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RESEARCH

# From Anomalous Dissipation Through Euler Singularities to Stabilized Finite Element Methods for Turbulent Flows

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## Abstract

It is well-known that kinetic energy produced artificially by an inadequate numerical discretization of nonlinear transport terms may lead to a blow-up of the numerical solution in simulations of fluid dynamical problems such as incompressible turbulent flows. However, the community seems to be divided whether this problem should be resolved by the use of discretely energy-preserving or dissipative discretization schemes. The rationale for discretely energy-preserving schemes is often based on the expectation of exact conservation of kinetic energy in the inviscid limit, which mathematically relies on the assumption of sufficient regularity of the solution. There is the (contradictory) phenomenological observation in turbulence that flows dissipate energy in the limit of vanishing molecular viscosity, an "anomalous" phenomenon termed dissipation anomaly or the zeroth law of turbulence. As already conjectured by Onsager, the Euler equations may dissipate kinetic energy through the formation of singularities of the velocity field. With the proof of Onsager's conjecture in recent years, a consequence for designing numerical methods for turbulent flows is that the smoothness assumption behind conservation of energy in the inviscid limit becomes indeed critical for turbulent flows. The velocity field rather has to be expected to show singular behavior towards the inviscid limit, supporting the dissipation of kinetic energy. Our main argument is that designing numerical methods against the background of this physical behavior is a strong rationale for the construction of dissipative (or dissipation-aware) numerical schemes for convective terms. From that perspective, numerical dissipation does not appear artificial, but as an important ingredient to overcome problems introduced by energy-conserving numerical methods such as the inability to represent anomalous dissipation as well as the accumulation of energy in small scales, which is known as thermalization. This work discusses stabilized $H^1$, $L^2$, and $H(\text{div})$-conforming finite element methods for incompressible flows with a focus on the energy-stability of the numerical method and its dissipation mechanisms to predict inertial dissipation. Finally, we discuss the achievable convergence rate for the kinetic energy in under-resolved turbulent flow simulations.

**Keywords** Anomalous energy dissipation · Finite-time Euler singularities · Large-eddy simulation · Navier–Stokes equations · Onsager's conjecture · Turbulence

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## 1 Motivation

In recent years, significant progress has been made from a mathematical perspective regarding the understanding of the phenomenon of anomalous energy dissipation, i.e. a non-vanishing dissipation rate in the limit $\nu \rightarrow 0$, and the occurrence of finite-time Euler singularities. The recent mathematical proof (Isett 2018) of Onsager's conjecture — providing a link between the occurrence of anomalous dissipation and singular behavior in terms of irregularities of the velocity field — is an important milestone in this context. It appears as if these topics are not yet sufficiently reflected in the numerically oriented literature on simulating turbulent flows and large-eddy simulation (LES).

Mathematical results suggest that the Euler equations are not per se energy-conserving (and time-reservible), but that the solution regularity is key to the question regarding the occurrence of kinetic energy dissipation in the inviscid limit. As an important message for researchers developing numerical discretization schemes, assumptions about the smoothness/regularity of the solution do not seem to be academic in the context of turbulence flows. Instead, turbulent flows can be expected to exhibit a regularity that is critical w.r.t. the occurrence of anomalous dissipation, involving singular behavior. The CFD community is familiar with singular behavior in the form of shocks in hypersonic flows (compressible regime). However, also incompressible flows in the inviscid limit (Euler equations) can be expected to show singularities. We denote such singular behavior as *turbulent singularities* in the following, in order to distinguish them from *shock singularities* for compressible flows. While shock singularities are characterized by (strong) singularities with discontinuities/jumps in certain quantities, turbulent singularities can be considered more *weak* in comparison, i.e. they have a higher degree of regularity or smoothness as compared to shock singularities, but *strong* enough to cause dissipation. According to the understanding stemming from Onsager's conjecture, turbulent singularities are characterized by a continuous velocity field with infinite gradient.

Implicit large-eddy simulation is well-established as a numerical technique to simulate turbulent flows in a numerically under-resolved setting. The term *implicit* describes that the numerical method itself accounts for the numerical dissipation present in turbulence, instead of adding physically motivated turbulence models (acting as energy sink) to the scheme. Numerical methods such as the MILES approach in the context of finite volume methods (Margolin and Rider 2002; Margolin et al. 2006) and stabilized methods or variational multiscale methods in the context of finite element methods (Bazilevs et al. 2007; Ahmed et al. 2017; Rasthofer and Gravemeier 2018) are well-known. A main motivation for the present work is to link implicit LES techniques with the topic of anomalous dissipation, singular behavior and the new mathematical insight on Onsager's conjecture. Important and remarkably early contributions in this otherwise rarely considered field are by Hoffman and Johnson (2008, 2010), a perspective we will discuss in more detail in this work.

Certainly, the phenomenon of anomalous energy dissipation has been internalized by the LES community in the sense that there is a need for representing/modeling dissipation in LES simulations. At the same time, the LES community seems to be divided regarding the question whether a numerical method should be able to represent backscatter, or whether backscatter is essentially a detrimental property of a numerical scheme due to its potentially de-stabilizing behavior. In terms of singular behavior, there still exist mysterious imaginations of finite-time singularities. In particular the link between dissipation and singular behavior appears to be blurred. It is still a widespread assumption that solutions to the Euler equations must preserve kinetic energy and will remain smooth. The imagination of finite-time singularities as an event (often termed blow-up) with catastrophic consequences for a numerical method, implying its unavoidable breakdown once such a singularity occurs, is not un-common, see e.g. Winters et al. (2018), Lee and Lee (2023). It seems as if the picture drawn currently in the literature contains further contradictions. To give one concrete example, the work by Hughes et al. (2000) on the variational multiscale method for LES expects a theoretical energy-conserving behavior for the convective term $(-\nabla \boldsymbol{u}, \boldsymbol{u} \otimes \boldsymbol{u})_\Omega$, while the need for stabilization terms is considered a peculiarity of continuous Galerkin discretizations rather than a consequence of the physical behavior related to the inertial dynamics of turbulent flows. In the literature on variational multiscale methods (Bazilevs et al. 2007; Ahmed et al. 2017; Rasthofer and Gravemeier 2018), the metric of energy stability and dissipation seems to be under-represented in the discussion of implicit LES methods as compared to multiscale aspects. Often, discretely energy-conserving numerical methods are designed without awareness of the phenomenon of anomalous energy dissipation (Charnyi et al. 2017; Coppola et al. 2019; Zhang et al. 2022). The aim of the present paper is to shed light on these contradictions and add another perspective to the discussion, with a main focus on the metric of energy-stability (versus energy-conservation).

The present work discusses the design of numerical methods for simulation of incompressible turbulent flows against the background of the novel mathematical insight described above. Of interest are conforming and non-conforming finite element methods for large-eddy simulation. The outline of this article is as follows. As a basis for subsequent discussions, we summarize the relevant theoretical background in Sect. 2. Section 3 discusses the numerical technique of large-eddy simulation as well as different philosophies and metrics employed within this wide field of research. Section 4 forms the core of the present work. We summarize and compare $H^1$, $L^2$, and $H(\text{div})$-conforming methods in terms of their stabilization mechanisms and their impact on consistency, mass conservation, and energy stability. Section 5 shows numerical results for the three-dimensional Taylor–Green problem at infinite Reynolds number, investigating the kinetic energy evolution and dissipation under mesh refinement. Section 6 uses a simple scaling argument with the goal to explain convergence rates in the kinetic energy observed in practical numerical simulations of turbulent flows. We close this article with a discussion and open questions in Sect. 7.

## 2 Theoretical Background

Hyperbolic balance laws in fluid dynamics have the fundamental characteristic that steep gradients may develop from smooth initial data. An important manifestation is the formation of small scale structures according to the dynamics of high-Re or infinite-Re turbulence, typically described by the mathematical model of the incompressible Navier–Stokes/Euler equations

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} + \nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{u}) - \nu \nabla^2 \boldsymbol{u} + \nabla p = \boldsymbol{0} \, , \\ \nabla \cdot \boldsymbol{u} = 0 \, , \end{aligned} \tag{1}$$

with $\nu = 0$ for the Euler equations. For a physical understanding of turbulent flows and, in particular, a characterization of flow structures and energy-transfer mechanisms in terms of the velocity gradient tensor, we refer to two recent contributions by Hoffman (2021), Johnson and Wilczek (2024) and references therein. By analyzing velocity gradient statistics and dynamics, the review article by Johnson and Wilczek (2024) argues there is strong evidence that the energy cascade from large to small scales is not driven by viscous dissipation but by inertial dynamics (strain-rate self-amplification and vortex stretching). State-of-the-art pictures of the turbulent energy cascade involve molecular dissipation as the final energy transfer mechanism during the energy cascade from large to small scales, acting as an energy sink for the above mathematical model of incompressible flow. Based on the observation that fluids typically involved in technical systems exhibit a finite, non-vanishing viscosity, the Navier–Stokes equations with viscous term might be considered more relevant than the limit case of the Euler equations from a physical point of view.

From a numerical point of view, i.e. when computing numerical approximations to the Navier–Stokes equations at high Reynolds number, however, the situation is different. The circumstance that irreversible effects such as dissipation of kinetic energy are expected to happen at length scales much smaller than what can be resolved by a numerical scheme renders the robust and accurate numerical solution of such problems an outstanding challenge. For a finite spatial resolution at very high Reynolds number, molecular dissipation through the viscous term is impossible because the small scales relevant for molecular dissipation are not resolvable.[1] Hence, the Euler equations are actually the mathematical model seen by the numerical discretization scheme. Due to finite resolution in numerical simulations, the computed solution can be thought of as a filtered or spatially coarse-grained velocity field, for which the viscous term and the associated molecular dissipation vanishes for $\nu \to 0$. Mathematical arguments for this reasoning are given in (Eyink 2024, Equations 3.4 and 3.5). The dynamics of turbulent flows with energy transfer to small scales driven by inertial dynamics on a macroscopic scale lets one expect that a numerical approach resolving only large structures down to a numerical cutoff length scale is a feasible approach to accurately compute quantities of engineering interest in turbulent flows.

## 2.1 Anomalous Energy Dissipation

For incompressible flows, the kinetic energy is given as

$$E = \frac{1}{2} \int_{\Omega} \boldsymbol{u} \cdot \boldsymbol{u} \, \mathrm{d}\Omega \, . \tag{2}$$

Regarding the rate of dissipation of kinetic energy in turbulent flows (considering e.g. homogeneous isotropic turbulence in a periodic box), there is widespread consensus to the theory that the dissipation rate does not tend to zero in the limit $\nu \to 0$, but takes a positive value with a finite amount of dissipation that is independent of $\nu$. This phenomenon was first described by Taylor (1935), a phenomenon also called inviscid/inertial dissipation or the zeroth law of turbulence. Also Kolmogorov's 1941 (K41) theory is based on the assumption of a non-vanishing dissipation rate in the inviscid limit that is independent of viscosity ($\epsilon = \mathcal{O}(U^3/L)$). Numerical results in Sreenivasan (1998), Kaneda et al. (2003) and experimental results in Pearson et al. (2002) indicate evidence of anomalous

[1] A statement we consider valid as long as the compute power available is not sufficient to conduct direct numerical simulations for challenging problems.

dissipation within numerical/experimental capabilities. According to recent review articles, anomalous dissipation appears to be well-accepted nowadays as an essential building block of turbulence theories (Eyink 2008; Dubrulle 2019).

### 2.2 Onsager's Conjecture–A Link Between Dissipation and Singular Behavior

Divergence terms in the differential form of a PDE, such as the nonlinear convective term in the incompressible Euler equations, suggest a physical conservation property. Integrating over the domain and applying Gauss' divergence theorem indicate that only boundary faces contribute to the evolution of the (kinetic) energy and that energy has to be conserved in case of periodic boundary conditions (p.b.c.),

$$\int_\Omega \nabla \cdot \boldsymbol{f}(\boldsymbol{u}) \mathrm{d}\Omega = \int_{\partial\Omega} \boldsymbol{f}(\boldsymbol{u}) \cdot \boldsymbol{n} \, \mathrm{d}\Gamma \overset{\text{p.b.c.}}{=} 0 \, . \tag{3}$$

This is precisely the argument used e.g. in (Guermond et al. 2004, Equation 2.3), where the authors write that the nonlinear term does not contribute to the global kinetic energy balance due to

$$\int_\Omega (\boldsymbol{u} \cdot \nabla \boldsymbol{u}) \cdot \boldsymbol{u} \, \mathrm{d}\Omega = \int_\Omega \boldsymbol{u} \cdot \nabla \left( \frac{1}{2} \boldsymbol{u}^2 \right) \mathrm{d}\Omega = \int_\Omega \nabla \cdot \left( \frac{1}{2} \boldsymbol{u}^2 \boldsymbol{u} \right) \mathrm{d}\Omega = 0 \, . \tag{4}$$

This relation relies on the essential assumption of *differentiable solutions*, which is often not sufficiently highlighted in the literature in this context.[2] Following the smoothness assumption nevertheless, incompressible viscous flows on periodic domains then obey the classical dissipation equation

$$\frac{\partial E(t, \nu)}{\partial t} = - \int_\Omega \nu \nabla \boldsymbol{u}^\nu : \nabla \boldsymbol{u}^\nu \, \mathrm{d}\Omega \, . \tag{5}$$

Equations (4) and (5) are often (mis-)interpreted as stating conservation of energy in the absence of viscosity, $\partial E(t, \nu = 0)/\partial t = 0$. For example, Duponcheel et al. (2008) write "Since there is no dissipation term, the energy should be conserved." and "The three-dimensional incompressible Euler equations are time-reversible.". The review article by Coppola et al. (2019) summarizes in the abstract that "The invariant character of quadratic quantities such as global kinetic energy in inviscid incompressible flows is a particular symmetry [...].". When it comes to the development of numerical methods for turbulent flows, another typical mis-interpretation of Eq. (4) is that the numerical discretization of the nonlinear convective term has to be discretely energy-preserving, an aspect discussed in detail in the present work. These mis-interpretations originate from the fact that the numerical PDE community has not been aware of the dissipation anomaly until recently, as explained e.g. in Eyink (2024). In the following, we want to briefly explain how the phenomenon of the dissipation anomaly is related to the regularity of the velocity field.

Onsager (1949) pointed out that conservation of energy in the inviscid limit is an invalid conclusion as the proof of the conservation of energy (according to Eq. 4) relies on a differentiable velocity field. To make this point more clear, we assume for now that the

[2] The attentive reader might be reminded of the words by G. C. Lichtenberg, "The most dangerous of all falsehoods is a slightly distorted truth.". In the following, we comment in more detail on typical mis-interpretations of Eq. (4).

regularity assumptions behind Eq. (4) are fulfilled and consider the limit $\nu \to 0$ in Eq. (5). Anomalous dissipation describes a non-vanishing dissipation rate in the inviscid limit,

$$\lim_{\nu\to 0} \frac{\partial E(t,\nu)}{\partial t} = -\lim_{\nu\to 0} \nu \int_\Omega \nabla \boldsymbol{u}^\nu : \nabla \boldsymbol{u}^\nu \, \mathrm{d}\Omega = -D(t) < 0 \,. \tag{6}$$

Note that a positive dissipation rate $D(t) > 0$ would imply that the enstrophy $\|\nabla \boldsymbol{u}^\nu\|_\Omega^2$ tends to infinity in the limit $\nu \to 0$. Hence, bringing the above naive result on the kinetic energy balance in agreement with the phenomenon of anomalous energy dissipation requires a velocity field that is non-smooth. This should raise concerns regarding the assumptions underlying the naive kinetic energy balance. While this simple derivation might serve as a descriptive illustration of Onsager's reasoning that energy dissipation in three-dimensional incompressible flows can take place for $\nu = 0$ by the formation of singularities of the velocity, we want to refer to important works by Duchon and Robert (2000) and Eyink (2024) for a comprehensive reasoning. By describing the spatial regularity of the velocity in terms of Hölder continuity[3], Onsager postulated an important result that formulates precisely the relation between the occurrence of anomalous dissipation and the spatial regularity of the velocity. This is today known as *Onsager's conjecture* (Eyink and Sreenivasan 2006)

- Energy is conserved if the velocity is Hölder continuous with Hölder exponent $> 1/3$
- Energy may be dissipated if the velocity is Hölder continuous with Hölder exponent $< 1/3$

To date, Onsager's conjecture has been proven by a sequence of mathematical contributions, see e.g. Eyink (1994), Constantin and Titi (1994), De Lellis and Székelyhidi (2014), Buckmaster et al. (2018), Isett (2018). It has first been shown that Hölder continuity with exponent $> 1/3$ implies conservation of energy. Later, dissipative weak Euler solutions have been constructed up to Onsager's critical regularity of 1/3 using convex-integration techniques. The critical case 1/3 appears to be still open, where we want to refer to the recent result in De Rosa and Inversi (2024). The critical Hölder exponent 1/3 separates the range $0 < \alpha < 1$ of singular behavior into *dissipative singularities* and *non-dissipative singularities*. Let us summarize here that today's understanding of turbulence clearly includes the occurrence of singularities (Dubrulle 2019; Eyink 2024).

## 3 Large-Eddy Simulation

The methodology dealing with the numerical simulation of high-Reynolds number flows in an under-resolved setting, including the inviscid limit, is commonly known as large-eddy simulation (LES). In our understanding, developing LES methods describes the art of finding a dissipation mechanism that is (i) strong enough to ensure numerical robustness and to produce physically-relevant *dissipative solutions* (see below) for high Reynolds numbers and (ii) weak enough to preserve high accuracy for smooth regions along

[3] A function $f : U \subset \mathbb{R} \to \mathbb{R}$ is Hölder continuous with exponent $\alpha$ if there exists $C > 0 \in \mathbb{R}$ such that it holds

$$|f(x) - f(y)| \le C|x-y|^\alpha \text{ for all } x, y \in U.$$

with a minimal dependency of numerical results on turbulence model parameters. The review article by Guermond et al. (2004) takes a mathematical perspective on LES and emphasizes the important results by Duchon and Robert (2000) on dissipative solutions. The authors conclude that LES should select such “dissipative solutions” or “suitable weak solutions” (i.e. weak solutions satisfying an energy balance inequality locally in a distributional sense) to obtain physically relevant solutions.

### 3.1 A Summary of Review Articles on LES

We surveyed review articles on large-eddy simulation with a particular focus on the aspect of energy conservation/dissipation of the numerical scheme. Different philosophies can be found in the literature:

- A group of review articles does not discuss the topic of energy-stability of discretization schemes explicitly or identify this as a central topic in LES, see Lesieur and Metais (1996), Piomelli (1999), Ferziger (2000), Fureby (2008), Piomelli (2014), Georgiadis et al. (2010). For example, Ferziger (2000) writes “Almost any method used in computational fluid dynamics can be applied to LES.”. The early review article by Piomelli (1999) mentions briefly the development of high-order energy-conserving schemes as future goal.
- Another group advocates kinetic-energy-conserving discretization schemes, see Moin (2002), Zhiyin (2015). Discrete energy-conservation is typically considered both robust and accurate, while upwind fluxes are considered inappropriate for LES.[4] Despite numerical robustness (i.e. no nonlinear blowup), we note that such a numerical method will not find *dissipative solutions* unless further/additional measures (such as sub-grid turbulence models) are taken.
- A group of articles explicitly addressing or highlighting the need for stable discretization schemes, see e.g. Drikakis et al. (2009). These articles can typically be associated to the paradigm of implicit large-eddy simulation, see also Margolin and Rider (2002), Margolin et al. (2006). Stabilized finite element methods (Hoffman and Johnson 2007) and related methods interpreted in the context of variational multiscale methods (Bazilevs et al. 2007; Principe et al. 2010; Codina et al. 2011) (see also the review articles by Gravemeier (2006), Ahmed et al. (2017), Rasthofer and Gravemeier (2018) on variational multiscale methods for incompressible turbulent flows) can also be associated to this category.

Interestingly, the early review article by Lesieur and Metais (1996) puts the occurrence of enstrophy blow-up/singularities and inertial dissipation into a context. However, the trust in numerical discretization techniques reliably predicting turbulent dissipation was very

[4] A typical prejudice in the LES community is that upwinding is too dissipative. Research on high-order methods in recent years has revealed that the order of numerical discretization schemes (high-order methods) strongly influence the amount of dissipation and the suitability of upwind fluxes. Hence, while upwinding is sometimes considered inappropriate for LES due to excessive dissipation (in particular for low-order discretizations), upwinding as the only stabilization mechanism in a numerical scheme might also not be *strong* or *dissipative* enough to ensure stability (in particular for high-order discretizations). This aspect appears to be blurred in the review articles on LES cited above; some indications can be found e.g. in Georgiadis et al. (2010).

limited (Lesieur and Metais 1996, Section 9). The review article by Bouffanais (2010) points to the importance of the interplay between LES modeling and numerical discretization schemes, but no connection to the need for dissipative discretization schemes is made in that work.

The developments in large-eddy simulation described above are also reflected in the ambivalence of how to judge the capability of LES techniques to represent backscatter, i.e., the transfer of energy from small to large scales. While some works strengthen the ability of a numerical method to represent this physical mechanism, a numerical failure of such schemes is often reported due to kinetic energy blow-up induced by negative dissipation. As a consequence, additional techniques such as averaging of certain quantities in homogeneous directions are typically required (Lesieur and Metais 1996). The article by Mason (1994) argues that a negative eddy viscosity (systematically enhancing resolved gradients) is not an adequate model for the physical process of backscatter. Guermond (2008) argues strongly that letting energy come back from under-resolved scales is not legitimate. Rasthofer and Gravemeier (2013) see one of the strengths of their multifractal sub-grid scale model in the ability to represent backscatter. Codina et al. (2011) argue similarly for the orthogonal subscales model with dynamic subscales. The recent review article by Johnson and Wilczek (2024) argues “that the incorporation of backscatter is not a necessary feature of accurate LES models.”.

### 3.2 Physical (Explicit) Versus Numerical (Implicit) Sub-Grid Modelling

The categorization of review articles on LES made above reflects an understanding gained over decades in our opinion, namely that the mathematical properties of numerical discretization schemes are of outmost importance for robust and reliable LES simulations, while the impact of turbulence modeling and the attempt to address the problem by even better sub-grid models has been realized to be limited more severely than expected in the early days of LES. We summarize important properties that an LES method should fulfill:

- Ensure robustness of the numerical simulation, i.e. avoid non-linear blow-up that causes a numerical simulation to terminate,
- Include mechanisms of dissipation in order for the LES method to be able to find *dissipative solutions*, and
- Realize a dissipation mechanism that does not dominate numerical approximation errors for problems (or sub-regions within a problem) with smooth solution.

Regarding a realization of these properties in the design of a numerical solver for turbulent flows, the literature distinguishes between two main approaches, namely physical (or explicit) LES versus numerical (or implicit) LES, even though a precise assignment to one of these groups might not always be possible, see e.g. Guermond et al. (2011), Dairay et al. (2017). Sticking to this binary distinction nevertheless, the LES community appears to be undecided whether one approach can generally be considered more appropriate than the other one.

**Physical (explicit) LES** On the one hand, it is a widespread concept that a computational method for LES should separate the numerical discretization method from the turbulence model accounting for the dissipation of energy. The typical procedure is to follow the design goal of constructing discretely energy-conserving schemes regarding the development of numerical methods, and to consider a proper treatment of turbulence as a topic to be addressed subsequently/separately. This approach tries to separate the two aspects of robustness and dissipation mentioned above. Representatives of this concept in recent years are so-called skew-symmetric (Coppola et al. 2019), EMAC (energy, momentum, angular momentum conserving) (Charnyi et al. 2017), etc. formulations as numerical discretization schemes along with a nonlinear viscosity model such as a (dynamic) Smagorinsky model (or more advanced techniques) as turbulence model. In the context of finite element methods discussed in the present work, explicit sub-grid models e.g. in the form of a (dynamic) Smagorinsky model are used in Hughes et al. (2000), Röhe and Lube (2010), Gravemeier et al. (2010), Lehmkuhl et al. (2019) in an $H^1$-conforming setting, and e.g. in Marek et al. (2015), Ferrer (2012) in an $L^2$-conforming setting. We further refer to Guermond et al. (2004), Guermond (2008) for a mathematical perspective on explicit LES.

**Numerical (implicit) LES** On the other hand, concerns are shared whether such a separation of disciplines is appropriate regarding the overall goal of simulating turbulent flows in a robust, accurate, and computationally efficient manner. Dissipation may be represented directly by the numerical scheme according to the concept of implicit large-eddy simulation. Boris et al. (1992) stated in a pioneering work on implicit large-eddy simulation: "I do not believe it is practical to separate the formulation of the LES problem from the numerical method used for its solution.". Research on implicit LES techniques has demonstrated that numerical schemes—due to their self-adaptive character (Drikakis 2003)—are able to correctly predict the physical dissipation rate. As an important result in this context, the work by Margolin et al. (2006) has shown that implicit LES results fulfill Kolmogorov's 4/5 law,[5] which describes that the energy dissipation rate is independent of the viscosity and is driven by the large scales of the flow. This might be seen as a demonstration that the numerical dissipation in implicit LES is not artificial or unphysical but controlled by resolved scales. In the context of finite element methods, the works by Hoffman et al. (2011), Fehn et al. (2022) indicate that inertial dissipation with grid-independent dissipation rates may be predicted by suitably stabilized (dis-)continuous Galerkin discretizations of the incompressible Euler equations. These results might be interpreted positively in the sense that they suggest there are *physics-compatible* numerical methods that are able to represent the phenomenon of anomalous dissipation (and other turbulence phenomena) without a physically motivated turbulence model. A detailed overview of finite element methods for incompressible flows that can be associated to the paradigm of implicit large-eddy simulation is given in Sect. 4.

### 3.3 On Energy-Conserving Numerics and the Trust in Numerical Dissipation

There appears to be limited trust regarding the physical correctness or the predictive character of implicit LES with inbuilt numerical dissipation. We see the reasons for this mainly twofold:

Firstly, inappropriate mathematical assumptions such as Eq. (4) formulate misleading design goals. Without further critical assessment, it seems to be natural to expect exact conservation of energy for the Euler equations. Several works on numerical discretization techniques for incompressible Euler flows aim at conservation of energy, see e.g. Charnyi et al. (2017), Coppola et al. (2019), Zhang et al. (2022). Even benchmark problems (Duponcheel

[5] For deviations of real turbulence from K41 theory, we refer e.g. to Küchler et al. (2023).

et al. 2008) have been defined that aim at benchmarking numerical methods against the design goal of exact energy conservation. One may justify this perspective in the sense that grid-converged results for inviscid three-dimensional flows with dissipative dynamics are typically not available, so the best one can do is benchmarking against energy conservation. The consequence is that (i) one is benchmarking against a (potentially) physically inconsistent result in conflict with the phenomenon of anomalous dissipation, and/or—even worse—that (ii) dissipative numerics are considered inaccurate or physically wrong.

Secondly, there seems be the expectation in parts of the literature that the numerical solution in space (and time) produced by energy-conserving discretization schemes for turbulent flows would be qualitatively accurate apart from the missing dissipation. Instead, such schemes suffer from a problem more severe than just under-estimating turbulent dissipation when applied to dissipative dynamics: they inherently lead to oscillatory solutions, a phenomenon also known as *thermalization*. The circumstance that energy-conserving numerical methods applied to hyperbolic problems are in itself plagued by small-scale noise has already been formulated by Leonard (1975): "Considerable 'damming up' of the turbulent energy in the large scales would occur, for example, if the unmodified equations were used with an energy-conserving finite-difference scheme on the advective term.". The cause are small structures close to the resolution limit of a discretization scheme, which exhibit large dispersion errors that cannot be dissipated by the numerical scheme as no energy can leave the system. These oscillations do not remain localized, but spread over an increasingly large part of the domain. This implication of discretely energy-conserving schemes is well understood for pseudo-spectral solvers of the Galerkin-truncated Euler equations, preserving kinetic energy in the discrete case, see Ray et al. (2011), Murugan et al. (2020) for the one-dimensional Burgers' equation with formation of shocks and (Cichowlas et al. 2005; Murugan and Ray 2022; Kolluru et al. 2022) for the three-dimensional Euler equations. Instead, one may consider it an important task of numerical methods for problems with dissipative hyperbolic dynamics to avoid such pollution effects and keep the numerical solution clean of small scale noise. However, this will be accompanied by (numerical) dissipation. There seems to be an inherent conflict to construct numerical methods for dissipative hyperbolic dynamics that are energy-conserving and at the same time free of thermalization. In our opinion, the one-dimensional Burgers' equation with formation of shocks appears to be most illustrative to convince oneself regarding these conflicting goals of both energy-conserving (or dissipation-free) and thermalization-free numerics (Fehn et al. 2022; Murugan et al. 2020). The charm of the one-dimensional Burgers equation over three-dimensional turbulence regarding such considerations is that dissipative solutions are—supported by the existence of analytical solutions for this 1D problem–generally accepted.

### 3.4 On the Notion of Physics-Compatible or Structure-Preserving Numerical Methods

The present contribution emerged from a conference talk by Fehn and Kronbichler (2023) held at ETMM14 in a minisymposium on "Physics-compatible numerical methods for fluid flow". In this section, we want to briefly comment on the notions of *physics-compatible* or *structure-preserving* numerical methods and how these terms are used (at the time of writing) by the scientific community in our experience. In a significant number of cases, we observed that these terms are used synonymously to discretely *energy/entropy-preserving* numerical schemes (typically realized e.g. by so-called skew-symmetric formulations). In the conference talk, the first author of the present work mentioned that seven talks in the minisymposium used the wording *energy/entropy-conserving* or *non-dissipative* in their title, while only two talks used the wording *energy/entropy-stable* or *dissipative* in their title. We shall therefore define what we mean by physics-compatible or structure-preserving methods for three-dimensional turbulence. The present work expects physics-compatible or structure-preserving numerical methods for incompressible turbulent flows to conserve mass and to fulfill an energy dissipation inequality (rather than a strict energy conservation property). We believe that inbuilt numerical dissipation mechanisms are desirable for *compatibility* with irreversible or dissipative hyperbolic dynamics. In contrast, a scheme fulfilling a discrete energy conservation property is per se not *compatible* with the physics of inertial energy dissipation. To strengthen the aspect of irreversibility or dissipation, we think a numerical method fulfilling an energy dissipation inequality (with the possibility of dissipation) is described best by the notion *physics-compatible* (or, alternatively, *dissipation-aware* as used in Fehn and Kronbichler 2023) rather than *structure-preserving*.

### 3.5 On Identifying Singularities by Energy Dissipation Arguments

The goal of identifying finite-time singularities in numerical simulations of Euler flows has a long history. The main categories of methods for singularity detection applied in numerical studies are the vorticity blow-up criterion according to the theorem by Beale et al. (1984) and the analyticity strip method by Sulem et al. (1983). The work by Luo and Hou (2014) reported numerical evidence of a finite-time singularity, results supported very recently by mathematical proofs in Chen and Hou (2022). To not make the present contribution exhaustive, we want to refer to the study by Fehn et al. (2022) and references therein for a more detailed overview of techniques and numerical studies. Here, we want to focus in particular on techniques exploiting the connection between singular behavior and anomalous dissipation according to Onsager's conjecture. First ideas to exploit this connection for blowup detection in numerical Euler simulations were formulated in Hoffman and Johnson (2008) termed *global blowup* criterion. In a similar direction, an *indirect approach* for the detection of dissipative singularities was proposed by Fehn et al. (2022), which exploits the connection between anomalous dissipation and singularities according to Onsager's conjecture. By the weak–strong uniqueness property (Wiedemann 2017), indirect evidence of a dissipative singularity is given by demonstrating convergence to a dissipative Euler solution according to this technique.

We want to emphasize that numerical blow-up of a non-robust numerical scheme is not indicative of physical blow-up. Instead, we consider numerical robustness a prerequisite for physical blow-up detection, i.e. numerical methods for hyperbolic terms describing inviscid flows (as well as under-resolved high-Re flows) should be able to deal with singular behavior without numerical blow-up. In our opinion, there is also the following mathematical argument supporting singularity detection by the indirect approach based on energy arguments (compared to direct blow-up criteria based on diverging velocity gradients). Numerical simulations necessarily operate in a filtered or coarse-grained regime with regularized velocity gradients. Now, considering *numerical convergence* as the tool of providing evidence for a certain hypothesis, it is principally feasible to obtain numerical convergence in the kinetic energy in a coarse-grained setting (due to the spectral distribution of energy in three-dimensional fluid dynamics where only the tail of the spectrum is affected by coarse-graining). In contrast, numerical convergence in maximum vorticity, a quantity affected by coarse-graining in an essential manner, is actually not feasible in a coarse-grained setting. This understanding is in line with the notion of *computability* of certain mean-value output of turbulent flows by weak Galerkin methods (Hoffman and Johnson 2006; Hoffman et al. 2011, 2015).

## 4 Stabilized Finite Element Methods for Incompressible Turbulent Flows

This section discusses finite element methods for incompressible turbulent flows. In the following, we take the perspective that the numerical method itself should be physics-compatible in terms of the capability to represent/predict inertial dissipation through inherent numerical dissipation mechanisms. Compared to physically motivated LES models, a main motivation for incorporating dissipation directly into the numerical discretization scheme (implicit LES) is to obtain a more accurate filter with the numerical resolution limit (e.g. the wavenumber $k_{1\%}$ defined in Moura et al. 2017) as close as possible to the Nyquist wavenumber $k_{hp,\mathrm{Ny}}$ (see Sect. 6 for a definition). The polynomial degree of the shape functions might be considered a parameter of the implicit LES approach. Motivated by results on dispersion-dissipation analysis, the work by Moura et al. (2017) suggests an improved resolution capability of high-order discontinuous Galerkin discretizations compared to low-order discretizations in a numerically under-resolved setting. (Fehn 2021, Figures 2.23, 2.24) confirmed these results for the viscous Taylor–Green vortex. These results foster a perspective where the polynomial degree of the method is chosen such that the implicit LES characteristics of the method are optimized. In comparison, mathematical results on regularization/filtering of the Navier–Stokes equations in the sense of explicit LES might indicate that the discretization length scale $h$ should be significantly smaller than the regularization/filtering length scale, see (Guermond and Prudhomme 2005, Section 4), such that the numerical resolution might be unnecessarily high for explicit LES. The general goal of a minimal dependency of flow results on turbulence model constants (or even the absence of turbulence model constants) serves as a further motivation to explore the technique of implicit LES.

### 4.1 On the Mechanism of Numerical Dissipation

In the above setting of implicit LES, numerical methods for turbulent flows should transport large structures accurately, but control and actively dissipate the smallest resolvable scales as a means to reflect the physics of turbulent fluid dynamics. We consider here (dis-)continuous Galerkin methods (potentially using a high-order polynomial basis on each element) with inbuilt mechanisms of numerical dissipation (typically called *stabilization* in variational methods). To qualify for implicit large-eddy simulation, small-scale structures close to the resolution limit, which cannot be transported accurately on a given grid, need to be damped by dissipative mechanisms. These dissipative mechanisms should only become active if a non-smooth numerical solution is detected locally. Residuals of the incompressible Navier–Stokes/Euler equations typically serve as indicators of non-smoothness. Suitable terms in the weak formulation that are based on these residuals, e.g. terms that are symmetric w.r.t. the test and solution functions and thus appear as quadratic terms in the energy dissipation equation, act as dissipation mechanisms. Having the picture in mind of a locally dissipative solution according to Duchon and Robert (2000), such a numerical approach should principally allow to find suitable dissipative weak solutions to the Euler equations. A numerical scheme with a finite-dimensional function space for the velocity, equipped with suitable dissipation mechanisms ensuring that the kinetic energy remains bounded, yields a solution with finite velocity gradients. Hence, the overall technique can be thought of as regularising the Euler equations without involving an explicit form of filtering/coarse-graining or physically motivated turbulence model. The chosen approach is thus in line with the philosophy of *implicit* turbulence modeling. According to the numerical studies by Hoffman et al. (2011) and Fehn et al. (2022), stabilized (dis-)continuous Galerkin methods might indeed be suitable candidates to predict the phenomenon of anomalous dissipation numerically and, thereby, qualify as physics-compatible numerical methods. Let us note that this discretization approach differs in its philosophy rather fundamentally from the goals persued e.g. in Charnyi et al. (2017), Zhang et al. (2022), where the authors aim at energy conservation on the discrete level.

### 4.2 Notation and Discrete Velocity/Pressure Spaces

We address the numerical solution of the incompressible Navier–Stokes/Euler equations on a domain $\Omega \in \mathbb{R}^d$, which is approximated by the computational domain $\Omega_h = \sum_{e=1}^{n} \Omega_e$ consisting of non-overlapping elements of hexahedral shape. In order to avoid technicalities related to the imposition of boundary conditions, we assume a periodic domain with $\partial\Omega_h = \emptyset$, i.e. there are only interior faces, where $\Gamma_h^{\text{int}}$ denotes the set of all interior faces. In addition to periodic boundary conditions, we assume a vanishing body force $\boldsymbol{f} = \boldsymbol{0}$ according to Eq. (1), in order to keep the formulation simple when studying aspects like energy stability. As usual, we denote by $(\cdot,\cdot)_{\Omega_e}$ and $(\cdot,\cdot)_{\partial\Omega_e}$ the $L^2$ inner product over an element $\Omega_e$ and its boundary $\partial\Omega_e$, respectively.

In the following, we denote the velocity function space by $\mathcal{V}_h^u$ and the pressure function space by $\mathcal{V}_h^p$. We denote by $L^2(\Omega)$ the Sobolev space of square-integrable functions and by $H^1(\Omega)$ the Sobolev space of square-integrable functions with square-integrable derivatives. In this article, the term finite element method shall denote velocity function spaces that are $H^1$-conforming (classical continuous Galerkin methods), $L^2$-conforming (so-called discontinuous Galerkin methods), and $H(\text{div})$-conforming (a hybrid version particularly relevant for incompressible flows allowing to construct discretization methods that lead to an exactly divergence-free velocity field). An $H(\text{div})$-conforming function space for the velocity is defined as

$$\mathcal{V}_h^{u,H(\text{div})} = \left\{ \boldsymbol{u}_h \in \left[L^2(\Omega_h)\right]^d \;:\; \nabla \cdot \boldsymbol{u}_h \in L^2(\Omega_h) \right\} \tag{7}$$

i.e. both the velocity $\boldsymbol{u}_h$ and its divergence $\nabla \cdot \boldsymbol{u}_h$ are in $L^2$. A practical implication of this rather abstract definition is that the velocity field is continuous in normal direction between elements, $\boldsymbol{u}_h^- \cdot \boldsymbol{n} = \boldsymbol{u}_h^+ \cdot \boldsymbol{n}$ or $\left[\boldsymbol{u}_h\right] \cdot \boldsymbol{n} = (\boldsymbol{u}_h^- - \boldsymbol{u}_h^+) \cdot \boldsymbol{n} = 0$. Due to the discontinuity of tangential velocity components in $H(\text{div})$-conforming methods, they need similar discretization concepts as the (fully) non-conforming discontinuous Galerkin methods based on the concept of numerical fluxes to connect the elemental sub-problems. From this perspective, $H(\text{div})$-conforming methods might be considered a sub-class of discontinuous Galerkin methods with a constrained velocity function space.

### 4.3 On the Interplay of Mass Conservation and Inf-Sup Stability

A fundamental aspect in the design of finite element methods for the incompressible Navier–Stokes/Euler equations is the choice of the velocity function space w.r.t. the pressure function space. The resulting formulation should be both inf-sup stable and mass-conserving, see the review article by John et al. (2017) for a comprehensive summary. The term inf-sup stability describes the phenomenon of spurious pressure modes in case the space of velocity test functions is not sufficiently rich to exclude such modes from the numerical solution. In the present work, we merely mention that for each class of velocity function space introduced above, the velocity function space shall be combined with a pressure function space such that an inf-sup stable scheme is obtained or, alternatively, the method shall be equipped with suitable pressure stabilization terms circumventing the inf-sup condition. For $H^1$ and $L^2$-conforming methods, inf-sup stability is typically achieved by selecting mixed-order polynomials of degree $p$ for the velocity and degree $p-1$ for the pressure.[6] However, the resulting scheme is then in general not exactly mass-conserving, because the pressure space is not rich enough to ensure an exact fulfillment of the mass conservation equation. For this reason, stabilization terms are typically added to the variational formulation for improved mass conservation, as detailed in the following. We will also discuss the role of $H(\mathrm{div})$-conforming methods as an optimal choice w.r.t. the fulfillment of both inf-sup stability and mass conservation. Finally, note that exactly mass-conserving methods are also known to exist for certain combinations of $H^1$-conforming velocity spaces with $L^2$-conforming pressure spaces, such as the Scott–Vogelius element for simplicial elements.

### 4.4 Motivation for Stabilized Methods

Straight-forward weak finite element formulations of the incompressible Navier–Stokes equations and in particular the nonlinear term do not lead to an energy-stable discretization scheme in an $H^1$-conforming setting, and similarly in an $L^2$-conforming setting with typical upwind or Lax-Friedrichs fluxes. This is due to the nonlinearity of the convective term and the fact that these function spaces typically do not lead to a pointwise or exactly divergence-free velocity field. Being of fundamental importance for the present work, we detail these steps in the following, where we will make use of the identity

$$\nabla \cdot \left(\boldsymbol{u}_h\left(\boldsymbol{u}_h \cdot \boldsymbol{u}_h\right)\right) = \nabla \cdot \boldsymbol{u}_h\left(\boldsymbol{u}_h \cdot \boldsymbol{u}_h\right) + 2\boldsymbol{u}_h \cdot \left(\left(\boldsymbol{u}_h \cdot \nabla\right)\boldsymbol{u}_h\right). \tag{8}$$

Considering a continuous Galerkin formulation with the nonlinear term written in convective form

$$c^e_{h,\mathrm{conv}}\left(\boldsymbol{v}_h, \boldsymbol{u}_h\right) = \left(\boldsymbol{v}_h, \left(\boldsymbol{u}_h \cdot \nabla\right)\boldsymbol{u}_h\right)_{\Omega_e}, \tag{9}$$

we obtain the contribution of this term to the discrete energy balance

[6] Note that the aspect of inf-sup stability can be seen in analogy to the staggered arrangement of velocity and pressure unknowns in finite volume methods, see e.g. Ferziger and Peric (2002).

$$\frac{\partial E_h}{\partial t} = \frac{\partial}{\partial t}\int_{\Omega_h} \frac{1}{2}\boldsymbol{u}_h \cdot \boldsymbol{u}_h \mathrm{d}\Omega = \left(\boldsymbol{u}_h, \partial \boldsymbol{u}_h/\partial t\right)_{\Omega_h} = -\sum_{e=1}^{n} c_h^e\left(\boldsymbol{u}_h, \boldsymbol{u}_h\right) + \dots \tag{10}$$

by replacing the test function $\boldsymbol{v}_h$ by $\boldsymbol{u}_h$

$$\begin{aligned} c^e_{h,\mathrm{conv}}\left(\boldsymbol{u}_h, \boldsymbol{u}_h\right) &= \left(\boldsymbol{u}_h, \left(\boldsymbol{u}_h \cdot \nabla\right)\boldsymbol{u}_h\right)_{\Omega_e} \\ &\overset{(8)}{=} -\frac{1}{2}\left(\nabla \cdot \boldsymbol{u}_h, \boldsymbol{u}_h \cdot \boldsymbol{u}_h\right)_{\Omega_e} + \frac{1}{2}\left(1, \nabla \cdot \left(\boldsymbol{u}_h\left(\boldsymbol{u}_h \cdot \boldsymbol{u}_h\right)\right)\right)_{\Omega_e} \\ &= -\frac{1}{2}\left(\nabla \cdot \boldsymbol{u}_h, \boldsymbol{u}_h \cdot \boldsymbol{u}_h\right)_{\Omega_e} + \frac{1}{2}\left(\boldsymbol{u}_h \cdot \boldsymbol{u}_h, \boldsymbol{u}_h \cdot \boldsymbol{n}\right)_{\partial\Omega_e} , \end{aligned} \tag{11}$$

where we reformulated the convective term by splitting it according to Eq. (8), resulting in a term to be transformed into a surface integral by Gauss' divergence theorem as well as a volume integral containing a residual vanishing in the continuous case, $\nabla \cdot \boldsymbol{u} = 0$. Let us note that integration-by-parts is unproblematic in the discrete case with space $\mathcal{V}_h^u$ also in the inviscid limit, as opposed to the continuous problem (with the discussion related to Eq. 4). The second term vanishes when summing over all elements due to the continuity of the discrete velocity $\boldsymbol{u}_h$ at element boundaries, while the first term does not vanish due to $\nabla \cdot \boldsymbol{u}_h \neq 0$ in general in the discrete case,

$$\sum_{e=1}^{n} c^e_{h,\mathrm{conv}}\left(\boldsymbol{u}_h, \boldsymbol{u}_h\right) = -\sum_{e=1}^{n} \frac{1}{2}\left(\nabla \cdot \boldsymbol{u}_h, \boldsymbol{u}_h \cdot \boldsymbol{u}_h\right)_{\Omega_e} \neq 0 \ . \tag{12}$$

Considering alternatively the divergence formulation of the convective term with the weak form

$$c^e_{h,\mathrm{div}}\left(\boldsymbol{v}_h, \boldsymbol{u}_h\right) = \left(-\nabla \boldsymbol{v}_h, \boldsymbol{u}_h \otimes \boldsymbol{u}_h\right)_{\Omega_e} , \tag{13}$$

the contribution to the energy balance is

$$\begin{aligned} c^e_{h,\mathrm{div}}\left(\boldsymbol{u}_h, \boldsymbol{u}_h\right) &= \left(-\nabla \boldsymbol{u}_h, \boldsymbol{u}_h \otimes \boldsymbol{u}_h\right)_{\Omega_e} = \left(-\left(\boldsymbol{u}_h \cdot \nabla\right)\boldsymbol{u}_h, \boldsymbol{u}_h\right)_{\Omega_e} = -\left(\boldsymbol{u}_h, \left(\boldsymbol{u}_h \cdot \nabla\right)\boldsymbol{u}_h\right)_{\Omega_e} \\ &= -c^e_{h,\mathrm{conv}}\left(\boldsymbol{u}_h, \boldsymbol{u}_h\right) \ . \end{aligned} \tag{14}$$

For both formulations, the discrete convective term might produce energy and thereby cause numerical instabilities, in particular when applied to problems at high Reynolds number and in a spatially under-resolved setting. This potential instability is a fundamental property of the incompressible Navier–Stokes equations and the discretization errors of the finite element spaces (leading to a velocity field that is not exactly divergence-free in general). Note that inexact numerical quadrature of the nonlinear convective term in a practical numerical solver is an independent source of instability (commonly termed *aliasing*).

Comparing Eqs. (11) and (14), it is easy to see that the discrete nonlinear term does not contribute to the energy balance when using the so-called *skew-symmetric* formulation

$$c^e_{h,\mathrm{skew}}\left(\boldsymbol{v}_h, \boldsymbol{u}_h\right) = \frac{1}{2}\left(c^e_{h,\mathrm{conv}}\left(\boldsymbol{v}_h, \boldsymbol{u}_h\right) + c^e_{h,\mathrm{div}}\left(\boldsymbol{v}_h, \boldsymbol{u}_h\right)\right) = \frac{1}{2}\left(\left(\boldsymbol{v}_h, \left(\boldsymbol{u}_h \cdot \nabla\right)\boldsymbol{u}_h\right)_{\Omega_e} - \left(\nabla \boldsymbol{v}_h, \boldsymbol{u}_h \otimes \boldsymbol{u}_h\right)_{\Omega_e}\right) . \tag{15}$$

The use of skew-symmetric formulations is a classical way to achieve discrete energy-conservation for nonlinear convective terms. For further details, we refer to review articles on this particular topic (Coppola et al. 2019). Discretely energy-conserving schemes constructed this way are attractive because they provide robust schemes and, therefore, appear superior as compared to schemes that bear the potential of nonlinear blow-up. However, as emphasized previously, the discrete energy-conservation property is not suitable as a metric to describe accuracy in case dissipative anomalies are present physically. For turbulent flows, energy-conserving skew-symmetric schemes can therefore not be expected to yield accurate results e.g. in agreement with the dissipation anomaly unless additional mechanisms of dissipation are used. Hence, we do not consider skew-symmetric formulations as the distinctive feature of a numerical LES approach in the present work, but rather as a certain feature or ingredient used in combination with the essential mechanisms of stabilization.

## 4.5 A Brief Summary of Methods

Figure 1 gives a brief overview of the development of finite element methods for the incompressible Navier–Stokes equations with suitable stabilization or dissipation mechanisms for the under-resolved simulation of turbulent flows (large-eddy simulation). From left to right, we categorize methods w.r.t. the velocity function space. For a detailed overview of the development of $H^1$-conforming methods, we refer to the review articles by Gravemeier (2006), Ahmed et al. (2017), Rasthofer and Gravemeier (2018). An overview of the development of $L^2$-conforming methods can be found in Fehn (2021), and of $H(\mathrm{div})$-conforming methods in Schroeder (2019). From left to right, this figure presents roughly the historical development of finite element methods for turbulent flows. For example, stabilized techniques developed for $H^1$-conforming methods have partly been adapted to $L^2$-conforming

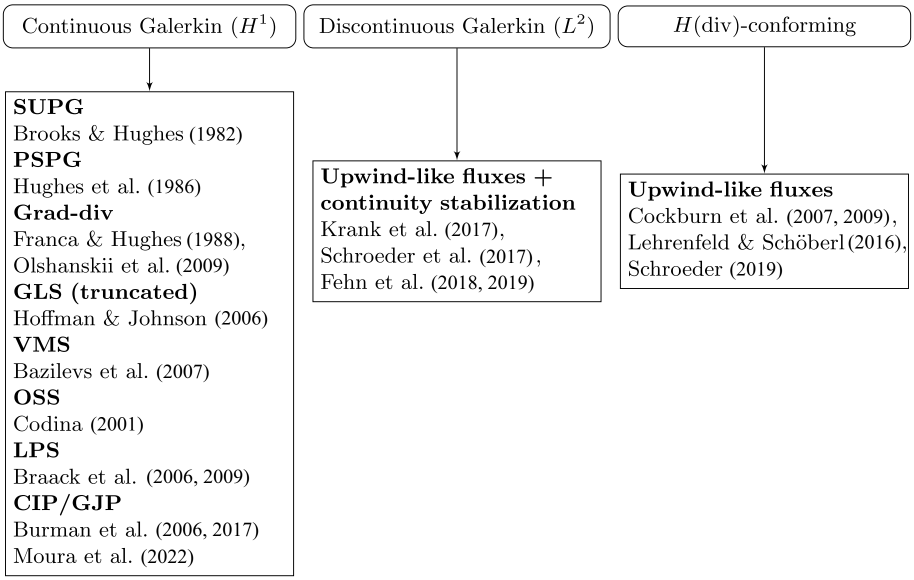


**Fig. 1** An overview of dissipation mechanisms / variational stabilization techniques for (high-order) (dis-)continuous Galerkin discretizations of incompressible flows. (Abbreviations: SUPG: streamline-upwind Petrov–Galerkin; PSPG: pressure stabilizing Petrov–Galerkin; GLS: Galerkin least squares; VMS: variational multiscale; OSS: orthogonal subscales; LPS: local projection stabilization; CIP: continuous interior penalty; GJP: gradient jump penalty)

methods developed later. Numerical fluxes and in particular upwind-like fluxes for convective terms used in $L^2$-conforming methods (and having their origin in the finite volume community), have been adapted to $H(\mathrm{div})$-conforming methods. As we show in the following, mathematical properties of the discretization schemes e.g. in terms of mass conservation and energy stability appear to be most sophisticated for the category of $H(\mathrm{div})$-conforming methods. However, we do not want to insinuate that this clarity and stringency in analytical results necessarily implies a higher level of sophistication or accuracy as a practical simulation tool for under-resolved turbulent flows. We rather consider this topic an open research question.

Stabilization techniques or dissipation mechanisms not discussed in the present work are spectral vanishing viscosity (SVV) methods, employed for continuous Galerkin methods e.g. in Karamanos and Karniadakis (2000), Kirby and Karniadakis (2002) and for discontinuous Galerkin methods e.g. in Manzanero et al. (2020). The recent work by Moura et al. (2022) argues that GJP is superior over SVV and competitive to upwind-like DG for under-resolved turbulent flow simulations regarding the balance of accuracy and robustness. We also do not discuss filtering techniques, employed for continuous Galerkin methods e.g. in Fischer and Mullen (2001), Fischer et al. (2002) and for discontinuous Galerkin methods e.g. in Hesthaven and Warburton (2008), and entropy-viscosity methods (Guermond 2008; Guermond et al. 2011).

### 4.6 Continuous Galerkin Methods ($H^1$-Conforming)

In this section, we discuss a family of stabilized continuous Galerkin discretizations of the incompressible Navier–Stokes equations, mainly of Galerkin least squares (GLS) type. These methods include the well-known streamline-upwind stabilization (for convection-dominated problems), pressure stabilization (for inf-sup stability problems), and grad-div stabilization (for improved mass conservation) as basic stabilization ingredients. Regarding the velocity/pressure function spaces, examples are the pair $Q_p/Q_p$ with pressure stabilization (where $Q_p$ denotes the polynomial basis of tensor degree $p$ on the elements), the Taylor–Hood pair $Q_p/Q_{p-1}$, or the pair $Q_p/P_{p-1}^{\mathrm{disc}}$ with polynomials of complete degree up to $p-1$ for the pressure, which is globally $L^2$-conforming. The Scott–Vogelius element for simplicial elements (with discontinuous pressure space) might also be attributed to this category of methods.

#### 4.6.1 Variational Formulation

Find $\boldsymbol{u}_h \in \mathcal{V}_h^u, p_h \in \mathcal{V}_h^p$ such that it holds

$$\sum_{e=1}^{n} \left( \left( \boldsymbol{v}_h, \frac{\partial \boldsymbol{u}_h}{\partial t} \right)_{\Omega_e} + c_h^e(\boldsymbol{v}_h, \boldsymbol{u}_h) + v_h^e(\boldsymbol{v}_h, \boldsymbol{u}_h) + g_h^e(\boldsymbol{v}_h, p_h) + s_h^e(\boldsymbol{v}_h, \boldsymbol{u}_h) \right) = 0\ , \quad (16)$$

$$-\sum_{e=1}^{n} d_h^e(q_h, \boldsymbol{u}_h) = 0\ , \quad (17)$$

for all $(\boldsymbol{v}_h, q_h) \in \mathcal{V}_h^u \times \mathcal{V}_h^p$. The forms corresponding to the different terms of the PDE (viscous, pressure gradient, velocity divergence) and an additional stabilization term $s_h^e$ are given as

$$\begin{aligned} v_h^e(\boldsymbol{v}_h, \boldsymbol{u}_h) &= \nu(\nabla \boldsymbol{v}_h, \nabla \boldsymbol{u}_h)_{\Omega_e} , \\ g_h^e(\boldsymbol{v}_h, p_h) &= -(\nabla \cdot \boldsymbol{v}_h, p_h)_{\Omega_e} , \\ d_h^e(q_h, \boldsymbol{u}_h) &= (q_h, \nabla \cdot \boldsymbol{u}_h)_{\Omega_e} , \\ s_h^e(\boldsymbol{v}_h, q_h, \boldsymbol{u}_h, p_h) &= \left(\hat{\boldsymbol{R}}_\mathrm{m}(\boldsymbol{u}_h, \boldsymbol{v}_h, q_h), \tau_\mathrm{M}^e \hat{\hat{\boldsymbol{R}}}_\mathrm{m}(\boldsymbol{u}_h, \boldsymbol{u}_h, p_h)\right)_{\Omega_e} + \left(\boldsymbol{R}_\mathrm{c}(\boldsymbol{v}_h), \tau_\mathrm{C}^e \boldsymbol{R}_\mathrm{c}(\boldsymbol{u}_h)\right)_{\Omega_e} , \end{aligned}$$

where the viscous term and the pressure gradient term have been integrated by parts, while the velocity divergence term is kept in its original form. For the convective term $c_h^e$, one of the forms discussed in Sect. 4.4 may be used. The stabilization term $s_h^e$ consists of a momentum stabilization term and a continuity stabilization term. The continuity stabilization term with $\boldsymbol{R}_\mathrm{c}(\boldsymbol{u}) = \nabla \cdot \boldsymbol{u}$ is the so-called grad-div stabilization term. In the context of variational multiscale methods, this stabilization term is interpreted as a model for the pressure sub-grid scales (Olshanskii et al. 2009; Bazilevs et al. 2007). Regarding the momentum stabilization, various formulations can be found in the literature w.r.t. the choice of the "residuals" $\hat{\boldsymbol{R}}_\mathrm{m}(\boldsymbol{u}, \boldsymbol{v}, q), \hat{\hat{\boldsymbol{R}}}_\mathrm{m}(\boldsymbol{u}, \boldsymbol{v}, q)$. Table 1 provides an overview of the different formulations for the momentum stabilization term discussed in the following.

From the perspective of Galerkin least squares (GLS) stabilization, it appears natural to consider the full residual on the test function and solution function

$$\hat{\boldsymbol{R}}_\mathrm{m}(\boldsymbol{u}, \boldsymbol{v}, q) = \hat{\hat{\boldsymbol{R}}}_\mathrm{m}(\boldsymbol{u}, \boldsymbol{v}, q) = \boldsymbol{R}_\mathrm{m}(\boldsymbol{u}, \boldsymbol{v}, q) = \partial \boldsymbol{v}/\partial t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{v} - \nu \nabla^2 \boldsymbol{v} + \nabla q . \quad (18)$$

***Remark 4.1*** Note that the temporal derivative applied to the test function gives rise to space–time formulations (Hughes et al. 2017), i.e. one would actually need an additional integral over time in Eqs. (16) and (17), which we omit here for ease of presentation. As noted in Hoffman et al. (2011, 2015), space-time formulations with time-dependent velocity test functions result in a significant increase in the number of unknowns. Moreover, second spatial derivatives applied to the test and solution function are needed in case of the incompressible Navier–Stokes equations with the viscous term, see also Table 1. These ingredients might not be straightforward to realize in every CFD software project, and may cause additional computational costs.

The complications mentioned in the above remark, in particular the time derivative of the test function, are partially overcome by truncating the residual on the test

**Table 1** Properties of different variants of the momentum stabilization term for stabilized continuous Galerkin discretizations of the time-dependent incompressible Navier–Stokes equations

| momentum stabilization term | Consistency | Energy stability | $\partial \boldsymbol{v}_h/\partial t, \nabla^2 \boldsymbol{v}_h$ | $\nabla^2 \boldsymbol{u}_h$ |
|---|---|---|---|---|
| $(\boldsymbol{R}_\mathrm{m}(\boldsymbol{u}_h, \boldsymbol{v}_h, q_h), \tau_\mathrm{M}^e \boldsymbol{R}_\mathrm{m}(\boldsymbol{u}_h, \boldsymbol{u}_h, p_h))_{\Omega_e}$ | ✓ | ✓ | ✓ | ✓ |
| $((\boldsymbol{u}_h \cdot \nabla)\boldsymbol{v}_h + \nabla q_h, \tau_\mathrm{M}^e \boldsymbol{R}_\mathrm{m}(\boldsymbol{u}_h, \boldsymbol{u}_h, p_h))_{\Omega_e}$ | ✓ | | | ✓ |
| $((\boldsymbol{u}_h \cdot \nabla)\boldsymbol{v}_h + \nabla q_h, \tau_\mathrm{M}^e((\boldsymbol{u}_h \cdot \nabla)\boldsymbol{u}_h + \nabla p_h))_{\Omega_e}$ | | ✓ | | |

function. This leads to a widely used *residual-based* formulation, see e.g. Bazilevs et al. (2007), Rasthofer and Gravemeier (2013),

$$\begin{aligned}\hat{\boldsymbol{R}}_{\mathrm{m}}(\boldsymbol{u},\boldsymbol{v},q) &= (\boldsymbol{u}\cdot\nabla)\boldsymbol{v} + \nabla q \;,\\ \hat{\hat{\boldsymbol{R}}}_{\mathrm{m}}(\boldsymbol{u},\boldsymbol{v},q) &= \boldsymbol{R}_{\mathrm{m}}(\boldsymbol{u},\boldsymbol{v},q) \;.\end{aligned} \tag{19}$$

***Remark 4.2*** This non-symmetric formulation uses the full residual for the solution function and corresponds to the original (steady-state) streamline-upwind and pressure-stabilizing Petrov–Galerkin stabilization (for which the abbreviations SUPG/PSPG are established in the literature).

In addition to the truncated residual on the test function, Hoffman and Johnson (2006), Hoffman et al. (2015) consider a truncated residual also on the solution function

$$\hat{\boldsymbol{R}}_{\mathrm{m}}(\boldsymbol{u},\boldsymbol{v},q) = \hat{\hat{\boldsymbol{R}}}_{\mathrm{m}}(\boldsymbol{u},\boldsymbol{v},q) = (\boldsymbol{u}\cdot\nabla)\boldsymbol{v} + \nabla q \;. \tag{20}$$

Table 1 compares the three variants of the momentum stabilization term in Eqs. (18), (19) and (20) w.r.t. different properties discussed in more detail in the following.

***Remark 4.3*** The formulation (20) achieves energy-stability for any positive stabilization parameter due to the symmetric design of the stabilization term, see also Sect. 4.6.4. However, this method is no longer *residual-based*, which has an important implication on consistency, see Sect. 4.6.2. Note that this method is also inconsistent for the special case of the unsteady incompressible Euler equations (which is the main target of the present work), since no time derivative is applied to the solution function in the momentum stabilization term.

***Remark 4.4*** A classical interpretation of SUPG methods (originally proposed in Brooks and Hughes 1982) appears to be the incorporation of upwind-mechanisms into $H^1$-conforming methods via Petrov–Galerkin methods, thereby avoiding wiggles that are well-known to form in an unstabilized method in the convection-dominated regime. More recent contributions (Hughes et al. 2017; Hoffman and Johnson 2006; Hoffman et al. 2011; Codina et al. 2011) seem to provide motivation for SUPG stabilization also from the perspective of energy dissipation, see Table 1 and Sect. 4.6.4.

***Remark 4.5*** The work by Röhe and Lube (2010) skips the momentum stabilization term and uses the grad–div stabilization term only, $s_h^e\left(\boldsymbol{v}_h, q_h, \boldsymbol{u}_h, p_h\right) = \left(\tau_{\mathrm{C}}^e \nabla\cdot\boldsymbol{v}_h, \nabla\cdot\boldsymbol{u}_h\right)_{\Omega_e}$. However, numerical results indicate that this numerical scheme leads to an accumulation of energy in small scales, which is why an additional Smagorinsky-like variational multiscale model is used in Röhe and Lube (2010). Similarly, the work by Gravemeier et al. (2010) uses grad–div stabilization and a Smagorinsky-like variational multiscale model, but additionally the PSPG term due to equal-order polynomial approximations, $\hat{\boldsymbol{R}}_{\mathrm{m}}(\boldsymbol{u},\boldsymbol{v},q) = \nabla q$. It is argued that the Smagorinsky-like model renders additional velocity stabilization superfluous (Gravemeier et al. 2010, Remark 2.3). In terms of turbulence modeling, these two works pick up the idea by Hughes et al. (2000) of a Smagorinsky-like multiscale model.

***Remark 4.6*** The works by Hughes et al. (1998, 2017) on variational multiscale methods distinguish between the so-called *smooth case* and the *rough case*, where the rough case refers to the circumstance that derivatives of the solution are discontinuous between elements for continuous Galerkin approximations. The rough case accounts for these discontinuities by additional face integrals with jump terms. However, the related works by Hughes et al. (2000), Bazilevs et al. (2007) on turbulent flows and the incompressible Navier–Stokes equations do not discuss the rough case in detail.

***Remark 4.7*** (Bazilevs et al. 2007, Section 4) argue that their (full) variational multiscale method is an advancement over classical stabilized finite element methods, since the variational multiscale method accounts not only for one cross-stress term, but for both cross-stress terms and also the Reynolds stress term (note that Eq. 19 shows the stabilization terms stemming from the first cross stress term, while we do not show here the additional stabilization terms stemming from the other cross-stress term and the Reynolds stress term). However, the terms with the temporal derivative and second spatial derivative to be applied to the test function are skipped in Bazilevs et al. (2007) without detailed explanation. In this sense, this (full) variational multiscale method is similar to classical stabilized methods. Similarly, the work by Rasthofer and Gravemeier (2013) considers only the classical SUPG/PSPG and grad–div stabilization terms, while the authors argue that the cross stress terms and Reynolds stress term are addressed by multifractal subgrid-scale modeling. For both works (Bazilevs et al. 2007; Rasthofer and Gravemeier 2013), energy-stability of the final LES approach appears to be unclear.

#### 4.6.2 Consistency

The standard Galerkin method without stabilization terms is derived as a weighted residual formulation using integration by parts. Hence, the consistency of the stabilized method is driven by the consistency of the stabilization terms. They are consistent if they are *residual-based*. The grad-div stabilization term is based on the continuity residual $\boldsymbol{R}_{\mathrm{c}}(\boldsymbol{u}) = \nabla \cdot \boldsymbol{u}$. Regarding the momentum stabilization term, consistency of the stabilized finite element method is given if $\hat{\hat{\boldsymbol{R}}}_{\mathrm{m}}(\boldsymbol{u}, \boldsymbol{v}, q) = \boldsymbol{R}_{\mathrm{m}}(\boldsymbol{u}, \boldsymbol{v}, q)$. As summarized in Table 1, Eqs. (18) and (19) result in a consistent method, while Eq. (20) results in general in an inconsistent method (only for the narrow case of the steady Euler equations, it would result in a consistent method).

Consistency has an important impact on achievable convergence rates for problems with smooth solution. The stabilization parameters typically scale like $\tau \sim h$ or $\tau \sim h^2$, implying that the rate of convergence is limited to low-order when using high-order polynomial approximations for velocity and pressure if the momentum stabilization term is not residual-based.

***Remark 4.8*** The stabilization term contains two stabilization parameters $\tau_{\mathrm{M}}^e, \tau_{\mathrm{C}}^e$. For stabilized methods, the general approach is to derive these stabilization parameters based on dimensional analysis, using scaling factors of order unity. To obtain physical units consistent with the other terms of the PDE, the stabilization parameters must have physical units of $\left[\tau^e_{\mathrm{M}}\right] = \mathrm{s}$ and $\left[\tau^e_{\mathrm{C}}\right] = \mathrm{m}^2/\mathrm{s}$, see also Ahmed et al. (2017). This aspect is not followed stringently in many works, see e.g. Hoffman and Johnson (2006), Hoffman et al. (2015), with the consequence that the results of a numerical simulation for the same flow problem with the same Reynolds number will be different when conducting the simulation in different physical units, or that the scaling factor has to be adjusted from one flow problem to another.

### 4.6.3 Mass Conservation

According to the weak formulation described above, the mass conservation equation in incompressible flows (divergence-free constraint) is fulfilled in a weak sense. From the weak continuity Eq. (17), we can derive the global mass-conservation equation by choosing $q_h = 1$ as pressure test function and performing integration-by-parts

$$0 = \sum_{e=1}^{n} \left(1, \nabla \cdot \boldsymbol{u}_h\right)_{\Omega_e} = \sum_{e=1}^{n} \int_{\partial\Omega_e} \boldsymbol{u}_h \cdot \boldsymbol{n} \mathrm{d}\Gamma = \int_{\partial\Omega} \boldsymbol{u}_h \cdot \boldsymbol{n} \mathrm{d}\Gamma \, , \tag{21}$$

i.e. the mass flux over all boundaries of the domain sums up to zero. Mass conservation in the above sense is not fulfilled on a single element (the reason behind is that for pressure functions in $H^1$ we cannot choose the pressure test function $q_h = 1$ on a single element and $q_h = 0$ on all other elements), unless an $L^2$-conforming pressure space is used (as briefly mentioned above and as summarized in Table 2). In particular, the method is not pointwise divergence-free, $\nabla \cdot \boldsymbol{u}_h \neq 0$. The grad-div stabilization term aims at improved mass conservation. We note that the above statements on mass-conservation are not affected by the chosen variant of the momentum stabilization term.

### 4.6.4 Energy Stability

Under the assumptions of vanishing body forces, periodic boundary conditions, and vanishing viscosity ($\nu = 0$), the following energy dissipation equation can be derived for the stabilized continuous Galerkin method (when using $c^e_h\left(\boldsymbol{v}_h, \boldsymbol{u}_h\right) = c^e_{h,\mathrm{conv}}\left(\boldsymbol{v}_h, \boldsymbol{u}_h\right)$)

**Table 2** Overview of mass conservation properties for different classes of function spaces (Abbreviations: SV: Scott–Vogelius; CR: Crouzeix–Raviart; RT: Raviart–Thomas; BDM: Brezzi–Douglas–Marini)

| mass conservation property | $H^1$ | $H^1/L^2(Q_p/P^{\mathrm{disc}}_{p-1})$ | $H^1/L^2$ (SV) | $L^2$ | $L^2$ (CR) | $H$(div) (RT/ BDM) |
|---|---|---|---|---|---|---|
| $\int_{\partial\Omega} \boldsymbol{u}_h \cdot \boldsymbol{n} \mathrm{d}\Gamma = 0$ | ✓ | ✓ | ✓ | | ✓ | ✓ |
| $\int_{\partial\Omega} \{\{\boldsymbol{u}_h\}\} \cdot \boldsymbol{n} \mathrm{d}\Gamma = 0$ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| $\int_{\partial\Omega_e} \boldsymbol{u}_h \cdot \boldsymbol{n} \mathrm{d}\Gamma = 0$ | | ✓ | ✓ | | ✓ | ✓ |
| $\int_{\partial\Omega_e} \{\{\boldsymbol{u}_h\}\} \cdot \boldsymbol{n} \mathrm{d}\Gamma = 0$ | | ✓ | ✓ | ✓ | ✓ | ✓ |
| $\nabla \cdot \boldsymbol{u}_h = 0$ (pointwise) | | | ✓ | | | ✓ |
| $\left[\boldsymbol{u}_h\right] \cdot \boldsymbol{n} = 0$ (point-wise) | ✓ | ✓ | ✓ | | | ✓ |

$$\frac{\mathrm{d}E_h(t)}{\mathrm{d}t} = \sum_{e=1}^{n} \Bigg( + \frac{1}{2}\big(\nabla\cdot\boldsymbol{u}_h, \boldsymbol{u}_h\cdot\boldsymbol{u}_h\big)_{\Omega_e} - \tau_{\mathrm{M}}^{e}\Big(\hat{\boldsymbol{R}}_{\mathrm{m}}(\boldsymbol{u}_h,\boldsymbol{u}_h,p_h), \hat{\hat{\boldsymbol{R}}}_{\mathrm{m}}(\boldsymbol{u}_h,\boldsymbol{u}_h,p_h)\Big)_{\Omega_e} - \tau_{\mathrm{C}}^{e}\big(\boldsymbol{R}_{\mathrm{c}}(\boldsymbol{u}_h), \boldsymbol{R}_{\mathrm{c}}(\boldsymbol{u}_h)\big)_{\Omega_e}\Bigg). \tag{22}$$

Note again that the convective term does not vanish since $\nabla\cdot\boldsymbol{u}_h \neq 0$ in the discrete case. It vanishes, however, when choosing a skew-symmetric formulation, Eq. (15).

From the above equation, it becomes clear that a symmetric choice of the momentum stabilization term has a dissipative character (see also the summary in Table 1), while energy stability w.r.t. this term would need further estimates and assumptions for other choices of the momentum stabilization term. The continuity stabilization term is by design quadratic and, therefore, represents a mechanism of numerical dissipation. Thus, the mathematical (mathe-matical) statement on energy stability becomes strict when considering a skew-symmetric formulation of the convective term and a symmetric variant of the momentum stabilization term

$$\frac{\mathrm{d}E_h(t)}{\mathrm{d}t} = \sum_{e=1}^{n} \Bigg( -\tau_{\mathrm{M}}^{e}\Big(\hat{\hat{\boldsymbol{R}}}_{\mathrm{m}}(\boldsymbol{u}_h,\boldsymbol{u}_h,p_h), \hat{\hat{\boldsymbol{R}}}_{\mathrm{m}}(\boldsymbol{u}_h,\boldsymbol{u}_h,p_h)\Big)_{\Omega_e} - \tau_{\mathrm{C}}^{e}\big(\boldsymbol{R}_{\mathrm{c}}(\boldsymbol{u}_h), \boldsymbol{R}_{\mathrm{c}}(\boldsymbol{u}_h)\big)_{\Omega_e}\Bigg) \leq 0. \tag{23}$$

From this equation, the interpretation of the stabilization terms as an implicit turbulence model becomes obvious, see also Hoffman and Johnson (2006), Hoffman et al. (2011). Table 1 highlights that formulations of the momentum stabilization term that are trivially energy-stable (due to a symmetric design of this term) make a compromise in terms of consistency, or need to deal with the time derivative and second spatial derivative of the test function. While this might explain why non-consistent, energy-stable formulations (Hoffman and Johnson 2006; Hoffman et al. 2011) or consistent stabilization variants with unproven energy-stability (Rasthofer and Gravemeier 2013; Bazilevs et al. 2007) are used as well, we see this need for compromises as one of the driving forces for the development of other $H^1$-conforming methods (such as the orthogonal subscales method discussed in Remark 4.9 or gradient jump penalty discussed in Sect. 4.9) as well as the development of $L^2$ and $H(\mathrm{div})$-conforming methods.

***Remark 4.9*** Codina et al. (2011) show that their orthogonal subscales method (in the context of variational multiscale methods) is dissipative (energy-stable) with numerical dissipation

$$\tau_{\mathrm{M}}^{e}\big(P_h^{\perp}\big((\boldsymbol{u}_*\cdot\nabla)\boldsymbol{u}_h + \nabla p_h\big), P_h^{\perp}\big((\boldsymbol{u}_*\cdot\nabla)\boldsymbol{u}_h + \nabla p_h\big)\big)_{\Omega_e} \tag{24}$$

for the inviscid limit $\nu = 0$ if so-called *quasi-static* subscales are used, where $P_h^{\perp}$ is the projection orthogonal to the finite element space $V_h$. This dissipation term seems to be designed very similarly to–and might be interpreted as a (consistent) multiscale variant of – the stabilization by Hoffman and Johnson (2006), Hoffman et al. (2011) based on Eq. (20). The proof of energy-stability for the case with *dynamic* subscales seems to be open.

### 4.7 Discontinuous Galerkin Methods ($L^2$-Conforming)

In this section, we describe a stablized discontinuous Galerkin method for the incompressible Navier–Stokes equations developed by Fehn et al. (2018), Fehn (2021). Inf-sup stability is assumed by choosing a Taylor–Hood-like element with polynomials of tensor degree $p$ for the velocity and tensor degree $p-1$ for the pressure.

#### 4.7.1 Variational Formulation

The variational formulation of the stabilized $L^2$-conforming method reads: Find $\boldsymbol{u}_h \in \mathcal{V}_h^u, p_h \in \mathcal{V}_h^p$ such that it holds

$$\left(\boldsymbol{v}_h, \frac{\partial \boldsymbol{u}_h}{\partial t}\right)_{\Omega_e} + c_h^e\left(\boldsymbol{v}_h, \boldsymbol{u}_h\right) + v_h^e\left(\boldsymbol{v}_h, \boldsymbol{u}_h\right) + g_h^e\left(\boldsymbol{v}_h, p_h\right) + a_{\mathrm{D},h}^e\left(\boldsymbol{v}_h, \boldsymbol{u}_h\right) + a_{\mathrm{C},h}^e\left(\boldsymbol{v}_h, \boldsymbol{u}_h\right) = 0\ , \tag{25}$$

$$-d_h^e(q_h, \boldsymbol{u}_h) = 0\ , \tag{26}$$

for all $(\boldsymbol{v}_h, q_h) \in \mathcal{V}_{h,e}^u \times \mathcal{V}_{h,e}^p$ and for all elements $e = 1, \dots, n$. Due to the discontinuity of the test functions, the weak formulation may be stated in an element-wise manner. The nonlinear convective term is discretized by the local Lax–Friedrichs flux

$$c_h^e\left(\boldsymbol{v}_h, \boldsymbol{u}_h\right) = -\left(\nabla \boldsymbol{v}_h, \boldsymbol{F}_\mathrm{c}(\boldsymbol{u}_h)\right)_{\Omega_e} + \left(\boldsymbol{v}_h, \boldsymbol{F}_\mathrm{c}^*(\boldsymbol{u}_h) \cdot \boldsymbol{n}\right)_{\partial\Omega_e}\ , \tag{27}$$

with $\boldsymbol{F}_\mathrm{c}^*(\boldsymbol{u}_h) = \{\{\boldsymbol{F}_\mathrm{c}(\boldsymbol{u}_h)\}\} + \frac{\Lambda}{2}[\![\boldsymbol{u}_h]\!]$ and $\boldsymbol{F}_\mathrm{c}(\boldsymbol{u}_h) = \boldsymbol{u}_h \otimes \boldsymbol{u}_h$, where $\Lambda$ is the stabilization parameter of the Lax–Friedrichs flux, see e.g. Fehn (2021). The viscous term is discretized by the symmetric interior penalty method (Arnold 1982) with stabilization parameter $\tau_\mathrm{SIPG}$

$$\begin{aligned} v_h^e\left(\boldsymbol{v}_h, \boldsymbol{u}_h\right) = \left(\nabla \boldsymbol{v}_h, \nu \nabla \boldsymbol{u}_h\right)_{\Omega_e} - \left(\nabla \boldsymbol{v}_h, \frac{\nu}{2}[\![\boldsymbol{u}_h]\!]\right)_{\partial\Omega_e} \\ - \left(\boldsymbol{v}_h, \nu\{\{\nabla \boldsymbol{u}_h\}\} \cdot \boldsymbol{n}\right)_{\partial\Omega_e} + \left(\boldsymbol{v}_h, \nu\ \tau_\mathrm{SIPG}\ [\![\boldsymbol{u}_h]\!] \cdot \boldsymbol{n}\right)_{\partial\Omega_e}\ . \end{aligned} \tag{28}$$

Central flux functions are used for the pressure gradient term and velocity divergence term

$$\begin{aligned} g_h^e\left(\boldsymbol{v}_h, p_h\right) &= -\left(\nabla \cdot \boldsymbol{v}_h, p_h\right)_{\Omega_e} + \left(\boldsymbol{v}_h, \{\{p_h\}\}\boldsymbol{n}\right)_{\partial\Omega_e}, \\ d_h^e\left(q_h, \boldsymbol{u}_h\right) &= -\left(\nabla q_h, \boldsymbol{u}_h\right)_{\Omega_e} + \left(q_h, \{\{\boldsymbol{u}_h\}\} \cdot \boldsymbol{n}\right)_{\partial\Omega_e}. \end{aligned}$$

The stabilization terms weakly enforcing the divergence-free constraint and normal continuity of the velocity are defined as

$$\begin{aligned} a_{\mathrm{D},h}^e(\boldsymbol{v}_h, \boldsymbol{u}_h) &= \left(\nabla \cdot \boldsymbol{v}_h, \tau_\mathrm{D}^e \nabla \cdot \boldsymbol{u}_h\right)_{\Omega_e}\ , \\ a_{\mathrm{C},h}^e(\boldsymbol{v}_h, \boldsymbol{u}_h) &= \left(\boldsymbol{v}_h \cdot \boldsymbol{n}, \tau_\mathrm{C}^f \left[\boldsymbol{u}_h\right] \cdot \boldsymbol{n}\right)_{\partial\Omega_e}\ , \end{aligned}$$

where the penalty factors $\tau_\mathrm{D}^e, \tau_\mathrm{C}^f$ are derived by dimensional analysis, see Fehn et al. (2018). The average and jump operators are defined as $\{\{a\}\} = (a^- + a^+)/2$, $[\![a]\!] = a^- \otimes \boldsymbol{n}^- + a^+ \otimes \boldsymbol{n}^+$ and $[a] = a^- - a^+$ with interior information $-$ on element $\Omega_e$ and exterior information $+$ as well as outward normal vector $\boldsymbol{n}$.

#### 4.7.2 Consistency

The method is consistent since (i) it is derived as a weighted residual formulation using integration-by-parts, (ii) all numerical fluxes are consistent, and (iii) the additional stabilization terms are based on residuals of the incompressible Navier–Stokes equations, $\nabla \cdot \boldsymbol{u}_h$ for the divergence penalty term and $\left[\boldsymbol{u}_h\right] \cdot \boldsymbol{n}$ for the continuity penalty term, which will vanish when replacing $\boldsymbol{u}_h$ by $\boldsymbol{u}$. The method including these stabilization terms is therefore naturally high-order accurate when using high-order polynomial approximations for velocity and pressure and for problems with sufficiently smooth solution (Fehn et al. 2018).

**Remark 4.10** In the mathematically oriented literature, stabilization parameters for $L^2$ -conforming methods are often chosen such that they have physical units inconsistent with the equations at hand (Akbas et al. 2018; Schroeder and Lube 2017; Guzmán et al. 2016; Montlaur et al. 2008). While the overall method can be considered a consistent discretization of the incompressible Navier–Stokes equations due to the residual structure of the stabilization terms, stabilization parameters of inconsistent physical units might in fact impact robustness and accuracy of the numerical method in practice.

#### 4.7.3 Mass Conservation

To study global mass conservation across the boundaries of the domain, we consider the weak continuity equation (26) and choose $q_h(\boldsymbol{x}) = 1$ as test function

$$0 = \sum_{e=1}^{n} \left( -\left(\nabla 1, \boldsymbol{u}_h\right)_{\Omega_e} + \left(1, \{\{\boldsymbol{u}_h\}\} \cdot \boldsymbol{n}\right)_{\partial\Omega_e} \right) = \int_{\partial\Omega_h} \{\{\boldsymbol{u}_h\}\} \cdot \boldsymbol{n} \mathrm{d}\Gamma \,. \tag{29}$$

The mass flux over interior faces between elements cancels since the numerical flux $\{\{\boldsymbol{u}_h\}\}$ is conservative and since $\boldsymbol{n}^+ = -\boldsymbol{n}^-$. The above equation states that global conservation of mass across the boundaries of the domain is fulfilled in the sense of the average velocity $\{\{\boldsymbol{u}_h\}\}$ (which depends on the interior solution $\boldsymbol{u}_h$ and prescribed boundary data on Dirichlet boundaries). In general, global mass conservation in terms of the interior velocity is not fulfilled, $\int_{\partial\Omega_h} \boldsymbol{u}_h \cdot \boldsymbol{n} \mathrm{d}\Gamma \neq 0$.

For the $L^2$-conforming case, the weak continuity equation holds also element-wise. We obtain the element-wise mass conservation equation

$$0 = -\left(\nabla 1, \boldsymbol{u}_h\right)_{\Omega_e} + \left(1, \{\{\boldsymbol{u}_h\}\} \cdot \boldsymbol{n}\right)_{\partial\Omega_e} = \int_{\partial\Omega_e} \{\{\boldsymbol{u}_h\}\} \cdot \boldsymbol{n} \mathrm{d}\Gamma \,. \tag{30}$$

Again, mass conservation over the boundaries of element is fulfilled in the sense of the average velocity $\{\{\boldsymbol{u}_h\}\}$, but not in the sense of the interior velocity, $\int_{\partial\Omega_e} \boldsymbol{u}_h \cdot \boldsymbol{n} \mathrm{d}\Gamma \neq 0$, see also Table 2. By performing integration-by-parts, we obtain an alternative weak continuity equation equivalent to Eq. (26)

$$0 = \left(q_h, \nabla \cdot \boldsymbol{u}_h\right)_{\Omega_e} - \frac{1}{2}\left(q_h, \left[\boldsymbol{u}_h\right] \cdot \boldsymbol{n}\right)_{\partial\Omega_e} \,. \tag{31}$$

From this equation, we can deduce that jumps in the normal velocity between elements $\left[\boldsymbol{u}_h\right] \cdot \boldsymbol{n} \neq 0$ (that occur in general for an $L^2$-conforming approach) as well as local divergence errors $\nabla \cdot \boldsymbol{u}_h \neq 0$ balance each other in a weak sense, with the consequence that the scheme is not locally or point-wise mass-conserving. This alternative continuity equation serves as a main motivation for the two stabilization terms $a^e_{\mathrm{D},h}, a^e_{\mathrm{C},h}$.

### 4.7.4 Energy Stability

Under the assumptions of vanishing body forces, periodic boundary conditions, and vanishing viscosity ($\nu = 0$), the following energy dissipation equation can be derived for the above stabilized $L^2$-conforming method

$$
\begin{aligned}
\frac{\mathrm{d}E_h(t)}{\mathrm{d}t} = &- \frac{1}{2}\left(\boldsymbol{u}_h \cdot \boldsymbol{u}_h, \nabla \cdot \boldsymbol{u}_h\right)_{\Omega_h} - \left(\nabla \cdot \boldsymbol{u}_h, \tau_{\mathrm{D}} \nabla \cdot \boldsymbol{u}_h\right)_{\Omega_h} \\
&+ \frac{1}{2}\left(\boldsymbol{u}_h^- \cdot \boldsymbol{u}_h^+, \left[\boldsymbol{u}_h\right] \cdot \boldsymbol{n}\right)_{\Gamma_h^{\mathrm{int}}} - \left(\left[\boldsymbol{u}_h\right] \cdot \boldsymbol{n}, \tau_{\mathrm{C},f} \left[\boldsymbol{u}_h\right] \cdot \boldsymbol{n}\right)_{\Gamma_h^{\mathrm{int}}} \\
&- \left( [\![\boldsymbol{u}_h]\!], \frac{\Lambda}{2} [\![\boldsymbol{u}_h]\!] \right)_{\Gamma_h^{\mathrm{int}}}.
\end{aligned}
\tag{32}
$$

The divergence and continuity stabilization terms have a dissipative character and aim to control the sign-indefinite terms stemming from the discrete convective term. The stabilization term of the Lax–Friedrichs flux also has a dissipative character. As noted in Fehn (2021), Fehn et al. (2019), the proof of energy-stability of this scheme appears to be open for finite stabilization parameters. In the context of implicit LES, the upwind term and the additional stabilization terms form the implicit turbulence model.

***Remark 4.11*** $L^2$-conforming formulations relying on upwind-like fluxes only (without additional stabilization terms $a^e_{\mathrm{D},h}, a^e_{\mathrm{C},h}$) as proposed e.g. in Hesthaven and Warburton (2008), Shahbazi et al. (2007) have been found to lack robustness for under-resolved simulations of turbulent flows (Fehn et al. 2018). The above scheme with additional stabilization terms $a^e_{\mathrm{D},h}, a^e_{\mathrm{C},h}$ has been demonstrated to exhibit promising properties in terms of energy-stability for under-resolved simulations of turbulent flows (Fehn et al. 2018).

***Remark 4.12*** The energy stability of the $L^2$-conforming formulation with artificial compressibility flux by Bassi et al. (2006, 2016), Massa et al. (2022) (and without stabilization terms $a^e_{\mathrm{D},h}, a^e_{\mathrm{C},h}$) appears to be unclear. The $L^2$-conforming approach for implicit LES by Ferrer (2017) is based on scaling the parameter of the interior penalty discretization of the viscous term (and using SVV in the periodic direction for typical turbulent flow examples with extruded two-dimensional geometry), while no divergence and continuity stabilization terms as presented here are used. Since the viscous term (and the associated stabilization) will vanish for $\nu = 0$, robustness of this scheme remains unclear in the inviscid limit.

***Remark 4.13*** Both $H^1$ and $L^2$-conforming methods seem to require stabilization of the mass conservation equation, where the divergence and normal-continuity stabilization terms of $L^2$-methods can be seen in analogy to the grad–div stabilization of $H^1$-methods (Akbas et al. 2018). Interestingly, $L^2$-conforming methods do not suffer from over-stabilization like $H^1$-conforming methods according to Akbas et al. (2018). The stabilization terms $a^e_{D,h}, a^e_{C,h}$ of $L^2$-methods mimick the constraints built directly into the function space of $H(\mathrm{div})$-methods discussed in Sect. 4.8. For this reason, the stabilization terms $a^e_{D,h}, a^e_{C,h}$ of $L^2$-methods might be interpreted as an $H(\mathrm{div})$-stabilization or a weak enforcement of $H(\mathrm{div})$-conformity (Fehn et al. 2019; Fehn 2021).

**Remark 4.14** As in the $H^1$-conforming case, sign-indefinite terms in the energy balance stemming from the convective term could potentially be healed by a "skew-symmetric" formulation. The DG community focusing on the compressible Navier–Stokes equations has developed *split-form* schemes based on a *collocation-basis* with a so-called *summation-by-parts* property, see e.g. Gassner (2013), Gassner et al. (2016), ensuring energy stability for suitable flux formulations. However, surprisingly few research in this direction has been conducted so far for the incompressible Navier–Stokes equations. One reason for this could be that such a formulation does not render the stabilization terms related to the continuity equation obsolete, despite achieving energy stability without these stabilization terms. According to our understanding, the structure of the incompressible Navier–Stokes equations with the divergence-free constraint as continuity equation requires these stabilization terms for improved mass conservation in an $L^2$-conforming setting. Further research in this direction could certainly bring further insight. Another reason (for not exploring split-form DG schemes in the incompressible case) could be that there exist problem-tailored finite element spaces for the incompressible case (see Sect. 4.8) that solve both problems of mass conservation and energy stability in a possibly more elegant way.

### 4.8 *H*(div)-Conforming Methods

This section discusses exactly mass-conserving methods by the use of problem-tailored function spaces. The mass conservation Eq. (31) and the energy dissipation Eq. (32) of the stabilized $L^2$ method serve as a main motivation for such methods. The $H(\mathrm{div})$-conforming velocity function space defined in Eq. (7) is combined with an $L^2$-conforming pressure function space. For ease of presentation, we may restrict the discussion in this section to a mesh consisting of undeformed (Cartesian) hexahedral elements (to avoid technicalities related to additional transformations such as the Piola transform required in the general case). If there holds the additional condition

$$\nabla \cdot \mathcal{V}^u_h \subseteq \mathcal{V}^p_h \tag{33}$$

between the velocity and pressure function spaces, it can be shown that the velocity is pointwise divergence-free, $\nabla \cdot \boldsymbol{u}_h \equiv 0$, as detailed below. For the hexahedral element shape considered in the present work, a prominent example fulfilling conditions (7) and (33) is the anisotropic Raviart–Thomas element for the velocity together with a discontinuous pressure space (for simplicial elements, see the Brezzi–Douglas–Marini element). The vectorial Raviart–Thomas velocity space is constructed such that velocity component $u_{h,i}$ has degree $p+1$ in coordinate direction $x_i$ and degree $p$ in the other $d-1$ directions. The scalar pressure space has uniform polynomial degree $p$ in all directions. These spaces might be considered an optimal choice w.r.t. inf-sup stability and mass-conservation, and we refer to e.g. Fehn et al. (2019) for more details and further references to the literature.

#### 4.8.1 Variational Formulation

The variational formulation of the $H$(div)-conforming method reads: Find $\boldsymbol{u}_h \in \mathcal{V}_h^u, p_h \in \mathcal{V}_h^p$ such that it holds

$$\sum_{e=1}^{n} \left( \left( \boldsymbol{v}_h, \frac{\partial \boldsymbol{u}_h}{\partial t} \right)_{\Omega_e} + c_h^e\left(\boldsymbol{v}_h, \boldsymbol{u}_h\right) + v_h^e\left(\boldsymbol{v}_h, \boldsymbol{u}_h\right) + g_h^e\left(\boldsymbol{v}_h, p_h\right) \right) = 0 \,, \tag{34}$$

$$- \sum_{e=1}^{n} d_h^e(q_h, \boldsymbol{u}_h) = 0 \,, \tag{35}$$

for all $(\boldsymbol{v}_h, q_h) \in \mathcal{V}_h^u \times \mathcal{V}_h^p$. The convective term $c_h^e\left(\boldsymbol{v}_h, \boldsymbol{u}_h\right)$ and the viscous term $v_h^e\left(\boldsymbol{v}_h, \boldsymbol{u}_h\right)$ are given by Eq. (27) and Eq. (28), respectively. The pressure gradient term and velocity divergence term are given as

$$\begin{aligned} g_h^e\left(\boldsymbol{v}_h, p_h\right) &= -\left(\nabla \cdot \boldsymbol{v}_h, p_h\right)_{\Omega_e} + \left(\boldsymbol{v}_h, \{\{p_h\}\}\boldsymbol{n}\right)_{\partial\Omega_e}, \\ d_h^e\left(q_h, \boldsymbol{u}_h\right) &= \left(q_h, \nabla \cdot \boldsymbol{u}_h\right)_{\Omega_e}. \end{aligned}$$

using a central flux formulation for the pressure gradient term as in the $L^2$-conforming case. Integration-by-parts of the velocity divergence term with corresponding flux function is superfluous in the $H$(div)-conforming case, due to the normal continuity of the velocity built into the function space. This can also be seen from continuity Eq. (31) for the (more general) $L^2$-conforming formulation, where the face integral is identically zero when restricting the $L^2$-conforming space to an $H$(div)-conforming space.

#### 4.8.2 Consistency

The $H$(div)-conforming space is a modification of the $L^2$-conforming space that is consistent with the incompressible Navier–Stokes equations. The $H$(div)-conforming approach is therefore also consistent, and naturally high-order accurate in a setting of sufficiently smooth solutions and high-order polynomial approximations.

#### 4.8.3 Mass Conservation

Condition (33) on the velocity and pressure function spaces together with the continuity Eq. (35) immediately leads to the result

$$\nabla \cdot \boldsymbol{u}_h = 0 \quad \forall \boldsymbol{x} \in \Omega_h \,, \tag{36}$$

i.e. the discrete velocity field is pointwise divergence-free (see also Table 2). This result shows that additional stabilization terms as used in the $L^2$-conforming case are not required here.

#### 4.8.4 Energy Stability

Under the assumptions of vanishing body forces, periodic boundary conditions, and vanishing viscosity ($\nu = 0$), an $H$(div)-conforming velocity space with pointwise divergence-free velocity ($\nabla \cdot \mathcal{V}_h^u \subseteq \mathcal{V}_h^p$) leads to the result

$$\frac{\mathrm{d}E_h(t)}{\mathrm{d}t} = -\left( [\![\boldsymbol{u}_h]\!], \frac{\Lambda}{2} [\![\boldsymbol{u}_h]\!] \right)_{\Gamma_h^{\mathrm{int}}} \leq 0 \,, \tag{37}$$

i.e. the scheme is provably energy-stable. It might be considered an advantage that this method does not involve stabilization parameters $\tau$ as in the $H^1$- and $L^2$-conforming case. From the perspective of dissipation mechanisms, only the jump of the velocity (in tangential direction) acts as an indicator of non-smoothness (with associated dissipation for $[\![\boldsymbol{u}_h]\!] \neq \boldsymbol{0}$) in the $H$(div)-conforming case. Additional divergence and continuity penalty terms would not have an effect, since $\nabla \cdot \boldsymbol{u}_h = 0$ and $\left[\boldsymbol{u}_h\right] \cdot \boldsymbol{n} = 0$ are not suitable as an indicator of non-smoothness in this case.

***Remark 4.15*** It appears to be an open question whether the pure upwind stabilization via face terms as dissipation mechanism is sufficient in the limit of high polynomial degrees $p \to \infty$, a limit in which this method tends to become a spectral method. Works on high-order discontinuous Galerkin methods with upwind stabilization for the compressible Navier–Stokes equations (Collis and Chang 2002; Chapelier et al. 2016; Flad and Gassner 2017; Manzanero et al. 2020) might point to the need for additional volume dissipation mechanisms, where a dynamic (or multiscale) Smagorinsky model has been a popular choice in the literature.

***Remark 4.16*** The Scott–Vogelius element for simplicial elements is also exactly mass-conserving, see Table 2. This (unstabilized) method with $H^1$-conforming velocity does, however, not have a mechanism of dissipation e.g. through upwinding as in $L^2$, $H$(div)-conforming methods or penalization of the divergence-free constraint as in $H^1$, $L^2$-conforming methods that are not exactly mass-conserving. This raises concerns regarding physics-compatibility of the unstabilized Scott-Vogelius element for three-dimensional Euler flows with dissipative hyperbolic dynamics.

### 4.9 Stabilization of Gradient Jumps Across Faces

The three categories of function spaces ($H^1, L^2, H$(div)-conforming) discussed so far differ w.r.t. the continuity of the discrete velocity field between elements. Another stabilization technique or dissipation mechanism relies on detecting non-smoothness of the velocity field in the sense of discontinuities of the velocity gradient across elements. Corresponding stabilization terms weakly enforce $C^1$-continuity and—if designed symmetrically—act as a dissipative mechanism. This method has been proposed by Douglas and Dupont (1976) for linear convection–diffusion problems considering conforming finite element methods (see e.g. Burman (2005) for non-conforming methods) and is today known as *continuous interior penalty*, *edge stabilization*, or *gradient jump penalty* method. In the context of incompressible flows, this technique has been proposed first for continuous Galerkin methods (Burman and Fernández 2007, Burman 2007), and later also for non-conforming methods considering Crouzeix–Raviart elements (Burman et al. 2006). In elementwise notation, this stabilization term reads

$$s^e_{\mathrm{GJP},h}\left(\boldsymbol{v}_h, \boldsymbol{u}_h\right) = \left(\nabla \boldsymbol{v}_h, \tau_{\mathrm{GJP}} \left[\nabla \boldsymbol{u}_h\right]\right)_{\partial\Omega_e} \tag{38}$$

for the full gradient. For continuous Galerkin methods, the gradient jump in tangential direction vanishes and only the normal gradient may be used, $s_h^e(\boldsymbol{v}_h, \boldsymbol{u}_h) = \left(\nabla \boldsymbol{v}_h \cdot \boldsymbol{n}, \tau_{\mathrm{GJP}}\left[\nabla \boldsymbol{u}_h\right] \cdot \boldsymbol{n}\right)_{\partial\Omega_e}$. A similar gradient jump penalty term can be used for the pressure to ensure inf–sup stability. Burman et al. (2006) design the convective term in an energy-conserving manner (Burman et al. 2006, Lemma 11). By the symmetric design of the gradient jump penalty term, this terms contributes to the kinetic energy balance

$$\frac{\mathrm{d}E_h(t)}{\mathrm{d}t} = \ldots - \left(\left[\nabla \boldsymbol{u}_h\right], \tau_{\mathrm{GJP}}\left[\nabla \boldsymbol{u}_h\right]\right)_{\Gamma_h^{\mathrm{int}}} \tag{39}$$

as a numerical dissipation mechanism or implicit turbulence model.

***Remark 4.17*** While multiple stabilization terms with different physical units are used in Burman and Fernández (2007), Burman (2007), a single stabilization term with consistent physical unit is used in Moura et al. (2022) penalizing the normal gradient jump. Even though gradient-jump penalty appears suitable also for $L^2$ and $H$(div)-conforming approaches, the practical implications of this stabilization approach in terms of robustness, accuracy, interaction with other stabilization terms, and the sensitivity of results on stabilization parameters appear to be open questions.

## 5 Numerical Results–Inviscid Taylor–Green Vortex Problem

In this section we present numerical results for the inviscid Taylor–Green vortex problem. The recent study by Fehn et al. (2022) has simulated the three-dimensional inviscid Taylor–Green problem using the stabilized $L^2$-conforming method presented in Sect. 4.7. A numerical convergence study has been performed, increasing the spatial resolution from $8^3$ up to $8192^3$ degrees of freedom per velocity component (defined w.r.t. $2\pi$-periodic box) using a fixed polynomial degree $p = 3$ of the shape functions on each hexahedral element. To reduce computational costs and minimize the number of unknowns, the problem has been simulated on 1/8th of the periodic box using symmetry boundary conditions. The time step size has been chosen according to the CFL condition and therefore reduces as $\Delta t \sim h$ under mesh refinement.

The results of this convergence study (Fehn et al. 2022) are shown in Fig. 2 in terms of the temporal evolution of the kinetic energy and the kinetic energy dissipation rate.

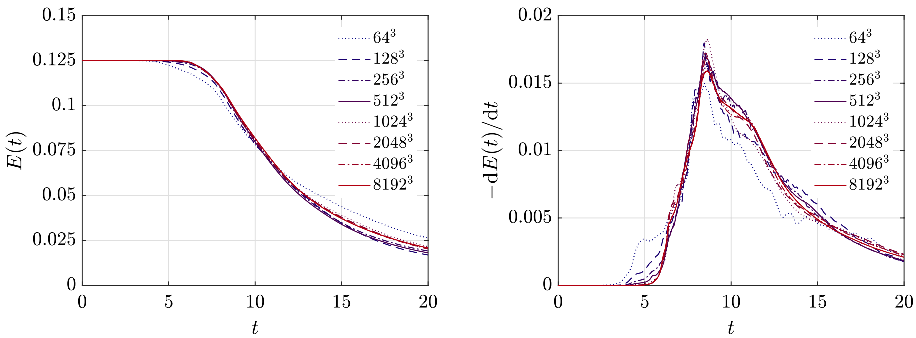


**Fig. 2** 3D inviscid Taylor–Green problem: temporal evolution of kinetic energy (left) and kinetic energy dissipation rate (right) for increasing resolution, reprinted with permission from Fehn et al. (2022)

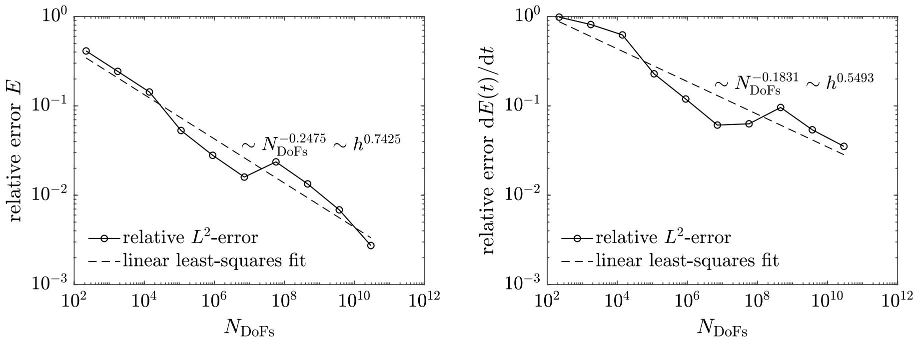


**Fig. 3** 3D inviscid Taylor–Green problem: relative $L^2$-errors over time of kinetic energy $E$ (left) and kinetic energy dissipation rate $\mathrm{d}E(t)/\mathrm{d}t$ (right) for resolutions of $8^3$ to $4096^3$ (measured against resolution of $8192^3$), reprinted with permission from Fehn et al. (2022)

Remarkably, a stable numerical solution has been obtained in all cases despite the circumstance that the result (32) is not strictly dissipative. In terms of anomalous dissipation of energy, these numerical results suggest grid-convergence to a solution with non-vanishing dissipation rate. These results are novel compared to studies reporting nonlinear blowup (Chapelier et al. 2012; Winters et al. 2018) or studies preferring exactly energy-conserving numerical solutions (Shu et al. 2005; Schroeder 2019; Lehmkuhl et al. 2019; Coppola et al. 2019) for this infinite-Re flow problem with laminar-turbulent transition. For a more quantitative assessment, Fig. 3 shows the relative $L^2$-errors over time (see Fehn et al. (2022) for a definition) of all the coarse resolution simulations against the finest resolution taken as reference. Figure 3 indicates a clear convergence trend to the fine resolution simulation at a relatively slow rate of convergence $< 1$, a topic we want to analyze and discuss in more detail in Sect. 6. We note that this convergence trend is obtained by purely numerical dissipation mechanisms, i.e. there is no physically motivated turbulence model involved in the simulations.

## 6 On Achievable Convergence Rates for the Kinetic Energy in Turbulent Flows

While we have mentioned above that stabilized finite element methods principally preserve high-order of accuracy of the numerical method when applied to problems with smooth solutions (Burman and Fernández 2007; García-Archilla et al. 2021), an important question is which rate of convergence we may expect for practical turbulence simulations, problems for which the solution can be considered non-smooth. (Guermond et al. 2004, Section 5.3.3) argue that "If one really wants to use spectral methods to do LES and if one really expects to achieve spectral accuracy, then the cutoff wavenumber should be chosen large enough for the large scales to be resolved.". As we discuss in this section, this expectation about high-order or spectral convergence in under-resolved turbulence simulations might be too optimistic. For the kinetic energy as the quantity of interest, we argue that one needs to resolve (almost) the whole spectrum of scales to observe high-order or spectral convergence and that LES is by definition in

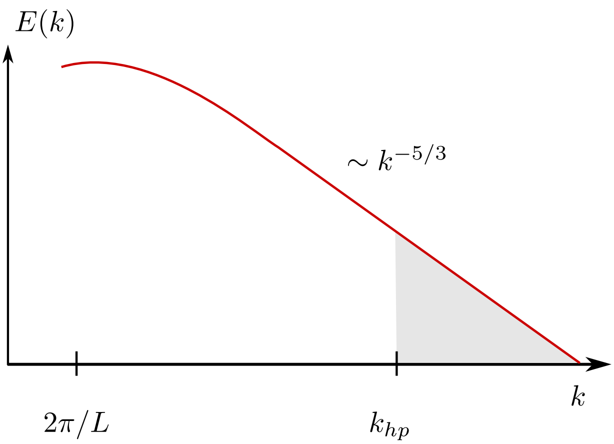


**Fig. 4** Kolmogorov energy spectrum $E(k) \sim k^{-5/3}$ (log-log plot) for infinite Reynolds number with illustration of unresolved scales (grey area)

conflict with high-order convergence. A recent work by García-Archilla et al. (2021) analyzed the theoretical rate of convergence in the kinetic energy for main types of finite element discretization methods for the incompressible Navier–Stokes equations in a high-Reynolds-number scenario (assuming sufficient regularity). To complement these results, we want to discuss an idea proposed independently in Burman (2007), Fehn and Kronbichler (2023) that aims to explain why the convergence rate in the kinetic energy is severely limited (essentially smaller than one independently of the formal approximation order of a high-order discretization scheme) if the regularity of the solution is typical of high-Re number or inviscid flows, which are by nature turbulent. By *turbulent*, we mean a solution involving a spectrum of spatial scales significantly exceeding the numerical resolution limit. For this purpose, we assume the kinetic energy in spectral space to follow the classical $-5/3$ decay (see Fig. 4) according to Kolmogorov's statistical theory of turbulence, a model we consider sufficient for this consideration. The argument based on the spectral decay of the kinetic energy as provided below has already been presented in (Burman 2007, Section 4) for flows with *multiscale features*, with the difference that the splitting into physical dissipation and *artificial dissipation* as in (Burman 2007, Equations 17 and 18) appears somewhat inappropriate for the case of three-dimensional turbulence with anomalous/inertial dissipation in the inviscid limit discussed here (where numerical dissipation is not *artificial* but instead aims to predict physical dissipation).

We describe the resolution limit of an *hp*-finite element discretization scheme by the wavenumber $k_{hp} < k_{hp,\mathrm{Ny}} := 2\pi/(\lambda_{\mathrm{Ny}}(h,p))$. The wave length $\lambda_{\mathrm{Ny}}(h,p)$ describes the minimum wavelength as two times the distance between discretization points and takes into account both the element length $h$ as well as the degree $p$ of the polynomial approximation on an element. The Nyquist wavenumber is typically written as $\lambda_{\mathrm{Ny}}(h,p) = 2\,h/p$ or $2\,h/(p+1)$ (typically, the former for continuous and the latter for discontinuous Galerkin schemes). Evidence for $k_{hp} < k_{hp,\mathrm{Ny}}$ is given by dispersion-dissipation analysis, see e.g. Moura et al. (2017) for upwind-like DG methods, where $k_{hp}$ comes closer to $k_{hp,\mathrm{Ny}}$ for higher polynomial degrees $p$ (i.e. high-order methods can be expected to be more accurate for the same number of unknowns). Then, the energy that can not be resolved by a discretization scheme is given as

$$E_{\text{trunc}} = \int_{k_{hp}}^{\infty} C\epsilon^{2/3} k^{-5/3} \mathrm{d}k > \int_{k_{hp,\mathrm{Ny}}}^{\infty} C\epsilon^{2/3} k^{-5/3} \mathrm{d}k$$
$$= 3/2\ C\epsilon^{2/3} k_{hp,\mathrm{Ny}}^{-2/3} = 3/2\ C\epsilon^{2/3} \left(\frac{2h/p}{2\pi}\right)^{2/3} .$$

Similarly, the overall energy with integral length scale $L$ is approximately given as

$$E = \int_{2\pi/L}^{\infty} C\epsilon^{2/3} k^{-5/3} \mathrm{d}k = 3/2\ C\epsilon^{2/3} \left(\frac{L}{2\pi}\right)^{2/3} .$$

It follows that the interpolation error of a finite element approximation in the kinetic energy is

$$\frac{E - E_{hp}}{E} = \frac{E_{\text{trunc}}}{E} \sim \left(\frac{2h/p}{L}\right)^{2/3} \sim (h/p)^{2/3} .$$

From this result one might conjecture that the $h$-convergence rate in the kinetic energy is limited by 2/3 (for a Kolmogorov spectrum of slope $-5/3$) in the limit $\mathrm{Re} \rightarrow \infty$, and also for all finite Reynolds numbers large enough so that the Kolmogorov length scale $\eta \ll \lambda_{\mathrm{Ny}}$. The argument presented here might contribute to the question “of whether or not there is a robust method with optimal convergence order for the kinetic energy” raised in García-Archilla et al. (2021). Even if such a method with robust error estimates can be constructed (under the assumption of sufficient regularity of the solution), the rate of convergence has to be expected to be $< 1$ even for formally high-order methods in the *turbulent* regime, i.e. in a regime where asymptotic rates of convergence cannot be reached due to the broad spectrum of spatial scales.

The above theoretical estimate could explain the numerical results in Fehn et al. (2022) for the 3D inviscid Taylor–Green problem, where the experimentally determined average convergence rate in the kinetic energy as an $L^2$-error over time (see Fig. 3) is close to the theoretical value of 2/3 estimated above.

## 7 Discussion and Open Questions

According to the argumentation in this work, the imagination that the convective term transports energy while energy can only be dissipated by the viscous term, does not hold in the inviscid limit. The same is the case when approaching high-Reynolds number flows numerically, for which a numerical simulation of finite resolution can be though of as behaving like an inviscid flow. In a numerical simulation, the required physical dissipation can not be represented by the numerical discretization of the viscous term. A discretization scheme with dissipation mechanisms stemming from the nonlinear convective term (or from additional variational stabilization terms) is in agreement with the continuous problem, which may in fact dissipate energy in the Euler limit. Following the technique known as implicit large-eddy simulation, a consistent numerical method for incompressible turbulent flow problems qualifies as physics-compatible if it conserves mass and is energy-stable with numerical mechanisms of dissipation to suppress high-wavenumber content, while a discretely energy-conserving method (without additional dissipation models) is per se not physics-compatible. The metric of *discrete conservation of energy* is not suitable to judge the accuracy of discretization schemes for turbulent flows, because:

- The metric is insensitive to the spatial resolution. Arbitrarily coarse spatial resolutions can achieve exact conservation of energy and, therefore, vanishing errors according to this metric, but it is generally accepted that coarse spatial resolutions yield poor approximations with large “true errors” (e.g. as compared to experimental evidence).
- Energy-conserving schemes cause thermalization, the solution is polluted by small-scale noise and an unphysical equipartitioning of energy across Fourier modes occurs. Quantities such as the maximum vorticity or the enstrophy (used to investigate finite-time singularities) get, therefore, falsified by the use of energy-conserving schemes. This can be expected to happen prior to the formation of potential finite-time singularities, i.e. at a time where the solution might still be expected to conserve energy from a theoretical perspective. The conception that energy-conserving schemes form a discretization scheme yielding a “clean solution” to investigate finite-time singularities lacks justification.

The metric of global energy-stability appears to be an established concept to assess the suitability of numerical methods for under-resolved turbulence simulations of incompressible flows. Mathematically, the proof of energy-stability for practical discretization schemes often consists of proving an energy-conserving behavior of a preliminary scheme as a first step, which then needs to be equipped with additional explicit or implicit mechanisms of dissipation. However, this procedure might appear artificial since exact conservation of energy is essentially based on the existence of a strong solution. Against the phenomenon of anomalous energy dissipation, it appears legitimate to raise the question whether there are other mathematical concepts that do not rely on this separation of concerns. Generally, one might formulate the goal that discretization schemes should be able to converge to *dissipative weak solutions*, in the sense that results for *computable quantities* in turbulent flows are in agreement with physical observations. It appears to be an open question whether the mathematical criteria for numerical discretization schemes should be formulated globally (in the sense of an integral over the domain), or locally. Hoffman et al. (2015) investigated a local version of energy stability in the sense of Duchon and Robert (2000). It appears to be an open question to which extent such statements carry more information than the global statements on energy stability. More research in this direction could bring further insight. In particular, we want to raise the question to which extent established global statements (e.g. on energy-stability) should be replaced and/or extended by local statements. Concepts of monotonicity and positivity of certain quantities appear to be established in the compressible Navier–Stokes/Euler community with *strong* singularities in the form of shocks. Is there an analogy for incompressible Navier–Stokes/Euler flows with *turbulent* singularities? Mathematically, a challenge appears to be proving convergence of discretization schemes such as the ones presented here to dissipative weak Euler solutions. Recently, the work by Lukáčová-Medvid’ová and Öffner (2023) addressed this problem for the compressible regime, and the work by Hajduk et al. (2024) for the incompressible regime. Likewise, higher-order numerical methods capable of reproducing the relevant invariant domains near shocks of the compressible Euler equations based on convex-limiting techniques have been developed (Guermond et al. 2018; Kuzmin 2020). Theoretically, an open question appears to be whether wild weak solutions (where current theoretical results point to the existence of infinitely many weak solutions) collapse to the same solution in terms of *computable quantities* of a turbulent flow, which are averaged in space and/or time, see also the recent essay by Eyink (2024).

In the present work, we discussed the three categories of $H^1$, $L^2$, and $H$(div)-conforming finite element methods for large-eddy simulation w.r.t. properties such as consistency, mass conservation, and energy-stability (or dissipation mechanisms). Dissipation mechanisms can be realized by volume and surface terms in the weak form. The incompressible Navier–Stokes equations with the continuity equation being as simple as $\nabla \cdot \boldsymbol{u} = 0$ allow for volume dissipation via penalization of the divergence-free constraint. This is not trivially applicable to the compressible Navier–Stokes equations, where volume dissipation mechanisms originating from explicit LES techniques have been explored frequently in the past (Collis and Chang 2002; Chapelier et al. 2016; Flad and Gassner 2017; Manzanero et al. 2020). Stabilization via orthogonal sub-scales is a volume dissipation mechanism preserving consistency of the method. Upwinding is a classical surface dissipation mechanism for $L^2$ and $H$(div)-conforming methods. Gradient-jump penalty can be described as a (surface) dissipation mechanism that appears generic with respect to the category of function space ($H^1$, $L^2$, $H$(div)) and with respect to the PDE model problem (e.g. incompressible vs. compressible flows), and therefore deserves further attention in our opinion. For certain high-order finite element methods, the interplay of both volume and surface dissipation mechanisms might be important to qualify as good implicit LES technique over a wide range of polynomial degrees $p$ of the shape functions. While the focus of the present work is on three-dimensional turbulent flows, we note that the recent study by von Wahl and Scott (2024) argues that numerical dissipation is also beneficial for two-dimensional flows (theoretically conserving energy in the Euler limit) to improve accuracy at coarse resolutions, while energy-conserving schemes would require very fine resolutions to achieve accurate results.

We also discussed that certain stabilizations would require space-time formulations, while a method-of-lines approach is often preferred in practical implementations for reasons of computational efficiency. For incompressible flows, so-called projection (or operator-splitting or fractional-step) methods are often used as efficient temporal discretization schemes (Kronbichler et al. 2021; Moxey et al. 2020; Fischer et al. 2022). Fehn et al. (2018), Fehn (2021) developed projection methods for stabilized $L^2$-conforming methods as discussed in this work. Piatkowski (2019) developed projection methods for $H$(div)-conforming methods. For $H^1$-conforming discretizations with symmetric gradient-jump stabilization, algebraic splitting methods (i.e. forming the pressure Poisson operator on the algebraic level after discretization in space) were developed in Burman et al. (2017), and operator-splitting methods with explicit treatment of viscous, convective, and stabilization terms (leading to a super-linear Courant condition) in Burman et al. (2023). Codina (2001) discusses projection methods for $H^1$-conforming methods with orthogonal sub-scales stabilization. The development of projection methods for $H^1$-conforming methods with residual-based stabilization (SUPG or variational multiscale) appears to be an open issue. State-of-the-art implementations of operator-splitting methods based on $H^1$-conforming discretizations (Moxey et al. 2020; Fischer et al. 2022) typically use filtering or spectral-vanishing viscosity methods to realize dissipative mechanisms rather than stabilized finite element methods.

Due to the rich spectrum of scales in turbulent flows, we expect the spatial convergence rate to be severly limited in under-resolved turbulence simulations. Linear dispersion–diffusion analysis is typically used to explain the cutoff behavior in spectral space of numerical schemes for large-eddy simulation, from which motivation is deduced for high-order numerical methods due to an improved numerical resolution limit $k_{hp}$ closer to $k_{hp,\mathrm{Ny}}$ than for low-order methods. The pre-conception that upwind schemes are inferior in accuracy compared to central fluxes or symmetric schemes seems to originate from low-order methods and seems to lack justification for high-order methods. In the limit of very high polynomial degrees, pure upwind stabilization might not be sufficient as dissipation mechanism. This supports the perspective that high-order methods provide the flexibility to choose the polynomial degree in a way that the dispersion-dissipation behavior appears optimal as an implicit LES technique, where intermediate polynomial degrees of $p = 2, ..., 7$ have been identified most favorable in the literature in this regard. Despite the limitations in spatial convergence rates in a fully turbulent state with rough solution, we consider the consistency and high-order capability of numerical discretization schemes (for problems with smooth solution) an important property. The main reason for this is the desire for parameter-free computational fluid dynamics solvers, which can be applied without a priori knowledge of a certain flow problem, which might involve regions of smooth solutions, laminar-turbulent transition and regions of fully turbulent, rough solutions. We think that comparative studies (done in a blinded setup) will help the LES community to identify the most promising techniques among the great variety of implicit and explicit LES approaches.

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## Declarations

**Conflict of interest** The authors have no Conflict of interest as defined by Springer, or other interests that might be perceived to influence the results and/or discussion reported in this paper.



## References

Ahmed, N., Rebollo, T.C., John, V., Rubino, S.: A review of variational multiscale methods for the simulation of turbulent incompressible flows. Arch. Comput. Methods Eng. **24**(1), 115–164 (2017)

Akbas, M., Linke, A., Rebholz, L.G., Schroeder, P.W.: The analogue of grad-div stabilization in DG methods for incompressible flows: Limiting behavior and extension to tensor-product meshes. Comput. Methods Appl. Mech. Eng. **341**, 917–938 (2018). https://doi.org/10.1016/j.cma.2018.07.019

Arnold, D.N.: An interior penalty finite element method with discontinuous elements. SIAM J. Numer. Anal. **19**(4), 742–760 (1982). https://doi.org/10.1137/0719052

Bassi, F., Botti, L., Colombo, A., Crivellini, A., Ghidoni, A., Massa, F.: On the development of an implicit high-order discontinuous Galerkin method for DNS and implicit LES of turbulent flows. Eur. J. Mech.- B/Fluids 55 Part **2**, 367–379 (2016)
Bassi, F., Crivellini, A., Pietro, D.D., Rebay, S.: An artificial compressibility flux for the discontinuous Galerkin solution of the incompressible Navier-Stokes equations. J. Comput. Phys. **218**(2), 794–815 (2006). https://doi.org/10.1016/j.jcp.2006.03.006
Bazilevs, Y., Calo, V., Cottrell, J., Hughes, T., Reali, A., Scovazzi, G.: Variational multiscale residual-based turbulence modeling for large eddy simulation of incompressible flows. Comput. Methods Appl. Mech. Eng. **197**(1–4), 173–201 (2007)
Beale, J.T., Kato, T., Majda, A.: Remarks on the breakdown of smooth solutions for the 3-D Euler equations. Commun. Math. Phys. **94**(1), 61–66 (1984). https://doi.org/10.1007/BF01212349
Boris, J.P., Grinstein, F.F., Oran, E.S., Kolbe, R.L.: New insights into large eddy simulation. Fluid Dyn. Res. **10**(4–6), 199 (1992). https://doi.org/10.1016/0169-5983(92)90023-P
Bouffanais, R.: Advances and challenges of applied large-eddy simulation. Comput. & Fluids **39**(5), 735–738 (2010). https://doi.org/10.1016/j.compfluid.2009.12.003
Braack, M., Burman, E.: Local projection stabilization for the Oseen problem and its interpretation as a variational multiscale method. SIAM J. Numer. Anal. **43**(6), 2544–2566 (2006). https://doi.org/10.1137/050631227
Braack, M., Lube, G.: Finite elements with local projection stabilization for incompressible flow problems, Journal of Computational Mathematics (2009) 116–147
Brooks, A.N., Hughes, T.J.: Streamline upwind/Petrov-Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations. Comput. Methods Appl. Mech. Eng. **32**(1), 199–259 (1982). https://doi.org/10.1016/0045-7825(82)90071-8
Buckmaster, T., De Lellis, C., Székelyhidi, L., Jr., Vicol, V.: Onsager's conjecture for admissible weak solutions. Commun. Pure Appl. Math. **72**(2), 229–274 (2018). https://doi.org/10.1002/cpa.21781
Burman, E.: A unified analysis for conforming and nonconforming stabilized finite element methods using interior penalty. SIAM J. Numer. Anal. **43**(5), 2012–2033 (2005). https://doi.org/10.1137/S0036142903437374
Burman, E.: Interior penalty variational multiscale method for the incompressible Navier-Stokes equation: Monitoring artificial dissipation. Comput. Methods Appl. Mech. Eng. **196**(41), 4045–4058 (2007). https://doi.org/10.1016/j.cma.2007.03.025
Burman, E., Ern, A., Fernández, M.A.: Fractional-step methods and finite elements with symmetric stabilization for the transient Oseen problem. ESAIM: Math. Modell. Numer. Anal. **51**(2), 487–507 (2017). https://doi.org/10.1051/m2an/2016028
Burman, E., Fernández, M.A.: Continuous interior penalty finite element method for the time-dependent navier-stokes equations: space discretization and convergence. Numerische Mathematik **107**, 39–77 (2007)
Burman, E., Garg, D., Guzman, J.: Implicit-explicit time discretization for Oseen's equation at high Reynolds number with application to fractional step methods. SIAM J. Numer. Anal. **61**(6), 2859–2886 (2023). https://doi.org/10.1137/23M1547573
Burman, E., Hansbo, P.: A stabilized non-conforming finite element method for incompressible flow. Comput. Methods Appl. Mech. Eng. **195**(23), 2881–2899 (2006)
Chapelier, J.-B., Plata, M.D.L.L., Lamballais, E.: Development of a multiscale LES model in the context of a modal discontinuous Galerkin method. Comput. Methods Appl. Mech. Eng. **307**, 275–299 (2016). https://doi.org/10.1016/j.cma.2016.04.031
Chapelier, J.-B., De La Llave Plata, M., Renac, F.: Inviscid and viscous simulations of the Taylor–Green vortex flow using a modal discontinuous Galerkin approach, in: 42nd AIAA Fluid Dynamics Conference and Exhibit, 2012, p. 3073. https://doi.org/10.2514/6.2012-3073
Charnyi, S., Heister, T., Olshanskii, M.A., Rebholz, L.G.: On conservation laws of Navier-Stokes Galerkin discretizations. J. Comput. Phys. **337**, 289–308 (2017). https://doi.org/10.1016/j.jcp.2017.02.039
Chen, J., Hou, T. Y.: Stable nearly self-similar blowup of the 2D Boussinesq and 3D Euler equations with smooth data, arXiv preprint arXiv:2210.07191 (2022)
Cichowlas, C., Bonaïti, P., Debbasch, F., Brachet, M.: Effective dissipation and turbulence in spectrally truncated Euler flows. Phys. Rev. Lett. **95**, 264502 (2005). https://doi.org/10.1103/PhysRevLett.95.264502
Cockburn, B., Kanschat, G., Schötzau, D.: A note on discontinuous Galerkin divergence-free solutions of the Navier-Stokes equations. J. Sci. Comput. **31**(1), 61–73 (2007). https://doi.org/10.1007/s10915-006-9107-7
Cockburn, B., Kanschat, G., Schötzau, D.: An equal-order DG method for the incompressible Navier-Stokes equations. J. Sci. Comput. **40**(1–3), 188–210 (2009)

Codina, R.: Pressure stability in fractional step finite element methods for incompressible flows. J. Comput. Phys. **170**(1), 112–140 (2001). https://doi.org/10.1006/jcph.2001.6725
Codina, R.: Stabilized finite element approximation of transient incompressible flows using orthogonal subscales. Comput. Methods Appl. Mech. Eng. **191**(39–40), 4295–4321 (2002)
Collis, S.S., Chang, Y.: The dg/vms method for unified turbulence simulation. AIAA paper **3124**, 2002 (2002)
Constantin, P., W. E E. S.: Onsager's conjecture on the energy conservation for solutions of Euler's equation. Commun. Math. Phys. **165**(1), 207 (1994)
Coppola, G., Capuano, F., de Luca, L.: Discrete Energy-Conservation Properties in the Numerical Simulation of the Navier-Stokes Equations. Appl. Mech. Rev. **71**(1), 010803 (2019). https://doi.org/10.1115/1.4042820
Dairay, T., Lamballais, E., Laizet, S., Vassilicos, J.C.: Numerical dissipation vs. subgrid-scale modelling for large eddy simulation. J. Comput. Phys. **337**, 252–274 (2017). https://doi.org/10.1016/j.jcp.2017.02.035
Douglas, J., Dupont, T.: Interior penalty procedures for elliptic and parabolic galerkin methods, in: Computing Methods in Applied Sciences: Second International Symposium December 15–19, 1975, Springer, pp. 207–216 (1976)
Drikakis, D.: Advances in turbulent flow computations using high-resolution methods. Prog. Aerosp. Sci **39**(6), 405–424 (2003). https://doi.org/10.1016/S0376-0421(03)00075-7
Drikakis, D., Hahn, M., Mosedale, A., Thornber, B.: Large eddy simulation using high-resolution and high-order methods. Philos. Trans. Royal Soc. A: Math. Phys. Eng. Sci. **367**(1899), 2985–2997 (2009). https://doi.org/10.1098/rsta.2008.0312
Dubrulle, B.: Beyond Kolmogorov cascades. J. Fluid Mech. **867**, P1 (2019). https://doi.org/10.1017/jfm.2019.98
Duchon, J., Robert, R.: Inertial energy dissipation for weak solutions of incompressible Euler and Navier-Stokes equations. Nonlinearity **13**(1), 249–255 (2000). https://doi.org/10.1088/0951-7715/13/1/312
Duponcheel, M., Orlandi, P., Winckelmans, G.: Time-reversibility of the Euler equations as a benchmark for energy conserving schemes. J. Comput. Phys. **227**(19), 8736–8752 (2008). https://doi.org/10.1016/j.jcp.2008.06.020
Eyink, G.L.: Energy dissipation without viscosity in ideal hydrodynamics I. Fourier Anal. Local Energy Trans. Phys. D: Nonlinear Phenomena **78**(3), 222–240 (1994). https://doi.org/10.1016/0167-2789(94)90117-1
Eyink, G.L.: Dissipative anomalies in singular Euler flows. Phys. D: Nonlinear Phenomena **237**(14), 1956–1968 (2008). https://doi.org/10.1016/j.physd.2008.02.005. (**euler Equations: 250 Years On.**)
Eyink, G.: Onsager's 'ideal turbulence' theory. J. Fluid Mech. **988**, P1 (2024). https://doi.org/10.1017/jfm.2024.415
Eyink, G.L., Sreenivasan, K.R.: Onsager and the theory of hydrodynamic turbulence. Rev. Mod. Phys. **78**, 87–135 (2006). https://doi.org/10.1103/RevModPhys.78.87
Fehn, N., Kronbichler, M., Munch, P., Wall, W.A.: Numerical evidence of anomalous energy dissipation in incompressible Euler flows: towards grid-converged results for the inviscid Taylor-Green problem. J. Fluid Mech. **932**, A40 (2022). https://doi.org/10.1017/jfm.2021.1003
Fehn, N., Wall, W.A., Kronbichler, M.: Robust and efficient discontinuous Galerkin methods for under-resolved turbulent incompressible flows. J. Comput. Phys. **372**, 667–693 (2018). https://doi.org/10.1016/j.jcp.2018.06.037
Fehn, N., Kronbichler, M., Lehrenfeld, C., Lube, G., Schroeder, P.W.: High-order DG solvers for under-resolved turbulent incompressible flows: A comparison of $L^2$ and $H(div)$ methods. Int. J. Numer. Meth. Fluids **91**(11), 533–556 (2019). https://doi.org/10.1002/fld.4763
Fehn, N., Kronbichler, M.: Dissipation-aware discontinuous Galerkin methods for incompressible turbulent flows, in: Proceedings of the 14th Ercoftac Symposium on Engineering, Turbulence, Modeling and Measurements (ETMM14), Barcelona, Spain, 2023
Fehn, N.: Robust and Efficient Discontinuous Galerkin Methods for Incompressible Flows, Phd thesis, Technische Universität München (2021)
Ferrer, E.: An interior penalty stabilised incompressible discontinuous Galerkin-Fourier solver for implicit large eddy simulations. J. Comput. Phys. **348**, 754–775 (2017). https://doi.org/10.1016/j.jcp.2017.07.049
Ferrer, E.: A high order discontinuous Galerkin–Fourier incompressible 3D Navier–Stokes solver with rotating sliding meshes for simulating cross-flow turbines, Ph.D. thesis, University of Oxford (2012)
Ferziger, J.H.: Direct and large eddy simulation of turbulence. Trans. Jpn. Soc. Mech. Eng. Ser. B **66**(651), 2754–2763 (2000)
Ferziger, J. H. Peric, M.: Computational methods for fluid dynamics. (2002)

Fischer, P., Kerkemeier, S., Min, M., Lan, Y.-H., Phillips, M., Rathnayake, T., Merzari, E., Tomboulides, A., Karakus, A., Chalmers, N., et al.: NekRS, a GPU-accelerated spectral element Navier-Stokes solver. Parallel Comput. **114**, 102982 (2022)

Fischer, P.F., Kruse, G.W., Loth, F.: Spectral element methods for transitional flows in complex geometries. J. Sci. Comput. **17**(1–4), 81–98 (2002)

Fischer, P., Mullen, J.: Filter-based stabilization of spectral element methods. Comptes Rendus de l'Académie des Sciences - Series I - Mathematics **332**(3), 265–270 (2001). https://doi.org/10.1016/S0764-4442(00)01763-8

Flad, D., Gassner, G.: On the use of kinetic energy preserving DG-schemes for large eddy simulation. J. Comput. Phys. **350**, 782–795 (2017). https://doi.org/10.1016/j.jcp.2017.09.004

Franca, L.P., Hughes, T.J.: Two classes of mixed finite element methods. Comput. Methods Appl. Mech. Eng. **69**(1), 89–129 (1988). https://doi.org/10.1016/0045-7825(88)90168-5

Fureby, C.: Towards the use of large eddy simulation in engineering. Prog. Aerosp. Sci. **44**(6), 381–396 (2008). https://doi.org/10.1016/j.paerosci.2008.07.003. (**large Eddy Simulation - Current Capabilities and Areas of Needed Research**)

García-Archilla, B., John, V., Novo, J.: On the convergence order of the finite element error in the kinetic energy for high Reynolds number incompressible flows. Comput. Methods Appl. Mech. Eng. **385**, 114032 (2021). https://doi.org/10.1016/j.cma.2021.114032

Gassner, G.J.: A skew-symmetric discontinuous Galerkin spectral element discretization and its relation to SBP-SAT finite difference methods. SIAM J. Sci. Comput. **35**(3), A1233–A1253 (2013). https://doi.org/10.1137/120890144

Gassner, G.J., Winters, A.R., Kopriva, D.A.: Split form nodal discontinuous Galerkin schemes with summation-by-parts property for the compressible Euler equations. J. Comput. Phys. **327**, 39–66 (2016). https://doi.org/10.1016/j.jcp.2016.09.013

Georgiadis, N.J., Rizzetta, D.P., Fureby, C.: Large-eddy simulation: Current capabilities, recommended practices, and future research. AIAA J. **48**(8), 1772–1784 (2010). https://doi.org/10.2514/1.J050232

Gravemeier, V.: The variational multiscale method for laminar and turbulent flow. Arch. Comput. Methods Eng. **13**(2), 249 (2006)

Gravemeier, V., Gee, M.W., Kronbichler, M., Wall, W.A.: An algebraic variational multiscale-multigrid method for large eddy simulation of turbulent flow. Comput. Methods Appl. Mech. Eng. **199**(13), 853–864 (2010). https://doi.org/10.1016/j.cma.2009.05.017. (**turbulence Modeling for Large Eddy Simulations**)

Guermond, J.-L.: On the use of the notion of suitable weak solutions in CFD. Int. J. Numer. Methods Fluids **57**(9), 1153–1170 (2008). https://doi.org/10.1002/fld.1853

Guermond, J.-L., Nazarov, M., Popov, B., Tomas, I.: Second-order invariant domain preserving approximation of the Euler equations using convex limiting. SIAM J. Sci. Comput. **40**(5), A3211–A3239 (2018). https://doi.org/10.1137/17m1149961

Guermond, J.-L., Oden, J.T., Prudhomme, S.: Mathematical perspectives on large eddy simulation models for turbulent flows. J. Math. Fluid Mech. **6**, 194–248 (2004)

Guermond, J.-L., Pasquetti, R., Popov, B.: From suitable weak solutions to entropy viscosity. J. Sci. Comput. **49**, 35–50 (2011). https://doi.org/10.1007/s10915-010-9445-3

Guermond, J.-L., Prudhomme, S.: On the construction of suitable solutions to the Navier-Stokes equations and questions regarding the definition of large eddy simulation. Physica D: Nonlinear Phenomena **207**(1), 64–78 (2005). https://doi.org/10.1016/j.physd.2005.05.014

Guzmán, J., Shu, C.-W., Sequeira, F.A.: H(div) conforming and DG methods for incompressible Euler's equations. IMA J. Numer. Anal. **37**(4), 1733–1771 (2016). https://doi.org/10.1093/imanum/drw054

Hajduk, H., Kuzmin, D., Lube, G.: Öffner, Locally energy-stable finite element schemes for incompressible flow problems: Design and analysis for equal-order interpolations in preparation (2024)

Hesthaven, J.S., Warburton, T.: Nodal discontinuous Galerkin methods: algorithms, analysis, and applications. Springer (2008). https://doi.org/10.1007/978-0-387-72067-8

Hoffman, J.: Energy stability analysis of turbulent incompressible flow based on the triple decomposition of the velocity gradient tensor. Phys. Fluids **33**(8), 081707 (2021)

Hoffman, J., Jansson, J., Jansson, N., De Abreu, R.V.: Towards a parameter-free method for high Reynolds number turbulent flow simulation based on adaptive finite element approximation. Comput. Methods Appl. Mech. Eng. **288**, 60–74 (2015)

Hoffman, J., Jansson, J., Vilela De Abreu, R.: Adaptive modeling of turbulent flow with residual based turbulent kinetic energy dissipation. Comput. Methods Appl. Mech. Eng. **200**(37), 2758–2767 (2011). https://doi.org/10.1016/j.cma.2010.11.016. (**special Issue on Modeling Error Estimation and Adaptive Modeling**)

Hoffman, J., Johnson, C.: A new approach to computational turbulence modeling. Comput. Methods Appl. Mech. Eng. **195**(23–24), 2865–2880 (2006)
Hoffman, J., Johnson, C.: Computational Turbulent Incompressible Flow. Springer (2007)
Hoffman, J., Johnson, C.: Blow up of incompressible Euler solutions. BIT Numer. Math. **48**(2), 285–307 (2008)
Hoffman, J., Johnson, C.: Resolution of d'Alembert's paradox. J. Math. fluid Mech. **12**(3), 321–334 (2010)
Hughes, T.J., Feijóo, G.R., Mazzei, L., Quincy, J.-B.: The variational multiscale method-a paradigm for computational mechanics. Comput. Methods Appl. Mech. Eng. **166**(1–2), 3–24 (1998)
Hughes, T.J., Franca, L.P., Balestra, M.: A new finite element formulation for computational fluid dynamics: V Circumventing the Babuška-Brezzi condition a stable Petrov-Galerkin formulation of the Stokes problem accommodating equal-order interpolations. Comput. Methods Appl. Mech. Eng. **59**(1), 85–99 (1986). https://doi.org/10.1016/0045-7825(86)90025-3
Hughes, T.J., Mazzei, L., Jansen, K.E.: Large eddy simulation and the variational multiscale method. Comput. Visualization Sci. **3**, 47–59 (2000)
Hughes, T.J.R., Scovazzi, G., Franca, L.P.: Multiscale and Stabilized Methods. John Wiley (2017)
Isett, P.: A proof of Onsager's conjecture. Annal. Math. **188**(3), 871–963 (2018)
John, V., Linke, A., Merdon, C., Neilan, M., Rebholz, L.G.: On the divergence constraint in mixed finite element methods for incompressible flows. SIAM Rev. **59**(3), 492–544 (2017). https://doi.org/10.1137/15M1047696
Johnson, P.L., Wilczek, M.: Multiscale velocity gradients in turbulence. Annu. Rev. Fluid Mech. **56**, 463–490 (2024)
Kaneda, Y., Ishihara, T., Yokokawa, M., Itakura, K., Uno, A.: Energy dissipation rate and energy spectrum in high resolution direct numerical simulations of turbulence in a periodic box. Phys. Fluids **15**(2), L21–L24 (2003). https://doi.org/10.1063/1.1539855
Karamanos, G.S., Karniadakis, G.E.: A spectral vanishing viscosity method for large-eddy simulations. J. Comput. Phys. **163**(1), 22–50 (2000)
Kirby, R.M., Karniadakis, G.E.: Coarse resolution turbulence simulations with spectral vanishing viscosity-large-eddy simulations (SVV-LES). J. Fluids Eng. **124**(4), 886–891 (2002)
Kolluru, S.S.V., Sharma, P., Pandit, R.: Insights from a pseudospectral study of a potentially singular solution of the three-dimensional axisymmetric incompressible Euler equation. Phys. Rev. E **105**, 065107 (2022). https://doi.org/10.1103/PhysRevE.105.065107
Krank, B., Fehn, N., Wall, W., Kronbichler, M.: A high-order semi-explicit discontinuous Galerkin solver for 3D incompressible flow with application to DNS and LES of turbulent channel flow. J. Comput. Phys. **348**, 634–659 (2017). https://doi.org/10.1016/j.jcp.2017.07.039
Kronbichler, M., Fehn, N., Munch, P., Bergbauer, M., Wichmann, K.-R., Geitner, C., Allalen, M., Schulz, M., Wall, W. A.: A next-generation discontinuous Galerkin fluid dynamics solver with application to high-resolution lung airflow simulations, in: Proceedings of the International Conference for High Performance Computing, Networking, Storage and Analysis, SC'21, Association for Computing Machinery (ACM), St. Louis, MO, USA, 2021, pp. 1–15
Kuzmin, D.: Monolithic convex limiting for continuous finite element discretizations of hyperbolic conservation laws. Comput. Methods Appl. Mech. Eng. **361**, 112804 (2020). https://doi.org/10.1016/j.cma.2019.112804
Küchler, C., Bewley, G.P., Bodenschatz, E.: Universal velocity statistics in decaying turbulence. Phys. Rev. Lett. **131**, 024001 (2023). https://doi.org/10.1103/PhysRevLett.131.024001
Lee, S., Lee, C.: Identification of a particle collision as a finite-time blowup in turbulence. Sci. Rep. **13**(1), 181 (2023)
Lehmkuhl, O., Houzeaux, G., Owen, H., Chrysokentis, G., Rodriguez, I.: A low-dissipation finite element scheme for scale resolving simulations of turbulent flows. J. Comput. Phys. **390**, 51–65 (2019). https://doi.org/10.1016/j.jcp.2019.04.004
Lehrenfeld, C., Schöberl, J.: High order exactly divergence-free hybrid discontinuous Galerkin methods for unsteady incompressible flows. Comput. Methods Appl. Mech. Eng. **307**, 339–361 (2016). https://doi.org/10.1016/j.cma.2016.04.025
De Lellis, C., Székelyhidi, L., Jr.: Dissipative Euler flows and Onsager's conjecture. J. Eur. Math. Soc. **16**(7), 1467–1505 (2014)
Leonard, A.: Energy cascade in large-eddy simulations of turbulent fluid flows. In: Frenkiel, F., Munn, R. (eds.) Turbulent Diffusion in Environmental Pollution, Vol. 18 of Advances in Geophysics, pp. 237–248. Elsevier (1975). https://doi.org/10.1016/S0065-2687(08)60464-1
Lesieur, M., Metais, O.: New trends in large-eddy simulations of turbulence. Annual Rev. Fluid Mech. **28**(1), 45–82 (1996)

Lukáčová-Medvid'ová, M., Öffner, P.: Convergence of discontinuous Galerkin schemes for the Euler equations via dissipative weak solutions. Appl. Math. Comput. **436**, 127508 (2023). https://doi.org/10.1016/j.amc.2022.127508

Luo, G., Hou, T.Y.: Potentially singular solutions of the 3D axisymmetric Euler equations. Proc. Natl. Acad. Sci. **111**(36), 12968–12973 (2014). https://doi.org/10.1073/pnas.1405238111

Manzanero, J., Ferrer, E., Rubio, G., Valero, E.: Design of a Smagorinsky spectral vanishing viscosity turbulence model for discontinuous Galerkin methods. Comput. & Fluids **200**, 104440 (2020). https://doi.org/10.1016/j.compfluid.2020.104440

Marek, M., Tyliszczak, A., Bogusławski, A.: Large eddy simulation of incompressible free round jet with discontinuous Galerkin method. Int. J. Numer. Methods Fluids **79**(4), 164–182 (2015). https://doi.org/10.1002/fld.4043. (**fld.4043**)

Margolin, L.G., Rider, W.J.: A rationale for implicit turbulence modelling. Int. J. Numer. Methods Fluids **39**(9), 821–841 (2002). https://doi.org/10.1002/fld.331

Margolin, L.G., Rider, W.J., Grinstein, F.F.: Modeling turbulent flow with implicit LES. J. Turbulence **7**, N15 (2006). https://doi.org/10.1080/14685240500331595

Mason, P.J.: Large-eddy simulation: A critical review of the technique. Q. J. Royal Meteorol. Soc. **120**(515), 1–26 (1994). https://doi.org/10.1002/qj.49712051503

Massa, F., Ostrowski, L., Bassi, F., Rohde, C.: An artificial equation of state based Riemann solver for a discontinuous Galerkin discretization of the incompressible Navier-Stokes equations. J. Comput. Phys. **448**, 110705 (2022). https://doi.org/10.1016/j.jcp.2021.110705

Moin, P.: Advances in large eddy simulation methodology for complex flows. Int. J. Heat Fluid Flow **23**(5), 710–720 (2002). https://doi.org/10.1016/S0142-727X(02)00167-4

Montlaur, A., Fernandez-Mendez, S., Huerta, A.: Discontinuous Galerkin methods for the Stokes equations using divergence-free approximations. Int. J. Numer. Methods Fluids **57**(9), 1071–1092 (2008). https://doi.org/10.1002/fld.1716

Moura, R.C., Cassinelli, A., da Silva, A.F., Burman, E., Sherwin, S.J.: Gradient jump penalty stabilisation of spectral/hp element discretisation for under-resolved turbulence simulations. Comput. Methods Appl. Mech. Eng. **388**, 114200 (2022). https://doi.org/10.1016/j.cma.2021.114200

Moura, R., Mengaldo, G., Peiró, J., Sherwin, S.: On the eddy-resolving capability of high-order discontinuous Galerkin approaches to implicit LES / under-resolved DNS of Euler turbulence. J. Comput. Phys. **330**, 615–623 (2017). https://doi.org/10.1016/j.jcp.2016.10.056

Moxey, D., Cantwell, C.D., Bao, Y., Cassinelli, A., Castiglioni, G., Chun, S., Juda, E., Kazemi, E., Lackhove, K., Marcon, J., Mengaldo, G., Serson, D., Turner, M., Xu, H., Peiró, J., Kirby, R.M., Sherwin, S.J.: Nektar++: Enhancing the capability and application of high-fidelity spectral/hp element methods. Comput. Phys. Commun. **249**, 107110 (2020). https://doi.org/10.1016/j.cpc.2019.107110

Murugan, S.D., Frisch, U., Nazarenko, S., Besse, N., Ray, S.S.: Suppressing thermalization and constructing weak solutions in truncated inviscid equations of hydrodynamics: Lessons from the Burgers equation. Phys. Rev. Res. **2**, 033202 (2020). https://doi.org/10.1103/PhysRevResearch.2.033202

Murugan, S. D., Ray, S. S.: On the thermalization of the three-dimensional, incompressible, Galerkin-truncated Euler equation, arXiv preprint arXiv:2209.05046 (2022)

Olshanskii, M., Lube, G., Heister, T., Löwe, J.: Grad-div stabilization and subgrid pressure models for the incompressible Navier-Stokes equations. Comput. Methods Appl. Mech. Eng. **198**(49–52), 3975–3988 (2009). https://doi.org/10.1016/j.cma.2009.09.005

Onsager, L.: Statistical hydrodynamics. Il Nuovo Cimento (1943-1954) **6**, 279–287 (1949)

Pearson, B.R., Krogstad, P.-Å., van de Water, W.: Measurements of the turbulent energy dissipation rate. Phys. Fluids **14**(3), 1288–1290 (2002). https://doi.org/10.1063/1.1445422

Piatkowski, S.-M.: A spectral discontinuous Galerkin method for incompressible flow with applications to turbulence, Ph.D. thesis, Ruprecht-Karls-Universität Heidelberg (2019)

Piomelli, U.: Large-eddy simulation: achievements and challenges. Prog. Aerosp. Sci. **35**(4), 335–362 (1999). https://doi.org/10.1016/S0376-0421(98)00014-1

Piomelli, U.: Large eddy simulations in 2030 and beyond, Philosophical Transactions of the Royal Society A: Mathematical. Phys. Eng. Sci. **372**(2022), 20130320 (2014). https://doi.org/10.1098/rsta.2013.0320

Principe, J., Codina, R., Henke, F.: The dissipative structure of variational multiscale methods for incompressible flows. Comput. Meth. Appl. Mech. Eng. **199**(13), 791–801 (2010). https://doi.org/10.1016/j.cma.2008.09.007. (**turbulence Modeling for Large Eddy Simulations**)

R. Codina, J. Principe, S. Badia, Dissipative structure and long term behavior of a finite element approximation of incompressible flows with numerical subgrid scale modeling, Multiscale Methods in Computational Mechanics: Progress and Accomplishments (2011) 75–93

Rasthofer, U., Gravemeier, V.: Multifractal subgrid-scale modeling within a variational multiscale method for large-eddy simulation of turbulent flow. J. Comput. Phys. **234**, 79–107 (2013). https://doi.org/10.1016/j.jcp.2012.09.013

Rasthofer, U., Gravemeier, V.: Recent developments in variational multiscale methods for large-eddy simulation of turbulent flow. Arch. Comput. Methods Eng. **25**(3), 647–690 (2018)

Ray, S.S., Frisch, U., Nazarenko, S., Matsumoto, T.: Resonance phenomenon for the Galerkin-truncated Burgers and Euler equations. Phys. Rev. E **84**, 016301 (2011). https://doi.org/10.1103/PhysRevE.84.016301

De Rosa, L., Inversi, M.: Dissipation in Onsager's critical classes and energy conservation in $BV \cap L^\infty$ with and without boundary. Commun. Math. Phys. **405**(1), 1–34 (2024)

Röhe, L., Lube, G.: Analysis of a variational multiscale method for large-eddy simulation and its application to homogeneous isotropic turbulence. Comput. Methods Appl. Mech. Eng. **199**(37–40), 2331–2342 (2010)

Schroeder, P.W., Lube, G.: Stabilised dG-FEM for incompressible natural convection flows with boundary and moving interior layers on non-adapted meshes. J. Comput. Phys. **335**, 760–779 (2017). https://doi.org/10.1016/j.jcp.2017.01.055

Schroeder, P. W.: Robustness of high-order divergence-free finite element methods for incompressible computational fluid dynamics, Ph.D. thesis, Georg-August-Universität Göttingen (2019)

Shahbazi, K., Fischer, P.F., Ethier, C.R.: A high-order discontinuous Galerkin method for the unsteady incompressible Navier-Stokes equations. J. Comput. Phys. **222**(1), 391–407 (2007). https://doi.org/10.1016/j.jcp.2006.07.029

Shu, C.-W., Don, W.-S., Gottlieb, D., Schilling, O., Jameson, L.: Numerical convergence study of nearly incompressible, inviscid Taylor-Green vortex flow. J. Sci. Comput. **24**(1), 1–27 (2005). https://doi.org/10.1007/s10915-004-5407-y

Sreenivasan, K.R.: An update on the energy dissipation rate in isotropic turbulence. Phys. Fluids **10**(2), 528–529 (1998). https://doi.org/10.1063/1.869575

Sulem, C., Sulem, P.L., Frisch, H.: Tracing complex singularities with spectral methods. J. Comput. Phys. **50**(1), 138–161 (1983). https://doi.org/10.1016/0021-9991(83)90045-1

Taylor, G. I. (1935): Statistical theory of turbulence, Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences 151 (873) 421–444.http://www.jstor.org/stable/96557

von Wahl, H., Scott, L.R.: Reliable chaotic transition in incompressible fluid simulations. Adv. Comput. Sci. Eng. **2**(3), 202–221 (2024). https://doi.org/10.3934/acse.2024011

Wiedemann, E.: Weak-strong uniqueness in fluid dynamics, arXiv preprint arXiv:1705.04220 (2017)

Winters, A.R., Moura, R.C., Mengaldo, G., Gassner, G.J., Walch, S., Peiro, J., Sherwin, S.J.: A comparative study on polynomial dealiasing and split form discontinuous Galerkin schemes for under-resolved turbulence computations. J. Comput. Phys. **372**, 1–21 (2018). https://doi.org/10.1016/j.jcp.2018.06.016

Zhang, Y., Palha, A., Gerritsma, M., Rebholz, L.G.: A mass-, kinetic energy- and helicity-conserving mimetic dual-field discretization for three-dimensional incompressible Navier-Stokes equations, part I: Periodic domains. J. Comput. Phys. **451**, 110868 (2022). https://doi.org/10.1016/j.jcp.2021.110868

Zhiyin, Y.: Large-eddy simulation: Past, present and the future. Chin. J. Aeronaut. **28**(1), 11–24 (2015). https://doi.org/10.1016/j.cja.2014.12.007

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