y) \otimes \mathcal{F}^\vee \otimes \mathcal{R}$ has no cohomology as it computes for any term $K_j \otimes \mathcal{F}^\vee \otimes \mathcal{R}$ in (8.25) where $j \in [0,6]$ the following:

$$\begin{aligned} H^\bullet(X, K_j \otimes \mathcal{F}^\vee \otimes \mathcal{R}) &= \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(X)}(\mathcal{O}, K_j \otimes \mathcal{F}^\vee \otimes \mathcal{R}) \\ &= \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(X)}(K_j{}^\vee \otimes \mathcal{F}, \mathcal{R}) \\ &= 0 \end{aligned} \tag{8.26}$$

The vanishing follows from combining the fact that the object $K_j{}^\vee \otimes \mathcal{F}$ lies in $\mathcal{D}$ and $\mathcal{R}$ lies in $\mathcal{D}^\perp$.
Thanks to the projection formula, we prepare

$$\iota_{\phi *}\iota_\phi{}^*(\mathcal{F}^\vee \otimes \mathcal{R}) = (\iota_{\phi *}\mathcal{O}_Y) \otimes \mathcal{F}^\vee \otimes \mathcal{R}. \tag{8.27}$$

Finally, we deduce the desired statement:

$$\begin{aligned} \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(X)}(\iota_\phi{}^*\mathcal{F}, \iota_\phi{}^*\mathcal{R}) &= \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(X)}(\iota_\phi{}^*\mathcal{O}_X, \iota_\phi{}^*(\mathcal{F}^\vee \otimes \mathcal{R})) && (8.28) \\ &= \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(X)}(\mathcal{O}_X, \iota_{\phi *}\iota_\phi{}^*(\mathcal{F}^\vee \otimes \mathcal{R})) && (8.29) \\ &\overset{(8.27)}{=} \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(X)}(\mathcal{O}_X, (\iota_{\phi *}\mathcal{O}_Y) \otimes \mathcal{F}^\vee \otimes \mathcal{R}) && (8.30) \\ &= H^\bullet(X, (\iota_{\phi *}\mathcal{O}_Y) \otimes \mathcal{F}^\vee \otimes \mathcal{R}) && (8.31) \\ &\overset{(8.26)}{=} 0 && (8.32) \end{aligned}$$

□

Lemma 8.1.4. *Assuming 8.1.1, the restriction $\iota_\phi{}^*\mathcal{R}$ is right orthogonal to any generator of our Lefschetz collection (8.5) on $\mathbf{D}^{\mathrm{b}}(Y)$.*

Proof. First, we show that $\iota_\phi{}^*\mathcal{R}$ is right orthogonal to every irreducible component of any generator of our Lefschetz collection (8.5) on $\mathbf{D}^{\mathrm{b}}(Y)$. In fact, every such object can be found (maybe as summand) in (8.6). Hence, we apply the previous Lemma 8.1.3. If necessary, we rely on the fact that the functor $\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(Y)}(-, \iota_\phi{}^*\mathcal{R})$ is additive.

Second, we conclude that $\iota_\phi{}^*\mathcal{R}$ needs to be right orthogonal to every generator as it is so for every of its irreducible components. □

Corollary 8.1.5. *Assuming 8.1.1, the restriction $\iota_\phi{}^*\mathcal{R}$ vanishes for any section ϕ.*

Proof. We deduce

$$\iota_\phi{}^*\mathcal{R} \overset{8.1.4}{\in} \left\langle \mathcal{O}_Y, \mathcal{Q}_Y; \mathcal{O}_Y(1), \mathcal{Q}_Y(1); \mathcal{O}_Y(2); \cdots; \mathcal{O}_Y(5) \right\rangle^\perp = \mathbf{D}^{\mathrm{b}}(Y)^\perp = 0.$$

□

Proposition 8.1.6. *Assuming 8.1.1, it holds $\mathcal{D}^\perp = 0$.*

We argue analogously as in [20, Lemma 7.6.].

Proof. We show that for any non-trivial $\mathcal{R}$ there is some ϕ such that the restriction ${\iota_\phi}^*\mathcal{R}$ does not vanish. Hence, it follows from the previous corollary 8.1.5 that $\mathcal{R}$ is not an object from $\mathcal{D}^\perp$.

Let us assume $\mathcal{R} \neq 0$. We find a maximal integer j such that the j^{th} cohomology of the cochain complex $\mathcal{R}$ does not vanish. This means,

$$0 \neq \mathcal{H}^j(\mathcal{R}) = H^j(X, \mathcal{R}) = \mathrm{Ext}^j(\mathcal{O}_X, \mathcal{R}) = \mathrm{Hom}_{\mathbf{D}^\mathrm{b}(X)}(\mathcal{O}_X, \mathcal{R}[j]). \tag{8.33}$$

Furthermore, we find a point $x \in \operatorname{supp} \mathcal{H}^j(\mathcal{R}) \subseteq X$. Now, we can choose a 2-dimensional subspace $\langle v_1, v_2 \rangle \subseteq V_9$ such that $x \in Y$ – i.e. we have $\langle v_1, w \rangle = 0$ and $\langle v_2, w \rangle = 0$ for any w from the 3-dimensional subspace of V corresponding to $x \in X = \mathrm{OGr}(3, V_9)$. Since the pullback functor ${\iota_\phi}^*$ is left exact, we see that j^{th} cohomology of the restricted cochain complex ${\iota_\phi}^*\mathcal{R}$ remains non-trivial – i.e. $\mathcal{H}^j({\iota_\phi}^*\mathcal{R}) \neq 0$. So, we deduce ${\iota_\phi}^*\mathcal{R} \neq 0$ as desired. □

8.2 The residual category of $OGr(3,9)$

Throughout this section, we assume that the subcategory $\mathcal{D}$, which as been introduced in the previous section 8.1, is full. We mutate the two objects $\mathcal{S}^{(2)}$ and $\mathcal{S}^{(2)}(1)$ to the far left position: First, we collect those objects in the starting block with maximal orbit length 5, namely the block $\mathcal{B} = \langle \mathcal{O}, \mathcal{U}^\vee, \wedge^2\mathcal{U}^\vee, \mathcal{S}^{(0)}, \mathcal{S}^{(1)}, \mathcal{M} \rangle$. Then we write the left mutations $\mathcal{R}^{(1)} := \mathbb{L}_{\mathcal{B}}(\,\mathcal{S}^{(2)}\,)$ and $\mathcal{R}^{(2)} := \mathbb{L}_{\langle \mathcal{B}, \mathcal{B}(1) \rangle}(\,\mathcal{S}^{(2)}(1)\,)$. This yields an exceptional collection

$$\begin{pmatrix} \mathcal{M} & \mathcal{M}(1) & \mathcal{M}(2) & \mathcal{M}(3) & \mathcal{M}(4) \\ \mathcal{S}^{(1)} & \mathcal{S}^{(1)}(1) & \mathcal{S}^{(1)}(2) & \mathcal{S}^{(1)}(3) & \mathcal{S}^{(1)}(4) \\ \mathcal{S}^{(0)} & \mathcal{S}^{(0)}(1) & \mathcal{S}^{(0)}(2) & \mathcal{S}^{(0)}(3) & \mathcal{S}^{(0)}(4) \\ \wedge^2\mathcal{U}^\vee & (\wedge^2\mathcal{U}^\vee)(1) & (\wedge^2\mathcal{U}^\vee)(2) & (\wedge^2\mathcal{U}^\vee)(3) & (\wedge^2\mathcal{U}^\vee)(4) \\ \mathcal{U}^\vee & \mathcal{U}^\vee(1) & \mathcal{U}^\vee(2) & \mathcal{U}^\vee(3) & \mathcal{U}^\vee(4) \\ \mathcal{O} & \mathcal{O}(1) & \mathcal{O}(2) & \mathcal{O}(3) & \mathcal{O}(4) \\ \mathcal{R}^{(2)} & & & & \\ \mathcal{R}^{(1)} & & & & \end{pmatrix}. \tag{8.34}$$

Hence, we obtain a semi-orthogonal decomposition of the form

$$\mathbf{D}^\mathrm{b}(X) = \langle \mathcal{D}^{(Res)}, \mathcal{D}^{(Rec)} \rangle$$

where $\mathcal{D}^{(Rec)} := \langle \mathcal{B}, \mathcal{B}(1), \cdots, \mathcal{B}(4) \rangle$ is the subcategory arising from the rectangular part and $\mathcal{D}^{(Res)}$ is the right orthogonal complement to $\mathcal{D}^{(Rec)}$ (the residual subcategory). Due to the assumed fullness, we conclude $\mathcal{D}^{(Rec)} = \langle \mathcal{R}^{(1)}, \mathcal{R}^{(2)} \rangle$.

We imitated the left mutations of $\mathcal{S}^{(2)}$ and $\mathcal{S}^{(2)}(1)$ numerically. This means, we considered $[\mathcal{S}^{(2)}]$ and $[\mathcal{S}^{(2)}(1)]$ as objects of $\mathrm{K}_0(X)$ and computed the left mutations $[\mathcal{R}^{(1)}]$ and $[\mathcal{R}^{(2)}]$ respectively on this level. Finally, we observed that the Gram matrix is of the form

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

which is in accordance with the conjecture that $\mathcal{D}^{(Res)}$ is equivalent to the bounded derived category of A_2-quivers.

8.3 Cases $OGr(3, 2n+1)$ for higher n

Let n be in the integer intervall $[5, 11]$ and therefore set $X = \mathrm{OGr}(3, V_{2n+1})$. We had some progress to extend the corresponding collections in 6.1 or 6.2 respectively on the level of $\mathrm{K}_0(X)$. This means, we say a bundle $\mathcal{E}$ is numerically exceptional if $\chi(\mathcal{E}, \mathcal{E}) = 1$, and we say for a pair that $\mathcal{E}''$ is numerically right orthogonal to $\mathcal{E}'$ if $\chi(\mathcal{E}'', \mathcal{E}') = 0$. With this notions in mind, we can show that there is a non-splitting G-equivariant extension arising by the irreducible components

$$\mathcal{U}^{\omega_1+2\omega_n}(-1), \qquad \mathcal{U}^{2\omega_n}(-1), \qquad \text{and} \qquad \mathcal{U}^{\omega_{n-1}}(-1)$$

which is numerically exceptional. It seems to be a counterpart of the missing link $\mathcal{M}^{(2)}$ as in (7.63). It extends numerically our corresponding collection from 6.1 or 6.2 respectively. Furthermore, we repeat an analogous mutation as in Lemma 7.1.19 on the level of $\mathrm{K}_0(X)$ and therefore construct numerically an analogous counterpart of $\mathcal{M}^{(3)}$.

The upper bound of considered cases $n \leq 11$ arise from our limits of computational power.

If one considers the difference between the expected maximal length of a full exceptional collection on $\mathbf{D}^{\mathrm{b}}(X)$ and the length of our collections in 6.1 or 6.2 respectively as in (6.3), then one observes that this gap grows with order $\mathcal{O}(n^2)$. This means at the present point in time that we have a lack of support by further exceptional objects if $n \geq 9$.

Bibliography

[1] Alexander A. Beilinson. "Coherent sheaves on Pn and problems of linear algebra". In: *Functional Analysis and Its Applications* 12 (1978), pp. 214–216. URL: https://doi.org/10.1007/BF01681436.

[2] Pieter Belmans. *Grassmannian.info – A periodic table of (generalised) Grassmannians*. 2024. URL: https://www.grassmannian.info (visited on 05/08/2024).

[3] Pieter Belmans, Alexander Kuznetsov, and Maxim Smirnov. "Derived categories of the Cayley plane and the coadjoint Grassmannian of type F". In: *Transformation Groups* 28.1 (May 2021), 9–34. ISSN: 1531-586X. DOI: 10.1007/s00031-021-09657-w. URL: http://dx.doi.org/10.1007/s00031-021-09657-w.

[4] Vladimiro Benedetti, Daniele Faenzi, and Maxim Smirnov. *Derived category of the spinor 15-fold*. 2023. arXiv: 2310.01090 [math.AG]. URL: https://arxiv.org/abs/2310.01090.

[5] Alexei Bondal and Dimitri Orlov. *Derived categories of coherent sheaves*. 2002. arXiv: math/0206295 [math.AG]. URL: https://arxiv.org/abs/math/0206295.

[6] Alexei Bondal and Dmitri Orlov. "Reconstruction of a variety from the derived category and groups of autoequivalences". In: *Compositio Mathematica* 125.3 (2001), 327–344. ISSN: 0010-437X. DOI: 10.1023/a:1002470302976. URL: http://dx.doi.org/10.1023/A:1002470302976.

[7] Raoul Bott. "Homogeneous Vector Bundles". In: *Annals of Mathematics* 66.2 (1957), pp. 203–248. ISSN: 0003486X. URL: http://www.jstor.org/stable/1969996 (visited on 09/12/2023).

[8] Christian Böhning. "Derived categories of coherent sheaves on rational homogeneous manifolds". In: *Doc. Math.* 11 (2006), pp. 261–331.

[9] Giordano Cotti, Boris Dubrovin, and Davide Guzzetti. *Helix Structures in Quantum Cohomology of Fano Varieties*. 2019. arXiv: 1811.09235 [math.AG]. URL: https://arxiv.org/abs/1811.09235.

[10] Boris Dubrovin. "Geometry and analytic theory of Frobenius manifolds". In: *Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998)*. Extra Vol. II. 1998, pp. 315–326.

[11] Daniele Faenzi and Laurent Manivel. "On the derived category of the Cayley plane II". In: *Proceedings of the American Mathematical Society* 143.3 (2015), pp. 1057–1074. ISSN: 00029939, 10886826. URL: http://www.jstor.org/stable/24507373 (visited on 05/08/2024).

[12] Anton Fonarev. "Full Exceptional Collections on Lagrangian Grassmannians". In: *International Mathematics Research Notices* 2022.2 (June 2020), 1081–1122. ISSN: 1687-0247. DOI: 10.1093/imrn/rnaa098. URL: http://dx.doi.org/10.1093/imrn/rnaa098.

[13] Anton Fonarev. "Minimal Lefschetz decompositions of the derived categories for Grassmannians". In: *Izv. Ross. Akad. Nauk Ser. Mat.* 77.5 (2013), pp. 203–224. ISSN: 1607-0046,2587-5906. DOI: 10.1070/im2013v077n05abeh002669. URL: https://doi.org/10.1070/im2013v077n05abeh002669.

[14] Lyalya A. Guseva. "On the derived category of IGr(3,8)". In: *Sbornik: Mathematics* 211.7 (July 2020), p. 922. DOI: 10.1070/SM9292. URL: https://dx.doi.org/10.1070/SM9292.

[15] Brian Hall. *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*. Springer International Publishing, 2015. ISBN: 9783319134673. DOI: 10.1007/978-3-319-13467-3. URL: http://dx.doi.org/10.1007/978-3-319-13467-3.

[16] Andreas Hochenegger. "Exzeptionelle Folgen in der torischen Geometrie". Dissertation. 2011. URL: http://dx.doi.org/10.17169/refubium-5723.

[17] Jens Carsten Jantzen. *Representations of algebraic groups*. eng. 2. ed. Mathematical surveys and monographs. Providence, RI: American Mathematical Society, 2003, XIII, 576 S. ISBN: 0-8218-3527-0 and 978-0-8218-4377-2 and 978-0-8218-3527-2.

[18] Michail Michailowitsch Kapranow. "On the derived categories of coherent sheaves on some homogeneous spaces". In: *Inventiones mathematicae* 92.3 (1988), pp. 479–508. URL: http://eudml.org/doc/143579.

[19] Alexander Kuznetsov. "Calabi–Yau and fractional Calabi–Yau categories". In: *Journal für die reine und angewandte Mathematik (Crelles Journal)* 2019.753 (Mar. 2017), 239–267. ISSN: 0075-4102. DOI: 10.1515/crelle-2017-0004. URL: http://dx.doi.org/10.1515/crelle-2017-0004.

[20] Alexander Kuznetsov. "Exceptional collections for Grassmannians of isotropic lines". In: *Proceedings of the London Mathematical Society* 97.1 (Mar. 2008), 155–182. ISSN: 0024-6115. DOI: 10.1112/plms/pdm056. URL: http://dx.doi.org/10.1112/plms/pdm056.

[21] Alexander Kuznetsov. "Homological projective duality". In: *Publications mathématiques* 105 (2005), pp. 157–220. URL: https://api.semanticscholar.org/CorpusID:645981.

[22] Alexander Kuznetsov. "Hyperplane sections and derived categories". In: *Izvestiya: Mathematics* 70.3 (June 2006), p. 447. DOI: 10.1070/IM2006v070n03ABEH002318. URL: https://dx.doi.org/10.1070/IM2006v070n03ABEH002318.

[23] Alexander Kuznetsov. *Semiorthogonal decompositions in algebraic geometry*. 2015. arXiv: 1404.3143 [math.AG]. URL: https://arxiv.org/abs/1404.3143.

[24] Alexander Kuznetsov and Alexander Perry. *Serre functors and dimensions of residual categories*. 2021. arXiv: 2109.02026 [math.AG]. URL: https://arxiv.org/abs/2109.02026.

[25] Alexander Kuznetsov and Alexander Polishchuk. "Exceptional collections on isotropic Grassmannians". In: *Journal of the European Mathematical Society* 18 (2011), pp. 507–574. URL: https://api.semanticscholar.org/CorpusID:118667572.

[26] Alexander Kuznetsov and Maxim Smirnov. "Residual categories for (co)adjoint Grassmannians in classical types". In: *Compositio Mathematica* 157.6 (May 2021), 1172–1206. ISSN: 1570-5846. DOI: 10.1112/s0010437x21007090. URL: http://dx.doi.org/10.1112/S0010437X21007090.

[27] Riccardo Moschetti and Marco Rampazzo. "Fullness of the Kuznetsov–Polishchuk Exceptional Collection for the Spinor Tenfold". In: *Algebras and Representation Theory* 27 (2024), pp. 1063–1081. DOI: 10.1007/s10468-023-10246-6. URL: https://doi.org/10.1007/s10468-023-10246-6.

[28] Aleksandr Novikov. *Lefschetz Exceptional Collections on Isotropic Grassmannians*. 2020.

[29] Dimitri Orlov. "Projective bundles, monoidal transformations, and derived categories of coherent sheaves". In: *Izvestiya: Mathematics* 41.1 (Feb. 1993), p. 133. DOI: 10.1070/IM1993v041n01ABEH002182. URL: https://dx.doi.org/10.1070/IM1993v041n01ABEH002182.

[30] Maxim Smirnov. "On the derived category of the adjoint Grassmannian of type F". 2023. arXiv: 2107.07814 [math.AG].